Symposium on
Applied Mathematics
dedicated to the
late Prof. Dr. R. Timman
Symposium on
APPLIED MATHEMATICS
dedicated to the late
PROF. DR. R. TIMMAN
Professor Reinier Timman (1917–1975)
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APPLIED MATHEMATICS
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PROF. DR. R. TIMMAN

Delft, the Netherlands, 11-13 January 1978

Edited by
A.J. Hermans
M.W.C. Oosterveld

1978
Delft University Press
Sijthoff & Noordhoff International Publishers
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Distributed by
Sijthoff & Noordhoff International Publishers
Postbus 66
Groningen
the Netherlands

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ISBN 90 286 0448 0
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FOREWORD

Professor Reinier Timman was born on May 6th, 1917, at Den Helder, North Holland. In 1934 he started his studies in mathematics and physics at the City University of Amsterdam, where he graduated in 1938. In the years just before the Second World War, it was practically impossible to find a job in the academic field. Timman succeeded to get an appointment at Fokker aircraft industry in 1939. During the war he was mainly engaged in studies in the field of aerodynamics. He succeeded to compose a doctoral thesis on the aerodynamical forces on vibrating wings, taking into account the compressibility of the air. The results of this research have been made public after the war, when his promotion took place. Because of this work he became convinced of the necessity to combine mathematics with physics and technique. This marked the rest of his life. He was very much interested in the essence of the phenomena and he greatly considered mathematics as a useful tool to achieve this. After a short stay at the research laboratory of the B.P.M. oil company, he joined the National Aerospace Laboratory in Amsterdam. In the five years which followed, he published a great number of major contributions in different fields of aerodynamics. He also worked with several young staff members and he stimulated them in a very special way. He gave courses in applied mathematics at the Mathematical Centre in Amsterdam. Many people have been inspired by him to use mathematical methods to solve problems of a technical or physical nature.

In 1952 they appointed him a professor in pure and applied mathematics and mechanics at the Delft University of Technology. With this appointment applied mathematics was introduced in Delft. Timman started a separate curriculum in applied mathematics and introduced the degree in mathematical engineering. An important part in this educational program is the education in physical and technical sciences. At the time Timman recognized the great future of the electronic computer and therefore computer orientated disciplines were added to the program.

At present every Dutch university offers possibilities to study applied mathematics. In more than twenty years Timman guided many young people towards their engineering degree. Twenty-six doctoral promotions took place under his supervision. Timman recognized every new field of interest in applied mathematics and introduced it at an early stage.

Besides his work at the University, he found the time to serve many research institutes by means of his scientific capacities. These institutes are situated not only in The Netherlands. His contacts abroad were just as valuable to him, especially the relation he had with the D.T.N.S.R.D.C. in Washington. He thought it necessary to have a close contact with real practice to fulfil his task at the University properly.
Quite recently he became interested in problems of a more social structure. The appearance of the first report to the Club of Rome was one of the motives to start research in this field. Because of his early death, he couldn't make a major contribution in this field, although the preliminary results were very promising. Somebody else will have to finish this task. With the death of Timman on November 9th, 1975, the scientific world has lost a man of great importance. This symposium is meant to present some of the scientific results obtained by his friends and his pupils who got so much inspiration in many discussions with him.

The Editors
Ladies and Gentlemen:

It is an honour to me to welcome you here at this moment as dean of the Mathematical Department of this University.

We are glad that so many scientists and friends have come from abroad and are now here.
It is a tribute to the man who had so many scientific contacts all over the world.

Reinier Timman has had an immense influence on the development of mathematics in this country.
Educated as a pure mathematician, he devoted his work to applied mathematics. His influence cannot be measured by the number of papers he left us. He was too modest to publish every problem solved by him. However many of the papers he wrote were starting points for new developments and the ideas that they contained were elaborated by many mathematicians.

As soon as Timman obtained the fundamental features of a solution of a problem his interest switched to other kinds of mathematical, physical or technological problems.
It is nearly impossible to describe completely all the areas where he introduced new ideas or methods, as every man knows who has worked with him. Many personal friends are here to witness this.

We are glad that Mrs Timman and her children are today our guests, surrounded by so many who were often her guest.

Finally I wish you all a good symposium devoted to the fields where Timman was working.
OPENING

As vice-chancellor of Delft University of Technology it is both an honour and pleasure to me to welcome you at this symposium. We are glad and proud to see how many people have come from often far-away places to attend this meeting. This symposium is organized in memory of Prof. Timman and for us the large attendance is proof that he was held in high esteem by the scientific world.

Reinier Timman was born on the 6th of May, 1917. From his early years nothing is known officially, but he must have been a bright and keen student, as already in 1938, at the age of only 21 he got his master's degree in mathematics, with philosophy and physics, at the University of Amsterdam. Educated as a pure mathematician, he devoted his whole life to the field of applied mathematics. First in industry, with the Dutch Aircraft Manufacturer Fokker, with the Shell International Petroleum Company and with the National Aerospace Laboratory and since 1952 at our University where he stayed till his untimely death in 1975, on the 9th of November.

It is hard to give due credit to the importance of Timman's ideas and work on the development of scientific teaching and research at the Delft University of Technology. His credo is best summed up by the title of his own doctor's thesis (Delft, 1946): "On the significance of mathematics for applied scientific research". More than anyone else Timman understood the art of making the results of mathematical work comprehensible and applicable to engineers. He did this moreover in such a charming way that he found many followers, who applied his ideas in design and construction. In this respect it is difficult to underestimate his beneficial influence on our engineering departments. For his own department of mathematics, the work done by him was even more important and had much larger effects as he was the originator of education in mathematical engineering, which gave a new impetus to the engineering community in this country. Prof. Dijkman, the present dean of this department has already told you more about it.

Although having a full-time position in Delft, Timman always found time for additional activities, as an advisor or visiting professor to the Netherlands Ship Model Basin, the Delft Hydraulics Laboratories, the Dutch State Mines, the David W. Taylor Naval Ship Research and Development Center in Washington, the University of Delaware and many others. He was member of the Royal Dutch Academy of Sciences and helped greatly in the establishment of our third University of Technology in Twente. During the last years of his life he was very active with work on Club of Rome type of problems for a vast area with Rotterdam as centre. Unfortunately he was unable to complete this study.
Personally I came to know Timman in 1951 when he was also chief of the department for Applied Mathematics at the Mathematical Centre in Amsterdam. For the western part of Holland, the dunes along the North Sea Coast have always been the major source for public water supplies. For many decades, however, much more water had been abstracted, than was naturally recharged by residual rainfall, with a sharp rise of the fresh-salt water interface as unavoidable result. Clearly this could not continue and many schemes for artificial recharge with water from the river Rhine were devised. This entailed the solving of many mathematical problems for which Timman's assistance proved to be invaluable. His lectures on the mathematical theory of flow through porous media will be remembered by many waterwork engineers with great gratitude.

Ladies and gentlemen, I could say much more, but quite a number of speakers are waiting. I hereby open this symposium and I wish you pleasant and fruitful discussions.

Rector Magnificus of the University of Technology, Delft.

Prof. L. Huisman
SESSION I

Chairman: Prof. Dr. E. van Spiegel,
University of Technology, Delft
1. LOOKING BEHIND

As a salute to Rein Jan Timman, teacher, colleague, neighbour, sparring partner, I have recorded five pieces of small talk between us.

Small talk has been the pepper and the salt in our relationship. I judged that I could give by this personal approach a better idea of Rein Jan's convictions and motivations than by an analysis of his achievements.

At least, I gain two advantages by this deliberate non-learned approach.

Firstly, it will make clear from the start that Rein Jan did not create, as so many do, a distinction between his work and daily practice, between thinking and doing. His vocation was creative thinking, but his proof of it the act following the idea. Also small talk with him was daily life to some purpose.

Secondly, by proceeding as I do, I will not risk to kill my feelings by a bookkeeping exercise of his work, which would anyhow not have met with his appreciation.

2. ON PROGRESSIVE AND CONSERVATIVE FORCES

-Mais, mon père, me dit-il, moi aussi je souhaite le bonheur des âmes.

-Non, mon ami, tu souhaites leur soumission.

André Gide 2)

We were discussing the balance between progressive and conservative forces in society. With a number of learned friends we had written a most subjective pamphlet coined "Work for the Future", which called for radical changes in quite some fields of human activity. We backed this pamphlet emotionally. We welcomed it as a political act. It opposed the blind conservatism of those who had heard of Malthus' thesis...
before, as it should do. But, it had left Rein Jan (and me) with some doubts and some problems.

The study manifested a serious lack of background knowledge, so that it could easily be attacked on single issues. That happened indeed, but not so much because of the weakness of some of its arguments. The simple fact that some of the writers were known at large as being conservative, some as being progressive, made it impossible for any politician to react without reservations. We suffered from the common human error to identify individuals with the circles in which they are frequently seen.

At that stage, Rein Jan and I had our brief encounter, the argument of which ran as follows.

- The comments on "Work for the Future" point to a paradox. Our institutions, be it nations, churches or universities, face a lasting dilemma. Non-conformism and diversity add to creativity and innovation, but they foster also instability, rebellion and revolt in the niches of the society. Conformism and dogmatism repress, at least for some time, these forces tending to potentially unstable situations. We have been pleading in our pamphlet for diversity, but we suggest such big remedies for the evils as we see them, that they will only work if everyone will conform to our objectives. That is incompatible with our call for diversity.

- You are trying the impossible, said Rein, if you want to eliminate a paradox. Our dilemma in the choice between diversity and uniformity cannot be solved in a unique way. I guess that the quality of our society is determined by the way in which we learn to live in harmony with such basic paradoxes.

- Right, but history shows that such a harmony will only exist as long as the movement of the masses is without momentum. Saint John could still be sure that the house of the Lord provided many shelters. But, Saint John conformed all shelters into one and the same blueprint, equal building blocks of the new imperium.

- This has all been said before, returned Rein. We are reformulating the laws of entropy for human behaviour. A large diversity, a high probability, so a small driving force and stabilizing friction. Hence, gradual and stable change. And vice versa. For progress we need concerted action, for stability in the course of progress sufficient friction. The question is not to be progressive or to be conservative. We need both, rebels and believers, pessimists and optimists, maçons and free-maçons.

My view on our state is pessimistic. Decline is conditioned, if we go on as we do. My view on individual man is optimistic. He is able to strike the right balance between the two opposing poles. Where conformism prevails, I favour to challenge it. It is the choice to keep the spirit alive of a never ending inquiry into all what we are doing.

I agree, rebellion can be the symbol of our belief (3). As long as rebels will testify, no statement will be too obvious, no act too trivial. Let us keep that spirit, which protects continence by heresy, alive (1).

3. ON SYSTEMS AS INSPIRED CHAOS

Tous les voyages ont un retour, sauf un. Sauf un. Notre cas à nous a le double avantage de nous stabiliser sans nous limiter, nous laissant le parfum d’aventure, d’enthousiasme et de nonchalance, qui m’est vital.

Albertine Sarrazin
I told Rein Jan one day how he had disappointed me years ago as a student. I had written him a vast letter on some scientific problem (for those who want to know: on the influence of coalescence and mixing in a chemical tank reactor stirring two immiscible liquids) and I had asked for his help. He found time for a discussion, as always, but the letter was lost in a mass of papers spread over his desk and over all the chairs of his room. I had to start my story right from the beginning. Only after an hour or so, I discovered from his reactions that he had read my letter. That alleviated the original disappointment, but nevertheless.

I will not go into the details of that working session. We did not come to any solution. The notes which I made of it proved to be full of unfinished thoughts and not without that kind of tricky errors, which can pester a student for longlasting hours. But, I came out of his room convinced that we were close and that I could do the rest by myself. It took me three months. Since that experience, I am very sensitive to any attempt by administrators to straightjacket teaching by more planning, even though only a few professors have that talent to use their freedom in teaching to that purpose. Rein Jan responded to my late complaint, that he had decided not to spend any longer time on the menial work of the scientific trade. The essential would be remembered, because it had been digested as essential. The rest could be forgotten. Half lifetimes of a few months for papers with little inspiration were to be accepted. They did not even deserve the effort to be filed.

- Okay, I said, that holds for a pioneering mind. That mind will invent anew if necessary. The restorative mind, however, has to reply on the cultivation of our heritage. On textbooks, retrieval systems and museum collections. That is where the student starts. Many will not outgrow that stage. For them, more science will only lead to more uncertainty and more information to more questions. Like an autocatalytic process.

- I agree, answered Rein, their tower of Babel will be but a bookshelf high. They will spend an increasing portion of their time restorating. Their world will indeed be textbooks rewritten from old ones. Change to some good has never been effected in that world. Progress is the outcome of a random walk by many individuals, each picking up what he sees as essential. Only if the individual will stick to his vocation, progress will be feasible, but nevertheless uncertain. That is why we sometimes fear the outcome.

That random character of change also explains why we may believe that the concepts of probability and statistics could be applied to social affairs, to economics and to ecology. Why should not we try to approach concepts as quality of life or well-being or quality of habitat in such a manner? Finding a way-out from the dead-lock, in which sociology finds itself so desperately.

Walking home, I had the afterthought that one condition for having the courage to explore new paths is having the grace of being naive in a convincing way.

4. THE CREATIVE REPRODUCTIVE VERSUS THE MIND.

Plaire est affaire de soins plus que de données.  

Albertine Sarrazin 4)

My first meeting with Rein Jan has been an oral examination. The subject was optimizing principles, a favourite one at
that time. And still so, if I see the work of control engineers on world models. If they could only find the one and unique goal function for the ideal world, they would know how to take the steering wheel.

He asked me to calculate the desired distributed heat input along a one-dimensional bar, insulated at one end and cooled at the other end. The objective seemed to be to keep the bar as close as could be to its uniform starting temperature.

I told him that I could not imagine a practical situation in which the simplest solution, i.e. to insulate the other end also, could not be effected. I think I was trying to save time.

Instead of cutting down that argument, which a student expects, Rein started to construct appropriate practical situations. After a while he gave up and challenged me to come up with a practical problem.

I suggested the chemical reactor tube from which heat had to be removed such that the yield of the reactor would be maximal. A fashion subject in my field at that time.

- Could that then practically be achieved in a continuous way, he promptly asked?

Of course, it could not. I had fallen into my own trap.

- So, he said, what would you do?

- Set up sections and treat each section uniformly.

- All right, he said, let us see how to modify the theory to deal with such a situation.

He did put some real effort into it, leaving me feeling like some classical chorus, making a sort of humming noises at the edge of the stage.

I recalled this experience to him many years later when we spoke about his students taking my examinations. They found I asked for too much physics and for too little math.

- The real struggle, said Rein, is to formulate a problem properly. Compared to that, solving a problem is a straightforward battle. One could never design a course to teach how to taste the very heart of a matter. It will be parboiled before it will be served. We have learned to taste that taste from the bits which our teachers left behind, offside the formal dish. One has to be glad if an examination provides such a snack.

- Yes, but students want to show what they have mastered already, not what they could master. That gives them the kick to take the next hurdle. A large part of education, as a large part of daily life, happens to be reproduction and imitation.

- The other, more important part is intuition and initiative. If we practice teaching according to design, the attitude around us will be to solve known problems by turning the prayer-wheel of known solutions. The only antidote for a teacher is to rely on his talent of improvisation, wherever possible.

A warm interest in people is both the basis and the outcome of that philosophy which orders one to do primarily what does come naturally. It is the zest for human intercourse without conditioned responses.

5. ON EDUCATING AND TRAINING

What I learned would never stick. I could not imagine that a priest needed so much Latin, philosophy and theology. He could better be a bit less sophisticated and a bit more practical. Specialism is a poor guide. A good one despises specialists.

Maurice Sachs
The university was in upheaval. It had been yielding passively for too long to the pressures of a changing environment. A large increase of scale and an unsane degree of specialization had infested its structure, revealing an unclear and hence, a misunderstood and a mistrusted management. The latter was the bigger evil, of course. It prevented cure.

Rein among other voiced his understanding of the situation. The school had to match again the different existing motivations. That only could save the element of personal education in a university which society since Robert de Sorbon has judged always on its performance as un haut école, as a training school (6). For that purpose we could not be too dogmatic about our tools. Student participation, workshops, block courses, etc. could all be considered.

More important than choosing the tools was that the university did restore the internal relations by itself, before the administrator, who tends to forget that laws can not direct desired human action (they can only prevent undesired acts), would confuse the issue.

- We face three basic questions which the politicians have to answer before considering legal reform, Rein volunteered. First, do students who qualify have the right to a study of their choice or the right to some guarantee of a job? Second, given limited means, will many qualify for a relatively short study or will only a few qualify for a long study? Third, will that study survey many subjects in a broad sense or only some subjects in depth?

Whatever the politicians will decide, they can leave it to the university to solve the educational problems arising from their choice. I favour freedom of study for many and not too wide ranging curricula. We are not able to forecast social demand nor individual development. Creating equal opportunities is the best bet, but not by reducing the study to an introduction - into - something. It is better to learn how to study than to survey the possible subjects of a possible study.

The best security for a job is a successful personal education, - the first nuclear reactor in the U.K. has been built by a biologist. High level achievements based on study alone are often unpractical, they better rely on motivations stemming from practice. The latter is even more true for general knowledge. There is no such knowledge without practical experience and without matured judgement. In short, for their own personal development, both the bright and the dull student should not shy away too long from practice (7). The first does not need to. The second will not improve by it, on the contrary. It is perhaps odd to say but if a personal education succeeds it is in spite of curricula and teachers.

- Since the haydays of Bologna, I rejected, the university has not longer focussed on student needs. It just cannot deal with many different values.

- In the following sequence: the statement of a problem, its analysis, the synthesis of an answer and the judgement thereof, only the first and the last part depend on personal value and motivation. One should not restrict the teaching university in its choice of problems, nor should one ask that university to reach consensus on issues of value judgement. If that is truthfully accepted, there need not be clashes of values disguised as scientific arguments.

Originally, the university made so rightly a distinction between disputationes solemnes, aiming at scientific judgement and disputationes de quolibet, dealing
3. J. Zamjatin, a professional engineer writing at the eve of the Russian revolution on entropy. In "Tomorrow".

4. In "Journal de prison".

5. In "The Sabbath, a report of an unbearable life".

6. A.B. Cobban, The medieval universities, Methuen,
    J. Bowen, A history of western education, Methuen.

7. Originally from Prof. Dr. G. Holst, Delft Institute of Technology, 1935.

8. Yukio Mushima in "Thurst for love".

9. Muriel Spark, from her collected short stories.
SESSION II

Chairman: Prof. F. Ursell,
University of Manchester
AMPLITUDE RELATION FOR TRAPPED WAVES AROUND A CIRCULAR ISLAND

by

A.J. HERMANS
UNIVERSITY OF TECHNOLOGY, DELFT
THE NETHERLANDS

ABSTRACT
In this paper an analysis of the trapping of waves around a circular island is presented. Recent literature provides a method to construct the phase of trapped waves. We extend this theory to the construction of the amplitude function. A useful tool hereby is the ray method.

Three regions can be distinguished. Close to the island there exists a region where waves are trapped. Also there exists a region at some distance to the island, where waves exist related to these trapped waves. Each region is bounded by a caustic. Between these caustic lines no wave solution exists. With the help of the ray method the amplitude of the waves is determined. Near the caustic uniform asymptotic solutions are constructed with the help of a boundary layer method. Between the two caustic a non-wave solution is obtained.

As a final result we present a relation between the amplitude in the two wave regions.

INTRODUCTION
In this lecture I would like to present an example of an engineering thesis of which the subject was initiated by a discussion we had with professor dr. R. Timman. The kind of research carried out by P.G.A. Maas may be considered as representative for the work that our engineering students carry out to fulfil the requirements to obtain their engineering degree. Timman asked me to supervise this particular subject because of my interest in wave propagation. The work may be seen as an extension of the paper of Shen, Meyer and Keller [2]. In this paper they considered the spectra of water waves in channels and around islands. S.M.K. used the ray method to show that there exist trapped waves for certain configurations such as channels, islands and sills with a variable seabed. Our attention was drawn to the case of a circular islands and variable seabed where for a certain range of frequencies two separate regions of waves may exist. Because this example is circular symmetric the wave regions are bounded by circles.

In a way there are waves trapped in the inner circle while outside the second circle waves propagate towards infinity (figure 1). S.M.K. derive their results with the help of the ray method by considering the phase function only.
The results are derived with the classical small-amplitude theory. Although, trapped waves are found in shallow water areas only, no use is made of shallow water approximations. The small parameter involved is the slope of the seabed. With the help of an intrinsic horizontal scale L characteristic of the wave pattern as a whole, which is unknown to begin with, and a scale K of the local wave number a large parameter $M = KL$ is defined. The analysis is based on a heuristic asymptotic approximation corresponding to the limit $M^{-1} \to 0$. 

First of all a region bounded by the island and a circular caustic around the island where the actual trapping takes place. Secondly a region like a ring around the island bounded by the caustic which belongs to the trapped waves and a second circular caustic. In this region no waves which match the trapped waves are possible. Thirdly a outer region bounded by the second caustic and extending toward infinity. In this region waves exist which are related to the trapped waves. S.M.K. do not give a relation between the waves in both wave region. The purpose of our research is to find an amplitude relation between the waves in the two wave regions. To find this relation we use the same ray method as done by S.M.K., however we proceed a step further by formulating and solving the first transport equation. We do not pay much attention to the determination of the phase function. This can be found in S.M.K.'s paper.

We just select a phase relation which leads to a wave which belongs to the possible spectrum. The transport equation will be solved. We do not include a uniform solution near the shore line because it is not relevant for our discussion. We prescribe the outgoing wave at a point where the depth is no yet zero. We may say we prescribe the outer solution of the solution near the shoreline.
where \( h = 0 \).

Our main concern is the behaviour near the two caustics. Because the caustics are lines of non uniformity we use one of the existing methods to construct asymptotic expansions near those lines. We shall use the boundary layer expansion method because of its physical significance. The uniform asymptotic methods such as developed by Ludwig need some information about the type of solution obtained by boundary layer methods as well. Therefore we stick to the boundary layer method.

In this paper we will show that starting with a trapped wave solution near the island, the waves in the outer region are outgoing.

In the next section we present a short review of the theory employed by Longuet-Higgings, which is based on the shallow water approximation. In section 3 we formulate the problem with the help of the small amplitude theory which is the base for our further analysis.

2. THE SHALLOW WATER APPROACH.

The shallow water approximation is based on the observation that the wavelength is large compared to water depth. It can be shown by means of an averaging procedure applied to the exact equation that the following formulation for the horizontal velocity components \( u, v \) and the surface elevation \( \zeta \) is permitted

\[
\frac{\partial u}{\partial t} = -\frac{\partial \zeta}{\partial x} \tag{2.1}
\]
\[
\frac{\partial v}{\partial t} = -\frac{\partial \zeta}{\partial y} \tag{2.2}
\]
\[
\frac{\partial \zeta}{\partial t} + \frac{1}{h} \frac{\partial}{\partial x} (hu) + \frac{1}{h} \frac{\partial}{\partial y} (hv) = 0 \tag{2.3}
\]

Horizontal lengths are made nondimensional by reference to a scale \( L \), but the vertical lengths, by reference to a water depth. The seabed is given by \( z = h(x,y) \). The functions \( u \) and \( v \) may be eliminated and a wave-equation for \( \zeta \) is obtained

\[
\zeta_{tt} - \frac{1}{L^2} \zeta_{xx} + \frac{1}{h} \nabla \cdot \nabla \zeta = 0 \tag{2.4}
\]

where the Laplacian \( \Delta \) and gradient operator \( \nabla \) are with respect to the horizontal variables \( x,y \).

For axially symmetrical seabed topographies, we introduce polar coordinates \( r, \theta \) and long-wave solution of the surface elevation of the form

\[
\zeta(r,\theta) = P(r) \exp[i\theta - i\omega t] \tag{2.5}
\]

Where the function \( P(r) \) is a solution of the equation

\[
\frac{d}{dr} \left( rh \frac{dP}{dr} \right) + (\omega^2 - (n/r)^2) P = 0 \tag{2.6}
\]

It is well known that the character of the solutions of this Sturm-Liouville equation is determined by the sign of the coefficient

\[
qu(r) = \omega^2 - n^2 h(r)/r^2 \tag{2.7}
\]

We notice that sufficiently far from the island all solutions are oscillatory because

\[
\lim_{r \to \infty} q(r) = \omega^2 > 0
\]

If we consider a submerged reef situation it can be shown that in many cases \( h(r)/r^2 \) is a monotone function. Hence only in an outer region waves occur. This situation changes drastically if \( h(r) \) becomes very small or zero for some finite value of \( r \). This is the situation if we consider an island.

Figure 2 shows that for certain values of \( \omega^2/n^2 \)

\( q > 0 \) for all values of \( r \) and for other value

of \( \omega^2/n^2 \) there exist two regions where \( q > 0 \) and a region where \( q < 0 \) in between.

In this case the ray pattern may look like the one in figure 1. Hence for small values of \( \omega/n \) the solution exhibit a ring of wave trapped near shore, separated from the outer wave region by a ring of damping. There are two caustics at which \( q(r)=0 \). The trapped waves are seen to ply around the island and for integer \( n \), existence of some resonant frequencies \( \omega \) thus become plausible. It was a very important observation of Longuet-Higgins that a new phenomenon enters oceanography. The annulus of damping is of finite width, and while the inner and outer wave motions decay exponentially with distance from the respective caustics, such decay cannot be strictly complete within a finite
distance. Accordingly, the trapped wave motion inside
the inner caustic and the progressive wave motion
beyond the outer caustic cannot be independent.
A trapped motion cannot exist without a progressive
motion outside the damping annulus, and the latter
motion must involve the radiation of wave energy
toward infinity. We may say that energy leaks
toward infinity. It can be shown that the waves are
related analytically and that for small values of
\( w/n \), where the annulus is very wide resonance
is an important factor in the wave phenomena around
islands. It is shown that for the case of a sill
the wave elevation may be written as series
expansions of Bessel functions

\[
\zeta = \sum_{n=-\infty}^{\infty} \exp(i(n\theta-ct)) \times \begin{cases} A_n J_n(k_1 r) & r < a \\ B_n H_n(k_2 r) + i^n J_n(k_2 r) & r > a \end{cases}
\]

where \( a \) equals the radius of the sill and \( k_1, k_2 \)
are the wave numbers belonging to the depths above
and outside the sill respectively. It turns out
that near resonance situations exist for certain
values of the parameter see figure 3.

Application to the situation of a circular island,
shows a similar near resonance behaviour and the
ray pattern of the diffracted wave has components
which look like the ray pattern of figure 1. The
question to be answered is whether the asymptotic
ray theory applied to the small amplitude equations
leads to an amplitude relation between the two
wave regions.

3. SMALL AMPLITUDE FORMULATION

We consider a circular island and a seabed which
has circular symmetry. For convenience we consider
a bottom topography with a negative slope. It has
been shown [2] that in that case we get not more
than two caustics. In cylindrical coordinates the
bottom is given as \( z = -h(r) \) with \( h(\alpha L) = 0 \) and
\( \frac{\partial h}{\partial r} = 0 \) if \( r > \alpha L \) (fig. 4).

The linearized differential equation and boundary
conditions for the potential function are

\[
\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\phi^2}{2} = 0, \quad \text{at } -h(r) < z < 0
\]

\[
\frac{\partial \phi}{\partial t} + g \frac{\partial \phi}{\partial z} = 0, \quad \text{at } z = 0 \tag{3.1}
\]

\[
\frac{\partial \phi}{\partial h} + \frac{2h}{\partial r} \frac{\partial \phi}{\partial r} = 0, \quad \text{at } z = -h(r)
\]

where \( \phi \) is the velocity potential with \( v = \text{grad } \phi \),
g is the gravitational acceleration, \( r > \alpha L \) and
\( 0 \leq \theta \leq 2 \).
The parameter \( L \) is a typical length parameter.

We shall prescribe the outgoing wave part at a point
along the shoreline. We keep in mind that the out­
going wave which we prescribe must be considered
as an outer solution of the solution which is uni­
formly valid along the shore line. We prescribe
the outgoing wave part as follows,
\[ \phi = \frac{L}{\sqrt{g}} \cosh \left( \frac{h}{L} \right) (k^2 - 2 L^2)^{-\frac{1}{2}} \left[ \frac{g}{2} \sinh \left( \frac{kh}{L} \right) \right] \] for \( r = a \), where \( h = 0 \).

This complicated form is chosen in such a way that the constants which have to be determined later on have a simple form.

The parameters in this formula will be explained later on.

We consider waves with frequency \( \omega \) and introduce dimensionless parameters,
\[ r^* = \frac{r}{L}, z^* = \frac{z}{L}, h^* = \frac{h}{L}, t^* = \frac{t}{L}, \]
\[ \phi = \omega^* L \]
and \( \theta(r,\theta,z,t) = \sqrt{\frac{L}{g}} \exp(-i \omega t) \phi(r^*,\theta,z^*,L) \)
where \( M = \omega^* L = \frac{w^*}{\sqrt{g}} \).

The parameter \( \epsilon \) is a typical seabed slope, which is small. The theory which we shall describe is consistent if \( \omega^* = O(1) \).

In the following formulas we drop the asterisks.

The equation and boundary conditions become,
\[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \cot \frac{\partial}{\partial \theta} + \frac{M}{2} \frac{\partial^2}{\partial z^2} = 0 \]
\[ \frac{\partial}{\partial z} = \frac{2}{M} \phi = 0, \text{ at } z = 0 \] (3.2)
\[ \frac{\partial}{\partial r} = \frac{3}{2} \frac{\partial}{\partial z} = \frac{3}{2} \frac{\partial}{\partial r} + M \frac{\partial}{\partial z} = 0, \text{ at } z = -h(r) \]

with boundary condition for \( r = a \)
\[ \phi = -\cosh(k(z+h))(k^2 - \epsilon_1)^{-\frac{1}{2}} \frac{M}{w^2} \sinh(kh)h^{\frac{1}{2}} \] (3.3)
for the outgoing wave part. Notice that \( \frac{w^2}{M} = O(1) \)

The ray method of J.B. Keller et al. provides us a tool to construct asymptotic solutions. The validity of these solutions is questionable from mathematical point of view because no proofs concerning the asymptotic character of these solutions can be given up to now. The method leads to formal asymptotic approximations.

According to the theory of Keller et al. we shall determine a phase function \( S \) as a function of \( r \) and \( \theta \). The amplitude function \( A \) will be a series expansion with respect to inverse powers of the large parameter \( M \). The first term of this expansion depends on \( r \) only, while the other terms depend on \( z \) as well.

The solution is supposed to be of the form:
\[ \phi = \sum_{j=0}^{\infty} A_{n,j} M^{-n} \] (3.4)

The summation over \( j \) arises because several wave components may be summarized at a certain point.

In this formula
\[ \phi_j = \phi_j(r,\theta) \; \; ; \; \; k = k(r,\theta), A_{0,j} = A_{0,j}(r) \]
\[ A_{n,j} = A_{0,j}(r,z) \text{ for } n > 0 \]

First of all we substitute (3.4) in the condition at \( z = 0 \) in (3.2). This leads to the following equation, where we omit the summation over \( j \) and disregard the index \( j \).

\[ k \sinh(kh) e^{i M S} \sum_{n=0}^{\infty} A_n M^{-n} - \sum_{n=0}^{\infty} A_n M^{-n} = 0 \]

We notice that \( \frac{w^2}{M} = O(1) \)

Selecting terms with the same order of magnitude we obtain
\[ k \tanh(kh) = \frac{w^2}{M} \] (3.5)
and \( A_{0,z} = 0 \) for \( z = 0 \) and \( n \neq 0 \)

From the dispersion relation (3.5) it follows that \( k \) is independent of \( \theta \). Substitution of (3.4) into the condition at the bottom leads to

\[ e^{i M S}(M \sum_{n=1}^{\infty} A_n M^{-n} + h_i M S \sum_{n=0}^{\infty} A_n M^{-n} + h_i n M S \sum_{n=0}^{\infty} A_n M^{-n}) = 0 \]

Equating terms of the same order equal to zero one obtain
\[ A_{1,z} + i h_i S A_{0,z} = 0 \]
\[ A_{n,z} + i h_i S A_{n-1} + h_i A_{n-2} = 0 \; \; \text{ for } n \geq 2 \] (3.6)
Substitution of (3.4) into the differential equation and equalizing powers of $M^n$ leads to
\[(\nabla S)^2 = k^2, \tag{3.7}\]
where $\nabla$ denotes the operator $\left(\frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta}\right)$.

and $i \cosh[k(z+h)]A_{iz} + 2ik \sinh[k(z+h)]A_{iz} = A_0 \nabla^2 S \cosh[k(z+h)] + \nabla S \nabla S \{ A_0 \cosh[k(z+h)] \} \tag{3.8}$

equations for $n > 2$ may be obtained as well.

The solution of (7) has the form
\[S_j(r, \theta) = \pm \int \{ k^2(c) - \frac{c_1^2}{c_2^2} \} \, dc \pm c_1 \theta \tag{3.9}\]
All possible combinations of the $\pm$ signs are valid solutions. Therefore $j = 1, 2, 3, 4$.

We select those trapped wave modes which move clockwise. (fig. 1)
This means the second $\pm$ possibility is taken as a plus sign and $j = 1, 2$.

The value of $c_1 > 0$ is selected according to S.M.K. in such a way that two caustics occur, this means two values of $c$ such that
\[k^2(c) - \frac{c_1^2}{c_2^2} = 0.\]

4. THE AMPLITUDE FUNCTION.

We shall determine the first term of the expansion for the amplitude function.

First we rewrite (3.8) in a different form,
\[- i A_{iz} \cosh^2[k(z+h)] = \]
\[= (2 \nabla S \nabla A_0 + A_0 \nabla^2 S) \cosh^2[k(z+h)] + \]
\[+ A_0 \nabla S \nabla (\cosh^2[k(z+h)]) \tag{4.1}\]
Remember that the $\nabla$ and $\nabla^2$ operators are two dimensional operators in the horizontal plane.

We now integrate with respect to $z$ and make use of the fact that
\[A_{iz} = 0 \quad \text{if} \quad z = 0.\]

This leads to the relation
\[i A_{iz} \cosh^2[k(z+h)] = \]
\[= \pm \int (2 \nabla S \nabla A_0 + A_0 \nabla^2 S + A_0 \nabla S \nabla S) \]
\[\pm k^{-1}(\cosh[k(z+h)] \sinh[k(z+h)]) - \cosh[kh] \sinh[kl]), \]
We now take $z = -h(r)$ and make use of the boundary condition at the bottom. This leads to
\[\nabla S \nabla A_0 + A_0 \nabla S = 0 \quad \tag{4.2}\]
with $A_0 = A_0 \frac{M}{w} \sinh^2(kh) + h$.

This is the well known transport equation.
Making use of (3.9) this equation is easily solved
\[A_0 = c_2(k r^2 - c_1^2)^{-\frac{1}{2}}\]
when $c_2$ is an integration constant and is still to be determined.

This leads to a general wave solution of the form
\[\phi = \cosh[k(z+h)] A_{01} e^{i M S_1} + \]
\[\cosh[k(z+h)] A_{02} e^{i M S_2} \tag{4.3}\]
where
\[A_{01} = c_2(k r^2 - c_1^2)^{-\frac{1}{2}}(\frac{M}{w} \sinh^2(kh) + h)^{-\frac{1}{2}} \]
and
\[A_{02} = c_2(k r^2 - c_1^2)^{-\frac{1}{2}}(\frac{M}{w} \sinh^2(kh) + h)^{-\frac{1}{2}}.\]

According to S.M.K. we are able to select the wave mode such that
\[k(r^2) r^2 - c_1^2 \quad \text{has two zero's} \quad (r = 8, \gamma)\]
In our case $a < b < \gamma$
where $r = a$ is the shore line.

The outgoing wave is given at $r = a$.
Therefor we first consider the region $a < r < b$.
We do not bother about the non uniformity at $r = a$.
Our main concern is the singularity at $r = b$.

Condition (2.3) for the outgoing wave leads to the outgoing wave solution,
\[\phi(r, \theta, \eta) = \cosh(k(\eta + h)) (k^2 r^2 - c_1^2)^{-\frac{1}{2}} e^{\frac{iM}{\omega} \sinh^2(kh) + h} \]
\[\exp\left(\frac{r}{\theta_a} \int (k^2(c) - \frac{c_1^2}{c_2^2}) \, dc\right), \] for $a < r < b \quad \tag{4.4}\]
If we proceed along an outgoing ray we observe that the expansion near the caustic becomes non uniform.
In the next section we describe the solution inside the boundary layer near the first caustic.

5. BOUNDARY LAYER EXPANSION

In the introduction we stated that the goal of this research is to find an amplitude relation for $a < r < \beta$ and $r > \gamma$.

Therefor we have to pass the caustics and the solutions must be known in the vicinity of the caustics. To find these solutions we choose the method of boundary-layer expansions.

We try to find solutions of (3.2) of the form

$$\phi = e^{i M \theta} A^*(r,z,M)$$

The differential equation for $A^*$ becomes

$$M^2 A_{zz}^* - \frac{c^2}{r^2} A^* + A_{rr}^* + \frac{1}{r} A^*_r = 0$$

We stretch the $r$-coordinate near the caustic $r = \beta$ as follows

$$r - \beta = M^u \rho$$

and write for $A^*$ by means of the expansion

$$A^*(r,z,M) = \cosh[k(z+h)](A_0(\rho) + M^u A_1(\rho,z) + \ldots)$$

The value $\beta$ is the value of the smallest zero of $f(r)$. It has the form as given in figure 6.

Hence we know that $f_{r}(\beta) < 0$ and $f_{r}(\gamma) < 0$.

The function $f(r)$ has to be expanded near $r = \beta$ as well

$$f(r) = f(\beta) + (r-\beta) f_{r}(\beta) + \ldots = M^u f_{r}(\beta) \rho + \ldots$$

The equation for $A_0$ has the form

$$A_0 \rho + f_{r}(\beta) \rho A_0 = 0$$

for $u = \frac{2}{3}$, which is the only relevant choice for $u$.

Because $A_0$ is a function of $r$ only it obeys the free surface and bottom condition automatically.

Equation (5.3) is an Airy equation with well known solutions [8]. The solution we are looking for has to be selected such that for $\rho < 0$ a wave solution exists which for large values of $|\rho|$ has the same "outgoing" part as the solution in the region outside the boundary layer, while for $\rho > 0$ the solution has to tend to zero exponentially for large values of $\rho$.

The solution which fulfills these requirements is

$$A_0(\rho) = c_4 \text{Ai}(\sqrt{|f_{r}(\beta)|} \rho)$$

All other choices of Airy function leads to the wrong exponential behaviour.

The boundary layer solution has the form

$$\phi = e^{i M \theta} \cosh[k(z+h)](c_4 \text{Ai}(\sqrt{|f_{r}(\beta)|} \rho) + \ldots)$$

This solution must be matched with (4.4) to obtain the value of the constant $c_4$.

We insert $r = \beta = M^\frac{2}{3} \rho$ in (4.4) and expand (5.5) for large values of $|\rho|$ for $\rho < 0$.

Comparing similar wave modes leads to
The wave "reflected" at the caustic has the form
\[
\psi_{III} = -\cosh[2(k(y)h(y)) + h(y)] \exp\left\{ \frac{1}{2} i \left[ 2 \frac{c_1^2}{c_2} \right] \int_\alpha^{r_1/2} \right\}.
\]
(5.6)

The solution will be matched with the outer expansion of the solution near the caustic at \( r = r_1/2 \).

Matching leads to
\[
c_6 = \frac{1}{2} \left( \frac{M}{r_{II}(\gamma)} \right) c_5 e^{\frac{1}{2} \frac{2n_1}{3}}.
\]
(7.4)

The wave solution in the outer region for \( r > \gamma \) has the general form:
\[
\psi_{IV} = -\cosh[2(k(y)h(y)) + h(y)] \exp\left\{ \frac{1}{2} i \left[ 2 \frac{c_1^2}{c_2} \right] \int_\alpha^{r_1/2} \right\}.
\]
(6.1)

The solution becomes singular again near the second caustic where \( r = \gamma \) and
\[
f_2(y) = k(y) - \frac{c_2^2}{\gamma^2} = 0
\]
with \( r_2(y) > 0 \).

7. OUTSIDE BOUNDARY LAYER AND OUTSIDE REGION

The outer boundary layer is treated similarly as the inner one. We introduce the stretched coordinate
\[
r - \gamma = M^2 \rho
\]
and we again assume the solution to be of the form
\[
\psi = \cosh[k(z+h)] e^{i M \rho} (A_0(\rho) + M A_1(\rho, z) + \ldots)
\]
(7.1)

where \( A_0(\rho) \) is a solution of the Airy equation when \( r_1(\gamma) > 0 \).

We may choose the solution of this equation as a combination of two of the three following Airy functions
\[
A_1\{-\rho \sqrt{r_2(\gamma)} \}, A_1\{-\rho \sqrt{r_2(\gamma)} e^{\frac{2n_1}{3}} \}
\]
The solution we select must match the solution in the transient region where \( \rho < 0 \) and \( \rho \) large.

The solution which matches correct is
\[
\psi = \cosh[k(z+h)] e^{-\frac{i M}{\gamma} \int_\alpha^{r_1/2} \right\}.
\]
(7.3)

Matching leads to
\[
c_6 = \frac{1}{2} \left( \frac{M}{r_{II}(\gamma)} \right) c_5 e^{\frac{1}{2} \frac{2n_1}{3}}.
\]
(7.5)

The solution which matches correct is
\[
\psi = \cosh[k(z+h)] e^{-\frac{i M}{\gamma} \int_\alpha^{r_1/2} \right\}.
\]
(7.6)

The last exponential function describes the exponential decay of the amplitude of the wave just inside the inner caustic to the wave just outside the outer caustic.

8. RESULTS

In this section we summarize the results. The question to be answered is, what is the amplitude attenuation factor between the waves in region I and region VI fig. 5.

In region I we constructed an outgoing wave of the form
In region VI outside both caustics we constructed an outgoing wave, which is a continuation of the foregoing result, of the form

\[ \psi IV = \frac{1}{w_0} \sinh (k(r)h(r) + h(r)) \frac{1}{2} \exp \left( iM \int \left( k^2(\sigma) - \frac{c_1^2}{\sigma^2} \right) d\sigma \right) \text{ if } \alpha < r < \beta \]

Because the function

\[ f(r) = k^2(r) - \frac{c_1^2}{r^2} \]

has negative sign on the interval \( \beta < r < \gamma \).

The exponential decay of the amplitude is mainly determined by the term

\[ \exp \left( -M \int_{\beta}^{\gamma} \frac{c_1^2}{\gamma^2} \left( -k^2(\sigma) \right)^2 d\sigma \right) \]

This result is no surprise to us, because similar results are obtained in problems in quantum mechanics [9] where the attenuation through a potential barrier is considered.

9. CONCLUDING REMARKS

In the proceeding sections we have derived an exponential decay factor. It will be clear that due to this relation an energy leak is present.

The actual leakage of energy is a non-stationary process and cannot described by the equations used in this paper. With the help of an artificial boundary value problem we have shown that an amplitude relation exists. The stationary value of the potential function at the boundary of the island makes the problem stationary. An indication about the time dependent leakage may be obtained by describing the potential function at \( t = 0 \) and looking at the development in time of this potential function. In this case we take as a boundary condition \( \frac{\partial \psi}{\partial n} = 0 \) for \( t \geq 0 \) if we consider an ideal vertical beach. For a sloping beach an other boundary condition has to be imposed and the energy loss due to breaking may be taken into account. However for the leakage phenomenon we have discussed here this it only means that a different boundary value is imposed near the beach.

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ABSTRACT
A procedure to calculate slow-drift oscillations of a ship in irregular beam sea waves is presented. The hydrodynamic boundary-value problem is formulated and solved correctly to second order in wave amplitude. The drift force and moment in oblique sea are also studied. The first order potential is found by using strip theory and solving two-dimensional Helmholtz-equation-problems for the cross-sections of the ship. The drift force and moment are obtained by a formula derived by Newman. The methods are compared with experiments and other methods used to predict drift force and moments. The results are in general satisfactory. Results for second order difference frequency force due to presence of two simultaneous waves with frequencies \( \omega_j \) and \( \omega_k \) are presented.

INTRODUCTION
Slow-drift oscillations of a moored structure in irregular waves may be an important problem. The large horizontal excursions that occur can cause large forces in anchor lines and limitations in drilling operation. The phenomena is commonly seen in model tests, but nobody seem to have a sufficiently reliable analytical tool to analyze the problem.

Hsu and Blenkarn (1) have given a simple explanation of the phenomena. They imagine the irregular wave system divided into approximate regular wave parts. In each regular wave part the structure will experience a constant horizontal drift force (and yaw moment). This is illustrated in figure 1 where the drift force in each "regular wave part" is indicated by an arrow. In this way a slowly varying excitation force is obtained. The magnitude is not large, but if the mean period is close to a natural period in yaw, sway or surge, a significant amplification may occur due to small damping in the system.

The drift force in regular waves is the important building brick in Hsu and Blenkarn's analysis of the slowly-varying drift force. The same is true in the approach by Remery and Hermans (2) and Newman (3). Different theories exist to predict drift forces in regular waves. For a ship in regular beam sea waves one may use the Maruo's formula (Maruo (4)). For a

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1) The University of Trondheim, Norwegian Institute of Technology, Trondheim, NORWAY
2) Det norske Veritas, Høvik, NORWAY
ship in regular head sea waves the method of Gerritsma and Beukelman (5) and Maruo (6) seems to give good results in many cases. For the oblique wave case the state of the art is less satisfactory. Maruo (4) derived a theory for the drift-force, but his work does not include numerical results for the oblique sea case. Newman (7) used a different approach than Maruo and, for the zero-speed case, derived the drift-yaw-moment and rederived the force results of Maruo. His numerical results were based on slender-body theory combined with long wave length assumption. Salvesen (8) derived a result where the final results were expressed in the framework of the ship-motion strip-theory. He showed partial agreement between theory and experiment for the driftforce while discrepancy between theory and experiment for the drift moment. Kaplan and Sargent (9) proposed to use the formula of Gerritsma and Beukelman (5) also for oblique sea. This is a simplification since the influence of roll, sway and yaw motion is neglected. However, the formula is simple and may be a practical tool to give a rough estimate of driftforces and moments on a ship in regular oblique sea waves. Kim and Chou (10) have proposed a method where the first order potential is obtained by strip theory and solving 2-D Laplace equation problems for the cross-sections of the ship. For a large-volume structure of any shape Faltinsen and Michelsen (11) presented a method based on three-dimensional source technique and a generalization of the formula of Newman (7). The method is applicable for any wave direction and finite water depth. It shows good agreement with experiments. The driftforce on a ship in regular beam sea waves and in the vicinity of another structure has been theoretically determined by Ohkusu (12). For certain frequencies and for a given distance between ship and structure he predicted a negative driftforce. This causes a slowly varying oscillation even in regular waves. It should be kept in mind that all methods presented above are based on potential theory. Viscous effects may be the most important contribution to mean wave forces on semisubmersibles and other small-volume structures. Huse (13) has pointed out that viscous effects may create a negative driftforce on semisubmersibles in beam sea. All the results are based on waves of moderate wave height. Waves close to breaking may change the result significantly (Longuet-Higgins(14)).

The Hsu and Blenkarn approach or Newman's approach of calculating slowdriftoscillations is a simplification. They disregard several nonlinear interaction terms between wave and structure. Breslin and Kim (15) have presented a different approach than Hsu and Blenkarn. They calculate second order transfer functions. But in the second order hydrodynamic calculation only the velocity square term in Bernoulli's equation is taken into account. This is too simple. It may yield quite erroneous answers.

In this paper a new procedure to calculate slowdriftoscillations of a ship in irregular beam sea wave is presented. The hydrodynamic boundary value problem is formulated and solved correctly to second order in wave amplitude. The first order problem is the well-known linear ship motion problem, and the second order problem contains the necessary slowdrift excitation forces. The second order potential satisfies Laplace equation with inhomogeneous boundary conditions on the free surface and body boundary. Green's theorem is used to derive a formula for the driftforce and slowly-varying horizontal force. Numerical results are presented.

The driftforce and moment in oblique sea are also studied. The first order potential is found by using strip theory and solving 2-D Helmholtz equation problems for the diffraction potential and 2-D Laplace equation problems for the forced motion
problems. The drift force and moment are obtained by Newman's formula (Newman (17)).

The method described above is numerically compared with other methods used to predict drift force and moments. Those methods are

a) Faltinsen's and Michelsen's method where the 1. order potential is obtained by 3-dimensional sink-source calculation.
b) Newman's formula where the 1. order potential is obtained by a long wave length slender body theory.
c) Maruo's formula for beam sea.
d) Salvesen's formula.
e) Kim and Chou's formula.
f) Gerritsma and Beukelman's formula.

SLOW DRIFT OSCILLATIONS OF A SHIP IN BEAM SEA

Consider a ship in longcrested irregular beam sea waves in infinite water depth. A cross-section of the ship is shown in fig. 2. Let us choose a coordinate system $(x,y)$ which is fixed with respect to the ship and coincide with the inertial system $(x,y)$ when the ship is at rest. $y = 0$ is in the mean free surface and positive $y$-axis is upwards. The $y$-axis goes through the center of gravity of the ship.

Let us assume the fluid to be incompressible and the fluid motion irrotational so that there exists a velocity potential $\phi$ which satisfies the Laplace equation

$$\nabla^2 \phi = 0.$$

The solution will be written as a series expansion in the wave amplitude, i.e. we write

$$\phi = \phi_1 + \phi_2 + \ldots$$

where $\phi_1$ is linear with respect to the wave amplitude and $\phi_2$ is quadratic with respect to the wave amplitude and so forth.

Let the incident waves propagate along the positive $x$-axis. The incident wave potential correct to first order in wave amplitude can be written as

$$\phi^i = \sum_{i=1}^{N} \frac{gA_i}{\omega_i^2} \sin(\omega_i t + \epsilon_i)$$

Here $g$ is the acceleration of gravity, $\omega_i$ the circular frequency of oscillation and $\nu_i$ the wave number of wave component no. $i$. $\omega_i$ and $\nu_i$ are connected through the dispersion relationship

$$\frac{\omega_i^2}{g} = \nu_i$$

The amplitudes $A_i$ of the wave components may be given by a wave spectrum and $\epsilon_i$ may be considered as random phase angles. $t$ is the time variable. It should be noted that our solution procedure will provide a second order correction to the incident wave potential.

Due to waves the ship will oscillate in heave $(\eta_3)$, sway $(\eta_2)$ and roll $(\eta_4)$. It is assumed that the ship has fore- and aft-symmetry so that the pitch, surge and yaw motion may be neglected. The ship has zero mean forward speed.

The velocity potential has to satisfy boundary conditions on the body surface and the seafloor. In addition necessary radiation conditions have to be imposed. The mathematical formulation will be presented below separately for the 1. order and the 2. order problem.

1. ORDER PROBLEM

It may be shown that the first order po-
The potential \( \phi \) satisfies the 2-D Laplace equation, i.e.
\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]
(4)
The free surface condition to first order in wave amplitude may be written
\[
\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial y} = 0 \quad \text{on } y = 0
\]
(5)
The boundary condition can be written as
\[
\frac{\partial \phi}{\partial n} = e_x \cdot \frac{dn}{dt} + e_y \cdot \frac{dn}{dt} + (\bar{x} e_y - \bar{y} e_x \cdot n) \frac{dn}{dt}
\]
(6)
on the average position \( \bar{S} \) of the oscillating body.

Here \( e_x \) and \( e_y \) are unit vectors along the \( \bar{x} \)- and \( \bar{y} \)-axis, respectively. Further \( n \) is the unit normal vector to the body surface. Positive direction is into the fluid.

The scattered waves must satisfy a radiation condition of outgoing waves. Further we require that the fluid velocity
\[
|\nabla \phi| \rightarrow 0 \quad \text{when } y \rightarrow -\infty
\]
(7)
The solution to the 1. order problem is not trivial, but is well established in the literature. The solution is for instance referred in Faltinsen (16) for one wave component, which may easily be generalized to a number of wave components. Faltinsen uses Lewisform technique to solve the problem. Another technique would be the Frank Closefit technique.

The potential \( \phi \) may be divided into one part that is symmetric and one part that is asymmetric with respect to the \( y \)-axis. That means we can write
\[
\phi = \phi_1^A + \phi_1^S
\]
(8)
where \( \phi_1^A \) is the asymmetric part and \( \phi_1^S \) is the symmetric part.

2. order problem
It may be shown that the second order potential \( \phi \) has to satisfy the free surface condition
\[
\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial y} = -2 \frac{\partial \phi}{\partial x} \frac{\partial^2 \phi}{\partial x \partial t} - 2 \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \left( \frac{\partial^2 \phi}{\partial x^2} + g \frac{\partial \phi}{\partial y} \right)
\]
on \( y = 0 \)
(9)
and the body boundary condition
\[
\frac{\partial \phi_1}{\partial n} = n_1^2 \left( \eta_1 \frac{dn}{dt} - \eta_4 \frac{\partial \phi}{\partial y} + (n_2 + \bar{y} n_4) \frac{\partial \phi}{\partial x} + (n_4 + \bar{x} n_4) \frac{\partial \phi}{\partial x} \right) + \eta_3 \left( -\eta_4 \frac{dn}{dt} - \frac{\partial \phi}{\partial x} + (n_2 + \bar{y} n_4) \frac{\partial \phi}{\partial x} - (n_4 + \bar{x} n_4) \frac{\partial \phi}{\partial x} \right)
\]
on \( \bar{S} \)
(10)
Here \( n_2 = e_x \cdot n \) and \( n_3 = e_y \cdot n \). There is not included any second order motion in the body boundary condition. This implies that our solution provides the excitation force for the second order motion.

Further \( \phi \) has to satisfy the Laplace equation
\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]
in the fluid. The conditions far away from the oscillating body will be dealt with later on. We are not interested in the total second order solution, only the part that produces a horizontal force and is slowly-varying in time. To calculate a horizontal force we will only need the asymmetric part of \( \phi \).

The free surface condition may be re-written as
\[
\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial y} = \sum_{i=1}^{N} \sum_{j=1}^{N} \left( h_{ij}^c (x) \sin (\omega_j - \omega_i) t + h_{ij}^s (x) \cos (\omega_j - \omega_i) t \right)
\]
(12)
+ terms that are either or both symmetric and oscillating with sum frequencies.

Here \( h_{ij}^S(x) \) and \( h_{ij}^C(x) \) are expressed entirely in terms of first order potentials. They follow from equation (9) and are presented in more detailed form by Faltinsen (17).

We are not interested in terms that are either or both symmetric and oscillating with sum frequencies and drop those terms, but still use the notation \( \phi \) on what is left. The free surface condition is generally complicated, but is somewhat less complicated when \( x \to \infty \).

The velocity potential \( \phi_1 \) can then be written

\[
\phi_1 = \sum_{i=1}^{N} \sum_{j=1}^{S} \left( -A_i e^{jy} \cos \omega_j x \sin (\omega_i t - \epsilon_i) + N_i^S e^{jy} \sin (\omega_i t - \epsilon_i x + \delta_i) + A_i e^{jy} \sin \omega_j x \cos (\omega_i t - \epsilon_i) + N_i^a e^{jy} \sin (\omega_i t - \epsilon_i x + \delta_i) \right)
\]

where \( N_i^S, \delta_i^S, N_i^a, \delta_i^a \) are determined from 1st order problem.

Further "s" indicates symmetric part and "a" indicates asymmetric part.

Using (13) we can now write the free surface condition (9) for \( x \to \infty \) as

\[
\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial y} = \sum_{i=1}^{N} \sum_{j=1}^{S} \left( \begin{array}{c} \left( B_{ij} \cos (\nu_j - \nu_i) x + C_{ij} \sin (\nu_j - \nu_i) x \right) \sin (\omega_j - \omega_i) t + \\
(D_{ij} \cos (\nu_j - \nu_i) x + E_{ij} \sin (\nu_j - \nu_i) x) \cos (\omega_j - \omega_i) t \end{array} \right) \sin (\omega_j - \omega_i) t = 0
\]

where terms that are either or both symmetric and oscillating with sum frequencies have been dropped and where \( B_{ij}, C_{ij}, D_{ij}, E_{ij} \) and \( E_{ij} \) may expressed in terms of known quantities. (See Faltinsen (17).) The asymmetric part of the body boundary condition (10) can be rewritten as

\[
\frac{\partial \phi}{\partial t} + \frac{3}{3S} \left( \frac{d n}{d t} \right) (\vec{y} + \vec{n}) \frac{\partial \phi}{\partial y} = \sum_{i=1}^{N} \sum_{j=1}^{S} \left( \frac{\partial \phi}{\partial x} \right) \sin (\omega_i t - \epsilon_i) \sin (\omega_j - \omega_i) t
\]

where \( \frac{\partial \phi}{\partial S} \) is the derivative along the body surface tangent and

\[
n_2 = \frac{2}{3S}, \quad n_3 = -\frac{2x}{3S}
\]

Equation (14) can formally be written as

\[
\frac{\partial \phi}{\partial t} = \frac{3}{3S} \sum_{i=1}^{N} \sum_{j=1}^{S} \left( f_{ij}^S \sin (\omega_j - \omega_i) t + f_{ij}^C \cos (\omega_j - \omega_i) t \right)
\]

We introduce complex quantities, i.e. we write

\[
\phi_1 = \sum_{i=1}^{N} \sum_{j=1}^{S} \phi_{ij} e^{-i(\omega_j - \omega_i) t}
\]

where it is understood that the real part has physical meaning. Further we impose the restriction that

\[
\phi_{ij} = 0 \quad \text{when } i > j
\]

This is just a question of organizing the double summation. Let us consider the case of \( j > i \) and then study what happens when \( j = i \).

Using equations (11), (12) and (16) we can write that \( \phi_{ij} \) must satisfy

\[
\frac{\partial^2 \phi_{ij}}{\partial x^2} + \frac{\partial^2 \phi_{ij}}{\partial y^2} = 0 \quad \text{in the fluid domain}
\]

\[
-(\omega_j - \omega_i)^2 \phi_{ij} + \frac{\partial \phi_{ij}}{\partial y} = h_{ij}(x) \quad \text{on } y = 0
\]

outside the mean position of the body
We can write
\[ h_{ij}(x) = h^C_{ij}(x) + h^S_{ij}(x) + i(h^S_{ij}(x) - h^S_{ji}(x)) \]  
(21)

Further
\[ \frac{\partial \phi_{ij}}{\partial n} = \frac{3}{8} f_{ij} \text{ on } S \]  
(22)

where
\[ f_{ij} = f^C_{ij} + f^S_{ij} + i(f^S_{ij} - f^S_{ji}) \]

Further \( \phi_{ij} \) is asymmetric and
\[ |\phi_{ij}| \to 0 \text{ when } y \to -\infty \]  
(23)

When \( x \to \infty \), \( \phi \) has to satisfy the free-surface condition (14). That means \( \phi_{ij} \) can be written as
\[ \phi_{ij} = \frac{e^{i(\omega_j - \omega_i)x}}{-\omega_j + \omega_i} \times \left( \frac{1}{f^C_{ij} + f^S_{ij} + i(f^S_{ij} - f^S_{ji})} \right) + i(\omega_j - \omega_i)^2 y/g \frac{i(\omega_j - \omega_i)^2 x/g}{e} \]  
(24)

when \( x \to \infty \). \( F_{ij} \) and \( G_{ij} \) are determined by \( D_{ij}, C_{ij}, B_{ij}, \) and \( E_{ij} \). (See Faltinsen (17).)

\( S_{ij} \) cannot be determined before the \( \phi_{ij} \) problem has been solved. The last term in (24) represents outgoing waves that are created by the body. The first term in (24) contains both incident and outgoing waves.

A solution for \( \phi_{ij} \) may be found, but we are not interested in \( \phi_{ij} \) in itself. It is the horizontal force due to \( \phi_{ij} \) that interests. In obtaining this force we will use Green's theorem and make use of the solution to a generalized forced sway problem, i.e. we will use the solution \( \psi \) to the following problem
\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \text{ in the fluid domain} \]  
(25)

\[ -\left(\omega_j - \omega_i\right)^2 \psi + g \frac{\partial \psi}{\partial y} = 0 \text{ on } y = 0 \]  
(26)

\[ \psi \to 0 \text{ as } e^{-i(\omega_j - \omega_i)x/g} \]  
when \( x \to +\infty \)

\[ |\nabla \psi| \to 0 \text{ when } y \to -\infty \]  
(28)

\[ \frac{\partial \psi}{\partial n} = n_2 \text{ on } S \]  
(29)

The solution of \( \psi \) is incorporated in the solution of the first order problem and can be considered known.

Let us now return to the \( \phi_{ij} \) problem. The horizontal force on the body due to \( \phi_{ij} \) can be written
\[ F^{ij} = -\pi i(\omega_j - \omega_i)^2 \int \phi_{ij} e^{-i(\omega_j - \omega_i)x} n_2 dy \]  
(30)

where positive normal vector \( n \) is into the fluid. We can write (see (29))
\[ \int_{S_{ij}} \phi_{ij} n_2 dS = \int_{S_{ij}} \frac{\partial \psi}{\partial n} dS \]  
(31)

We now apply Green's theorem, which means that
\[ \int_{S_{ij}} \frac{\partial \psi}{\partial n} dS = \int_{y = 0} \phi_{ij} n_2 ds \]  
(32)

Here \( S \) is a closed surface which we let consist of \( S \), a part of the free surface \( S_F \), vertical control surfaces \( S_0 \) and \( S_{\infty} \) far away from the body and a horizontal control surface \( S_B \) far down in the fluid. This is shown in figure 3 where \( a \) is

\[ \text{FIG. 3 POSITION OF BODY AT A CERTAIN INSTANT} \]
x-coordinate of $S_\infty$ and $-a$ is the x-coordinate of $S_{\infty}$. Further $b$ is the beam at waterline.

In expression (32) $\frac{\partial}{\partial n'}$ is the derivative along the normal to the boundary surface and $n'$ is positive out of the fluid, i.e.

$$\frac{\partial}{\partial n'} = \frac{\partial}{\partial n'}/n'$$
on $\bar{S}$

It is now possible to show that

$$F_{ij} = 2\rho \left(\omega_j - \omega_i\right)e^{-i\left(\omega_j - \omega_i\right)t}$$

$$\left[-\int \psi \frac{d(f_{ij})}{S^2} \right] + \lim_{a \to \infty} \frac{a}{b/2} \left(\psi \frac{d(f_{ij})}{x}\right) - \lim_{a \to \infty} \frac{a}{b/2} \left(\psi \frac{d(f_{ij})}{x}\right)

$$

$$\frac{\partial}{\partial n'} \left(\omega_j - \omega_i\right)^2 x/g

$$

$$i(\omega_j - \omega_i)^2 x/g

$$

$$F_{ij} = 2\rho \left(\omega_j - \omega_i\right)e^{-i\left(\omega_j - \omega_i\right)t}$$

$$\left[-\int \psi \frac{d(f_{ij})}{S^2} \right] + \lim_{a \to \infty} \frac{a}{b/2} \left(\psi \frac{d(f_{ij})}{x}\right) - \lim_{a \to \infty} \frac{a}{b/2} \left(\psi \frac{d(f_{ij})}{x}\right)

$$

$$\frac{\partial}{\partial n'} \left(\omega_j - \omega_i\right)^2 x/g

$$

$$i(\omega_j - \omega_i)^2 x/g

$$

where $\frac{b^2}{g^2}$ and $\gamma$ are determined by

$$\psi = \frac{e^{iy}}{\left(\omega_j - \omega_i\right)^2 x/g} \frac{b^2}{g^2}$$

$$\text{when } x \to \pm \infty.

By examining the solution procedure for the forced sway-problem it is possible to show that

$$F_{ij} = 0 \text{ when } i = j$$

For more details, see Faltinsen (17).

We have here only evaluated one contribution to the slowly varying horizontal driftforce. To derive the other terms, let us first consider Bernoulli’s equation i.e.

$$p = -\rho v - \frac{\partial l}{\partial t} - \frac{1}{2}(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 + p_0$$

where $p$ is the fluid pressure and $p_0$ atmospheric pressure. By Taylor expansion we can write the pressure on the body in terms of the values on the mean position of the body

$$p = -\rho v - \frac{\partial l}{\partial t} - \frac{1}{2}(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 + p_0$$

This is correct to second order in wave amplitude. In figure 4 is shown the position of the oscillating body at a certain instant.

![FIG. 4 POSITION OF BODY AT A CERTAIN INSTANT](image-url)

The first term in (37) is the hydrostatic pressure which causes the following horizontal force on the body

$$F_{2H} = \frac{\rho g b/2}{x - b/2} - \frac{\rho g b/2}{y = 0}$$

By integrating the second term $-\rho \frac{\partial l}{\partial t}$ m
in (37) over the mean wetted surface we obtain the added mass and damping coefficients. Since it is a first order term we have to be careful in the integration and take proper account of the changing wetted surface of the body.

We can write the force component in the $x$-direction (not $x$-direction) due to $\frac{3\phi}{\partial t^3}$ correct to second order as

$$
-\rho \frac{3\phi}{\partial t^3} \left\{ \left( -\frac{1}{g} \frac{3\phi}{\partial t^1} \right)_{x=b/2}^{y=0} \right. \\
\left. - \left( \eta + b/2 \eta \right) \right\} - \\
\left( \eta - b/2 \eta \right) \\
\left( \eta \right) \\
\left( \eta \right)
$$

(39)

We are however, interested in the force component in the $x$-direction (not $x$-direction). Keeping only second order terms the added mass and damping terms in sway and roll will not cause any slowly varying force in the $x$-direction. This is not true for the added mass and damping force in heave. We can write the vertical added mass and damping force in the $y$-direction (not $y$-direction) as

$$
- A_{33} \frac{d^2 n_1}{d t^2} - B_{33} \frac{d n_1}{d t} \\
= \frac{d^2 n_1}{d t^2} - \frac{d n_1}{d t}
$$

(40)

This force has a component in the $x$-direction which correct to second order in wave amplitude can be written as

$$
\eta _n \left( A_{33} \frac{d^2 n_1}{d t^2} + B_{33} \frac{d n_1}{d t} \right)
$$

(41)

The slowly varying horizontal force can now be obtained by summing the expression for $F_{ij}$ (see (33)) over $i$ and $j$ and by adding contributions from equations (38), (41) and (42).

$$
The slowdriftexcitation force and the driftforce can be written as$$
The driftforce is

\[ F_{SV} = \sum_{i=1}^{N} A_i^2 \sum_{j=1}^{N} A_j \{ F_{ij} \cos(\omega_j - \omega_i) t + F_{ij} S \sin(\omega_j - \omega_i) t \} \]

(43)

A degree of ambiguity exist in the coefficients \( F_{ij}^C \) and \( F_{ij}^S \) when \( i \neq j \). We could for example impose the restriction that \( F_{ij}^C \) and \( F_{ij}^S \) are equal to zero when \( i > j \). Another possibility is to require that \( F_{ij}^C = F_{ji}^C \) and \( F_{ij}^S = -F_{ji}^S \) when \( j > i \). This was done by Newman (3) and in the presentation of our results we will follow the same procedure.

\( F_{ij}^C \) and \( F_{ij}^S \) depend on the phaseangles \( \xi_i \) and \( \xi_j \), but we could rewrite equation (43) in terms of second order transfer-functions.

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**DRIFTFORCES AND SLOWDRIFTOSCILLATIONS IN OBLIQUE SEA**

The procedure presented above may be generalized to oblique sea. There would be differences in the solution procedure. For instance it is not longer appropriate to state that the 1. order potential satisfies a 2-D Laplace equation. It is only the forced velocity potentials that satisfies a 2-D Laplace equation. The incident wave potential satisfies a 2-D Helmholtz equation. The same is true for the diffraction potential as long as the wave propagation direction is not close to either head sea or following sea. In those cases a more significant three-dimensional effect is present (Faltinsen (18)). We have not tried to generalize the beam-sea procedure yet. It is important first to examine numerically in detail our complicated beam sea procedure versus the much simpler procedure by Hsu and Blenkarn (1) and Newman (3), where the driftforces and driftyawmoments in regular waves are the important building brick in establishing a theory for slowdriftooscillations. Presently we have developed a method according to Hsu and Blenkarn to calculate slowdriftooscillations of a ship in oblique sea. In doing this we have developed a new procedure to calculate driftforces and yawmoments which will be presented below.

According to Newman (7) we can write the horizontal driftforce and driftyawmoment on a ship in regular sinusoidal waves in infinite water depth as

\[ F_x = \frac{\partial^2}{\partial \theta^2} \int_0^\pi |H(\theta)|^2 \cos \theta \, d\theta + \frac{1}{2} \partial^1 \partial \theta \ \xi_a \cos \theta \ \text{Re}\{H(\pi+\theta)\} \]

(45)

\[ F_y = \frac{\partial^2}{\partial \theta^2} \int_0^\pi |H(\theta)|^2 \sin \theta \, d\theta + \frac{1}{2} \partial^1 \partial \theta \ \xi_a \sin \theta \ \text{Re}\{H(\pi+\theta)\} \]

(46)

\[ M_z = -\frac{\partial}{\partial \theta} \int_0^\pi |H(\theta)|^2 H'(\theta) \, d\theta + \frac{1}{2\partial} \partial^1 \partial \theta \ \xi_a \ \text{Im}\{H'/(\pi+\theta)\} \]

(47)

where \( H'(\pi+\theta) \) is to be interpreted as

\[ \left[ \frac{d}{d\theta} H(\theta) \right]_{\theta=\pi+\theta} \]

Expression (45), (46) and (47) are slightly different from Newman's expressions due to different expressions for the incident wave potential.

\( F_x \) and \( F_y \) are the driftforce components along the \( x-\) and \( y-\)axis, respectively and \( M_z \) the driftyawmoment about the \( z-\)axis.

The coordinate system used in this chapter do not degenerate to the two-dimensional coordinate system used in the last chapter. A right-handed coordinate system \((x,y,z)\) fixed with respect to the mean position of the ship is used, with posi-
tive z vertically through the centre of gravity of the ship and the origin in the plane of the undisturbed free surface (see figure 5). Further k is the wave number, ω the circular frequency of oscillation, ε the wave amplitude and θ the angle between the x-axis and the propagation direction of the incident waves. θ = 0° corresponds to head sea and θ = 90° corresponds to beam sea. It is assumed the ship has zero mean forward speed.

The asterisk denote the complex conjugate. H(θ) is the Kochin function which is given by

\[ H(\theta) = \int \frac{3 \phi^*_B}{3n} \exp(kz + ikx\cos\theta + ikysin\theta) \, dS_B \]

Here \( \phi_B e^{-i\omega t} \) is the velocity potential due to the presence of the ship. That means we may write the total velocity potential as

\[ \phi_B e^{-i\omega t} = \int \frac{3 \phi^*_B}{3n} \exp(kz + ikx\cos\theta + ikysin\theta) + \phi_B e^{-i\omega t} \]

where the first part is the incident wave potential. The integration in equation (48) is over the mean body surface \( S_B \).

Further \( 3/3n \) is the derivative along the normal vector of the body surface (positive normal direction is out of the fluid). Due to the body boundary condition we can write

\[ \frac{3 \phi^*_B}{3n} = - \frac{3 \phi^*_B}{3n} + U_n \]

in equation (48). Here

\[ \frac{3 \phi^*_B}{3n} = \eta_x \sin\theta \ \omega \ \epsilon \ \exp(kz \cos\theta + kysin\theta) - \]

in \( \omega \ \epsilon \ \exp(kz \cos\theta + kysin\theta) \)

and

\[ U_n = -i\omega n_2 (\bar{\eta}_2 - z\eta_2^* + x\eta_x^*) - \]

in \( \omega n_3 (\bar{\eta}_3 + y\eta_y^* - x\eta_x^*) \)

Further \( n_2 \) and \( n_3 \) are unit normal components along the y- and z-axis respectively and

\[ \eta_i = \bar{\eta}_i e^{-i\omega t} \quad i = 1, 6 \]

are the six modes of motion so that \( \bar{\eta}_2, \bar{\eta}_3, \bar{\eta}_4, \bar{\eta}_5, \bar{\eta}_6 \) are the complex amplitudes of sway, heave, roll, pitch and yaw respectively (see figure 5).

The velocity potential due to the presence of the body \( \phi_B e^{-i\omega t} \) may be approximated in different ways. Newman (7) used slender body theory combined with a long wave length assumption which is quite limited in application. In this paper we make a short wave length assumption. In the forced motion case this leads to the strip theory commonly used in ship motion calculation.

We write

\[ \phi_B e^{-i\omega t} = \sum_{j=2}^{6} \phi_j \eta_j + \phi_0 e^{-i\omega t} \]

The contribution from the surge motion \( \eta_1 \) are neglected due to the slenderness of the ship. The other forced velocity potentials \( \phi_i, \ i = 2, 6 \) are calculated using strip theory and solving 2-D Laplace equation problems for the cross-section of
the ship. The boundary value problems are of the same form as described for the 1.
order velocity potential in the beam sea case. The Lewis form technique or the
Frank Closefit method may be used to solve the problem. The so-called diffraction
potential $\psi e^{-i\omega t}$, i.e. the correction to the incident wave potential when the ship
is restrained from oscillating is a much more difficult task to evaluate. In the
head sea case and for high frequencies which in practice means for a wave length
to ship length ratio less than about 0.6,
the analysis has to be pursued along the
lines carried out by Faltinsen (18). This
involves solving the Helmholtz equation
away from the bow region. Faltinsen's
procedure is not valid near the bow, but
Maruo (19) has presented a solution which
he claims is uniformly valid along the
ship length. The following sea case is
not principally any different from the
head sea case. For heading angles signifi-
cant different from 0° and 180° the
diffraction potential may be solved by
strip theory and solving 2-D Helmholtz
equation problems along the ship. (Ursell
(20) and (21)). Ursell's solution breaks
down when $\beta \to 0^\circ$ and 180°. When $\beta = 90^\circ$
it reduces to a 2-D Laplace equation. The
Ursell equation is used in this paper and
the computation is carried out by a com-
puter program developed by Troesch (22).
Numerical calculations are presented in
the next chapter.

NUMERICAL CALCULATIONS
The two methods described above have been
compared with other methods used to pre-
dict driftforces and moments. Results for
second order difference frequency force
due to presence of two simultaneous waves
with frequencies $\omega_1$ and $\omega_2$ are also presen-
ted. The other methods used to predict
driftforce and moments are:

a) Faltinsen's and Michelsen's method
(Faltinsen and Michelsen (11)). In this
method a 3-dimensional sink-source tech-
nique is used to calculate the 1. order
potential. The method is applicable to
any structure, any wave heading and to
finite water depth. The expressions for
driftforce and yawmoment are a generali-
zation of Newman's expressions (Newman
(7)).

b) Newman's long wave length method where
the 1. order potential is obtained by long
wave length slender body theory and
Newman's expressions (7) are used.

c) Maruo's beam sea formula (Maruo (4)).
According to Maruo the driftforce in beam
sea can be written as

$$F_y = \frac{\rho g}{2} \int |A|^2 dx$$

(54)

where the integration is over the ship
length and $|A|^2$ is the amplitude of reflec-
ted wave for each cross-section. The
reflected wave is both due to the swaying,
heaving and rolling of a ship and the
diffraction effect of the restrained ship.
Both the forced motion potentials and the
diffraction potential will be calculated
by Lewisform technique. The calculations
are described in detail by Faltinsen (16).
The formula (54) is sometimes used for
oblique sea. The rational explanation for
doing that is questionable.

d) Salvesen's formula (Salvesen (8)).

e) Kim and Chou's formula Kim and Chou
(10) use equation (54) for oblique sea and
find the reflected wave amplitude by solv-
ing 2-D Laplace equation problems for the
diffraction problem. The Frank Closefit
method is used.

f) Gerritsma and Beukelman's formula (5)
The formula, which is derived for head sea,
is a simple expression that gives good
results in many cases. The rational basis
for the formula is somewhat vague as it is
based on the relative motion hypothesis.
The two methods presented in this report will be denoted Faltinsen's formula and Newman-Helmholtz formula.

Description of models used
Three different hull forms were used in the numerical computations; a circular cylinder with \( L \times B \times d = 120 \text{ m} \times 20 \text{ m} \times 10 \text{ m} \) and roll radius of gyration equal to 6 m, a Series 60 ship \( (C_p = 0.6) \) and a loaded tanker. The transverse metacentric height of the circular cylinder is 0.027 m (which is unrealistic low). The centre of gravity is in the mean position of the free surface. The principal dimensions of the tanker are presented in table 1. The hull geometry is shown in figure 6.

Information about Series 60 ships may be found in Todd (23).

TABLE 1: MAIN PARTICULARS OF 130,000 DWT TANKER

<table>
<thead>
<tr>
<th>PARAMETER</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length between perpendiculars</td>
<td>285.60 (m)</td>
</tr>
<tr>
<td>Beam</td>
<td>46.71 (m)</td>
</tr>
<tr>
<td>Depth</td>
<td>20.35 (m)</td>
</tr>
<tr>
<td>Draught fore</td>
<td>13.82 (m)</td>
</tr>
<tr>
<td>Draught aft</td>
<td>13.82 (m)</td>
</tr>
<tr>
<td>Draught mean</td>
<td>13.82 (m)</td>
</tr>
<tr>
<td>Displacement</td>
<td>154980.00 tons</td>
</tr>
<tr>
<td>Centre of gravity, longitud. from ( % )</td>
<td>+ 6.46 (m)</td>
</tr>
<tr>
<td>Centre of gravity, transv. from baseline</td>
<td>11.03 (m)</td>
</tr>
<tr>
<td>Metacentric height</td>
<td>8.97 (m)</td>
</tr>
<tr>
<td>Pitch/yaw radius of gyration</td>
<td>71.40 (m)</td>
</tr>
<tr>
<td>Roll radius of gyration</td>
<td>16.35 (m)</td>
</tr>
<tr>
<td>Natural pitch period</td>
<td>9.80 (s)</td>
</tr>
<tr>
<td>Natural roll period</td>
<td>12.80 (s)</td>
</tr>
</tbody>
</table>

Note: Centre of gravity, longitud. from \( \% \): + means forward of or station 10.

Results from numerical calculations of driftforces
Cylinder
Figure 7 shows driftforces on a circular cylinder in beam sea. The results are presented as horizontal driftforce nondimensionelized by \( \rho g \tau_{a}^{2} \). \( L \) as a function of \( \lambda/L \) where \( \lambda \) is the wave length of the incident wave. The asymptotic value for
small $\lambda/L$ is 0.5 which follows from equation (54) and the fact that the wave is totally reflected for small wave length. Both results for restrained ship and free-floating ship are presented. We note a marked influence of the body motions. The Maruo's beam sea formula, the Faltinsen's formula and Newman-Helmholtz' formula agrees quite well. The three formulas are quite different in form, but they are all correct to second order in wave amplitude. We note that the Newman-Helmholtz formula may give somewhat higher values than 0.5 for the nondimensionalized force. According to Maruo's beam sea formula equation (54) this correspond to a higher reflected wave amplitude than the incident wave amplitude. This is unphysical and the results must be due to numerical approximations.

The values calculated by Salvesen's formula are considerably lower than values calculated by the other three methods mentioned above. But it should be noted that Salvesen used the assumption that the velocity potential due to the presence of the body ($\phi_B$) is much smaller than the incident wave potential. This is questionable for small wave lengths. Asymptotically for small $\lambda/L$ the amplitude of $\phi_B$ is equal to the amplitude of $\phi$. This explain the general tendency that Salvesen's method diverge more and more from the other three methods for decreasing wave lengths.

**Series 60 ship**

Calculations of driftforces on a Series 60 ship ($C_B = 0.6$) in beam sea is presented in figure 8. Both Maruo's beam sea formula and Newman-Helmholtz formula are used. The values are in close agreement except in the vicinity of $\lambda/L = 0.9$ which is close to resonance in roll. This may be explained in the following way. The two formulas are derived using conservation of energy assuming no energy loss due to viscous effects. But viscous effects are very significant in the calculation of roll around resonance. See Salvesen, Tuck and Faltinsen (24), which is the method used to calculate 1. order potential.

In figure 9 are presented results for the longitudinal driftforce component in wave heading 60°. The agreement between the Newman-Helmholtz formula and the Newman-long-wave-length formula is quite poor for smaller wave length. The same tendency to disagreement is seen in figures 10 to 12. In figure 10 is presented transverse driftforce component for wave heading 60° and in figure 11 and 12 longitudinal and transverse driftforce component for wave heading 45°. In figure 10 are also analytical results by Kim and Chou (10).
and Salvesen (8) in addition to experimental results by Ogawa (25) presented. Ogawa's results are for Series 60, \( C_b = 0.7 \), but previous experience indicates that the influence of block coefficient is not significant. We note a good agreement between Newman-Helmholtz formula, Ogawa's experimental result and Kim and Chou's formula. Salvesen's results for a Mariner hull with block coefficient 0.61 is significant smaller than the results by Newman-Helmholtz formula. The same tendency was seen for the circular cylinder and can be explained in the same way as for the circular cylinder.

In figure 13 and 14 are presented results for drift yaw moment on Series 60 ship (\( C_b = 0.6 \)) for wave headings 45° and 60°. We note that the experimental values by Ogawa agree to some degree with Newman-Helmholtz formula and Kim and Chou's formula, while both the results by Newman's long wave length formula and Salvesen's formula disagree significantly with the experimental values.
Loaded tanker

Results for transverse drift force component on the loaded tanker in beam sea are presented in figure 15. We note that Newman-Helmholtz formula agrees well with Maruo's beam sea formula except around roll resonance. This has been explained earlier to be due to viscous effects. The results by Faltinsen and Michelsen's formula are expected to be the most accurate of the three methods. We note quite good agreement between Newman-Helmholtz formula and Faltinsen-Michelsen formula.

Results for longitudinal drift force component and drift yaw moment in beam sea are presented in figure 16. Keeping in mind that the quantities are quite small, we note quite good agreement between Faltinsen-Michelsen's formula and Newman-Helmholtz formula.

Results for longitudinal and transverse drift force components as well as drift yaw moment for wave heading 45° are presented in figure 17 and 18. The difference between Newman-Helmholtz formula and Faltinsen-Michelsen formula is greater than in beam sea.

The Newman-Helmholtz formula breaks down in head sea. The reason is that the solution of the 1st order potential by Helmholtz equation breaks down for head sea (Troesch (22)). The results by the
method in head sea are presented in figure 19. In the same figure are presented

results by Faltinsen and Michelsen's formula and Gerritsma and Beukelman's formula. The agreement between these two methods are not satisfactory.

Gerritsma and Beukelman's formula gives generally satisfactory results for added resistance of a ship at Froude number different from zero. But it should be noted that the quantities calculated here are quite small and hence more subjected to influence from surge motion and the bluntness of the bow which are neglected in Gerritsma and Beukelman's formula.

The first order motions are important quantities in the calculations of drift-forces and moments. It is therefore of interest to compare the calculation of 1. order motion by a three-dimensional sink-source method (Faltinsen and Michelsen method) and a two-dimensional method (strip theory) which is the basis of the Gerritsma and Beukelman's method. These calculations are shown in figure 20 to 24. The calculations are for heading angles 45° and 90° and are performed for the same loaded tanker as mentioned earlier in the text. We note that surge, sway, heave, pitch and yaw predicted by two-dimensional method (strip theory) and three-dimensional method agrees quite well. In general this is true for roll motion, too. But we note
the large discrepancy around roll resonance. The reason is that the three-dimensional method does not include roll-viscous damping while the strip method does.

**Calculation of second order difference frequency force**

The second order difference frequency force due to presence of two simultaneous waves with frequencies \( \omega_j \) and \( \omega_i \) were calculated. The circular cylinder described above was selected. Beam sea was assumed. The nondimensionalized wave frequencies \( \omega_i \) and \( \omega_j \) were chosen among the numbers 0.59, 0.79, 0.84, 0.95 and 1.12. The phase angles \( \xi \) and \( \eta \) were chosen equal to zero. The results are presented as \( F_{ij}^C/(\rho g L) \) and \( F_{ij}^S/(\rho g L) \) in table 2 and 4. \( F_{ij}^C \) and \( F_{ij}^S \) are defined by equation (43). Newman (3) assumed that \( F_{ij}^C \) could be approximated by \( F_{ii}^C \) and that \( F_{ij}^S \) was small when \( \omega_j \) were close to \( \omega_i \). Our results indicates that Newman's approximative way of calculating slow driftoscillations is reasonable. But a final conclusion cannot be made before more extensive investigations.
for other cross-sections have been performed.

In Table 3 and 5 are presented the contribution to $F_{ij}^C$ and $F_{ij}^S$ from the second order potential, which means the contribution from equation (33). This contribution is often quite small for the particular structure studied here.

<table>
<thead>
<tr>
<th>$\omega_j/\sqrt{g}$</th>
<th>$F_{ij}^C/(\rho gL)$</th>
<th>$F_{ij}^S/(\rho gL)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.59</td>
<td>0.01</td>
<td>-0.01</td>
</tr>
<tr>
<td>0.72</td>
<td>0.00</td>
<td>-0.03</td>
</tr>
<tr>
<td>0.84</td>
<td>-0.03</td>
<td>0.03</td>
</tr>
<tr>
<td>0.95</td>
<td>0.01</td>
<td>-0.01</td>
</tr>
<tr>
<td>1.12</td>
<td>0.02</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

Table 2: Numerical Calculation of $F_{ij}^C/(\rho gL)$ for Circular Cylinder in Beam Sea

<table>
<thead>
<tr>
<th>$\omega_j/\sqrt{g}$</th>
<th>$F_{ij}^C/(\rho gL)$</th>
<th>$F_{ij}^S/(\rho gL)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.59</td>
<td>0.13</td>
<td>0.06</td>
</tr>
<tr>
<td>0.72</td>
<td>0.10</td>
<td>0.00</td>
</tr>
<tr>
<td>0.84</td>
<td>0.02</td>
<td>0.00</td>
</tr>
<tr>
<td>0.95</td>
<td>0.06</td>
<td>0.00</td>
</tr>
<tr>
<td>1.12</td>
<td>0.10</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Table 3: Contribution to $F_{ij}^C/(\rho gL)$ From Second Order Potential

<table>
<thead>
<tr>
<th>$\omega_j/\sqrt{g}$</th>
<th>$F_{ij}^C/(\rho gL)$</th>
<th>$F_{ij}^S/(\rho gL)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.59</td>
<td>-0.16</td>
<td>-0.17</td>
</tr>
<tr>
<td>0.72</td>
<td>-0.12</td>
<td>-0.17</td>
</tr>
<tr>
<td>0.84</td>
<td>-0.04</td>
<td>-0.09</td>
</tr>
<tr>
<td>0.95</td>
<td>0.01</td>
<td>0.09</td>
</tr>
<tr>
<td>1.12</td>
<td>-0.01</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 4: Numerical Calculation of $F_{ij}^S/(\rho gL)$ for Circular Cylinder in Beam Sea

CONCLUSIONS

A method to predict slow-drift oscillations and driftforces on a ship in beam sea is presented. It is found that the method predicts driftforces satisfactory. Results for second order difference frequency force due to presence of two simultaneous waves with frequencies $\omega_j$ and $\omega_i$ are presented. The results indicate that Newman's approximative way of calculating slow-driftoscillation is reasonable. But a more extensive investigation is necessary before a final conclusion can be made.

In oblique sea the driftforces and drift-yawmoment are calculated by a new procedure denoted Newman-Hemholtz formula. The procedure shows satisfactory results as long as the wave headings are not close to either head-sea or following sea.

ACKNOWLEDGMENT

This study has been financially supported by the Royal Norwegian Council of Scientific and Industrial Research (NTNF) and by Det norske Veritas.

REFERENCES


NOMENCLATURE

\((x, y)\) Two-dimensional coordinatesystem fixed with respect to the ship. See figure 2

\((x, y)\) Two-dimensional coordinatesystem fixed in space. See figure 2

\((x, y, z)\) Three-dimensional coordinatesystem fixed in space. See figure 5

\(\phi\) Velocity potential

\(\phi_1\) is linear with respect to wave amplitude

\(\phi_2\) is quadratic with respect to wave amplitude and so forth (see equation 1)

\(\phi_I\) Incident wave potential correct to 1. order in wave amplitude (see equation 2)

\(g\) Acceleration of gravity

\(A_1\) Amplitude of first order incident wave components, (see equation 2)

\(\omega_1\) Circular frequency of oscillation of first order incident wave components (see equation 2)

\(v_i = \frac{\omega^2_1}{g}\)

\(\varepsilon_i\) Random phase angles of first order incident wave components (see equation 2)

\(t\) time variable

\(n_i\) \(n_2\) is sway, \(n_3\) is heave, \(n_4\) is roll, \(n_5\) is pitch and \(n_6\) is yaw.

\(e_x, e_y\) Unit vectors along the \(x\)- and \(y\)-axis

\(\vec{n}\) Unit normal vector to the body surface. Positive direction is into the fluid

\(\phi_1\) Asymmetric part of first order potential (see equation 8)

\(s\) Symmetric part of first order potential (see equation 8)

\(n_2 = e_x \cdot \vec{n}\)

\(n_3 = e_y \cdot \vec{n}\)

\(\phi_{ij}\) See equation (17)

\(P_x, P_y\) Driftforce-components along the \(x\)-axis and \(y\)-axis of the three-dimensional coordinate system \((x, y, z)\)

\(M_2\) Driftyawmoment with respect to \(z\)-axis

\(k\) Wave number of regular 1. order waves

\(\zeta_a\) Wave amplitude of incident 1. order regular waves

\(\beta\) Angle between the \(x\)-axis and the propagation direction of the incident waves (\(= 0^\circ\) head sea)

\(H(\theta)\) Kochin function (see equation 48)

\(L\) Length between perpendiculars

\(B\) Beam

\(d\) Draught

\(\lambda\) Wave length of 1. order regular incident waves

\(\rho\) Mass density of water
SUMMARY. The theory of optimal processes is concerned with the problem of finding conditions for a solution of a set of differential equations, depending on control-variables, such that a functional attains its maximum value. Moreover the solution is subject to constraints. A "maximum-principle" which has to be satisfied by the control of an optimal process was proved in 1958 by PONTRYAGIN and his coworkers; therefore this principle is known as Pontryagin's maximum-principle, though HESTENES proved this principle as early as 1950 in a technical report published by Rand-Corporation. PONTRYAGIN's proof of the maximum-principle is complex. TIMMAN tried to find another way to derive conditions for optimal processes and succeeded in 1966 to develop simple variational means which lead to the maximum-principle [1]. In this paper we have in view to show the power of TIMMAN's philosophy concerning problems of optimal control, governed by ordinary or partial differential equations.

1. INTRODUCTION.

Optimal control theory has recently obtained the attention of many mathematicians and is one of the major areas of applied mathematics today. Though fairly complete results are obtained for systems governed by ordinary differential equations much work remains to be done; moreover systems governed by partial equations constitute an important area of research nowadays. In the sixties TIMMAN became involved in optimal control problems and succeeded to find conditions for solutions of optimization problems by simple variational methods [1]. In the following we try to point out his ideas by applying a variational approach to problems of optimal control. Let us first consider an example; let a set of consecutive reactions

\[ X_1 + X_2 \rightarrow \text{decomposition products} \]

be described by the differential equations

\[
\frac{dx_1}{dt} = -k_1(u)f_1(x_1)
\]

\[
\frac{dx_2}{dt} = k_1(u)f_1(x_1) - k_2(u)f_2(x_2)
\]

where:

- \( x_1(t) \) and \( x_2(t) \) denote the concentrations of \( X_1 \) and \( X_2 \),
- \( k_1(u) \) and \( k_2(u) \) have the Arrhenius form
  \[ k_1 = c_1 e^{-E_1/RT}, \quad k_2 = c_2 e^{-E_2/RT} \]
  depending on the temperature \( u(t) \),
- \( f_1 \) and \( f_2 \) are non-negative increasing functions.
- \( X_2 \) is the desired product, so the problem is to find the optimal temperature profile \( u(t) \) which maximizes

\[
x_2(T) - x_2(0) - \int_0^T g(u)\,dt
\]

where the integral represents the cost of operation.
and T is the reaction-period. From the differential equations we have that 
\[ x_2(T) - x_2(0) = \int_0^T [v_k_1(u)f_1(x_1) - k_2(u)f_2(x_2)]\,dt \]
Thus the problem is to determine the control u(t) such that the initial concentrations \( x_1(0) \) and \( x_2(0) \) define a solution of the differential equations which solution describes the course of the optimal reaction for which the integral 
\[ T \int_0^T [v_k_1(u)f_1(x_1) + k_2(u)f_2(x_2)]\,dt. \]
attains its minimum value. 

The problem of the calculus of variations is another example of an optimal control problem. This problem might be formulated as the problem of finding a control u(t) such that the initial state \( x(T_0) \) defines a solution of the equation 
\[ \frac{dx}{dt} = f(x,u,t) \]
which solution corresponds to the minimum value of an integral 
\[ T_1 \int_0^{T_1} F(x,u,t)\,dt = \int_0^{T_1} F(x,u,t)\,dt. \]

2. ORDINARY DIFFERENTIAL EQUATIONS; THE MAXIMUM PRINCIPLE.

Let us consider a system with state 
\[ x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \]
depending on the time t, which state satisfies a set of ordinary differential equations 
\[ \frac{dx}{dt} = f(x,u,t) \tag{1} \]
depending on a control 
\[ u(t) = (u_1(t), u_2(t), \ldots, u_n(t)). \]

We are interested in piecewise continuous admissible controls such that corresponding solutions of the differential equations satisfy the conditions 
\[ x(T_0) = X_0 \text{ (initial state)} \]
\[ x(T_1) = X_1 \text{ (final state)} \tag{2} \]

Among these controls the optimal control minimizes an integral 
\[ T_1 \int_0^{T_1} F(x,u,t)\,dt. \]
Moreover the control u(t) is subject to the condition that it must belong to some admissible region U. The functions F and f are assumed to be sufficiently smooth. Let us denote the minimum value of the integral by \( I(X_1,T_1) \); clearly this value depends on the final state \( X_1 \) and the final time \( T_1 \).

To find conditions for the optimal control we consider a piecewise continuous admissible variation \( \delta u(t) \) of the optimal control u(t) such that the variation \( \delta x(t) \) of the optimal state x(t) corresponds to a neighboring state, which means that \( |\delta x(t)| \) is uniformly small on \([T_0,T_1] \). If \( \delta u(t) \) is such that 
\[ \frac{\delta}{\delta u(t)} \int_0^{T_1} |\delta u(t)|\,dt \]
is small then x(t) + \( \delta x(t) \) is a neighboring state of the optimal state x(t).

Now consider the difference 
\[ V = \int_0^{T_1} \delta F\,dt \]
where 
\[ \delta F = F(x + \delta x, u + \delta u, t) - F(x,u,t) = F(x + \delta x, u + \delta u, t) - F(x, \delta x, u, t) + F(x, \delta x, u, t) - F(x,u,t) \]
If we neglect terms of the second order and if we add the vanishing sum 
\[ \sum_{i=1}^n \delta P_i \delta x_i - \delta F \]
to \( \delta F \), where the functions \( P_1, P_2, \ldots, P_n \) will be specified next, we find for \( \delta F \): 
\[ \delta F = F(x + \delta x, u + \delta u, t) - F(x, \delta x, u, t) + \sum_i \frac{\partial F}{\partial x_i} \delta x_i + \sum_{i=1}^n \delta P_i (\delta x_i - \delta F) \]
From 
\[ \sum_i \delta P_i \delta x_i = \sum_i \frac{\partial F}{\partial x_i} \delta x_i - \delta F \]
and 
\[ \delta F_i = f_i(x + \delta x, u + \delta u, t) - f_i(x, \delta x, u, t) + \frac{\partial f_i}{\partial x_i} \delta x_i \]

it follows that 
\[ \delta F = F(x + \delta x, u + \delta u, t) - F(x, \delta x, u, t) \]
from which expression we take the definition of the functions \( P_1, P_2, \ldots, P_n \). We define these functions to be such that 
\[ P_1 - \frac{\partial F}{\partial x_1} + P_2 \frac{\partial F}{\partial x_2} = 0, \]
from which we obtain the adjoint equations
\[ \dot{p}_i = \frac{\partial F}{\partial x_i} - \sum \frac{\partial f_j}{\partial x_i}, \quad i = 1, 2, \ldots, n \quad (3) \]

and that
\[ \int_{T_0}^{T_1} \dot{x}_i(T_1) = \delta_i, \]
where \( \delta_i \) denote the first variation of \( I \) due to the variation \( \delta x(T_1) \). Then it follows from
\[ v - \delta_1 I \geq 0 \]
that
\[ T_1 \int_{T_0}^{T_1} \left[ H(x+\delta x, u+\delta u, p, t) - H(x, u, p, t) \right] dt \leq 0 \]
where
\[ H = - F + \sum P_i f_i \]

Let finally \( \delta u(t) \) be zero outside an interval \([t-\Delta, t+\Delta]\)
and let \( \Delta \) tend to zero. Then one finds by applying a mean-value theorem that
\[ H(x, u + \delta u, p, t) \leq H(x, u, p, t) \]
which inequality expresses the maximum-principle: the optimal control maximizes the Hamiltonian \( H \) on the admissible region \( U \).

3. ENDCONDITIONS, TRANSVERSALITY-CONDITIONS.

Thusfar we considered the "fixed end-point problem" with fixed final state \( x_1 \) and fixed final time \( T_1 \).
Let us now consider problems where the final state is assumed to belong to some region in \((x, t)\)-space. Then the endcondition \( x(T_1) = x_1 \) no longer holds.
In order to find the appropriate endconditions for the adjoint equations we first notice that we have the relations
\[ \dot{x}_i = p_i, \quad i = 1, 2, \ldots, n \]
\[ \dot{H} = H \]
for the derivatives of \( I \). The first relation follows from the endcondition we imposed on the adjoint equations and the second relation we find easily by the following argument: a small variation \( \delta x = \delta x(t) \) along the optimal trajectory from \((X_0, T_0)\) to a point \((X, T)\) in \((x, t)\)-space effects the variation
\[ \delta I = I(x+\delta x, t+\delta t) - I(x, t) \]
\[ \delta I = \delta I / \delta x + \delta I / \delta t \]
up to terms of the second order. From \( \delta I = \delta x(t) \) we therefore obtain the "Hamilton-Jacobi equation"
5. PARTIAL DIFFERENTIAL EQUATIONS.

Let us consider a system with state $z(x,y)$, which state satisfies a partial differential equation

$$Lz = f(x,y,z,p,q,u),$$

where

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y},$$

$$L = \frac{\partial}{\partial x} L_1 + \frac{\partial}{\partial y} L_2,$$

$$L_1 = a \frac{\partial^2}{\partial x^2} + b \frac{\partial^2}{\partial y^2},$$

$$L_2 = c \frac{\partial^2}{\partial x^2} + d \frac{\partial^2}{\partial y^2} \eta$$

Thus

$$L = a \frac{\partial^2}{\partial x^2} + 2 \eta \frac{\partial^2}{\partial x \partial y} + \frac{\partial^2}{\partial y^2},$$

where $a, b, c, \eta$ are constants. The differential equation depends on a control-function $u(x,y)$.

Let $R$ be the region $0 \leq x \leq X$

$0 \leq y \leq Y$

with boundary $\partial R$.

Boundary-conditions will be specified later on. We are interested in a control $u(x,y)$ which is piecewise continuous in each of the variables $x, y$ and which minimizes the functional

$$J = \int \int_R F(x,y,z,p,q,u) \, dx \, dy + \int \int_{\partial R} G(x,y,z,u) \, ds,$$

where $s$ denotes the arc-length of the boundary $\partial R$.

Moreover the control $u(x,y)$ is subject to the condition that it must belong to some admissible region $U$.

The functions $F$, $G$ and $f$ are supposed to be sufficiently smooth. To an admissible variation $\delta u(x,y)$ of the optimal control $u(x,y)$ correspond variations $\delta z(x,y)$, $\delta p(x,y)$, $\delta q(x,y)$ of $z(x,y)$, $p(x,y)$, $q(x,y)$.

The problem is assumed to be well posed in the sense that, if boundary-conditions do not depend on the control, $\delta z$, $\delta p$ and $\delta q$ depend continuously on $\delta u$. This means that a variation $\delta u$ such that

$$\int_0^x |\delta u(\xi,y)| \, d\xi,$$

$$\int_0^y |\delta u(x,\eta)| \, d\eta$$

are small implies a neighbouring state of the optimal state; a neighbouring state is such that $|\delta z|, |\delta p|, |\delta q|$ are uniformly small on $R$.

Now consider the difference

$$V = \int \int_R \delta F \, dx \, dy + \int \int_{\partial R} \delta G \, ds \geq 0,$$

where

$$\delta F = F(x,y,z+\delta z,p+\delta p,q+\delta q,u+\delta u) - F(x,y,z,p,q,u) + \delta z \frac{\partial F}{\partial z} + \delta p \frac{\partial F}{\partial p} + \delta q \frac{\partial F}{\partial q}$$

and

$$\delta G = G(x,y,z+\delta z,u+\delta u) - G(x,y,z,u) + \delta z \frac{\partial G}{\partial z}$$

up to terms of the second order. To $\delta F$ we add the term

$$\psi(x,y)(L\delta z - \delta f)$$

where the function $\psi$ will be specified next. Now

$$\psi_{L\delta z} = \delta z \frac{\partial}{\partial z} \psi + \frac{\delta z}{\partial x} \frac{\partial}{\partial x} \psi + \frac{\delta z}{\partial y} \frac{\partial}{\partial y} \psi$$

and

$$\psi(\delta z,p,q) = \psi_{L\delta z} - \delta z \frac{\partial}{\partial z} \psi \frac{\partial}{\partial z} \psi + \delta p \frac{\partial}{\partial p} \psi \frac{\partial}{\partial p} \psi + \delta q \frac{\partial}{\partial q} \psi \frac{\partial}{\partial q} \psi$$

If we replace $F \delta z + F \delta p + F \delta q$ by

$$F \delta z + \frac{\partial}{\partial z} \left( F \delta z + \frac{\partial}{\partial p} \left( F \delta p + \frac{\partial}{\partial q} \left( F \delta q \right) \right) \right)$$

the following integrals appear in $V$

$$\int \int \frac{\partial}{\partial x} \left( \psi_{L\delta z} - \delta z \frac{\partial}{\partial z} \psi \frac{\partial}{\partial z} \psi + \delta p \frac{\partial}{\partial p} \psi \frac{\partial}{\partial p} \psi + \delta q \frac{\partial}{\partial q} \psi \frac{\partial}{\partial q} \psi \right) \, dx \, dy$$

$$+ \int \int \frac{\partial}{\partial y} \left( \psi_{L\delta z} - \delta z \frac{\partial}{\partial z} \psi \frac{\partial}{\partial z} \psi + \delta p \frac{\partial}{\partial p} \psi \frac{\partial}{\partial p} \psi + \delta q \frac{\partial}{\partial q} \psi \frac{\partial}{\partial q} \psi \right) \, dx \, dy$$

which integrals reduce to the line-integrals

$$- \int \int \delta z \frac{\partial}{\partial x} \left( \psi_{L\delta z} - \delta z \frac{\partial}{\partial z} \psi \frac{\partial}{\partial z} \psi + \delta p \frac{\partial}{\partial p} \psi \frac{\partial}{\partial p} \psi + \delta q \frac{\partial}{\partial q} \psi \frac{\partial}{\partial q} \psi \right) \, dx \, dy$$

$$+ \int \int \delta z \frac{\partial}{\partial y} \left( \psi_{L\delta z} - \delta z \frac{\partial}{\partial z} \psi \frac{\partial}{\partial z} \psi + \delta p \frac{\partial}{\partial p} \psi \frac{\partial}{\partial p} \psi + \delta q \frac{\partial}{\partial q} \psi \frac{\partial}{\partial q} \psi \right) \, dx \, dy$$

Thus we arrive at the expression

$$V = \int \int \delta H \, dx \, dy$$

$$+ \int \int \delta z (L\delta z - H \delta z + \frac{\partial}{\partial x} \left( \frac{\partial}{\partial p} \left( F \delta p + \frac{\partial}{\partial q} \left( F \delta q \right) \right) \right) \, dx \, dy$$

*) Sufficiently smooth hyperbolic problems.

$(a = \gamma = 0)$ and problems which are linear in $z$ and its derivatives are well posed problems.
\[-\int (\psi L_2 \delta z - \delta z (L_2 \psi - F_q + \psi f_q)) dx \]
\[+ \int (\psi L_1 \delta z - \delta z (L_1 \psi - F_p + \psi f_p)) dy \]
\[+ \int G \delta z ds \]

where $H$ denotes the function

$H = -F + \psi f$

and where

$\Delta H = F(x,y,z+\delta z,p+\delta p,q+\delta q,u+\delta u) - F(x,y,z,p,\delta p,q,u) - G(x,y,z+\delta z,u+\delta u) - G(x,y,z,u)$.

From this expression for $V$ we take the definition of the function $\psi$; we define this function to be such that the adjoint equation

$\Delta \psi = H(x,y,z+\delta z,p+\delta p,q+\delta q,u) - F(x,y,z+p,\delta p,q,u)$

is satisfied. Then $V$ reduces to

$V = -\int \Delta H \, dx \, dy + \int \Delta G \, ds$

This fundamental expression for $V$ will be the starting-point for our observations in the next sections.

6. DISTRIBUTED CONTROL.

Let us put $G=0$ and let $L$ be the hyperbolic operator

$L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$

Thus $a = -\gamma = 1$ and $\beta = 0$. As for boundary-conditions we consider mixed conditions:

For $0 \leq x \leq X$ and $q(x,0)$ have prescribed values while for $0 \leq y \leq Y$ and $z(x,y)$ have prescribed values.

Hence $\delta z$, $\delta p$ and $\delta q$ vanish along $OX$, while $\delta z$ and $\delta q$ vanish along $OY$ and $XP$.

From

$\psi L_1 \delta z - \delta z L_1 \psi = \psi \delta p - \psi_x \delta z$

and

$\psi L_2 \delta z - \delta z L_2 \psi = -\psi \delta q + \psi_y \delta z$

it follows that

$V = -\int \Delta H \, dx \, dy$

\[\int Y \psi \delta p \, dy + \int X \psi \delta p \, dy \]

\[+ \int \psi \delta q - (\psi_y \Delta z) \, dx \geq 0 \]

Imposing the conditions

$\psi = 0$ along $OY$

$\psi = 0$ along $XP$

$\psi = 0$ along $Y P$

on the adjoint equation we thus find that

$\int \Delta H \, dx \, dy \leq 0$

from which inequality we obtain by applying a local admissible variation $\delta u(x,y)$ at some point $(x,y)$ of $R$ that the optimal control maximizes the Hamiltonian $H$ on the region $U$.

Now let us again put $G=0$ and let $L$ be the parabolic operator

$L = \frac{\partial^2}{\partial x^2}$

Thus $a = 1$ and $\beta = \gamma = 0$. We assume $z(x,y)$ to be given along $O X$, $O Y$ and $X P$. Hence $\delta z$ and $\delta p$ vanish along $O X$ while $\delta z$ and $\delta q$ vanish along $O Y$ and $X P$. From

$\psi L_1 \delta z - \delta z L_1 \psi = \psi \delta p - \psi_x \delta z$

$\psi L_2 \delta z - \delta z L_2 \psi = 0$

we obtain that

$V = -\int \Delta H \, dx \, dy$

\[\int Y \psi \delta p \, dy + \int X \psi \delta p \, dy \]

\[+ \int \psi \delta q - (\psi_y \Delta z) \, dx \geq 0 \]

Imposing the conditions

$\psi = 0$ along $O Y$

$\psi = 0$ along $XP$

$\psi = 0$ along $Y P$

on the adjoint equation we find that

$\int \Delta H \, dx \, dy \leq 0$

from which inequality it again follows that the optimal control maximizes the Hamiltonian $H$ on the region $U$. 
Let us lastly put $G=0$, consider the parabolic operator
\[ L = \frac{\partial^2}{\partial x^2} \]
and assume $z(x,0), p(0,y)$ and $p(x,y)$ to be prescribed. Hence $\delta z$ and $\delta p$ vanish along $O X$ while $\delta p$ vanishes along $O Y$ and $X P$. It then follows that
\[ V = - \int_R \int_R \Delta H \, dx \, dy \]
\[ + \int_0^Y (\psi_x + H_p) \, 8z \, dy - \int_X (\psi_x + H_p) \, 8z \, dy \]
\[ - \int_Y H_q \, 8z \, dx \geq 0 \]

Hence, imposing the conditions
\[ \psi_x + H_p = 0 \quad \text{along } O Y \]
\[ \psi_x + H_p = 0 \quad \text{along } X P \]
\[ H_q = 0 \quad \text{along } Y P \]
on the adjoint equation we again obtain that the optimal control maximizes the Hamiltonian $H$ on the region $U$.

7. BOUNDARY-CONTROL.

Let us put $F=0$ and let us consider an elliptic equation which is linear in $z$ and its derivatives. Thus $a = \gamma = 1$ and $\beta = 0$. Hence
\[ \psi_L \psi z - \delta z L_1 \psi = \psi \delta p - \psi \delta z \]
\[ \psi L_2 \psi z - \delta z L_2 \psi = \psi \delta q - \psi \delta z \psi . \]

Along the boundary $\partial R$ the normal derivative of $z(x,y)$ is supposed to be prescribed by
\[ \frac{\partial z}{\partial n} = g(x,y,z,u) \]
where $n$ denotes the outer normal. $\delta z$ is assumed to depend continuously on $\delta u$. From
\[ \delta g = \delta g + g_z \delta z \]
where
\[ \delta g = g(x,y,z + 8z, u + 8u) - g(x,y,z, u) \]
we obtain that
\[ V = \int \int \Delta G \, ds + \int \int G_z \, 8z \, ds \]
\[ - \int_0^Y (\psi \delta q - \delta z(\psi_y + H_q)) \, dx \]
\[ + \int_0^Y (\psi \delta p - \delta z(\psi_x + H_p)) \, dy \]
\[ = \int \int (\Delta G + \psi \Delta g) \, ds \]

from which expression we take the definition of the boundary condition for the adjoint function:
\[ h_z + \psi_y + H_q = 0 \quad \text{along } O X \]
\[ h_z + \psi_x + H_p = 0 \quad \text{along } O Y \]
\[ h_z - \psi_x - H_p = 0 \quad \text{along } X P \]
\[ h_z - \psi_y - H_q = 0 \quad \text{along } Y P \]

where
\[ h = G + \psi g \]

By applying a local admissible variation $8u$ at some point of the boundary $\partial R$ we then find that the optimal boundary-control minimizes $h$.

REMARKS.

In a similar way other systems may be dealt with. Even more complicated problems of optimal control may be tackled by the variational approach as well. For applications we refer to [3] - [7].

REFERENCES.


SESSION III

Chairman: Dr. W.E. Cummins,
David W. Taylor Naval Ship
Research and Development Center, Washington
NONLINEAR ACOUSTICS

by

L. VAN WIJNGAARDEN
TENEXE UNIVERSITY OF TECHNOLOGY
ENSCHDE, THE NETHERLANDS

ABSTRACT
Standing waves and progressive waves in compressible media are, when of sufficiently small amplitude, governed by the acoustic approximation of the equations of gasdynamics, subjected to linearised boundary conditions. This approximation breaks down in a number of circumstances even at small amplitudes of the excitation. Considered in this paper is the breakdown of the acoustic approximation when in standing waves in pipes and ducts the driving frequency is near a resonance frequency. Certain nonlinear aspects have to be incorporated in the analysis of waves under these circumstances. In closed pipes shock waves appear in the solutions of the equations of motion near resonance. The dissipation in these shock waves rather than in the thermal and viscous boundary layers along the wall of the pipe, balances the work done by the driving motion, e.g. that of a piston. Since formally the equations of gravity waves over water of small depth are the same as for one dimensional gasdynamics, the analysis can also be used for studying the motion e.g. in anti roll tanks in ships. Next resonant motion in open pipes is considered. Here the situation is much more complicated. The flow inside the pipe interacts in various ways with the outside atmosphere. One of these interactions is the radiation of acoustic energy, another the formation and subsequent shedding of vortices at the open end. These mechanisms as well as the boundary layer dissipation are discussed.

1. INTRODUCTION
The generation and propagation of acoustic waves is a familiar phenomenon in nature and in society. Noise is a frequent, though not always welcome, companion of our technical activities, like driving a car or flying an airplane. Most of us prefer the sound waves produced by musical instruments like the flute and others.

In many of these applications of acoustics the sound is connected with flow of some kind, aero­dynamic noise. We shall restrict to waves passing through a medium which is otherwise at rest. In terms of gas dynamics we can define acoustics as embracing waves and oscillations of such small amplitude that their dynamics can be studied
analytically with linearized equations, in other words oscillatory flows with small Mach number. In a number of circumstances such basically linear waves cannot be determined analytically by wholly linear means. Some aspects have to be taken into consideration which call for nonlinear methods. Such circumstances occur when in pipes and tubes waves are excited by "small" causes (which in itself would make us expect equally small disturbances) and the frequency is near a resonant frequency. We will discuss in the following a number of the complicated problems, both of fluid mechanical and of applied mathematical nature, which one meets in these types of flow. From the point of view of fluid dynamics we need elements of gasdynamics, of boundary layer theory, of vorticity dynamics and general properties of wave propagation.

We have come together here in a meeting to commemorate the late Professor Timman. He made significant contributions in all the aspects of fluid flow which I just mentioned. Timman's thesis (Timman 1946), was an important contribution to the understanding of the influence of compressibility on the oscillations of airfoils. Later he made significant contributions to the calculation of boundary layers, both two- and threedimensional. His lifelong interest in waves of all kinds and their propagation is known to all of us. He used to consider these and other pieces of understanding as means for the solution of problems in science and technology. Either he was led to consider such a problem by his penetrating insight in the development of technology or the problem was posed to him during his many discussion with researchers both in the universities and in industrial laboratories.

More than two years ago Rein Timman died, but he will remain alive in the hearts of his many friends. He has been to me a friend, colleague and teacher and in some way he was all three of this at the same time. With feelings of love and gratitude I dedicate this paper to the memory of Rein Timman.

2. ACOUSTIC WAVES IN PIPES AND TUBES; RESONANCE. Imagine, as in Figure 2.1, a piston which executes small oscillations at one end of a pipe, whereas the other end is closed (Figure 2.1a) or open to the atmosphere (Figure 2.1b).

As a result of the motion of the piston standing waves will be generated in the gas in the pipe. The amplitude of pressure and velocity in these waves can be fairly accurately calculated from the inviscid linearized equations of gasdynamics. These lead, for the velocity $u$ to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2},$$

(2.1)

in which $c_0$ is the sound velocity and $x$ and $t$ denote distance, from the end of the pipe, and time respectively. Equation (2.1) has solutions in the form $u = f(x - c_o t)$, right going waves, and $u = f(x + c_o t)$ left going waves. Once $u$ is known the pressure relative to the undisturbed pressure, $p - p_0$, follows from the equation of motion

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0.$$

For a right going wave, where $\partial u/\partial t = -a_o \partial^2 u/\partial x^2$ we have

$$p - p_0 = \rho_0 a_o u,$$

(2.2)

whereas for a left going wave

$$p - p_0 = -\rho_0 a_o u.$$

(2.3)

A standing wave can be obtained by superposition of a left and a right going wave.
For the closed pipe in Figure 2.1 a where the boundary condition at \( x = 0 \) is

\[
\begin{align*}
  x &= 0; \
  u &= 0,
\end{align*}
\]

we obtain

\[
\begin{align*}
  u &= A [\cos(n(x/a_0 - t)) - \cos(n(x/a_0 + t))] \
  p - p_o &= A [\cos(n(x/a_0 - t)) + \cos(n(x/a_0 + t))],
\end{align*}
\]

for the standing wave with velocity amplitude \( x \) and angular frequency \( n \). In the case of the open pipe the boundary condition at \( x = 0 \) is in the acoustic limit

\[
\begin{align*}
  x &= 0; \
  p &= p_o,
\end{align*}
\]

whence we obtain for the open pipe

\[
\begin{align*}
  u &= \tilde{A} [\cos(n(x/a_0 - t)) + \cos(n(x/a_0 + t))], \
  p - p_o &= \tilde{A} [\cos(n(x/a_0 - t)) - \cos(n(x/a_0 + t))].
\end{align*}
\]

It follows from (2.6) and (2.9) that a wave of compression is reflected as a wave of compression from a closed end and as a wave of expansion from an open end. At the end \( x = L \) we have from (2.5) - (2.6) and (2.8) - (2.9)

\[
\begin{align*}
  \text{closed end} & \quad u = 2A \sin \omega t \sin \frac{\pi L}{a_0} \
  \text{open end} & \quad u = 2\tilde{A} \cos \omega t \cos \frac{\pi L}{a_0},
\end{align*}
\]

\[
\begin{align*}
  p - p_o &= \tilde{A} \sin \omega L \sin \frac{\pi L}{a_0} \quad (2.10) \
  p - p_o &= 2\tilde{A} \cos \omega t \cos \frac{\pi L}{a_0} \quad (2.11)
\end{align*}
\]

The amplitudes \( A \) and \( \tilde{A} \) finally follow from the prescribed motion at \( x = L \), where the piston is located. When this executes a harmonic motion with frequency \( \omega \) and amplitude \( \delta \), we have

\[
\begin{align*}
  \text{closed end} & \quad A \sim \frac{\delta \omega}{\sin \omega L/a_0}, \
  \text{open end} & \quad \tilde{A} \sim \frac{\delta \omega}{\cos \omega L/a_0}.
\end{align*}
\]

For a closed end the amplitude becomes infinite when

\[
\omega = \frac{n \omega_0}{L}, \quad n = 1, 2, \ldots
\]

The corresponding wave length is twice the pipe length \( L \), at the lowest resonant frequency

\[
\omega_0 = \frac{\omega}{2L}.
\]

The corresponding relation for the resonant frequency in an open ended tube is

\[
\frac{n}{\omega} = \frac{2L}{\omega_0}, \quad n = 0, 1, 2, 3
\]

at the lowest of which

\[
\omega_0 = \frac{2\omega}{L}.
\]

The wavelength is four times the pipe length.

Apparently our approximate solution (2.10) - (2.13) which can be considered as the first term of a perturbation series in terms of ascending powers of the Mach number \( \delta a/a_0 \), breaks down at these wave lengths. The reason for this is that at resonance a nonzero amount of work is done by the piston. Take for example the open ended pipe. According to (2.12) and (2.13) \( u \) lags, at the piston, \( 90^\circ \) behind the pressure \( p - p_o \) and on the average no work is done by the piston in a cycle. Above resonance \( u \) runs ahead of \( p - p_o \) by \( 90^\circ \) and again the work exerted by the piston on the gas is zero.

Right at resonance \( u \) and \( p - p_o \) are in phase at the piston and work is done on the gas. Nothing is there to consume this work and therefore the amplitude continues to grow. Real gases are viscous and conduct heat. Both these properties are disregarded in our theory and it suggests itself to inspect in what amount the work done by the piston can be dissipated by viscosity and heat conduction. These are not important in the whole flow, but only in boundary layers at the wall, first described by Stokes (see e.g. Rayleigh 1945). With kinematic viscosity \( \nu \) (for air at room temperature \( \nu = 1.5 \times 10^{-6} \text{ m}^2/\text{s} \)) these layers have a thickness of order \( (\nu/\nu_0)^{1/3} \) which is with \( \omega = a_0/L \approx 10^2 \) about 1 mm.

In practice this is much less than the diameter of the pipe. The velocity distribution near the wall can be calculated by using the boundary layer approximation to the linearized Navier-Stokes equations.

With \( y \) as co-ordinate normal to the wall and \( u_B \) denoting the velocity in the boundary layer, we have

\[
\frac{\partial p}{\partial y} = 0,
\]

\[
\frac{\partial^2 u_B}{\partial y^2} = -\frac{1}{\rho_o} \frac{\partial p}{\partial x} + \frac{\nu^2}{\rho_o} \frac{\partial^2 u_B}{\partial y^2},
\]

with

\[
\begin{align*}
  y &= 0; \
  u_B &= 0 \
  u &= u(x, t)
\end{align*}
\]

\( u(x, t) \) being the previously derived inviscid velocity distribution (2.5) or (2.8).

The thermal boundary layer is thicker by a factor \( \text{Pr}^{-1/3} \), where \( \text{Pr} \) is the Prandtl number. For air \( \text{Pr} \) is 0.7 and therefore the thermal boundary layer has approximately the thickness of the viscous boundary layer.
With a view to (2.16) the pressure $p(x,t)$ inside the boundary layer follows from the corresponding expressions (2.6) and (2.9) for the pressure. The calculation of the temperature distribution proceeds in a similar way. The dissipation is

$$\phi = \int \left[ \mu \left( \frac{\partial u}{\partial y} \right)^2 + \lambda \left( \frac{\partial T}{\partial y} \right)^2 \right] \, dv,$$

(2.19)

$\mu$ being the dynamic viscosity and $\lambda$ the thermal conductivity. The integration is over the volume occupied by the boundary layer. The ingredients for the calculation of $\phi$ in the general case can be found in Rayleigh (1945).

For the particular case of oscillations in a pipe we have (Temkin 1968), $R$ being the radius of the pipe and $L$ the length,

$$\phi = \frac{1}{2} \rho_0 (u_{\text{max}})^2 \left( \frac{2\pi}{L} \right)^4 \pi R^2 \left( 1 + \frac{1}{Pr} \right)^2.$$

(2.20)

In this expression $u_{\text{max}}$ denotes the maximum velocity in the inviscid outer flow (e.g. $2X$ in (2.8)) and $\gamma$ is the ratio of specific heats of the employed gas in the pipe. To assess the role of dissipation in determining $u_{\text{max}}$ we may compare $\phi$ with the work $W$ done by the piston per unit time. It follows directly from either of the combinations (2.5)-(2.6) and (2.8)-(2.9) that (recall that $u$ and $p - p_0$ are in phase at the piston at resonance)

$$W = \rho_0 a_0 u_{\text{max}} \Delta R R^2.$$

(2.21)

Requiring that $W$ in (2.21) balances $\phi$ in (2.20), gives

$$\frac{u_{\text{max}}}{a_0} \propto \frac{R}{L} \left( \frac{\rho_0 a_0^2}{\nu} \right)^{\frac{1}{2}}.$$

(2.22)

For a closed pipe with $L = 1$ m, $R = 0.05$ m, we have with air (at room temperature $a_0 = 343$ m/s, $\nu = 15 \times 10^{-6}$ m$^2$/s) $\gamma = 538 \, s^{-1}$ (from 2.15) and we find

$$\frac{R}{L} \left( \frac{\rho_0 a_0^2}{\nu} \right)^{\frac{1}{2}} = 300 \, \delta.$$

It follows from (2.22) that for amplitudes $\delta$ larger than $3 \times 10^{-4}$ m, the velocity in the pipe is more than 10% of the velocity of sound. For a $\delta$ of $5 \, \text{mm}$ $u_{\text{max}}$ would be 3 times $a_0$.

This is much larger than is observed in the laboratory (see sections 3 and 5). It is clear that unless $\delta$ is very small other mechanisms are at work. An obvious one is the radiation of acoustic energy, of course only in the case of open pipes. The acoustic radiation from open pipes was studied in great detail by Lord Rayleigh in his "The Theory of Sound", Rayleigh (1945). He obtained experimentally the important result that, due to acceleration and deceleration of ambient air, the first resonance frequency is not precisely (2.15) but has to be corrected with $\Delta a$, given by

$$\Delta a = - \alpha \delta R.$$

(2.23)

A familiar interpretation is as follows: With (2.23) and $\Delta L = 0.6R$, we can to the second order in $\Delta L/L$, write

$$a = a_0 \frac{L + \Delta L}{L}.$$

(2.24)

Thus, the resonance frequency is not based on the true length of the pipe, but on the length of a fictitious one larger than the actual one by an amount $\Delta L$, the mass in the additional length representing the amount of the ambient air participating in the motion. Theoretically, the problem of acoustic radiation from an unflanged pipe was solved exactly (that is to say in the context of inviscid linearized equations) in a famous paper by Levine and Schwinger (1948).

Mathematically we have to do with a boundary value problem of mixed type for Helmholtz' equation. Levine and Schwinger solved this difficult problem by making use of the, at that time quite modern, method of Wiener and Hopf. What they found, is this: Write in complex notation for

$$p - p_0 = e^{i \omega x} \left( a_0 - r e^\omega \right).$$

(2.25)

The coefficient $r$ (which is complex) is the reflection coefficient, unity in case the boundary condition is just $p = p_0$ at $x = 0$, which follows from comparison of (2.25) with (2.9).
Now put
\[ r = \left| r \right| e^{2i\pi l a_0}, \] (2.26)
with which (2.25) becomes
\[ p - p_0 = e^{i\alpha l a_0} \{ e^{i\alpha l a_0(x+1)} - \left| r \right| e^{-i\alpha l a_0(x+1)} \}. \]
Comparison with (2.9) shows that for \( |r| = 1 \) this corresponds with a completely passive atmosphere \((p = p_0 \text{ at } x = 0)\) however with the effective origin in \( x = -1 \).
Levine and Schwinger find in the limit \( R/L \to 0 \), with which we have to do in general
\[ \frac{1}{L} = 0.6133 \frac{R}{L}, \] (2.27)
confirming the experimental value (see 2.23) obtained by Rayleigh. Also Levine and Schwinger obtained an expression for \( |r| \).
This differs from unity because the sound waves do not propagate unidirectionally from the mouth of the pipe. The effect of nonuniform radiation, becomes stronger for larger values of \( L/R \).
Using the results of Levine and Schwinger and equating the net radiated energy to the work done by the piston we obtain
\[ \frac{u_{\text{max}}}{a_0} = \frac{2a_0}{(\frac{2\pi}{L})^2}, \] (2.28)
or, with (2.15), under resonance,
\[ \frac{u_{\text{max}}}{a_0} = \frac{\pi}{2L} \left( \frac{2\pi}{2R} \right)^2. \] (2.29)
Like with viscosity, this means small \( u_{\text{max}}/a_0 \) for small \( \delta/L \) but it is clear that with \( L/R \sim 10^3 \) and \( \delta/L \sim 10^{-3} \), (2.29) gives unrealistically large values for \( u_{\text{max}}/a_0 \).
With viscous and thermal dissipation and acoustic radiation, linear mechanisms (by which we mean mechanisms that result in \( u_{\text{max}} \sim \delta \)) are exhausted. We have to turn to nonlinear phenomena, because linear treatments predict amplitudes for which nonlinear terms in the full, nonlinearized, equations are far from negligible.
When the acoustic waves which make up standing waves like in (2.5) - (2.9) have finite amplitude we may imagine them as simple waves.
In fact, in the first approximation beyond the linear one, the waves under discussion share some properties with simple waves, see section 5.
The velocity of propagation of a simple wave depends on the perturbation that it carries. To be specific, a velocity amplitude \( \dot{u} \) propagates in a simple wave with velocity
\[ u + a = a_0 + \frac{r+1}{2} \dot{u}. \] (2.30)
This means that the time necessary for a certain amplitude \( \dot{u} \) between \( 0 \) and \( u_{\text{max}} \) to travel up and down the pipe is different for each \( \dot{u} \). Therefore, as was pointed out by Seymour and Mortell (1973b), whereas in the linear approximation the whole wave is in resonance (when there is resonance) with finite amplitude only a small part of the wave can be in phase with the excitor. This will lower the resonance peak and at the same time lead to a resonance band rather than resonance at discrete values of \( \alpha \), like in (2.14) and (2.15).
One way to incorporate nonlinear terms is to construct a perturbation series, e.g. for the velocity \( u \), in terms of a suitable small parameter like \( \alpha \delta / a_0 \). One soon finds out that such an expansion is singular because like in corresponding problems for ordinary nonlinear differential equations, secular terms are generated.

**Figure 2.2**
Characteristics in linear waves—broken lines and in slightly nonlinear waves—solid lines. Although the latter deviate only slightly from a straight line, the differences in \( t \) become large eventually.
A very fruitful suggestion was made by C.C. Lin (Lin 1954), namely that perturbations like the one under consideration are regular when the expansion is done with the characteristics as coordinates.

In figure 2.2 this is made plausible. Drawn are a number of characteristics in a slightly nonlinear flow. Although the characteristics deviate only slightly from straight lines (broken lines in the Figure) the corresponding values of \( t \) for given \( x \) values differ ever more from the linear values. Lin applied his idea to a number of problems in gasdynamics but it took another ten years before it was applied to nonlinear acoustic waves near resonance.

Since in many ways the acoustic oscillations in closed pipes are different from those in open pipes, we will now discuss first the closed pipe and turn to open pipes in section 5.

3. RESONANT OSCILLATIONS IN CLOSED PIPES.

![Figure 3.1](image)

Shock waves, \( AB \) and \( BC \), in resonant oscillations in a closed tube. Drawn are also some rightgoing characteristics, \( \alpha = \) const., and left going characteristics, \( \beta = \) constant. The pressure in \( x=x_1 \) as function of \( t \) is sketched in Figure 3.2\(^a\), in \( x=0 \) in Figure 3.2\(^b\).

The treatment of nonlinear oscillations in the spirit of Lin's (1954) suggestion starts with writing the dependent variables \( u, p, x \) and \( t \) as perturbation series in terms of the characteristic curves as independent variables. With characteristics \( \alpha \) and \( \beta \) and using the velocity of sound \( a \) rather than \( p \), we have for example

\[
\begin{align*}
  u &= cu_1(\alpha, \beta) + \varepsilon^2 u_2(\alpha, \beta) + \ldots \\
  a &= a_0 + \varepsilon a_1(\alpha, \beta) + \varepsilon^2 a_2(\alpha, \beta) \\
  x &= x_0 + \varepsilon x_1(\alpha, \beta) + \varepsilon^2 x_2(\alpha, \beta) + \ldots \\
  t &= t_0 + \varepsilon t_1(\alpha, \beta) + \varepsilon^2 t_2(\alpha, \beta) + \ldots
\end{align*}
\]

In these expansions \( \varepsilon \) is a small parameter representative for the Mach number of the flow. The equations of motion are in terms of \( \alpha \) and \( \beta \)

\[
\frac{3x}{3\alpha} = (u - a) \frac{3t}{3\alpha},
\]

along the characteristics \( \beta = \) constant, and

\[
\frac{3x}{3\beta} = (u + a) \frac{3t}{3\beta}
\]

along the characteristics \( \alpha = \) constant. We can label \( \alpha \) with the value of \( t \) at the intersection with \( x = 0 \) and \( \beta \) with the value of \( a \) at the intersection of the \( \beta \) characteristic under consideration with \( x = 0 \). A compression wave steepens by nonlinearity. Because with a closed end a compression wave is reflected as a compression wave a shock wave is eventually formed, travelling up and down the tube, like in Figure 3.1. Experimentally shock waves were observed near resonance by Lettau (1939) and Saenger and Hudson (1960). The latter authors as well as Betchov (1958) did some theoretical work to account for the strength and other properties of these shock waves. In particular Betchov (1958) using some sound physical arguments and making some good guesses remarkably well explained the waves as sinusoidal acoustic waves separated by shock waves. A more complete theory using Lin's method was given by Chu and Ying (1963). They dealt with oscillations excited by a periodic heat flux in the gas, in phase with the lowest resonance frequency. The magnitude of the parameter \( \varepsilon \) was chosen as

\[
\varepsilon = \delta^{1/2}
\]
on the grounds that the dissipation in the shock wave which is proportional to \( E^3 \) must balance the work done on the gas by the exciting agency and which is proportional to \( \dot{\theta}E \). Because the shocks are supposed to be weak, the flow is still isentropic. Of course, the shocks have to obey the shock relations. The system of equations is completed by the requirement of periodicity and the boundary conditions, expressed in terms of \( \alpha \) and \( \beta \). A disadvantage of this type of methods, like the hodograph method, is that in many problems the boundary conditions become rather awkward. This is not so here, because as Lin (1954) pointed out in connection with a similar problem, the zeroth term is a state of rest.

For example in the \( x,t \) plane we have

\[
x = \delta \sin \omega t + L \tag{3.8}
\]

\[
u = \alpha \cos \omega t. \tag{3.9}
\]

In the lowest approximation we have from (3.3) - (3.6)

\[
x_0 = \frac{\alpha}{2} (\beta - \alpha) \tag{3.10}
\]

\[
t_0 = \frac{1}{4} (\alpha + \beta) \tag{3.11}
\]

Hence, from (3.3), (3.7), (3.8) and (3.10), in the lowest approximation the piston is in

\[
\beta = \alpha + 2L/\alpha_0, \text{ where } u = 0.
\]

In the next approximation \( x_1 \) and \( t_1 \) are zero (from 3.8 and 3.9). This means that when in the first approximation the piston is in

\[
\beta = \alpha + 2L/\alpha_0 + \epsilon v(a),
\]

we have

\[
\nu(a) \frac{\partial x}{\partial \beta} + x_1 (a, a + 2L/a_0) = 0 \tag{3.12}
\]

\[
u_1 (a, a + 2L/a_0) = 0 \tag{3.13}
\]

(3.12) and (3.13) are conditions in the first approximation.

The calculation carried out by Chu and Ying are lengthy but straightforward and resulted in predictions of shock paths and strength for given exciting forces. The work by Chu and Ying has, perhaps because of the rather involved calculations, remained since a little bit in the shadow of that by Chester (1964) who gave a theory of nonlinear oscillations in a different way, making implicitly use of characteristics but arriving at results more rapidly. Chester (1964) also included viscous effects in his calculations. Chester (1964) starts with the equation of motion for one dimensional waves as given by Lighthill (1956) in his celebrated article "Viscosity in waves of finite amplitude"

\[
\frac{du}{dt} + \frac{2u}{3} \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \right) \frac{\partial u}{\partial x} = \frac{3}{2} \frac{\partial^2 u}{\partial x^2} \tag{3.14}
\]

\[
s = \nu(4/3 + \gamma^{-1} \frac{\partial}{\partial t}). \tag{3.15}
\]

In the right hand side of (3.14) viscous normal stresses are taken into account. The effect of viscosity as it manifests itself in the wall boundary layer is chiefly the mass flow from the boundary layer into the main flow. Denoting the velocity normal to the wall in the boundary layer with \( v_b \) we have approximately,

\[
v_b = \int_0^s \frac{\partial u_b}{\partial x} \, dy,
\]

\( u_b \) being the \( x \) component of the velocity in the boundary layer, as given by (2.17). By applying a Laplace transformation with respect to \( t \) on (2.16) - (2.18) and solving for the Laplace transform \( \tilde{v}_b \), we obtain

\[
\tilde{v}_b = -\tilde{u} \exp \left( -\left( \frac{s}{\nu} \right)^{1/2} y \right),
\]

\( s \) being the transform variable, and

\[
\tilde{v}_b = \int_0^u \frac{\partial v_b}{\partial x} \, dy.
\]

Transformation to physical variables then gives

\[
v_b = \left( \frac{\nu}{\nu} \right)^{1/2} \frac{\partial u}{\partial x} \left( x, t - \xi \right) \frac{dr}{\xi^2} \tag{3.16}
\]
Actually, also the density changes by temperature changes in the boundary layer must be taken into account. The result is that the righthand side of (3.16) is multiplied with $(1+(y-1)/Pr)^{\frac{2}{3}}$.

Using (3.16), corrected in this way, in the continuity equation and eliminating $p$ in (3.14) gives

$$
\frac{\partial u}{\partial t} + \frac{2a}{\gamma-1} \left( u \frac{2a}{\gamma-1} \right) = \frac{\partial}{\partial x} \left( u \frac{2a}{\gamma-1} \right)
$$

$$
\frac{\partial^2 u}{\partial x^2} + \frac{3}{2} \frac{2u}{\gamma-1} + \frac{ka_o}{\gamma-1} \int_0^\infty \frac{2u}{\delta x} (x,t-\xi) \xi^{-\frac{3}{2}} d\xi,
$$

(3.17)

$$
k = \frac{2}{\gamma-1} \frac{1}{(y-1)/Pr}.
$$

(3.18)

Equation (3.17) was given in this form by Chester (1964). The quantity $a$ is the difference between the sound speed and the equilibrium value $a_0$. In the inviscid and linear approximation the right hand side would be zero, because it summarizes the nonlinear and viscous and thermal contributions. Although these effects are small, the wave being essentially a linear wave, they are necessary to completely determine $u$ and $a$. Chester (1964) solved (3.17) by successive approximation, the first one being given by

$$
\begin{align*}
\frac{u_1 + 2a_1}{\gamma-1} &= 2a_0 f_1(t-x/a_o) \\
\frac{u_1 - 2a_1}{\gamma-1} &= 2a_0 f_2(t-x/a_o).
\end{align*}
$$

(3.19)

(3.20)

Because of the boundary condition $u=0$ at $x=0$ we have $f_1=f_2=f$. The next approximation $u_2$ can be expressed in terms of $f$ and $f'$. To the second order in $f$ Chester (1964) obtains from inserting (3.19) and (3.20) and similar expressions for $u_2$ and $a_2$ into (3.17)

$$
\begin{align*}
\frac{\partial \cos \theta}{\partial t} &= -2a_0^{-1} \tan \left( \frac{\partial}{\partial t} f(t-L/a_o) \right) \\
\times (y+1)^2 f(t-L/a_o) f'(t-L/a_o) + 2a_0^{-1} f''(t-L/a_o) \\
+ kl \int_0^\infty f'(t-L/a_o-\xi) \xi^{-\frac{3}{2}} d\xi.
\end{align*}
$$

(3.21)

This is a complicated equation to solve, except in some special cases. Far from resonance, where $\tan \frac{\theta}{\gamma-1} = 0$, the classic linear solution is recovered when nonlinearity and viscosity are neglected.

The latter are important however near resonance where $\tan \frac{\theta}{\gamma-1} = 0$. The term with $\theta$ is in general unimportant. It smooths out shock waves which appear as discontinuities when this term is neglected. Chester solved the remaining equation in the asymptotic case of very thin Stokes boundary layers. The solutions obtained contain shock waves under certain conditions, and qualitatively agree with the measured pressure profiles observed by Saenger and Hudson (1960).

However, the parameters in the experiments by Saenger and Hudson did not fall in the region covered by Chester's asymptotic solution. Keller (1976) made a detailed numerical and analytical study of Chester's equation for conditions which better agree with Saenger and Hudson's experiments. He concluded to good agreement of Chester's theory with experiment. Experiments on standing oscillations in closed pipes were more recently carried out by Cruikshank (1972). His experiments cover a wide range of parameter values including both Chester's asymptotic solution and Keller's numerical solution.

There is agreement in the overall features of the solution. However there are significant differences in details between the experimental and theoretical solutions. Neither Cruikshank (1972) nor Keller (1976) found an explanation for these. An interesting and important special case of (3.24) is the equation which remains when viscosity is neglected.

$$
\frac{\partial \cos \theta}{\partial t} = -2a_0^{-1} \tan \left( \frac{\partial}{\partial t} f(t-L/a_o) \right) f'(t-L/a_o) f''(t-L/a_o)
$$

(3.22)

Chester (1964) made a detailed analysis of this equation. Far from resonance there are continuous solutions but when

$$
\frac{\theta}{\gamma-1} < \frac{\pi}{2} (y+1)^{\frac{1}{2}}
$$

no continuous solution exist. The solutions are discontinuities (shock waves), riding on continuous curves like in Figure 3.2.

$x$ the tan occurs because Chester writes $\tan(\theta/\gamma-1) = \tan(L/a_o)$ thereby achieving that his equation contains the classic solution far from resonance.
where

\[ f(\beta) = \varepsilon f_1(\beta) + \varepsilon^2 f_2(\beta) + \ldots \] (3.26)

\[ g(\alpha) = \varepsilon g_1(\alpha) + \varepsilon^2 g_2(\alpha) + \ldots \] (3.27)

\( \varepsilon \) being given by (3.23).

The variables \( x \) and \( t \) are expanded in series like in (3.3) and (3.4). By requiring that the travel time up and down the tube is the given period \( 2\pi/a \), and applying the boundary conditions at both ends Jimenez finds for \( f_1 \)

\[ 2\int_0^\pi \frac{\sin \alpha}{\cos \alpha} - \frac{1}{2} \int_0^{\pi+1} f_1(\xi) d\xi = \frac{1}{2} (\gamma+1) f_1(-f_1') = \sin \alpha. \] (3.28)

This is an equation very similar to Chester's equation (3.22) and in fact it can be shown (Jimenez 1973) that they are identical.

In a closed pipe we can always define \( f_1 \) such that it has zero mean and then the term with the integral in (3.28) is zero. In closing this section we look at (3.28) in case the solution contains a shock.

From (3.24) and (3.25) it follows that, because with a view to the boundary condition \( u = 0 \) at \( \alpha = \beta \), we have \( f = g \) and hence

\[ u_1 = f_1(\alpha) - f_1(\beta), \] (3.29)

\[ a_1 = \frac{\gamma-1}{2} (f_1(\alpha) + f_1(\beta)). \] (3.30)

When the solution of (3.28) contains a shock wave we cannot integrate over one period. We can integrate however (3.28) at \( x = 0 \) where \( \alpha = \beta = t \) from the back (+ in Figure 3.2b) of one shock wave to the front (-) of the shock wave one period ahead, giving

\[ \int_0^{2\pi} \sin \alpha d\alpha = 2\pi \gamma (f_1' + f_1'') (f_1'' - f_1') = 0. \] (3.31)

Because \( (f_1' - f_1'') \) is proportional to the strength of the shock (see 3.30) we must have

\[ \int_0^{2\pi} \sin \alpha d\alpha = 2\pi \gamma (f_1' + f_1'') (f_1'' - f_1') = 0. \] (3.32)
Now, since in this approximation the shock is supposed to be weak, we may expect that its path is approximately a characteristic and that the velocity of propagation is (see e.g. Whitham 1974)

\[ V = a_0 + \frac{c}{2} \left( (u_1 + a_1)^+ - (u_1 + a_1)^- \right). \]  

(3.33)

For a shock travelling, see Figure 3.1., from A to B along an \( \alpha \) characteristic, we have, using (3.29) and (3.30), from (3.35)

\[ V = a_0 + \frac{c}{4} \left( f_1^+ (a_S) + f_1^- (a_S) \right) + \frac{c^2}{2} f_1'(\theta). \]  

(3.34)

Along the shock we write

\[ V = \frac{dx}{dt} \text{ shock} \]

and with

\[ t_1 = \frac{1}{2} (\theta + \alpha), \]

and

\[ \gamma_A = a_S, \quad \gamma_B = a_S + \frac{2\pi}{\alpha}, \]

we obtain from (3.34) upon integration

\[ L = n_0 (t_B - t_A) + \int \frac{c}{4} \left( f_1^+ + f_1^- \right) \frac{dx}{dt} + \frac{2\pi}{\alpha} \int \frac{c^2}{2} f_1'(\theta) \frac{dx}{dt} dt. \]  

(3.35)

The integral is zero because \( f \) has zero mean. Likewise for the shock travelling along a \( \beta \) characteristic from B to C:

\[ L = n_0 (t_C - t_B) + \int \frac{c}{4} \left( f_1^+ + f_1^- \right) \frac{dx}{dt} \]

(3.36)

Now

\[ t_C - t_B = \frac{2\pi}{\beta} - \frac{2\pi}{\alpha} - \frac{c_0}{c_0}. \]  

(3.37)

Addition of (3.35) and (3.36) gives with help of (3.37) and the relation

\[ n_0 = \frac{\gamma \alpha}{L}, \]

\[ -2 \frac{c^2}{c_0} n_0 + \frac{\gamma \alpha}{L} \left( f_1^+ + f_1^- \right) = 0 \]

which is identical with (3.32).

The weak shock relations are therefore satisfied. The shock rides with constant strength on a sinusoidal perturbation.

The resultant pressure disturbance as a function of time looks like sketched in Figure 3.2, with the corresponding location \( x = x_1 \) indicated in Figure 3.1.

We may conclude that there is a fairly complete understanding of the phenomena near resonance in closed tubes.

4. NONLINEAR OSCILLATIONS IN ANTI-ROLL TANKS

We interrupt the treatment of nonlinear acoustics for a short excursion to an interesting application of the foregoing in ship hydrodynamics. Often cargo ships are equipped for stabilisation with anti-roll tanks. In essence these are vessels filled with water the oscillations of which exert when properly designed moments on the ship which counteract moments excited by waves.

Schematically the motion of the water is like in Figure 4.1. In fluid mechanical terms we have to do with shallow water waves. The equations valid for such wave motions are for shallow enough water formally identical with the equations of one-dimensional gas dynamics. The device works most effectively when the exciting moment resonates with a natural frequency of the water in the container. On basis of the analogy with gasdynamics we may expect then hydraulic jumps or bores on the surface of the water. These jumps were observed indeed in experiments by Verhagen and Van Wijngaarden (1965), who also made calculations of the surface elevation of the liquid by applying the theory of Chu and Ying (1963). Fairly good agreement was obtained as is shown in Figure 4.2, taken from their paper.
Figure 4.2.
Surface elevation \( \eta \) as a function of time for liquid in a container under resonance. (From Verhagen and Van Wijngaarden 1965) The undulations on the measured elevation curves led Chester (1968) to include dispersion in the analysis.

---

Chester (1968) when looking at these results saw that the deviation between calculated and measured profiles is primarily due to dispersion which is present in the actual wave but not in the theory described in section 3. In Chester (1968) he extended his 1964 theory to include dispersion as well. In Chester and Bones (1968) experiments are reported conducted with the purpose to verify experimentally the predictions made by the new theory which includes dispersion. Good agreement was obtained.

5. RESONANT OSCILLATIONS IN OPEN TUBES.
We have seen that in closed tubes amplitude dispersion and shock formation solves the dilemma of how oscillations are kept small near resonance. With open tubes these nonlinear phenomena are far less dominant. To see this we again develop a perturbation scheme in the spirit of Seymour and Mortell (1973\textsuperscript{a}, 1973\textsuperscript{b}) and Jimenez (1973).

---

Figure 5.1.
Characteristics \( a = \text{const.} \) and \( \beta = \text{const.} \) for oscillating flow in open pipe.
Let us write with reference to the diagram in Figure 5.1:

\[
\frac{2(a-a_0)}{\sqrt{\nu}} = 2g(a),
\]

\[
\frac{2(a-a_0)}{\sqrt{\nu}} = 2f(\beta).
\]
Along the characteristics and for the zeroth approximation, we have respectively
\[ \frac{\partial x}{\partial \beta} = (u+a) \frac{\partial t}{\partial \beta}, \quad (5.3) \]
\[ \frac{\partial x}{\partial \alpha} = (u-a) \frac{\partial t}{\partial \alpha}. \quad (5.4) \]

The coordinates \( x \) and \( t \) are expanded as
\[ x = x_0(\alpha,\beta) + x_1(\alpha,\beta) + x_2(\alpha,\beta) + \ldots \quad (5.5) \]
\[ t = t_0(\alpha,\beta) + t_1(\alpha,\beta) + t_2(\alpha,\beta) + \ldots \quad (5.6) \]

The Riemann invariants \( f \) and \( g \) are supposed to be comparable in magnitude and small with respect to the velocity of sound \( a_o \).

The quantities \( x_1 \) and \( t_1 \) are assumed to be of order \( \delta \), \( x_2 \) and \( t_2 \) of order \( \delta^2 \) and so on. The characteristics are labelled such that
\[ a = \delta : t = \alpha, \quad x = 0 \quad (5.7) \]

To the lowest order one obtains upon substitution of (5.5) and (5.6) into (5.3) and (5.4), taking account of (5.7)
\[ x_0 = \frac{1}{2} a_0(\delta-a) \quad (5.8) \]
\[ t_0 = \frac{1}{2}(\delta+a). \quad (5.9) \]

In the next approximation to (5.3) and (5.4) we have, using (5.1) and (5.2)
\[ \frac{\partial x}{\partial \beta} = \gamma \frac{1}{2} g(\alpha) - \frac{3}{2} \gamma f(\beta) \frac{\partial t}{\partial \beta} + a_0 \frac{\partial t_1}{\partial \beta}, \quad (5.10) \]
\[ \frac{\partial x}{\partial \alpha} = \gamma \frac{1}{2} f(\beta) + \frac{3}{2} \gamma g(\alpha) \frac{\partial t}{\partial \alpha} - a_0 \frac{\partial t_1}{\partial \alpha}. \quad (5.11) \]

The boundary conditions on \( x_1 \) and \( t_1 \) are
\[ x_1 = 0 \quad \text{at } \alpha = \delta \]
\[ t_1 = 0 \quad \text{at } \alpha = \delta. \quad (5.12) \]

Upon integration we obtain
\[ x_1 = \frac{1}{8} (\gamma+1)(\alpha-a)(g(\alpha) - f(\beta)) \int_0^\beta g(\xi) - f(\xi) d\xi \]
\[ t_1 = \frac{1}{8} (\gamma+1)(\alpha+a)(f(\alpha) + g(\beta)) \int_0^\beta f(\xi) + g(\xi) d\xi \]

Taking together these expressions and the corresponding (5.8) and (5.9) for the zeroth approximation, we obtain, using (5.5) and (5.6)
\[ x-a_0 t = a_0(\delta + \frac{1}{2} \gamma g(\alpha) - \frac{3}{4} \gamma f(\xi) d\xi, \quad (5.13) \]
\[ x t_a_0 t = a_0(\delta + \frac{1}{2} \gamma f(\beta) x - \frac{3}{4} \gamma g(\xi) d\xi. \quad (5.14) \]

In the zeroth approximation the characteristics are straight. This is no longer the case in the first approximation. For constant \( \alpha \) for instance the line in the \( x, t \) plane given by (5.13) differs from a straight line through the term \( (3-\gamma)/4 \int_0^\beta f(\xi) d\xi \) which varies with \( \beta \). Only for \( \gamma = 3 \), a physically unrealistic value, the characteristics are straight, a convenient coincidence which has been used to advantage by Keller (1977).

Let us now consider a characteristic \( \alpha = a \). The time \( t_A \) is by definition \( \frac{L}{a_0} \) and the time \( t_B \) is from (5.13)
\[ t_B - t_A = \frac{L}{a_0} \int_0^a (1 - \frac{1}{2} \gamma \frac{g(\xi)}{a_0}) + \frac{3}{4} \gamma \int_0^b f(\xi) d\xi. \quad (5.15) \]

The functions \( f \) and \( g \) are periodic with zero mean. The time difference \( t_B - t_A \) is by definition equal to the period \( 2\pi/a \) of the motion and therefore the integral is zero, whence
\[ t_B - t_A = \frac{L}{a_0} \int_0^a (1 - \frac{1}{2} \gamma \frac{g(\xi)}{a_0}). \quad (5.16) \]

This is an interesting result, because it tells us that whereas an \( \alpha \) characteristic is not quite a simple wave, the time of arrival at the other end of the pipe is for a disturbance only dependent on the value of \( \alpha \) that carries.

In fact, the time difference \( t_B - t_A \) is the same as it would be for a simple wave carrying the velocity disturbance \( g(\alpha) \).

Repeating the process for the characteristic which is labelled \( \beta = \alpha + 2\pi/a \) and which is nearly a "\( \alpha \)" simple wave, we obtain
\[ t_B - t_A = \frac{L}{a_0} \int_0^{2\pi/a} g(\xi) - f(\xi). \quad (5.16) \]

In the linear approximation it takes a time \( 2L/a_0 \) for a wave to travel once through the pipe and back.
In the approximation one order beyond the linear this time is shorter by an amount \((y+1)L/Za\) \((\sim\delta)\). The quantity \((g-f)\) is, as is readily inferred from (5.1) and (5.2) proportional to the excess pressure at \(x=0\) \((\sim\delta)\). To the approximation involved in (5.16) this is zero and therefore \(g=f\). This means that the travel time is for all wavelets equal to the time \(2L/a_o\) and that amplitude dispersion plays no role in this approximation, in contrast to closed pipes where \(f=-g\) and the travel time is shorter for a compressive part and longer for a rarefied part of the wave. Nonlinearity of the kind discussed here becomes therefore important in the next approximation, where it leads to a amplitude of oscillation of order \((\delta/L)^{1/3}\). This dependence on \(\delta/L\) has been shown first by Collins (1971) and has also been found as part of the results in the more complicated analyses by Seymour and Mortell (1973) and Jimenez (1973). These authors developed the equations in the next order of approximation in combination with a model for the events at the mouth of the pipe represented by

\[
x = 0 \quad p - p_o = -ju. \tag{5.17}
\]

With \(j=0\) we have the classical acoustic condition \(p - p_o = 0\), whereas \(j = 0\) holds for a closed pipe. An open pipe may be characterized by

\[
j << 1. \tag{5.18}
\]

The boundary condition

\[
x = L + \delta \sin \alpha t : \quad u = \Omega \cos \alpha t \tag{5.19}
\]

gives together with (5.17) in the next approximation (Seymour and Mortell 1973, Jimenez 1973)

\[
\left((1 + \frac{\alpha - \alpha_0}{\alpha_0}) b f^2 \frac{\alpha - \alpha_0}{\alpha_0} + \int_0^{\alpha/\alpha_0} f^2 ds\right) \frac{df}{dn} = -\delta /\cos n
\]

\[
-\frac{2j}{\Omega} f. \tag{5.20}
\]

Here \(f\) is the dimensionless Riemann invariant, \(n\) stands for \(\alpha t\), and \(b\) is a constant.

Some special features of solutions follow immediately: i) for \(n = n_0\) and \(j = 0\) the left hand side of (5.20) is of order \(f^3\) and therefore \(f \sim (\delta/L)^{1/3}\), ii) for \(j = 0\) and \((\alpha - \alpha_0)/\alpha_0\) of unit order we have the classical solution \(f \sim (\delta/L)\). The general behavior of solutions of (5.20) however is rather complicated. Very interesting from a mathematicatical point of view, see e.g. the recent article by Seymour and Mortell (1975), less important for practical applications, I am afraid, because experiments by Sturtevant (1974), Van Wijngaarden and Wormgoor (1974) do not in general support (5.17) and the idea that amplitude dispersion governs the flow. It seems that the relation between \(p\) and \(u\) at the mouth of the pipe cannot be described by a relation as simple as (5.17). For one thing, the nonuniform radiation as found by Levine and Schwinger (1948) and discussed in Section 2, cannot be expressed like (5.17) and is ignored in both the work of Seymour and Mortell (1973, 1975) and Jimenez (1973).

Van Wijngaarden (1968) noted that in all likelihood the flow at the mouth of the pipe is not reversible in the sense that the streamline pattern is at outflow like at inflow with reversed direction.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5.2}
\caption{Figure 5.2 Illustrates equations (5.21): With a sharp edge the flow is like in Fig.5.2a at inflow and like in Fig.5.2b at outflow, according to Van Wijngaarden (1968).}
\end{figure}
On the contrary, at inflow the flow is as if a sink is located at the mouth of the pipe (Figure 5.2a) and at outflow the fluid separates from the wall and issues as a jet (Figure 5.2b). Van Wijngaarden (1968) proposed according to this the boundary condition.

\[
\begin{align*}
    x=0 & \quad u_e > 0 \quad p-p_0 = \frac{\rho}{2} \left( \frac{\partial u}{\partial x} \right)^2 + \rho u_0^2 e \\
    x=0 & \quad u_e < 0 \quad p-p_0 = 0,
\end{align*}
\]  

(5.21)

where \( u_e \) denotes the homogeneous velocity of the oscillating flow at the exit.

The first line in (5.21) needs some comment. It stems from requiring the conservation of momentum of the fluid enclosed by the surface \( \Gamma \) in Figure 5.2a. The first term on the right hand side actually represents the "Levine and Schwinger" effect and the value of \( c \) must be chosen such that the resonance frequency is according to the relation (2.24) and (2.23). It is the effect of the \( \partial p/\partial t \) term in Bernoulli's theorem, averaged over the cross-section of the pipe. The second term would be \( \frac{1}{2} \rho u_e^2 \) when Bernoulli's theorem would hold everywhere. The absence of the factor \( j \) in the right hand side reflects the presence of a separated zone near the edges. Keller (1977) in discussing (5.21) assumed that this relation should be

\[
\begin{align*}
    u_e > 0 & \quad p-p_0 = \frac{\rho}{2} \left( \frac{\partial u}{\partial x} \right)^2 + \rho u_0^2 e \\
    u_e < 0 & \quad p-p_0 = 0
\end{align*}
\]  

(5.22)

This is not the case when the flow separates at the edges. However, as pointed out by Van Wijngaarden and Desselhorst (1977), (5.22) holds when the flow does not separate at inflow and this is the case when the edge is sufficiently round like sketched in Figure 5.3.

The application of (5.21) at the open end leads in a perturbation analysis like that of Chu and Ying to the following equation for the quantity \( f \) comparable with \( f \) in (5.20), (Van Wijngaarden 1968)

\[
\frac{c' L_0}{a_0} + \frac{(\alpha-\alpha_0)}{\rho_0} \frac{\partial f}{\partial n} + \frac{1}{2} \frac{f}{f} \frac{2\alpha_0}{a_0} \cos n.
\]  

(5.24)

The constant \( c' \) is proportional to \( c \) in (5.21) and (5.22). It follows from (5.24) that the boundary condition at \( x=0 \) determines the magnitude of \( f \). In particular, at resonance \( f \approx (\text{6/}L)^2 \).

Keller (1977) compared solutions of (5.24) with experiments by Sturtevant (1974). Unfortunately a large part of these experiments were done at \( \alpha \) values were shock waves appear in the solution, predicted in Seymour and Mortel1 (1973b). From the general features of the experimental results Keller concluded that the boundary condition (5.21), while being not entirely satisfactory, means a great deal of improvement over Jimenez's condition (5.17). A similar conclusion was reached by Van Wijngaarden and Wormgoor (1974), who pointed out that presumably the formation of vortices at inflow and the subsequent shedding at outflow is an important mechanism, which should be taken into consideration.
Disselhorst (see for a brief summary Van Wijngaarden and Disselhorst 1977) in a forthcoming thesis investigates oscillations in open pipes with round edges like in Figure 5.3. First of all because at low $\delta/L$ the flow will be completely governed by radiation from the end and by dissipation in boundary layers, which opens the possibility to accurately assess these mechanisms quantitatively, second because with a round edge boundary layer separation takes place only at outflow and not at inflow. It may be expected therefore that at sufficiently large $\delta/L$, when dissipation is relatively unimportant, the boundary condition (5.22) is applicable.

When acoustic radiation and boundary layer dissipation, discussed in Section 2, determine the flow Disselhorst obtains for the quantity $Q$, defined by

$$Q = \frac{(p_p - p_o) a_o}{\gamma p_o a_o^2},$$

(5.25)

$P_p$ being the maximum pressure at the piston, the expression

$$Q = \left(\frac{a_o}{\gamma p_p a_o^2} \right)^2 + \left(\frac{a_o}{\gamma p_p a_o^2} \right)^4 \left(1 - \frac{x - 1}{p_o} \right)^{-1}. \quad (5.26)$$

On the other hand from solution of (5.24) at resonance it follows that

$$Q^2 = \frac{2a_o}{\gamma p_o} \left(\frac{R}{\delta}\right). \quad (5.27)$$

When using the boundary condition (5.22) this changes into

$$Q^2 = \frac{4a_o}{\gamma p_o} \left(\frac{R}{\delta}\right). \quad (5.28)$$

In Fig. 5.4, are drawn as solid or broken lines for pipes of different lengths the asymptotes given by (5.26) and (5.28). Also are presented the experimental values obtained by Disselhorst during systematic experiments with pipes of different length under resonance.

\[ Q = (p_p - p_o) \frac{a_o}{\gamma p_o a_o^2} \text{ as a function of } \delta/L \text{ for pipes with a round edge of different length under resonance (Van Wijngaarden and Disselhorst 1977).} \]

<table>
<thead>
<tr>
<th>Theory</th>
<th>Experiment</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>o</td>
<td>1.085 m</td>
<td></td>
</tr>
<tr>
<td>o</td>
<td>1.833 m</td>
<td></td>
</tr>
<tr>
<td>o</td>
<td>3.329 m</td>
<td></td>
</tr>
<tr>
<td>o</td>
<td>6.321 m</td>
<td></td>
</tr>
</tbody>
</table>
It follows from inspection of this Figure that the agreement between theory and experiment is very good for round edges.

An example of such a vortex, while developing in the mouth of the pipe, is shown in Figure 5.6.

Systematic experiments were carried out also with pipes with a sharp edge. Some results representing $Q$ as a function of $\delta/D$ are shown in Figure 5.5. Obviously there is no region for low $\delta/D$ where $Q$ is independent on $\delta$ like with the round edge. On the other end of the $\delta/D$ regime is drawn the asymptote given by equation (5.27).

The experimental results tend to follow this asymptote, which confirms the findings by Van Wijngaarden en Wormgoor (1974). There remains to explain the behaviour for lower values of $\delta/D$. Disselhorst made visible the flow in the exit by heating the air and making use of the differences in optical refraction index produced by the density differences. Observations in twodimensional flow with sharp edges clearly showed the formation of vortices during inflow and the subsequent shedding of vortices at outflow.

$Q = (p_p - p_o) a / \rho o \delta$ as function of $\delta/D$ under resonance for a pipe with a sharp edge. Theory; o experiment. For comparison the corresponding results for a pipe of the same length with a round edge are given also.

The incorporation of vortex formation and shedding poses another interesting fluid mechanical problem.
Figure 5.7.
Scheme for calculations: During inflow vortices with strength \( \Gamma(t) \) are located at distance \( S(t) \) of the nearest edge in a two dimensional model of the flow.

The flow in the vicinity of the edge can be considered as incompressible. Because acoustic radiation can be dealt with separately, see Section 2, it suffices to find a relation between the pressure \( p_e \) in the homogeneous flow in the pipe and the pressure \( P_0 \) far away.

The two dimensional version of the problem, which at least can provide a qualitative answer to the actual three dimensional problem, can be solved, making use of conformal mapping, when the strength \( \Gamma \) of the vortices and their location, indicated with the distance \( S \) from the nearest edge in Figure 5.7, are known. One relation between \( u_e \), \( \Gamma \) and \( S \) is provided by the Kutta condition which should hold at the edges. Another relation comes from consideration of the rolling up process of the vortices and the trajectories of the effective centres. Results of calculations by Disselhorst will be published in his forthcoming thesis.

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Timman R. 1946 Beschouwingen over de luchtkrachten op trillende vliegtuigen waarbij in het bijzonder rekening wordt gehouden met de samendrukbaarheid van de lucht. Thesis Delft.


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FLUID MECHANICAL ASPECTS IN VAPOUR DEPOSITION PROCESSES.

by

W.P.A. JOOSEN
PHILIPS RESEARCH LABORATORIES
EINDHOVEN, THE NETHERLANDS

ABSTRACT
The two-dimensional mass diffusion problem in an arbitrary velocity and temperature field is solved by a perturbation method. Formulae for the rate of growth are derived for a diffusion-limited process, including the effect of thermal diffusion. For a surface-controlled process expressions for the growth rate are obtained assuming some simplifying conditions.

NOMENCLATURE

t
$\bar{r}=\bar{r}(\bar{x}, \bar{y})$
$\bar{x}, \bar{y}$
$\bar{u}, \bar{v}$
$T$
$\bar{p}$
$\rho$
$\lambda$
$\gamma$
$C_{ps}$
$D$

- time variable
- space vector
- space coordinates
- length of reactor
- velocity components of gas mixture
- temperature of gas mixture
- pressure of gas mixture
- mass density of gas mixture
- viscosity of gas mixture
- coefficient of heat conduction of gas mixture
- gas constant
- heat capacity of gas mixture
- diffusion coefficient

- partial pressure of chemical compound
- partial pressure of chemical compound at the entrance
- mass density of chemical compound
- equilibrium mass density
- molecular mass of chemical compound
- Boltzmann constant
- thermal diffusion factor
- chemical reaction coefficient
- flux of chemical compound in the y-direction
- Reynolds number
- Prandtl number
- Schmidt number
1. INTRODUCTION

The rate of growth of a layer deposited in a horizontal reactor depends on various parameters. One of the effects that are very difficult to quantify is the influence of the velocity and temperature field on the diffusion and deposition process.

Several authors have proposed theories in order to explain this effect. The theories can be distinguished according to the character of the velocity field: fully developed flow /1/ or boundary layer flow /2,3,4/. All of these studies fail in describing adequately the behavior in the entrance region of the reactor because the effect of the actual entrance conditions on the flow cannot be taken into account very easily /5,6,7/.

Therefore the starting point for the present study will be an arbitrary velocity field and temperature field. The mass diffusion problem will be solved by a perturbation method on the condition that the diffusion layer is thin compared to the flow and the temperature boundary layer.

2. FORMULATION OF THE PROBLEM

The geometrical configuration and reference frame of a two-dimensional chemical reactor is represented in Figure 1. A gaseous chemical compound diffuses through the carrier gas towards the hot substrate at \( y = 0 \), where it is decomposed and a solid layer is deposited. The mean gas flow is in the \( x \)-direction and the velocity components are \( \bar{u} \) and \( \bar{v} \). The temperature of the substrate \( T_s \) and the reactor wall \( T_o \) remain constant, whereas the temperature of the gas at the entrance \( \bar{x} = 0 \) is assumed to be equal to \( T_o \).

The mass concentration of the chemical compound in the carrier gas is assumed to be small and constant at the entrance:

\[
\rho_1 = \rho_{10}
\]

Fig. 1. Geometry of the system

The differential equations and boundary conditions that determine the concentration field can be expressed in the following dimensionless quantities:

\[
x = \bar{x}/H, \quad \bar{y} = \bar{y}/H, \quad \tau = U_s t/H,
\]

\[
u = \bar{u}/u_s, \quad v = \bar{v}/u_s, \quad \rho = \bar{\rho}/\rho_s, \quad \mu = \bar{\mu}/\mu_s
\]

\[
v = \mu/\rho, \quad c = \bar{c}/c_s, \quad D = \bar{D}/D_s, \quad \theta = T/T_s,
\]

\[
f = T_o/T_s, \quad p = \bar{p}/\rho_s T_s,
\]

\[
j = RT_s H/D_s, \quad \delta = D_s/U_s H, \quad \gamma = k/m_1 R_s y = D_s H K_d
\]

\[
\lambda_s = \lambda_s / \rho_s c_s, \quad \lambda_r = v_s / K_s, \quad R_e = U_s H / \nu_s,
\]

\[
\rho_e = \rho_s \rho_r, \quad \gamma = \nu_s / D_s,
\]

\[
\delta = D_s / U_s H, \quad \beta = k / m_1, \quad R_s y = D_s H K_d
\]

The scaling factors are denoted by a subscript \( s \) and refer to values of the quantities at the substrate. The scaling velocity \( U_s \) is defined as:

\[
U_s = \left( \frac{\Delta p H}{\rho_s L} \right)^{1/2}
\]

where \( \Delta p \) is the pressure difference between two points that are a distance \( L \) apart in the \( x \)-direction. Neglecting terms of order \( \rho_1^2 \) and \( U_s^2 / RT_s \) the governing differential
equations and boundary conditions can be written as follows:

**Equation of State:**

\[
\rho = \frac{1}{\theta}, \quad p_1 = \rho \theta_1
\]

**Continuity Equation:**

\[
\frac{\partial p}{\partial t} + \frac{\partial \rho v}{\partial x} = 0
\]

**Mass Diffusion Equation**

\[
\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial \rho v}{\partial y} = \frac{1}{\theta} \left( \frac{\partial}{\partial x} \left( D \frac{\partial \rho}{\partial x} + \frac{1}{\rho} \frac{\partial \rho}{\partial y} \right) \right)
\]

**Boundary Conditions:**

\[
x = 0, \quad p = p_1, \\
y = 0, \quad \theta = \theta_1
\]

3. **Solution of the Problem**

The equation (2.3) will be solved by means of a perturbation method. A series expansion can be derived with respect to the small parameter \( \delta \). The solution consists of an inner solution which is valid near to the boundary \( y = 0 \) and an outer solution which is valid in the flow far away from the boundary. A new small parameter is introduced:

\[
\varepsilon = \delta^{1/3}
\]

The first term in the outer expansion of the pressure \( q \) is trivial:

\[
q = 1 + O(\varepsilon)
\]

In order to obtain the inner solution of \( q \), the \( y \)-coordinate is stretched:

\[
y = \varepsilon z
\]

The functions \( u, v, \theta \) and \( D \) can be expanded in a Taylor series e.g.:

\[
u = u' + \frac{\varepsilon}{2} u'' + \ldots
\]

where \( u' \) is the symbol for \( \frac{\partial u}{\partial y}(x,0) \), etc. The order of magnitude of the derivatives \( u(n) \) and the other functions is assumed to be of order unity. This is justified because it can be assumed that the diffusion layer thickness \( \delta_D \) is smaller than the velocity boundary layer thickness \( \delta_v \) and the temperature boundary layer thickness \( \delta_T \):

\[
\delta_v \leq \frac{1}{Re}, \quad \delta_T \leq \frac{1}{Re}, \quad \delta_D \leq \frac{1}{Re} \quad \text{for} \quad Sc > 1
\]

Inserting all this in the differential equation and neglecting terms of \( \varepsilon^2 \) and higher, the diffusion equation becomes:

\[
\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial}{\partial x} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial x} \right) = \varepsilon \left( \frac{\partial^2}{\partial x^2} \left( \frac{1}{\rho} \frac{\partial \rho}{\partial x} \right) \right)
\]

\[
s = 0, \quad q = 1, \\
y = \infty, \quad q = 1, \\
y = 0, \quad q = 0.
\]
This equation can be simplified by introducing the following new coordinates and functions:

\[ \begin{align*}
\tau &= s(u')^{\frac{1}{2}} , \\
\sigma &= \int_{u'}^{x}(x_1) \, dx_1, \\
g(\sigma) &= \theta'/(u')^{\frac{1}{2}} , \\
h(\sigma) &= u'''/(u')^{3/2}
\end{align*} \]

and using the Taylor expansion of the continuity equation (2.2):

\[ \frac{\partial^2 q}{\partial x^2} - \frac{\partial q}{\partial x} = \varepsilon \left[ \frac{2}{x} ( \frac{\partial h}{\partial x} + \frac{1}{2} \frac{\partial h}{\partial \sigma} + \frac{\partial q}{\partial \sigma} \right] + \frac{\partial^2 g(\sigma)}{\partial \sigma^2} (\tau \frac{\partial q}{\partial \tau}) + 0(\varepsilon^2) \]

The next step is to expand the partial pressure \( q \) as follows:

\[ q = q_0 + \varepsilon q_1 + \text{higher order terms} \quad (3.2) \]

After substituting this in the differential equation and boundary conditions a boundary value problem for \( q_0 \) and \( q_1 \) arises:

\[ \begin{align*}
\frac{\partial^2 q_1}{\partial x^2} - \frac{\partial q_1}{\partial x} &= 0; \quad \sigma=0, x=0: \quad q_0 = 1 , \\
\tau=0: \quad q_1 = 0 , \\
\frac{\partial^2 q_1}{\partial x^2} - \frac{\partial q_1}{\partial x} &= \bar{F}(\sigma, \tau); \quad \sigma=0, x=0: \quad q_1 = 0
\end{align*} \]  

where

\[ \begin{align*}
\bar{F}(\sigma, \tau) &= 2 \varepsilon A(\sigma) \frac{\partial q_0}{\partial \sigma} + \left[ B(\sigma) - \frac{\partial^2 \bar{E}(\sigma)}{\partial \sigma^2} \right] \frac{\partial q_0}{\partial \tau} , \\
A(\sigma) &= \frac{1}{2} h(\sigma) - \frac{\partial D}{\partial \sigma} g(\sigma), \\
B(\sigma) &= -g(\sigma) \left( \frac{\partial D}{\partial \sigma} \sigma - 1 \right).
\end{align*} \]

The mass flux in the y-direction at \( y=0 \) then becomes

\[ j = -\left( u' \frac{1}{2} \varepsilon \frac{\partial q_0}{\partial \tau} + \varepsilon \frac{\partial q_1}{\partial \tau} \right) \quad (3.6) \]

After the substitutions

\[ \eta = \frac{2}{3} \tau^{3/2} \quad \text{and} \quad q = \eta^{1/3} \]

the expression \( \frac{\partial^2 q}{\partial x^2} - \frac{\partial q}{\partial x} \) transforms to

\[ \left( \frac{3}{2} \right)^{2} \eta \left( 2 \frac{\partial q}{\partial \sigma} + \frac{1}{3} \frac{\partial w}{\partial \eta} - \frac{1}{9} \frac{\partial w}{\partial \eta} \frac{\partial w}{\partial \sigma} \right). \]

It is easy to see that the solution of the equation (3.3) satisfying the boundary conditions is given by

\[ q_0 = \eta^{2} \int_{0}^{1} e^{-\sigma t} J(\sigma t) d\sigma. \]

For the value of the flux at \( y=0 \) we find

\[ \frac{\partial q_1}{\partial \tau} = \left( \frac{3}{2} \right)^{2} \eta \int_{0}^{1} \frac{1}{3} \frac{\partial q_0}{\partial \sigma} \frac{\partial w}{\partial \eta} - \frac{1}{9} \frac{\partial w}{\partial \eta} \frac{\partial w}{\partial \sigma} \].

The second-order term of \( q \) can be derived using a Green's function:

\[ \begin{align*}
q_1 &= -\left( \frac{3}{2} \right)^{2} \eta \int_{0}^{1} T(\sigma, \sigma, \sigma) \, d\sigma, \\
T(\sigma, \sigma, \sigma) &= \frac{1}{2} \eta \int_{0}^{1} R(\sigma, \sigma) \, d\sigma
\end{align*} \]

By inserting \( q_0 \) into the expression (3.11)
it can be proved that

\[ \mathbf{R}(\eta, 0) = \frac{2}{\Gamma(\frac{4}{3})} \int_0^\infty \left( (3\alpha - 1)A(\sigma) + B(\sigma) + \frac{3}{4}\eta^2 \int_0^\sigma d\sigma \right) e^{-\sigma} \left\{ \frac{2}{3} - \frac{2}{3} \eta^2 \right\} d\eta \]

\[ = \frac{1}{3} \left( \frac{\eta^3}{\Gamma(\frac{4}{3})} \right) \left[ B(\sigma) - \frac{3}{4}\sigma^2 (A(\sigma) + \frac{\eta}{\sigma} \eta^2 \int_0^\sigma d\eta) \right] e^\frac{-\eta}{\sigma} \]

\[ = \frac{2}{\Gamma(\frac{4}{3})} \left[ B(\sigma) - \frac{3}{4}\sigma^2 (A(\sigma) + \frac{\eta}{\sigma} \eta^2 \int_0^\sigma d\eta) \right] e^\frac{-\eta}{\sigma} \]

(3.12)

The behaviour of the function \( T \) has been thoroughly studied by Sutton /7/. The calculation of \( q_1 \) for an arbitrary point \((\eta, \sigma)\) is a rather complicated matter. It is possible, however, to derive a simple formula for the mass transfer at \( \eta = 0 \).

From /7/ it follows that:

\[ \lim_{\eta \to 0} \frac{1}{2} \eta^3 \frac{\partial q_1}{\partial \eta} = \frac{1}{2} \mathbf{R}(\frac{1}{3}, 0) \int_0^\sigma \mathbf{T} \frac{1}{2} \mathbf{R}(\frac{4}{3}, 0) \left[ \mathbf{B}(\sigma) - \frac{3}{4}\sigma^2 (A(\sigma) + \frac{\eta}{\sigma} \eta^2 \int_0^\sigma d\sigma) \right] \]

\[ = \frac{1}{3} \left( \frac{\eta^3}{\Gamma(\frac{4}{3})} \right) \left[ B(\sigma) - \frac{3}{4}\sigma^2 (A(\sigma) + \frac{\eta}{\sigma} \eta^2 \int_0^\sigma d\sigma) \right] e^\frac{-\eta}{\sigma} \]

(3.13)

Using this expression and carrying out some integrations the result becomes

\[ \frac{\partial q_1}{\partial \tau} (\sigma, 0) = - \frac{1}{2} \frac{1}{\Gamma(\frac{4}{3})} \int_0^\sigma \left[ \mathbf{B}(\sigma) + \frac{3}{4}\sigma^2 (A(\sigma) + \frac{\eta}{\sigma} \eta^2 \int_0^\sigma d\sigma) \right] \]

\[ - \frac{\partial q_1}{\partial \sigma} \frac{\partial B}{\partial \sigma} \]

(3.14)

If \( A \) and \( B \) are constant or if \( \sigma = \) constant the expression reduces to

\[ \frac{\partial q_1}{\partial \tau} (\sigma, 0) = - \frac{1}{2} \frac{1}{\Gamma(\frac{4}{3})} \int_0^\sigma \left[ \mathbf{B}(\sigma) + \frac{3}{4}\sigma^2 (A(\sigma) + \frac{\eta}{\sigma} \eta^2 \int_0^\sigma d\sigma) \right] \frac{\partial q_1}{\partial \tau} \]

\[ = - (0.34243B - 0.05442A) \]

\[ = 0.342430(\alpha-1+0.841077 \frac{\partial q_1}{\partial \sigma}) + 0.02721h. \]

(3.15)

For the rate of growth of the layer deposited at \( y = 0 \) we obtain

\[ G(x) = - \frac{1}{\rho_{is}} \frac{\partial q_1}{\partial y} \]

\[ = \frac{P_{te} - P_{te}}{\mathbf{R}_{te} \mathbf{H}_0} \left( \frac{u'}{v} \right) \]

\[ = \frac{1}{\mathbf{R}^{\frac{1}{3}} \left( \frac{2}{3} \mathbf{G} + \frac{1}{3} \mathbf{G} \frac{\partial q_1}{\partial y} (\sigma, 0) \right)} \]

(3.16)

where \( \rho_{is} \) is the mass density of the solid deposited chemical compound.

4. SURFACE-CONTROLLED PROCESS

If the reaction kinetics at the growing interface is relevant in the deposition process the full boundary condition (2.1) has to be applied. In order to determine the influence of the parameters \( \gamma \) and \( p_{te} \), we will solve in this section the first order problem, neglecting the thermal diffusion effect \( (\alpha = 0) \). Defining the parameters

\[ q = \frac{P_{te} - P_{te}}{P_{te} - P_{te}} , \quad \omega = \frac{\gamma}{\epsilon} , \quad \sigma' = (u')^\frac{1}{2} \]

(4.1)

the problem can be formulated as

\[ \frac{\partial q}{\partial \tau} - \tau \frac{\partial q}{\partial \sigma} = 0 ; \quad \sigma = 0, \tau = \omega \Rightarrow q = 1 \]

\[ \tau = 0 ; \quad \omega \frac{\partial q}{\partial \tau} = q. \]

(4.2)

Introducing the coordinate

\[ \eta = \frac{3}{2} \left( \frac{1}{2} \right) \]

and subtracting the solution for the diffusion-limited problem

\[ q = \frac{2}{\tau} \eta^3 \int_0^\sigma \left( \frac{2}{3} \eta^3 \int_0^\sigma \left( \frac{2}{3} \eta^3 \int_0^\sigma d\sigma \right) \right) \]

we obtain the boundary conditions for \( Q \):

\[ \sigma = 0, \quad \eta = \infty \Rightarrow Q = 0 \]

\[ \eta = 0 ; \quad \omega \left( \frac{2}{3} \right) \eta^3 \int_0^\sigma \left( \frac{2}{3} \eta^3 \int_0^\sigma d\sigma \right) \]

(4.3)
The function $Q$ can be expressed as

$$Q = \frac{1}{2n} \frac{1}{\pi a} \int \eta \frac{1}{\eta} \exp(-\frac{\eta^2}{\frac{1}{4}a}) \ .$$

$$K_0(\frac{\eta}{\frac{1}{2}a}) d\eta \tag{4.4}$$

It can be proved that

$$\lim_{n \to 0} \frac{1}{\pi a} \int \eta \frac{1}{\eta} \exp(-\frac{\eta^2}{\frac{1}{4}a}) d\eta = -\frac{1}{\pi a} \int \frac{1}{\eta} \exp(-\frac{\eta^2}{\frac{1}{4}a}) d\eta \ .$$

$$\lim_{n \to 0} Q = -\frac{1}{\pi a} \int \frac{1}{\eta} \exp(-\frac{\eta^2}{\frac{1}{4}a}) d\eta \ .$$

$Q$ satisfies the first boundary condition (4.3). The unknown function $f$ can be obtained by application of the second boundary condition, which leads to the integral equation:

$$\int_0^\infty \phi(\alpha z) e^{-\frac{1}{3}z} dz = 1 \tag{4.5}$$

where

$$f(2/\sqrt{a}) = \phi(z) \ , \ m = \frac{3}{\omega^2} \frac{(1)}{\Gamma(1)}$$

If $\sigma'(\sigma)$ is an arbitrary function this equation has to be solved numerically.

It is possible for at least two functions $\sigma'$ to derive an analytical solution.

If we use a velocity distribution that exists around a semi-infinite plate in an infinite medium, the function $\sigma'$ becomes

$$\sigma' = \sigma_0 \frac{1}{\sigma}$$

where $\sigma_0$ is a constant.

This was done by Levich in /2/ and the result becomes

$$\phi(z) = \exp(-3m \frac{2}{2a} \frac{1}{z^3})$$

$$Q(\sigma, \sigma') = \frac{1}{\Gamma(\frac{1}{3})} \int_0^\infty \exp(-3m \frac{2}{2a} \frac{1}{z^3}) \frac{1}{z^3} dz \ .$$

The second example that can thus be solved is the case of a fully developed flow, where $\sigma$ is constant.

By expanding $\phi$ in a power series,

$$\phi(z) = \sum_k a_k z^3$$

the integral equation can be solved to obtain

$$\phi(z) = \frac{3}{\Gamma(\frac{1}{3})} \int_0^\infty \exp(-3m \frac{1}{z^3}) dt$$

and

$$Q(\sigma, \sigma) = 3 \int_0^\infty e^{-n^2 \frac{1}{3} t} A(t) dt, \tag{4.7}$$

where $A(t)$ is the Airy function and

$$n = \frac{\Gamma(\frac{1}{3})}{\omega^2} \sigma'^{-\frac{2}{3}}$$

The growth rate of the deposited layer then becomes

$$G = \frac{D_s P_{15}}{R T s \rho_{15}} \frac{1}{\gamma} Q(\sigma, \sigma) \tag{4.8}$$
5. SOME NUMERICAL RESULTS

The expression for the rate of growth of the deposited layer is derived for an arbitrary velocity and temperature field. In order to compare adequately theoretical and experimental results accurate data are needed concerning the behaviour of the velocity and temperature profile in the boundary layer. Values for the temperature gradient can be found in the literature /9/.

However, reliable velocity measurements in a strong temperature gradient over the boundary layer are not available. Nevertheless, some qualitative conclusions can be drawn from the formulae (3.15) and (3.16). Considering the first order term only, the velocity gradient can be expressed in terms of the growth function. It turns out that

\[ u' = \frac{2}{3} g_0^2 g_0(x) \int_0^x g_0(x) \, dx, \]

where

\[ g_0(x) = \frac{RT}{D_S P_1} \frac{1}{1 - \frac{1}{3} \Gamma \left( \frac{1}{3} \right) G(x)} \]

From this it can be concluded that a constant rate of growth is attainable if the velocity gradient shows a linear behaviour:

\[ u' = \frac{2}{3} g_0 x. \]

This can be realized by using a special laminar flow element at the entrance and a variation of the cross-section of the reactor.

If the second-order term is taken into account a deviation from the linear behaviour will occur.

In order to obtain a rough estimate of the second-order term, the asymptotic expression (3.15) can be used. Assuming

\[ a = 1, \quad \frac{dD}{d\varepsilon} = 2, \quad P_x = 1, \quad \varepsilon = 0.2, \quad R = 5. \]

and assuming that the thickness of the velocity and temperature boundary layer is

\[ \Delta \approx 0.5 \]

it follows that

\[ g = h = \Delta - \frac{1}{3} \]

and

\[ j = 4.29 \left( \frac{\Delta}{x} \right)^{1/3} + 0.22. \]

The second-order term is much smaller than the first-order term if

\[ x << 100 \Delta \approx 20 \]

It should be emphasized that the expressions (3.14) and (3.15) are valid only provided the latter condition is fulfilled. Because the inner expansion is derived from a first-order differential equation in \( x \), each individual term of the expansion does not satisfy the boundary condition at infinity

\[ \lim_{x \to \infty} j = 0 \]

The complete series of the expansion will ensure this. Therefore a truncated series is valid only for a finite value of the coordinate \( x \).

In order to assess the influence of the parameter \( \gamma \), the growth function for a fully developed flow has been computed. The first-order approximation of \( \gamma \) is represented in figure 2 for \( \gamma = 0 \) and \( \gamma \) having an arbitrary value.

\[ \frac{RT}{D_S P_1} G \left( \frac{p_x - p_0}{p_x} \right) \]

\[ \frac{1}{10 \times 3} \]

Fig. 2. Rate of growth for diffusion-limited process -- -- -- --

and for surface-controlled process - - - - -
The difference between the two curves is less than 20 per cent if
\[ x > 700 \gamma^3 \]
From this it can be deduced that for small values of \( \gamma \) the effect of the chemical reaction kinetics is limited to the region just behind the entrance of the reactor.

6. CONCLUSION
The mass diffusion problem has been solved for an arbitrary flow and temperature field, using a perturbation method in which it is assumed that the Schmidt number \( Sc \) is large. The effect of the chemical reaction kinetics on the deposition of the solid layer at the substrate appears to be mainly important just behind the entrance of the reactor. The first-order approximation of the rate of growth of the deposited layer depends on the temperature and the tangent to the velocity profile at the substrate. In deriving the second-order approximation the thermal diffusion effect and the temperature dependence of the diffusion coefficient are taken into account. This second-order term contains not only the tangent to the temperature profile but also the curvature of the velocity profile.

The growth function can be evaluated once the temperature and velocity in the boundary layer is known. For the isothermal flow problem in the entrance region of a channel, velocity calculations are available, depending on the entrance conditions. \( /4,5,6/ \). It seems possible without too much effort to extend these results to the non-isothermal problem provided the Prandtl number is high. Unfortunately this is not the case in a gas flow where the thickness of the velocity boundary layer and the thermal boundary layer are of the same order of magnitude. The solution in that case is much more complicated because the flow and temperature equations have to be solved simultaneously. \( /12/ \).

Moreover, the flow can become unstable as a consequence of the large temperature gradient and concentration gradient. A complete numerical calculation is then necessary to yield the required results. Of course, an alternative way of applying the formulae arrived at in this article is to use experimental data for the velocity and temperature.

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OPTIMAL CONTROL PROBLEMS IN RADIATION AND SCATTERING

by

T. S. Angell and R. E. Kleinman
Mathematics Department, University of Delaware
Newark, Delaware, U.S.A.

Dedicated to the memory of
R. Timman
Teacher and Friend

ABSTRACT

The problem of determining the surface current on a closed piecewise smooth curve which optimizes some functional of the far field is considered. In particular, the existence of square integrable Dirichlet data on the boundary which optimizes power radiated in angular sectors is proven. Moreover, a Galerkin method for approximating the optimal solution is developed and estimates of the rates of convergence are presented. The results are applied in the specific example of a circular cylinder and numerical evaluation of both the optimal surface current and far field are given.

INTRODUCTION

The antenna synthesis problem is usually posed in one of two ways. The first addresses the problem of finding an aperture distribution which produces a far field best approximating a prescribed pattern. The second involves determining the location and feed characteristics of a finite array of elementary sources which produce a radiation pattern which best realizes some property e.g. maximum directivity. Most work in the first area has been concerned with planar apertures (e.g. Bouwkamp and de Bruijn [1], Rhodes [2]) although apertures in cylinders have also been treated (e.g. Wait and Householder [3], Einarsson et al [4]). Deschamps and Cabayan [5] present a good overview of the mathematical problem and call attention to some dangers due to ill-posedness.

The present paper attempts to treat a combination of the two synthesis approaches in both a rigorous mathematical theory and its practical numerical consequences. Specifically, a closed curve in two dimensions (infinite cylinder of general cross section) is considered as a radiating structure. For given Dirichlet data on the curve (perfectly conducting cylinder with E-polarized field) there is associated a radiated far field. Some functional of the far field is to be maximized and the boundary data which
achieves this is sought in some class of admissible controls. In a sense the entire boundary is the "aperture" but instead of attempting to achieve a prescribed radiation pattern, the quantity to be optimized is the power radiated in specified angular sectors.

A careful statement of the problem is given in section 2 where the relation between the radiation problem and the boundary value (scattering) problem is shown. The existence of an optimal control in any closed convex control set in $L^2$ is proven in section 3. It is also shown that the search for this optimal solution may be confined to the boundary of the control set. The practical problem of constructing an optimal solution is treated in section 4 where the solutions are shown to be eigenvalue-eigenvector pairs associated with a compact operator. Moreover a Galerkin approximation to the eigenvalue problem is developed with specific estimates of the rates with which the approximate solutions converge to the exact solutions. These results are applied in section 5 to the specific case when the curve is a circle and the radiated power is to be maximized in an angular sector. Some sample numerical results are given. While the approximate and numerical results treated the case when the control set was the unit ball in $L^2$, meaningful optimization problems may be posed for other control sets for which the existence theory of section 3 applies. A specific example where the controls are bounded above and below by continuous functions is considered in the Appendix. It is shown in that case that the solution is a limit of bang-bang controls.

2. STATEMENT OF THE PROBLEM

Let $\Gamma$ be a simple closed curve consisting of piecewise $C^2$ arcs in $\mathbb{R}^2$, let $\Omega$ be the unbounded region determined by $\Gamma$, and let $\Omega^c$ denote the complementary region. For a more complete specification of the geometry see Leis [6] and Vekua [7]. In particular, the curve will have a unit normal $\hat{n}(q)$ except at finitely many points.

The vector $\hat{n}$ will vary continuously on each smooth arc and will be directed into the region $\Omega$. Choose the origin of a rectangular coordinate system in the interior of $\Omega^c$ and let $(r, \theta)$ be the polar coordinates of an arbitrary point $p \in \Omega$. The situation is pictured in Figure 1.

The Dirichlet problem for the Helmholtz equation in the exterior domain consists of finding a function $u$ such that

\begin{align}
  \text{i)} & \quad (\nabla^2 + k^2)u = 0, \quad p \in \Omega \\
  \text{ii)} & \quad u = h, \quad p \in \Gamma \\
  \text{iii)} & \quad \lim_{r \to \infty} \int_{S_r} \frac{|2u|}{2r} - iku |^2 dS = 0
\end{align}

where $S_r$ is a circle of radius $r$ lying entirely in $\Omega$. This problem has a unique solution for $h \in C(\Gamma)$ where $C(\Gamma)$ is the class of continuous functions on the boundary $\Gamma$ [16], [7]). Moreover this solution may be represented in the form

\begin{equation}
  u(p;h) = \int_{\Gamma} h(q) \frac{3}{2} G(p,q) (p,q) d\Gamma, \quad p \in \Omega
\end{equation}

where $G(p,q)$ is the Dirichlet Green's function for the exterior Helmholtz equation (e.g. Vekua [7] and Stakgold [8]). The Green's function has the form

\begin{equation}
  G(p,q) = \frac{1}{4} R_0^{(1)} (k|p-q|) + g(p,q)
\end{equation}

where $g(p,q) \in C^2(\Omega)$ (set of twice continuously differentiable functions in $\Omega$) and $R_0^{(1)}$ is the Hankel function of the first kind, the fundamental solution of the Helmholtz equation consistent with the radiation condition in (2.1 iii). Since $\Gamma$ is bounded one may employ the asymptotic properties of $G(p,q)$ for large $r$, viz.
\[ G(p, q) = \frac{e^{ikr}}{r^{1/2}} \hat{G}(\theta, q) + O\left(\frac{1}{r}\right), \quad (2.3) \]

to represent the solution \( u \) in the far field as
\[ u(p; h) = \frac{e^{ikr}}{r^{1/2}} \int \frac{h(q)}{r} \frac{\partial}{\partial q} \hat{G}(\theta, q) d\Gamma + 0\left(\frac{1}{r}\right). \quad (2.4) \]

The function \( \hat{G}(\theta, q) \) is regular for \( q \in \Gamma \) and, since \( \Gamma \) is piecewise smooth in the sense specified above, will have a piecewise continuous normal derivative which is also square integrable. The representation 2.4 may be used as a defining relation for the far field radiation pattern in terms of the boundary data \( h \). Specifically, define the far field coefficient \( f \) by the relation
\[ f(\theta) := \int_{\Gamma} h(q) \frac{\partial}{\partial q} \hat{G}(\theta, q) d\Gamma \quad (2.5) \]
in which case, with 2.4,
\[ u = \frac{e^{ikr}}{r^{1/2}} f(\theta) + O\left(\frac{1}{r}\right). \quad (2.5) \]

Note that this definition of the far field coefficient differs from that in Bowman et al. [9] by a factor \( \sqrt{2} \pi^{-1/4} \). The regularity of the function \( \hat{G} \) guarantees that the kernel \( \frac{\partial}{\partial q} \hat{G}(\theta, q) \) defines a linear operator \( K \) on \( L^2(\Gamma) \) where \( L^2(\Gamma) \) is the Hilbert space of functions defined on \( \Gamma \) and square integrable with respect to arc length. The operator \( K \) is compact. In fact it is Hilbert Schmidt, since \( \frac{\partial}{\partial q} \hat{G}(\theta, q) \in L^2((0, 2\pi) \times \Gamma) \). Thus we can rewrite 2.5 simply as
\[ f = Kh. \quad (2.7) \]

In what follows, we will denote the inner product in \( L^2(0, 2\pi) \) by \( (\cdot, \cdot) \) and the norm by \( ||\cdot||_{L^2} \); those in \( L^2(\Gamma) \) by \( (\cdot, \cdot)_{\Gamma} \) and \( ||\cdot||_{\Gamma} \).

Let \( Q(r; h) \) denote the power flux through a circle of radius \( r \) containing in its interior the curve \( \Gamma \) on which \( u \) has boundary values \( h \). Then
\[ Q(r; h) = \int_{\Gamma} |u|^2 ds = \int_0^{2\pi} |f(\theta)|^2 d\theta + O\left(r^{-1}\right) \quad (2.8) \]

Denoting the far field radiated power as
\[ Q(h) = \lim_{r \to \infty} Q(r; h) = \int_0^{2\pi} |f(\theta)|^2 d\theta \quad (2.9) \]
it follows that
\[ Q(h) = ||Kh||^2. \quad (2.10) \]

If \( A \) is a measurable subset of the interval \([0, 2\pi]\) then we denote the far field power flux through \( A \) by \( Q_A(h) \) where
\[ Q_A(h) = \int_A \alpha(\theta) |f(\theta)|^2 d\theta = \int_0^{2\pi} \alpha(\theta) |Kh|^2 d\theta. \quad (2.11) \]

Here \( \alpha(\theta) \) is the characteristic function of the set \( A \). For a fixed \( A \), \( Q_A(h) \) defines a functional on \( L^2(\Gamma) \).

These preliminary remarks allow us to pose a meaningful optimization problem. We consider the boundary data \( h \), suitably restricted, to be at our disposal and ask for those \( h \) which are optimal with respect to some criterion expressed in terms of the induced far field. Specifically, for a given closed, bounded, convex subset \( U \) of the Hilbert space \( L^2(\Gamma) \), called the class of admissible controls, find \( h_0 \in U \) for which \( Q_A(h_0) \) is an absolute maximum over \( U \). In other words, we seek \( h_0 \in U \) satisfying
\[ Q_A(h_0) \geq Q_A(h), \quad \text{for all } h \in U. \quad (2.12) \]

Observe that
\[ |Q_A(h)| \leq \int_0^{2\pi} |\alpha(\theta)| |Kh|^2 d\theta \leq \int_0^{2\pi} |Kh|^2 d\theta \]
\[ \leq ||K||^2 ||h||^2_{\Gamma} \quad (2.13) \]
hence for bounded \( U \), \( Q_A(h) \) is uniformly bounded. Note also that
\[ Q_A(h) = (\alpha Kh, Kh) = (K^* \alpha Kh, h)_\Gamma \quad (2.14) \]
where \( K^* \) is the operator adjoint to \( K \) which maps \( L^2(0, 2\pi) \to L^2(\Gamma) \). Explicitly
\[ K^*f = \int_0^{2\pi} f(\theta) \frac{2}{3n_q} \hat{G}(\theta, q) d\theta \quad (2.15) \]
where \( \hat{G} \) denotes complex conjugate (cf. 2.5, 2.7). If \( A \) is the entire interval \([0, 2\pi]\) then \( Q_A(h) = Q(h) = (K^* \alpha Kh, h)_\Gamma \). For convenience, introduce the notation
\[ R := \mathbb{K}^* c \mathbb{K}, \quad \text{where} \quad (\mathbb{R}h)(q) = \left( 0^\frac{1}{3 n q} \right) \tau_{1} G(\theta, q) \int_{\Gamma} h(q_1) \frac{1}{3n q_1} \, dq_1, \quad \text{and note that} \quad R \text{ maps } L^2(\Gamma) \to L^2(\Gamma). \]

We emphasize that there is no requirement on the set \( \mathbb{R} \) other than that it be a measurable subset of \( \mathbb{R} \). This flexibility will permit the simultaneous treatment of a number of problems of practical interest, e.g., the optimization of bi-directional antennas.

3. \textbf{EXISTENCE OF OPTIMAL CONTROLS}

In this section, we prove the existence of an optimal control \( h_0 \in U \subset L^2(\Gamma) \) which maximizes the functional

\[ Q_\alpha(h) = \int_{0}^{2\pi} a(\theta) \left| (\mathbb{K}h)(\theta) \right|^2 \, d\theta \quad (3.1) \]

where \( \mathbb{K} \) is defined in (2.5) and (2.7).

Let \( U \) be a closed, bounded and convex subset of \( L^2(\Gamma) \). Since \( L^2(\Gamma) \) is separable, the weak relative topology on the unit sphere is a metric topology [Dunford and Schwartz, [10]]. Hence the weak relative topology on any sphere is metric and, since \( U \) is bounded (and hence contained in some sphere), the weak relative topology on \( U \) is metric. Therefore if \( f : L^2(\Gamma) \to \mathbb{R}, f_{|U} \) is weakly continuous provided, \( u_n \to u \) weakly in \( U \) implies \( f_{|U}(u_n) \to f_{|U}(u) \) in \( U \).

\textbf{Lemma 3.1:} The functional \( Q_\alpha \) defined by (3.1) is weakly continuous on \( U \).

\textbf{Proof:} The remarks above imply that \( Q_\alpha \) will be weakly continuous on \( U \) provided \( h_n \to h \) in \( U \) implies \( Q_\alpha(h_n) \to Q_\alpha(h) \). But the operator \( \mathbb{K} : L^2(\Gamma) \to L^2(0,2\pi) \) is compact and hence \( h_n \to h \) weakly in \( U \) implies \( h_n \to h \) weakly in \( L^2(\Gamma) \) and hence \( \mathbb{K}h_n \to \mathbb{K}h \) strongly in \( L^2(0,2\pi) \). But

\[ Q_\alpha(h_n) - Q_\alpha(h) = \left| \right| \mathbb{K}h_n \left| \right|^2 - \left| \mathbb{K}h \right|^2 \left| \right| d\theta \]

\[ \leq \left| \left| \mathbb{K}h_n \right|^2 - \left| \mathbb{K}h \right|^2 \right| \]

\[ \leq \frac{1}{\left| \left| \mathbb{K}h_n \right|^2 - \left| \mathbb{K}h \right|^2 \right|} \]

Since \( \mathbb{K}h_n \to \mathbb{K}h \) strongly in \( L^2(0,2\pi), \left| \mathbb{K}h_n - \mathbb{K}h \right| \to 0 \) hence \( \lim_{n \to \infty} \left| Q_\alpha(h_n) - Q_\alpha(h) \right| = . \]

Now the existence of an optimal control follows:

\textbf{Theorem 3.1:} If \( U \) is a closed, bounded, convex subset of \( L^2(\Gamma) \) then there exists \( h_0 \in U \) such that

\[ Q_\alpha(h_0) = \sup_{h \in U} Q_\alpha(h). \]

\textbf{Proof:} Since \( U \subset \subset L^2(\Gamma) \) is closed and convex it is weakly closed (Dunford and Schwartz [10]). Since \( L^2(\Gamma) \) is reflexive, \( U \) is in fact weakly compact. The preceding lemma shows that \( Q_\alpha \) is a weakly continuous function on \( U \) and hence \( Q_\alpha \) attains its least upper bound on \( U \).

Both the optimal control \( h_0 \) and the optimal value of the functional, \( Q_\alpha(h_0) \), are not known a priori. It is known however (Theorem 3.1) that \( Q_\alpha \) does have a maximum.

Constructive methods usually proceed by forming maximizing sequences, that is, a sequence \( \{h_n\} \subset U \) such that \( Q_\alpha(h_n) \to Q_\alpha(h_0) \). The process produces approximations to both an optimal control, \( h_0 \), and the optimal value of the functional, \( Q_\alpha(h_0) \).

The following lemma, due essentially to Vainberg, suggests that any search for an optimal solution may be confined to the boundary of \( U \).

\textbf{Lemma 3.2:} Let \( U \) be a closed bounded convex subset of a Hilbert space \( H \) having non-empty interior. If \( F : U \to R \) is weakly continuous, then the image under \( F \) of the boundary of \( U \) is dense in \( F(U) \).

For the proof, the reader is referred to Vainberg ([11], p.77.) In our situation, we can improve this result as follows:

\textbf{Theorem 3.2:} Let \( U \) be a closed, bounded, convex subset of \( L^2(\Gamma) \). Then the functional \( Q_\alpha \) takes its optimal value at a point of the boundary of \( U \).
Proof: Recall the form of the cost functional (2.14, 2.16)

\[ Q_a(h) = (K_a^*Kh,h)_\Gamma = (Rh,h)_\Gamma. \]

Since the control set \( U \) is convex, if \( h_1 \) and \( h_2 \in U \) then \( h_\lambda = \lambda h_1 + (1-\lambda)h_2 \in U \) for \( 0 \leq \lambda \leq 1 \) and a straight-forward but tedious calculation shows that

\[ Q_a(h_\lambda) = (\lambda^2 - \lambda)Q_a(h_1 - h_2) + \lambda Q_a(h_1) + (1-\lambda)Q_a(h_2) \]

Moreover for any \( h \in L^2(\Gamma) \),

\[ Q_a(h) = \int_0^{2\pi} a(\theta) |Kh|^2 d\theta \geq 0 \]

and, since \( \lambda^2 - \lambda \leq 0 \), it follows that

\[ Q_a(h_\lambda) \leq \lambda Q_a(h_1) + (1-\lambda)Q_a(h_2) \]

i.e. the functional \( Q_a(h_\lambda) \) is convex.

Suppose now that \( h_0 \in U \) is an optimal solution of the problem 2.12. Then, since \( U \) is closed, bounded, and convex, \( U \) is identical with the convex hull of its boundary points. This means that \( h_0 \) may be written as

\[ h_0 = \sum_{i=1}^r a_i h_i, \quad h_i \in \text{bdry } U, \quad a_i \geq 0, \quad i = 1, \ldots, r, \]

and \( \sum_{i=1}^r a_i = 1 \). The convexity of \( Q_a(h) \) then implies that

\[ Q_a(h_0) = Q_a\left(\sum_{i=1}^r a_i h_i\right) \leq \sum_{i=1}^r a_i Q_a(h_i) \]

\[ \leq \max_{1 \leq i \leq r} \left\{ Q_a(h_i) \right\} \sum_{i=1}^r a_i \leq \max_{1 \leq i \leq r} \left\{ Q_a(h_i) \right\}. \]

Since \( h_0 \) maximizes \( Q_a \), there is some point \( h^b \in \text{bdry } U \) such that \( Q_a(h_0) = Q_a(h^b) \).

These results become useful only if they reduce the candidates for optimal solutions involves controls bounded point-wise by continuous functions. In that case a different procedure for reducing the solution candidates must be employed and a sub-optimal bang-bang result can be obtained.

Details are given in the Appendix.

In this section we have established the existence of an optimal control \( h_0 \) in any closed bounded convex subset of \( L^2(\Gamma) \).

A question may be raised as to the relationship between far field patterns attainable with continuous data and those attainable with \( L^2 \) data. The problem posed in 2.1 concerned \( h \in C(\Gamma) \) and unique solvability of that problem does not guarantee that a solution \( h_0 \in L^2(\Gamma) \) of the optimization problem is the boundary value of some wave function (solution of 2.1). However if we denote by \( m_a \) the optimal value of \( Q_a \) over \( V := U \cap C(\Gamma) \) then we can say immediately that \( m_a \leq Q_a(h_0) \) where \( Q_a(h_0) \) is the optimal value of \( Q_a \) over the larger set \( U \).

Moreover, under certain circumstances it may be possible to approximate the optimal solution \( h_0 \in U \), by an element \( h_0^\varepsilon \in V \) in the sense that given \( \varepsilon > 0 \) we may find \( h_0^\varepsilon \in V \) such that both \( ||h_0 - h_0^\varepsilon||_\Gamma \leq \varepsilon \) and \( ||Q_a(h_0^\varepsilon) - Q_a(h_0)|| < \varepsilon \). Since, in our problem, the map \( Q_a \) is a continuous functional on \( L^2(\Gamma) \), this approximation question is settled in the affirmative provided \( V \) is dense in the class of admissible controls.

This latter condition depends on the nature of the set \( U \). In particular, we have the following:

Lemma: If \( U \) has a non-empty interior, then \( V = U \cap C(\Gamma) \) is dense in \( U \).

Proof: It is necessary to show that if \( u \in U \), every (relative) neighborhood of \( u \) contains points of \( V \). If \( u \) is an interior point of \( U \), then there is a neighborhood, \( N(u) \), of \( u \) which contains a ball \( S_\varepsilon(u) \) for \( \varepsilon \) sufficiently small. But \( C(\Gamma) \cap N(u) \supseteq C(\Gamma) \cap S_\varepsilon(u) \neq \emptyset \). On the other hand, since \( U \) is convex with non-empty interior, \( U = c^*(\text{int } U) \) and hence every (relative) neighborhood of a boundary
point will intersect \( C(\Gamma) \) since it will contain a neighborhood of some interior point.

Remark: The condition that \( U \) have a non-empty interior is sufficient, but certainly not necessary, to guarantee that \( C(\Gamma) \cap U \) is dense in \( U \). To cite one example, if \( \psi_0 \) and \( \psi_1 \) belong to \( C(\Gamma) \) and we define the closed bounded convex set \( \mathcal{V} \) by

\[
\mathcal{V} = \{ u \in L^2(\Omega) | \psi_0(s) \leq u(s) \leq \psi_1(s) \text{ a.e. on } \Gamma \}
\]

then \( \psi \) has an empty interior, as shown in the appendix. However, an application of Lusin's Theorem will show that \( V \) is dense in \( \mathcal{V} \).

On the other hand, simple examples show that some condition is necessary. For example, if \( C \) is an arc in the plane then \( \mathbb{R}^2 \setminus C \) is dense in \( \mathbb{R}^2 \) but \( (\mathbb{R}^2 \setminus C) \cap C = \emptyset \).

4. APPROXIMATION OF OPTIMAL SOLUTIONS

In this section we discuss the approximation of solutions of the optimization problem presented in section 2 in the particular case when the control domain \( U \) is taken to be the unit sphere in \( L^2(\Omega) \). The optimization problem is now (cf. 2.14):

\[
\text{find } h_0 \in S := \{ h \in L^2(\Gamma) | \|h\|_{L^2}^2 = 1 \}
\]

such that

\[
Q_\alpha(h_0) = \sup_{h \in S} (R h, h) = \sup_{h \in S} (\alpha^2 h, h) = \lambda_0
\]

(4.1)

Since \( S \) is closed bounded and convex, Theorem 3.1 guarantees the existence of a solution and Theorem 3.2 shows that a solution exists on the boundary of \( S \). Hence it is sufficient to seek

\[
h_0 \in \text{bdry } S := \{ h \in L^2(\Gamma) | \|h\|_{L^2}^2 = 1 \}
\]

such that

\[
Q_\alpha(h_0) = \sup_{\|h\|_{L^2} = 1} (R h, h) \implies \lambda_0 = \lambda_0
\]

(4.2)

Solutions of this problem are characterized in the following theorem:

**Theorem 4.1:** If \( \lambda_0 \) is the largest eigenvalue of \( R \) and \( h_0 \) is a corresponding normalized eigenvector then

\[
\sup_{\|h\|_{L^2} = 1} (R h, h) = (R h_0, h_0) = \lambda_0
\]

(4.3)

**Proof:** \( K \) is compact (cf. 2.7) hence \( K^* \) is also compact as is \( R = K^* a K \). Moreover \( R \) is self adjoint since \( R^* = (K^* a K)^* = K^* a K^* \) and \( a^* = a \) (the characteristic function is real) and \( K^* = K \). This implies that the spectrum of \( R \) is discrete and real with zero as the only accumulation point. Moreover, the set of eigenvectors associated with any non-zero eigenvalue spans a subspace of finite dimension. It then follows (e.g. Stakgold, [8], p.188-190) that

\[
\sup_{\|h\|_{L^2} = 1} (R h, h) = |\lambda_0|
\]

(4.4)

where \( \lambda_0 \) is the eigenvalue of \( R \) with largest absolute value. Furthermore \( a(\theta) = a^2(\theta) \) (since \( a(\theta) \) is the characteristic function of a measurable subset of \((0, 2\pi)\)) hence

\[
(R h, h) = (K a Kh, h) = (h K, h) \geq 0
\]

(4.5)

Thus \( R \) is positive (although not necessarily positive definite) and its spectrum is non-negative. Therefore

\[
\sup_{\|h\|_{L^2} = 1} (R h, h) = \lambda_0
\]

If \( h_0 \) is a normalized eigenfunction associated with \( \lambda_0 \), then

\[
(R h_0, h_0) = (\lambda_0 h_0, h_0) = \lambda_0
\]

(4.7)

and the theorem is proved.

Note that \( \lambda_0 \) may be an eigenvalue with multiplicity greater than one and, in general, we do not expect uniqueness of the optimal control.

This theorem shows that an optimal solution of the problem (4.1) is found by determining the maximum eigenvalue and a corresponding eigenvector of the operator \( R \). To compute the maximum eigenvalue \( \lambda_0 \) and an associated eigenvector we employ
Galerkin's method. A complete discussion of this and related methods is given by Krasnoselskii et al. [12] and we only sketch the relevant facts.

We consider a family of subspaces $\{M_n\}_{n=1}^\infty$ of $L^2(\Gamma)$ having the property that for every $h \in L^2(\Gamma)$, $\inf_{u \in M_n} \|h-u\| \to 0$ as $n \to \infty$. Such a sequence of subspaces is said to be ultimately dense in $L^2(\Gamma)$. For each $n$ denote by $P_n$ the orthogonal projection associated with the subspace $M_n$, i.e. $P_n : L^2(\Gamma) \to M_n$. Galerkin's method determines approximate eigenvalues and eigenfunctions from the equation

$$P_n(\lambda u - Ru) = u, \quad u \in M_n.$$  \hspace{1cm} (4.8)

or equivalently

$$\lambda u = P_n u, \quad u \in M_n \quad \text{(4.9)}$$

Since $R$ is compact, $P_n R$ is compact on $L^2(\Gamma)$ and hence on $M_n$ for all $n$. In fact, considered as an operator on $M_n$, $P_n R$ is self adjoint. To see this note that $P_n u = u$ for all $u \in M_n$ and, for $u \in M_n$, $v \in M_n$,

$$(P_n R u, v) = (u, (P_n R)^* v). \quad \text{(4.10)}$$

But $(P_n R)^* v \in M_n$ hence

$$(P_n R u, v) = (u, P_n (P_n R)^* v) = (u, P_n R P_n^* v). \quad \text{(4.11)}$$

and since $R$ and $P_n$ are both self adjoint

$$(P_n R u, v) = (u, P_n R v). \quad \text{(4.12)}$$

Moreover $P_n R$ is a positive operator on $H_n$ since $R$ is positive on $L^2(\Gamma)$ and hence on $M_n$. It follows that the spectrum of $P_n R$ is real, positive and, denoting the maximum eigenvalue of $P_n R$ by $\lambda_{0,n}$, we have the error estimate (see [12], p. 281).

$$|\lambda_{0,n} - \lambda_0| \leq \sqrt{2} \| (I - P_n) R \|_\Gamma. \quad \text{(4.13)}$$

Furthermore

$$\| (I - P_n) R \|_\Gamma \to 0 \quad \text{as} \quad n \to \infty. \quad \text{(4.14)}$$

This follows because $|P_n u - u| \to 0$ as $n \to \infty$ and $u \in L^2(\Gamma)$ thus $P_n \to I$ strongly.

Moreover $R$ is compact so $R(S^1) = K^* a K(S^1)$ is relatively compact. Thus $P_n u + u$ uniformly in $R(S^1)$ and

$$\| P_n R - R \|_\Gamma = \sup_{\| u \|_1 \leq 1} \| P_n Ru - Ru \|_\Gamma \to 0 \quad \text{as} \quad n \to \infty.$$  \hspace{1cm} (4.15)

Remark: Estimates on the rate of convergence in specific situations may be obtained from detailed knowledge of the operator $R$. This is done explicitly in the example treated in the next section.

5. EXAMPLE - A cylindrical antenna

In this section we apply the preceding results in the particular case when $\Gamma$ is a circle of radius $a$ and $\alpha$ a sub-interval $[0, \theta_0] \subset [0, 2\pi]$. The optimal solution $h_0$ will then be the boundary values of that solution of the exterior Dirichlet problem for the Helmholtz equation for which the power radiated in the angular sector $[0, \theta_0]$ is maximized over a class of admissible boundary values. This control set is chosen to be the unit ball in $L^2(\Gamma)$ i.e.

$$S^1(\Gamma) = \left\{ h | \| h \|_1 = \int_0^{2\pi} |h(\theta)|^2 d\theta \right\} \quad \text{(5.1)}$$

In this example, of course, there is no difference between $L^2(0, 2\pi)$ and $L^2(\Gamma)$ however we attempt to preserve the distinction in order for the example to be instructive in cases when the spaces are not identical. The presence of the factor $a$ in the element of arc length on $\Gamma$ helps to make this distinction.

It is convenient to choose the family of subspaces $\{M_n\}_{n=1}^\infty$ to be the spaces of trigonometric polynomials, i.e.,

$$M_n = \left\{ u | u = \sum_{m=-n}^n c_m e^{im\phi} \right\}. \quad \text{(5.2)}$$

and since $\{e^{im\phi}\}_{m=-\infty}^\infty$ is a basis for $L^2(0, 2\pi)$ and also, in this case, for $L^2(\Gamma)$ then $h \in L^2(\Gamma) \Rightarrow h = \sum_{m=-\infty}^\infty c_m e^{im\phi}$, i.e. $h$ may be represented as a Fourier series

The projection onto $M_n$ is merely

$$P_n h = \sum_{m=-n}^n c_m e^{im\phi}. \quad \text{(5.3)}$$
The Dirichlet Green’s function for the circle is well known, e.g. Bowman (et al. [9], p. 112)

\[ G(p,q) = G(r,\theta,p,\phi) = \begin{array}{l}
\frac{i}{4} \sum_{j=-\infty}^{\infty} \frac{J_{|j|}(kr)H_{|j|}^{(1)}(kr)}{H_{|j|}^{(1)}(ka)} e^{ij(\theta-\phi)}
\end{array} \]

where \( r< = \min(r,p) \), \( r> = \max(r,p) \), \( J_{|j|} \) is the Bessel function and \( H_{|j|}^{(1)} \) is the Hankel function of the first kind. On the circle \( \rho = a \),

\[ \delta_{\rho=a} = \frac{\delta_{\rho=a}}{2\pi a} \]

and, with the Wronskian relation for cylindrical functions, it follows that

\[ \frac{3}{2\pi a} G(p,q) = \frac{1}{2\pi a} \sum_{j=-\infty}^{\infty} \frac{H_{|j|}^{(1)}(kr)}{H_{|j|}^{(1)}(ka)} e^{ij(\theta-\phi)} \]

The representation of a solution of the Dirichlet problem in terms of its boundary values \( h \) is, cf. (2.2)

\[ \frac{3}{2\pi a} = \left[ \int_0^{2\pi} h(\phi) \frac{1}{2\pi a} \sum_{j=-\infty}^{\infty} \frac{H_{|j|}^{(1)}(kr)}{H_{|j|}^{(1)}(ka)} e^{ij(\theta-\phi)} d\phi \right] \]

were it permissible to employ the asymptotic form

\[ H_{|j|}^{(1)}(kr) = \sqrt{\frac{2}{\pi kr}} e^{-\frac{|j|\pi i}{2}} + 0(\frac{1}{r}) \]

for all \( j \), then the far field equation (2.4) would become

\[ u(p,h) = \frac{e^{ikr}}{r^{1/2}} \cdot \frac{e^{-\pi i j/4}}{\pi \sqrt{2\pi k}} \int_0^{2\pi} h(\phi) \sum_{j=-\infty}^{\infty} \frac{e^{ij|j|\pi/2+ij(\theta-\phi)}}{H_{|j|}^{(1)}(ka)} d\phi + 0(\frac{1}{r}) \]

and adopting the notation

\[ a_j := \frac{e^{-\pi i j/4 - i|j|\pi/2}}{\pi \sqrt{2\pi k} H_{|j|}^{(1)}(ka)} \]

equation 2.7, which defines the far field coefficient, becomes

\[ f := Kh = \int_0^{2\pi} h(\phi) \sum_{j=-\infty}^{\infty} a_j e^{ij(\theta-\phi)} d\phi \]

Actually if \( h \) is restricted to \( M_n \), it follows that Equation 5.7 can be rewritten as

\[ u = \frac{1}{2\pi a} \int_0^{2\pi} h(\phi) \sum_{j=-\infty}^{\infty} \frac{H_{|j|}^{(1)}(kr)}{H_{|j|}^{(1)}(ka)} e^{ij(\theta-\phi)} d\phi \]

in which case it is permissible to employ the asymptotic form 5.8. The equation 5.11 is valid and defines the operator

\[ K : M_n \rightarrow L^2(0,2\pi), \text{ for any } n. \]

This operator is a map of \( L^2(\Omega) \rightarrow L^2(0,2\pi) \), is compact, and in fact is Hilbert Schmidt, since

\[ \int_0^{2\pi} \int_0^{2\pi} \sum_{j=-\infty}^{\infty} a_j e^{ij(\theta-\phi)} d\phi = (2\pi)^2 \sum_{j=-\infty}^{\infty} |a_j|^2 \]

The latter series converges since

\[ |a_j|^2 = \frac{2\pi a^2}{2\pi^2} \sum_{j=-\infty}^{\infty} \frac{1}{(j-1)!} \left( \frac{1}{r} \right)^2 \]

this estimate on the growth of \( a_j \) follows from its definition 5.10 and the asymptotic form of the Hankel function for large order.

Since \( a \) is the interval \( (0,\theta_o) \), its characteristic function is

\[ \alpha(\theta) = \begin{cases} 1, & 0 \leq \theta \leq \theta_o \\ 0, & \theta_0 < \theta < 2\pi \end{cases} \]

and the far field power flux, 2.11, becomes

\[ Q_{\alpha}(h) = \int_0^{\theta_0} \int_0^{2\pi} h(\phi) \sum_{j=-\infty}^{\infty} a_j e^{ij(\theta-\phi)} d\phi \frac{2\pi}{\pi} \]

where

\[ (Rh,h) = \int_0^{2\pi} (Rh)(\phi_1) h(\phi_1) d\phi_1 \]

and

\[ Q_{\alpha}(h) = \int_0^{\theta_0} \int_0^{2\pi} \sum_{j=-\infty}^{\infty} \frac{e^{ij(\theta-\phi)-i\lambda(\theta-\phi)}}{\lambda} d\phi d\theta \]

\[ = \frac{1}{\lambda^2} \int_0^{2\pi} \sum_{j=-\infty}^{\infty} \frac{e^{ij(\theta-\phi)}}{\lambda} d\phi \]

The optimization problem we seek to solve is that of finding \( h_0 \in S^1 \) (see 5.1) such that \( Q_{\alpha}(h_0) \geq Q_{\alpha}(h) \) for all \( h \in S^1 \). The solution according to Theorem 4.1 is an
eigenvalue-eigenvector pair corresponding to the largest eigenvalue of \( R \). That is, if \( R h_0 - \lambda_0 h_0 = 0 \) and \( \lambda_0 \geq \lambda_1 \geq \lambda_2 \ldots \geq 0 \) where \( \{ \lambda_i \} \) are the eigenvalues of \( R \) then

\[
Q_{\alpha}(h_0) = \lambda_0 \geq Q_{\alpha}(h) \quad h \in \mathbb{S}^1.
\]

(5.19)
The solution is found via Galerkin's method by finding approximate eigenvalue-eigenvector pairs \( (\lambda_0, n, h_0, n) \) in \( M_\infty \) of the projection of the eigenvalue equation (4.8). Explicitly if \( h_0 \in M_\infty \) has the finite Fourier representation

\[
h_n = \sum_{m=-n}^{n} c_m^{(n)} e^{i m \phi}
\]

then

\[
P_n R h_n = \sum_{m=-n}^{n} \sum_{j=-n}^{n} a_j^m \delta_{m,j} c_j^{(n)} e^{i m \phi}
\]

(5.20)

and

\[
Q_{\alpha}(h_n) = (P_n R h_n, h_n) = \sum_{m=-n}^{n} \sum_{j=-n}^{n} a_j^m \delta_{m,j} c_j^{(n)} c_m^{(n)}
\]

(5.21)

In matrix form, define

\[
B^{(n)} := (4 \pi^2 a_j^m \delta_{m,j})
\]

and let \( h_n \) be represented by the column vector of its Fourier coefficients

\[
h_n = (c_j^{(n)})
\]

(5.22)

Then 5.22 becomes

\[
Q_{\alpha}(h) = h_n^T B^{(n)} h_n.
\]

(5.23)
The complex conjugate of an eigenvector corresponding to the largest eigenvalue \( \lambda_0, n \) of the Hermitian matrix \( B^{(n)} \) has components which are the Fourier coefficients of an optimal solution, \( h_0, n \) of the approximate optimization problem and the eigenvalue \( \lambda_0, n \) is the maximum value of the functional \( Q_{\alpha}(h_n) \).

This numerical problem has been solved by standard techniques for a number of values of \( ka \) and \( \theta_0 \). Once the optimal \( h_0, n \) is found it can be inserted in equation 5.11 to determine the optimal radiation pattern and this has also been done. Figures 2 and 3 present the absolute value of the optimal

\[
FIGURE 2. Optimal Solution - ka = 1, \theta_0 = \pi/6
\]

--- far field
--- surface current

\[
FIGURE 3. Optimal Solution - ka = 2, \theta_0 = \pi/6
\]

--- far field
--- surface current

surface current, \( |h_0, n| \), and the magnitude of the associated radiation pattern, \( |f| = |R h_0, n| \) as a function of angle. For \( n = 1 \), the difference between successive approximations

\[
\sum_{j=-n}^{n} |c_j^{(n+1)} - c_j^{(n)}|^2
\]

was less than .001.

Some idea of the rate of convergence is obtained from the bound 4.13,

\[
|\lambda_0 - \lambda_0, n| \leq \sqrt{2} ||(I-P_n)R||_F
\]

In the present case \( R h \) is given by 5.17 and if \( h \) is represented by its Fourier series,

\[
h(\phi) = \sum_{n=-\infty}^{\infty} c_n e^{i m \phi}
\]

then

\[
(R h)(\phi_1) = \sum_{m=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} c_{ij} a^i_k \delta_{i,j} e^{i m \phi}
\]

(5.24)
and

\[ (Rh)(\phi_1) - (P_n Rh)(\phi_1) = \frac{2\pi}{a} \sum_{j=-\infty}^{\infty} \left( \frac{a_j}{|k|n} \phi_1 \right) j \phi_1 \]  

(5.25)

then

\[ |Rh - P_n Rh|_\Gamma = \left( \frac{(2\pi)^3}{a} \right) \sum_{|k|n \in \mathbb{Z}^+} |a_j| \left( \sum_{j=-\infty}^{\infty} \left( \frac{a_j}{|k|n} \phi_1 \right) j \phi_1 \right)^{1/2} \]  

(5.26)

With the definition 5.18 it may be shown that

\[ \delta_{j,k} = \begin{cases} \left( \frac{(j-k)\theta_0}{2} \right) & j \neq k \\ \theta_0 & j = k \end{cases} \]  

(5.27)

hence \[ |\delta_{j,k}| \leq \theta_0 \] and 5.26 becomes

\[ |(I-P_n)Rh|_\Gamma \leq 2\pi \theta_0 \sqrt{\sum_{j=-\infty}^{\infty} |a_j|^2 \frac{\sin \left( \frac{(j-k)\theta_0}{2} \right)}{(j-k)^2} \frac{|k_{j}\phi_1|^2}{|k_{j}|^2}} \]  

(5.28)

If \[ |h|_\Gamma = 1 \] then \[ \sum_{j=-\infty}^{\infty} |c_j|^2 = 1 \] and with Schwartz' inequality

\[ |(I-P_n)R|_\Gamma = \sup_{|h|_\Gamma = 1} |(I-P_n)Rh|_\Gamma \leq 2\pi \theta_0 \sqrt{\sum_{j=-\infty}^{\infty} |a_j|^2 \sum_{j=-\infty}^{\infty} \frac{|c_j|^2}{|j|^2} \frac{|k_{j}\phi_1|^2}{|k_{j}|^2}} \]  

(5.29)

Since \[ \sum_{j=-\infty}^{\infty} |a_j|^2 \] converges and \[ a_{\ell} = a_{-\ell} \] this may be written as

\[ |(I-P_n)R|_\Gamma \leq C \left( \sum_{j=-\infty}^{\infty} |a_j|^2 \right)^{1/2} \]  

(5.30)

where \[ C \] is some constant independent of \( n \). With 5.14 it may be shown that

\[ \sum_{j=-\infty}^{\infty} |a_j|^2 = \frac{\left( \frac{1}{2} \right)^{2|k_n|+2} \left[ 1 + O \left( \frac{1}{(k_n+1)^2} \right) \right]}{2 \pi \kappa_k} \]  

(5.31)

\[ \leq \frac{1}{2 \pi \kappa_k} \frac{(k_n+2)!}{(n!)^2} e^{\frac{1}{4} \left( 1 + \frac{1}{n} \right)} \]  

hence we obtain the estimate

\[ |(I-P_n)R|_\Gamma \leq C_1 \left( \frac{ka}{2} \right)^{n+1} \frac{1}{n!} \left[ 1 + O \left( \frac{1}{n} \right) \right] \]  

(5.32)

\[ APPENDIX - A control set with empty interior. \]

We have shown in Theorem 3.2 that the search for an optimal solution to the problem of maximizing the functional \( Q_u \) over a closed bounded and convex set \( U \) of control functions in \( L^2(\Gamma) \) may be confined to the boundary of the set \( U \). In some cases, as we have seen in section 4, this restriction to the boundary of \( U \) suggests a convergent numerical procedure for approximating the optimal solution. In other cases, it may either reduce the number of candidates for the optimal control or it may afford additional insight into the nature of an optimal solution.

In this appendix, we will consider an example of a control set which has empty interior (see the remarks at the end of section 3). Here we are able to show that there exists a maximizing sequence of extreme points of the control set and we can characterize these extreme points as bang-bang controls.

Consider the control set

\[ \Psi = \{ u \in L^2(\Gamma) | \psi_0(s) \leq u(s) \leq \psi_1(s), \text{ a.e. in } \Gamma \} \]

where \( \psi_0, \psi_1 \in C(\Gamma) \). It is easy to see that \( \Psi \) is convex. Indeed, for any \( u, v \in \Psi \) and for any \( \lambda, 0 < \lambda < 1 \),

\[ \psi_0(s) = \lambda \psi_0(s) + (1-\lambda) \psi_1(s) \leq \lambda \psi_1(s) + (1-\lambda) \psi_1(s) = \psi_1(s) \]

However, every point of \( \Psi \) is a boundary point.

To see this, take any \( u \in \Psi \), let \( \epsilon > 0 \) be given and choose \( M > \sup_{s \in \Gamma} \psi_1(s) + |\inf_{s \in \Gamma} \psi_0(s)| \). Let \( E \subset \Gamma \) be measurable and satisfy the condition:

\[ 0 < u(E) < (\epsilon/M)^2 \]

Define \( v \in L^2(\Gamma) \) by

\[ v(s) = \begin{cases} u(s) & \text{on } E \\ M + u(s) & \text{on } \Gamma \setminus E \end{cases} \]

Then certainly, \( v \notin \Psi \) since, on \( E \),
\[ v(s) = M + u(s) > \sup_{\Gamma} \psi_1(s) \]
\[ + |\inf_{\Gamma} \psi_0(s)| + u(s) > \sup_{\Gamma} \psi_1(s). \]

On the other hand,
\[ \left( \int_{\Gamma} |u(s) - v(s)|^2 \, d\Gamma \right)^{1/2} = M(\mu(E))^{1/2} < \varepsilon. \]

Hence every fundamental neighborhood of \( u \) meets \( L^2(\Gamma) \). Since \( \mathcal{V} \) is convex it can have no isolated points and so every point of \( \mathcal{V} \) is a boundary point.

Clearly, no additional information is gained here by the knowledge that \( Q_\alpha \) will take on its optimal value on the boundary of \( \mathcal{V} \). In order to reduce the number of candidates for optimality, we need a result analogous to the lemma of Vainberg which is applicable to the present situation.

To this end, we will consider, instead of the boundary of the control set \( \mathcal{U} \), the set \( \mathcal{U}_{\text{ext}} \), of extreme points. We recall, that a point of a convex set is called an extreme point if it is not an internal point of a line segment contained in the set. The Krein-Milman theorem (see [10]) asserts that a compact convex subset of a locally convex topological vector space is the closure of the convex hull of its extreme points. Using this fact, we can show that, for direct methods, it is sufficient to construct maximizing sequences in \( \mathcal{U}_{\text{ext}} \).

**Proposition:** Let \( \mathcal{U} \), the class of admissible controls, be a closed, bounded, and convex subset of \( L^2(\Gamma) \) and let \( h_0 \) be an optimal solution of the problem (3.1). Then, there exists a (maximizing) sequence of extreme points \( \{h_n^e\} \) such that
\[ Q_\alpha(h_n^e) = Q_\alpha(h_0). \]

**Proof:** Since \( \mathcal{U} \) is bounded in \( L^2(\Gamma) \), it is weakly relatively compact in \( L^2(\Gamma) \).

In fact, it is weakly compact since strongly closed convex subsets of a Hilbert space are weakly closed.

Let \( \mathcal{U}_{\text{ext}} \) be the set of extreme points of \( \mathcal{U} \). It follows from the Krein-Milman theorem that \( \mathcal{U} = \text{co} \mathcal{U}_{\text{ext}} \) and hence that there exists a sequence \( \{h_n\} \subset \text{co} \mathcal{U}_{\text{ext}} \) such that \( h_n \to h_0 \) strongly (and so also weakly) in \( L^2(\Gamma) \). Hence, by the weak continuity of \( Q_\alpha \) on \( \mathcal{U} \), \( Q_\alpha(h_n) \to Q_\alpha(h_0) \).

Moreover, for each \( n, n = 1,2, \ldots \), we may write
\[ u_n = \sum_{k=1}^{m_n} a_k^n h_k^n, \quad \sum_{k=1}^{m_n} a_k^n = 1; \]

where \( h_k^n \in \mathcal{U}_{\text{ext}}, k = 1, \ldots, m_n, n = 1,2, \ldots \).

Reasoning as before, for each \( n \), we have that, for some \( k_n, 1 \leq k_n \leq m_n \),
\[ Q_\alpha(h_n) \leq Q_\alpha(h_{k_n}^n) \leq Q_\alpha(h_0), \quad n = 1,2, \ldots, \]

where \( h_{k_n}^n \in \mathcal{U}_{\text{ext}} \). Hence, the sequence \( \{h_{k_n}^n\} \) is a maximizing sequence of extreme points of \( \mathcal{U} \).

**Remark:** The proof of the proposition does not depend on the particular form of the functional \( Q_\alpha \) but only on its weak continuity and convexity. Nor does it depend on the choice of \( L^2(\Gamma) \) as the underlying Hilbert space.

The proceeding proposition affords no further information when applied to the case in which \( \mathcal{U} \) is the unit sphere in \( L^2(\Gamma) \) since the boundary, in that case, coincides with the set of extreme points. For the particular case of the set \( \mathcal{V} \), however, we can identify the set \( \mathcal{V}_{\text{ext}} \) and can show that this set is weakly compact in \( L^2(\Gamma) \). Hence, the proceeding proposition is therefore applicable and the optimal value for \( Q_\alpha \) can be approximated by values of \( Q_\alpha \) taken at points of \( \mathcal{V}_{\text{ext}} \). It is interesting to note that the characterization contained in the following proposition identifies the extreme points of \( \mathcal{V} \) as "bang-bang" controls. That is, the control functions in \( \mathcal{V}_{\text{ext}} \) take on only the extreme values \( \psi_0(s) \) or \( \psi_1(s) \) at almost all points \( s \in \Gamma \). In the following, \( \chi_M \) denotes the characteristic function of...
the measurable set $M$ and $\mu(M)$ denotes its measure.

**Proposition:** Let $u \in \mathcal{V}$. Then $u$ is an extreme point of $\mathcal{V}$ if and only if $u = x_{E_0} \psi_0 + x_{E_1} \psi_1$ a.e. where $E_0 \cap E_1 = \emptyset$, $\mu(E_0 \cup E_1) = \mu(\Gamma)$.

**Proof:** Suppose $u$ is not of the prescribed form. Then there exists an $\varepsilon > 0$ and a measurable set $E_2 \subseteq \Gamma$, $\mu(E_2) > 0$, such that $\psi_0(s) < u(s) - \varepsilon < u(s) < \psi_1(s) + \varepsilon$. Define the functions $u_1$ and $u_2$ by

$$u_1(s) = \begin{cases} u(s), & s \in \Gamma \setminus E_2 \\ u(s) - \varepsilon, & s \in E_2 \end{cases}$$

$$u_2(s) = \begin{cases} u(s), & s \in \Gamma \setminus E_2 \\ u(s) + \varepsilon, & s \in E_2 \end{cases}$$

Then clearly, $u(s) = \frac{1}{2}(u_1(s) + u_2(s))$ a.e. on $\Gamma$ with $u_1, u_2 \in \mathcal{V}$. So $u$ is not an extreme point of $\mathcal{V}$. Every extreme point, therefore, must have the form $u = x_{E_0} \psi_0 + x_{E_1} \psi_1$ a.e. for a suitable choice of $E_0$ and $E_1$.

Conversely, suppose that $u = x_{E_0} \psi_0 + x_{E_1} \psi_1$ a.e. and suppose further that $u = \lambda u_1 + (1-\lambda) u_2$, $0 < \lambda < 1$, for some $u_1, u_2 \in \mathcal{V}$. We show that, for $i = 0, 1$, $u_i = \psi_i$ on $E_i$ and hence $u$ cannot be expressed as a proper convex combination of other points of $\mathcal{V}$. To see this, suppose $s \in E_0$. Then

$$u(s) = x_{E_0} \psi_0(s) + x_{E_1} \psi_1(s) = \psi_0(s).$$

Hence

$$\lambda u_1(s) + (1-\lambda) u_2(s) = \lambda \psi_0(s) + (1-\lambda) \psi_0(s)$$

and, since $0 < \lambda < 1$ and $u_i \geq \psi_0$ a.e., we have that

$$\lambda(u_1(s) - \psi_0(s)) + (1-\lambda)(u_2(s) - \psi_0(s)) = 0$$

a.e.

This implies that, for almost all $s \in E_0$, $u_1(s) = \psi_0(s)$ and $u_2(s) = \psi_0(s)$. The corresponding result for $s \in E_1$ completes the proof.

In order to show that a maximizing sequence can be constructed from the set of "bang-bang" controls (the extreme points of $\mathcal{V}$) we need only show that the set $\mathcal{V}$ is weakly compact in $L^2(\Gamma)$.

**Theorem:** If $u^0$ is an optimal solution of the problem (3.1) (with $\mathcal{V}$ as the class of admissible control functions) then there exists a sequence of bang-bang controls which forms a maximizing sequence.

**Proof:** Clearly, $\mathcal{V}$ is bounded in $L^2(\Gamma)$ by $||\psi_i||$. It is also, as we have seen, convex. To prove that $\mathcal{V}$ is weakly compact, it suffices to show that it is closed in $L^2(\Gamma)$. But any sequence $\{u_n\} \subseteq \mathcal{V}$ such that $u_n \rightarrow u$ in $L^2(\Gamma)$ contains a subsequence which converges pointwise almost everywhere to $u$ and hence the defining inequalities are satisfied by the limit $u$ almost everywhere on $\Gamma$.

**Remark:** We cannot conclude, from the arguments above, that the optimal control itself is a bang-bang control; only that a "sub-optimal" bang-bang control exists. Lions [14] has established, by different arguments, a strict bang-bang result for systems governed by elliptic and parabolic operators and under the restriction that the boundary of the region is analytic. The techniques presented there, however, give no information about the distribution of switching points (the points of $\Gamma$ at which the control should be switched between $\psi_0$ and $\psi_1$). Here, if we can actually construct a maximizing sequence, then, at least for a sub-optimal control, we will know the location of the switch points.

**ACKNOWLEDGMENT:** This work was supported by the United States Air Force Office of Scientific Research under Grant 74-2634. Part of this work was carried on while one of the authors (TSA) was a guest at the School of Mathematics, Georgia Institute of Technology, whose congenial reception was much appreciated. The authors wish to acknowledge the work of Kirk Jordan and Gary Custis in programming the numerical calculations.
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VORTICITY IN TWO-FLUID HYDRODYNAMICS

BY

J.A. GEURST

PHILIPS RESEARCH LABORATORIES
EINDHOVEN, THE NETHERLANDS

ABSTRACT

The non-dissipative two-fluid equations of Landau are derived from a Hamiltonian density by suitably choosing pairs of canonically conjugate variables. At the same time the equations are generalised to the case of non-vanishing superfluid vorticity. In the derivation use is made of Clebsch transformations. The superfluid vorticity is shown to be transported by the mass velocity. A Lagrangian variational principle is deduced from the Hamiltonian formalism.

1. INTRODUCTION

It is well known that below the λ-point (T_λ = 2.17K at saturated vapour pressure) liquid 4He exhibits some peculiar thermo-hydrodynamic properties. It can, for example, flow through fine capillaries and narrow constrictions without any detectable friction. This means that during flow no pressure difference is observed under isothermal conditions. This last restriction is essential since a small temperature gradient is accompanied by a large pressure gradient under equilibrium conditions (fountain pressure). On the other hand experiments show that a pile of oscillating disks entrains liquid helium. This can only be explained by assuming that below T_λ liquid helium possesses a finite viscosity. Another peculiar property of 4He below T_λ is its extremely high thermal conductivity, which may be many orders of magnitude larger than that of 4He above the λ-point.

These and other phenomena caused London (1938) and Tisza (1938) to consider 4He below T_λ as a composition of two fluids, viz. a non-viscous superfluid component which can flow without friction through fine capillaries, and a normal-fluid component which accounts for the viscous properties. This normal component is thought to carry the entropy of the fluid. Landau (1941) was the first to give a kinetic interpretation of the two-fluid model. We shall summarize it in the form given by Kronig [1].

At absolute zero 4He is considered as a non-viscous fluid. At higher temperatures some elementary types of motion, designated as phonons and rotons, are excited in the fluid. These so-called elementary excitations may have a non-vanishing drift velocity \( \vec{V} \) with respect to the non-viscous carrier fluid. The local velocity of the carrier fluid is denoted by \( \vec{V}_g \). Writing
down the kinetic balance equations for the energy and momentum of the excitations and using statistical expressions for the thermodynamical functions, one can derive the following equations which are valid for low velocities of the carrier fluid as well as of the excitations:

\[
\frac{\partial S}{\partial t} + \nabla \cdot (S\mathbf{v}_n) = 0, \quad (1.1)
\]

\[
\frac{\partial S}{\partial t} + \mathbf{v}_0 \cdot \nabla S = -SVT. \quad (1.2)
\]

Here \( S \) denotes the entropy density of the excitations, \( \mathbf{v}_n = \mathbf{v}_s + \mathbf{w} \) is the local velocity of the excitations, \( \mathbf{v}_0 \) designates their local momentum density with respect to the carrier fluid and \( T \) is the absolute temperature. The quantity \( \mathbf{v}_0 \) can be written as \( \mathbf{v}_0 = \rho_n \mathbf{w} \), where \( \rho_n \) is the mass density of the elementary excitations. The total momentum density \( \mathbf{j} \) is given by

\[
\mathbf{j} = \rho \mathbf{v}_s + \mathbf{v}_0 = \rho \mathbf{v}_s + \rho_n \mathbf{w} = \rho \mathbf{v}_s + \rho_n \mathbf{w}_n
\]

with \( \rho = \rho_s + \rho_n \). The quantity \( \rho_n \) is interpreted as the mass density of the superfluid component, i.e. that part of the fluid that is not associated with the elementary excitations. Equations (1.1) and (1.2) should be supplemented by the equations expressing the conservation of total mass and that of total momentum, given respectively by

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}_s) = 0, \quad (1.3)
\]

\[
\frac{\partial}{\partial t} (\rho \mathbf{v}_s + \rho_n \mathbf{w}) = -\nabla \mu. \quad (1.4)
\]

We recall that equations (1.1) to (1.4) are only valid in the case of low velocities. They constitute 8 equations for the determination of the 8 unknowns \( \mathbf{v}_s, \mathbf{w}, \rho, \) and \( T \) as functions of space and time. The equations should, of course, be supplemented by appropriate initial and boundary conditions. Note that \( \rho_n \) is determined thermodynamically. From equations (1.1) to (1.4) there follow the equations of motion

\[
\frac{\partial \mathbf{v}_s}{\partial t} = -\frac{1}{\rho} \nabla \mu - \frac{\partial P}{\partial \rho} \frac{\partial S}{\partial \rho} VT. \quad (1.5)
\]

Equation (1.5) is usually written as

\[
\frac{\partial \mathbf{v}_s}{\partial t} = -\mathbf{v}_s \cdot \nabla \mu. \quad (1.7)
\]

where \( \mu \) is the specific thermodynamic potential or specific free enthalpy.

It is an immediate consequence of (1.7) that \( \frac{\partial}{\partial t} \mathbf{v}_s \cdot \mathbf{v}_s = 0 \). Assuming that at some instant in the past \( \mathbf{v}_s \cdot \mathbf{v}_s = 0 \), we have at every instant \( \mathbf{v}_s = 0 \).

Within the approximation of low velocities eq. (1.2) can be written as

\[
\frac{\partial \mathbf{A}}{\partial t} = -VT, \quad (1.9)
\]

where

\[
\mathbf{A} = \mathbf{v}_0 - \frac{\mathbf{v}_s}{S} \mathbf{v}_s. \quad (1.10)
\]

Reasoning similar to that given for \( \mathbf{v}_s \) yields

\[
\mathbf{v}_s \cdot \mathbf{A} = 0. \quad (1.11)
\]

The quantity \( \mathbf{A} \), which plays an important part in the extension of the equations of motion to the non-linear case, was introduced only recently by Putterman [2] and Saffman [3]. From (1.8) and (1.11) it follows that \( \mathbf{v}_s \) and \( \mathbf{A} \) may be represented by the potentials \( \phi \) and \( \alpha \) according to

\[
\mathbf{v}_s = \nabla \phi \quad \text{and} \quad \mathbf{A} = \nabla \alpha. \quad (1.12)
\]

Integration of (1.7) and (1.9) then gives

\[
\frac{\partial \phi}{\partial t} + \mu = 0 \quad (1.13)
\]

and

\[
\frac{\partial \alpha}{\partial t} + T = 0. \quad (1.14)
\]

Constants of integration have been absorbed into the functions \( \phi \) and \( \alpha \), respectively.

Having obtained the equations of motion in the linear approximation of low velocities we shall first generalise these equations in the next section on the assumption that \( \mathbf{v}_s \cdot \mathbf{v}_s = 0 \) and \( \mathbf{v}_s \cdot \mathbf{A} = 0 \).

This will be achieved by showing that they are equivalent to a set of canonical equations derived from a Hamiltonian density. This set of canonical equations then constitutes the extension of the equations of motion to the non-linear case. In the non-linear case, \( \mathbf{v}_s \) and \( \mathbf{A} \) are
not necessarily curl-free. The next step is therefore to extend the equations of motion to the case that \( \nabla \times \mathbf{V} = 0 \) and \( \nabla \mathbf{A} = 0 \). The Hamiltonian formalism proves very useful for that purpose. If it is assumed that \( \nabla \times \mathbf{V} = 0 \), the equations obtained coincide with the non-linear Landau two-fluid equations. A Lagrangian variational principle is formulated in section 3. Expressions are given for the energy flux, the momentum density and the stress tensor associated with the Lagrangian density of the variational field. The variational principle can be transformed into its reciprocal form. Details of this transformation will be given elsewhere.

2. HAMILTONIAN FORMALISM

On account of Galilean invariance the energy density \( H \) of superfluid helium is given by

\[
H = \frac{1}{2} \rho \mathbf{v}_s^2 + \mathbf{J}_0 \cdot \mathbf{V}_s + U(\rho, S, J_0),
\]

(2.1)

where \( U(\rho, S, J_0) \) denotes the internal energy density. The thermodynamic potential \( \theta \), the absolute temperature \( T \) and the normal-fluid density \( \rho_n \) are determined by \( U \) according to

\[
dU = \mu d\rho + T ds + \rho_n dJ_0.
\]

(2.2)

Let us first consider the case that \( \nabla \times \mathbf{V}_s = 0 \) and \( \nabla \mathbf{A} = 0 \). The vector fields \( \mathbf{V}_s \) and \( \mathbf{A} \) may then be represented by the potentials \( \phi \) and \( \alpha \) according to (1.12). When we substitute (1.12) in (2.1), the energy density \( H \) becomes a Hamiltonian density \( H(\rho, \phi, S, \alpha) \), if we assume that \(-\rho\) is the variable canonically conjugate to \( \phi \) [5] and that \(-S\) is the variable canonically conjugate to \( \alpha \) [6]. To justify this choice we show that the resulting canonical equations reduce to the equations (1.1), (1.3), (1.13) and (1.14) in the approximation of low velocities. The canonical equations for the conjugate variables \( \phi \) and \(-\rho\) read

\[
\frac{\partial \phi}{\partial t} = \frac{\delta H}{\delta (-\rho)} \quad \text{and} \quad \frac{\partial (-\rho)}{\partial t} = -\frac{\delta H}{\delta \phi},
\]

(2.3)

where the functional derivatives are given by

\[
\frac{\delta H}{\delta (-\rho)} = \frac{\partial H}{\partial (\rho)} \quad \text{and} \quad \frac{\delta H}{\delta \phi} = -\mathbf{v} \cdot \frac{\partial H}{\partial (\mathbf{V}_s)}.
\]

(2.4)

It immediately follows that

\[
\frac{\partial \phi}{\partial t} + \mu + \frac{1}{2} \mathbf{v}_s^2 = 0
\]

(2.4)

and

\[
\frac{\partial \alpha}{\partial t} + \mathbf{V} \cdot (\rho \mathbf{V}_s + \mathbf{S} \mathbf{A}) = 0.
\]

(2.5)

From

\[
\frac{\partial \alpha}{\partial t} + T + \mathbf{V}_n \cdot \mathbf{A} = 0
\]

(2.7)

and

\[
\frac{\partial \mathbf{V}_n}{\partial t} + \mathbf{V} \cdot (\rho \mathbf{V}_s + \mathbf{S} \mathbf{A}) = 0.
\]

(2.8)

Equations (2.4), (2.5), (2.7) and (2.8) clearly reduce to (1.13), (1.3), (1.4) and (1.1) in the linear case of low velocities. We may therefore consider the first set of equations as an extension of the second set in the non-linear case of velocities of finite magnitude.

It lies at hand to use the same procedure for extending the applicability of the equations further to the case that the \( \mathbf{V}_s \) and \( \mathbf{A} \) fields are no longer curl-free. To that end it is necessary to generalise the representations (1.12) for \( \mathbf{V}_s \) and \( \mathbf{A} \). This can be achieved by introducing Monge potentials \( \psi, \chi \) and \( \beta, \gamma \) according to

\[
\mathbf{V}_s = \mathbf{V} \phi + \mathbf{V} \psi \chi
\]

(2.9)

and

\[
\mathbf{A} = \mathbf{V} \alpha + \mathbf{V} \beta \gamma.
\]

(2.10)
It is immediately clear that

\[ \nabla \times \mathbf{V}_s = \mathbf{V}_{\chi} \times \mathbf{V}_\chi \]  
\[(2.11)\]

and

\[ \nabla \mathbf{A} = \nabla \times \mathbf{V}_\gamma. \]  
\[(2.12)\]

The vortex lines of the \( \mathbf{V}_s \) field are therefore the intersections of the surfaces \( \psi = \text{constant} \) and \( \chi = \text{constant} \) and the vortex lines of the \( \mathbf{A} \) field are the intersections of the surfaces \( \beta = \text{constant} \) and \( \gamma = \text{constant} \). When (2.9) and (2.10) are introduced into (2.1) the energy density becomes the Hamiltonian density \( H(\rho, \mathbf{V}_s, S, \mathbf{V}_{\alpha}, \mathbf{V}_{\beta}, \mathbf{V}_{\gamma}) \), if we assume that \( -\rho \mathbf{V} \) is canonically conjugate to \( x \)\(,y\) and \( -S \mathbf{S} \) canonically conjugate to \( \gamma \) \(,\beta\).

The canonical equations pertaining to \( \chi \) and \( -\mathbf{V} \) are given by

\[ \frac{\partial \mathbf{V}_s}{\partial t} + \nabla \cdot \mathbf{V}_s \mathbf{V}_\chi = 0 \]  
\[(2.13)\]

and

\[ \frac{\partial}{\partial t}(\rho \mathbf{V} + \mathbf{V}(\rho \mathbf{V} \mathbf{V})) = 0, \]
\[(2.14)\]

where

\[ \rho \mathbf{V} = \rho \mathbf{V}_s + S \mathbf{A} \].

For the canonical equations relating to \( \gamma \) and \( -S \mathbf{S} \) we obtain

\[ \frac{\partial \mathbf{V}_\gamma}{\partial t} + \mathbf{V}_n \cdot \mathbf{V}_\gamma = 0 \]  
\[(2.15)\]

and

\[ \frac{\partial}{\partial t}(S \mathbf{S}) + \mathbf{V}(S \mathbf{S} \mathbf{V}_n) = 0. \]  
\[(2.16)\]

The canonical equations (2.4) and (2.7) are changed into

\[ \frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}(\mathbf{V}_s \mathbf{V}) + \mathbf{u} + \frac{1}{2}v^2 = 0 \]  
\[(2.17)\]

and

\[ \frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}(\mathbf{V}_s \mathbf{V}) + \mathbf{T} + \mathbf{V}_n \cdot \mathbf{A} = 0 \]  
\[(2.18)\]

while eqs. (2.5) and (2.8) remain unchanged. Equations (2.17) and (2.18) may be considered as Clebsch transformations. They are generalisations of the Bernoullian theorems (2.4) and (2.7) and constitute integrated forms of the equations of motion of the carrier fluid and the excitations, respectively. From (2.5) and (2.14) it follows that

\[ \frac{\partial \mathbf{V}_s}{\partial t} + \nabla \cdot \mathbf{V}_s \mathbf{V}_s = 0. \]  
\[(2.19)\]

In the same way (2.8) and (2.16) yield

\[ \frac{\partial \mathbf{V}_\gamma}{\partial t} + \mathbf{V}_n \cdot \mathbf{V}_\gamma = 0 \]  
\[(2.20)\]

From the Clebsch transformation (2.17) and the equations (2.13) and (2.19) we immediately derive the equation of motion for the carrier fluid

\[ \frac{\partial \mathbf{V}_s}{\partial t} + \mathbf{V}(\mathbf{V}_s \mathbf{V}_s) = \mathbf{V} \times (\nabla \times \mathbf{V}_s). \]  
\[(2.21)\]

Similarly, from the Clebsch transformation (2.18) and equations (2.15) and (2.20), we obtain the equation of relative motion for the excitations

\[ \frac{\partial \mathbf{V}_\gamma}{\partial t} + \mathbf{V}(\mathbf{V}_n \mathbf{A}) = \mathbf{V}_n \times (\nabla \times \mathbf{V}_\gamma). \]  
\[(2.22)\]

When \( \nabla \times \mathbf{V}_s = 0 \), the equations of motion (2.21) and (2.22) reduce to the non-dissipative Landau equations known from two-fluid hydrodynamics (see, for example, [2]). The Landau equations, however, are derived in an entirely different way (see [4]).

When \( \nabla \times \mathbf{V}_s \neq 0 \), equations (2.21) and (2.22) constitute natural generalisations of the Landau equations.

Two circulation theorems of the Kelvin type are associated with the generalised Landau equations (2.21) and (2.22). They are given respectively by

\[ \frac{d}{dt} \oint \mathbf{V}_s \cdot d\mathbf{F} = 0 \]  
\[(2.23)\]

and

\[ \frac{d}{dt} \oint \mathbf{A} \cdot d\mathbf{F} = 0. \]  
\[(2.24)\]

Formula (2.23) expresses the fact that the circulation of the velocity \( \mathbf{V}_s \) of the carrier fluid is constant when moving with the mass velocity \( \mathbf{V} \), while (2.24) states that the circulation of \( \mathbf{A} \), a quantity connected with the relative velocity of the excitations, is constant when moving with the velocity \( \mathbf{V}_n \) of the excitations. The circulation theorem for the relative motion of the excitations was derived by Saffman [3] and Putterman [2]. These
authors used
\[ \frac{\partial \vec{V}_g}{\partial t} + \nabla (\mu + \frac{1}{2} \vec{V}_g^2) = \nabla x (\nabla \vec{V}_g) \]

instead of (2.21) as the equation of motion for the superfluid. They therefore had to assume that \( \nabla \vec{V}_g = 0 \) in order to arrive at a correct circulation theorem for the relative motion of the excitations.

The circulation theorems (2.23) and (2.24) however, have unrestricted validity.

The circulation theorem (2.24) can be derived by noting that (2.22) is equivalent to
\[ \frac{\partial \vec{V}}{\partial t} + (\vec{n} \cdot \nabla) \vec{A} + \nabla \vec{V}_n \cdot \vec{A} + \nabla T = 0. \] (2.25)

Since
\[ \frac{d}{dt} \phi \cdot d\Gamma = \phi \frac{d}{dt} \vec{A} \cdot d\Gamma + \phi \vec{A} \cdot \frac{d}{dt} d\Gamma = \phi (\frac{\partial \vec{A}}{\partial t} + (\vec{n} \cdot \nabla) \vec{A} + \nabla \vec{V}_n \cdot \vec{A}) \cdot d\Gamma = \phi \nabla T \cdot d\Gamma = 0 \]

according to (2.25). The proof of (2.23) proceeds in a similar manner.

It follows that isolated \( \vec{V}_g \) vortices move with the local mass velocity \( \vec{V} \) and isolated \( \vec{A} \) vortices with the velocity \( \vec{V}_n \) of the excitations.

3. VARIATIONAL PRINCIPLE

With a Hamiltonian density \( H(\psi_\sigma, \nabla \psi_\sigma, \pi_\sigma) \), where the functions \( \pi_\sigma(x_k,t) \) are canonically conjugate to the functions \( \psi_\sigma(x_k,t) \), we can associate a variational principle with the Langrangian density
\[ L(\psi_\sigma, \nabla \psi_\sigma, \pi_\sigma) \] (see, for example, [9]).

Half of the canonical equations, viz.
\[ \frac{\partial \psi_\sigma}{\partial t} = \frac{\delta H}{\delta \pi_\sigma} \] or \[ \frac{\partial \pi_\sigma}{\partial t} = -\frac{\partial H}{\partial \psi_\sigma} \] (3.1)

serve as side conditions for expressing the functions \( \pi_\sigma \) in terms of the functions \( \psi_\sigma \), while the other half of the canonical equations, viz.
\[ \frac{\partial \psi_\sigma}{\partial t} = -\frac{\delta H}{\delta \psi_\sigma} \] or
\[ \frac{\partial \pi_\sigma}{\partial t} = -\frac{\partial H}{\partial \psi_\sigma} + \frac{\partial}{\partial x_k} \frac{\delta H}{\delta (\psi_\sigma / \partial x_k)} \] (3.2)

coincide with the Euler-Lagrange equations of the variational principle. Note that the summation convention for repeated indices is used. The Lagrangian density \( L \) is connected with the Hamiltonian density \( H \) by the Legendre transformation

\[ L(\psi_\sigma, \nabla \psi_\sigma, \pi_\sigma) = \frac{\partial L}{\partial t} \frac{\partial^2 L}{\partial \psi_\sigma^2} - \frac{\partial L}{\partial \psi_\sigma} \frac{\partial^2 L}{\partial \pi_\sigma \partial \psi_\sigma} - H(\psi_\sigma, \nabla \psi_\sigma, \pi_\sigma). \] (3.3)

The variational principle is expressed by
\[ \int_{t_1}^{t_2} dt \int dV L = 0, \] (3.4)

where
\[ \psi_\sigma(x_k,t_1) = \psi_\sigma(1)(x_k), \quad (x_k \in \Omega) \]
and
\[ \psi_\sigma(x_k,t_2) = \psi_\sigma(2)(x_k), \quad (x_k \in \Omega) \] (3.5)

and
\[ \psi_\sigma(x_k,t) = \psi_\sigma(0)(x_k,t), \quad (t_1 < t < t_2, x_k \in \Omega). \]

It is easily established that
\[ \frac{\partial L}{\partial \psi_\sigma} = -\frac{\partial H}{\partial \psi_\sigma}, \] (3.6)

\[ \frac{\partial L}{\partial (\psi_\sigma / \partial x_k)} = \pi_\sigma, \] (3.7)

\[ \frac{\partial L}{\partial (\psi_\sigma / \partial x_k)} = -\frac{\partial H}{\partial (\psi_\sigma / \partial x_k)} \] (3.8)

According to classical field theory [9] it is possible to associate an energy flux density \( Q_k \) with the \( \psi_\sigma \) field. This energy flux density is given by
\[ Q_k = \frac{\partial}{\partial t} \frac{\partial L}{\partial \psi_\sigma} \frac{\partial^2 L}{\partial (\psi_\sigma / \partial x_k)} \] (3.9)

The equation for conservation of energy can now be written as
\[ \frac{\partial H}{\partial t} + \frac{\partial Q_k}{\partial x_k} = 0. \] (3.10)

In addition, a momentum density \( \vec{Q}_k \) and a
stress tensor $T_{jk}$ can be associated with the field. They are given by

$$G_k = - \frac{\partial}{\partial x_k} \left( \frac{\partial \psi}{\partial \psi_0} \right) - \frac{2L}{3(\psi_0 \psi_3)} \tag{3.11}$$

$$T_{jk} = - \frac{\partial}{\partial x_k} \left( \frac{\partial \psi_0}{\partial \psi_3} \right) + L \delta_{jk} \tag{3.12}$$

and are connected by the equation

$$\frac{\partial G_k}{\partial t} + \frac{\partial T_{jk}}{\partial x_j} = 0, \tag{3.13}$$

which expresses the conservation of momentum of the field.

We want to force the Hamiltonian density $H$ for superfluid helium which was considered in the preceding section into the general scheme of classical field theory just summarized. To that end we have to choose as generalised momenta $\pi_i$ canonical variables the spatial gradients of which do not enter into the Hamiltonian density $H$. This selection has already been made in the preceding section. We therefore put

$$\pi_1 = \phi, \pi_2 = \chi, \pi_3 = \alpha, \pi_4 = \gamma, \tag{3.14}$$

The Lagrangian density $L$ is given according to (3.3) by

$$L = \sqrt{\psi_0} \left[ \frac{1}{2} \left( \frac{\partial \psi}{\partial \psi_0} \right)^2 - \frac{1}{2} \left( \frac{\partial \psi}{\partial \psi_3} \right)^2 - \frac{1}{2} \left( \frac{\partial \psi}{\partial \chi} \right)^2 - \frac{1}{2} \left( \frac{\partial \psi}{\partial \alpha} \right)^2 - \frac{1}{2} \left( \frac{\partial \psi}{\partial \gamma} \right)^2 + \frac{1}{2} \left( \frac{\partial \psi}{\partial \phi} \right)^2 \right]$$

$$- H(\psi_0, -\psi_0, \psi_3, -\psi_3, \psi_1, -\psi_1, \psi_2, -\psi_2, \psi_4, -\psi_4) \tag{3.15}$$

Here

$$H(\psi_0, -\psi_0, \psi_3, -\psi_3, \psi_1, -\psi_1, \psi_2, -\psi_2, \psi_4, -\psi_4)$$

is obtained by introducing (2.9) and (2.10) into (2.1).

According to (3.1), the functions $-\psi, -\psi_0, -\psi_3, -\psi_1, -\psi_2, -\psi_4$ are expressed in terms of

$$\frac{\partial \psi}{\partial \phi}, \frac{\partial \psi}{\partial \chi}, \frac{\partial \psi}{\partial \alpha}, \frac{\partial \psi}{\partial \gamma}, \psi_0, \psi_3, \psi_1, \psi_2, \psi_4,$$

by means of

$$\frac{\partial \psi}{\partial \phi} = \frac{\partial \psi}{\partial \chi} + \mu + \frac{1}{2} v^2 = 0, \tag{3.16}$$

$$\frac{\partial \psi}{\partial \chi} + v \cdot n = 0. \tag{3.17}$$

The equations (3.16) constitute one half of the canonical equations obtained in section 2. It is understood that the vector functions $v_s, \chi, \psi_0$ and $v$ are given by the expressions

$$\chi = v_0 + \beta v_2, \tag{3.18}$$

$$\chi = v_0 + \beta v_2, \tag{3.19}$$

$$v = v_s + \frac{\beta}{\alpha} \tag{3.20}$$

The Euler-Lagrange equations of the variational principle now read

$$\frac{\partial}{\partial t} \left( \rho \psi \right) + v \cdot (\rho \psi v) = 0, \tag{3.17}$$

$$\frac{\partial}{\partial t} \left( \rho \psi_0 \right) + v \cdot (\rho \psi_0 v) = 0, \tag{3.18}$$

$$\frac{\partial}{\partial t} \left( \rho \psi_3 \right) + v \cdot (\rho \psi_3 v) = 0, \tag{3.19}$$

This set of equations coincides with the remaining half of the canonical equations obtained in section 2.

The pressure $p$ may be defined by (see [2])

$$p = \left( \frac{\partial}{\partial t} \rho \psi \right) \psi_0 \psi_3 \psi_1 \psi_2, \tag{3.20}$$

where $V$ is the volume of the liquid. It follows that

$$p = \rho \mu - U + TS + \rho n (\chi - v_s)^2. \tag{3.21}$$

It is easily established by means of (2.17), (2.18) and (3.20) that the Lagrangian density as given by (3.15) equals the pressure:

$$L = p.$$
The variational principle can therefore be formulated as
\[ \int_{t_1}^{t_2} dt \int_{\Omega} dV \quad p = 0 \] (3.22)
with end and boundary conditions given according to (3.5) by
\[ \phi(x_k, t_i) = \phi(i)(x_k) \]
\[ x(x_k, t_i) = x(i)(x_k) \quad i = 1, 2 \]
\[ a(x_k, t_i) = a(i)(x_k) \]
\[ y(x_k, t_i) = y(i)(x_k) \]
and
\[ \phi(x_k, t) = \phi(0)(x_k, t) \]
\[ x(x_k, t) = x(0)(x_k, t) \quad t_i < t < t_2 \]
\[ a(x_k, t) = a(0)(x_k, t) \quad x_k \in \Omega \]
\[ y(x_k, t) = y(0)(x_k, t) \]
Formula (3.22) constitutes a generalisation of Bateman's principle [10] to two-fluid hydrodynamics.

We finally derive explicit expressions for the energy flux density \( \mathcal{E} \), the momentum density \( \mathcal{M} \) and the stress tensor \( T_{jk} \).

Using (3.8), (3.9) and (3.15) we find that
\[ \mathcal{E} = \rho (\mu + \frac{1}{\rho} V^2) \mathbf{v} + S(T + \mathbf{v} \cdot \mathbf{A}) \mathbf{v} \] (3.23)
An expression equivalent to (3.23) has already been given by Khalatnikov [4] for the case of a curl-free superfluid velocity field. It was, however, obtained in an entirely different way. From (3.7), (3.8), (3.11), (3.12) and (3.15) we obtain
\[ \mathcal{E} = \rho \mathbf{v} \] (3.24)

and
\[ T_{jk} = \rho v_{sk} v_{kj} + S A_{k} v_{nj} + \rho \delta_{jk} \] (3.25)
Formula (3.24) states that the momentum density equals the mass flux density. An expression for the stress tensor equivalent to (3.25) can be found in [4]. It is again derived in an entirely different way.

The variational principle (3.22) can be transformed into its reciprocal form. Details of this transformation will be given elsewhere.

REFERENCES

SESSION IV

Chairman: Prof. Dr. A.I. van de Vooren,
University of Groningen
ABSTRACT
Wave radiation by a slender body is discussed, with emphasis on the scale of the characteristic wavelength in relation to the disparate width and length of the body. The discussion is illustrated by reviewing the analysis of oscillatory forced motions of a slender floating body in otherwise calm water. Two complementary regimes are considered, where the wavelength is long (comparable to the body length) or short (comparable to the beam). The limitations of these two theories suggest the need for a unified approach, where geometric slenderness is assumed without restricting the wavelength. A possible approach to the unified theory is outlined in general terms, and illustrated in detail by considering the simpler physical problem of axisymmetric acoustic radiation by a slender body in an unbounded medium.

1. INTRODUCTION
The field of slender-ship theory is a logical development from the earlier analysis of slender bodies in aerodynamics. However a significant complication results for ships and other floating slender bodies, due to the extra length scale associated with the characteristic wavelength ($\lambda$), which must be related to the disparate length ($L$) and beam ($B$) of the vessel. An analogous situation exists for
acoustic radiation in an unbounded medium. With the fundamental geometric slender­ness parameter $B/L = \varepsilon \ll 1$, we seek an asymptotic solution valid to leading order in $\varepsilon$. For this purpose it generally is necessary to restrict the order of magnitude of the wavelength to one of two complementary regimes. For long waves, comparable to the body length, $\lambda/L = O(1)$, whereas in the short­wavelength case $\lambda/B = O(1)$. (Some workers reserve the term "slender-body theory" or "slender-ship theory" for the long-wavelength regime. A more liberal definition is adopted here, the word "slender" being applied only to the geometry of the body.)

In the long-wavelength regime, interactions are significant between adjacent sections of the body, but wave effects are absent from the near field close to the body surface. For the ship-motion problem, the importance of transverse gradients in the inner region is such that the linear free-surface boundary condition degenerates to the "rigid-wall" condition of zero vertical velocity. This greatly simplifies the theory, but also restricts its domain of applicability to sub-resonant frequencies.

In the short-wavelength regime wave effects are present in the near field, but one recovers a simple strip theory without interactions between sections. In one sense this result is very satisfying, as it provides some rigor to the otherwise empirical strip theory of ship motions. However, the consistent leading­order theory is again oversimplified for practical purposes. In particular, the hydrodynamic effects of the ship's forward speed are absent completely from the leading-order theory, as suggested by Vossers (1962) and confirmed by Joosen [1964] and Ogilvie and Tuck (1969). No fundamental difficulties are anticipated in adopting the suggested unified approach to include forward-speed effects.

2. THE SHIP-MOTION PROBLEM
The problem of interest here is to analyse the motions of a ship, or more generally of any elongated vessel floating on the free surface. Cartesian coordinates $(x,y,z)$ are defined with $z=0$ the plane of the undisturbed free surface, and the direction of the positive $z$-axis upward. It is convenient to nondimensionalize and orient these coordinates such that the longitudinal body axis occupies the segment $(0,1)$ of the $x$-axis. The motions of the body and surrounding
fluid are assumed to be oscillatory in time, with the complex factor $e^{i\omega t}$ suppressed. The motions are assumed also to be of small amplitude by comparison to the wavelength and body dimensions. Thus a linearized boundary-value problem can be justified, after making the usual assumption that the fluid is ideal and incompressible.

The fluid velocity vector is expressed as the positive gradient of a potential $\phi(x)$, which is governed throughout the fluid domain by the Laplace equation

\[ \nabla^2 \phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = 0. \]  

(1)

The two principal boundary conditions are on the wetted surface of the body and on the free surface. The appropriate boundary condition on the body surface $S$ is

\[ 3\phi/3n = V_n, \]

(2)

where $V_n$ is the specified normal velocity. In the linearized theory, (2) is imposed on the mean position of $S$, and the free-surface boundary condition takes the form

\[ K\phi - \phi_z = 0, \text{ on } z = 0, \]

(3)

with $K = \omega^2 L/g$. For deep water the fluid motion should vanish as $z \to -\infty$, and $K = 2\pi L/\lambda$ is the (nondimensional) wavenumber of the radiated waves.

This boundary-value problem is completed by imposing the radiation condition

\[ R^{-1/2} e^{-iKR}, \text{ as } R = (x^2 + y^2)^{1/2} \to \infty. \]

(4)

Alternatively, the radiation condition (4) can be deleted if the steady-state oscillatory time-dependence is replaced by a suitable initial-value problem, with a state of rest at $t = -\infty$. A convenient approach is to retain the time-dependence $e^{i\omega t}$, but with a complex frequency $\omega$ having a vanishingly small negative part,

\[ \omega = \omega - i0. \]  

(5)

From the definition of the wavenumber it follows that

\[ K = K - i0. \]  

(6)

We shall use (6) in place of (4), since this avoids the asymptotic analysis to verify the far-field wave form. Furthermore, (5) and (6) can be utilized in the acoustic problem of Section 3.

The body geometry is characterised by a length $L=1$, beam ($B$) and draft ($T$), such that $(B,T) = O(\epsilon)$. Our approach is based on the method of matched asymptotic expansions. Thus, we define an inner region adjacent to the body surface, where $(y,z) = O(\epsilon)$, and an outer region where $(y,z) = O(1)$. By assumption there exists an overlap region $\epsilon << (y,z) << 1$, where the solutions obtained separately in the inner and outer regions are both valid and can be matched. This technique is described by Van Dyke (1975), and was first applied to slender ships by Tuck (1963).

In the inner region, a coordinate-stretching argument can be used to show that gradients in the transverse plane ($\partial/\partial y, \partial/\partial z$) are $O(\epsilon^{-1})$, and dominate the longitudinal derivative $\partial/\partial x = O(1)$. Thus (1) reduces to the two-dimensional Laplace equation

\[ \phi_{yy} + \phi_{zz} = 0, \]

(7)

with the error a factor $1+O(\epsilon^2)$.

The solution in the inner region is of the form

\[ \phi(x,y,z) = \phi^{(2D)}(y,z;x) + f(x). \]  

(8)
Here \( \phi^{(2D)} \) denotes the solution of a two-dimensional boundary-value problem, and \( f(x) \) is a trivial solution of (7) which is determined ultimately by matching to the outer solution.

In the outer region there is no preferred direction, and all three components of the gradient operator are of the same order of magnitude. Thus, the three-dimensional Laplace equation (1) holds in the outer region, without simplification. From geometric considerations, the components of the unit normal vector \( n \) directed into the body surface are

\[
\begin{align*}
    n_x &= O(\epsilon), \\
    (n_y, n_z) &= O(1).
\end{align*}
\]

The estimates (9-10) must be modified near the bow and stern unless these ends are sharply pointed. Hereafter we restrict the body geometry such that (9-10) are uniformly valid along the length of the ship. Furthermore, a longitudinal source distribution will be utilized subsequently to derive the outer solution, and it will be necessary to restrict the source strength near the ends. We shall assume that the source strength is zero at the body ends, especially in connection with various partial integrations; a stronger condition may be necessary, particularly if the solution is to be uniformly valid near the ends. This question is discussed by Ursell (1962), but is not of practical importance in the leading-order analysis of vertical ship motions.

It is obvious that the body boundary condition (2) is applied in the inner region, and the radiation condition (4) is valid only in the outer region. The free-surface condition (3) must be applied in both the inner and outer regions, and the manner in which this is done depends on the order of magnitude of the wavelength \( \lambda \) or wavenumber \( K \).

**Long Wavelengths, \( K=O(1) \)**

If \( K=O(1) \), the dominance of transverse gradients in the inner region leads to the "rigid" free-surface condition

\[
\phi_z = 0 \quad \text{on} \quad z = 0,
\]

in place of (3). By reflection about the plane \( z=0 \), \( \phi^{(2D)} \) is the solution of a two-dimensional flow exterior to the wetted contour \( C \) of the ship and its image \( \tilde{C} \) above \( z=0 \), with prescribed normal velocity \( V_n \) on \( C \) and the same normal velocity on \( \tilde{C} \).

From conservation of mass \( \phi^{(2D)} \) will include a source strength proportional to the net flux associated with the normal velocity \( V_n \) on \( C \). Thus for heave and pitch

\[
\phi^{(2D)} = \frac{1}{2\pi} \sigma(x) \log r,
\]

\[
r \equiv (y^2 + z^2)^{1/2} \gg \epsilon.
\]

The source strength \( \sigma(x) \) is the product of the local beam and vertical velocity.

The remaining motions of the ship will not result in a net source strength and

\[
\phi^{(2D)} = \frac{1}{2\pi} \bar{\mu}(x) \cdot \nabla \log r, \quad r \gg \epsilon.
\]

For sway and yaw the dipole moment \( \bar{\mu}(x) \) can be related to the added mass of \( C + \tilde{C} \).

An arbitrary "constant" \( f(x) \) can be added to \( \phi^{(2D)} \), and this constant contributes to the pressure in the inner region adjacent to the body surface. This affects the vertical force on the profile \( C \), but not the horizontal force. This constant is significant for pitch and heave, and must be determined by matching (12) with the outer solution. By comparison, for sway and yaw, the constant can be neglected and the local solution in the inner region is given simply by the two-dimensional potential of the form (13).
Hereafter we consider only pitch and heave. In the outer region a distribution of three-dimensional wave sources is required, governed by the complete free-surface condition (3). The necessary source potential or Green function is well known, cf. Wehausen and Laitone (1960, eq. 13, 17). For a source of complex time-dependence $e^{i\omega t}$, situated at the point $(\xi,0,0)$, the velocity potential can be expressed in the form $G(x-\xi,y,z)$, where

$$G(x,y,z) = \int_0^\infty k'dk' \left[ k'(x^2+y^2)^{1/2} \right] e^{k'z}.$$  \hspace{1cm} (14)

Note that (14) satisfies the (three-dimensional) Laplace equation (1), and free-surface condition (3). In view of (6) the appropriate contour of integration is above the pole at $k'=K$.

With this choice of contour (14) can be shown by asymptotic expansion to satisfy the radiation condition (4). Near the source point, (14) is dominated by the singularity

$$G = \int_0^\infty dk' \left[ k'(x^2+y^2)^{1/2} \right] e^{k'z} = (x^2+y^2)^{-1/2}.$$ \hspace{1cm} (15)

In the outer region far from the surface of the body, the potential can be expressed as a source distribution on the body axis with unknown strength $\sigma(x)$, in the form

$$\phi = -\frac{1}{4\pi} \int_0^1 \sigma(\xi) G(x-\xi,y,z) d\xi .$$ \hspace{1cm} (16)

In our subsequent analysis it will be assumed that $\sigma$ is a regular function, which vanishes at the body ends.

The inner expansion of (16), for small $(y,z)$, is required for matching with the inner solution. This nearfield approximation of (16) can be derived in a systematic manner using Fourier transforms, following Ursell (1962). With an asterisk (*) used to denote the Fourier transform with respect to $x$, we define

$$\phi^*(y,z;k) = \int_{-\infty}^{\infty} e^{ikx} \phi(x,y,z) dx ,$$ \hspace{1cm} (17)

and similarly for other functions of $x$. By the convolution theorem

$$\phi^* = \frac{1}{4\pi} \sigma^* G^* ,$$ \hspace{1cm} (18)

where $G^*$ is the Fourier transform of (14).

After substitution of the integral representation for $J_0$, the Fourier transform of (14) is given in the form

$$G^* = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \int_0^{2\pi} e^{ik'z} \sin k'y \sin k'y d\phi dk'dx .$$ \hspace{1cm} (19)

The $x$-integral in (19) yields a delta function. After an application of generalized harmonic analysis it follows that

$$G^* = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[ \frac{1}{2} \left( k^2 + k'^2 \right) + ikxy \right] \frac{dk'}{k^2-k'^2} \right] \frac{d\phi}{2\pi} .$$ \hspace{1cm} (20)

where

$$k = k' \sin \theta .$$

An expansion of (20) in modified Bessel functions of argument $(|k|r)$ is derived by Ursell (1962). For small values of this argument the leading-order terms are

$$G^* (r \sin \theta, -r \cos \theta;k) = -2 \log \frac{1}{2} |k' r| \left[ \frac{\pi}{2} \cos^{-1} \left( k' r \right) \right]$$

$-2Y + \frac{2K}{(k^2-k'^2)^{1/2}} \left\{ \cos^{-1} \left( k' r \right) \right\}$

for $\left\{ |k| \geq k' \right\} .$ \hspace{1cm} (21)
Ursell's expression is conjugate to (21), due to the opposite convention for the complex time dependence, and includes higher-order terms proportional to $Kr$.

The inverse transform of (21) can be evaluated, following Ursell (1962). After a second application of the convolution theorem, the inner expansion of the outer solution can be expressed in the alternative forms

$$
\phi = \frac{1}{2\pi} \sigma(x) \log r 
+ \frac{1}{4\pi} \int_0^1 \sigma'(\xi) L_0(x-\xi) \, d\xi 
+ \frac{K}{4\pi} \int_0^1 \sigma(\xi) \cdot L_1(Kx-K\xi) \, d\xi ,
$$

(22a)

or

$$
\phi = \frac{1}{2\pi} \sigma(x) \left[ \log (Kr) + \gamma + \pi i \right] 
+ \frac{1}{4\pi} \int_0^1 \sigma'(\xi) L_2(Kx-K\xi) \, d\xi
$$

(22b)

Here the kernels $L_n$ are defined as

$$
L_0(u) = -\log (2|u|) \, \text{sgn}(u),
$$

(23a)

$$
L_1(u) = \frac{\pi}{2} \left[ H_0(|u|) + Y_0(|u|) + 2i J_0(|u|) \right],
$$

(23b)

$$
L_2(u) = \left[ -\log 2|u| - \gamma - \pi i + \int_0^{|u|} L_1(u') du' \right] \frac{\text{sgn}(u)}{u},
$$

(23c)

and, in the usual notation, $H_0$, $Y_0$, and $J_0$ denote the Struve and Bessel functions of order zero. The last term in (22a) vanishes as $K \to 0$, leaving the result for a slender body in an infinite fluid with the kernel (23a) governing longitudinal interactions along the body. The last term in (22a) can therefore be interpreted as the free-surface correction, and the corresponding kernel (23b) can be derived in a direct manner by subtracting (15) from (14), following the approach of Newman (1964). The equivalence of (22a) and (22b) follows in a straightforward manner by partial integration with the restriction that the source strength vanishes at the body ends.

The outer and inner solutions are matched by equating (8), (12) and (22). The terms proportional to $\log r$ match automatically, since the same source strength has been anticipated for the two solutions. The remaining terms, of order one, determine the "constant" $f(x)$ in the inner solution as the sum of the last two terms in (22a). It follows that the inner solution (8) can be expressed in the alternative forms

$$
\phi = \phi^{(2D)}(y,z; x) + \frac{K}{4\pi} \int_0^1 \sigma(\xi) L_1(Kx-K\xi) \, d\xi ,
$$

(24a)

or

$$
\phi = \phi^{(2D)}(y,z; x) + \frac{1}{2\pi} \sigma(x) \left[ \log(K) + \gamma + \pi i \right] 
+ \frac{1}{4\pi} \int_0^1 \sigma'(\xi) L_2(Kx-K\xi) \, d\xi
$$

(24b)

The potential $\phi^{(2D)}$ is the solution of the two-dimensional Laplace equation (7), the rigid free-surface condition (11), and the body-boundary condition (2). The arbitrary constant in this solution is determined uniquely by the requirement that (12) holds at large distances from the body section, with an error $O(1)$. The source strength $\sigma$ is determined by continuity, as twice the product of the local waterplane beam and vertical velocity. The only effects of the free surface are on the second term of (24a).

There are two practical objections to the above theory. One is that waves exist only in the outer region. The other is that the leading-order force opposing the motion of the ship is hydrostatic, proportional to the waterplane area and hence $O(\epsilon)$. By comparison, the hydrodynamic force and the inertial force due to the mass of the ship are $O(\epsilon^2)$. Thus, to leading order, there is no resonance
of the pitch and heave motions. A simple dimensional argument can be used following Newman (1977), to show that the resonant frequencies are of order \((g/B)^{1/2} = O(\epsilon^{-1/2})\).

These defects in the long-wavelength slender-body theory can be overcome by including higher-order terms in \(\epsilon\), following Newman and Tuck (1964). Alternatively we can revise our assumption regarding the order of magnitude of the frequency, and treat the short-wavelength problem where \(\lambda/L = O(\epsilon)\) or \(K = O(\epsilon^{-1})\).

Short Wavelengths, \(K = O(\epsilon^{-1})\)

With \(K = O(\epsilon^{-1})\), both terms in the free-surface boundary condition (3) are of the same order of magnitude in the inner region. Thus we anticipate significant wave effects in the inner region adjacent to the body. The same is true in the outer region, with the assumption that radiated waves are present, of wavenumber \(K\), and that the gradient is proportional to \(K\). Thus, in the short-wavelength regime, the free-surface condition (3) applies in both regions.

Since the boundary-value problem for the outer solution is unchanged from the long wavelength case, the solution can be constructed in an identical manner. Restricting our attention to vertical motions of ships with symmetry about the plane \(y=0\), the outer solution is given by a distribution (16) of three-dimensional sources (14), and the Fourier transform of the outer solution is given by (18). Only the transform \(G^*\) must be reevaluated, with \(K = O(\epsilon^{-1})\). (Throughout we assume that \(k = O(1)\), or that longitudinal variations of the quantities transformed are governed by the ship’s length as opposed to the wavenumber. This assumption is not valid in the far field, where radiated waves of large wavenumber \(K\) propagate in all radial directions. However the Fourier transformed quantities are to be used only in the matching region where this assumption is more appropriate.)

For \(K>>1\), the integral (20) for \(G^*\) can be approximated by deforming the contour of integration, following Ogilvie and Tuck (1969) and Faltinsen (1971). With branch cuts of the square-root function on the imaginary axis between \(ix\) and infinity, the contour of integration can be deformed to the upper (lower) half plane for positive (negative) values of \(y\). The resulting integral is \(O(1/K)\), and the dominant contribution is from the residue. (Recall from (6) that the imaginary part of \(K\) is a small negative quantity, and thus the poles of (20) are adjacent to the real axis in the second and fourth quadrants.) The contribution from the residue gives

\[
G^* = -2\pi i \exp \left[ \frac{Kz-i|y|}{(k^2-K^2)^{1/2}} \left(1-\frac{k^2}{K^2}\right)^{1/2} \right] (25)
\]

Using the last form of (25), which is valid for \(K>>|k|\), the inverse transform of (18) is given in the form

\[
\phi = \frac{1}{4\pi i} \exp(Kz-iK|y|). \quad (26)
\]

This is the appropriate inner expansion of the outer solution for the short-wavelength case.

The inner solution is governed by the two-dimensional Laplace equation (7), the boundary condition (2), and the free-surface condition (3). Since the matching requirement (26) is mathematically identical to a two-dimensional radiation condition, the desired inner solution corresponds to the forced motion of the profile \(C\), with the flow constrained to the plane \(x = \text{constant}\). This problem has been studied extensively, for a variety of body profiles. A compendium of results is given by Vuyts (1968). Various computer programs now exist for solving this problem with arbitrary body profile \(C\), as described by Chapman (1977).

With the inner solution specified in this
manner, the problem is completely solved. Since the free-surface condition (3) does not allow an arbitrary constant to be added to the inner solution, no interaction exists between different sections of the ship. Thus we recover the strip theory of ship motions, with the flow at each section of the ship independent of the shape or motion at adjacent sections.

As noted in the Introduction, the last conclusion is cause for mixed feelings. On the positive side, it provides a rational basis for the strip theory, which originally was introduced to ship hydrodynamics in a heuristic manner. On the other hand, we began with a fully three-dimensional problem and it is disappointing to find that in the final result three-dimensional effects are absent from the inner solution. Moreover, if forward speed is included, as in Ogilvie and Tuck (1969), the leading-order solution in the inner region does not depend on the ship's forward speed.

As in the analogous situation for the long-wavelength theory, one may overcome the deficiencies of the simple strip theory by including higher-order terms in $\varepsilon$. In effect this is done in all strip theories of ship motion where forward-speed effects are included. Ogilvie and Tuck (1969) derive a consistent higher-order theory, including terms of relative order $\varepsilon^3$ but neglecting $O(\varepsilon)$. Included in the neglected terms are contributions to the force and moment proportional to the square of the forward speed, which are assumed generally to be of practical importance.

This situation is not entirely satisfactory. As has been stated aptly by Ogilvie (1976) in the context of wave-resistance theory, "we are more likely to make practical progress by developing a better linear theory than by devising second-order corrections to an inadequate linear theory".

A Composite Approximation

An "interpolation" between the two theories outlined above has been constructed by Maruo (1970), in a similar manner to the additive composition of matched asymptotic expansions. First we note that the inner solutions for $K=O(1)$ and $K=O(\varepsilon^{-1})$ possess a common "overlap" expansion. Thus the formal approximation of (24) for large $K$ is equal to the approximation of the strip-theory potential for small $K$. The former result is obtained simply by noting that the kernel $L_2$, defined by (23c), vanishes for large values of its argument. Thus the large-$K$ approximation of (24) is given by

$$ \lim_{K \to \infty} \phi_{(K=O(1))} = \phi_{(2D)} \left[ \frac{1}{2\pi} \sigma(x) \log(K) + \gamma + \pi i \right]. $$

The small-$K$ approximation of the strip-theory potential can be obtained by expressing the solution as a source plus wave-free potentials, following Ursell (1968), and noting that the second term of (27) is the low-frequency limit of the source potential, minus the $\log(r)$ term in $\phi_{(2D)}$.

Following the rules for additive composition, we add the two inner solutions and subtract their common limit (27). It follows that

$$ \phi = \phi_{\text{strip}} + \frac{K}{4\pi} \int_0^1 \sigma(\xi) L_1(Kx-K\xi)d\xi $$

$$ - \frac{1}{2\pi} \sigma(x) \left[ \log \left( K \right) + \gamma + \pi i \right] $$

$$ = \phi_{\text{strip}} + \frac{1}{4\pi} \int_0^1 \sigma(\xi) L_2(Kx-K\xi)d\xi. $$

This is a composite inner solution, with the property that it contains the $K=O(1)$ and $K=O(\varepsilon^{-1})$ results as limiting cases. However (28) does not satisfy the inner boundary-value problem for all values of $K$, since the interaction terms added to

* Computations by Maruo are reported in discussion of Chang (1977). A similar approach is described by Tuck (1966).
the strip-theory potential \( \phi_{\text{strip}} \) violate the free-surface condition (3).

In the interpolation theory of Maruo (1970), the interaction terms of (28) are multiplied by a higher-order factor \((1+Kz)\). As a result the free-surface condition (3) is satisfied. However, the uniformity of this correction is doubtful for large \( K \), and a troublesome term which results is ignored in Maruo's computation of the hydrodynamic pressure force.

A Unified Solution

To overcome the deficiencies of the long- and short-wavelength theories, it is logical to seek a "unified" theory which embraces both as special cases but is valid throughout the wavenumber regime \( O(1) K O(\varepsilon^{-1}) \). To ensure that all leading-order ingredients of the two limiting problems are included, the guiding principle is that boundary-value problems are formulated in the inner and outer regions such that for all relevant values of the wavenumber no terms of leading order in the slenderness parameter are neglected.

Proceeding on this basis for the ship-motion problem, the outer problem is unchanged, but in the inner region both terms of the free-surface condition (3) are retained in the "inconsistent" manner suggested by Ogilvie (1974). The strip-theory potential of the short-wavelength regime is a particular solution of the resulting inner problem. However for arbitrary wavelengths it is not possible to match this solution with the outer potential. Indeed, for long wavelengths, an arbitrary additive constant must be allowed in the inner solution, as in (8).

Since the free-surface condition (3) does not permit such a constant to be added to the inner solution, a nontrivial homogeneous solution is required.

The only solutions of the homogeneous problem in the inner region are the velocity potentials of the diffraction problem, for scattering of incident waves moving past the fixed body in the positive and negative directions. The appropriate solution symmetrical about \( y=0 \) is composed of two equal and opposite incident waves, or an "incident standing wave". This homogeneous solution violates the two-dimensional radiation condition but, from (26), that objection applies only for short wavelengths. As a result, we anticipate that the contribution to the inner potential from the homogeneous solution must vanish in the short-wavelength regime. More generally, a non-zero contribution from the homogeneous solution is both necessary and appropriate to match with the outer solution.

Thus, the unified slender-body theory for ship motions in calm water is based on retaining the complete free-surface condition (3) in both the inner and outer regions. The inner problem is simplified principally by imposing the two-dimensional Laplace equation (7), which is valid for all wavenumbers provided waves in the inner region are parallel to the \( x \)-axis. The inner solution is assumed to consist of a linear combination of the two-dimensional strip-theory potential, corresponding to forced motion of the body, and of the two-dimensional homogeneous solution for diffraction of incident standing waves by the fixed body. For the outer solution a longitudinal distribution (16) of three-dimensional sources (14) is assumed. The resulting inner and outer solutions can be matched, in a suitable overlap region \( B<y<L \).

This unified theory of ship motions can be compared to the modified strip theory of Grim (1960). Grim solves the two-dimensional problem with appropriate singularities, and assumes a corresponding distribution of three-dimensional singularities with the same coefficients. To correct the resulting error in the body boundary condition, due to the change from two to three dimensions, a longitudinal diffraction solution is introduced. Here the incident waves are associated with radiation along the hull from the three-
dimensional wave source. Grim anticipates that this "longitudinal diffraction problem" can not be realized physically, as has been confirmed mathematically by Ursell (1975).

Aside from the lack of a systematic formalism based on asymptotic approximations, and the introduction of additional assumptions by Grim to simplify his computations, the principal distinction with the present unified theory is that we supplement our inner solution with a diffraction potential for transverse incident waves. The resulting solution is strictly two-dimensional, and well defined both mathematically and physically. The expected longitudinal interactions along the hull arise indirectly from matching with an outer solution.

The analysis of ship motions based on the unified slender-body theory is now in progress, both with and without forward speed. Rather than pursuing either study here in detail, I shall describe instead a simpler but analogous problem where the same approach can be illustrated more explicitly.

3. AN ANALOGOUS ACOUSTIC PROBLEM
Let us consider a slender axisymmetric body, of radius \( r = r_0(x) \), which radiates acoustic waves in the surrounding medium. With the usual assumptions of linearized acoustics, the Laplace equation (1) is replaced by the reduced wave equation

\[
(\nabla^2 + K^2) \phi = 0 ,
\]

where \( K = \omega L/c \) is the nondimensional wavenumber and \( c \) is the speed of sound. The body boundary condition (2) is unchanged, and the radiation condition of outgoing waves at infinity can be replaced by the complex wavenumber (6).

For simplicity we assume axisymmetric forced motions of the body, with normal velocity \( V(x) \). The leading-order boundary condition on the body takes the form

\[
\phi_x = -V(x) , \quad \text{on} \quad r = r_0(x) .
\]
of the Hankel function, the transform of the outer solution is

\[ \phi^* = \frac{1}{4\pi} a^* \left( \frac{2}{(r^2-K^2)^{1/4}} \right) \]

for \(|k| < K|\).

(36)

Matching now can be performed in the transformed domain, by equating (35) and (36) for a suitable range of \( r \) in the overlap domain \( r_o < r < 1 \). As in Section 2, we assume that \( k = O(1) \), and \( \varepsilon < kr < 1 \) in the matching region.

**Short Wavelengths, \( K = O(\varepsilon^{-1}) \)**

For short waves, where \( K > |k| \), the second form of (36) applies. After expanding for small values of \(|k|/K\)

the transformed outer solution is

\[ \phi^* = \frac{1}{4\pi} a^* \left[ H_0^2(Kr) \right] \]

\[ \quad + \frac{1}{2} \left( k^2 r^2/Kr \right) H_1^2(Kr) \]

(37)

where the error is a factor \( 1 + O(k^2 r^2/K^2) \).

Equations (35) and (37) can be matched to leading order with the results

\[ a_1^* = 0 \]

(38)

\[ a_2^* = \frac{1}{4} i\sigma^* \]

(39)

and the error is a factor \( 1 + O(k^2 r/K) \).

Transforming (38-39) determines the coefficients

\[ a_1 = 0 \]

(40)

\[ a_2 = \frac{1}{4} i\sigma \]

(41)

Thus, the inner solution for short wavelengths is given to leading order by

\[ \phi = a_2(x) \frac{H_0^2(Kr)}{K H_1^2(Kr)} \]

(42)

where the last equality follows from the boundary condition (30). This is the "strip theory" result, with the inner solution identical to the two-dimensional radiation from a circular cylinder. Since \( a_1 = 0 \), the two-dimensional radiation condition is satisfied in an analogous manner to the strip theory of ship motions.

**Long Wavelengths, \( K = O(1) \)**

For \( (k, K) = O(1) \), and small \( r \), the Hankel functions in (35) and (36) can be approximated from the leading terms in their infinite-series representations. Matching the two transformed solutions after this is done gives the result

\[ (a_1^* + a_2^*) \left( 1 - \frac{1}{4} K^2 r^2 \right) + \frac{2}{i} i(a_1^* - a_2^*) \left[ 1 - \frac{1}{4} K^2 r^2 \right] \]

\[ \left[ \log \left( \frac{1}{2} Kr \right) + \gamma + \frac{1}{2} K^2 r^2 \right] \]

\[ \equiv \frac{1}{2\pi} \sigma^* \left[ \log \left( \frac{1}{2} Kr \right) + \gamma + \frac{1}{2} \log (|k^2/K^2 - 1|) \right] \]

\[ - \frac{1}{4} (k^2 - K^2) r^2 + \left[ \gamma + i/2 \right] \quad \text{for} \quad |k| > K \]

(43)

Here \( \gamma = 0.577... \) is Euler's constant, and the error in (43) is a factor

\( 1 + O((kr)^2, (kr)^3) \). To leading order for small \( r \), it follows that

\[ a_1^* + a_2^* = \frac{1}{4\pi} \sigma^* \left[ \log (k^2/K^2 - 1) \right] \]

\[ \log (1 - k^2/K^2) + \pi i \]

(44)

\[ a_1^* - a_2^* = - \frac{1}{4} i\sigma^* \]

(45)

where the error is a factor

\( 1 + O((kr)^2, (kr)^3) \).

The inverse transform of (45) is simply
The inverse transform of the factor in braces in (44) can be related to the exponential integral, after a contour integration in the complex k-plane with branch cuts on the real axis for $|k| > k$. Another application of the convolution theorem then gives the transform of (44),

$$a_1 - a_2 = \frac{1}{4} i\sigma \quad \text{(46)}$$

Here, the kernel $L$ is defined by

$$L(u) = E_i(|u|) \text{sgn}(u) \quad \text{(48)}$$

and

$$E_i(z) = \int_0^\infty \frac{e^{-zt}}{zt} \, dt \quad \text{(49)}$$

is the exponential integral.

After solving (46) and (47) for the coefficients $a_1$ and $a_2$, and substituting in (32), the inner solution can be expressed in the form

$$\phi = \frac{1}{4} i\sigma(x) H_2^0(Kr)$$

$$+ \frac{1}{4\pi} J_0(Kr) \int_{\xi} \sigma'(|\xi|) L(Kx-K\xi) \, d\xi. \quad \text{(50)}$$

Finally, if the Bessel functions in (50) are expanded for small $Kr$, it follows that

$$\phi = \frac{1}{2\pi} \sigma(x) \left[ \log \left( \frac{1}{2} Kr \right) + \gamma + \frac{\pi i}{2} \right]$$

$$+ \frac{1}{4\pi} \int_{\xi} \sigma'(|\xi|) L(Kx-K\xi) \, d\xi. \quad \text{(51)}$$

At this stage the boundary condition (30) can be used to determine the source strength

$$\sigma(x) = -2\pi r_o(x) V(x), \quad \text{(52)}$$

and the inner solution is completed.

The outer solution and far-field radiation patterns can be derived by substituting (52) in (33). This aspect of the long-wavelength problem has been studied by Pond (1966), who deduces the source strength (52) in a global manner without consideration of the inner solution for the velocity potential. Chertock (1975) has presented a more general slender-body theory for the long-wavelength case, starting with an integral equation for the exact three-dimensional solution and using physical arguments to approximate the three-dimensional singularity distribution. No attempt is made to reduce the inner solution to the form (51).

For very long wavelengths, $K << 1$, the kernel (48) can be approximated from the infinite-series expansion of (49). In the limit $K = 0$, (51) tends to the classical inner solution for an unbounded incompressible fluid,

$$\phi = \frac{1}{2\pi} \sigma(x) \log r$$

$$+ \frac{1}{4\pi} \int_{\xi} \sigma'(|\xi|) \log(2|\xi|) \text{sgn}(\xi-x) \, d\xi. \quad \text{(53)}$$

This is consistent with the corresponding limit of (22).

A Composite Approximation

The inner solutions (42) and (51) are valid in the mutually exclusive regimes $K = O(\epsilon^{-1})$ and $K = O(1)$, respectively. These solutions possess a common approximation in the "overlap region" $1 << K << \epsilon^{-1}$, which follows by assuming $Kr < 1$ or $K > 1$ in (42) or (51), respectively. Thus a composite solution can be derived by addition of (42) and (51), and subtraction of (54), in the form

$$\phi \approx - r_o V \left[ \log \left( \frac{1}{2} Kr \right) + \gamma + \frac{\pi i}{2} \right], \quad \text{(54)}$$

and the outer solution and far-field radiation patterns can be derived by substituting

$$\phi = \frac{1}{K} \frac{V(x)}{H_1^2(2Kr)}$$

in (55).
with $\sigma$ defined by (52). The analogous formula for the ship-motion problem is (28). Once again the composite solution is correct in the two limiting regimes where $K=O(1)$ or $O(e^{-1})$, but the validity is not uniform throughout the intermediate range of wavenumbers since the last term in (55) violates the wave equation.

A Unified Solution

The deficiency in (55) can be overcome simply by observing that the results of our derivation for the long-wavelength case, up to and including (50), are consistent with the corresponding short-wavelength results. Thus, for $|k|/K<<1$, the Fourier transforms (44-45) yield (38-39). Similarly, since the integral in (47) tends to zero for $K>>1$, (46-47) yield the coefficients (40-41) and the inner solution (50) is dominated by the first term, proportional to $H_0^{(2)}$, in agreement with (42).

From this standpoint the inner solution (50) apparently is valid for all wavenumbers $K=O(e^{-1})$, provided the source strength is determined in accordance with the boundary condition (30). This gives an integro-differential equation for the source strength,

$$i\sigma(x) H_1^{(2)}(K_0(x))$$

$$+ \frac{1}{\pi} J_1(K_0(x)) \int_0^1 \sigma'(\xi) L(K_0 - K\xi) d\xi = \frac{4}{K} V(x).$$

(56)

To confirm that (50) is the desired "unified" solution, we note that this velocity potential satisfies the two-dimensional wave equation (31) and the boundary condition (30) in the inner field; the corresponding outer solution (33) satisfies the three-dimensional wave equation (29) and the appropriate radiation condition. The only question which remains is to confirm that (33) and (50) can be matched for all possible wavenumbers, in some appropriate overlap region close to the body in the outer problem and far from the body in the inner problem.

The overlap region is expected to exist for a range of $r$ such that $e<<r<<1$, or when $r=O(e^{a})$, $0<a<1$. Since the Fourier-transform parameter $k=O(1)$, $kr=o(1)$. However the value of $Kr$ in the overlap domain is not restricted to this range. In particular, when the body is radiating short waves, the matching must be carried out in the far-field of the inner solution, where $Kr>>1$.

Since the matching condition (43) neglects a factor $1+O(Kr)^4$, and the unified inner solution (50) derives from this condition, the possibility exists that (43) is not valid throughout the range of all wavenumbers $K=O(e^{-1})$, in spite of the fact that it yields correct results in the limit $K=O(e^{-1})$. This question can be resolved by noting that the error in (43) is a factor $1+O(Kr)^4$, whereas the error in the short-wavelength results (30-39) is a factor $1+O(k^2r/K)$. Requiring these errors to be of equal magnitude gives a condition for transition between the short- and long-wavelength matching relations,

$$k^2r/K = (Kr)^4.$$

(57)

If $r=O(e^a)$, the transitional wavenumber is

$$K = e^{-\gamma a/3},$$

(58)

and the error in matching is a factor $1+O(e^{a/3})$. Thus the matching error will be $O(e)$ if the overlap region is defined by $r=O(e^{\gamma a/3})$ at the transitional wavenumber $K = e^{-\gamma a/3}$. A more complete analysis is necessary to prove that the full solution is accurate to $O(e)$ with this particular choice of the overlap region. Nevertheless it is apparent that the relative
error in the unified inner solution (30) is $o(1)$ for all wavenumbers.

The unified solution (30) can be compared with the composite approximation (35). The latter result is equivalent to replacing the Bessel function $J_o(Kr)$ by its limit of unity, in the second term of (50), and determining the source strength $\sigma$ in each term of (50) from (42) and (52) respectively.

ACKNOWLEDGEMENT
Preparation of this lecture has been supported by the Office of Naval Research, Contract N0014-76-C-0365, and by the National Science Foundation, Grant 10846.

REFERENCES


SESSION V

Chairman: Dr. J.H. Greidanus,
Fokker, Amsterdam
END EFFECTS IN SLENDER-SHIP THEORY

by

T. FRANCIS OGILVIE
THE UNIVERSITY OF MICHIGAN
ANN ARBOR, MICH., U.S.A.

ABSTRACT

The theory of the slender ship is based generally on the theory of the slender body in aerodynamics, but there are some important differences. One of the most important of these is in the nature of "end effects," especially those arising from the ship bow. Incomplete understanding of such end effects has led to incorrect conclusions both in favor of and against slender-ship theory. The paper includes discussion of several slender-ship end effects that arise in steady-forward-motion and/or unsteady-motion problems. Emphasis is placed on physical concepts and their mathematical modeling.

NOTATION

\( g \): acceleration of gravity
\( h_o \): amplitude of incident waves
\( J_\nu(z) \): Bessel function of order \( \nu \)
\( L \): ship length
\( n \): unit normal vector on hull surface, directed into the hull
\( n_i \): components of \( n \)
\( r \): \((x^2+y^2)^{1/2}\)
\( t \): time
\( U \): steady forward speed
\( x, y, z \): Cartesian coordinates

\( \alpha \): wedge half-angle
\( \beta \): heading of incident waves with respect to \( x \) axis
\( \epsilon \): slenderness parameter
\( \zeta \): wave amplitude
\( \theta \): \( \arctan (y/x) \)
\( \kappa \): \( g/U^2 \), a wave number
\( \nu \): \( \omega^2/g \), a wave number
\( \rho \): density of water
\( \sigma \): source strength or density
\( \phi \): velocity potential
\( \omega \): frequency in radians per second
INTRODUCTION

For the most part, a ship has the attribute of a "slender body," namely, that the cross-section shape and size change gradually in the lengthwise direction, and so one expects that the flow pattern will change smoothly and gradually in the longitudinal direction. This suggests the basic assumption of slender-ship theory: Gradients of flow variables are much smaller in the longitudinal direction than in the transverse directions.

Because there are generally waves (ambient and/or ship-generated) present near a ship and such waves may themselves have large associated gradients in the longitudinal direction, the basic assumption has to be modified: The effect of a slender ship on any existing waves is gradual in the longitudinal direction, in that gradients of the amplitude and phase of the waves are small in the lengthwise direction. For example, consider a ship at zero speed in ambient waves. Let the waves be described in terms of the velocity potential

\[ \phi(x,y,z,t) = (gh/\omega)e^{i(\omega t - vx \cos \beta - vy \sin \beta)}. \]

The ship generates waves that might be expressed approximately by the potential

\[ \phi_d(x,y,z,t) = [gh(x,y)/\omega] e^{i(\omega t - vx \cos \beta - vy \sin \beta - \delta(x,y))}, \]

where \( h(x,y) \) and \( \delta(x,y) \) vary "slowly" with respect to \( x \), the longitudinal coordinate.\(^\dagger\) Such an assumption may be valid even though \( \delta \phi_d/\delta x \) is not small in any useful sense.

However, even such a modified slender-ship concept is not valid near the ends of the ship, not even, for example, for a ship with a "fine bow" (in the sense that a naval architect uses the term). The limitations on slender-ship theory imposed by end effects are much more severe than those generally encountered in aerodynamic slender-body theory.

It is an understatement to say that end effects in slender-ship theory are poorly understood. Actually, they are so misunderstood that they have led to incorrect conclusions and even to the complete rejection of slender-ship theory by many investigators. For example, it has frequently been stated that slender-ship theory predicts negative wave resistance under certain conditions. Such a conclusion is based on an incorrect treatment of end effects. It can be shown that one of the possible expressions for the wave resistance of a slender ship is the following:

\[ R = \frac{2}{\kappa} \int \frac{K^2 dk}{(k^2 - \kappa^2)^{1/2}|\sigma^*(k)|^2}, \]

where \( \kappa = g/U^2 \) and \( \sigma^*(k) \) is the Fourier transform of the far-field slender-ship source density, \( \sigma(x) \). This expression obviously cannot be negative. The error leading to the incorrect conclusion is based on the use of another expression equivalent to the above in which the actual source density (not its Fourier transform) appears, the density function being manipulated incorrectly at the ship ends. (See Ogilvie (1977).)

Even when formulas such as the above are used correctly, unacceptable results can occur. For example, a ship with a fine wedge-shaped bow has an infinite wave resistance according to the above formula, since the integral does not exist in that case. This result arises because the usual methods for determining the far-field source density, \( \sigma(x) \), do not adequately allow for the conditions peculiar to the flow around a ship bow. This is a special problem of slender-ship theory that must be resolved before predictions from the theory can be made useful.

All of the present-day "strip theories" for predicting ship motions and wave loads on ships in waves are based on the concept of slenderness, as described above, regard-
less of whether their creators explicitly say so. And slender-ship theory has had its greatest success in the prediction of ship motions. No one should dispute this success, for it has been demonstrated with experiments many times. But one must recognize the element of good luck that contributed to that success: End effects are not very important in predicting the total force and moment on a ship in waves, and thus inaccuracies in predicting end effects do not matter much. However, these inaccuracies are more significant in computing quantities that depend on the distribution of force on the ship, and so predictions of bending moment, shear force and so on are not so good as predictions of total force and moment. Predictions of wave-induced excitation of vibration ("springing") fail badly for the same reason, as do predictions of relative motion between, say, the bow and the water surface. Satisfactory solutions of such problems still await the development of a better understanding of end effects of a slender ship in incident waves.

There have been recent attempts to develop rational analyses of ship maneuvering problems based on the application of slender-ship concepts. Some of the results have been auspicious, even suggesting that the usual phenomenological approach to ship maneuvering may be supplanted in the foreseeable future. However, in at least one situation, there are critical end effects that indicate serious difficulties still to be resolved: For a ship maneuvering in shallow water, the net force and moment on the ship are sensitive to the shape of the bow. This becomes especially important when the ship moves near a channel wall or other obstacle; the predicted effects of bank suction and bank cushion can be reversed in sign by small changes in what is assumed about the flow around the bow.

In this paper, we shall be most concerned with identifying the problems that we describe collectively as "end effects." In some cases, progress in solving these problems can be reported, although that is, unfortunately, not the general situation. For the most part, we are still trying to determine the proper assumptions for developing valid mathematical models.

First I shall discuss problems of a ship in incident waves. Only the zero-speed case will be considered. Then steady-forward-motion problems will be taken up. The maneuvering problem just mentioned will not be discussed, since it appears to represent an entirely new kind of end effect from the others considered here. One may refer to Beck (1977).

**SLENDER SHIP IN INCIDENT WAVES**

**Formulation.** First we state the general linearized problem of a ship in incident waves. Since we do linearize the problem from the beginning, we can formulate separately the problem of a ship fixed in waves and the problem of a ship oscillating in any of its rigid-body or vibration modes of motion. Let the longitudinal axis of the fixed ship be aligned with the x axis, the z axis oriented vertically upwards and the y axis horizontally to starboard. The surface of the ship can be described by an equation of the form

\[ F(x,y,z) = 0. \]  

Everything will be assumed so that a velocity potential exists. The incident waves will be given by the potential

\[ \phi_t(y,z)e^{i(\omega t - x\cos \beta)} \]

\[ = (gh_0/\omega)e^{-\omega z}e^{i(\omega t - x\cos \beta - y\sin \beta)}, \]

where \( g \) is the acceleration of gravity, \( h_0 \) is the incident wave amplitude, \( \omega \) is the radian frequency of the incident waves, \( v = \omega^2/g \) is the corresponding wave number, and \( \beta \) is the angle at which the incident waves propagate, measured with respect to the positive x axis. The corresponding wave elevation at any point is given by $\phi_t(y,z)$ actually by the real part only.
Let the potential for the diffraction waves be given by \([\text{the real part of}]\)
\[ \phi(x,y,z) e^{i(\omega t - vx \cos \beta)}; \]
the corresponding free-surface elevation will be given by
\[ \zeta(x,y) e^{i(\omega t - vx \cos \beta)}. \]

The dynamic and kinematic conditions on the free surface take on the following linearized forms:
\[ g \xi(x,y) + i \omega \xi(x,y,0) = 0, \]
\[ i \omega \xi(x,y) - \phi_z(x,y,0) = 0, \]
which can be combined into the usual form,
\[ i \omega \phi - \phi_z = 0 \text{ on } z = 0. \]

The body boundary condition is
\[ n_1 \phi_x - i n_1 \phi \cos \beta + n_2 \phi_y + n_3 \phi_z = O(\varepsilon^{-1}), \]
\[ 0(\varepsilon^{-1}) 0(1) 0(\varepsilon^{-1}). \]

Here the unit vector normal to the hull surface, directed into the hull, is given by
\[ n = (n_1, n_2, n_3) = V F / |V F|. \]

The orders of magnitude of the terms in (7) have been indicated, based on the usual assumptions of ship-motion theory:
\[ u = O(\varepsilon^{-1}), \quad \omega = O(\varepsilon^{-1/2}); \]
\[ n_1 = O(\varepsilon), \quad n_2, n_3 = O(1); \]
\[ \phi_x = O(\phi), \quad \phi_y, \phi_z = O(\varepsilon^{-1}). \]

On the right-hand side of (7), the orders of magnitude must be multiplied by the order of magnitude of \( \phi_\theta \).

In ordinary slender-body theory, we keep only the leading-order terms in (7) to provide an approximate boundary condition in the near field, that is,
\[ n_1 \phi_y + n_3 \phi_z = 0 \]
\[ = O(\phi_\theta \varepsilon^{-1}). \]

Using (10) and (11), we conclude that \( \phi = O(\phi_\theta) \). This implies that the diffraction wave is comparable in magnitude to the incident wave near the body. This is entirely reasonable, since the diffraction wave motion must cancel the normal velocity component of the incident wave motion on the hull.

The potential for the diffraction wave must satisfy the Laplace equation. Thus the governing equation for \( \phi(x,y,z) \) is
\[ \phi_{xx} - 2i v \phi_x \cos \beta - (v \cos \beta)^2 \phi + \phi_{yy} + \phi_{zz} = 0. \]

The lowest-order terms, when their sum is set equal to zero, give us a Helmholtz equation for \( \phi \). This is the equation that will be used in what follows.

The head-sea difficulty. The boundary-value problem posed in (6), (12) and (13) (leading terms only), supplemented by a radiation condition, can be solved without any difficulty in principle if \( \beta \neq 0 \), as shown by Troesch (1975). However, as \( \beta + 0 \), the amplitude of the diffraction waves is found to approach infinity everywhere, even for fixed amplitude of the incident waves. Ursell (1968) has shown that there is no physically acceptable solution of the above problem for \( \beta = 0 \).

We have few theorems proving the existence of solutions in water-wave problems, but we have no reason physically to expect that the problem of a ship in head waves has no solution. The difficulty leading to this paradox lies in the formulation of the approximate problem. If a ship is very, very slender, one usually expects that the solution from section to section should vary slowly and that the solution at any particular section should not depend strongly on what happens at far removed sections. But this is not the case for head seas. The disturbance from the bow accumulates with the diffraction disturbances produced further aft in such a way that the total incident wave at a section far from the bow is much smaller than the original incident waves. Faltingsen (1971) made the first progress in analyzing this phenomenon. He
obtained a consistent asymptotic solution in which the first term for the diffraction wave in the near field is simply the negative of the incident waves, that is,

\[ \phi \sim -\phi_s. \]  

(14)

In other words, the incident wave is completely canceled out in the first approximation. The interesting results then come from the second term in the asymptotic expansion for the near-field solution. Faltinsen showed that the total wave amplitude near the hull decreases as \( x^{-1/2} \) when \( x \to \infty \), \( x \) being measured positively from the bow toward the stern. This asymptotic result is valid, of course, only if the hull is very long, with parallel middle body, but Faltinsen showed fairly good agreement with experimental data when hull length was just twice the wavelength of the incident waves.

Since Faltinsen's essential contribution was to show how total wave amplitude depends on distance from the bow, it is perhaps surprising to note that his solution is not valid near the bow itself. It is, as mentioned, valid as \( x \to \infty \). In fact, one could move the origin of coordinates a finite amount, and the change in his results would be of higher order, in principle. Ursell (1975) showed that Faltinsen's near-field approximation is valid in a region that grows in width as \( x^{1/2} \) when \( x \to \infty \), but it is strictly valid only in this asymptotic sense. Inasmuch as the wave amplitude approaches zero as \( x \to \infty \) with \( y \) fixed, we can say that there is a "shadow" region created by the ship. Ursell described this situation by saying that the waves are diffracted away from the ship.

Grim (1962) gave a physical interpretation of this phenomenon many years before Faltinsen produced an actual solution. Waves are generated by the ship in such a way as to cancel the normal velocity component of the incident waves on the hull; these ship-generated waves propagate in all directions, but those components that move in the same direction as the incident waves tend to reinforce each other, whereas those propagating in other directions interfere destructively with each other. Accordingly, at any section not too near the bow, the incident waves and the diffraction waves arriving from the bow portion already partially cancel each other, and so the amplitude of additional diffraction waves that must be generated is smaller and smaller toward the stern.

In Faltinsen's analysis, as indicated in (14), the first approximation to the diffraction wave system is just the negative of the incident wave system. This cannot be true very close to the bow, of course. Thus his analysis is not valid in such a region. This fact manifests itself in his next approximation: The second-order wave amplitude is infinite at the bow. Anyone who has worked with the method of matched asymptotic expansions recognizes this immediately as a signal that the lowest-order approximation is not correct locally.

Maruo and Sasaki (1974) produced what is apparently an inconsistent asymptotic expansion to obtain a solution without this defect. They assumed that

\[ \phi \sim C(x)e^{vz} + \phi(x,y,z) \]

(15)

in the first-order near-field expansion. If just the first term had been retained, the problem would have reduced to Faltinsen's, with the function \( C(x) \) turning out to be a constant. Retaining both terms, however, and substituting (15) into the body boundary condition, one obtains the approximate condition

\[ \nabla n_1 e^{vz} + n_2 \phi_y + n_3 \phi_z \sim -\nabla n_1 \phi_s. \]

(6)

The first term on the left-hand side has the same form as the right-hand side (see (2)); both terms can be interpreted locally as representing the normal component of an incident wave. Their net effect must be offset by the flow corresponding to \( \phi \). But the term containing \( C(x) \) really represents part of the diffraction wave system.
It turns out that \( C(x) \) depends on the solution \( \phi \) only in the region between the bow and the section \( x \) under consideration. (In fact, Maruo and Sasaki obtained a Volterra integral equation for \( C(x) \), which they solved numerically.) This agrees with Grim's physical picture in which the diffraction waves coming from the bow direction reinforce each other, while those coming from the stern direction suffer so much destructive interference that they effectively cancel each other out. The solutions presented by Maruo and Sasaki appear to be more reasonable near the bow than Faltinsen's. In particular, they have eliminated the infinite wave amplitude at the bow. Therefore, one is inclined to believe that they have identified the proper second-order effects to include in the first-order solution so as to eliminate the unrealistic behavior that Faltinsen predicted. It is not clear whether they have included all second-order effects, and so the matter is not settled in any mathematical sense.

It may be noted that Maruo and Sasaki's predictions of pressure near the waterline of the ship show an especially large improvement over Faltinsen's predictions, whereas the two predictions do not differ much for the keel region. The reason is not known.

One can come close to Maruo and Sasaki's statement of the problem through a consistent analysis, but the necessary assumptions appear to be so contrived as to make such an approach hardly credible. To be specific, one can assume that \( \frac{\partial a}{\partial x} = O(\varepsilon^{-1/3}) \) in a region in which \( x = O(\varepsilon^{1/3}) \).

If the body is pointed, the transverse dimensions are then \( O(\varepsilon^{1/3}) \) in this bow region, and so it is reasonable to assume that \( \frac{\partial a}{\partial y} \) and \( \frac{\partial a}{\partial z} \) are both \( O(\varepsilon^{-1/3}) \). These assumptions do lead to a body boundary condition such as (16). But one quickly finds it necessary to distinguish among quantities that differ in order of magnitude by \( \varepsilon^{1/6} \). While there is nothing wrong with this in principle, and it can lead to a perfectly acceptable mathematical analysis, I cannot imagine that anyone would trust the results in a practical application. After all, we do not really want solutions that are useful only as \( \varepsilon \to 0 \); we want to be able to apply the solutions to problems involving real ships. I would personally prefer an asymptotically inconsistent solution that makes sense physically to such a highly contrived, albeit consistent, solution.

A wedge-like bow in head seas. In discussing the head-sea difficulty above, I used an approximation to the body boundary condition in which terms containing the component \( n_1 \) of the unit normal vector on the hull were neglected. From Equation (7), we can see that this is generally a reasonable thing to do. However, it is not satisfactory if the hull is wall-sided and has large draft. This is seen easily if we rewrite the right-hand side of (7) as follows:

\[
\{ -\nu n_1 + i\nu n_3 \} \{ g_{\theta} / \omega \} e^{i\omega t} . \tag{17}
\]

(Recall that \( \beta = 0 \) for head-seas.) In the extreme case of a body of infinite draft, we have \( n_3 = 0 \), and the term containing \( n_3 \) gives the only contribution on the right-hand side of (7). If we consider intermediate cases (finite draft), the contribution from the \( n_3 \)-term is always formally of higher order than the contribution from the \( n_1 \)-term. Nevertheless, it is evident that, as draft increases indefinitely, the contribution from the \( n_3 \)-term must decrease to the point where it is negligible compared to the formally higher-order contribution. For practical purposes, the separation by formal orders of magnitude becomes meaningless at some point.

Since both terms in (17) may have to be considered, it is worthwhile to look at each in some detail. We have already done this...
for a case in which only the first term is retained: There is an ever-widening shadow region, growing as $x^{1/2}$. This conclusion is valid if the hull extends for a long distance with constant cross-section. It is not valid if the beam of the hull continues to increase. (The extreme case would be a flat wedge-shaped dock of infinite extent.) I do not know of any general results for such a case.

If we retain only the second term in (17), we can concentrate on the simplified problem of a wall-sided body of infinite draft. Again there are two special cases that can provide some helpful information:

(i) The sides of the body become parallel at some finite distance from the bow, in which case the disturbance from the bow region disperses in all directions and, far from the bow, one must observe just the original incident waves traveling along the flat sides of the body, undisturbed by the presence of the body. This is in contrast to the situation for the flat-body problem described by Grim and analyzed by Faltinsen and Ursell, in which there is a growing shadow region even though the body is aimed directly head-on into the incident waves.

(ii) The sides of the body form a vertical wedge for a great distance from the bow (apex), in which case some very different conclusions can be drawn. In fact, this problem can be solved explicitly, and the results are worthy of more detailed comment. So far as I know, this is the only problem of this kind that can be solved analytically and exactly, and so we should obviously try to learn as much as possible from the solution.

Let the geometry be as shown in Figure 1. The body surface is given by the equation $y = \pm x \tan \alpha$. Let $\epsilon \equiv \alpha/\pi$. For the moment, this is not necessarily a small quantity. The complete potential (sum of incident and diffraction waves) can be expressed as follows:

$$\Phi(x, y, z, t) = e^{i\omega t + vz} \left[ \frac{1}{1 - \epsilon} J_1(vz) + \frac{2}{1 - \epsilon} \sum_{m=1}^{\infty} e^{i\pi m/2(1-\epsilon)} J_m/(1-\epsilon) (vz) \cos [m\pi/(1-\epsilon)] \right].$$

(A procedure for deriving this solution is given, for example, by Stoker (1957).) This form of the solution is valid everywhere in the fluid domain, but it is useful only if $v\epsilon$ is a small quantity. In particular, an integral expression for the solution will be used when we consider the wave motion far from the origin.

It will be useful to express this solution in terms of a surface distribution of sources over the centerplane, $y = 0$, $x > 0$. This is easily accomplished by computing from (18) the jump in normal velocity on the centerplane. We obtain the following result:

$$\sigma(x, z, t) = \frac{4}{(1 - \epsilon)^2} e^{i\omega t + vz}$$

$$\times \sum_{m=1}^{\infty} m e^{i\pi m/2(1-\epsilon)} J_m/(1-\epsilon) (vz) \sin [m\pi/(1-\epsilon)].$$

This distribution of sources generates the velocity field of the diffraction waves only, since there is no discontinuity in the incident-wave velocity field. It does produce the diffraction wave motion exactly (it is not a thin-body approximation). Of course, it represents a fictitious flow field in the region actually occupied by the wedge.

If we now assume that $\epsilon$ is a very small quantity and that $v\epsilon$ is also very small, we can approximate the above source
distribution:

\[ \sigma(x, z, t) = -2\pi i e^\omega t + v z (v x / 2) e^{i \omega t} + \ldots \]  

If \( \epsilon \) is indeed very small, the source density starts from a value of zero at \( x = 0 \) and very rapidly rises to the value \(-2\pi i e^{i \omega t + v z} \), after which its value varies very slowly.

Next we note that we can obtain a "geometrical optics" solution, that is, a solution based on a ray approach to the wave motion. From this point of view, we expect to find reflected waves in the region \( r > 0, (\pi - 2\alpha) < |\theta| < (\pi - \alpha) \). The reflected waves must have the same frequency and wavelength as the incident waves but they propagate in the direction prescribed by Snell's Law of optics. It is easy then to show that the corresponding potential for the complete solution is

\[ \phi(x, y, z, t) = e^{v z e^{i (\omega t - v x)}} \times \begin{cases} 
1 + e^{-iv(x (\cos 2\alpha - 1) - iv|y|\sin 2\alpha)} & \text{in } \pi - 2\alpha < |\theta| < \pi - \alpha,
\end{cases} \]

\[ \phi(x, y, z, t) = e^{v z e^{i (\omega t - v x)}} \text{ in } |\theta| < \pi - 2\alpha. \]  

This solution satisfies the free-surface condition, the body boundary condition, and the radiation condition. It does not satisfy the Laplace equation on the boundary between the two regions defined in (21). We can expect this approximate solution to be valid in a region very far from the wedge apex but close to the wedge surface.

The geometrical-optics solution can also be used to define a source distribution on the centerplane. The result is:

\[ \sigma(x, z, t) = -2\pi i e^{i \omega t} e^{i v z (v x / 2)} e^{i \omega t} + \ldots \]  

(20)

It should be noted that this result is questionable in one respect. In order to discuss the potential at all in the region occupied by the body, we must continue it analytically from the fluid region. From the reflection principle for analytic functions, there is a simple relationship between the potential on the \( x-z \) plane and on the planes bounding the regions of reflection waves (the planes given by \( |\theta| = \pi - 2\alpha \)). On the latter, we can expect a very complicated motion, involving the transition from the region of reflection waves to the region of no reflection waves. This complication does not show up in (22), since only the geometrical-optics solution was used to obtain the analytic continuation.

Ursell, in unpublished work, has worked out asymptotic estimates of the centerplane source distribution corresponding to the exact solution. He shows that, for \( vx \) very large, the source distribution can be written:

\[ \sigma(x, z, t) = e^{v z e^{i (\omega t - v x)}} \times \begin{cases} 
-\sin 2\alpha F(2v x \sin^2 \alpha) & \text{for } \alpha \text{ small, positive,}
\end{cases} \]

\[ + (2v x / \pi x)^{1/2} (1 - \cos \alpha) e^{i \pi / 4} + \ldots \]  

(23)

where

\[ F(\alpha) \approx \begin{cases} 
2\pi i \alpha^2 + (\pi / \alpha)^2 e^{i \pi / 4} & \text{for } \alpha \text{ large, positive.}
\end{cases} \]

The quantity \( \alpha \) must be replaced by \( 2v x \sin^2 \alpha \), which is a product of a large quantity, \( 2v x \), and a small quantity, \( \sin^2 \alpha = \alpha^2 = \pi^2 \epsilon^2 \). The product may be small or large, and so we need both estimates above, even though we consider only the case of large \( v x \).

If \( v x \) is so large that \( 2v x \sin^2 \alpha \) is also large ( \( 1/v x = O(\epsilon^2) \)), we find that

\[ \sigma(x, z, t) \approx e^{v z e^{i \omega t}} \begin{cases} 
-4\pi i e^v e^{-i v x \cos 2\alpha} & \text{for } \alpha \text{ small, positive,}
\end{cases} \]

\[ + (2v / \pi x)^{1/2} e^{-i v x + i \pi / 4} \]  

(25)

The first term is seen to be identical to the result obtained from the geometrical-optics solution, given in (22). The second term on the right-hand side of (25) is formally of the same order of magnitude as the first if we require only that \( 1/v x = O(\epsilon^2) \), but it is of higher order if \( 1/v x = O(\epsilon^2) \).

The other case included in Ursell's result requires that \( v x \epsilon^2 = O(1) \) as \( \epsilon \to 0 \), that is, \( v x \) is large but the product \( v x \epsilon^2 \) is small. We find then that
\[ \sigma(x,z,t) = e^{\dfrac{\nu z}{z}} e^{\dfrac{i(\omega t - \nu k)}{\nu} \left[ -2\pi i \nu \right] + 4\pi^2 \nu z (2
\nu x)/4} = e^{-i\nu \theta} \] (26)

If, for example, \( \nu x = O(\epsilon^{-1}) \), the first term is \( O(\epsilon) \) and the second term is \( O(\epsilon^{3/2}) \).

This result leads to two interesting comparisons: (i) The amplitude of the source density, as given by the first term, is one-half of the amplitude much farther from the apex, as given in Equation (25). There is also a slight difference in the periodicity along the \( x \) axis: Here the periodicity is essentially the same as in the incident wave, whereas farther away the periodicity is characteristic of the reflected wave. (ii) The first term here "matches" with the result for small \( \nu x \), as given in Equation (20). This matching is realized in the sense of the method of matched asymptotic expansions. The real part of (26) (first term only) is plotted in Figure 2, along with the result from Equation (20). (In both, the factor \( e^{i \omega t + \nu x} \) is suppressed.)

The fact that the approximations for large and small values of \( \nu x \) match in a satisfactory manner suggests that more complicated problems can be solved by formulating a near field around the bow and a far field (far from the bow) and matching the solutions in these two regions. In fact, the matching may very well be trivial, in the sense that each of the partial solutions can be found without reference to the other.

It is ironic that, for the wedge problem discussed above, we have a complete analytic solution but we have no approximate method yet that might be useful in problems of less restrictive geometry. Some kind of slender-body analysis ought to be feasible, but I have not found it, nor has anyone else so far as I know.

We have now considered two extreme cases, the difference between them depending on which term in (17) (the right-hand side of the body boundary condition) is retained. In many practical problems, one may reasonably expect the two terms in (17) to have more or less comparable effects, notwithstanding their formally different orders of magnitude. For a body of large but finite draft, \( n_3 \) is large on the bottom, but the exponential factor in (17) makes the product negligible. The quantity \( n_1 \) is presumably much smaller than \( n_3 \), but it may well have its largest values near the level of the free surface, where the exponential factor has a value close to unity. It seems to me to be preferable to include both terms at once in the analysis, even though this is mathematically inconsistent.

Whether such a hybrid problem can in fact be solved is quite another matter. Since we do not have approximate methods for solving both extreme cases, we may have to wait yet a while for a method of solving the hybrid problem.

Nearby head seas. In introducing the head-sea difficulty, I mentioned that the general problem for short oblique waves can be solved by slender-body theory but that the amplitude of the diffraction waves approaches infinity as the heading angle approaches zero. If we divide the diffraction problem into two parts, one of which is odd with respect to \( y \) and the other even, we find that this behavior is characteristic of just the even part of the solution. In

![Figure 2. Source density at wedge apex.](image-url)
fact, the amplitude of the odd part of the diffraction waves approaches zero in the limit, which it obviously must do, since the odd part of the incident waves approaches zero. The only real problem concerns the even part.

The fact that the even part of the diffraction-wave amplitude approaches infinity as heading approaches zero suggests that the order of magnitude of the head-sea solution is lower than that of the oblique-sea solution. This is indeed the case, as one can show by comparing, say, Faltinsen's head-sea solution and Troesch's oblique-sea solution. (However, one must be careful to ensure that the same assumptions are being made in the two cases with respect to wavelength.)

This fact then suggests what will be necessary to solve the problem of a slender ship in nearly head seas. One must relate the heading angle, $\beta$, to the slenderness parameter, $\varepsilon$, and develop an asymptotic solution for the nearly-head-sea case, the solution having the property that the first term of the expansion rapidly approaches zero as $\beta$ increases from zero. Only in this way can it match with the higher-order oblique-sea solution.

Laplace vs. Helmholtz equation. Newman (1970) pointed out that one can obtain two apparently identical Khaskind-type formulas for the force on a ship in waves, one formula requiring that a 2-D potential be found satisfying the Laplace equation and the other requiring a 2-D solution of the Helmholtz equation. The two formulas were obtained in a slightly different way by Ogilvie (1974). The 2-D solutions that are needed relate to forced-motion problems, even though it is the force on a ship fixed in waves that is to be found. (This is the usual situation in using Khaskind formulas.)

Newman and Ogilvie showed that the two formulas were formally equivalent except for "end effects." In fact, if the body ends are sufficiently slender (pointed perhaps), even the end effects vanish. Thus there is no fundamental mathematical paradox in the existence of the two formulas.

However, this is not a practical resolution of the difficulty. Solutions of the 2-D Laplace equation and solutions of the 2-D Helmholtz equation do not give the same hydrodynamic force on a body, at least not in the manner that has generally been proposed for the practical use of the two formulas. The difficulty is that the end effects are quite real and nonnegligible, but they cannot be handled by the usual methods of slender-body theory or strip theory.

Fortunately, the answers given by the two formulas are not grossly different for the few cases in which they have been compared. But that does not mean that such luck will always hold as one or the other formula is used in problems involving conditions quite different from those that have been checked. This is a matter of end effects quite similar to some of the others that have been mentioned earlier here.

Numerical 3-D solutions. Eventually, it will be necessary to develop programs that are as efficient as possible for solving these wave problems by numerical methods. Some such programs already exist, of course, but they are cumbersome and expensive to use, and they provide little general guidance of the kind that an engineer needs in a design problem. When a particular design has been worked out, existing numerical methods may perhaps be used to predict hydrodynamic force or pressure for that special case. But it is not generally feasible to make parametric studies with such programs.

SLENDER SHIP IN STEADY FORWARD MOTION

There are two major defects in the classical slender-ship steady-motion theory, as developed by Vossers (1960), Maruo (1962) and Tuck (1963): (a) There is no simple mechanism for including the diffraction by the ship of the ship-generated waves. (b)
The procedure for determining the density of sources on the longitudinal axis of the ship is completely inadequate for representing end effects with acceptable accuracy.

The first of these defects is shared with thin-ship theory. The only way of removing the defect from thin-ship theory appears to be to solve the Kelvin-Neumann problem, which is a formidable task. In principle, it appears that a simpler procedure might be available for slender ships, but the difficulties encountered by Reed (1975) indicate that a fix-up for slender-ship theory is not simple either. Reed's results clearly point in the right direction, but his solution fails at the body ends in somewhat the same way that Faltinsen's does for the incident-wave problem. Moreover, Reed's procedure is itself not simple, either in its justification or in its implementation.

Reed effectively incorporates into his slender-ship model the diffraction of the ship-generated transverse waves by the ship itself. There must be a similar phenomenon with respect to the diverging waves too, which he does not include. Again, thin-ship theory is deficient in the same respect.

The second defect of slender-ship theory, relating to end effects, is even more serious in that it not only causes a loss of accuracy in the predictions but it can even make nonsense of an otherwise reasonable analysis. This defect can probably be removed, and, in fact, it has been for certain simple cases. But no analysis has been developed for handling general cases.

In this section, I shall first discuss briefly how diffraction of the diverging waves can be analyzed by slender-body theory. There is no need to go into much detail, since there have been several papers published on this subject in the last few years. One may question whether this is even an "end effect," but I shall include it as such. The diffraction of transverse waves is in no sense an end effect (except inasmuch as the corresponding analysis breaks down at the bow of the ship), and it will not be discussed further here.

Following that, I shall discuss some ideas on the nature of the breakdown of slender-ship theory at the ends, especially in the far-field view. This discussion will be rather conjectural in nature.

Finally, there will be a short synopsis of some recent work on predicting what happens just ahead of the bow of a slender body.

The bow-wave problem. The first attempt to use slender-body theory for obtaining an accurate description of the bow-wave system was made by Cummins (1956), several years before the development of what I call "classical slender-ship theory." Unfortunately, Cummins' work was generally neglected for fifteen years, but it was a clear precursor of much work during the early 1970s.

Cummins started with a linearized steady-motion problem in the classical manner described by, for example, Lamb (1932). Then he assumed that the governing equation in the neighborhood of the hull could be taken as a 2-D Laplace equation in the transverse planes. He required that the potential should satisfy the body boundary condition on the precise location of the body surface, rather than, say on the centerplane.

The use of the 2-D Laplace equation in the transverse planes can be interpreted in this problem most easily by replacing the longitudinal coordinate, \( x \), by a time variable, \( t \). Consider a transverse plane fixed to the fluid at infinity, with the ship passing through the plane. The bow cuts the plane at \( t = 0 \). As the ship moves forward, successive crossplanes cut the fixed section as time increases. We study the motion in this fixed section as a function of time. When the bow first cuts the plane, an impulsive transverse motion is imparted to the fluid, causing
waves to be created for all later time, much as waves are created impulsively in the Cauchy-Poisson problem. As the ship passes through this plane, more and more waves are created in the same way, and the total motion at any instant is the sum of all of the motion created by the parts of the ship that have already passed through the fixed plane. In addition to causing an impulsive motion, the ship cross-section at time \( t \) interacts with the waves already created by the forward sections. It is this interaction that is equivalent to a diffraction of the waves created earlier.

It is important to ask whether such a time-dependent 2-D motion has any relevance to the steady 3-D motion around a ship. It can be shown (see Ogilvie (1975)) that, if the disturbance created by a moving point source in three dimensions is analyzed by the method of stationary phase, the part of the result that represents the diverging-wave system is mathematically equivalent to the disturbance created in two dimensions by an impulsive point source; it is necessary only to replace the space coordinate \( x \) by a time variable \( Ut \). (The time-dependent motion must also be analyzed by the method of stationary phase in order to establish the equivalence.) One might expect this to happen in view of the facts that (i) diverging waves in the 3-D problem spread out in such a way that wave crests (or troughs) are parabolic close to the source, that is, they are given by curves of the form \( x^2/y = \text{constant} \), and (ii) points of constant phase in the 2-D time-dependent problem move outwards in such a way that \( y = \text{constant}.t^2 \), where \( y \) is measured laterally in both cases. The equivalence is not just qualitative.

Furthermore, in the case of the point source moving in three dimensions, if the source is shallow enough, the diverging waves dominate the entire wave motion in the neighborhood of the track of the source. This was shown carefully by Ursell (1960) for the case of a moving pressure point. It clearly is not true if the source is deeply submerged, but it should be approximately true for some distance downstream if the submergence is very small.

The implications of this equivalence are startling. If a moving point source in three dimensions is very shallow, the fluid motion near its track can be analyzed by the methods of slender-body theory. (This would be utter nonsense if the free surface were not present and, in fact, dominant in its effects.) If slender-body theory can be used to describe the motion behind a single 3-D point source, it ought to be even better in describing the motion near a longitudinal line of sources or even near a spatial distribution of sources that is clustered near a line close to the free surface. This concept will be used in a bold (and perhaps questionable) way when we discuss how one might investigate the disturbance just ahead of a slender body.

Cummins did not attempt to justify his approach on the basis of some kind of systematic perturbation analysis.\(^\text{†}\) However, Ogilvie (1967, 1972) did this, first by considering that speed (or Froude number) is very high, then later by pointing out that there are some physical reasons to consider that the flow near a ship bow is, in effect, a high-Froude-number system. In the first paper (1967), it was found that a problem just like Cummins' problem arises naturally if the speed is very high, except that there seemed to be no compelling argument for linearizing the free-surface condition. (This is actually the problem that was later solved by Chapman (1976), who, like Cummins, did not use a systematic perturbation analysis.) Later, Ogilvie (1972) proposed that near a ship bow the flow variables have longitudinal gradients that are larger than is usually assumed in slender-body theory. In particular, if it is assumed that the operator \( \partial/\partial x \) has an

\(^\text{†}\)It should be noted that he also introduced the transverse wave system, but he was not quite successful in building a complete analysis including both wave systems. This had to await the development of new perturbation methods, including the method of matched asymptotic expansions.
order-of-magnitude effect that is $O(\epsilon^{-1/2})$ in a bow region with longitudinal dimensions that are $O(\epsilon^{1/2})$, the lowest-order near-field problem near the bow is precisely Cummins' [linear] problem. In the usual way, he assumed that transverse gradient operators have an effect that is $O(\epsilon^{-1})$, so that the 3-D Laplace equation reduces to the 2-D equation in the cross-planes, at least for the first-order problem. This unusual stretching has the effect of allowing a formal linearization of the free-surface conditions. One can arrive at the same situation simply by assuming very large Froude number, as in Ogilvie (1967), and then arbitrarily linearizing the free-surface conditions.

Ogilvie's procedure was applied to a flat ship by Tuck (1973) and Baba (1974) and to a yawed flat plate by Hirata (1972). In each of these works, as in Ogilvie's (1972) original work, the boundary condition on the body was arbitrarily transferred to a plane surface. Daoud (1975) has solved the full problem in which the linearized free-surface conditions are used but the body condition is applied precisely on the body surface. It should be noted that this is a consistent approach, not to be compared with, say, the Kelvin-Neumann problem, which is an inconsistent modification of thin-ship theory.

Figure 3 shows the wave contours predicted on the basis of the simplest version of Ogilvie's (1972) model. Partly because of the nature of the model and partly because of the extra simplification resulting from transferring the body boundary condition to the centerplane, one set of curves such as those shown gives the results for any Froude number, any wedge angle, and any draft; the values of $Z$ corresponding to the curve are universal values in this sense, although it should be noted that the two coordinate scales are made nondimensional in different ways, depending on Froude number and draft.

Figure 4 shows Daoud's corresponding results for a special case: $F = U/(gH)^{1/2} = 1.5$, wedge half-angle $= 7.5^\circ$. Because diffraction effects are included here, the contours are different for each Froude number and each wedge angle, and so this is just a sample result. The outline of the wedge is also shown. (It does not look like $7.5^\circ$ because of the different coordinate scales.) The bow wave has obviously been shifted forward from the location predicted by Ogilvie. However, the shift is not so great as Standing's (1974) experiments indicate that it should be. This discrepancy is understandable when one considers that Daoud, like Ogilvie, assumes that there is no disturbance whatever ahead of the bow.

These bow-flow analyses are all basically near-field solutions, the near field being a region with length that is $O(\epsilon^{1/2})$ and transverse dimensions that are $O(\epsilon)$. Such solutions ought to be matched to a far-field solution representing a line-distribution of sources. Presumably, the usual rules for determining the density of sources will be modified in the neighborhood of the bow. Such a matching process has not yet been reported, although a particular case will be considered below.

Chapman's (1976) analysis of the yawed-plate problem deserves special mention here. His basic assumptions are quite similar to those of Ogilvie et al. However, he used the full nonlinear free-surface condition, a fact that must make his work almost unique. He also computed linearized and second-order results. He found that linear and fully nonlinear models gave only small differences in the side force on the plate, although his predictions of actual wave contours show greater differences. The second-order results were useless, presumably because singularities implied in the linear analysis cause higher-order singularities in the second-order problem. Chapman (1975) also solved some unsteady-motion problems by a similar technique, although in this case he used the linearized free-surface conditions.
Figure 3. Contour map of nondimensional wave height near a wedge bow (Ogilvie's approximation).

Figure 4. Contour map of nondimensional wave height near a wedge bow. (Daoud's approximation).

Wedge half-angle = 7.5°, $F_H = 1.5$.

Breakdown of slender-ship theory at the ends. For a body of uniform draft and wedge-shaped waterplanes, the far-field line-source density in ordinary slender-ship theory is given by the broken line in Figure 5. It has three discontinuities, corresponding to the three discontinuities in cross-section area of the body. If this source distribution is used to compute the wave resistance of the body (even by the correct formula given in the Introduction), a ridiculous answer is obtained: The wave resistance is infinite.

The solid curve in Figure 5 represents a modified line-source distribution that yields the same value of wave resistance as is given by thin-ship theory for this particular hull form. A complete discussion of the modification, including a derivation, has been given by Ogilvie (1977), and so only a few comments need to be made here.
Figure 5. Line-source distribution for a double-wedge hull

For a body such as this, thin-ship theory is not very accurate, but neither does it give a value of wave resistance that is grossly wrong. The modified line-source distribution in Figure 5 has been constructed to give the same wave resistance as thin-ship theory\(^1\) and so it cannot be grossly wrong either. But the broken-line distribution gives infinite wave resistance. The major difference between the two source distributions is that one has discontinuities and the other does not. This is a crucial difference. The smooth curve could be distorted to be arbitrarily close to the discontinuous curve, but it would still correspond to a finite wave resistance.

It is well known that a single pressure point on the free surface generates a wave system with infinite wave resistance. From a physical point of view, the pressure point produces large-amplitude short-wavelength waves close to the track. In fact, these short waves are so large that an infinite force is required for producing them.

A smooth source distribution at the level of the free surface can be considered as a superposition of many point sources, each of which produces waves like the waves behind a pressure point. But these waves all interfere with each other destructively, so that the total amount of wave energy left behind per unit time is finite. Any smooth distribution of sources on a line accomplishes this.

On the other hand, it is evident that the waves generated by the sources in a discontinuous distribution do not interfere with each other in such an effective way. The short waves near the track still have such a large amplitude that only an infinitely large force can produce them. Therefore such a discontinuous distribution of sources on a line cannot in any sense represent the disturbance caused by a real body.

The problem then is to find the correct way to smooth the source density function that comes from ordinary slender-body theory so that the wave resistance is not only finite but even accurate. The procedure described by Ogilvie (1977) is not a general one, for it gives only the value of wave resistance that is predicted by thin-ship theory. One might as well use thin-ship theory itself if no better procedure can be obtained. Current work by Daoud may well lead to the finding of such a better procedure, and this would be a significant improvement over thin-ship theory, for it would reflect the effect of the ship's diffraction of its own waves, at least in the bow region.

The source distributions in Figure 5 relate to a ship that extends from \(x = 0\) to \(x = L\). The modified source distribution evidently extends beyond the two ends of the body. This requires some careful interpretation. There are no sources in the fluid ahead of or behind the body. But when one views the ship from the far field it appears that there must be such sources ahead of and behind the body. In the near field, on the other hand, one has to deal with the actual body and the fluid around it; there is no line of sources at all. These two views must be reconciled by a matching process.

The difference between the two source distributions shown in Figure 5 is, in a sense, a higher-order quantity that one might ordinarily neglect. Suppose that the

\(^1\)Actually, if the potential corresponding to this line of sources and the thin-ship potential are analyzed by the method of stationary phase, they give exactly the same wave systems.
order of magnitude of the source strength is $O(\sigma)$, where $\sigma$ has a simple dependence on $c$. Then the difference between the ordinates of the two curves is also $O(\sigma)$. However, this difference is significant only over a longitudinal distance that is $O(c^{1/2})$, whereas the total body length is $L = O(1)$. Thus, at a finite distance, one would expect the effect of the difference to be of higher order than the total effect of the source distribution. (This would certainly be true for a slender body in an infinite fluid.) However, such a conclusion is not consistent with the unique wave behavior associated with the presence of the free surface. The net effect of the difference between the source distributions is not of higher order.

One further implication of the above line of reasoning should be noted. Inasmuch as we can represent the wave-generation properties of a centerplane distribution of sources in terms of an equivalent line distribution of sources on the free surface, we should also be able to represent the wave-generation properties of a submerged point source in terms of an equivalent line distribution on the free surface. In fact (see Ogilvie (1977)), one can show that, if a source of strength $\phi_0$ is located at $(0,0,z_0)$ (with $z_0 < 0$), the waves generated some distance away are the same as those generated by a distribution of sources

$$\phi_0[\kappa/4\pi|z_0|]^{1/2}e^{\kappa x^2/4z_0},$$

where $\kappa = g/U^2$, this distribution lying on the $x$ axis at the level of the free surface. This distribution is symmetrical fore and aft of the location of the actual source, the amount of spreading depending strongly on the submergence, $|z_0|$, of the point source. This spreading out shows that the usual picture of the Kelvin wave system bounded by a wedge-shaped region of half-angle $19^\circ 28'$ is not so clear as is usually thought to be the case; the apex of the wedge cannot be defined precisely.

**Fluid motion ahead of the bow.** In the earlier discussion of the bow-wave problem, it was pointed out that all of the investigators mentioned there had assumed that there was no disturbance at all ahead of the bow. This is not true, of course, except possibly in the asymptotic sense that such a disturbance might be of higher order than the disturbance alongside the bow. In order to study the upstream disturbance numerically, Kaiho (1977) formulated a problem in which the forward disturbance ought to be relatively large. His analysis nevertheless depends strongly on the ideas presented earlier in connection with the bow-flow problem.

If a body is slender enough, especially in the bow region, one may well expect the disturbance ahead to be very small. But this is not generally true for a nonslender body. Consider, for example, a surface-piercing strut of large aspect ratio. At some distance below the free surface, the flow around the strut should approximate the flow around a 2-D vertical strut in an infinite fluid, and so there is just as large a disturbance of the fluid ahead of the strut as there is behind it. Of course, the presence of the free surface modifies these conditions greatly, but still there must be a considerable disturbance forward of the leading edge.

Kaiho formulated the problem of such a strut as follows: Let the complete potential be represented as the sum of three terms,

$$\phi(x,y,z) = Ux + \phi_0(x,y,z) + \phi(x,y,z) \quad (27)$$

where $\phi_0$ satisfies the free-surface condition

$$\phi_0 = 0 \quad \text{on } z = 0, \quad (28)$$

the exact body boundary condition and the Laplace equation in three dimensions. Since the full potential satisfies the free-surface condition

$$\phi_{xx} + \kappa \phi_z = 0 \quad \text{on } z = 0,$$
the potential \( \phi(x,y,z) \) satisfies
\[
\phi_{xx} + \kappa \phi_x = - \phi_z \quad \text{on} \quad z = 0. \tag{29}
\]
It also satisfies a homogeneous Neumann condition on the body surface, and Kaiho assumes that it satisfies a Laplace equation in two dimensions (the transverse planes). Except for the last assumption, the formulation of the problem is entirely conventional, the subdivision into two partial potentials being arbitrary but certainly not incorrect.

The justification for requiring that \( \phi \) satisfy the 2-D Laplace equation follows from Equation (29) and the earlier discussion of bow-flow problems. The right-hand side of (29) can be interpreted as if there were a pressure field applied on the free surface, the pressure distribution being such that \( p_x(x,y) \) is proportional to \( \phi_z(x,y,0) \). Then the potential \( \phi \) represents the fluid response to such a pressure distribution (with a correction for the presence of the body). Since the fluid motion near and behind a pressure point is given approximately by a potential function satisfying the 2-D Laplace equation, Kaiho requires that \( \phi \) satisfy such an equation, rather than the full 3-D equation.

It would have been completely incorrect to set up a slender-body model for the direct solution of the strut problem, since the flow around the strut at considerable depths would have been represented as being approximately two-dimensional in the wrong set of planes. The flow down deep is nearly two-dimensional in the horizontal planes \( z = \text{constant} \), whereas the slender-body flow is nearly two-dimensional in the transverse planes \( x = \text{constant} \). In the procedure adopted by Kaiho, the correction potential, \( \phi \), simply vanishes rapidly with depth, and the fact that it represents a 2-D flow with a wrong orientation does not produce significant errors.

There is another respect in which Kaiho's mathematical model is not strictly correct, although again it does not appear to cause serious consequences: In order to solve the \( \phi \) problem, he must obtain the disturbance in the neighborhood of the body due to the entire upstream distribution of \( \phi_z \) (the effective pressure distribution). The 2-D problem that is formulated for \( \phi \) is not a valid problem statement far downstream of a pressure point or at locations laterally far removed from the track of a pressure point. However, the disturbance function \( \phi_z \) varies so smoothly in most of the upstream region that no net effect is produced near the body except from a small region directly ahead of the body, where \( \phi_z \) does indeed vary rapidly with respect to \( x \) and \( y \). Once again, it appears that the predicted disturbance must have some characteristics that are wrong in principle, but the errors are mostly so small that they do not matter. If the disturbance just ahead of the body (and its effect) is predicted correctly, the rest of the forward disturbance can be ignored — or even treated in an incorrect way, just so long as no net downstream effects are predicted.

The advantage of Kaiho's method is in the fact that he needs to solve only one fully 3-D boundary-value problem, and that is the \( \phi_0 \) problem, which does not involve any wave motion. The wavelike effects all enter with the \( \phi \) problem. Like the bow-

![Figure 6. Side-force coefficient on yawed strut AR = 0.5, tan \( \alpha \) = 0.08](image)
flow problems discussed earlier, the problem is partly two-dimensional and partly three-dimensional: The governing equation is the 2-D Laplace equation, but the solutions in the transverse planes are related through the free-surface condition, (29).

Figure 6 shows Kaiho's calculations of the lift (side-force) coefficient on a vertical surface-piercing strut of aspect ratio (AR) 0.5 moving with an angle of attack \(\arctan 0.08\). Also shown are the corresponding experimental results of Van den Brug (1971). The comparison is very good except at low Froude number, where Kaiho's model cannot be expected to yield valid predictions.

Figure 7 shows Kaiho's predicted wave contours for the same strut for the case of Froude number equal to 1.0. The scale for wave elevation is 100/L.

An important restriction on Kaiho's approach is identical to the restriction on the bow-flow solutions discussed previously: The solution includes only diverging waves, and so the solution is valid at a small distance downstream of the bow, where "small distance" means a distance considerably less than the wavelength of the transverse waves, \(2\pi U^2/g\). Extending the solution farther downstream will require matching it with a solution that includes fully 3-D wave effects. As already noted, this has not been done completely even for the somewhat simpler bow-flow problem, and so of course it has not been done for Kaiho's procedure either.

Limitations of these analyses. There are two general limitations in all of the foregoing analyses: (1) two-dimensional governing equations are used, and (2) the free-surface conditions are linearized (with

Figure 7. Contour map of nondimensional wave height near a strut. AR = 0.5, F = 1.0.
the sole exception of Chapman’s (1976) work). Perhaps the consequences of these limitations are obvious, but they should be noted nevertheless.

The use of the 2-D Laplace equation places serious constraints on the body geometry that can be considered. For a body in an infinite fluid, the 2-D Laplace equation can be used near the nose only if the body is pointed, and even in such a case it can be used only to obtain a first approximation. I have argued that the use of the 2-D equation is somewhat more justified in the free-surface problem, mainly because the fluid motion near the bow is dominated by the divergent waves, which can be described approximately by 2-D solutions in the transverse planes close behind the the cause of the disturbance. Nevertheless, it is evident that such an argument cannot be pushed too far. Certainly a blunt-nosed body cannot be treated in this way. Neither can a body that extends far into the water, for example, a strut, unless a modification such as Kaiho’s is made.

There are two physical aspects to the linearization question:

(a) There is the obvious difficulty that a stagnation point ought to appear somewhere on the nose of the body, and the fluid disturbance in the neighborhood of such a point is in no sense small. Thus one generally expects to find there a singularity in the first-order asymptotic solution. In the bow-flow problems considered, it is interesting to note that this singularity is weaker than expected: For a symmetrical fine wedge, the slope of the free surface is infinite at the leading edge, but the elevation is not. (In fact, the elevation is zero.) I am not sure that I can say that such a result ought to have been anticipated, but the predicted shape of the bow wave does agree rather well with the observed shape. (See Ogilvie (1972).)

(b) It is questionable whether all of the nonlinear terms in the free-surface conditions can legitimately be ignored. Transverse gradients of the flow variables in the diverging-wave region may be very large, in fact, large enough to make nonlinear terms such as \( \phi_y \phi_y \) (in the kinematic condition) comparable in magnitude with the linear terms, \( \nabla \phi \) . The error incurred may very well be comparable with the error that would result if the body boundary conditions were transferred to the \( y = 0 \) plane or to the \( z = 0 \) plane, notwithstanding the fact that a mathematically consistent problem can be formulated in which these two errors are in principle of different orders of magnitude. Chapman’s (1976) analysis could be used to investigate this point, since he alone includes all of the effects in question, and so the individual simplifications could be made separately. It may also be noted that the importance of the nonlinear terms in the free-surface conditions has recently been stressed by Baba (1977) and by Inui and Kajitani (1977), who are attacking similar problems from quite a different point of view.

**Possibilities for future analysis.** The point in the development of all slender-body theory is to take advantage of the gradual change of shape along most of the length of a body (such as a ship). This smooth geometry presumably causes smooth changes in the flow field around the body, so that simplified boundary-value problems can be formulated in place of the exact global problem. But the assumptions generally fail completely near the ends. The physical quantities of interest may not be sensitive to this failure in the end regions, and in such cases we can proceed to use simple slenderness concepts. But some important physical quantities are sensitive to what happens at the ends, and our existing theories fail to yield useful results. We may improve the situation somewhat by some of the devices discussed in this paper, but there is clearly a limit to how much can be accomplished by such fix-up methods.

The most obvious approach to achieving further progress is to abandon simplified models such as the slender body altogether and use a large digital computer to solve
the complete problem at hand. Numerous attempts are being made to do just this, as was evident at the recent Second International Conference on Numerical Hydrodynamics (Berkeley, California, September 1977), and some investigators are achieving considerable success in this endeavor.

However, it also seems that a compromise should be possible between these extremes. The fact remains that a ship is slender for the most part, and to ignore this fact is to discard a possible means of achieving solutions that are relatively economical and perhaps also more perspicuous than brute-force computer solutions. Clearly, some part of the boundary-value problem must be solved by purely numerical methods. This is already a recognized fact in existing slender-body methods of solution, in which typically the 2-D problems in the transverse planes are solved numerically. The question is whether end effects can also be treated separately by numerical methods without the necessity to extend the computation to the entire body all at once.

REFERENCES


THE CAVITATING PROPELLER AND ITS PRESSURE FIELD; APPLICATION OF FREE STREAMLINE THEORY

by

L. NOORDIJ
NETHERLANDS SHIP MODEL BASIN
WAGENINGEN, THE NETHERLANDS

ABSTRACT

A model to describe the velocity potential of a cavitating propeller is considered. Strip-wise quasi-steady application of free streamline theory leads to a source distribution on the blades, accounting for the time-dependent behaviour of the cavity geometry. The model is used to study the relation between the pressure field of the propeller and the structure of the wake in which the propeller operates and to show the differences in effects between cavitating and non-cavitating propellers.

1. INTRODUCTION

Ship hydrodynamics (including propulsion) offers many problems which are interesting for applied-mathematicians. Mathematical models have been developed for ship motions and wave making. Much attention has been paid to calculation procedures for the viscous resistance. A brief review of these problems has been given by Timman [1]. Also many aspects of propulsion lend themselves to mathematical treatment. Lifting line and lifting surface theories have been developed for the design and analysis of the conventional, marine propeller (widely applied and usually located in the ship's wake).

The problem dealt with in this paper is hydrodynamic hull excitation due to a cavitating propeller. Cavities are created when dynamical conditions in the liquid are such as to produce relative low pressures at certain places (Batchelor, [2]). This can occur on and near a propeller. A typical example of such cavitation is shown in Fig. 1.

Each blade of the propeller can be considered as a lift generating wing. A schematic representation of a propeller is given in Fig. 2.

Fig. 1. A cavitating propeller

Fig. 2. Schematic representation of a propeller in a cylindrical co-ordinate system \( (r, \theta, X) \). \( w \) is the propeller angular speed and \( V \) the speed of advance.

When a blade moves with respect to the surrounding fluid, circulation will occur around the blade, initiated by the so-called Kutta condition at the trailing edge. Due to circulation a lower pressure will occur on one side (suction side) of the blade and a higher pressure on the other (pressure side). Thus circulation is associated with a lift force (loading) on the blade which contributes to the propeller thrust. Say the ship is propelled in this way at a speed of advance \( V \). (For further details on propulsion the reader is referred to a relevant textbook).

The pressure on the suction side of the blade
can locally (usually close to the leading edge) decrease so far that the liquid can no longer sustain the relatively low pressure and a vaporous cavity appears. See Fig. 2. The conditions under which cavitation starts will not be discussed here. A review on cavitation inception can be found, e.g., [3]. Here it is assumed that a cavity appears on the blade for a given combination of depth of immersion of the blade, angular and advance velocities \( \omega \) and propeller geometry.

A propeller operating behind a ship will certainly not experience homogeneous flow. Due to the inhomogeneous flow the circulation around the blade (loading) will change periodically and inherently the cavity volume on the blade. Now both loading and growing and collapsing cavities on the propeller blades induce velocity fluctuations in the liquid. In this way both are a source of hydrodynamic hull excitation. Besides these, the displacement of the blades and the attached cavities will also contribute to the hull excitation. However, as shown experimentally by Van Manen [4] and later on by many other investigators, the contribution due to growing and collapsing cavities to the hull forces is quite often dominant, especially when cavitation appears as a sheet (Fig. 1). This contribution is governed by the rate of change of the cavity volume (see [5]), and thus by the wake of the ship in which the propeller operates.

In order to be able to design ship hulls and propellers in a way that ships and crews suffer least from hydrodynamic hull excitation, model experiments and a theoretical analysis have to be performed. The theory for the excitation due to a non-cavitating propeller was derived by Pohl [6] and Breslin [7]. A review can be found in a paper by Breslin [8] and in the book by Tsay [9]. In this theory the loading of the propeller is represented by a system of line sources. The thickness or displacement of the blade is accounted for by a source-sink distribution. Some preliminary work on a theory for a cavitating propeller was done by Huse [5]. Huse showed that, by adopting various artificial cavity volumes, the pressure field surrounding the propeller can be altered significantly due to the growth and collapse of the cavity on a propeller blade. The cavity was represented by sources, along a curved line, with time-dependent strength. This strength is proportional to the rate of change of the local cavity-cross section. The theory of Huse, however, assumes a known cavity geometry. Noordzij [10] and [11] developed a theory to calculate the geometry of the cavity, i.e. the cavity on the propeller blade as a function of the angular position in the wake. With help of this theory the effect of cavity geometry on the hull pressure was analysed. Noordzij [11] showed that a number of line sources distributed along the blade instead of a single line source led to an essential improvement in the calculated hull pressure compared with that given in [10] and the experimental results. This is illustrated in Fig. 3.

![Fig. 3. Various contributions to the calculated first harmonic of the hull pressure [11]. The shaded column is obtained from [10]. The experimentally obtained pressure amplitude is also given.](image-url)

In this figure the calculated first harmonic of the periodical fluctuation of the hull pressure is shown (adopted from [11]). The shaded column represents the contribution to the amplitude when the cavity is represented by a single line source. Besides, the effect of cavitation is clearly demonstrated.

To determine the cavity geometry on a propeller blade the general problem of unsteady cavitation needs to be solved. The actual phenomena are extremely complicated involving an interplay of viscosity, surface tension, diffusion and heat effects. The translation of the problem into a mathematical model is very difficult. It is not surprising therefore, in order to proceed and to obtain a mathematical model for engineering purposes, that some simplifications are introduced. The first is to consider the propeller rotating in an unbounded medium with a given velocity distribution. Secondly the flow relative to the propeller blade sections is assumed to be quasi-steady. This approximation is justified for those cases where the reduced frequency is small (see [12]). This frequency is defined as the ratio of the length of the propeller blade in rotational direction and the wave length of the velocity field (fractions of \( \pi \) times the propeller diameter). Further, velocities normal to coaxial cylindrical surfaces being small, except for the propeller blade tip region, the flow components tangentially to these surfaces can be treated as two-dimensional. With these
simplifications results of a two-dimensional potential flow model for the propeller blade section can be applied. The cavity is considerèd as a region of constant pressure which adheres to the blade section. The free streamline theory, which will be applied in linearized form, provides the tool to determine cavity length extending along the blade in rotational direction. (See Geurst [13] and [14], Geurst and Verbrugh [15], and Noordzij and Officier [16]). Also the cavity thickness distribution over the blade can be found ([11] and [16]). For a particular blade position in the wake the cavity length and thickness distribution can be calculated as a function of the radial co-ordinate. This procedure can be repeated for each blade position in the wake. In this way a description for growth and collapse of the cavities on the propeller blades is obtained and the source strength representing this dynamic cavity behaviour can be determined. With help of the calculated source strength the pressure and total excitation force on a hull can be computed exactly. This can be done with the so-called Chertock formulas. Their application to the calculation of the excitation force of the hull is shown by Vorus ([17] and [18]) for a non-cavitating propeller. This theory can easily be extended for a cavitating propeller when the source strength representing the cavity is given. This theory will not be dealt with in this paper.

In those cases where we like to illustrate the effect of the cavitation on hydrodynamic excitation, an idealized ship's hull, e.g. flat plate, will be taken.

Besides the constraint of low reduced frequency, there is another for cavitation: the rate of change of the cavity volume on the blade should be small compared with the rate of change of a free collapsing cavity. Otherwise "history effects" have to be included, i.e. the cavity geometry for a particular blade position depends on the geometry in previous positions.

In this paper attention is paid to the velocity potential for a cavitating propeller. How the resulting hull force depends on the wake field components is investigated. The above mentioned contribution to the excitation of rigid neighbouring boundaries is illuminated. It is revealed that its dominance not only results from the increased pressure amplitude, but also from the reduced phase differences of the pressure fluctuations between different places on the hull due to the source-like behaviour of the cavity. The singular behaviour of the cavity length, resulting from the free streamline theory when the cavity approaches the trailing edge of the blade, is investigated. By modifying the expression for the length in a semi-empirical way this singular behaviour is suppressed. Finally, the magnification of the pressure field due to reflection connected with the presence of the propeller blade very near the cavity is dealt with.

2. THE VELOCITY AND PRESSURE FIELD OF A CAVITATING PROPELLER

We shall consider the contribution of growing and collapsing cavities to the pressure field. This contribution is obtained from the velocity potential, which describes the induced velocities due to the cavities on the propeller blades. In order to derive this potential attention has to be paid to the description of propeller and cavity geometry.*

The scheme of notations and co-ordinate systems are given in Fig. 4.

![Fig. 4. Scheme of notation and co-ordinate system for the propeller, with diameter D, and cavity geometry.](image)

The propeller and cavity geometry are given in a cylindrical co-ordinate system \((r,X,\phi)\). The blade section shown in Fig. 4 will be characterized by its camberline, pitch angle \(\beta\) and chordlength \(C\). These are functions of \(r\).

The pitch angle

\[
\beta = \tan^{-1}P/2\pi r,
\]

where \(P\) is the pitch angle and \(r\) is the local radius.

*A detailed description of propeller geometry can be found in [19].
The blade section is considered locally two-dimensional and the flow relative to the section is assumed to be steady. The flow at a "large" distance of the section $U$ makes an angle $\alpha^*$ with the nose-tail line of the section. This is illustrated in Fig. 5.

**Fig. 5.** Schematic presentation of a cross section of a cavitating propeller blade. The cross section is placed in a uniform stream, its nose-tail line making an angle $\alpha^* \lt \rho$ with $U$.

$U$ and $\alpha^*$ are both functions of $r$ and $\phi$. $V_a$ is the axial component of $U$. The velocity component in rotational direction is set equal to $2\pi n r$; $n$ is the rate of rotation ($=\omega/2\pi$). $\alpha^*$ and $U$ are obtained by means of a propeller theory, either a lifting line or a lifting surface procedure applied for a given wake velocity field of the ship.

For a certain combination of $\alpha^*$, $U$ and depth of submergence of the blade section, the pressure at the suction side of the blade section attains to vapour pressure and cavitation appears, extending from the leading edge along the suction side. Either the cavity covers the section partially, this will be called partial cavitation, or extends behind the section, this will be called full or super cavitation. A measure for the occurrence of cavitation in similar circumstances is the cavitation index, defined as

$$\sigma^* = \frac{p_0 - p_{av}}{\frac{1}{2} \rho U^2} \quad (2.1)$$

where $p_0$ is the undisturbed pressure at the depth of submergence and $p_{av}$ is the vapour pressure. The critical value of $\sigma^*$ at which cavitation appears depends on the flow conditions and blade section geometry. For a description of these effects the reader is referred to [3] and [20].

The blade section is considered as a slender body. The angle of incidence $\alpha^*$ is small. For the flow around such a body Tulin [21] developed a linearized cavity theory. Geurst [11], [14] and [15] extended this theory to cambered foils, with the deviation of camberline to noise-tail line being small. He calculated the length of the cavity, area of the cavity and also lift, drag and moment as a function of the camberline function, $\alpha^*$ and $\sigma^*$. With help of the results of Geurst the cavity thickness on the blade as a function of the chordwise co-ordinate can be calculated as shown by Noordzij [11]. The expressions for cavity length $i$, cavity area $A$ and cavity thickness $t$ are given in Appendix A.

For each radial position on the blade, the distribution of $t$ can be calculated. This procedure has to be repeated for each desired blade position $\phi$. Hence the way $t$ evolves as a function of $\phi$ is found. The velocity potential is derived from $3\pi/36$ by means of a source proportional to $3\pi/36$ and not by a dipole as argued by Voros [17]. With a dipole no normal velocities of the cavity boundary relative to the blade can be found. A dipole distribution can be used for the displacement effect contributing to the velocity potential; this will not be discussed in detail in this paper. The reader is referred to, e.g. [18]. The expression for the velocity potential for the growing and collapsing cavity is a double integral over the potential of an infinitesimal source strength equal to

$$2\pi n \int_0^{\frac{\pi}{\phi}} r dr d\phi \frac{1}{\cos \beta} \quad (2.2)$$

as shown in [11].

Summing over all the blades we obtain for the potential $\phi$

$$\phi = \frac{n}{2} \int_0^{\frac{\pi}{\phi}} D/2 r^2 2\pi R^2 D \cos \beta d\Omega = \frac{n}{2} \int_0^{\pi} D \cos \beta d\Omega M = 1 \quad (2.3)$$

where $D_1$ is the distance between a point $A$ on the cavity (see Fig. 4) and the point which is considered. This distance

$$D_1 = (X - x)^2 + (Y - y)^2 + (Z - z)^2$$

with $Z$ is the number of blades, $M = 1(1)Z$ and $(R,X,Z)$ is the cylindrical co-ordinate system for a field point.

With help of Bernoulli's equation the pressure $p$ can be derived from the potential $\phi$. The pressure

$$p = -\frac{\rho}{2} \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \cdot \frac{V}{V_x} + \frac{\partial}{\partial y} \cdot \frac{V}{V_y} + \frac{\partial}{\partial z} \cdot \frac{V}{V_z} \right) \quad (2.5)$$

In the frame of the ship with a steadily rotating source this becomes, dropping the constant $\frac{1}{2} V^2$,

$$p = 2\pi n \int_0^{\frac{\pi}{\phi}} \frac{1}{2} (2X^2 + Y^2 + Z^2 + \frac{2X}{R} + \frac{2Y}{R}) \quad (2.6)$$

For practical circumstances the second term can
be neglected. Substitution of (2.3) into (2.6) yields

\[ p = \sum_{M=1}^{n} \frac{\beta_{2}}{\cos \theta_{2}} \int_{R_{x}}^{\frac{D_{2}}{2}} \int_{\frac{2\zeta_{1}}{D_{1}}}^{1} \frac{r dr d\phi}{\rho \sin \theta_{1}}. \tag{2.7} \]

Huse [5] and Noordzij [10] assumed \( D_{1} \) to be constant over the integration interval \((\theta_{1}, \theta_{2})\). In that case the integration in (2.3) or (2.7) over \((\theta_{1}, \theta_{2})\) can be carried through resulting into

\[ \phi = \sum_{M=1}^{n} \frac{D_{2}}{2} \int \frac{S}{D_{1}} dr, \tag{2.8} \]

where \( S \) is the sectional area of the cavity given in Appendix A. The error introduced by this approximation is of order \((R/\rho D_{1})\). \( \rho \) is a representative length on the blade. Obviously, only for those cases where \( \rho \) is very small with respect to the ratio \( D_{1}/R \), this order of magnitude is sufficiently smaller than unity. Otherwise, as mentioned in the introduction and illustrated in Fig. 3, the error due to this approximation is considerable. Taking a cavity near the propeller blade tip, \( r_{c}=D/2 \), \( D_{1} \) is almost independent of \( r \) and (2.8) can be further simplified to

\[ \phi = \frac{D_{1} \sqrt{\frac{2 \rho V_{0}}{\rho D_{1}}} \frac{\beta_{2}}{\cos \theta_{2}}}{D_{1}}. \tag{3.1} \]

with \( V_{0} \), the volume of the cavity on blade \( M \). This expression illustrates that the rate of change of the cavity volume determines the pressure field.

3. THE FORCE ON AN IDEALIZED HULL

The rate of change of the cavity volume depends on the propeller geometry and the wake field in which the propeller operates. We shall investigate how, for a given propeller, the resulting excitation force depends on the wake field.

The excitation force can be found by application of the Chertock formulas. This method was applied to a ship with a non-cavitating propeller by Voros ([17] and [18]). To illustrate the role of cavitation we represent the ship’s hull by an idealized hull as shown in Fig. 6.

The hull is represented by a flat plate extending from \( x=-\infty \) to \( x=+\infty \) with a width \( 2b \). For this idealized hull Breslin [22] presented the general expression for the force induced by a source \( S \), at radius \( r \), rotating at angular frequency \( 2\pi n \)

\[ F_{n}(\phi) = \rho 2 \pi n r^{2} \int_{0}^{\frac{2\pi}{\phi}} (S \int_{M=0}^{n} \cos \theta_{M}) \left( \frac{2}{D_{2}} \right) \frac{\beta_{2}}{\cos \theta_{2}} dr d\phi, \tag{3.2} \]

and sum over the blades, which gives

\[ F = \sum_{M=1}^{n} \frac{\beta_{2}}{\cos \theta_{2}} \frac{D_{2}}{2} \frac{\beta_{2}}{D_{1}} (\frac{2}{D_{2}}) \frac{\beta_{2}}{\cos \theta_{2}} \tag{3.3} \]

The force \( F \) is dimensionless with \( \frac{\rho \pi n^{2} a^{4}}{D_{2}} \), and obtain, adopting the same notation,

\[ F = \frac{\beta_{2} \frac{2}{D_{2}} \frac{\beta_{2}}{\cos \theta_{2}}}{\frac{D_{2}}{2} \frac{\beta_{2}}{D_{1}} (\frac{2}{D_{2}}) \frac{\beta_{2}}{\cos \theta_{2}}} \tag{3.4} \]

where

\[ \phi = \frac{\beta_{2}}{\frac{2}{D_{2}} \frac{\beta_{2}}{D_{1}} (\frac{2}{D_{2}}) \frac{\beta_{2}}{\cos \theta_{2}}} \tag{3.5} \]

We make the force \( F \) dimensionless with \( \frac{\rho \pi n^{2} a^{4}}{D_{2}} \), and obtain, adopting the same notation,

\[ F = \frac{\beta_{2} \frac{2}{D_{2}} \frac{\beta_{2}}{\cos \theta_{2}}}{\frac{D_{2}}{2} \frac{\beta_{2}}{D_{1}} (\frac{2}{D_{2}}) \frac{\beta_{2}}{\cos \theta_{2}}} \tag{3.6} \]

To illustrate the effect of cavitation we just consider \( m=0 \), the first term in the Fourier expansion in (3.1),

\[ F = C_{1} \frac{\beta_{2} \frac{2}{D_{2}} \frac{\beta_{2}}{\cos \theta_{2}}}{\frac{D_{2}}{2} \frac{\beta_{2}}{D_{1}} (\frac{2}{D_{2}}) \frac{\beta_{2}}{\cos \theta_{2}}} \tag{3.7} \]

with

\[ C_{1} = \frac{2}{(b^{2}+h^{2})^{\frac{1}{4}}-h} \tag{3.8} \]
For $S$ independent of $r$ (3.5) can be integrated yielding

$$F(\Omega) = C_1 \frac{\pi}{2} \frac{S_{\Omega}^2}{\Omega^2} \frac{Vol_M}{\pi^2}.$$  \hspace{1cm} (3.7)

The formula displays the role of the growth and collapse of the cavity attached to the propeller blade. The way $Vol_M$ varies with $\Omega$ depends on the wake of the ship. By measuring $Vol_M$ and $F$, an idea is obtained about the (integrated) effect of the wake.

Assuming the volume to be known as a periodic function of $\Omega$, we apply Fourier expansion and obtain

$$Vol_M = \sum_{k' = 0}^\infty E_{k'} \cos k' \Omega + \sum_{k' = 0}^\infty F_{k'} \sin k' \Omega. \hspace{1cm} (3.8)$$

Summing over the number of blades yields

$$= \sum_{k' = 0}^\infty \left( E_{k'} \cos k' \Omega + F_{k'} \sin k' \Omega \right). \hspace{1cm} (3.9)$$

Only those terms contribute in (3.9) for which $k' = k\tau, k = (1), \ldots$

This yields

$$Z \sum_{k = 0}^\infty \left( E_k \cos k \Omega + F_k \sin k \Omega \right). \hspace{1cm} (3.10)$$

Substitution in (3.7) gives

$$F = C_1 \sum_{k = 0}^\infty \left( E_k \cos k \Omega + F_k \sin k \Omega \right). \hspace{1cm} (3.11)$$

It can be observed that the amplitudes $E_k$ and $F_k$ are magnified by a factor $k^2$, indicating that higher harmonics can contribute to a large extent in the excitation force. This was also found in experiments, (23), for ships suffering from vibrations.

We shall now focus attention to equation (3.4) and derive an expression to illustrate the effect of the wake and propeller geometry. (3.4) can be integrated in chord-wise (0) direction and be written as

$$F = \frac{1}{2} \sum_{M = 1}^\infty \sum_{m = 0}^{2k} (s_m \cos m \Omega) + \sum_{M = 1}^\infty \sum_{m = 0}^{2k} (s_m \cos m \Omega) \frac{Vol_M}{\pi^2} + ms_M \sin m \Omega \frac{dVol_M}{d\Omega},$$  \hspace{1cm} (3.11)

with $s_M$ the sectional area (dimensionless) of the cavity at blade $M$. The first term in the right-hand side of (3.11) will be considered further. We assume the propeller blade to be partially covered with a cavity. Linearization of (A.6), with $a_1 = 0$, for small cavity length $k$ gives

$$k = 64 \left( \frac{a_1}{s_M} \right)^2,$$  \hspace{1cm} (3.12)

where $a_1 / s_M$ is assumed to be of the same order of magnitude as $s_M / s_M$. We set $a_1 = s_M$ and $s_M = \sigma_2$. For the cavity area $s_M$, on a blade section with chordlength $2C/D$ we obtain from linearization of (A.17) and using (3.12), dropping the asterisk,

$$s = \frac{5r^2}{32} \frac{C}{D} \sigma_2 (8a_1)^4.$$  \hspace{1cm} (3.13)

Substitution of (3.13) into (3.11) yields

$$F = C_2 \sum_{M = 1}^\infty \sum_{m = 0}^{2k} \frac{1}{2} (s_m \cos m \Omega \frac{1}{\pi^2} \cos m \Omega) \frac{dVol_M}{d\Omega},$$  \hspace{1cm} (3.14)

where

$$C_2 = \frac{5r^3}{64}.$$  \hspace{1cm} (3.15)

(3.13) shows that $s \neq 0$ for $a_1 \neq 0$. This is an oversimplification with respect to what is observed in experiments. Cavitation will occur for a certain critical value of $\sigma$ and $\Omega$ respectively. This can be simulated by multiplying the term between brackets in (3.14) with a gate function representing the range of angles in which the propeller blade cavitates. The range will be a function of $r$. For convenience we assume this range symmetrical with respect to $\Omega = 180^\circ$ and represent the gate function by its Fourier expansion

$$= \sum_{s = 0}^\infty I_s \cos s \Omega,$$  \hspace{1cm} (3.16)

Substitution in (3.14) yields

$$F = C_2 \sum_{M = 1}^\infty \sum_{m = 0}^{2k} \frac{1}{2} (s_m \cos m \Omega \frac{1}{\pi^2} \cos m \Omega) \frac{dVol_M}{d\Omega},$$  \hspace{1cm} (3.17)

We take for $\Omega$ the value as defined in Fig. 5.

$$\alpha = 2 \tan^{-1} \frac{V_a(r, \Omega)}{2\pi \Omega}$$  \hspace{1cm} (3.18)

The cavitation number

$$\eta(r, \Omega) = \frac{P_a + pgh - p_g \cos \Omega \frac{1}{\pi^2}}{\mu U^2}.$$  \hspace{1cm} (3.19)

$P_a$ is the pressure at the free surface and $g$ is the gravity force per unit mass. Now we introduce $\sigma_0$, the cavitation number defined at propeller shaft submergence, based on the rotation rate

$$\sigma_0 = \frac{P_a + pgh - P_g}{\frac{1}{2} \rho (U^2)}.$$  \hspace{1cm} (3.20)

Substitution of (3.18) into (3.17) gives

$$\eta = \frac{P_a + pgh - P_g}{\frac{1}{2} \rho (U^2)}.$$  \hspace{1cm} (3.21)
\[ c(r, \phi) = \frac{a^2 g r \cos \phi}{(nD)^2} \]  

(3.19)

\[ U, \text{ defined in Fig. 5, is} \]

\[ U = \left( \frac{(2\pi n)^2 V_a^2}{r} \right)^{\frac{1}{2}} \]  

(3.20)

Considering \( V_a / 2\pi n < 1 \) and \( \sigma_a (nD)^2 / 2gr \cos \phi \gg 1 \), we obtain from (3.15), with help of (3.16), (3.19) and (3.20):

\[ F = \frac{C_C}{r} \sum_{m=0}^{\infty} \cos m \phi \cos \phi M \cos M \]

(3.21)

where

\[ G_m(r, \phi, M) = \left( \frac{\sigma_a \cos \phi}{(nD)^2} + 3J \left( \frac{D}{2n} \right)^2 \frac{V_a^2}{V_a^2} \right) \]

(1-\( \frac{D}{2n} \)) \[ V_a^2 \]

(3.22)

and \( J = \frac{V}{nD} \) is the coefficient of advance. \( P/D \) is called the pitch ratio. \( V_a \), the axial component of \( U \), is assumed to be known and will be represented in a Fourier series of which henceforth the symmetrical contribution is considered

\[ \left( \frac{V_a}{V} \right) = \sum_{k=0}^{\infty} A_k \cos k \phi \]  

(3.23)

Substitution of (3.23) into (3.21), gives rise to a complicated expression which will not be shown here. It will appear that no straightforward conclusion can be drawn to which extent the various harmonic components of \( V_a / V \) contribute to the total force due to

(i) the non-linear dependence of \( G \) on \( (V_a / V) \),

(ii) the gate function for cavitation and (iii) the rotation of the source. To illustrate this we look at the coefficients of the first harmonic \( (2\phi) \) and second harmonic \( (2\phi) \) of \( F \). To this end we collect first some terms contributing to the coefficient of \( \cos 2\phi \):

\[ 2 \pi \left( \frac{D}{2n} \right)^2 \sum_{k=0}^{\infty} A_k B_k \frac{1}{2n} \left( \frac{2\pi n}{2n} \right)^2 \sum_{m=0}^{\infty} \frac{1}{(nD)^2} \frac{B_m}{B_m} \frac{Z_m}{Z_m^{1-1}} \]

(3.24)

With help of (3.24) and (3.25) assuming different gate functions and wake field velocity distributions, for example, the number of wake fields harmonics which have to be accounted for in excitation calculation procedures can be determined. Also the effect of propeller geometry can be investigated.

4. COMPARISON OF THE HULL PRESSURE OF A CAVITATING AND A NON-CAVITATING PROPELLER.

The pressure field due to a collapsing cavity is described by means of the velocity potential of sources. The effect of displacement and of loading can be described by means of dipole and vortex distributions. In this section we shall investigate the phase relation of the induced pressure on the hull for a cavitating as well as for a non-cavitation propeller.

We consider the following idealized problem - for the growing and collapsing cavity, rotating at radius \( r \), we take a source of strength \( 2\pi n \sigma_0 V \phi \),

- for the displacement effect of the propeller blade we take a dipole, rotating at the same radius \( r \), with strength \( 2\pi n A (1 - \cos \phi) \).

The coefficient \( \sigma \) represents the effect of non-homogeneous flow. \( A \) is a representative cross sectional area of the blade. (The effect of loading will result in an expression for the induced pressure very similar to that for the displacement. The loading, therefore, will not be considered further). The pressure will be calculated, see Fig. 7, on a solid boundary at a distance \( R \) from the axis of rotation. The solid boundary extends to infinity.

For simplicity we calculate the pressure at \( X=0 \). The distance between the source or dipole and a point on the solid boundary

\[ D_2 = \left( x^2 + r^2 - 2rx \cos (\phi - \theta) \right)^{\frac{1}{2}} \]  

(4.1)

In the expression for the potential of a source and dipole we have powers of \( 1/D_2 \). \( 1/D_2 \) can be written in terms of Bessel functions of
Fig. 7. Scheme of notation for a source or dipole at radius \( r \) at a distance \( \theta_2 \) from the point on the solid boundary.

For the source, Morse and Feshbach [241, 242],

\[
\frac{1}{D_2} = \sum_{m=0}^{\infty} m e_m \cos(m-\theta) \int_0^1 r J_m(\rho r) \, d\rho,
\]

(4.2)

where \( e_0 = 1 \) and \( e_m = 2, \ m = 1, 2, \ldots \). (4.2) can be written as

\[
\frac{1}{D_2} = \sum_{m=0}^{\infty} m \frac{1}{n/\sqrt{R}} Q_{m-\frac{1}{2}} r^2 \cos(m-\theta),
\]

(4.3)

\( Q_{m-\frac{1}{2}} \) is a Legendre function.

The potential for the source is, without solid boundary,

\[
\phi_s = \frac{2}{3} \frac{\text{Vol} \ 1}{D_2}.
\]

(4.4)

The potential for the dipole

\[
\phi_d = -\frac{\pi r^2 A(1 - \sin^2 \theta)}{D_2}.
\]

(4.5)

The pressure for a source or dipole rotating with angular frequency \( \omega (= 2 \pi n) \)

\[
P = \frac{2 \pi}{n} \frac{\omega}{n}.
\]

(4.6)

For the source we have from (4.3), (4.4) and (4.6)

\[
P_s = -\frac{\pi \eta^2}{6} \frac{2}{3} \frac{\text{Vol} \ m_0}{r^2} \sum_{m=0}^{\infty} \frac{1}{n/\sqrt{R}} Q_{m-\frac{1}{2}} \cos(m-\theta) + \frac{1}{n/\sqrt{R}} Q_{m-1} \sin(m-\theta).
\]

(4.7)

For the dipole we obtain

\[
P_d = \frac{\pi \eta^2}{6} \frac{2}{3} \frac{\text{Vol} \ m_0}{n/\sqrt{R}} Q_{m-1} \sin(m-\theta) + \frac{\pi \eta^2}{6} \frac{2}{3} \frac{\text{Vol} \ m_0}{n/\sqrt{R}} Q_{m-1} \cos(m-\theta).
\]

(4.8)

\( Q_{m-\frac{1}{2}} \) can be written in a series expansion. The expansion is given in Appendix B.

The leading term in the series expansion for \( p_s \), \( m = 1 \) equation (B.5), is

\[
P_s = -\frac{\pi \eta^2}{6} \frac{2}{3} \frac{\text{Vol} \ 1}{r^2} \frac{1}{(r^2 + \rho^2)^{3/2}} \frac{1}{(r^2 + \rho^2)^{1/2}} + O \left( \frac{1}{r^2 + \rho^2} \right)^4.
\]

(4.9)

Taking \( r/R < 1 \) and expanding (4.9) in powers of \( r/R \)

\[
P_s = -\frac{\pi \eta^2}{6} \frac{2}{3} \frac{\text{Vol} \ 1}{r^2} \left( \frac{1}{(1 + \frac{1}{2}r/R + \frac{1}{2}r/R)^{1/2}} \right.
\]

\[+ O \left( \frac{1}{r^2 + \rho^2} \right)^4 \right) \sin(m-\theta) + \frac{1}{n/\sqrt{R}} Q_{m-1} \cos(m-\theta).\]

(4.10)

The leading term in the series expansion for \( p_d \), \( m = 1 \) equation (B.6), is

\[
P_d = \frac{\pi \eta^2}{6} \frac{2}{3} \frac{\text{Vol} \ 1}{r^2} \frac{1}{n/\sqrt{R}} Q_{m-1} \sin(m-\theta) + \frac{1}{n/\sqrt{R}} Q_{m-1} \cos(m-\theta).\]

(4.11)

Again taking \( r/R < 1 \) and expanding (4.11) in powers of \( r/R \)

\[
P_d = \frac{\pi \eta^2}{6} \frac{2}{3} \frac{\text{Vol} \ 1}{r^2} \frac{1}{n/\sqrt{R}} Q_{m-1} \sin(m-\theta) + \frac{1}{n/\sqrt{R}} Q_{m-1} \cos(m-\theta).
\]

(4.12)

To obtain the pressure at the solid boundary both \( p_s \) and \( p_d \) have to be multiplied by \( 2 \). By integrating these over a specific area, the total force on that area is found. The force due to the source can be larger since \( p_s \) does not depend on \( \epsilon \), whereas \( p_d \) does. For a given \( n \) all values of \( p_s \) at various locations on the hull have the same phase. The values of \( p_d \) do not have the same phase. This was experimentally observed by Jonk and van der Kooij [26]. Furthermore it can be concluded that for a fixed source and dipole strength \( p_s \) shows a weak dependence on \( r/R \), whereas \( p_d \) shows a much stronger dependence.

5. CONSIDERATION ON THE RESULTS OF FREE STREAMLINE THEORY

For given flow properties and for given blade section geometry, characterized by \( (a^*, b^*) \) and camberline function respectively, the cavity extent along the blade section and cavity cross-sectional area can be found with help of the free streamline theory ([13] and [21]). In Appendix A the expressions for cavity length and cavity area are given for partial as well as super cavitation.

In this section we shall focus attention to the behaviour of cavity length \( l \) as a function of \( a^*/b^* \). Furthermore the dependence of cavity...
length and cavity area $s$, respectively, on $a^2$ is considered. For this the expressions as derived in Appendix A will be used.

In Fig. 8, cavity length $l$ as a function of $a^2/a^*$ is shown for an angle of incidence $a^* = 0.25^\circ$, $a^* = 2^\circ$ and a value of the camber of $0.02$.

![Fig. 8. Cavity length $l$ as a function of $a^2/a^*$ for $a^* = 0.25^\circ$, $a^* = 2^\circ$ and camber $= 0.02$.](image)

In Fig. 9, cavity area $s$ as a function of cavity length $l$ for $a^* = 0.25^\circ$, $a^* = 2^\circ$ and camber $= 0.02$ is shown.

![Fig. 9. Cavity area $s$ as a function of cavity length $l$ for $a^* = 0.25^\circ$, $a^* = 2^\circ$ and camber $= 0.02$.](image)
6. SOME RESULTS OF AN IMPROVED MODEL FOR CAVITY GEOMETRY.

An improved model for cavity geometry is adopted in order to obtain a better fit with the experimental results for cavity length. Looking at experimental results of Meijer [27] for a partially cavitating hydrofoil and at the results reported by Hanaoka [28] for partial as well as super cavitation a continuous curve for \( \frac{\lambda}{\theta} \) could be expected. With help of a curve fitting procedure for the experimental results a better approximation for \( \lambda \) as a function of \( \frac{\alpha^p}{\alpha} \) can be obtained. In this section we investigate the effect of this curve fitting on the theoretical model as used by Geurst. Curve fitting was applied by Hanaoka [28] who tried to fill up the "gap" between the theory for partial cavitation and super cavitation. This was done by relaxing the closure condition for the cavity, i.e. a cavity with a width at the rear unequal zero (open cavity) was adopted. The cavity width was adjusted in such a way that the theoretical value of the cavity length \( \lambda \) is close to the experimental values and a continuous single-valued function for \( \lambda \) is obtained. The results of this procedure are illustrated in Fig. 10. for a cavitating flat plate under an angle of incidence \( \alpha \).

In this figure the improved function for \( \lambda \) is given (solid line). The dashed curve represents \( \lambda \) obtained from the theory of Geurst (closed cavity). The open cavity model is a reasonable approximation of what is observed in experiments. By assuming an open cavity in the theory of Geurst values of \( \alpha_1 \), in (A.6) and (A.19), unequal zero are introduced. \( \alpha_0 \) (open cavity) means mathematically that a source is placed in the wake of the cavity. The same procedure as used by Hanaoka will be followed: the source strength \( \alpha_1 \) is adjusted such that the theoretical value of \( \lambda \) is close to experiments.

First we look at the open partial cavity. The open cavity is schematically shown in Fig. 11.

In Appendix C the expressions for \( \alpha_0 \) and the source strength \( \alpha_1 \) are given which are obtained by application of the above mentioned procedure. No attempts has been made to find an optimal fit with the experiments. Hence, the derived functions can be improved in this respect. This is beyond the scope of this paper. In Fig. 12 \( \lambda \) as a function of \( \frac{\alpha^p}{\alpha} \) is given with \( \alpha =2^\circ \) and camber equal to .02. Some experimental results of Meijer are also shown.

![Fig. 10](image1)

![Fig. 11](image2)

![Fig. 12](image3)
For completeness, in Fig. 12 resulting from the closed cavity model is presented. As mentioned in Appendix C for \( \alpha^* \), \( \alpha \) would fall below zero. This would lead to a negative cavity width \( \tau_p \). To avoid this \( \alpha \) is set equal to zero for these values of \( \tau \) and the solutions for the closed cavity should be used. In Appendix C the expression for cavity area \( S \) is also presented.

In Fig. 13 \( 2S/\alpha^* \) is given for \( \alpha^* = 0.25 \), \( \alpha^* = 2 \) and camber is 0.02.

The singular behaviour near \( \tau = 1 \) disappears, at least for not too small values of \( \alpha^* \). For small values of \( \alpha^* \) \( S \) becomes negative. This behaviour could be suppressed by placing a dipole in the wake of the cavity and adjusting the dipole strength such that \( S \) would be close to experiments. With no such experiments available this procedure can not be conducted. For \( \alpha^* = 0 \) the resulting expression for cavity length describes a dependence of \( \tau \) on \( \alpha^* \) which is not correct from a physical point of view. The expression predicts an increasing cavity length for increasing cavitation number. So we conclude for \( \alpha^* = 0 \) no partial cavity can be predicted.

Now we look at super cavitation. The open cavity is schematically shown in Fig. 14.

The expressions for \( \alpha/\rho \) and source strength \( \alpha \) are given in Appendix C. The source strength is adjusted such that the function for \( \tau \) is a continuous function at \( \tau = 1 \) (pass into the solution for the open partial cavity) and tends to the solution for the closed cavity for \( \tau \to \infty \) (at least for zero camber). Again no optimal curve fitting was applied. In Fig. 15 \( \tau \) as a function of \( \alpha^*/\alpha^* \) is given for \( \alpha^* = 2 \) and camber is 0.02.

In the figure the solution for the closed cavity is also shown (dashed curve). In Appendix C the expression for cavity area \( S \) is derived. In Fig. 16 \( 2S/\alpha^* \) is given for \( \alpha^* = 0.25 \), \( \alpha^* = 2 \) and camber is 0.02.

For completeness the solution for the closed cavity is given (dashed curve). As shown in Appendix C, cavity area of the open cavity is a continuous function at \( \tau = 1 \). However, \( S \) can still attain negative values for small values of \( \alpha^* \). Furthermore for \( \alpha^* > 0 \) near unity \( S \) decreases with increasing \( \alpha^* \). Beyond a certain value of \( \alpha^* \) (depending on magnitude of camber and \( \alpha^* \)) \( S \) will increase with \( \tau \). The latter behaviour of \( S \), as a function of \( \tau \), is what we expect it to be. For \( \alpha^* = 0 \) an expression for cavity length as a function of \( \alpha^* \) and camber is obtained which gives for \( \tau > 1.6 \) positive values for \( S \). Furthermore \( S \) and \( \tau \) increase with decreasing \( \alpha^* \). So we conclude for \( \alpha^* = 0 \) only super cavitation will occur for low enough values of \( \alpha^* (0 < \alpha^* < 0.41) \).

In Fig. 16 a part of the curve for the open partial cavity is also given. It can be observed that \( 2S/\alpha^* \) is a discontinuous function at \( \tau = 1 \). The requirement of continuous derivative can be fulfilled by placing a dipole in the wake of the cavity.
7. SOURCE LOCATED AT A SOLID WALL

In the calculation procedure for the pressure fluctuations induced by a cavitating propeller, the source describing the growth and collapse of the cavity is treated as a free source. The source, however, is located on the propeller blade. In order to find out whether or not the approximation by a free source, i.e. no effect of the propeller, is justified, we shall estimate the error which is introduced by the neglect of the effect of the propeller blade.

The following approximations are made. The propeller blade is considered to be a 2-dimensional strip of width C (See Fig. 17). The cavity is represented by a line source of strength m, located at $z_0$. Usually the source is located very near the strip. A point in the z-plane is characterized by the radius $|z|$ and angle $\delta$.

The expression for the complex potential, which describes the flow around the strip due to the source can be derived from the complex potential for the flow around an elliptical cylinder due to a source, see Milne-Thomson [29]. The complex potential

$$\phi_1 = m \log z - m \log (z - z_0) - m \log (z^2 - a^2)$$

where

$$a = C/2,$$

$$z = \frac{1}{2} (z + (z^2 - a^2)^{1/2}),$$

and

$$z_0 = \frac{1}{2} (z_0 + (z_0^2 - a^2)^{1/2}).$$

$z_0$ is the complex conjugate of $z_0$.

Without the strip the complex potential is

$$\phi_2 = m \log (z - z_0).$$

We shall look at some special values of $\phi_1$.

When $z_0$ is real and greater a (7.1) yields

$$\phi_1 = m \log (z - z_0).$$

Hence, the potential due to a source located on the real axis outside the strip is equal to the potential for the flow without the strip.

When $z_0 = a$, we have

$$\phi_1 = m \log (z - a).$$

For $z_0$ real and smaller a (7.1) becomes,

$$\phi_1 = m \log \left(\frac{1}{2} (z + (z^2 - a^2)^{1/2}) + \frac{1}{2} (z_0 + (z_0^2 - a^2)^{1/2})\right).$$

To estimate the effect of the strip the ratio of the pressure resulting from (7.6) and (7.5) will be calculated for $z/|z| > a$.

First we set $z_0 = 0$, i.e. a source in the centre of the strip. Then the ratio becomes

$$\frac{1}{1 + \frac{2a}{z^2}} \sin \delta.$$
Hence, for \( y > 0 \), the effect of the strip is an augmentation with respect to the pressure of the free source. A practical value of \( \frac{2a}{L} \) is about \( \frac{1}{4} \). So the effect of the strip is non-negligible. For most practical circumstances, however, the source is located near the edge of the strip. To estimate the augmentation for this case we set \( a_s = a - d \) and \( d/a << 1 \). Then the ratio is

\[
\frac{1}{2} \frac{2a}{L} \left( 1 - \left( \frac{d^2}{a} \right)^{1/2} \right) \sin \theta.
\]

This leads to a much smaller augmentation than the one given by (7.7).

We conclude, considering the approximations, that for cavities distributed along the leading edge of the propeller blade, the effect of the propeller is negligible.

8. CONCLUDING REMARKS

Some remarks can be made about future work. An improvement can be introduced in the free streamline theory when three-dimensional effects are taken into account. This can be done in a way as proposed by Fahner [30] or Tulin and Heu [31]. The latter introduced also the effect of thickness of the foil; especially the effect of round-nose foils. In this way an improved approximation for cavity geometry can be obtained and consequently the excitation force can be calculated more accurately.

REFERENCES

APPENDIX A

PARTIAL CAVITATION

In Fig. 18 the partial cavity with length 2l on a blade section of chordlength 2 is schematically represented. The camberline function f is approximated by a parabolic arc

\[ f(x) \equiv \lambda^1(1-x^2). \]  

(A.1)

The camber is \( \lambda^{1/2} \).

For the flow as defined in Fig. 18 Geurst and Verbrugh [15] derived a complex velocity function \( w(z) \) in the \( z=(x+iy) \)-plane. \( u \) is the component of the velocity parallel to the \( x \)-axis and \( v \) the velocity parallel to the \( y \)-axis. At infinity

\[ v = \frac{\sin \alpha}{(1-\beta^2)^{1/2}} = \alpha, \]  

(A.2)

and

\[ u = \frac{\cos \alpha}{(1+\beta^2)^{1/2}} - 1 = -\frac{\alpha}{2}. \]  

(A.3)

By expanding this complex function \( w \) in powers of \( \frac{1}{z} \) near \( z=\infty \) and applying the boundary conditions, expressions for \( \alpha \) and cavity area \( S \) are found. Adopting the notation of [15] the expansion can be written as

\[ w = a_0 + ib_0 + \frac{a_1 + ib_1}{z} + \frac{a_2 + ib_2}{z^2} + \cdots \]  

(A.4)

From the boundary condition at \( z=\infty \) it follows that

\[ a_0 = \frac{\alpha}{2}, \]  

and

\[ b_0 = -\frac{\alpha}{2}. \]  

(A.5)

Equating the other coefficients of (A.4) to those of the expansion of \( w(z) \), \( a_1, b_1, \) etc. are found as function of \( \alpha \) for given \( \alpha \) and camberline function. For \( a_1 \), see [15],

\[ 4a_1 = 2a(1+\sin \frac{\alpha}{2})\cos \frac{\alpha}{2} - \frac{\alpha}{2} \left( 1-\sin \frac{\alpha}{2} \right) \sin \frac{\alpha}{2} + \]  

\[ + 2 \lambda^1 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}, \]  

(A.6)

with

\[ \lambda = \cos^2 \frac{\gamma}{2}. \]  

(A.7)

\[ a_1 \] follows from the closure condition. This will be discussed below.

The thickness of the cavity

\[ \gamma = g - f, \]  

(A.8)
where \( \frac{dg}{dx} = -\text{Im}(w) \) at \( y = +0 \).

(A.8)

With this \( \tau \) becomes

\[
\tau = \int \frac{X}{\text{Im}(w)} \, dx^1 - \lambda^1(1-x^2) ; \quad y = +0.
\]

(A.9)

The thickness of the cavity can be obtained numerically with help of the expression for \( w(z) \) from [15]. This is shown by Noordzij [11].

On the cavity (\( y = +0 \))

\[
v^* = \frac{dg}{dx}.
\]

and on the wetted part of the profile (\( y = +0 \) as well as \( y = -0 \))

\[
v^- = \frac{df}{dx}.
\]

Hence we can write for \( \tau \)

\[
\tau = \int \frac{v^*}{\text{Im}(w)} \, dx^1 - \int \frac{v^-}{\text{Im}(w)} \, dx^1.
\]

(A.10)

For \( x > 2l-1 \) : \( v^* = v^- \).

Setting \( \tau = \tau_p \) at \( x = 2l-1 \), \( \tau_p \) becomes

\[
\tau_p = \int \frac{1}{(v^* - v^-)} \, dx^1.
\]

(A.11)

By taking a contour \( c_1 \) in the \( z \)-plane, as indicated in Fig. 19,

we have

\[
\oint v \, dz = \tau_p
\]

or

\[
\text{Im} \oint v \, dz = \tau_p.
\]

(A.12)

The theory of residues states that

\[
\oint v \, dz = 2\pi i \sum \text{residues},
\]

in the interior of \( c_1 \). Following Gearst, we consider the region exterior to \( c_1 \), including the point at infinity as the region of application for the theory of residues.

Then

\[
\oint v \, dz = -2\pi i \text{Im}(w)_{z=\infty}.
\]

(A.13)

With (A.4) this yields

\[
\text{Im}(-2\pi i \text{resw})_{z=\infty} = 2\pi a_1.
\]

Finally we obtain for \( \tau_p \) with (A.12) and (A.13)

\[
\tau_p = 2\pi a_1.
\]

Now we use an closure condition: \( \tau_p = 0 \), hence \( a_1 = 0 \). Inserting this in (A.6) we have an expression for \( \lambda \) as a function of \( \alpha \), \( \sigma \) and \( \lambda^1 \).

The area of the cavity

\[
S = \int (g-f) \, dx^1 = \tau_p (2l-1) + \int x^1(v^* - v^-) \, dx^1.
\]

(A.14)

This can be written as

\[
S = \tau_p (2l-1) - \text{Im} \oint v \, dz.
\]

(A.15)

Applying the expansion (A.4) and the theorem of residues to the region outside the contour gives

\[
S = \tau_p (2l-1) - 2\pi a_2.
\]

(A.16)

Equating the coefficient \( a_2 \) to the coefficient in the expansion of \( w(z) \) as given in [15] yields for the dimensionless cavity area

\[
S = \frac{\tau_p (2l-1)}{16} \left( 2x(1+\sin^2 \frac{\lambda}{2})(1-3\sin^2 \frac{\lambda}{2}) \sin \frac{\lambda}{2} \cos \frac{\lambda}{2} + \right.
\]

\[
+ \frac{3}{2} (1-\sin^2 \frac{\lambda}{2}) (-1-3\sin^2 \frac{\lambda}{2} + 2\sin ^2 \frac{\lambda}{2} + 6\sin^3 \frac{\lambda}{2})
\]

\[
+ 2\lambda^1 \cos \frac{\lambda}{2} (1+\sin^2 \frac{\lambda}{2} - 4\sin^3 \frac{\lambda}{2}) \right).
\]

(A.17)

SUPER CAVITATION

In Fig. 20 the super cavity with length \( \lambda \) on a blade section of chordlength 1 is schematically represented.

Fig. 20. Schematic representation of a cross section of a fully cavitating propeller blade with chordlength 1. The cross section is placed in a uniform stream \( U \) making an angle \( \alpha \) with \( U \).
The camberline function \( f \) is approximated by a parabolic arc
\[
f(x) = \lambda x(1-x). \tag{A.18}
\]
The camber is \( \lambda/4 \).

Completely analogous to the procedure for the partial cavity, we have for \( a_1 \) in the case of supercavitation, using the results of Geurst [14] and Noordzij and Officier [16]
\[
8a_1 \cos^2 \frac{\theta}{2} = \lambda \sin^2 \frac{\lambda}{2} + 2 \cos \frac{\lambda}{2} (1-\sin^2 \frac{\lambda}{2}) + \sin^2 \frac{\lambda}{2} (1-\sin^2 \frac{\lambda}{2}), \tag{A.19}
\]
with
\[
a = 1/\cos^2 \frac{\theta}{2}. \tag{A.20}
\]

Again \( a_1 \) follows from the closure condition.

For the thickness \( \tau \) we have
\[
\tau = \int_0^1 \frac{dx}{1-\lambda x(1-x)}, \quad 0 < x < 1,
\]
and
\[
\tau = \int_0^1 \frac{dx}{1-\lambda x(1-x)}, \quad 1 < x < 1. \tag{A.21}
\]

With help of the complex velocity \( w \) the thickness \( \tau_n \) of the cavity at the rear becomes
\[
\tau_n = \text{Im} \int \frac{w}{dz}, \tag{A.22}
\]
where the contour of integration \( C_2 \) is given in Fig. 21.

[Fig. 21. Contour \( C_2 \) in the complex \( z \)-plane.]

Then the theorem of residues applied to the region outside \( C_2 \) yields
\[
\tau_n = 2na_1. \tag{A.23}
\]

For the closed cavity \( a_1 = 0 \) and (A.19) represents a relation between \( \ell, \alpha, \sigma \) and \( \lambda \) for the supercavity.

The area of the cavity
\[
S = \tau a - 2na_2. \tag{A.24}
\]

With the closure condition \( \tau_n = 0 \) this becomes
\[
S = -2na_2. \tag{A.25}
\]

\( a_2 \) can be obtained from Geurst [14] and Noordzij and Officier [16]. The resulting expression for \( S \) is
\[
S = \frac{\tau a_0}{16} \csc^2 \frac{\theta}{2} + \frac{1}{32} \cos^2 \frac{\lambda}{2} (1-\sin^2 \frac{\lambda}{2}). \tag{A.26}
\]

APPENDIX B

The Legendre function \( Q_n(z) \), for \(|z| < 1\), can be written as series expansion, see Abramowitz and Stegun [25],
\[
Q_n(z) = \frac{\Gamma(n+1)}{2^{n+1} \Gamma(n+\frac{1}{2})} \left( \frac{1}{z} \right)^{n+1} \sum_{j=2}^{n+1} \frac{P(j+\frac{1}{2}, j+\frac{1}{2}, \nu, \frac{1}{2}, \nu, \frac{1}{2})}{z^j}.
\]

(\( B.1 \))

where \( \Gamma \) is the Gamma function and \( P \) the hypergeometric function
\[
P(a,b,c;z) = \frac{\Gamma(c) \Gamma(a+b-c)}{\Gamma(a) \Gamma(b)} \sum_{j=0}^{n} \frac{\Gamma(a+n) \Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.
\]

(\( B.2 \))

Expanding \( Q \) in powers of \( \frac{1}{z} \) we find by substituting (\( B.2 \)) into (\( B.1 \)),
\[
Q_n = \frac{\Gamma(n+\frac{1}{2})}{2^{n+1} \Gamma(n+\nu+\frac{1}{2})} \left( \frac{1}{z} \right)^{n+1} \sum_{j=2}^{n+1} \frac{P(j+\frac{1}{2}, j+\frac{1}{2}, \nu, \frac{1}{2}, \nu, \frac{1}{2})}{z^j} + \ldots.
\]

(\( B.3 \))

with \( v = m+\frac{1}{2} \) (\( B.3 \)) becomes
\[
Q_{m+\frac{1}{2}} = \frac{\Gamma(m+\frac{1}{2})}{2^{m+1} \Gamma(m+1)} \left( \frac{1}{z} \right)^{m+1} \sum_{j=2}^{m+1} \frac{P(j+\frac{1}{2}, j+\frac{1}{2}, m+\frac{1}{2})}{z^j} + \ldots.
\]

(\( B.4 \))

For \( m = 0 \)
\[
Q_{\frac{1}{2}} = \frac{\pi}{\sqrt{2} \sqrt{\pi}} \left( \frac{1}{z} \right)^{\frac{3}{2}} + \ldots,
\]
and \( m = 1 \)
\[
Q_1 = \frac{\pi}{4\sqrt{2} \sqrt{\pi}} \left( \frac{1}{z} \right)^{\frac{5}{2}} + \ldots.
\]

(\( B.5 \))

(\( B.6 \))

APPENDIX C

(For the scheme of notations : see Appendix A).

OPEN PARTIAL CAVITY

The expression for \( \ell \) as a function of \( a, \sigma \) and \( \lambda \) is
\[
\ell = \frac{5a}{4} \cos \frac{\lambda}{2} (1-\sin^2 \frac{\lambda}{2}), \tag{C.1}
\]
where

\[16\]
From (A.6) and (C.1) \( a_1 \) becomes

\[
a_1 = \frac{1}{2} a (1 - \sin \frac{\gamma}{2} - \frac{3}{10} \cos \frac{\gamma}{2} (2 - \sin \frac{\gamma}{2} (1 + \sin \frac{\gamma}{2})))
\]

\[
+ \frac{1}{5} \sin \frac{\gamma}{2} \cos \frac{\gamma}{2} (\cos \frac{\gamma}{2} - \frac{3}{5} \cos \frac{\gamma}{2} + \frac{1}{5} \cos^2 \frac{\gamma}{2} \sin \frac{\gamma}{2} - \frac{1}{5} \sin \frac{\gamma}{2}).
\]

(C.2)

For \( t = 1 \) \((\gamma = 0)\): \( \frac{a}{\alpha} = \frac{1}{5} \alpha^{-1} \),

(C.3)

and \( a_1 = a / 2 \).

When \( t \to 0 \), \( a_1 \to 0 \).

The dimensionless cavity area for a foil of chordlength 2 is

\[
\frac{25}{\alpha^2} (2 \cos \frac{\gamma}{2} - 1) a_1 / \alpha + \frac{1}{5} \left( \frac{1}{2} \cos \frac{\gamma}{2} (1 - \cos \frac{\gamma}{2} \sin^2 \frac{\gamma}{2})
\]

\[
+ \frac{3}{5} \sin \frac{\gamma}{2} \cos \frac{\gamma}{2} (\cos \frac{\gamma}{2} - \frac{3}{5} \cos \frac{\gamma}{2} + \frac{1}{5} \cos^2 \frac{\gamma}{2} \sin \frac{\gamma}{2} - \frac{1}{5} \sin \frac{\gamma}{2})
\]

\[
+ \frac{1}{10} \sin \frac{\gamma}{2} \cos \frac{\gamma}{2} \left( \sin \frac{\gamma}{2} - \frac{1}{10} \cos \frac{\gamma}{2} \sin \frac{\gamma}{2} \right).
\]

(C.4)

For \( t \to 0 \), \( S \to 0 \). The other limit \( t \to 1 \) gives

\[
\frac{25}{\alpha^2} = \frac{9}{16} \left( \frac{1}{2} - \frac{1}{2} \right).
\]

(C.5)

OPEN SUPER CAVITY

The expression for \( t \) as a function of \( a, \phi \) and \( \lambda \) is

\[
a \frac{\phi}{\alpha} = \frac{5}{2} \alpha \frac{H(\phi)}{10 \alpha + \lambda (\cos \frac{\gamma}{2} + \frac{1}{2} \cos \frac{\gamma}{2} \sin \frac{\gamma}{2})} \cos \frac{\gamma}{2},
\]

where

\[
H = 1 - \frac{3}{10} \cos \frac{\gamma}{2} (1 - \sin \frac{\gamma}{2}).
\]

From (A.19) and (C.6) \( a_1 \) becomes

\[
a_1 = \frac{1}{4 \cos \frac{\gamma}{2} H} \left( \left( \frac{1}{2} \sin \frac{\gamma}{2} \right) \right.
\]

\[
+ \frac{1}{10} \sin \frac{\gamma}{2} \cos \frac{\gamma}{2} \left( \sin \frac{\gamma}{2} - \frac{1}{10} \cos \frac{\gamma}{2} \sin \frac{\gamma}{2} \right) (1 - \cos \frac{\gamma}{2})).
\]

(C.7)

For \( t = 1 \)

\[
a \frac{\phi}{\alpha} = \frac{a}{10 \alpha + \lambda},
\]

(C.8)

and

\[
a_1 = a / 4.
\]

(C.9)

For \( t = \infty \), \( a_1 = 0 \). Without any further experimental evidence on conditions for large \( t \) we put on for this value of \( a_1 \).

The dimensionless cavity area \( S \) becomes

\[
\frac{25}{\alpha^2} = \frac{4 a_1}{4 \cos \frac{\gamma}{2} H} + \frac{1}{4 \cos \frac{\gamma}{2} H} \left( -1 - 3 \sin \frac{\gamma}{2} + \sin \frac{\gamma}{2} \right)
\]

\[
+ 5 \sin \frac{3 \gamma}{2} - 2 \sin \frac{5 \gamma}{2} + \frac{1}{8 \cos \frac{\gamma}{2} H} \left( 2 \cos \frac{\gamma}{2} - 2 \sin \frac{\gamma}{2} \right)
\]

\[
+ \frac{1}{4 \alpha} \left( 1 + 4 \sin \frac{\gamma}{2} - 2 \sin \frac{3 \gamma}{2} \right).
\]

(C.10)

For \( t = 1 \)

\[
\frac{25}{\alpha^2} = \frac{9}{8} - \frac{1}{16 \alpha}.
\]

(C.11)

Comparing \( S \) given in (C.5) with \( S \) given in (C.11), we see that cavity area is a continuous function keeping in mind that (i) \( S \) of (C.5) has to be divided by a factor 4 and (ii) \( \lambda^1 = \lambda / 2 \).

\( a / c \) from (C.8) equals \( a / c \) from (C.3) taking into account that the expression for the partial cavity is found for a hydrofoil of chordlength 2. Also the cavity thickness at the rear for \( t = 1 \) is a continuous function.
THE ASYMPTOTIC APPROACH TO THE
THEORY OF LIFTING SURFACES.

by

TH. VAN HOLTEN
DELFT UNIVERSITY OF TECHNOLOGY
THE NETHERLANDS

SUMMARY
The paper presents a review of the present status of "matched asymptotic expansion" techniques, applied to the analysis of lifting surfaces such as aircraft wings, propeller ducts, helicopter rotors, tipvane windturbines, etc.

Some 15 years ago it was recognized that the classical lifting line theory of wings due to Prandtl could be considered as the first-order matched asymptotic expansion solution of a singular perturbation problem, where the perturbed quantity is the two-dimensional solution, and the inverse aspect ratio of the wing $A^{-1}$ is the perturbation parameter. This recognition in principle opened the possibility to systematically refine and extend the theory far beyond its existing limits of validity and applicability.

The first attempt to refine lifting line theory was published by Van Dyke in his well known book "Perturbation methods in fluid mechanics". The author of the present paper, in close cooperation with the late professor Timman, in recent years commenced work on the further development of asymptotic theory applied to lifting surfaces. A new element in this more recent work was the combination of the matched asymptotic expansion technique with the theory of the acceleration potential. The combination of these two theories made it possible to considerably simplify asymptotic theory, as well as to overcome a few problems which were hitherto inherent in asymptotic analyses. It is argued in the paper that asymptotic theory might in the future become a competitor of the present day numerical approaches, although much work still remains to be done.
INTRODUCTION.

The terminology "asymptotic theory" is used in the present paper as an abbreviation for "matched asymptotic expansion solution of a singular perturbation problem". The problem considered is the calculation of the pressure distribution on the surface of aircraft wings, rotorblades, propellerducts, etc. From the physical point of view it is intuitively obvious that there must be a strong similarity between the flow around a purely two-dimensional aerofoil (i.e. the section of an infinitely long wing) and the section of an airplane wing having finite length. However, it is also obvious that differences must exist, i.e. perturbations of the two-dimensional solution. Therefore, the theory of lifting surfaces may be approached from the point of view of "perturbation" theory.

A systematic way to determine the nature and magnitude of the perturbations is the formalism of the "asymptotic expansion". It is assumed for instance that the pressure distribution in a vertical streamwise plane through a wing section may be approximated by the series:

$$P_{\text{section}} \approx P_{\text{two-dim}} + \frac{1}{A} P_1 + \frac{1}{A^2} P_2 + \ldots$$

for $A \to \infty$.

where $A$ denotes the aspect ratio (= span/chord ratio b/c) of the wing. The field $P_{\text{two-dim}}$ is the purely two-dimensional solution, whereas the other terms $P_1, P_2, \ldots$ describe the way in which the pressure field becomes two-dimensional when the aspect ratio grows larger and larger. Since $P_1, P_2, \ldots$ will be determined in the limit $A \to \infty$, the series is an asymptotic expansion, strictly valid only for infinitely large aspect ratios. Nevertheless, one hopes that the asymptotic series also yields a useful approximation in the case of finite values of $A$, so that in that case the terms $P_1/A, P_2/A^2, \ldots$ express the actual deviations from the two-dimensional solution.

In practice this hope is nearly always fulfilled.

Closed form solutions are obtained of the chordwise pressure distribution on the wingsection. This fact, combined with the fact that there is only a relatively small deviation from the pressure on the geometrically corresponding two-dimensional aerofoil, makes it easy to interpret the asymptotic solution as the pressure field of an "effective" two-dimensional aerofoil whose geometry and/or working conditions differ slightly from the geometry of the actual wingsection. In fact, the theory is a generalization of Prandtl's well known classical lifting line theory, where three-dimensional wingsections are compared with two-dimensional aerofoils at an effective angle of attack. New, more accurate asymptotic theories also provide analytical transformation relations for the effective camber and thickness distributions of the comparison aerofoil. In unsteady flows, one obtains the "effective mixture" of pitching and heaving motions to which an aerofoil should be subjected in order to be comparable with the section of a 3-d wing. The use of an "effective" two-dimensional comparison aerofoil is more than just a complicated way to express the asymptotic results. The asymptotic analysis is mostly performed under the assumption of small perturbations in the flow (linearized theory). The results, when stated in terms of an effective aerofoil lead to an obvious de-linearization: the wingsection is then analyzed by using non-linear analytical results or even experimental results of the "effective" aerofoil.

Finally, the adjectives "singular" and "matched" which were also used above indicate that the nature of the perturbations depends on whether one is an observer close to the wing or at larger distances. If one looks at the problem on the scale of the wingchord, the limit $A = b/c \to \infty$ means that the span becomes infinitely large, and the wingsection becomes two-dimensional.
On the other hand, if one looks on the scale of the span, the limit $b/c \to 0$ implies that the chordlength $c$ shrinks to zero. This implies that one is dealing then with a finite line as the degeneration of the wingshape. Correspondingly, different asymptotic series solutions have to be used, and these different series have to be "matched".

Prandtl's classical lifting line theory (discussed on page 6), applicable to the straight aircraft wing, was really an asymptotic theory, although it was originally not recognized as such. Neither was it derived using the asymptotic formalisms. Because of its inherent deficiencies, Prandtl's theory has almost completely been superseded nowadays by numerical lifting surface methods. It was only after the introduction of computer methods, when it was realized by Van Dyke et al. that the classical theory, if formalized as an asymptotic theory, could be greatly improved and extended, and would have the potential to become a very practical research tool alongside numerical methods. At the Delft University of Technology rather systematic research has been done during the past few years on this subject. The work started with an investigation of helicopter rotors, published as the author's dissertation written under supervision of Timman. The helicopter blade is interesting because it is one of the few practical cases where lifting line concepts are still in widespread use. Later, other configurations were studied such as higher-order theories of swept wings, unsteady wings, propeller ducts and tip-vanes. The paper presents a review of the work done at the Delft University of Technology on asymptotic theory.

**SUMMARY OF THE FLOW EQUATIONS USED.**

The basic governing equations for incompressible inviscid flows have been established already in the 18th century by Leonhard Euler, who developed the well-known Euler equations expressing the conservation of momentum in the fluid, as well as the equation of continuity expressing the conservation of mass. These equations take the form:

1. $\text{div} \, \mathbf{V} = 0$ (continuity)
2. \[
\frac{DV}{Dt} = \frac{\partial V}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = - \nabla p
\]
   (Euler eq.)

where $\mathbf{V}$ is the velocity vector, $p$ the pressure and $\rho$ the fluid density.

In the following use will be made of two different methods of solving this set of equations:

- a. the method of the velocity potential;
- b. the method of the acceleration potential.

### Velocity potential

The velocity potential $\phi(x,y,z)$ is defined as a scalar function such that

$$\mathbf{V} = \nabla \phi$$

(3)

It is easily shown that such a scalar function $\phi(x,y,z)$ always exists in flow regions where the rotation of the fluid particles is zero:

$$\nabla \times \mathbf{V} = 0$$

(4)

Substitution of (3) into (1), yields Laplace's equation for $\phi$:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

(5)

In the theory of lifting surfaces the velocity field of a so-called vortex-singularity is an extremely important elementary solution of Laplace's equation. The velocity field associated with this type of singularity is given by:

$$\mathbf{V} = \frac{\Gamma}{4\pi} \frac{d\mathbf{s} \times \mathbf{r}}{r^3} \quad \text{(Biot and Savart)}$$

(6)

where $\mathbf{V}$ is the velocity in a point $P$, $\Gamma$ denotes the vortex element having "strength" $\Gamma$ and length $ds$, and $\mathbf{r}$ is the radius vector from the vortex element to
the point $P$. Eq. (6) is the analogon of Biot and Savart's formula for the magnetic field induced by an element of a conducting wire carrying an electric current. Just as in the latter case, one may construct a velocity field satisfying given boundary conditions, using a suitable configuration of vortex lines.

The reason why the vortex type singularity is of great importance in the theory of lifting surfaces is the fact that it represents an external force. In aerodynamics there is an analogon of the Lorentz-force in a magnetic field. The external force $dF$ represented by a vortex element $\Gamma \, ds$ is given by the so-called Kutta-Youkowski theorem:

$$dF = - \rho \, \Gamma \, ds \times V_{rel} \tag{7}$$

where $V_{rel}$ is the velocity relative to the vortex element $\Gamma \, ds$.

**Acceleration potential.**

The method indicated above is the classical method already initiated by Lagrange and Laplace in the 19th century. A perhaps less well known, but equally important approach was later introduced in 1936 by Prandtl. The quantity $-\frac{E}{\rho}(x,y,z)$ was called the "acceleration potential" of the flow, since according to Euler's equation the gradient of $-p/\rho$ equals the acceleration of the fluid particles. Writing $V = U + V'$ and $p = p_o + p'$ where $U$ is the undisturbed velocity (taken to be independent of the space- and time coordinates) and $V'$ is the perturbation velocity, linearization of Euler's equation leads to

$$3V'/\partial t + (U \cdot \nabla)V' = -\frac{1}{\rho} \text{grad } p' \tag{8}$$

which yields, on taking the divergence of all the terms of (8) and applying the continuity equation $\text{div } V' = 0$, the Laplace equation for $p'$:

$$\text{div } \text{grad } p' = 0 \tag{9}$$

The important elementary solution of (9) in the case of lifting surfaces is the field of a surface element on which pressure dipoles are distributed.

Such a pressure dipole distribution represents a discontinuity in pressure between the two sides of the surface element, and consequently an external force. Because the pressure field of a surface distribution of dipoles equals the field of a pressure vortex around the circumference, one may use again Biot and Savart's formula to calculate the pressure gradient and therefore (using eq. (8)) the fluid acceleration in the space around the force-carrying surface element. The velocity in a point of the field is found by integrating the acceleration of a particle of air coming from far upstream. The convective part of the acceleration in (8), in which $U$ occurs instead of $V$, shows that in linearized theory during this integration the particle's trajectory may be approximated by its straight, unperturbed trajectory. Boundary conditions must accordingly be applied to flat surfaces, parallel to the undisturbed flow.

**BRIEF HISTORICAL REVIEW OF THE ASYMPTOTIC ANALYSIS OF AIRCRAFT WINGS.**

The physical nature of the perturbation:

**Lanchester's model of the aircraft wing.**

In the beginning of the 20th century, soon after the birth of aviation technology, the necessity arose to get better insight into the flow around airplane wings. Kutta had already made potential flow calculations for two-dimensional aerofoils, but these results did not appear to have sufficient relation to the observed characteristics of three-dimensional wings. The influence of the wingshape has first been studied qualitatively by Lanchester. Although at that time the theory of the acceleration potential was not yet in existence, Lanchester's description of the flow is a qualitative forerunner of the
pressure theory. In the pressure theory one would describe lifting forces as pressure dipole distributions over the surfaces supporting the external forces.

Fig. 1. Pressure-dipole representation of two-dimensional aerofoil.

As a simple example, fig. 1 shows schematically the flux lines of the pressure gradients associated with a two-dimensional aerofoil. When air particles move through this field, the direction of their acceleration is opposed to the direction of the pressure gradient. It is interesting to note that this way of describing the flow around an aerofoil is almost identical to the model developed by Lanchester, which in his case was based almost entirely on physical intuition (ref. 1): "The fluid particles, which are gradually influenced by the plane while passing through the field of force established around it, will receive an upward acceleration as they approach the aerofoil, and will have an upward velocity as they encounter its leading edge. While passing instead under or over the aerofoil, the field of force is in the opposite direction, viz., downward, and thus the upward motion is converted into a downward motion. Then, after the passage of the aerofoil, the air is again in an upwardly directed field, and the downward velocity imparted by the aerofoil is absorbed".

The magnitude of the pressure gradients on the surface of the aerofoil itself is thus dictated by the requirement that air particles arriving at the nose of the aerofoil must be turned exactly along the aerofoil.

In fig. 2 the flow about a three-dimensional lifting surface is depicted schematically. A description can again be given by referring to Lanchester (ref. 1): "... the lines of force being no more constrained to lie in parallel planes would diverge, some portion of them escaping, as it were, and passing around the tips of the aerofoil laterally. The fluid traversing these lateral regions will then have upward momentum communicated to it during the whole time that it is in these regions, and will be finally left in a state of upward motion. The fluid traversing instead, the middle region, crossed by the aerofoil, will receive as in the preceding case, an upward acceleration before encountering the leading edge of the wing, a downward acceleration while passing under or over the aerofoil, and again an upward acceleration after the passing of the aerofoil. But here the upward and downward momentum will no longer balance each other, as owing to the lateral spread of the ascending field forward of the aerofoil the upward velocity communicated to the fluid before and after the passage of the wing is less than the downward velocity imparted to it during the passage of the wing. Consequently the portion of the fluid traversing the middle region will be ultimately left with some residual downward momentum; while, as noted, the fluid passing laterally around the wing on both sides has received an upward momentum".

Fig. 2. Pressure-dipole representation of three-dimensional wing.
The description given by Lanchester is a qualitative perturbation analysis: it explains how the two-dimensional flow is perturbed, as soon as the aspect ratio (span/chord) of the wing is no longer infinite as in the two-dimensional case.

The first asymptotic solution: Prandtl's lifting line theory.

As a response to the needs of aircraft designers, Prandtl succeeded during the period 1910 to 1915 to develop a very useful quantitative theory of the lift and drag on straight airplane wings.

Prandtl's analysis started from the flow model which is depicted in fig. 3 and which was already explained by Lanchester. His fundamental ideas were:

a). The flow around a wing section is bound to be very similar to the flow of a two-dimensional aerofoil, at least if the aspect ratio is reasonably large. Therefore, the wing section characteristics may be treated by two-dimensional theory.

b). In the first assumption the "spreading of the force field due to three-dimensional effects" recognized by Lanchester is neglected. However, this effect will have a much larger influence on the velocity field, because it is then integrated from infinitely far upstream of the wing to the wing itself. Therefore, the most important effect of the finite aspect ratio must be the downwash-phenomenon. Although the downwash reaches its full value only far downstream of the wing, it develops gradually and exists already to some extent at the wing location. Its effect is bound to be "felt" by the wing sections as a reduction of their "effective" angle of attack. The downwash also causes drag, even in a completely inviscid fluid. This may be understood more clearly by considering the similar case of a water-skier. The skier generates, like the airplane, dynamic lift by imparting downward momentum to the water. The downwash causes the effect that the water-skier must continuously be pushed against the slope of a "water-hill".

Clearly, in order to calculate the lift-and drag characteristics of a finite wing, one must be able to determine quantitatively the value of the downwash at every wing section. For this purpose Prandtl modelled the flow as in fig. 4.
The lift of the wing is thought to be concentrated along a line: the so-called lifting line, mathematically represented by a "bound vortex" fixed with respect to the wing. The downwash far behind the wing will hardly be affected by this idealization. Therefore one may also expect that the downwash exactly at the lifting line yields an approximation for the sought "real" downwash. These downwash velocities are calculated by Biot and Savart's law from the configuration of vortices leaving the wing. Actually, one has to assume a sheet of vorticity instead of two tipvortices (in analogy with Kirchhoff's law for electrical currents, every spanwise change in strength of the bound vortex is accompanied by a trailing vortex filament). One may, under the assumption of small perturbation velocities in the flow, take the trailing vortex sheet as a flat sheet, stretching infinitely far downstream behind the wing, parallel to the undisturbed flow. In this way Prandtl was able to derive an integral equation from which the lift distribution along the wing may be calculated.

The present author has worked, in close cooperation with Timman, in the same field. Several configurations were considered, e.g. a higher-order theory of the unsteady wing, the swept wing, the helicopter blade (which is a combination of the latter two cases) and the propeller duct (refs. 6-12). During the course of this work, a few additions to the general theory were also made:

a) Major simplifications in the asymptotic approach to lifting surface theory were possible by using the theory of the acceleration potential instead of the more usual velocity potential;

b) A view on the matching process was developed which differs slightly from the usual view and enables to explain some pieces of mysticism - as they were called by Van Dyke - in earlier work;

c) A physical interpretation was given of the differences between the classical intuitive lifting line theories and the more formal asymptotic theory. In fact, a rather fundamental error was discovered in the usual interpretation of Prandtl's theory.

It was Timman's opinion that asymptotic methods can be fruitfully used only if one has a good qualitative understanding of the physical processes that are the subject of the quantitative asymptotic treatment. In what follows, the emphasis is accordingly placed on general concepts and physical interpretations instead of the mathematics involved. The interested reader will be able to find the mathematical details in the references mentioned.

The Modern View of Lifting Line Theory.

The Near Field.

In the modern theory one starts from the consideration that the pressure field in the vicinity of the wingsurface has two different characteristic length scales: in spanwise direction the characteristic length is the span b, in chordwise direc-
tion it is the chordlength $c$.

As a second step the Laplace equation is rewritten in terms of the non-dimensional coordinates $x/c$, $y/c$ and $z/b$ (fig. 5):

$$\frac{\partial^2 p}{\partial (x/c)^2} + \frac{\partial^2 p}{\partial (y/c)^2} = -\frac{1}{\lambda^2} \frac{\partial^2 p}{\partial (z/b)^2} \quad (10)$$

The partial derivatives occurring in this equation are all of the same order of magnitude, keeping in mind the physical assumption introduced above. One concludes that the pressure field satisfies a two-dimensional Laplace-equation if one considers large aspect ratio's ($\lambda \to \infty$) and if terms of an asymptotic order $O(\lambda^{-2})$ are neglected. The pressure distribution in a vertical, streamwise plane through a wingsection will thus, near the wingsurface, correspond to the field of a two-dimensional flat-plate aerofoil which, as will be explained, is given by:

$$\frac{(-p)}{\frac{\lambda}{2} u^2} \quad 2D, \text{flat plate} = -\frac{C_L(z)}{\pi} \cdot \frac{\sin \varphi}{\cosh n + \cos \varphi} \quad (12)$$

This field satisfies the two-dimensional Laplace equation. It has along the chord (i.e. for $\eta=0$) a value $\partial p/\partial y = 0$, except at the very leading edge ($\eta=0$, $\varphi=\pi$) where the field becomes singular. According to the Euler equation (8) this means that particles of air reaching the aerofoil experience an infinite vertical acceleration at the leading edge. In other words the streamlines are kinked there. After the flow has been deflected at the leading edge there is no further curvature of the streamlines because of $\partial p/\partial y=0$. The pressure field (12) represents a lift on the section

$$L(z) = C_L(z) \frac{\lambda}{2} u^2 \pi c \quad (13)$$

Its value is still unknown, however, because the lift coefficient $C_L$ is not known yet. The coefficient $C_L$ should be chosen such that the leading edge singulari-
ty becomes of such a magnitude that the flow is turned exactly parallel to the wingchord at the leading edge. We are thus tempted to use again Euler’s equation (8) in order to determine the vertical perturbation velocity on the wing chord (say at the mid-chord point). This must be equated to \(-U_0a(z)\) where \(a(z)\) is the angle of incidence of the wingchord with respect to the undisturbed flow. In other words, the function \(C_t(z)\) might be calculated from the condition:

\[
\frac{\partial V}{\partial x} (0,0,z) = -\frac{1}{\rho U^2} \int_{-\infty}^{0} \frac{\partial p}{\partial y} (x,0,z) \, dx = -a(z) \tag{14}
\]

However, \(\partial p/\partial y\) calculated from eq. (12) is an approximation valid only in the near-field of the wing. The integral (14) cannot be calculated, unless an approximation has been found for the complete pressure field, valid also at large distances.

**The far field.**

The far field is defined as the field at distances of the order of the wing span \(b\) from the wing surface. In this region the physical assumptions for the near field are no longer valid because the characteristic length scale of the far pressure field will be equal to the span \(b\) in all directions. However, looking at our problem on the scale of the span, another simplification may be introduced by noting that the limit \(A = b/c \to \infty\) means that the chord length \(c\) shrinks to zero. This means that the far field in the asymptotic approximation will become the field of a line along which pressure singularities are distributed.

Again neglecting all effects of \(O(A^{-2})\), the far field turns out to be given by a line of pressure dipoles along the mid-chord line of the wing, symbolized by \((\frac{\partial}{\partial y})^3d,\ dip'\). The common field.

For both the near and far field of the wing asymptotic representations have been found strictly valid only in the asymptotic limit \(A \to \infty\). In practice, however, because the asymptotic results are always applied to cases of finite \(A\), the near and far field are just regarded as approximations found in a convenient way. No such approximation can be found for the intermediate region: the physical assumptions leading to the simplification of the field equations close to the wing surface cannot be valid in the intermediate region, and neither is the degeneration of the boundary contour into a line valid there. In other words, for finite \(A\) one cannot assume the existence of an "overlap" region where both approximations are simultaneously valid. The only way then, to find an approximation for the complete pressure field is to find a suitable interpolation expression which bridges the gap between near and far field and smoothly merges with the approximate solutions of the near and far field. Such a uniformly "valid" field may be found by summing the near and far pressure field, and subtracting a correction field, which will be called the "common field". The correction field must be chosen such that far from the wing it cancels the near field to the required order of accuracy so that only the far field remains. Close to the wing surface, the correction field has to cancel the far field, so that only the near field remains there. Denoting the complete pressure field thus obtained by \(p_{\text{composite}}\), we have the structure

\[
p_{\text{composite}} = p_{\text{near}} + p_{\text{far}} - p_{\text{common}} \tag{15}
\]

whereas it should be required that

\[
p_{\text{common}} = (p_{\text{near}})^{r=\text{order } b} \tag{16}
\]

and also

\[
p_{\text{common}} = (p_{\text{far}})^{r=\text{order } c} \tag{17}
\]
Combining (16) and (17) yields the matching condition:

\[(P_{\text{near}})^{r}\text{-order } b \sim (P_{\text{far}})^{r}\text{-order } c \quad (18)\]

where the symbol \(\sim\) denotes identity up to a certain specified order of accuracy, which in the considered case is an accuracy to include effects of an asymptotic order \(O(A^{-1})\). Now one may consider \(p\) in eq. (12) as the field of a two-dimensional pressure dipole distribution along the wingchord. It is not surprising therefore, that at large distances the near field behaves like the field of a discrete two-dimensional dipole. This is symbolized here by writing eq. (16) like \(P_{\text{common}} = P_{2D,dip}'\). The latter is also consistent with (17), since at small distances from a three-dimensional dipole line, the character of the field becomes predominantly that of a two-dimensional pressure dipole. By a suitable choice of the strength of the dipole distribution along the three-dimensional dipole line which represents the far field, one can thus satisfy the matching condition (18). A quantitative treatment is given in e.g. ref. 9.

The resulting integral equation.

All the terms in (15) have now been determined in terms of the function \(C_{\ell}(z)\), and the integral (14) may be evaluated. The contribution of \(P_{\text{near}}\) is \(-C_{\ell}(z)/2\pi\), as in two-dimensional aerofoil theory. The contribution of the second term in (15) is symbolically written as \(-v_{1}(z)/U\).

One may show that the velocity perturbations associated with a three-dimensional pressure dipole line are identical with the velocity field of a bound vortex line and its associated trailing vortex sheet. (fig. 4). Consequently, the contribution of \(P_{\text{far}}\) to the integral (14) equals exactly, what is in Prandtl's theory called the induced downwash \(v_{1}\) (with a minus-sign).

The velocity field of a two-dimensional pressure dipole equals, accordingly, the velocity field of a two-dimensional vortex. Because in eq. (14) the velocity is evaluated exactly at the mid-chordline, i.e. at the place where the three-dimensional dipole line as well as the two-dimensional dipole are situated the contribution of the third term of (15) to the integral (14) is zero. Equation (14) thus yields:

\[
\begin{align*}
\frac{\mathbf{V}}{U}(0,0,z) &= (\frac{\mathbf{V}}{U})_{\text{near field}} + (\frac{\mathbf{V}}{U})_{\text{far field}} + (\text{common part}) \\
&= \frac{C_{\ell}(z)}{2\pi} - \frac{v_{1}(z)}{U} = -\alpha(z),
\end{align*}
\]

which is Prandtl's classical integral equation, stating that a wing section behaves like a two-dimensional aerofoil placed at an effective angle of attack \(\alpha - v_{1}/U\).

Some failures of classical lifting line theory.

Prandtl's lifting line theory has, during the many decades of its existence, proved to be of great value. It provided reasonably accurate estimates of the lift and drag of straight wings. Even more important: it provided qualitative insight into the flow phenomena. The success of lifting line theory naturally led to attempts to apply similar concepts to other configurations such as swept wings, vibrating wings, propellers, helicopter rotors and propeller ducts.

Propellers in axial flow.

Some of the attempts mentioned above were highly successful. This was the case with the propeller. The blade sections are treated like two-dimensional aerofoils, placed at an effective angle of attack. The effective angle of attack differs from
the geometrical angle by the influence of the slipstream effects. Goldstein (ref. 13) was able, for the particular case of optimum propeller loading, to analyze quantitatively these slipstream effects, by calculating the induced velocity associated with the spiral-shaped vortex sheets (fig. 7) downstream of the propeller. Lock (ref. 14) later made use of Goldstein's solution, and devised an approximate method of analysis for arbitrary propeller blades, by combining Goldstein's solution with momentum theory.

Fig. 7. Vortex sheets propeller.

The flow around a propeller blade is a rather extreme situation, and the success of the lifting line concepts under these circumstances makes it the more surprising that all of the other attempts mentioned above failed altogether.

Swept wing

Fig. 8 shows schematically the problem encountered in the case of the swept wing. The trailing vorticity is not perpendicular to the lifting line and may be decomposed in two components: one sheet involving vorticity perpendicular to the lifting line, and another sheet composed of vorticity parallel to the line. It is the latter sheet that causes the downwash to become infinitely large, which is physically clearly in error. An ad hoc solution has been found by Weissinger (ref. 15) who calculates the downwash not along the lifting line, but along the 3/4-chord line of the wing, where it is not singular. It is then required that the bound vortex must be of such a strength that the flow becomes tangential to the wingsurface along the 3/4-chord line. Weissinger's method has never been well founded from the theoretical point of view, until it was shown very recently (ref. 9) that the 3/4-chord method is a particular formulation of the 2nd order asymptotic solution for the lifting surface. Weissinger's formulation is, however, incomplete and consequently yields the liftdistribution along the wingspan with good accuracy (fig. 9) but does not yield information about drag or pitching moment, nor about the surface pressure distribution. In the complete asymptotic solution all this information is contained.

Fig. 8. Vortex sheet of swept wing.

Fig. 9. Calculated $C_L$ of rectangular wings, acc. to ref. 95.
Other ad hoc methods have also been designed, e.g. by Küchemann (ref. 16). Nevertheless, practically the only way available to analyze swept wings is the use of numerical lifting surface methods. In principle, one divides the wingsurface into a large number of panels each carrying a pressure dipole distribution ("doublet-lattices"). The dipole-strengths are determined by requiring the flow to become tangential to the wingsurface in a number of points. Refinements are also often used (ref. 17) making use of continuous dipole distributions on the surface.

**Vibrating wings.**

Under unsteady conditions, the strength of the bound vorticity representing a wingsurface is variable in time. Consequently, so-called shed vorticity is carried away with the flow, downstream of the wing. The shed vorticity forms a sheet, which similar to the swept wing causes the downwash to be singular along the bound vortex if a lifting line solution is attempted. A theory analogous to Weissinger's 3/4-chord method is not available for unsteady flow. This means, that only two approaches have been open to the aero-elastic analyst seeking to determine unsteady airloads:

a) "strip-theory", in which each wingsection is treated like an isolated two-dimensional aerofoil whose characteristics are not influenced at all by the remainder of the wing;
b) Complete lifting surface theory, such as a "doublet-lattice" approach.

**Helicopter rotors**

The blades of a helicopter rotor in forward flight encounter both yawed-flow as well as unsteady effects (fig. 10). No wonder that any lifting line approach fails when it is based on the classical concepts of two-dimensional blade section characteristics and induced downwash effects. The traditional analysis has made use of very crude momentum concepts to determine the effective angles of attack of the blade sections.

**Unsteady effects**

1. Pitching motion blade
2. Time-varying angle of attack
3. Time-varying relative velocity
4. Time-varying sweep angle

With the advent of high-speed computing equipment, the vortex wake could be represented by a network of discrete vortex elements. Using Biot and Savart's theorem, the induced downwash may be calculated in points outside the vortex elements. But then of course uncertainties arise as to what is the best distribution of the vortex elements, both time- and spanwise relative to the points where the induced downwash is calculated. The problem of the helicopter is too complicated to make in practice use of lifting surface analyses.

**Propeller ducts**

In the case of a relatively short propeller duct, one might again think that Prandtl's concepts could be applied in order to gain at least a first estimate of the required duct length and shape. The duct sections look very much like aerofoils, and one might get a rough idea about the mass flow...
through the duct and the pressure distribution on its surface if one would know the effective angle of attack and the effective flow velocities encountered by the sections. The self-induction effects of a duct are known to be very strong. In analogy with lifting line theory, the self-induced velocities which determine the effective working conditions of the duct sections, might then be determined by considering the self-induction of a discrete vortex-ring. The reason why such an analysis fails is immediately clear: the self-induced velocity of a vortex-ring is singular at the ring itself. For this reason, one has no alternative other than to treat the duct as a ringshaped lifting surface. Küchemann and Weber were the first to calculate numerically some of the most important characteristics of ducts (ref. 18), and they published tables with results for a number of different duct parameters.

**REMEDIES PROVIDED BY THE ASYMPTOTIC VIEWPOINT.**

The fundamental role played by the common part.

The failure of Prandtl's fundamental ideas to provide a satisfactory analytical model of swept wings, vibrating wings, etc., was unexpected, inexplicable, even somewhat mysterious. Of course, it is easy enough to trace what is mathematically the trouble. However, from the physical point of view, it is really inexplicable why for instance the sections of a swept wing resist to be treated as two-dimensional aerofoils subject to a downwash velocity which alters their effective angle of attack.

In fact, aerodynamicists are so convinced that there is a relation between wing sections and two-dimensional aerofoils, that they do make use of it. For instance, the design of wings invariably starts by considering the two-dimensional characteristics of the sections to be used. One still tends to lean heavily on two-dimensional section aerodynamics when judging numerical lifting surface results, although such concepts like two-dimensional characteristics are strictly lifting line concepts.

Perhaps the greatest achievement of asymptotic wing theory is the reconciliation of on the one hand physical intuition and on the other hand the principles upon which an actual analysis is based.

In the asymptotic theory briefly outlined in chapter 4 the far field consisted of the field of a pressure dipole line, straight and perpendicular to the flow. At least, this was the model for the straight wing. The dipole-line would have a kinked V-shape for the swept wing, and would be circular in the case of the duct. When calculating the induced velocity by integrations of the type shown in eq. (14), the integrand would not be integrable, were it not for the fact that the "common part" consisting of a two-dimensional dipole, eliminates the strong singularity of the integrand. The common part always performs this same "trick", irrespective of the shape of the dipole-line.

Translated into vortex terminology (fig. 11), this means that $v_4$ is the velocity at a point $P$ due to the lifting vortex and its associated trailing vorticity together with the velocity due to a two-dimensional
vortex of equal local strength but with opposite direction passing through the same point P. Naturally, this does not affect the quantitative results in steady flow: the contribution of the two-dimensional vortex to \( v_i \) is zero. Things are very different, however, when we come to consider unsteady flow.

In the case of unsteady lift, we should again take for \( v_i(z,t) \) at the point P the velocity due to the lifting vortex line (having a wake of trailing as well as shed vorticity) and add to this the velocity due to the two-dimensional vortex, which is now also accompanied by shed vorticity (fig. 12). Again, the strength of this two-dimensional vortex varies in time in the same way as the bound vorticity at P, although it is of opposite direction.

It will be clear that this definition of induced velocity does not lead to infinite values of \( v_i(z,t) \). One of the obstacles preventing the use of classical lifting-line concepts in unsteady flow has thus been traced back to a wrong interpretation of Prandtl's steady flow model.

Unsteady lifting line theory.

Having solved the problem of the singular values of downwash, it is now a relatively simple matter to set up a lifting line theory for the wing in unsteady flow. Consider for instance a straight, flat-plate wing, whose sections execute a harmonic motion in torsion around their mid-chord point (fig. 13):

\[
a(z,t) = a_o(z) \cos \omega t
\]  

(21)

Fig. 12. Definition of unsteady induced velocity at P: sum of contribution of vortex systems A and B.

The vertical velocity of particles of air moving along the chord of a wingsection is:

\[
v(x,0,z,t) = a_o(z) \omega \sin \omega t \cdot x - U a_o(z) \cos \omega t
\]  

(22)

and the corresponding vertical acceleration is:

\[
\frac{DV}{Dt} = a_o(z) \omega^2 \cos \omega t \cdot x + 2U a_o(z) \cdot \omega \sin \omega t
\]  

(23)

which should, according to Euler's equation (8) equal \(- \frac{1}{p} \frac{\partial p}{\partial y}\) on the wingsurface. The solution of the two-dimensional Laplace equation satisfying this boundary condition is, formulated in terms of the elliptical
coordinates \((n, \varphi)\) defined by eq. (11):

\[
\frac{P}{\frac{1}{2} \rho U^2} = - \frac{\sin \varphi}{\cosh n + \cos \varphi} +
\]

\[
+ 4 \alpha_0(z) k \sin \omega t e^{-\eta} \sin \varphi +
\]

\[
+ \frac{1}{2} \alpha_0(z) k^2 \cos \omega t e^{-2\eta \sin 2\varphi}.
\]

(24)

where \(k\) is the reduced frequency \(k = \frac{\omega C}{2U}\).

The first term, i.e. the flat-plate pressure field, has an indexed coefficient \(C_{\lambda 1}(z,t)\), in order to indicate that \(C_{\lambda 1}\) is just one part of the total lift coefficient. Another contribution to the lift is given by the second term.

If we had been considering a purely two-dimensional aerofoil, eq. (24) would have been the complete pressure field. The function \(C_{\lambda 1}(z,t)\) might then have been determined by calculating, using Euler's eq. (8), the vertical velocity \(v(0,0,z,t)\) at the mid-chord point and by equating this to \(- U_0(z) \cos \omega t\). This case has been worked out, and \(C_{\lambda 1,2D}\) may be found in e.g. ref. 19. Let us denote the corresponding pressure field by \((P_{2D,pitching})\).

In the three-dimensional case actually considered, the near field of a wingsection may thus be written like:

\[
P_{\text{near}} = \frac{\Delta C_{\lambda 1}(z,t)}{\frac{1}{2} \rho U^2} \frac{\sin \varphi}{\cosh n + \cos \varphi} +
\]

\[
+ \frac{(-P_{2D,pitching})}{\frac{1}{2} \rho U^2}.
\]

(25)

The complete pressure field of the wing is, according to an asymptotic analysis similar to that of the steady wing, given by:

\[
P = \frac{\Delta C_{\lambda 1}(z,t)}{\frac{1}{2} \rho U^2} \frac{\sin \varphi}{\cosh n + \cos \varphi} +
\]

\[
+ \frac{(-P_{2D,pitching})}{\frac{1}{2} \rho U^2} + \frac{P_{\text{far}} - P_{\text{common}}}{\frac{1}{2} \rho U^2}.
\]

(26)

where \(P_{\text{far}}\) is again the field of a line-distribution of dipoles, and \(P_{\text{common}}\) is the field of a two-dimensional dipole.

We must now apply the tangency condition for the vertical velocity along the mid-chord line:

\[
v(0,0,z,t) = - U_0(z) \cos \omega t.
\]

The first term in (26) is the pressure field of a two-dimensional flat-plate aerofoil, stationary with respect to an inertial frame of reference, but having a variable lift. The value of \(v(0,0,z,t)\) contributed by this pressure field should thus be calculated according to the two-dimensional theory for an aerofoil in a gust field, and is symbolically written as:

\[
\frac{v}{U}(0,0,z,t) = f_{2D,gust}(C_{\lambda 1}(z,t))
\]

Applying the tangency condition, one thus obtains

\[
v(0,0,z,t) = f_{2D,gust}(\Delta C_{\lambda 1}(z,t)) -
\]

\[
- \alpha_0(z) \cos \omega t + \frac{v_i}{U}(z,t) = - \alpha_0(z) \cos \omega t
\]

(28)

where \(v_i\) is the induced velocity contributed by the far field and common part together. Inverting eq. (28) yields:

\[
\Delta C_{\lambda 1}(z,t) = C_{\lambda 2D,gust} - \frac{v_i}{U}(z,t)
\]

(29)

where \(C_{\lambda 2D,gust}\) denotes the functional relationship between the time-varying gust angle \(\nu g/U\) at the mid-chord of a stationary two-dimensional aerofoil and its time-varying lift (see e.g. ref. 19). The induced velocity thus acts as a self-induced gustfield for the wing. The unsteady lifting line theory finally takes the form:

\[
\Delta C_{\lambda 1}(z,t) = C_{\lambda 2D,pitching} - v_i(z,t)
\]

(30)

where \(C_{\lambda 2D,pitching}\), denotes the functional relationship between the time-varying angle of incidence and the time-varying lift, as
calculated using the two-dimensional theory of pitching aerofoils (see again ref. 19). The induced velocity \( v_i \) is the velocity caused by the total lift of the wing sections, i.e. the lift due to pitching as well as the lift due to the effective gust velocities.

Obviously, an analogous expression can be derived for the case of a heaving motion. The above described theory has not yet been worked out further to make actual calculations.

**Lifting line theory of the swept wing.**

The swept wing is an interesting application of asymptotic theory, since it appears that even in the first-order lifting line theory more effects are involved than a mere reduction of the effective angle of attack of the wing sections. In fact, the streamwise sections should be treated like two-dimensional aerofoils with an effective camber, the magnitude of which depends on the spanwise liftslope.

As a first step in the analysis, it is necessary to introduce a non-orthogonal coordinate system \((x', y', z')\) sketched in Fig. 14. The \( z' \)-axis of this system lies along the mid-chord line of the winghalf considered, and the \( x' \)-axis is parallel to the undisturbed flow.

Next, non-dimensional coordinates are introduced, defined by:

\[
\begin{align*}
x' &= \frac{x - z \tan \lambda}{c/2} \\
y' &= \frac{y}{c/2} \\
z' &= \frac{z}{b/2} \\
\end{align*}
\]  

\( (31) \)

In terms of the latter system, Laplace's equation reads:

\[
(1 + \tan^2 \lambda) \frac{\partial^2 p}{\partial x'^2} + \frac{\partial^2 p}{\partial y'^2} + 2 \tan \lambda \frac{\partial^2 p}{\partial x'^2 \partial z'} + \frac{1}{A^2} \frac{\partial^2 p}{\partial z'^2} = 0 \quad (z \geq 0) \]  

\( (32) \)

where \( A \) denotes again the aspect ratio \( b/c \) (for simplicity the wing is here assumed to have a constant chord) and \( \lambda \) is the sweep angle of the wing.

The pressure field \( p \) is written in the form of an asymptotic series:

\[
p = p_{\text{two-dim}} + \frac{1}{A} p_1 + \frac{1}{A^2} p_2 + \ldots \quad \text{for } A \to \infty.
\]

Substituting the asymptotic series into the Laplace equation (32) and multiplying successively by \( A, A^2, \) etc., one finds on taking each time the limit \( A \to \infty \) the following equations to be satisfied by \( p_k \):

\[
(1 + \tan^2 \lambda) \frac{\partial^2 p_0}{\partial x'^2} + \frac{\partial^2 p_0}{\partial y'^2} = 0
\]  

\( (33) \)

\[
(1 + \tan^2 \lambda) \left( \frac{\partial^2 p_1}{\partial x'^2} - \frac{\partial^2 p_1}{\partial y'^2} \right) = 2 \tan \lambda \frac{\partial^2 p_0}{\partial x'^2 \partial z'}
\]  

\( (34) \)

\[
(1 + \tan^2 \lambda) \left( \frac{\partial^2 p_2}{\partial x'^2} + \frac{\partial^2 p_2}{\partial y'^2} \right) = 2 \tan \lambda \left( \frac{\partial^2 p_1}{\partial x'^2 \partial z'} - \frac{\partial^2 p_0}{\partial z'^2} \right)
\]  

\( (35) \)
Clearly, a simple further coordinate transformation \( y^* = y^*/\cos \alpha \) transforms (33) into a two-dimensional Laplace equation, and (34) into a two-dimensional Poisson equation, whose right hand side is known once (33) has been solved. The third equation may be treated similarly. Working out the first two equations (ref. 10) and matching with a far field consisting of a kinked dipole line finally shows:

\[
P_{\text{near}} = - \frac{C_{1}^{2} (z^*)}{4 \rho U^2} \frac{\sin \psi}{\cosh \eta + \cos \psi} - \frac{1}{\lambda} \sin 2 \alpha \frac{d C_{1}^{2} / dz^*}{\cosh \eta} \eta \sinh \eta \sin \psi + \frac{2}{3 \lambda} \int (36)
\]

where the elliptical coordinates \( \eta \) and \( \psi \) are defined by the transformation formulae:

\[
x^* = \cosh \eta \cos \psi, \quad y^* = \sinh \eta \sin \psi \quad (37)
\]

The field \( P_{\text{near}} \) given above is the pressure distribution in a vertical, streamwise plane through the wingsection. Even if the streamwise sections are geometrically similar to flat-plate aerofoils, there is an effective camber effect which is represented by the second term in (36). The effect is that of parabolic camber, and is proportional to the slope of the wing lift in spanwise direction. The effective camber is the cause of the well-known "unsweeping" of the isobars on a swept wing in the tip regions. The second-order theory, based on eq. (35) has also been worked out, taking into account camber and taper effects as well. The latter theory has not been published yet, and neither has it already been implemented to get comparisons with lifting surface results.

AN EXAMPLE OF HIGHER-ORDER ASYMPTOTIC ANALYSIS: TIPVANES AND DUCTS.

In the examples given above, the effects of \( O(\alpha^{-2}) \) were neglected. However, higher accuracy may be achieved by taking these higher order effects into account, as was already shown by Van Dyke (ref. 3). The results of higher-order lifting line theory come close in accuracy to the results of fully numerical lifting surface theories, as was shown in fig. 9 where some results are given for the rectangular flat-plate wing.

Another example will now be given. The example deals with the analysis of so-called tipvanes. This is a case where the asymptotic approach has proved to be of great practical value. Furthermore, in this particular case the theory was confirmed both by some numerical results which happened to be available, as well as by experiments.

It may be well to explain first of all the physical situation from which the research arose. The so-called "tipvane"-system is being studied by several research groups in the world (among which a group of the Delft University of Technology), as a possible way to increase the efficiency of wind-energy. Tipvanes are small auxiliary wings
mounted on the bladetips of windturbines (fig. 15) in such a way that during rotation a large radial, inward directed, lifting force acts on them. They thereby increase the massflow through the turbine by a factor of 4 or 5. The physical explanation of this phenomenon is given in the figs. 16 and 17. Fig. 16 shows a situation of very small rotational speed.

The tipvanes deflect small amounts of air, bounded by tipvortices, radially outwards. When the tipvane speed is increased (fig. 17), the separate "downwash" regions meet each other whereas the tipvortices begin to cancel each other. The result is a diffuser type of flow which is associated with a contraction of the streamlines in front of the turbine disc.

Evidently, one of the first questions that has to be answered is, how large the mass flow augmentation is as a function of the tipvane lift. Let us consider linearized theory, where the trailing vortex sheets emanating from the tipvanes lie on a straight cylindrical surface with radius $R$ and are carried away in axial direction with the undisturbed velocity $U$. Now let us also assume for a moment that the lift on the tipvanes is constant along the span, so that the trailing vorticity is concentrated in two discrete tipvortices. These tipvortices will trace out helical paths on the cylindrical surface (fig. 18).
upstream vane-tips has equal strength but opposite direction compared with the vorticity coming from the downstream vane-tips. The two vortex cylinders associated with the upstream and downstream vane-tips cancel each other, except for a band of vorticity with a width equal to the span of the tipvanes (fig. 19). A more careful analysis given in ref. 20 shows that the same is true for the case of general spanwise liftdistribution along the tipvanes.

It is thus seen that, apart from the type of boundary conditions, the analytical model has now become identical to that of a propeller duct. The problem was attacked by assuming that the chord/diameter ratio $c/D$ of the virtual duct is small ($c/D \rightarrow 0$), so that an asymptotic analysis could be attempted. In terms of circular cylinder coordinates ($\rho$, $\psi$, $z$), Laplace's equation for the velocity potential $\phi$ reads in the case of circular symmetry:

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \phi}{\partial \rho} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

(38)

Substituting $\rho = R(1 + \frac{y}{c/2} \frac{c}{D})$ where $y = r - R$, eq. (38) may be written

$$\frac{\partial^2 \phi}{\partial \rho^2} + \frac{c}{D} \left(1 - \frac{c}{D} \frac{y}{c/2} + \ldots\right) \frac{\partial \phi}{\partial \rho^2} + \frac{\partial^2 \phi}{\partial \rho^2} = 0$$

(39)

on condition that $c/D$ is small, and that we only consider the part of the field close to the duct surface (the near field), where $\frac{y}{c/2}$ and $\frac{\rho}{c/2}$ are of order unity. The coordinates $x$ and $y$ used here, are Cartesian coordinates fixed to the chord of the duct, as in fig. 6. Since the partial derivatives occurring in (39) are all of the same order of magnitude in the near field, the equation shows that, for very small $c/D$, the perturbation potential $\phi$ satisfies a two-dimensional Laplace equation in the near field. This result is not unexpected: looking at our problem on the scale of the duct chord $c$, $c/D \rightarrow 0$ means that $\phi$ is increased indefinitely so that indeed a two-dimensional strip problem remains. This leads us to assume an asymptotic series solution for the near field potential $\phi_{\text{near}}$:

$$\phi_{\text{near}} = \phi_0 + \frac{c}{D} \phi_1 + \left(\frac{c}{D}\right)^2 \phi_2 + \ldots$$

(40)

for $c/D \rightarrow 0$ (40)

where $\phi_0$ is the purely two-dimensional field obtained in the limit $c/D = 0$, whereas the other terms describe the way in which the field becomes two-dimensional when $c/D \rightarrow 0$. It will appear later that also terms occur behaving like $c/D \ln(c/D)$, $\ln(c/D)$, etc. for $c/D \rightarrow 0$. For convenience, such terms have not been written explicitly in (40), but are assumed to be included in the corresponding terms having an asymptotic behaviour like $c/D$, etc. Substituting the assumed type of solution (40) into eq. (39) and equating terms of an equal order finally shows that $\phi_0$, $\phi_1$ and $\phi_2$ satisfy the following equations:
The boundary conditions are such that a velocity discontinuity is required corresponding to the vortex strength, whilst the radial velocity should be continuous through the surface of the ring. The equations can be solved relatively easily (ref. 12). It appears that to the order \( O(C/D) \) the near field (the pure two-dimensional stripsolution) matches with a discrete vortex-ring representing the far field. The higher-order solution matches with a dipole-ring. It leads to additional axial velocities in the vicinity of the vortex band.

The highest order solution determined was the solution to order \( O((C/D)^2) \) which matches with a quadrupole ring and adds radial correction velocities to the near-field solution. In this way it was possible to obtain analytical formulae for the mass flow induced by the vortex cylinder. For instance, an elliptical spanloading on the tipvanes causes an incremental velocity \( \Delta V \):

\[
\Delta V = \frac{\Gamma}{2\pi R} \left( \ln\left(\frac{R}{C/4}\right) + \frac{1}{2}\right) \text{ to order } (C/D)^1
\]

when averaged over the disc plane. Here \( \Gamma \) is the total vortex strength of the vorticity band. This result was confirmed very well by experiments.

In order to give a comparison of the higher-order asymptotic results with numerical results, the figures 20 through 25 have been prepared.

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Figs. 20. Comparison asymptotic theory (-) and numerical results of ref. 18 (--). First Birnbaum distribution; axial velocity.

Figs. 21. Idem; \( 1^{st} \) Birnbaum distribution; radial velocity.

These figures show for the first three Birnbaum distributions the axial velocity \( v_a \) and the radial velocity \( v_r \) on the vorticity surface, in comparison with the tabulated values of ref. 18. The axial velocities shown do not include the compo-
nent $\pm \gamma(\theta)$. It appears that, although the asymptotic theory was developed under the assumption $c/D \ll 1$, the asymptotic theory does not lead to large errors until $c/D$ reaches the order of magnitude unity.

This is really a very clear illustration of what may be achieved by applying the asymptotic formalisms, considering that in the past any "Prandtl-type" theory for the propeller duct or solenoid was regarded
impossible due to singularities. At least it was regarded worthless because of the dominating self-induction effects encountered in this particular case.

CONCLUSIONS.

What the practical user of theoretical aerodynamics really wants to have available, is a range of methods, where simplicity and accuracy may be exchanged for each other. Existing gaps in the range of aerodynamic methods may be filled in by asymptotic theories. For instance, lifting line theories of the swept and/or vibrating wing can now be constructed. Asymptotic theories furthermore have the advantage of providing qualitative insight into the solutions. Finally, the higher-order asymptotic theories come close in accuracy to relatively refined numerical methods, whereas they still offer the advantage of closed-form solutions.

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ON THE KERNEL FUNCTION OF LINEARIZED LIFTING SURFACE AND
ACTUATOR SURFACE THEORY

by

J.A. SPARENBERG
UNIVERSITY OF GRONINGEN
THE NETHERLANDS

INTRODUCTION

When the author started in 1958 at the NSMB to study lifting surface theory for ship screws, he had the privilege to have contact with Rein Timman about this subject. It was not difficult, using the ideas developed by Prandtl in /1/, to set up a linearized lifting surface theory for a propeller blade of zero thickness using a layer of pressure dipoles at the blade /2/. By substituting the pressure dipole field in the linearized equations of motions the velocity field induced by this pressure field could be determined. This velocity field is the Green function or the kernel for the integral equation which has to be solved in case of given geometry of the blade. By surrounding the singularity of the pressure field by a small geometrical surface and writing down the pressures on it and the flux of momentum through it, it can be shown that at that place an external force is acting at the fluid. However, still the feeling remained that although the results were correct a real understanding in what happened failed. It is the intention of this paper, which in nature is a didactical one because it does not yield new results, to use the external force a priori in the equations of motion and to find as a result the pressure dipole and the velocity field by means of solving these equations. By this the lifting surface theory looks like a contact problem in the theory of elasticity where unknown external force actions are used more frequently than in hydrodynamics. Also the relationship between linearized actuator surface theory and linearized lifting surface theory becomes more clear. The latter one being in essence a special case of the first one.

The author hopes that this contribution is in the spirit of Rein Timman who combined successfully several disciplines of science by which deeper understanding is promoted.

2. SOLUTION OF THE EQUATION OF MOTION.

We will use a Cartesian coordinate system (x,y,z) with respect to which the undisturbed fluid at infinity is at rest. We could have
introduced as well an incoming parallel flow however then the formulas will become complicated and it is more easy to apply a simple Galilei transformation to our results afterwards. The linearized equation of motion of the incompressible and inviscid fluid reads
\[ \frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho} \text{grad} \, p + \frac{1}{\rho} \vec{F} \]  
(2.1)
where \( \vec{v} \) is the disturbance velocity with components \( u, v \) and \( w \), \( t \) is the time, \( \rho \) the specific density and \( \vec{F}(x,y,z,t) \) is an external force field per unit of volume confined to a bounded region of space \( B \). The equation of continuity is
\[ \text{div} \, \vec{v} = 0. \]  
(2.2)
Application of the operator divergence to both sides of (2.1) yields
\[ \frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2} = \text{div} \, \vec{F}. \]  
(2.3)
The unique solution of this Poisson equation, that vanishes at infinity is
\[ p = -\frac{1}{4\pi} \iiint_{V} \frac{\text{div} \, \vec{F}}{R} \, d\xi \, d\eta \, d\zeta, \]  
(2.4)
where
\[ \vec{R} = (x-\xi, y-\eta, z-\zeta), \quad R = |\vec{R}|, \]  
(2.5)
and \( \xi, \eta \) and \( \zeta \) are variables of integration.
By partial integration we find
\[ p = \frac{1}{4\pi} \iiint_{B} \frac{\vec{F} \cdot \vec{R}}{R^3} \, d\xi \, d\eta \, d\zeta. \]  
(2.6)
Application of the operation rotation to both sides of (2.1) yields
\[ \text{rot} \, \frac{\partial \vec{v}}{\partial t} = \frac{1}{\rho} \text{rot} \, \vec{F}. \]  
(2.7)
The unique solution of this equation (/3/, page 165) that vanishes at infinity and satisfies condition (2.2) is
\[ \frac{\partial \vec{v}}{\partial t} = -\frac{1}{4\pi \rho} \text{rot} \, \int_{B} \frac{\vec{F}}{R} \, d\xi \, d\eta \, d\zeta. \]  
(2.8)
When we take the point \((x,y,z)\) outside the force region \( B \) it follows from (2.8) by partial integration and integration with respect to time
\[ \vec{v}(x,y,z,t) = -\frac{1}{4\pi \rho} \int_{t_0}^{t} \int_{B} \frac{\vec{F}}{R^3} \, d\xi \, d\eta \, d\zeta \, dt, \quad (x,y,z) \notin B, \]  
(2.9)
where we assumed that the fluid is at rest at \( t=t_0 \) and that the force field is switched on at \( t=t_0 \). Because outside the force region the velocity field is free of rotation there exists a potential function \( \phi(x,y,z,t) \) with
\[ \vec{v} = \text{grad} \, \phi, \quad (x,y,z) \notin B. \]  
(2.10)
From (2.9) we find
\[ \phi(x,y,z,t) = \frac{1}{4\pi \rho} \int_{t_0}^{t} \int_{B} \frac{\vec{F} \cdot \vec{R}}{R^3} \, d\xi \, d\eta \, d\zeta \, dt, \quad (x,y,z) \notin B. \]  
(2.11)
From the foregoing the following conclusions can be made. When the force field is switched off it follows from (2.6) that the pressure becomes zero everywhere. From (2.8) it follows that then the acceleration of the fluid particles becomes zero, hence the velocity field becomes independent of time. Of course these results are a consequence of the neglect of viscosity and the linearization of the equation of motion. In potential theory the field of a dipole is given by
where \( \tilde{A} / \) is the strength of the dipole or doublet and its axis is in the direction \( \tilde{A} \), this means that the positive values of its field are in that direction. Hence the pressure given in (2.6) is a superposition of pressure dipoles distributed over the region \( B \) everywhere in the local force direction. The potential \( \phi(2.11) \) is a superposition of the potentials of sink-source doublets, where the line from sink to source is in the local force direction.

We can make several specializations of the force distribution. For instance we can concentrate the time dependent force field on a finite part of a surface

\[
H(x,y,z,t) = 0, \quad (2.13)
\]

by this we have the linearized theory for an actuator surface. In the special case that \( H \) is independent of time and the force field is perpendicular to \( H \) we can use a time dependent force field to represent a lifting surface moving along \( H=0 \).

Also it is possible to concentrate the force field in a point (singular force) which moves in a general way through space. Then we put

\[
\tilde{F}(x,y,z,t) = \tilde{F}(t) \delta(x-\tilde{x}(t)) \delta(y-\tilde{y}(t)) \delta(z-\tilde{z}(t)), \quad (2.14)
\]

where \( \delta \) is the Dirac delta function and the sufficiently smooth line \( L=(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)), \quad t_0 \leq t, \) is the path along which moves the point of application \( Q \) of the singular force \( \tilde{F}(t) = (f_x(t),f_y(t),f_z(t)) \) (Figure 2.1). The magnitude of this force is denoted by \( f(t) = \tilde{F}(t) \). By (2.14) our previous results (2.6), (2.9) and (2.11) change into

\[
p = \frac{1}{4\pi} \frac{\tilde{F} \cdot \tilde{R}}{R^3} \quad (2.15)
\]

\[
\vec{v} = -\frac{1}{4\pi \rho} \int_{t_0}^{t} \frac{\tilde{F} - 3 \tilde{A} \cdot \frac{\tilde{r}R}{R^3}}{R^5} dt,
\]

\[
(x,y,z) \neq (\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)), \quad (2.16)
\]

\[
\phi = -\frac{1}{4\pi \rho} \int_{t_0}^{t} \frac{\tilde{F} \cdot \tilde{R}}{R^3} dt,
\]

\[
(x,y,z) \neq (\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)). \quad (2.17)
\]

Now the pressure is represented by the field of one pressure dipole at the point \( Q \) of application of the singular force \( \tilde{F}(t) \). For the velocity potential many possibilities can happen of which we mention some.

When the direction \( \tilde{j} \) (unit vector) of the singular force \( \tilde{F}(t) \) and its point of application \( Q=(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) \) are independent of time we find that the velocity potential \( \phi \) has the simple form

\[
\phi = -\frac{1}{4\pi \rho} \int_{t_0}^{t} f(t) dt \cdot \frac{\tilde{r}R}{R^3}, \quad (2.18)
\]

hence \( \phi \) is the velocity potential of a sink-source doublet of which the strength is time dependent. Another case occurs when both \( \tilde{F}(f) \) and \( Q \) are time dependent while the velocity \( V=(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) \) is sufficiently smooth and satisfies the condition

\[
V = |\tilde{V}| > \tilde{V} > 0, \quad (2.19)
\]

where \( V \) is some positive number. Along \( L \) we introduce a length \( Q=(\tilde{x}(t),\tilde{y}(t),\tilde{z}(t)) \)

Fig.2.1. The force \( f(t) \) moving along the line \( l \).
parameter $s$. Then by (2.19) it is possible to write for the value of $s$ at the point $Q$.

$$s = s(t), \quad t = t(s), \quad s_0 = s(t_0). \tag{2.20}$$

In the following we can use the time or the length along $L$ as the variable on which quantities depend.

Hence we can rewrite (2.16) and (2.17) as

$$\begin{align*}
\vec{v}(x,y,z,t) &= -\frac{1}{4\pi \rho} \int_{s_0}^s \left( \frac{t}{\sqrt{R^2}} - \frac{3R \hat{r} \cdot \hat{R}}{\sqrt{R^2}} \right) \, ds, \quad (x,y,z) \notin L, \tag{2.21} \\
\phi(x,y,z,t) &= -\frac{1}{4\pi \rho} \int_{s_0}^s \frac{t}{\sqrt{R^2}} \, ds, \quad (x,y,z) \notin L, \tag{2.22}
\end{align*}$$

where the integrations are along $L$.

We now give a simple interpretation of (2.22) in terms of vorticity. Consider a small flat vortex ring of area $dS$ and strength $\Gamma$ at the point $Q = (\xi, \eta, \zeta)$ (Figure 2.2). The smallness of the ring is with respect to the other dimensions of the problem namely the radii of curvature of $L$ and the length of $L$. The direction of the vorticity $\Gamma$ is coupled with a right hand screw to its locally induced velocities. We erect at the centre of this ring the unit normal $\hat{n}$, related to $L$ by a right hand screw. The potential $d\phi(x,y,z,t)$ of this vortex has the value (3/ page 170)

$$d\phi = -\frac{\Gamma}{4\pi} \frac{\hat{n} \cdot \hat{R}}{R^3} dS = -\frac{\Gamma}{4\pi} \frac{\cos \alpha}{R^2} dS, \tag{2.23}$$

where $\alpha$ is the angle between $\hat{n}$ and $\hat{R}$. From this it follows that we can consider (2.22) as a superposition of small ring vortices around the line $L$ enclosing an area $dS$, perpendicular to $\hat{r}(s)$, connected to the direction of $\hat{r}(s)$ by a right hand screw and of strength

$$\Gamma = \frac{1}{\rho} \frac{f(s)}{V(s)} ds. \tag{2.24}$$

More precisely we have to consider the limits $ds \to 0$ and $dS \to 0$ for the velocity potential induced by these ring vortices in order to obtain the velocity potential induced by the moving singular force $\vec{f}$.

Note that it is not at all essential that the small ring vortices are circular; they can have other shapes for instance rectangular.

We can split the vector function $\vec{f}(t)$ uniquely into two parts

$$\vec{f}(t) = \vec{h}(t) + \vec{g}(t), \tag{2.25}$$

where $\vec{h}(t)$ is perpendicular to $L$ and $\vec{g}(t)$ is tangent to $L$. Because the theory is linear we can add the velocities induced by these moving forces. In the next two sections we will discuss $\vec{h}(t)$ and $\vec{g}(t)$ separately.
3. THE SINGULAR FORCE PERPENDICULAR TO ITS VELOCITY

We now will elaborate further the picture of the vorticity left behind by the singular force, in this section with respect to the force perpendicular to its velocity. Suitable for this is the representation of the wake by small closed rectangular vortices in the plane perpendicular to the force. This vortex distribution follows also from the one behind a short lifting line of varying intensity moving through space.

![Fig. 3.1. Vortex representation of a moving force perpendicular to its velocity.](image)

At the place where the force is acting we consider three mutually orthogonal unit vectors. The vector \( \hat{i} \) tangent to \( L \) in the direction of increasing \( s \), the vector \( \hat{j} \) along \( \hat{h} \) and the vector \( \hat{k} \) perpendicular to both, so that \( \hat{i}, \hat{j}, \hat{k} \) forms a right-handed system. Then we replace the force \( \hat{h} \) by a bound vortex of strength \( \Gamma(s) \), of length 2\( \varepsilon \) in the direction of \( \hat{k} \). The strength of this vortex follows from

\[
\rho \Gamma(s) V(s) 2 \varepsilon = h(s), \quad h = |\hat{h}|
\]

and because \( \hat{h} \) is the force exerted on the fluid, \( \Gamma \) is with a right hand screw in the negative \( \hat{k} \) direction. In the neighbourhood of \( L \) we have two lines \( L_1 \) and \( L_2 \) which have the representation

\[
(\xi(s), n(s), \zeta(s)) = \varepsilon \hat{k}(s),
\]

where the \( +(-) \) sign belongs to \( L_1(L_2) \), along which move the tips of the short lifting line.

By the length parameter \( s \) on \( L \) we have also a parameter \( s \) on \( L_1 \) and \( L_2 \), however, this is no longer a length parameter.

Along \( L_1 \) we have a tip vortex of strength

\[
\frac{h(s)}{\rho V(s)} \frac{1}{2} \varepsilon,
\]

with a right hand screw in the +\( s \) direction and along \( L_2 \) we have a tip vortex of the same strength however in the -\( s \) direction. For \( s=s_0 \) we have a starting vortex of strength (3.3) in the +\( k \) direction. At last we have distributed vorticity along \( L \) in the +\( k \) direction of strength

\[
\frac{1}{\rho} \frac{d}{ds} h(s) \frac{1}{V(s) 2 \varepsilon},
\]

per unit of length in the \( s \) direction. These four types of vorticity are such that i) the desired force \( \hat{h}(s) \) is induced and ii) the vortex field is free of divergence. It can also be formed by rectangular closed vortices mentioned in the first paragraph of this section. We now show that this vortex field in the limit \( \varepsilon \to 0 \) induces the same velocity field as is given in (2.21).

In order to do this we use the law of Biot and Savart, which states the following. Consider a line element \( ds \hat{i} \) at the point \( (\xi, n, \zeta) \) where \( \hat{i} \) is a unit vector, with a vortex of intensity \( \Gamma \), coupled with a right hand screw to \( \hat{i} \). Then the velocity \( \vec{v} \) induced by this vortex element is

\[
\vec{v} = \frac{\Gamma}{4\pi} \left( \hat{i} \times \frac{\hat{k}}{R^3} \right) ds.
\]

where * denotes the cross product.

By (3.5) the tip vortex along \( L \), induces at the point \( (x,y,z) \) while the force is at \( s=s_1 \), the velocity
\[ \mathbf{v}_{l_1}(x,y,z,s_1) = \]

\[ \frac{1}{4\pi\mu} \int_{s_o}^{s_1} \frac{h(s)}{V(s)} \left( \frac{\hat{r} + \epsilon \frac{d\hat{r}}{ds}}{R - \epsilon k} \right) \cdot \frac{(R + \epsilon k)}{R - \epsilon k} \cdot ds. \]  

(3.6)

Analogously the tip vortex along \( L_2 \) induces the velocity at \((x,y,z)\)

\[ \mathbf{v}_{l_2}(x,y,z,s_1) = \]

\[ -\frac{1}{4\pi\mu} \int_{s_o}^{s_1} \frac{h(s)}{V(s)} \left( \frac{\hat{r} - \epsilon \frac{d\hat{r}}{ds}}{R - \epsilon k} \right) \cdot \frac{(R + \epsilon k)}{R - \epsilon k} \cdot ds, \]  

(3.7)

when the force is at \( s=s_1 \). Hence adding these two contributions and taking the limit \( \epsilon \to 0 \),

\[ \mathbf{v}_{l_1} + \mathbf{v}_{l_2} = \]

\[ \frac{1}{4\pi\mu} \int_{s_o}^{s_1} \frac{h(s)}{V(s)} \frac{d}{ds} \left( \left( 1 + \lambda \frac{d}{ds} \right) \right) \cdot \frac{(R - \lambda k)}{(R - \lambda k)} \cdot ds = \]

\[ = \frac{1}{4\pi\mu} \int_{s_o}^{s_1} \frac{h(s)}{V(s)} \cdot 3 \cdot \frac{(R - \lambda k)}{R} \cdot \hat{r} + \frac{1}{R^3} \frac{d}{ds} \hat{r} \cdot \hat{r} \cdot ds \]

\[ - \frac{1}{R} \hat{r} \cdot \hat{k} \cdot ds. \]  

(3.8)

By partial integration we find

\[ \int_{s_o}^{s_1} \frac{h(s)}{V(s)} \frac{1}{R^3} \frac{d}{ds} \hat{r} \cdot \hat{r} ds = \hat{r} \cdot \frac{R}{R^3} \int_{s_o}^{s_1} \frac{h(s)}{V(s)} ds \]

\[ + \int_{s_o}^{s_1} \left( \frac{d}{ds} \frac{h(s)}{V(s)} \right) \frac{1}{R^3} \frac{\hat{r} \cdot \hat{r}}{R^3} ds \]

\[ + \int_{s_o}^{s_1} \frac{h(s)}{V(s)R^3} \hat{r} \cdot \hat{k} ds \]

\[ \int_{s_o}^{s_1} \frac{h(s)}{V(s)} \frac{1}{R^3} \frac{d}{ds} \hat{r} \cdot \hat{r} ds = \]

\[ -3 \int_{s_o}^{s_1} \frac{h(s)}{V(s)} \frac{(R \cdot \hat{r})}{R^3} \hat{r} \cdot \hat{k} ds. \]  

(3.9)

Hence

\[ \mathbf{v}_{l_1} + \mathbf{v}_{l_2} = \frac{1}{4\pi\mu} \int_{s_o}^{s_1} \frac{h(s)}{V(s)} \hat{r} \cdot \hat{r} ds \]

\[ + \int_{s_o}^{s_1} \frac{h(s)}{V(s)R^3} \hat{r} \cdot \hat{k} ds \]

\[ + \frac{1}{4\pi\mu} \int_{s_o}^{s_1} \frac{h(s)}{V(s)} \frac{d}{ds} \frac{h(s)}{V(s)} \frac{1}{R^3} \hat{r} \cdot \hat{r} ds \]

\[ + \int_{s_o}^{s_1} \frac{h(s)}{V(s)R^3} \hat{r} \cdot \hat{k} ds. \]

(3.10)

The contributions from the starting vortex and the bound vortex representing the force, are

\[ \left( - \frac{1}{4\pi} \frac{h(s)}{V(s)} \frac{1}{2\epsilon} \frac{\hat{r} \cdot \hat{k}}{R^3} \right) \int_{s_o}^{s_1} \frac{h(s)}{V(s)} ds \]

\[ = - \frac{1}{4\pi} \frac{h(s)}{V(s)R^3} \hat{r} \cdot \hat{k} ds. \]  

(3.11)
At last the contribution of the distributed vorticity of strength \(3.4\) in the \(+k\) direction is
\[
\frac{1}{4\pi\mu} \int_{s_0}^{s_1} \frac{d}{ds} \left( \frac{f(s)}{V(s)} \right) \frac{\hat{k} \cdot \hat{R}}{R^3} ds. \tag{3.12}
\]
Hence by adding (3.10), (3.11) and (3.12) we find for the total velocity induced by the vorticity distributions
\[
\vec{v} = \frac{1}{4\pi\mu} \int_{s_0}^{s_1} \frac{h(s)}{V(s)} \left[ \frac{\hat{R} \cdot \hat{k}}{R^3} + \frac{3}{R^2} \hat{i} \cdot \hat{R} \right. \\
\left. \frac{3}{R^2} \frac{\hat{R} \cdot \hat{k}}{R^3} - \frac{2}{R^3} \hat{i} \cdot \hat{R} \right] ds. \tag{3.13}
\]
The question is now if the velocity (3.13) is equal to the velocity given in (2.21). Comparing the integrands in both formulas we have to prove
\[
\frac{1}{4\pi\mu} \int_{s_0}^{s_1} \frac{d}{ds} \left( \frac{f(s)}{V(s)} \right) \frac{\hat{k} \cdot \hat{R}}{R^3} ds. \tag{3.14}
\]
This can be done most easily by taking scalar products of both sides with \(\hat{i}, \hat{j}\) and \(\hat{k}\) successively. In each of these three cases equality is clear, hence (3.14) is correct and the elementary time dependent vorticity drawn in Figure 3.1 can be considered as to be induced by the moving singular force perpendicular to its velocity.

4. THE SINGULAR FORCE IN THE DIRECTION OF ITS VELOCITY.

In the case of a singular force \(\vec{g}(t)\) in the direction of its motion we write (2.22) as
\[
\phi(x,y,z,t) = -\frac{1}{4\pi\rho} \int_{s_0}^{s(t)} \frac{g(s)}{V(s)} \frac{\cos \alpha}{R^2} ds, \tag{4.1}
\]

Because these types of forces are useful in propulsion theory we reckon \(g(s)\) to be positive in the negative \(s\) direction. The angle \(\alpha\) is then defined as the angle between the tangent to \(L\) pointing in the negative \(s\) direction and \(\hat{R}\).

We now rewrite (4.1) as
\[
\phi = \frac{1}{4\pi\rho} \int_{s_0}^{s(t)} \frac{g(s)}{V(s)} \frac{d}{ds} \left( \frac{1}{R} \right) ds. \tag{4.2}
\]
By partial integration we obtain
\[
\phi = \frac{1}{4\pi\rho} \int_{s_0}^{s(t)} \frac{g(s)}{V(s)} \frac{1}{R(s)} - \frac{g(s)}{V(s)R(s)} + \\
- \int_{s_0}^{s(t)} \frac{1}{R} \frac{d}{ds} \left( \frac{g(s)}{V(s)} \right) ds, (x,y,z) \in L. \tag{4.3}
\]
This formula can be given a simple interpretation. It is well known that a source placed at \((\xi, \eta, \zeta)\), which yields a unit volume of fluid per unit of time, has the velocity potential
\[
-\frac{1}{4\pi} \frac{1}{R}. \tag{4.4}
\]
Hence (4.3) has the following meaning. On the line \(L\) we have a source distribution of strength
\[
\frac{1}{\rho} \frac{d}{ds} \left( \frac{g(s)}{V(s)} \right) = \frac{1}{\rho} \frac{d}{dt} \left( \frac{g(t)}{V(t)} \right). \tag{4.5}
\]
At the starting point \(s=s_0\) we have a starting source of strength
\[
\frac{g(s_0)}{\rho V(s_0)}, \tag{4.6}
\]
at the point where the force acts, we have a source of strength
\[
\frac{-g(s)}{\rho V(s)}. \tag{4.7}
\]
hence a sink. When the force field is switched off at $t=t_e$ for $s=s_e$, there remains an ending source of strength (4.7) as follows from the remark below (2.11).

The velocity field follows from (4.3) by

$$
\hat{v}(x,y,z,t) = \frac{1}{4\pi \rho} \left( \frac{g(s)\hat{R}(s)}{V(s)R^3(s)} + \frac{g(s)\hat{R}(s)}{V(s_o)R^3(s_o)} \right) + \int_{s_o}^{s} \frac{\hat{R}(s)}{R^3(s)} \frac{d}{ds} \left( \frac{g(s)}{V(s)} \right) ds, \quad (x,y,z) \notin L.\quad (4.8)
$$

It seems that by the representation of the potential function by a combination of the potentials of a source distribution along $L$ (4.5) and two sources (4.6) and (4.7) the divergence of the flow is no longer zero, hence we don't satisfy (2.2). By integrating (4.5) along $L$ and adding to the result (4.6) and (4.7) it is seen that the total divergence in the fluid is zero. However, also locally the divergence of the velocity field is zero. What we have left out of consideration is the flow inside the narrow vortex tube around $L$ which follows from the vortex representation given in section 2. Because the shape of the small vortex rings around $L$ is irrelevant we choose them circular with radius $b$, they are locally perpendicular to $L$ in the case of force $g$. We find from (2.23) for the vortex strength $\gamma$ per unit of length along $L$

$$
\gamma(s) = \frac{1}{\eta b^2} \frac{g(s)}{V(s)},\quad (4.9)
$$

where $V(s) \geq V(0)$. This vorticity is reckoned positive when it induces a flow in the negative $s$ direction. When $b \to 0$ the velocities of the fluid inside the tube increase and become approximately $\hat{v}(s) = \gamma(s)$ reckoned positive in the negative $s$ direction. Hence the fluid transport per unit of time inside the tube of vanishing diameter, in the negative $s$ direction becomes

$$
\gamma(s) \cdot b^2 = \frac{g(s)}{\rho V(s)}.\quad (4.10)
$$

This singular mass transport along $L$ clearly meets the distributed sources (4.5) and the concentrated ones (4.6) and (4.7) in such a way that also locally no divergence occurs.

5. CONCLUDING REMARKS

The velocity field given in (2.21) yields the kernel function for linearized lifting surface and actuator surface theory. For instance consider a wing of prescribed geometry placed in a homogeneous flow with velocity $U$ parallel to the $x$ axis (Fig. 5.1).

![Fig. 5.1. Lifting surface in a homogeneous flow.](image)

Then we can consider the equivalent problem of a wing moving with velocity $U$ in a fluid at rest, which started at $x = +\infty$ and arrived at some time at the position drawn in Figure 5.1. We replace this wing by a distribution of elementary forces $f$ parallel
to the y axis of which the magnitude is equal to the still unknown pressure difference $\Delta p(x,z)$ between upper and lower side of the wing, multiplied by an element of area,

$$|\vec{f}(x,z)| = \Delta p(x,z) \, dx\,dz. \quad (5.1)$$

The total velocity induced by the pressure distribution at the wing, then follows from integrating over the planform the elementary velocities (2.21) induced by the elementary forces. Next by comparing the y component of this velocity to the local angle of incidence of the wing, the well known lifting surface integral equation is derived.

The same holds for the blades of a screw propeller of a ship where the line L of Figure 2.1 becomes helicoidal /2/. For the use of (2.21) in linearized actuator disk theory we refer to /4/.

REFERENCES


LATE PAPERS
SLENDER-BODY THEORY FOR LOW-REYNOLDS-NUMBER FLOWS

THEODORE Y. WU
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA 91125

Slender-body theory for low-Reynolds-number flows differs distinctly from its counterpart at high Reynolds number in several aspects. When all the inertial effects are negligible, an effective construction of solution requires a suitable distribution of fundamental singularities known as Stokeslet and its derivatives. Further, the Stokeslets have a long-range effect on the flow field, thereby inducing a mutual interaction between different parts of the body. To enlighten the general method of solution, the fundamental problem of a triaxial ellipsoid is examined with new results of alternative representations of the exact solution by surface distribution or centerplane distribution of flow singularities. Construction of solution by these means for the general case leads to integral equations of either the first or the second kind, which can be solved by numerical computations.

Some special cases in which the inertial effects become effective are explored; the result shows that they require a new approach to obtain solutions of uniform validity. Comments are also made on applications of the theory to microbiology, rheology of suspension, aerosol physics, environmental pollutants, and problems in other disciplines.

1. INTRODUCTION

Of the many fields in which Professor R. Timman's teaching and work have made an impact, his continuous interest in ship hydrodynamics was particularly important and his views have been greatly appreciated as to have shed much needed light on difficult problems of long standing. In his (1974) paper "Small parameter expansives in ship hydrodynamics" delivered at the Tenth ONR Symposium on Naval Hydrodynamics, Professor Timman chose a triaxial ellipsoid to elucidate a few fundamental problems of free-surface flows that had been of especial interest to the scientific community. That work of his exhibited his ingenuity in taking a full benefit of a known exact solution of a typically general problem to achieve a deeper understanding of some new problems — in that case the wedge-shaped bow problem of Ogilvie's (1972) and the horizontal cylinder problem of Herman's (1974), for both of which Professor Timman illuminated the main features of their solutions with a synthesis between the two approaches. Like many of Professor Timman's colleagues and friends, I have enjoyed from time to time the wonderful experience of exploring with Rein some
parts of fluid mechanics, or philosophy and humanity, and I would like to renew with you my memory on some of his advices. In his (1975) Keynote Address at the First International Conference on Numerical Ship Hydrodynamics, Professor Timman noted with emphasis that to obtain an accurate solution of the potential flow around a ship will be essential for both wave-field calculations and boundary-layer calculations. To overcome this primary problem he shared with many mathematicians and hydrodynamists his insight that Green's integral equation method would be a fruitful and effective approach. Along this general direction he also pointed out the need of caution in handling the integral equations of the first kind that arise from the centerplane-singularity method and its variations. These are but a few examples to show how enormously benefited we are from the privilege of knowing our beloved and respected friend and mentor Reinier Timman.

These thoughts have supplied stimuli in my attempt to extend in this paper the scope of study from the potential flow limit to the other extremity of vanishingly small Reynolds number. Although the physics is drastically changed by the presence of predominant viscous effects (and diminishing inertial effects) in the latter case, it has become clear to me that the various methods of solution, representations of the flow variables, optimum execution of analysis and numerical computation, etc., have many features in common for both cases. It also appears that a mathematical triumph made in one case is almost always beneficial for the other. It is in this sense that I hope this paper may complement Professor Nick Newman's and Professor Francis Ogilvie's on slender ship and other contributor's lectures during this Symposium in dedication to Professor Timman.

Before I proceed to discuss the problem in more detail, it is worthwhile to say a few words about the general features underlying this broad class of phenomena. In studies of microbiology, rheology of suspension, aerosol physics, environmental science, and flows through membrane and porous bodies, the fluid invariably has suspended microscopic particles or organisms. Their movement has at least three important features in common: (i) First, the motion is generally characterized by two (or more) small Reynolds numbers, one based on the translational velocity and the other on the rotational or oscillatory component of motion, say

$$\text{Re}_U = \frac{U l}{v}, \quad \text{Re}_\omega = \frac{\omega b^2}{v}, \quad (1)$$

where $U$ is the translational velocity of a body of length $l$, $\omega$ the angular velocity or oscillatory frequency of a body segment of dimension $b$ proper to the motion, and $v$ the kinematic viscosity of the medium. For very slender bodies, two translational Reynolds numbers based on the longitudinal and transverse body dimensions are found by Chwang and Wu (1976) to be indispensable. When all the specific Reynolds numbers are small ($< 1$), the viscous effects become dominant while the inertial effects, of both the fluid and the body, are negligible. However, the problem becomes more challenging when the Reynolds numbers have values in mixed orders.

(ii) The velocity field due to the motion of a body segment has the longest range known in fluid mechanics, and the motion is often of finite amplitude. For slender bodies, such as the flagella and cilia of micro-organisms and fibrillar pollutants in the atmosphere, the recent research has been primarily directed to improve the theory for evaluating more accurately the body-curvature effects on the motion of such particles, as discussed by Lighthill (1975, 1976) and Johnson (1976).

(iii) In a broad sense, movement of micro-organisms can be regarded as self-propelled, i.e. without aid of any extraneous force and moment of force, whereas the motion of inanimate small particles can be caused by external forces (e.g. gravity and electromagnetic forces) and by departure of the primary flow from a uniform state. In either case the physical state may be prescribed as

$$\Sigma \text{forces} = 0 \quad \text{and} \quad \Sigma \text{moments} = 0,$$

(2)

provided the external forces and moments, if any, are included while neglecting all the inertial forces as may be justified. When an external force is playing a role, the resulting motion of a body drags a large bulk of fluid with it, causing the fluid velocity to fall off in proportion to the inverse distance from the body. In contrast, the
fluid at some moderate distances away from a self-propelling micro-organism will already feel no trace of a net force and moment, although a distribution of forces and moments is always involved in forming the propulsive mechanism. These features combine to make the hydromechanical problem both interesting and challenging.

For the general case of a slender flexible body moving through a viscous fluid in arbitrary manner at low Reynolds number, the problem and the construction of its solution can become much enlightened from a close examination of the exact solution of a representative problem. It is therefore fitting that we start with the fundamental problem of a triaxial ellipsoid.

2. THE FUNDAMENTAL PROBLEM OF TRIAXIAL ELLIPSOIDS

The triaxial ellipsoid has played a fundamental role as a featuristic representative of three-dimensional bodies in the development of theories of attraction, potential flow, electrostatics, low-Reynolds-number flow, elasticity and other fields of mathematical physics. In view of their close analogies one may wonder if the structure of solutions to corresponding problems in these fields are interrelated. Indeed, owing to the special shape and symmetry of ellipsoids there exists a simple relationship between the potential flow past an ellipsoid in arbitrary motion and the gravity field of a homogeneous ellipsoid; its discovery can be traced back to Green (1833, p. 56 footnote; Mathematical Papers, p. 317; see also Wu and Chwang, 1974, p. 105). This remarkable relationship immediately facilitates solutions to potential flows involving ellipsoids by tapping on the resources of theory of attraction that dates back to Newton and the classic works of Gauss, Rodrigues, Dirichlet, Maclaurin and other most eminent mathematicians of the time.

In the development of the hydromechanics of Stokes flow of a viscous fluid, the problem of translation of an ellipsoid was solved by Oberbeck (1876), and the problem of rotation about one of its axis by Edwardes (1892) and Gans (1928, with a correction of the numerical factor in Edwardes result of the hydrodynamic moment). The motion of ellipsoidal particles suspended in a viscous fluid subject to a nonuniform primary flow (which is a linear function of the coordinates) and the corresponding effect on the bulk viscosity was evaluated by Jeffery (1922). In these investigations use was made of the ellipsoidal coordinates and ellipsoidal harmonics, in terms of which Oberbeck (1876) was the first to notice the analogy between the Stokes flow and the gravity field of the ellipsoid together with the electric field of a charged ellipsoid (or the gravity field of an ellipsoidal shell), as we shall also identify. The main purpose of our considering these classical results here is to convert the resulting expression into one in vector form which is then readily applicable to our subsequent study of the general case of a slender body performing arbitrary movement.

Insofar as the inertial effects are negligible, the low-Reynolds-number flow of a viscous fluid satisfies the Stokes equations:

\[ \nabla \cdot \mathbf{u} = 0, \]
\[ \nabla p = \mu \nabla^2 \mathbf{u} + f(x, t), \]

where \( \mathbf{u} \) denotes the flow velocity, \( p \) the pressure, \( \mu \) the dynamic viscosity coefficient (\( \mu = \mu v \), \( \rho \) being the constant fluid density), and \( f \) represents a generalized distribution of extraneous force which may depend on the radius vector \( x \) and time \( t \). In the absence of any external force, \( f \) vanishes within the flow region but may represent a distribution of forces and their moments within the body immersed in the fluid. This will be assumed to be the case.

The general solution of the Stokes equations (3) and (4) may be expressed as

\[ \mathbf{u} = 2\mathbf{x} - \nabla (x \cdot \mathbf{q}) + \nabla \varphi_c, \]

where

\[ p = -2\mu \nabla \cdot \mathbf{q}, \]
\[ \nabla^2 \varphi = 0, \] and \( \nabla^2 \varphi_c = 0. \]

This expression may be called the Oberbeck potential representation (of Stokes flow) in attribute to A. Oberbeck (1876, in which this representation was formulated; it is in complete analogy to the Papkovich-Neuber representation for the displacement vector of a linear elastic body, found independently by Papkovich (1932) and Neuber (1934). A special case of interest is given by \( \mathbf{q} = \nabla \times (x \varphi_c) \), which is a vector perpendicular to \( \mathbf{x} \) and is a
harmonic vector for arbitrary harmonic function $\phi_0$, and hence

$$u = 2 \nabla \phi \times \nabla \phi_c - (\nabla^2 \phi_0 = \nabla^2 \phi_c = 0), \tag{6}$$

the accompanying pressure field being identically zero. This complementary solution is due to Borchardt (1873) who derived it for the heat and elasticity studies. These general forms of solution, being expressed in terms of arbitrary harmonic functions, provide an effective means for identifying the analogy between Stokes flow problems and theory of attraction.

We can now proceed to investigate the Oberbeck potentials $\phi$ and $\phi_c$ for the Stokes flow past a triaxial ellipsoid

$$E: \sum_{i=1}^{3} \frac{x_i^2}{a_i^2} = 1 \quad (a_1 \geq a_2 \geq a_3), \tag{7}$$

which performs a translation with arbitrary velocity $\mathcal{U}$ and a rotation with arbitrary angular velocity $\Omega$ about an axis passing through the body centroid, the surrounding unbounded fluid being otherwise at rest. The no-slip boundary condition then requires that

$$u = \mathcal{U} + \Omega \times x \quad (x \in E). \tag{8}$$

The fluid at infinity will be assumed to remain undisturbed. Since the equations are linear, the problems of translation and rotation can be treated separately.

2.1 TRANSLATION OF AN ELLIPSOID

When $\Omega = 0$, the solution $(u, p)$ satisfying (3) and (4) for $x$ outside $E$ and the boundary condition $u = \mathcal{U}$ on the surface $S$ of $E$ has been given by Oberbeck (1876) in terms of the Oberbeck potentials as

$$\phi_1(x) = 2\pi a_1 a_2 a_3 \alpha_1 \int \frac{d\lambda}{\Delta(\lambda)} \quad (l = 1, 2, 3), \tag{9}$$

and the lower limit $\lambda(x)$ is the ellipsoidal coordinate of $x$, which is the algebraically largest root of

$$\frac{3}{\Sigma} \frac{x_i^2}{a_i^2} = 1. \tag{12}$$

$\lambda = 0$ corresponds to the ellipsoid $E$. Here $(\phi_1, \phi_2, \phi_3)$ are the Cartesian components of $\phi$. This solution also satisfies the condition $u = \mathcal{U}$ on $S$ if

$$\alpha_i = \frac{U_i}{2\pi(A_0 + A_1 a_1^2)}, \quad \beta_i = -\alpha_i^2 (i = 1, 2, 3), \tag{13}$$

where

$$A_0 = a_1 a_2 a_3 \int_0^{\infty} \frac{d\lambda}{\Delta(\lambda)}, \quad A_1 = a_1 a_2 a_3 \int_0^{\infty} \frac{d\lambda}{(a_1^2 + \lambda)\Delta(\lambda)}. \tag{14}$$

To ascertain the analogy of the above potentials $\phi_1$ and $\phi_c$ with those of gravity origin, we begin with the gravity potential of a homogeneous ellipsoid, of the same shape and size as $E$, expressed in terms of the Poisson integral

$$\Phi(x) = \int \frac{1}{R} d^3\xi \quad (R = |x - \xi|), \tag{15}$$

with the uniform mass density suitably normalized. For an interior point $(x \in E)$, the volume integral in (15) can be reduced, by elementary integration (see, e.g. MacMillan 1930), to the representation of Gauss (1813) and Rodrigues (1815),

$$\Phi(x) = \pi \left\{ A_0 \sum_{i=1}^{3} A_i x_i^2 \right\} = \pi a_1 a_2 a_3 \int_0^{\infty} \frac{d\lambda}{(1 - \Sigma i=1 \frac{x_i^2}{a_i^2 + \lambda}) \Delta(\lambda)} \quad (x \in E). \tag{16}$$

At an exterior point, (15) may assume Dirichlet's representation:

$$\Phi(x) = \pi a_1 a_2 a_3 \int_0^{\infty} \frac{d\lambda}{(1 - \Sigma i=1 \frac{x_i^2}{a_i^2 + \lambda}) \Delta(\lambda)} \quad (x \notin E). \tag{17}$$

This expression of Dirichlet for the exterior solution can readily be derived by applying the 'Splendid Theorem' of Maclaurin (as so referred
to by Thomson and Tait 1883):

The gravity potentials at the same \( x \) exterior to two individual ellipsoids of confocal shapes:

\[
\sum_{i=1}^{3} \left( \frac{x_i}{a_i} \right)^2 = 1, \quad \sum_{i=1}^{3} \left( \frac{x_i}{a_i'} \right)^2 = 1, \quad a_i^2 = a_i + \lambda, \quad i = 1, 2, 3 \tag{18}
\]

are equal if they have the same total uniform mass, that is

\[
a_1a_2a_3 = m'a_1' a_2' a_3', \tag{19}
\]

\( m' \) being the ratio of the mass density of \( E' \) to the mass density of \( E \).

A comparison between (10) and (17) immediately shows that

\[
\varphi_i = -\frac{\sigma_i}{R} \quad \text{or to the surface-doublet representation (see Wu & Chwang 1974) as}
\]

\[
\varphi_c(x) = \int_S (\mathbf{r} \cdot \mathbf{n}) \frac{\partial}{\partial n_x} \left( \frac{1}{R} \right) dS_x \tag{20a}
\]

where the usual summation convention is adopted. We thus see that \( \varphi_c \) represents the potential of a volume distribution of doublets of uniform density \( \mathbf{p} = (\beta_1, \beta_2, \beta_3) \) throughout the ellipsoid \( E \). This volume-doublet representation can also be converted to the following surface-source representation (by applying the divergence theorem)

\[
\varphi_c(x) = -\int \frac{n \cdot \mathbf{p}}{R} dS_x, \tag{20b}
\]

or to the surface-doublet representation (see Wu & Chwang 1974) as

\[
\varphi_c(x) = \int_S (\mathbf{r} \cdot \mathbf{n}) \frac{\partial}{\partial n_x} \left( \frac{1}{R} \right) dS_x \tag{20c}
\]

Here \( n \) denotes the unit outward normal to \( S \).

To find the gravitational or electric analog of \( \varphi_i \) given by (9), we first note that if we continue this harmonic function into the interior of \( E \) by letting \( \varphi_i = \text{const} = 2\pi\alpha_i A_0 \) for \( x \) within \( E \), then \( \varphi_i \) is continuous throughout the entire space. But the first derivative of this extended \( \varphi_i \) field is discontinuous along the normal direction at the surface \( S \) of \( E \), this \( \varphi_i \) thus represents a surface source distribution of density (strength per unit area of \( S \))

\[
\sigma_i = \frac{1}{4\pi} \frac{\partial \varphi_i}{\partial n_x} \frac{\partial \varphi_i}{\partial n_y} = \frac{1}{4\pi} \frac{\partial \varphi_i}{\partial n_x} \frac{\partial \varphi_i}{\partial n_y} \quad \text{at} \quad (n = 0) \tag{21a}
\]

where

\[
N(x) = \mathbf{r} \cdot \mathbf{n} = \left( \frac{3}{i=1} \sum \frac{x_i^2}{a_i^2} \right) - \frac{1}{3} \quad (x \in S) \tag{21b}
\]

is the distance from the center to the plane tangent to \( S \) at \( x \). Consequently, \( \varphi_i \) has the alternative expression

\[
\varphi_i = -\int \frac{\sigma_i}{R} dS_x = \alpha_i \int \frac{n \cdot \mathbf{r}}{R} dS_x. \tag{22}
\]

In view of the fact that \( \varphi_i \) is constant on \( S \) and falls off like \( |x|^{-1} \) at infinity, \( \varphi_i \) is obviously analogous to the electric potential of a charged ellipsoidal conductor held at a specified potential, \( \varphi_i \) being the corresponding surface charge distribution. We further note that this potential is also analogous to the gravity potential of a homoeoid which has the same shape as \( S \) of the ellipsoid. A homoeoid, by definition, is a massive shell of vanishing thickness, obtained by the limiting process of compressing together two adjacent similar (and concentric) ellipsoidal surfaces which contain a material homogeneously distributed throughout the process (see, e.g., Kellogg 1953, p. 191).

Having converted the original expressions for \( \varphi_i \) and \( \varphi_c \) into vector form as given by (20) and (22), we observe that these representations can be subject to further variations which are useful for guiding constructions of solution for the general case of arbitrary body shape and movement.

(i) Surface-singularity representation

By substituting (20) and (21) in (5), we obtain for the velocity field due to a translating ellipsoid as

\[
\mathbf{u} = \int_S \left( n \cdot \frac{\mathbf{r}}{R} \right) U_S(x - \mathbf{r}, \mathbf{a}) dS_x - \nabla \cdot \left( \mathbf{r} \cdot \mathbf{n} \right) \frac{U_S(x, \mathbf{a})}{R} dS_x, \tag{23}
\]

where \( U_S(x, \mathbf{a}) \) denotes the Stokeslet defined by

\[
U_S(x, \mathbf{a}) = \mathbf{a}/r + (\mathbf{a} \cdot \mathbf{r}) x/r^3 \quad (r = |x|), \tag{24}
\]

\( U_S \) is the fundamental solution of (3) and (4) corresponding to the singular point force

\[
\mathbf{f}_S = 8\pi \rho \mathbf{a} \delta(x), \tag{25}
\]

\( \delta(x) \) being the three-dimensional Dirac function (see e.g., Batchelor 1967; Chwang & Wu 1975).
The first integral in (23) therefore signifies a surface distribution of Stokeslets, pointed in the \( \mathbf{a} \) direction and with their magnitude varying with \( \xi \) in proportion to \( (n \cdot \xi) \), or the distance from the center to the tangent plane at \( \xi \) on \( S \). This illustrates the trend of the Stokeslet to build up at the points of greatest curvature, just like the way electric charges are distributed over an ellipsoidal conductor. The second integral in (23) represents a surface distribution of mass source with its density dependent on both \( \mathbf{a} \) and \( \mathbf{b} \). It is easily shown that the total source strength is zero. With \( \mathbf{a} \) and \( \mathbf{b} \) already determined, (23) is the surface-singularity representation of the exact solution for \( u \).

(ii) Centerplane-singularity representation

By applying Maclaurin’s theorem, the integral representation (20a) of \( \varphi_c \) can be transformed into a volume distribution of doublets throughout an ellipsoid, \( E' \), which is interior and confocal to \( E \):

\[
E': \sum_{i=1}^{3} \frac{(x_i/a_i)^2}{k_i} = 1, \quad (a_i)^2 = 1 + \lambda_o (a_3^2 < \lambda_o < 0),
\]

with the result

\[
\varphi_c(x) = \frac{\lambda_1 a_2 a_3}{a_1 a_2 a_3} \beta_1 \int_{E'} \frac{\partial}{\partial x_i} \left( \frac{1}{r} \right) d^3 \xi. \tag{27}
\]

In the limit as \( \lambda_o \to a_3^{-2} \), \( E' \) collapses to the focal ellipse

\[
E_o: \frac{x_1^2}{a_0^2} + \frac{x_2^2}{b_0^2} = 1, \quad (a_0) = a_1 - a_3, \quad (b_0) = a_2 - a_3,
\]

and the corresponding limit of (27) is found as

\[
\varphi_c(x) = \int_{E_o} \beta_1^*(\xi, \eta), \quad \frac{\partial}{\partial x_i} \left( \frac{1}{R} \right) d^3 \eta \quad (R_o = |x - \xi|, \xi = (\xi, \eta, 0)), \tag{29a}
\]

where

\[
\beta_1^*(\xi, \eta) = 2 \beta_1 \frac{a_1 a_2 a_3}{a_0 b_0} \left( 1 - \frac{\xi^2}{a_0^2} - \frac{\eta^2}{b_0^2} \right)^{\frac{1}{2}} \quad (i = 1, 2, 3). \tag{29b}
\]

Thus in this ultimate limit, the doublet distribution of \( \varphi_c \) over the focal ellipse at the centerplane of \( E \) assumes the ellipsoidal density \( \beta_1^*(\xi, \eta) \).

For the homoeoid potential \( \varphi_1 \) given by (9), or equivalently by (22), it can be shown that the Maclaurin Theorem, in a corollary form, is also applicable to confocal homoeoids so that, as an alternative to (22).

\[
\varphi_1(x) = \frac{a_1 a_2 a_3}{a_1 a_2 a_3} \alpha_1 \int_{S'} \frac{n \cdot \xi}{R} dS_\xi \tag{30}
\]

where \( S' \) is the surface of \( E' \) prescribed by (26). The truth of this result can be established by the argument that if two confocal homoeoids, \( S \) and \( S' \), of equal masses (within their infinitesimally thin shells) are formed by removing the same amount of mass from each of the two confocal ellipsoids, \( E \) and \( E' \), which are also of equal masses, then the result follows since the attractions are thus reduced by the same amount at any \( x \) exterior to both \( E \) and \( E' \). Alternatively, if we write the potential of the homoeoid \( S' \) as

\[
\varphi_1(x) = a_1^* \int_{S'} \frac{n \cdot \xi}{R} dS_\xi = 2 \pi a_1^* a_2^* a_3^* \alpha_1^* \int_{E} \frac{d\lambda'}{\omega(\lambda')},
\]

then by changing the integration variable to \( \lambda = \lambda^* + \lambda_o \) and using (26) we obtain by comparing this \( \varphi_1^* \) with (9), that \( \varphi_1^*(\xi)/\varphi_1(x) = a_1^* a_2^* a_3^* \alpha_1^*/a_1 a_2 a_3 \alpha_1 \), from which (30) follows if we require \( \varphi_1^*(\xi) = \varphi_1(x) \).

Now, in the limit as \( \lambda_o \to -a_3^2 \), the homoeoid \( S' \) tends to the focal ellipse \( E_o \), and the corresponding limit of \( \varphi_1 \) can be derived from (30) by expanding \( R \) about \( \xi_3 = 0 \) and converting the integration over \( S' \) to one over \( E_o \) by

\[
\frac{n \cdot \xi_3}{R_o} dS_\xi = \pm d\xi_1 d\xi_2 d\xi_3 = \pm dS_\eta,
\]

where the \( \pm \) sign conforms with the sign of \( \xi_3 \), with the results:

\[
\varphi_1(x) = \int_{E_o} \alpha_1^*(\xi, \eta), \quad \frac{d\xi}{d\eta} = (\xi, \eta, 0), \tag{31a}
\]

where

\[
\alpha_1^*(\xi, \eta) = 2 \alpha_1 \frac{a_1 a_2 a_3}{a_0 b_0} \left( 1 - \frac{\xi^2}{a_0^2} - \frac{\eta^2}{b_0^2} \right)^{\frac{1}{2}}. \tag{31b}
\]

Finally, the centerplane-singularity representation of \( u \) is obtained by substituting (29) and (31)
in (5), with the result (after some simple manipulation)
\[ u = \int \frac{U_0(x - \xi \alpha)}{E_0(1 - \frac{\xi^2}{a_o^2} - \frac{\eta^2}{b_o^2})} \, d\xi \, d\eta + \nabla \cdot \left\{ \rho_o \int \left( 1 - \frac{\xi^2}{a_o^2} - \frac{\eta^2}{b_o^2} \right)^{-\frac{1}{2}} \, d\xi \, d\eta \right\}, \]
(32a)
where
\[ \alpha_o = 2 \frac{a_1 a_2 a_3}{a_0 b_o^3} \alpha, \quad \beta_o = - \frac{2}{a_0} \alpha_o, \quad (32b) \]
\[ \alpha = (\alpha_1, \alpha_2, \alpha_3) \] being given by (13). The first integral above represents a centerplane distribution of Stokeslets, which are now unidirectional along \( \alpha_o \) and with an inverse ellipsoidal density over the focal ellipse. This Stokeslet distribution is thus singular along the boundary curve of \( E_o \). The second integral in (32) signifies a centerplane distribution of doublets which are directed opposite to \( \alpha_o \) and have an ellipsoidal density. We may note that as \( a_o \to 0 \), the ellipsoid \( E \) shrinks to its focal ellipse \( E_o \), \( \alpha_o \) actually tends to a finite limit (see (13) and (14)) and \( \beta_o \) vanishes like \( a_o^2 \). Thus in this limit only the Stokeslet distribution remains, as should be expected.

The expression (32) includes the oblate spheroid as a special case, for which \( a_1 = a_2 > a_3 \), and \( a_o = b_o \).

(iii) **Focal-axis distribution for prolate spheroids**
For a prolate spheroid \((a_1 > a_2 = a_3)\), the focal ellipse \( E_o \) further reduces to the focal axis, 

\[-c < x_1 < c, \quad c = a_o = a_1 - a_3 \frac{b_o}{a_o}. \]
The expression for \( u \) corresponding to (32) can be obtained for this case by a procedure similar to the above by expanding \( (x - \xi) \) and \( R \) in (32) about \( \eta = 0 \), then integrating with respect to \( \eta \), and subsequently letting \( b_o \to 0 \). The final result reads
\[ u = \int U_0(x - \xi \alpha_1) \xi_1 \xi_2 \xi_3 \, d\xi_1 \, d\xi_2 \, d\xi_3 + \nabla \cdot \left\{ \rho_c \int \left( 1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2} \right)^{-\frac{1}{2}} R \, dR \right\}, \]
(33a)
where \( R = |x - \xi_1 \xi_2| \), and
\[ \alpha_c = 2\pi(a_1 a_2^2/c) \alpha, \quad \beta_c = -\frac{1}{2} a_2^2 \alpha_c. \]
(33b)
This result agrees with the solution previously obtained by Chwang and Wu (1975).

### 2.2 ROTATION OF AN ELLIPSOID

For a triaxial ellipsoid in pure rotation, so
\[ u = \Omega \times \xi (\xi \in S), \]
the solution given by Edwardes (1892) can be recast in terms of the Oberbeck potentials as
\[ \varphi_1 = \gamma_{ijk} \frac{\partial \Phi}{\partial x_j} - \gamma_{ikj} \frac{\partial \Phi}{\partial x_k} \quad \text{and} \quad \varphi_c = 0, \]
(34)
where \( \Phi(\xi) \) is given by (17), or equivalently by (15) for \( \xi \notin E \), and \( (i, j, k) \) assume any even permutations of \( (1, 2, 3) \). We note that in general
\[ \gamma_{ijk} \neq \gamma_{ikj} \] Substituting (34) in (5) yields
\[ u_1 = \varphi_1 - x_1 \frac{\partial \varphi}{\partial x_1} \quad (l = 1, 2, 3). \]
(35)
This solution satisfies the condition \( u = \Omega \times \xi \) on \( S \)
\[ \gamma_{ijk} = \frac{a_j^2 \Omega}{2\pi(a_1 A_i + a_j A_j)} \quad (i \neq j \neq k), \]
(36)
where \( A_i \) is given by (14).

Different variations of solution representation for the above \( u_1 \) can be obtained by the same procedure as in the previous case. Since the present solution involves only one harmonic function \( \Phi \), and its representations by volume distribution, centerplane distribution, etc. can be found in the previous section, the corresponding detailed expressions for \( u \) will be omitted here. We shall only remark that in the case of centerplane-singularity representation, the singularities involved are of three types, namely the stresslet, the rotlet and the doublet (for their definition, see (43) below; also Chwang & Wu 1975).

The above discussion has presented various possible representations of the exact solution to the triaxial ellipsoid problem. Of these expressions the centerplane-singularity version appears to be especially suitable for application to the general case of slender body movement since it is already in vector-tensor form and can readily be put in a fluid-frame or a body-frame of reference. The following discussion of general slender-body theory, however, will be limited in one aspect to the case of a slender body with only circular cross sections.
3. GENERAL SLENDER-BODY THEORY

The classical slender-body theory for Stokes flow is based on the pioneering work of Gray and Hancock (1955). Several modifications of it have been proposed by different authors for possible improvement (for some recent reviews, see Brennen & Winet 1977; Wu 1977). Although the theory is attractive for its simplicity in application, it is still regarded as somewhat too crude for making more refined studies by our modern standard. The main reasons for its insufficient accuracy are because it neglects the end effect (due to the presence of body ends) and the body-centerline-curvature effect when the body takes a finite departure from a stretched-straight position during its movement.

To overcome these drawbacks Johnson (1976) has developed an integral equation method for the general case of a slender body with circular cross section $b(s)$ of arbitrary but small gradient along the arc length $a$ of the body centerline:

$$x = x_0(s, t) \quad (-1 \leq s \leq 1)$$

except for its spheroidal-shaped ends:

$$b = e(t^2 - s^2)^{1/2} \quad \text{for} \quad (1 - s^2/t^2) \ll 1, \quad e = b_o / t \ll 1,$$

$b_o$ being a typical body radius. The movement of $x_0$ is assumed arbitrary for a flexible body, but the centerline is assumed to be inextensible. At each point $x_0$ on the centerline we introduce the tangential, normal, and binormal base vectors $(e_s, e_n, e_b)$ as shown in Fig. 1, i.e.

$$e_s = x_0', \quad e_n = a x_0'', \quad e_b = e_s \times e_n,$$  

where the prime denotes differentiation with respect to $s$, and $a(s, t)$ is the centerline radius of curvature, $a = |x_0''|^{-1}$.

A material point on the body surface has a translational velocity $\dot{x}_0 = \partial x_0 / \partial t$, a rotation of centerline with angular velocity $\Omega = e_s \times \dot{x}_0$, and a surface spin about centerline with angular velocity $\Omega = e_s \times e_b$, $\Omega(s, t)$ being arbitrary. Thus the no-slip boundary condition requires that on the body surface

$$u = \dot{x}_0 + b(s)(e_s \times e_n) + e_b \times \dot{x}_0 \times e_b,

= V_s e_s + V_n e_n + V_b e_b,$$  

where

$$\xi = x_0 \cos \psi + e_b \sin \psi.$$

The solution of the resulting Stokes flow can be represented by the distribution of necessary fundamental singularities on the body centerline as follows:

$$c

\int \left( U_S(R; \omega_1) + U_D(R; \omega_2) + U_R(R; \omega_3) + U_m(R; \omega_4) + \frac{3}{2} (\omega_1 \cdot \omega_2) \frac{R}{R^3} \right) \, ds,$$

where

$$\omega = x - x_0(s, t), \quad R = |R|.$$  

$U_S$ denotes a Stokeslet defined by (24), $U_D$ the velocity of a source, $U_R$ a doublet, $U_Q$ a rotlet, $U_{SS}$ a stresslet, and $U_{QQ}$ a quadrupole, as defined by

$$U_m(R; \omega) = -m \nabla \frac{1}{R}, \quad U_D(R; \omega) = \nabla \cdot (R/|R|),$$

$$U_R(R; \omega) = \nabla \times (R/|R|), \quad U_{SS}(R; \omega) = -3 (\omega \cdot R) R/|R|^3.$$  

The limits of integration, $\pm c$, are given by the generalized foci $c = (l^2 - b^2)^{1/2}$.

To apply the boundary condition, the integrand in (41) is expanded in the neighborhood of the body
(i.e. for \( r_1 < \ell \)) at each station \( s \), then applying condition (40), we obtain, upon neglecting the terms of \( O(a \epsilon \log \epsilon) \), the following integral equation for \( \varphi \):

\[
V_0(s, t) = \alpha_0(s, t) I_0 + \int_{-c}^{c} K_0(R_0 \varphi) ds' \quad (\theta = s, n, b),
\]

(44)

where

\[
L_{s} = 2[2 \log(2/\kappa) - 1], \quad L_{n} = L_{b} = 2 \log(2/\kappa) + 1,
\]

\[
K_0(R_0 \varphi) = \frac{\alpha_0(s', t) \cdot \varphi(s', t) \cdot R_0 R_{0}'}{R_0^3} - \frac{D_0 \alpha_0(s, t)}{|s - s'|},
\]

\[
D_0 = 2, \quad D_n = D_b = 1,
\]

\[
\varphi(s, t) = \alpha_0(s, t) + \alpha_0(s, t) \cdot \frac{R_0 e_s + R_0 e_n + R_0 e_b}{|s - s'|} \cdot \frac{R_0}{R_0}.
\]

In (44) the basis vectors \( e_s, e_n, e_b \) assumes their values at \( s \) where \( V_0(s, t) \) is specified.

The integral equation (44), in spite of its appearance, is actually of the first kind since the free term (the first on the right) exactly cancels the contribution from the last term of the kernel \( K_0 \).

The present expression for the kernel \( K_0 \) is obtained by a uniformly valid composite expansion. This integral equation can be solved numerically by several existing methods, including, of those already tested out, the direct matrix inversion and a few iteration schemes.

These numerical methods have been applied by Johnson (1977) to evaluate sectional force coefficients of a flagellum performing a planar or a helical wave motion, possibly of finite amplitude. An interesting example given by Johnson is the planar flagellar motion as modeled by a slender spheroidal body whose centerline movement, when expressed in a reference frame translating with the propulsive velocity of the organism - \( U_0 \), is written as

\[
\mathbf{x}_0 = (x(s, t), -a \cos(ks - f(t)), 0) \quad (-\ell < s < \ell),
\]

(45)

in which \( x(s, t) \) is determined by the inextensibility condition. This prescribes a sinusoidal wave along the centerline, of wavelength \( \Lambda = 2\pi/k \), the corresponding wave form in the physical plane

being somewhat fuller about crests and troughs, and the wavelength, \( \lambda \), somewhat shorter \((4a < \lambda < \Lambda, \ a \) being the amplitude). The function \( f(t) \) is so chosen that the phase velocity, \( C \), in the \(+x\)-direction is a constant. This class of motion was found quite capable of describing some experimental data, including those for the spermatozoa Chaetopterus variopedatus. Figure 2 shows some numerical result of the integral equation (44) for the dimensionless normal force per unit arc length, \( f_n/\mu U \), of Chaetopterus at about the observed phase-velocity to propulsive-velocity ratio \( C/U = 5 \). It is of interest to note the salient features of the result that \( f_n \) depends appreciably on both \( s \) and \( t \), and is quite noticeably affected by the presence of body ends at \( s = \pm \ell \). Similar features have been found for the tangential force \( f_s \). In contrast, theoretical predictions by the Gray-Hancock formula, namely,

\[
f_n = \mu C_n V n', \quad f_s = \mu C_s V s',
\]

(46)

where

\[
C_n = 4\pi/(\log(2\ldots^{1/2}) \ldots, \quad C_s = \frac{1}{2} C_n,
\]

do not have these features, and often underestimate the correct result by 20% or more as indicated in Fig. 2 for a typical case.

Another interesting result is shown in Fig. 3, in which the average thrust predicted by using the solution of (44) for the flagellum of Chaetopterus is plotted versus the number of flagellar waves as the flagellum is hypothetically being lengthened, other factors being equal. The body-end effect

![Fig. 2 Comparison of the normal force per unit length at three time instants in equal progression intervals for a typical case.](image-url)
and body-curvature effect on propulsive thrust are vividly displayed as the thrust dips to a local minimum as the number of flagellar waves passes through each integer. We note that Chaetopterus, which has an observed value of $n = 1.25$, apparently operates in a 'favorable' region in regard to the variable $n$. It could be more than a coincidence that many different species of spermatozoa have been observed to have a value of their flagellar wave-number $n$ between 1.25 and 1.5 (Brennen & Winet 1977).

![Graph](image)

**Fig. 3** Variation of the average thrust with the flagellar wave number.

4. INERTIAL EFFECTS

Inertial effects will arise to cause changes in sectional force coefficients when any of the characteristic Reynolds numbers such as those listed in (1) becomes no longer small compared to unity. Such situations may occur in motions of large microscopic particles and organisms or in experiments with mechanical models whose corresponding Reynolds numbers cannot all be kept small in practice.

A clear explanation of how inertial effects can arise may be sought from a critical study of the translation of an elongated prolate spheroid for the case when the relevant Reynolds numbers are of mixed order in magnitude. Corresponding to the spheroid's translational velocity,

$$\mathbf{U} = U_1 \mathbf{e}_x + U_2 \mathbf{e}_y,$$

we introduce the Reynolds numbers

$$R_{a1} = U_1 a / \nu, \quad R_{b1} = U_1 b / \nu \quad \text{(for longitudinal motion)}, \quad (48a)$$

$$R_{a2} = U_2 a / \nu, \quad R_{b2} = U_2 b / \nu \quad \text{(for transverse motion)}, \quad (48b)$$

where $a$ and $b$ denote the semi-major and semi-minor axes as before. We shall assume $R_{b1} \ll 1$ and $R_{b2} \ll 1$, but leave $R_{a1}$ and $R_{a2}$ arbitrary. It may seem perplexing to note that it is necessary to consider $R_{a2}$ even when the motion is transverse to the major axis. The reason is because while one body segment exerts a force on the fluid due to its lateral motion, the inertial effect becomes comparable to the viscous effect at such large distances (from that body segment) that the local Reynolds number (based on the distance) is of order 1 or greater, and such a region includes some distant parts of the body when the body is sufficiently long.

The solution of this problem can be obtained based on the singular-perturbation technique, using the Stokes approximation for the inner field and the Oseen approximation for the outer field, and having the inner and outer expansions of the solution suitably matched for an intermediate region. Following this approach, an asymptotic solution for the transverse motion case has been provided by Chwang & Wu (1976) for the drag force $D_2$ of the spheroid (given here in coefficient form) as

$$C_{D_2} = \frac{D_2}{6\nu U_2 a} = \frac{8}{5} e \alpha_2', \quad (49)$$

where

$$\frac{1}{\alpha_2'} = \alpha_2^{-1} - 2[\log(\frac{1}{2} e a_2) + \gamma + E_1(\frac{1}{2} e a_2)]$$

$$\alpha_2 = 2 e^2[2e + (3e^2 - 1)] \log \frac{1 + e}{1 - e}$$

$$E_1(t) = \int_1^\infty e^{-t} \frac{dt}{t},$$

$e$ is the eccentricity of the spheroid and

$$\gamma = 0.5772...$$

is Euler's constant. As $R_{a2} \rightarrow 0$, $\alpha_2' \rightarrow \alpha_2$, and $D_2$ agrees in this limit with the Stokes flow solution of Oberbeck (1876). In the other extremity when $R_{a2} \rightarrow \infty$, the corresponding $D_2$ approaches the solution of Oseen (1910) for a two-dimensional cylinder. The numerical results of $D_2$ for intermediate values of $R_{a2}$ are shown in Fig. 4 for several small values of $R_{b1}$. This gradual transition from the Stokes limit to the Oseen limit thus provides a clear physical picture and explanation of the manifestation of
the 'Stokes paradox' known in viscous flow theory.

\[ \frac{C_{D1}}{6\pi\mu \frac{a}{b}} = \frac{8}{3} e^{\alpha_1}, \]  

where

\[ \alpha_1 = \alpha_1 - 2[\log(1 + e_1^2) + e_1 - 1], \]

\[ e_1 = e^{R_{a_1}^2}. \]

and \( R_{a_1}^2 \) is subject to two conditions, namely, \( R_{b_1} < 1 \) and \( R_{b_1} \leq R_{a_1} \); consequently \( R_{a_1} \) has an upper bound which we take

\[ R_{a_1} \leq R_{a_1}^* = 0.4 \frac{a^2}{b^2}. \]

This new result is illustrated in Fig. 5 which indicates the increasing departure of \( C_{D1} \) from its Stokes' limit (shown with arrows) with increasing \( R_{a_1} \) until it reaches the estimated bound (50b).

Fig. 4 Variation of the Stokes, the Oseen and the uniformly valid drag coefficient \( C_{D1} \) with the Reynolds numbers \( R_{a_1} \) and \( R_{b_2} \).

Fig. 5 Variation of the Stokes and the present drag coefficient \( C_{D1} \) with the Reynolds number \( R_{a_1} \) for various axis ratio \( a/b \).

From these two fundamental cases we realize that it is important to examine the inertial effects of significance in making theoretical predictions and interpreting experimental results. Development of the general slender-body theory under this amplified scope is still in its early stage, but its necessity should be fully recognized.

ACKNOWLEDGMENTS

I feel very much honored having this privileged opportunity to dedicate this work in fond memory to Professor Dr. Reinier Timmon whose broad influence has left a lasting illumination in the field of Applied Science.

Warmest thanks are due from me, as from all those present or to come, to the Department of Mathematics of the Delft University of Technology and all the Sponsoring Organizations for bringing forth this Symposium on Applied Mathematics.

This work was jointly supported by the U. S. Office of Naval Research, the National Science Foundation and the Army Research Office.
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ABSTRACT

Two aspects of maneuverability theory are discussed, the formulation of the equations of motion and the use of a planar-motion mechanism to measure constants and functions occurring in the linearized equations. It is shown that a certain proposed maneuver with a planar-motion mechanism allows experimental determination of the coefficients and functions occurring in the linearized maneuvering equations in the regions important for practical ship maneuvering and that this can be routinely done in only about twice the time required for a resistance test.

COORDINATE SYSTEMS

We shall use the same coordinate system used in the papers cited above. $O_{0} x, y, z_{0}$ is fixed in space with $O_{0} x_{0}, y_{0}$ being the undisturbed water surface and $O_{0} z_{0}$ directed downward. The system $Oxyz$ is fixed in the ship with $Oxz$ containing the longitudinal centerplane, $Oyz$ the midship section, and $Oxy$ the undisturbed water surface if the ship is at rest. $Ox$ points toward the bow. Both systems are right-
handed. They are shown in Figure 1. The center of mass of the ship is at \((x_G, 0, z_G)\).

**EQUATIONS OF MOTION**

We shall suppose that trim, squat and heel of the ship do not affect noticeably its surge, sway or yaw motion. If \(\mathbf{X}_O(t), \mathbf{Y}_O(t), \mathbf{Z}_O(t)\) are the coordinates of \(O\) in the frame \(O_O x O_O y O_O z_O\), then we may take \(\mathbf{Z}_O(t) = 0\). Let \(\psi(t)\) be the angle between \(O_x\) and \(O_O\) and \(\delta(t)\) the angle between \(O_x\) and the vector \((\mathbf{X}_O, \mathbf{Y}_O, 0)\). The water is assumed to be inviscid, the flow irrotational and the water surface smooth except for disturbances caused by surge, sway or yaw motion of the ship.

It is customary to write the equations of motion in the system \(Oxyz\) even though this has the disadvantage of being an accelerating system. Let \((\mathbf{U}, \mathbf{V}, 0)\) be the components of the velocity of \(O\) referred to \(Oxyz\); i.e.,

\[
\mathbf{U} = \dot{X}_O \cos \psi + \dot{Y}_O \sin \psi, \\
\mathbf{V} = -\dot{X}_O \sin \psi + \dot{Y}_O \cos \psi.
\]  

Then the equations are easily shown to be the following:

\[
m\left[\ddot{\mathbf{U}} - \dot{\mathbf{V}} + x_4 \dot{\mathbf{V}}^2\right] = \int p n_x \, dS + X_E, \\
m\left[\ddot{\mathbf{V}} + \dot{\mathbf{U}} + x_4 \dot{\mathbf{U}}^2\right] = \int p n_y \, dS + Y_E, \\
I_4 \dddot{\mathbf{U}} + m x_4 \left(\dddot{\mathbf{V}} + \dot{\mathbf{U}} \dot{\mathbf{V}}\right) = \int (p n_x - y n_y) \, dS + N_E.
\]

where \(m\) is the mass of the ship, \(I_4\) its moment of inertia about \(Oz\), \(X_E\) and \(Y_E\) external forces resolved along the axes \(Ox\) and \(Oy\) and \(N_E\) an external moment about \(Oz\). Here \(S\) is the wetted surface of the ship, \(p\) is the pressure and the components of the normal are along \(Ox\) and \(Oy\).

Equations (2) are exact except for the assumptions already mentioned. The difficulties lie in the integrals over \(S\): neither \(p\) nor \(S\) are known. Finding them means, of course, solving a nonlinear problem of hydrodynamics, and since this seems beyond doing analytically, some approximation must be made. Ideally the approximation should be part of a systematic scheme and should not restrict the path of the ship. In particular, one would like to avoid a scheme that allowed perturbations only about a straight-line path and constant velocity, although this is also a case of interest. Perhaps an assumption like

\[
\mathbf{U} = U(t) + \varepsilon U^{(1)}(t), \\
\mathbf{V} = V(t) + \varepsilon V^{(1)}(t),
\]

would be adequate, where \(\varepsilon\) is a small parameter. Then the curvature of the path is also \(O(\varepsilon)\):

\[
\kappa = \frac{\mathbf{U} \cdot \dot{\mathbf{V}} - \dot{\mathbf{U}} \cdot \mathbf{V} + (\mathbf{U}^2 + \mathbf{V}^2) \dot{\psi}}{[\mathbf{U}^2 + \mathbf{V}^2]^{3/2}} = \varepsilon \left[\frac{U^{(1)} \cdot \dot{V}^{(1)} + V^{(1)} \cdot \dot{U}^{(1)}}{U^{(2)} + V^{(2)}}\right] + O(\varepsilon^2).
\]

Since the frame \(Oxyz\) is fixed in the ship, certain developments from classical hydrodynamics of a body moving in an unbounded fluid can be extended to the case where there is a free surface. In particular, if \(S\) is the surface of the ship, \(S_F\) the free surface, and \(\Sigma\) a control surface, and if \(\phi(x,y,z,t)\) is the velocity potential referred to the moving coordinate system, then the force is given by

\[
F = -\rho \frac{d}{dt} \int \phi n dS - \int p g y n dS + \int \rho (\frac{1}{2} \mathbf{V} \cdot \mathbf{V}^2 - \mathbf{V} \cdot \mathbf{V}) dS.
\]

There is also an analogous equation for the moment \(\tilde{K}\) [for a derivation of both see Faltinsen, 1977]. Here \(\mathbf{V}\) is the velocity of the fluid referred to \(Oxyz\) and should not be confused with \(\mathbf{U}\) and \(\mathbf{V}\) of equations (1) to (4).

The real difficulties arise from the presence of the free surface, for the linearized condition takes the following complicated form in the frame \(Oxyz\):

\[
[-\mathbf{U} + y \dot{\mathbf{V}}] \phi_{xx}(x,y,0,t) + 2[-\mathbf{U} + y \dot{\mathbf{V}}] \phi_{xy} + \\
[-\mathbf{V} - x \dot{\mathbf{U}}] \phi_{yy} + 2[-\mathbf{V} - x \dot{\mathbf{U}}] \phi_{yt} + \\
2[-\mathbf{V} - x \dot{\mathbf{U}}] \phi_{tt} + \phi_{tt} + 
\]
The boundary condition on the surface of the ship is, of course, simple in the system Oxyz: If the hull surface is described by \( y = \pm f(x,z) \), then \( \nabla \cdot \mathbf{v} \) becomes

\[
\frac{\partial}{\partial x} f_x(x,z) + \frac{\partial}{\partial z} f_z(x,z) = \left( \frac{\partial}{\partial x} f_x(x,z) + \frac{\partial}{\partial z} f_z(x,z) \right) \cdot \mathbf{n}.
\]

Is there any hope of solving such a problem? Perhaps not, but that is not really our intention. What we should like to be able to do at present is to be able to find out something about the functional form of the pressure integrals on the right-hand side of (2). One may then attempt to devise means of measuring constants and functions that would otherwise be determined by solving boundary-value problems. Although I believe that this program can be carried through, I must unfortunately retreat to a simpler one mentioned earlier, that of small perturbations about a straight-line path with constant velocity.

This problem has been treated by Cummins (1962), by Ogilvie (1964) and by W.-C. Lin (1966) in a form most relevant to present needs. Lin has formulated the problem as a perturbation problem and has derived the equations determining the first-order terms. C.-L. Guo (unpublished) has carried this further and has determined also the form of the second-order terms. The general form of the pressure integrals according to the first-order theory has already been given in Frank et al. (1976). They turn out to be sums of terms like the following:

\[
\mu \mathbf{v}^{(0)} + b \mathbf{v}^{(1)} + \int_0^{\infty} \mathbf{v}^{(0)}(t-\tau) \mathbf{N}(\tau) d\tau.
\]

What should one expect for second-order terms? One might assume that they would be something like

\[
\mathbf{a} \mathbf{v}^{(1)} + \mathbf{b} \mathbf{v}^{(2)} + \mathbf{c} \mathbf{v}^{(1)}(t-\tau) \mathbf{N}(\tau) + \mathbf{d} \mathbf{v}^{(1)}(t-\tau) \mathbf{N}(\tau) d\tau + \ldots
\]

with perhaps some terms vanishing because of symmetries. One does get terms like this, but also others that can't be contained in this formalism without allowing \( \delta \)-functions. That is, one also obtains terms like

\[
\mathbf{b} \mathbf{v}^{(0)}(t) + \int_0^{\infty} \mathbf{v}^{(0)}(t-\tau) \mathbf{M}(\tau) d\tau + \int_0^{\infty} \mathbf{v}^{(0)}(t-\tau) \mathbf{P}(\tau) d\tau,
\]

Neither Lin's nor Guo's investigations include the effect of shed vorticity, but one knows that this is also an important ingredient in the determination of the pressure integrals. It is, in fact, difficult to include this, for, as far as I am aware, there is no completely satisfactory model of an 'exact' irrotational flow about a ship that includes shed vorticity. Consequently, one does not know how one should be improving a first-order approximation with which one has started.

In spite of such difficulties with the formulation of an exact model, there doesn't seem to be much doubt about the form of the first-order equations. These will be taken in the same form as in the paper by Frank et al. (1976):

\[
\begin{align*}
(\mathbf{m} + \mu \mathbf{xx}) \mathbf{v}^{(0)} + \beta \mathbf{xx} \mathbf{v}^{(0)} + \\
+ \int_0^{\infty} \mathbf{v}^{(0)}(t-\tau) \mathbf{N}(\tau) d\tau &= X(\mathbf{e}) t, \\
(\mathbf{m} + \mu \mathbf{yy}) \mathbf{v}^{(0)} + \beta \mathbf{yy} \mathbf{v}^{(0)} + \\
+ \int_0^{\infty} \mathbf{v}^{(0)}(t-\tau) \mathbf{N}(\tau) d\tau + (\mu \mathbf{yy} \mathbf{m} \mathbf{x} + \mathbf{x}) \mathbf{v}^{(0)} t + \\
+ (\beta \mathbf{yy} \mathbf{m} \mathbf{x} \mathbf{y} + \mathbf{x}) \mathbf{v}^{(0)} t + \\
+ (\mu \mathbf{yy} + \beta \mathbf{yy} \mathbf{x} + \mathbf{x}) \mathbf{v}^{(0)} + \\
+ \int_0^{\infty} \mathbf{v}^{(0)}(t-\tau) \mathbf{N}(\tau) d\tau + (\mathbf{I} \mathbf{yy} + \mu \mathbf{yy}) \mathbf{v}^{(0)} + \\
+ (\beta \mathbf{yy} + \beta \mathbf{yy} \mathbf{x} + \mathbf{x}) \mathbf{v}^{(0)} + \\
+ \int_0^{\infty} \mathbf{v}^{(0)}(t-\tau) \mathbf{N}(\tau) d\tau + (\mathbf{I} \mathbf{yy} + \mu \mathbf{yy}) \mathbf{v}^{(0)} + \\
+ (\beta \mathbf{yy} + \beta \mathbf{yy} \mathbf{x} + \mathbf{x}) \mathbf{v}^{(0)} + \\
+ \int_0^{\infty} \mathbf{v}^{(0)}(t-\tau) \mathbf{N}(\tau) d\tau = N(\mathbf{e}) t,
\end{align*}
\]

where \( r = \mathbf{v} \), and \( X(\mathbf{E}) \), \( Y(\mathbf{E}) \), \( N(\mathbf{E}) \) are components of the external force and moment. From now
on we shall consider only these equations
and shall drop the bothersome superscript$^{(1)}$.
Furthermore, we shall also deal only with
the equations for $v$ and $r$.

The constants $\mu$ and $\beta$ and the functions $N$
are properties of the hull shape but may
depend also upon the Froude number $U/gL$.
In principle they can be determined
theoretically. However, our concern will
be primarily with their experimental de­
termination.

Let us suppose that we have already de­
termined these constants and functions and
that we wish to determine $v$ and $r$
for given external force and moment, $Y_E$ and
$N_E$, and initial conditions, $v(0)$ and $r(0)$.
This would appear to be a natural situation
for the application of the Laplace trans­
form, say,

$$f(s) = \int_0^\infty f(t)e^{-st}dt$$

The equations for $\tilde{v}$ and $\tilde{r}$ are then the
following:

$$\begin{align*}
[(m + \mu_\gamma)\gamma + \beta_\gamma + \tilde{N}_\gamma(s)]\tilde{v}(s) + \\
[\gamma + \tilde{N}_\gamma(s)]\tilde{r}(s) = \\
\tilde{Y}_E(s) + (m + \mu_\gamma)\tilde{v}(0) + (\mu_\gamma + m\gamma')\tilde{r}(0),
\end{align*}$$

$$\begin{align*}
[(\mu_\gamma + \mu)\gamma + \beta_\gamma + \tilde{N}_\gamma(s)]\tilde{v}(s) + \\
[(I_0 + \mu)\gamma + \beta_\gamma + \tilde{N}_\gamma(s)]\tilde{r}(s) = \\
\tilde{N}_E(s) + (\mu_\gamma + m\gamma')\tilde{v}(0) + (I_0 + \mu_\gamma)\tilde{r}(0).
\end{align*}$$

It is evident that, if we could determine
the four coefficients of $\tilde{v}(s)$ and $\tilde{r}(s)$
in the equations above, this would be
sufficient for determination of $\tilde{v}$ and $\tilde{r}$,
and we need not try to find the several
constants and functions separately.

As we shall see later on, it is possible
to determine these coefficients by use of
a planar-motion mechanism (PMM). In fact,
this is not the direction that has been
taken in the use of a PMM. Instead one
has taken the Fourier transform:

$$\hat{f}(\omega) = \int_0^\infty f(t)e^{-i\omega t}dt = \hat{f}_v(\omega) - i\hat{f}_r(\omega)$$

of the equations (7), obtaining the follow­
ing equations for $\hat{v}(\omega)$ and $\hat{r}(\omega)$:

$$\begin{align*}
[i\omega(\mu_\gamma + \mu_\gamma') + \beta_\gamma + \tilde{N}_\gamma(\omega)]\hat{v}(\omega) + \\
[i\omega(\mu_\gamma + \mu_\gamma') + \beta_\gamma + \tilde{N}_\gamma(\omega)]\hat{r}(\omega) = \\
\tilde{Y}_E(\omega) + \tilde{N}_E(\omega) + \tilde{N}_E(\omega),
\end{align*}$$

Since the quantities in (9) are complex,
there are now eight clusters of constants
and functions that are needed for the
determination of $\hat{v}$ and $\hat{r}$. These clusters
may also be determined by use of a PMM,
as was described in Frank et al. (1976).

The use of the Fourier instead of the
Laplace transform was not, I believe, the
result of a considered choice. Rather, it
was a consequence of assumptions concerning
the nature of the equations of motion and
the envisaged use of the PMM. The class­
ical equations of motion contain no con­
volution integrals, which is equivalent to
assuming that the history of the motion
plays no role in the determination of $v$
and $r$. This implies further that there
are only constants to be determined, and
not also functions $N(t)$. For this purpose
certain forced periodic motions of a model
may be used. The procedure is described
briefly in Frank et al. (1976, section 5)
and in more detail elsewhere. Once it
was discovered that the 'constants' in the
equations depened upon the frequency $\omega$
of the periodic motion, it was a natural
next step to consider the Fourier trans­
of the equations of motion. However, the
maneuvres introduced in Frank et al. for
the purpose of determining the eight clus­
ters of constants and functions were not
periodic and there was no fundamental
reason for analyzing them by Fourier anal­
ysis except that it permitted comparison
of results obtained by the new maneuvers
introduced there with those obtained by
frequency-by-frequency testing.

PLANAR-MOTION MECHANISM (PMM)

A PMM is a mechanism that permits one to impart a given motion in one plane, here the \((x,y)\) plane, to a model at the same time that it is moving forward. The device is attached to a towing carriage and two rods, separately controllable, are attached to the model at a forward point \(B\) and an after point \(S\). As the rods are being moved, their motion is recorded and also the force in each rod. A schematic drawing of a PMM is shown in Figure 2. The PMM was apparently introduced as an experimental tool in ship maneuvering problems in a paper by Horn and Walinski (1958) and practically simultaneously and independently by M. Gertler and A. Goodman. One may find a detailed description in Goodman (1960).

\[
\begin{align*}
\Psi & \equiv \frac{1}{2d} (\ddot{y}_B + \ddot{y}_S) - \frac{1}{2d} (\dot{y}_B - \dot{y}_S), \\
\beta & \equiv \frac{1}{2U} (\ddot{y}_B + \ddot{y}_S) - \frac{1}{2d} (\dot{y}_B - \dot{y}_S),
\end{align*}
\]

These expressions may now be substituted into the second and third equations in (7) as a means for determining the coefficients. However, one can equally well form the Laplace or Fourier transforms (if they exist) of the functions in (10) and substitute into (8) or (9), respectively. This has the obvious advantage that, if one is solving (7) by means of one of these transforms, one has determined the coefficients of the unknown transforms, \(\tilde{\Psi}\) and \(\tilde{\beta}\) or \(\tilde{\nu}\) and \(\tilde{\gamma}\), in the form needed. In the case of the Fourier transform one needs an extension of the usual definition of the Fourier transform if the function in question is not absolutely integrable. In the situation that may occur here, when the function becomes constant for \(t \geq t_0\), the extension is simple and well known:

\[
\tilde{f}(\omega) = \int_{0}^{t_0} f(t) e^{-i\omega t} dt - i\omega \int_{t_0}^{\infty} f(t) e^{-i\omega t} dt.
\]

We have said nothing about the choice of the imposed motions, but it is evident from either (8) or (9) that at least two independent ones are required if two equations are to determine four unknowns (complex if (9) is used). A first convenient choice is clearly \(\dot{y}_B = \dot{y}_S\), for this makes \(r = 0\). The choice \(\dot{y}_B = -\dot{y}_S\) does not have the same simplifying effect, but is also convenient to impose. For the present we shall not specify them further, except to require that \(\dot{y}_B(0) = \dot{y}_S(0) = 0\), so that \(\nu(0) = r(0) = 0\). After substituting these motions into (8) one finds the following expressions for determining the four coefficient clusters:

\[
\begin{align*}
\Psi(s) & = \frac{2}{y_B(s)} - \frac{2}{y_S(s)} + \frac{2}{y_B(s)} - \frac{2}{y_S(s)}, \\
\tilde{\Psi}(s) & = \frac{2}{y_B(s)} - \frac{2}{y_S(s)} + \frac{2}{y_B(s)} - \frac{2}{y_S(s)}.
\end{align*}
\]
In the second pair, the measured $Y_B$ and $Y_S$ will, of course, be different from those in the first pair.

The analogous formulas for the Fourier transform, equations (9), have already been given in Frank et al. (1976, section 4). However, in order to discuss certain aspects of the formulas, it will be convenient to repeat them here. They are written in matrix and column-vector form:

$$
Y_B = Y_S : \ddot{\psi} = \omega \dot{\psi} B , \quad \dot{\psi} = 0 ,
$$

$$
\begin{bmatrix}
\beta_{yy} + \hat{N}_{yy_c} \\
(\omega \mu_{yy} + \hat{N}_{yy_y})
\end{bmatrix} = \frac{1}{\omega [\dot{\psi} B]^2} \begin{pmatrix}
\dot{\psi}_{Bs} & -\dot{\psi}_{Bc} \\
-\dot{\psi}_{Be} & \dot{\psi}_{Bs} + \dot{\psi}_{Sc}
\end{pmatrix} ,
$$

$$
\begin{bmatrix}
\beta_{yy} + \hat{N}_{yy_c} \\
(\omega \mu_{yy} + \hat{N}_{yy_y})
\end{bmatrix} = \frac{d}{\omega [\dot{\psi} B]^2} \begin{pmatrix}
\dot{\psi}_{Bs} - \dot{\psi}_{Bc} \\
-\dot{\psi}_{Be} & \dot{\psi}_{Bs} + \dot{\psi}_{Sc}
\end{pmatrix} ,
$$

STABILITY DERIVATIVES

The classical equations of maneuverability, analogous to the second and third equations of (7), are usually written in the following form:

$$
\begin{align*}
(m - Y_c) \ddot{v} - Y_c \dot{v} - (Y_c - mx_d) \ddot{r} - (Y_c - mU) r &= Y_E , \\
-(N_c - mx_d) \ddot{v} - N_c \dot{v} + (I_c - N_c) \ddot{r} - (N_c - mx_d) U &= N_E .
\end{align*}
$$
Here it is understood that the symbols $Y_v$, $Y_r$, etc. represent constants that may vary with Froude number $U/\sqrt{gL}$ but are otherwise independent of the superposed motion represented by $v$ and $r$. They do depend upon the hull form. One may, of course, take either the Laplace or Fourier transforms of these equations and compare the coefficients of $\hat{v}$ and $\hat{r}$ or $\tilde{v}$ and $\tilde{r}$ with those in (8) or (9) respectively. This is done for the Fourier transform in equation (45) of Frank et al. (1976) (but read $-\mu_{yy}$ for $\mu_{yy}$ in the equation for $N_s$).

One may do the same thing for the Laplace transform, but without as complete a separation of the stability derivatives. One finds the following:

$$\begin{align*}
- Y_v s - Y_v &= \mu_{yy} s + \hat{N}_v(s) + \beta_{yy}, \\
- Y_r s - Y_r &= \mu_{rr} s + \hat{N}_r(s) + \beta_{yy}, \\
- N_v s - N_v &= \mu_{yy} s + \hat{N}_v(s) + \beta_{yy}, \\
- N_r s - N_r &= \mu_{rr} s + \hat{N}_r(s) + \beta_{yy}.
\end{align*}$$

Here there is no automatic separation of the "static" and "dynamic" derivatives, as was achieved with the Fourier transform by taking real and imaginary parts. Equations (8) do not indicate a need for further separation. Furthermore, inasmuch as the equations (14) are not well founded and in any case presuppose that the derivatives are constants, the left and right sides of (15) are not really consistent and there is probably not much to be gained by pursuing the matter further.

The principal interest in results like (15) is in being able to use them for comparing new measurements with earlier ones. Although the earlier measurements with a PMM gave results allowing direct comparison with the Fourier-transformed coefficients, this does not seem to be the case here. There do exist formulas allowing one to express $\hat{N}$ in terms of $\hat{N}_c$ or $\hat{N}_s$, namely,

$$\begin{align*}
\hat{N}(s) &= \frac{2}{\pi} \int_0^\infty \hat{N}_c(\omega) \frac{d\omega}{s^2 + \omega^2} \\
&= \frac{2}{\pi} \int_0^\infty \hat{N}_s(\omega) \frac{d\omega}{s^2 + \omega^2}.
\end{align*}$$

But these don't seem very useful for this purpose.

**CHOICE OF $Y_B$**

Up to now the only requirement for $Y_B$ has been that the transforms exist. Let us now consider the effect of some special choices.

In Frank et al. the choice was a movement that can be typified by

$$Y_B(t) = \begin{cases} 1 - \cos \frac{2\pi}{a} t, & 0 \leq t \leq \frac{2\pi}{a}, \\ 0, & t > \frac{2\pi}{a}. \end{cases}$$

For this function

$$\begin{align*}
\hat{Y}_B(\omega) &= \frac{a^2}{\omega(\omega^2 - a^2)} \left[ \sin \frac{2\pi}{a} \omega + i 2\sin \frac{2\pi}{a} \omega \right] \\
&= \hat{Y}_{BC} - i \hat{Y}_{BS}, \\
\hat{Y}_B(s) &= \frac{a^2}{s(s^2 + a^2)} \left[ 1 - \exp -\frac{2\pi}{a} s \right].
\end{align*}$$

One may confirm that $\hat{Y}_{BC} = 0$ for $\omega = 1/2a, 2/3a, 2a, 5a, \ldots$, and $\hat{Y}_{BS} = 0$ for $\omega = a, 2a, 3a, \ldots$. Evidently $|\hat{Y}_B| = 0$ for $\omega = 2a, 3a, \ldots$. Also,

$$\hat{Y}_B(\omega) = \frac{2\pi}{a} - i \frac{2\pi}{a} s \omega + O(\omega^3).$$

$\hat{Y}_B(s)$ does not vanish and

$$\hat{Y}_B(s) = \frac{2\pi}{a} - \frac{1}{2} \left( \frac{2\pi}{a} s \right)^2 + O(s^3).$$

Examination of (13) shows immediately that one may expect difficulties with analysis of data in the neighborhood of $\omega = 0, 2a, 3a, \ldots$, for even though in principle there may be compensating zeros in the
numerators, these will be hidden by experimental noise. Inspection of (12) shows that one may expect the same thing to happen near $s = 0$, but apparently not at other values of $s$. The difficulties with the Fourier transform were mentioned in Frank et al. and are evident in the figures, especially near $\omega = 0$. Since the most important region for practical ship maneuvers is near $\omega = 0$, such difficulties throw some doubt upon the usefulness of the method.

Let us, however, consider a second maneuver, one that may be typified by

$$\gamma_B(t) = \begin{cases} 1 - \cos at, & 0 \leq t \leq \frac{\pi}{a}, \\ 2, & t \geq \frac{\pi}{a}. \end{cases} \tag{21}$$

For this maneuver

$$\tilde{\gamma}_B(\omega) = \frac{\tilde{\gamma}_B(\omega)}{\omega^2 + \omega^3} \left[ \sin \frac{\pi \omega}{a} + i 2 \cos \frac{2\pi \omega}{\omega} \right]$$

$$= -i \frac{\gamma_B}{a} - \frac{\pi}{a} + O(\omega), \tag{22}$$

$$\tilde{\gamma}_B(s) = \frac{\frac{\gamma_B}{s}}{s(s^2 + \omega)} \left[ 1 + \exp -\frac{\pi s}{a} \right]$$

$$= \frac{\gamma_B}{s} - \frac{\pi}{a} + O(s).$$

Here $|\gamma_B| = 0$ for $\omega = 3a, 5a, \ldots$, and $\tilde{\gamma}_B$ does not vanish for $s$, as in the first maneuver (17). It is evident that this choice of $\gamma_B$ postpones the first zero of $|\gamma_B|$ and also makes the zeros less frequent. Furthermore, the singularity at $\omega = 0$ just balances the singularities on the right-hand sides of (13a) and (13b) and in the first terms on the right-hand sides of (13c) and (13d). The singularities in the second terms on the right-hand sides of (13c) and (13d) will be dealt with presently. It is also evident that the singularity in $\tilde{\gamma}_B$ at $s = 0$ just balances those in (12a) and (12b) and in the first terms of the right-hand sides of (12c) and (12d), leaving the second terms still to be dealt with.

**Behavior near $s = 0$ or $\omega = 0$**

In the last section we have considered two special maneuvers and have found that the second one offers definite advantages, especially in the neighborhood of $s = 0$ or $\omega = 0$. An actual maneuver, especially if it is done manually, can only approximate (21) so that it will be useful to examine the behavior of (12) and (13) near $s = 0$ and $\omega = 0$, respectively. For this purpose it will be useful to have a series expansion for the transform of a function similar to (21). Let us denote it by $f(t)$. An example

**Figure 3**

is shown in Figure 3, and, as shown in the figure, $f(t) = f(t_0)$ for $t \geq t_0$. It is then easy to confirm that

$$\tilde{f}(s) = \int_0^{t_0} f(t) e^{-st} dt + \frac{f(t_0)}{s} e^{-st}$$

$$= f_{t_0} + \left[ \int_0^{t_0} f(t) dt - t_0 f(t_0) \right] +$$

$$+ \left[ - t_0 f(t_0) + \frac{1}{s} f(t_0) \right] s + \ldots \tag{23}$$

If the extended Fourier transform (11) is used, one need only replace $s$ by $i\omega$. Let us now consider the case $\gamma_B = \gamma_c$ and suppose that $\gamma_B(t)$ is something like $f(t)$ in Figure 3. We may then expect $\gamma_B(t)$ and $\gamma_c(t)$ to behave somewhat as in Figure 4 (but not both the same, of course).

**Figure 4**
The value $t_0$ should be chosen large enough so that $Y_B = Y_S = 0$ for $t \geq t_0$.

The notations $Y_B^0, Y_B^{1}, \ldots, Y_B^0, Y_B^{1}, \ldots$ are defined as in (23). Let us substitute the series into the first two equations of (12). After some manipulation one finds

$$
(m + \mu_{yy})s + \beta_{yy} + \tilde{N}_{yy}(s) =

= \frac{1}{Y_B^0} \left( Y_{B_1} + Y_{S_1} + s \left[ Y_{B_2} + Y_{S_2} \left( Y_{B_1} + Y_{S_1} \right) \right] + \ldots \right),
$$

(24)

$$
(\mu_y + m_{x_a})s + \beta_{yy} + \tilde{N}_{yy}(s) =

= \frac{d}{Y_B^0} \left( Y_{S_1} - Y_{B_1} + s \left[ Y_{S_2} - Y_{B_2} \left( Y_{B_1} - Y_{S_1} \right) \right] + \ldots \right).
$$

The formulas for the Fourier-transformed equations (13) are, of course, the same if $s$ is replaced by $iw$ and $N(s)$ by $\tilde{N}(w)$. One can then further separate real and imaginary parts.

Next consider the case $Y_B = -Y_S$, again with $Y_B$ similar to Figure 3. Now we may expect $Y_B$ and $Y_S$ also to behave somewhat like Figure 3. Consequently the $Y_B^0$ and $Y_S^0$ terms will be present. When the series expansions are substituted into the last two equations of (12), one finds

$$
(\mu_{yy} + m_{x_a})s + \beta_{yy} + mU + \tilde{N}_{yy}(s) =

= \frac{1}{s} \left( Y_{B_0} + Y_{S_0} \right) +

= \frac{d}{Y_B^0} \left( Y_{B_1} + Y_{S_1} - Y_{B_1} \left( Y_{B_0} + Y_{S_0} \right) \right) +

= \frac{d}{Y_B^0} \left( Y_{B_2} + Y_{S_2} - Y_{B_2} \left( Y_{B_1} + Y_{S_1} \right) \right) +

= \frac{d}{Y_B^0} \left( Y_{B_3} + Y_{S_3} - Y_{B_3} \left( Y_{B_2} + Y_{S_2} \right) \right) +

= \frac{d}{Y_B^0} \left( Y_{B_4} + Y_{S_4} - Y_{B_4} \left( Y_{B_3} + Y_{S_3} \right) \right) +

= \frac{d}{Y_B^0} \left( Y_{B_5} + Y_{S_5} - Y_{B_5} \left( Y_{B_4} + Y_{S_4} \right) \right) +

= \frac{d}{Y_B^0} \left( Y_{B_6} + Y_{S_6} - Y_{B_6} \left( Y_{B_5} + Y_{S_5} \right) \right) +

+ \frac{Y_{B_1} + Y_{S_1}}{Y_B^0} \left( Y_{B_0} + Y_{S_0} \right) + \ldots +

+ \frac{Y_{B_2} - Y_{B_1}/Y_B^0}{Y_B^0} \left( Y_{B_1} + Y_{S_1} \right) + \ldots +

+ \frac{U}{s} \left[ (m + \mu_{yy})s + \beta_{yy} + \tilde{N}_{yy}(s) \right],
$$

(25)

Once again the analogous formulas for (13) are easily obtained.

Although the maneuver used for $Y_B$ has avoided a singularity at $s = 0$ in the case $Y_B = Y_S$, i.e. in (24), this does not seem to be the case for $Y_B = -Y_S$, i.e. in (25). However, let us examine the quantities in square brackets that are the coefficients of $U/s$. If we expand each in a series, we find

$$
\beta_{yy} + \tilde{N}_{yy}(0) + O(s),
$$

(26)

$$
\beta_{yy} + \tilde{N}_{yy}(0) + O(s),
$$

respectively. Let us now examine equations (7) and suppose that $t \geq t_0$. Then $r = 0$ and $\dot{v} = 0$. If initial transients may be
considered negligible in the convolution integrals, we find
\[
\left[ \beta_{yy} + \int_0^\infty N_{yy}(t) \, dt \right] \nu(t) = 0
\]
\[
= \left[ \beta_{yy} + \tilde{N}_{yy}(0) \right] \frac{-U}{d} = Y_{bo} + Y_{s0},
\]
\[
= \left[ \beta_{yy} + \tilde{N}_{yy}(0) \right] \frac{-U}{d} = d(Y_{bo} - Y_{s0}),
\]
or
\[
U \left[ \beta_{yy} + \tilde{N}_{yy}(0) \right] = \frac{-d}{Y_{bo}} (Y_{bo} + Y_{s0}),
\]
\[
U \left[ \beta_{yy} + \tilde{N}_{yy}(0) \right] = \frac{-d}{Y_{bo}} (Y_{bo} - Y_{s0}).
\]

It is now evident that these terms will just cancel the first terms on the right-hand sides of (25), thus removing the singularity.

If all measurements were perfect, this cancellation would happen automatically when the results from the measurements used in (24) were combined with those in (25). Unfortunately, one cannot depend upon this, and if one uses (25) straightforwardly with results from (24), one is likely to find singular behavior at \( s = 0 \) (or \( \omega = 0 \)). In order to avoid this, one can, in (25a) for example, drop the first singular term, retain the next two terms, and replace the last by
\[
U \left[ m + \mu^*_{yy} + \tilde{N}_{yy}(s) - \tilde{N}_{yy}(0) \right].
\]

This in turn can be estimated from the series in (24a).

The ability of the considered maneuver to enable determination of the various coefficient clusters down to \( s = 0 \) or \( \omega = 0 \) is one of the noteworthy advantages of this procedure for using a PMM. In the conventional frequency-by-frequency method of using a PMM, one is seriously limited in the low-frequency range by wall reflections, and yet this is the most important range for computation of practical ship trajectories. Even with this short maneuver there were difficulties with reflections at slower speeds of testing.

In order to show that the described procedure really work there are reproduced in Figures 5, 6, 7 and 8 four graphs from as yet unpublished work of D. J. Loeser. These show (in conventional notation) \( Y_{v'}, Y_{v''}, Y_{r'} \) and \( Y_{r''} \) as functions of \( \tau = wU/g \) for a self-propelled Mariner at Froude number 0.155. \( H/d \) is the ratio of water depth to ship draft. In the first three one can see that (13) gives results practically down to \( \omega = 0 \), whereas for \( Y_{r''} \) the curves had to be stopped below \( \tau = 0.15 \) because of the singular behavior near \( \omega = 0 \). However, by using the procedure discussed above it was possible to find the values to which each curve should tend as \( \omega \) approaches zero.
RELATION BETWEEN EQUATIONS (14) AND EQUATIONS (7)

Although it is known that the conventional equations (7) cannot be correct in principle, since many experiments have shown that the various coefficients $Y_v, Y_p$, etc. are not independent of the superposed motions, it has been also shown that they give rather satisfactory predictions of ship paths for usual rudder commands. Furthermore, calculations by Fujino (1975) and others have confirmed that for certain simple maneuvers such as putting the rudder hard over and moving into a turning circle there is practically no discernible difference between the trajectories computed by (7) and by (14).

In section 7 of Frank et al. it was shown that if the ship motion were "quasi-steady" (of almost constant velocity), a term defined more precisely there, then one could indeed approximate solutions of (7bc) by solutions of equations of the form of (14). Furthermore the approximation is uniform. In the discussion to Frank et al. Carl Scragg noted that if the solutions of (7bc) were of "almost constant acceleration", which can be defined in a precise fashion, then these solutions can be uniformly approximated by solutions of different equations of the form (14), equations that had been proposed by Fujino in his discussion as being more appropriate. How does one resolve this seeming paradox, for it doesn't seem possible that solutions of (7bc) can be approximated by solutions of two different sets of equations like (14)?

To simplify the discussion let us take $r = 0$ and consider only (7b):

$$\left( m + \mu_{yy} \right) \ddot{v} + \beta_{yy} \dot{v} + \int_0^\infty v(t - \tau) N_{yy} \tau \, d\tau = Y_v(t)$$  \hspace{1cm} (29)

The two competing approximate equations of the form of (14) are the following, in order of introduction above:

$$\left( m + \mu_{yy} \right) \ddot{v} + \left[ \beta_{yy} + \int_0^\infty N(t) \tau \, d\tau \right] \dot{v} = Y_v(t),$$  \hspace{1cm} (30a)

$$\left[ m + \mu_{yy} - \int_0^\infty N(t) \tau \, d\tau \right] \ddot{v} + \left[ \beta_{yy} + \int_0^\infty N(t) \tau \, d\tau \right] \dot{v} = Y_v(t).$$  \hspace{1cm} (30b)

The proofs consist in letting $v_1(t)$ be an approximate solution of (29) (i.e., either quasi-steady or of almost constant acceleration) and $v_2(t)$ a solution of (30a) or (30b) respectively and showing that $|v_1(t) - v_2(t)|$ can be made as small as one pleases by ever more stringent imposition of the chosen condition. The proofs appear not to use the form of $Y_v(t)$ at all, but this must be, of course, an illusion. Indeed, it is just the form of $Y_v(t)$ that determines whether a solution of (29) is quasi-steady or of almost constant acceleration. If $Y_v$ is appropriate to the first, then (30a) provides a good approximation, if to the second, then (30b). Thus there
seems to be no paradox, but also no clear-cut decision. However, the computations of Fujino and others mentioned above do seem to indicate a close agreement between solutions of (29) and (30b) as well as for the analogous more complete equations with $r \neq 0$.

CONCLUDING REMARKS
In view of the apparent satisfactoriness of the set of equations (14) for practical maneuvers, one may ask why one should bother with the more complicated equations (7). As far as the linearized equations are concerned, there may be no reason for doing so. However, when one tries to improve the accuracy of the equations, the situation may change. The fundamental assumptions used in deriving (14) are demonstrably false. To continue to apply them in deriving higher-order approximations may lead to equations not only of incorrect form, but to predictions diverging substantially from the correct ones. Of course, one may still be lucky and not have this happen, but if true this fact should be established.

The procedures described earlier for using the PMM for the determination of the various coefficient clusters occurring in the equations of motion are independent of these considerations. The second maneuvers described earlier are a much more efficient method of using a PMM for this purpose than are the conventional sinusoidal maneuvers. They take less time by far to produce the same amount of information, and they can give information in the important low-frequency region that is not accessible to conventional testing methods because of wall reflection, unless the tank is very wide. Thus determination of the stability coefficients could easily become part of routine model tests, taking roughly twice the time required for the resistance tests. Moreover, they could be conducted in towing tanks of moderate size.

ACKNOWLEDGEMENT
Much of the work described above has been supported by General Hydromechanics Research Program of the David Taylor Ship Research and Development Center under contract N00014-75-C-0275 or by the National Science Foundation under grant.

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SHIP MOTIONS IN LARGE WAVES

by

NILS SALVESEN
David W. Taylor Naval Ship Research and Development Center
Bethesda, Maryland 20084

ABSTRACT

A second-order theory for predicting heave and pitch motions has been derived. It is assumed that the ship is slender and/or that the frequency is low so that to the first order the motions are governed by hydrostatic-restoring and Froude-Krilov forces. The incident-wave system is represented by a nonlinear Stokes wave. It is shown that the final nonlinear equations for the motions consist of two sets of coupled equations. One set is identical to the existing conventional strip-theory equations and the other set contains second-order terms which are products of the first-order motions and hydrostatic-restoring and Froude-Krilov forces. Numerical results are given for the first- and second-order exciting forces for three typical ship sections.

Reinier Timman was an innovative mathematician and philosopher whose impact upon the field of applied mathematics and theoretical ship hydrodynamics, in particular, will long be remembered. But Reinier was more than a great mathematician, he was a warm person with a compassionate interest in other people and a unique ability to inspire and encourage others. It is with deep admiration and respect that I dedicate this paper to Reinier Timman’s memory.

1. INTRODUCTION

Linear Theories

The well-known 1953 paper of St. Denis and Pierson [1] started a new era in the prediction of ship motions by showing how the statistical responses of a ship in irregular seas can be obtained by linear superposition of the responses in regular waves if the seaway is represented by an energy spectrum. In 1952 there existed no theoretical or numerical method which had useful accuracy for predicting the regular wave responses.

The first motion theory suitable for engineering computations was the strip theory of Korvin-Kroukovsky and Jacobs [2] which was published in 1957. This theory predicts the heave and pitch motions of a ship advancing at constant forward speed in regular head waves. In strip theory a sectional two-dimensional approximation is used for predicting the hydrodynamic coefficients. This approach is justifiable for slender hull forms at relatively high frequencies. In spite of the fact that the hydrodynamic coefficients are predicted only accurately in the high frequency range, strip theory predicts the heave and pitch motions with reasonable accuracy also in the low frequency range. The reason for this is that in the low frequency range the motions are dominated by the hydrostatic restoring forces. Here we shall not go into any details with regard to the forward speed but only state that several critical assumptions are applied resulting in reduced accuracy of the Korvin-Kroukovsky/Jacobs strip theory for increasing ship speeds.

A more general strip theory was developed in 1970 by Salvesen, Tuck, and Faltinsen [3]. This theory predicts the heave,
pitch, sway, roll, and yaw motions as well as the wave-induced loads for a ship advancing with arbitrary heading in regular waves. Ship designers in many parts of the world are today using computer programs based on this theory and on the statistical irregular-sea method of St. Denis and Pierson [1] to predict the motions and loads which ships experience in different sea conditions. The most obvious limitation to this approach is that linearity is assumed for all aspects of the problem. This means that at least the wave slopes, the ship responses, and the ship generated waves must all be relatively small. There are other limitations due to assumptions used in strip theory; the most important are the high-frequency and forward-speed assumptions. It is assumed in the Salvesen, Tuck, and Faltinsen strip theory that in addition to the hull form being slender, the frequency of encounter is high, \( \omega \gg U(d/\delta x) \), so that the three-dimensional speed-dependent free-surface condition can be reduced to a two-dimensional speed-independent free-surface condition. It is also assumed that the interaction between the steady-state Kelvin waves and the oscillatory motions can be disregarded. By applying these assumptions Salvesen, Tuck, and Faltinsen have shown that there are some additional forward-speed terms which were not included in the Korkin-Kroukovsky-Jacobs theory; however, even when these speed terms are included, the strip theory does not have the same accuracy at the higher speeds as it has at low and moderate speeds. Due to the high-frequency assumption the hydrodynamic coefficients are inaccurate in the low frequency range. As already pointed out these inaccuracies in the hydrodynamic coefficients have little effect on the vertical motions which are dominated by the hydrostatic forces in the low frequency range. However, for horizontal sway and yaw motions, for which there is no hydrostatic restoring, it is believed that the high-frequency assumption may result in large errors in the computed motions in the low frequency range. It is difficult to make general statements on the magnitude of these errors since there exist few comparisons between theoretical predictions and experimental data for the sway and yaw motions. It should be noted that at very low encounter frequencies, in following and quartering seas, even the correct order of magnitude of the sway and yaw responses cannot be predicted since according to strip theory these motions will approach infinity as the frequency of encounter goes to zero.

Very recently, M.S. Chang [4] developed a new computational method for predicting the six-degree-of-freedom motions of a ship advancing at constant speed with arbitrary heading in regular waves. In this method three-dimensional oscillating Kelvin sources which are distributed over the hull surface are used. The source strength is found by satisfying the body-boundary condition at the mean undisturbed location of the hull. Therefore, the only assumptions used within the potential flow formulation are that the free-surface conditions are linearized and that the body condition is satisfied at the mean position. The interaction between the steady-state wave-resistance perturbation potential and the oscillatory potential is included. As far as I can see this is the complete, ultimate linear theory for ship motions, and considerable improvements over strip theory can be expected at least in the high-speed range and for the sway and yaw motions at low encounter frequencies. A general evaluation of the improved accuracy of this promising method is needed.

In the linear ship-motion theories it is assumed that ship displacements are small relative to ship dimensions so that the body-boundary condition can be satisfied at the mean undisturbed position of the hull. This assumption is violated by the bow motions of most ships even in moderately-steep head waves. Consider, for example, a Mariner hull advancing at a Froude number, \( F = 0.20 \) in head waves with wave length, \( \lambda = 1.20L \). From [5] we know that the amplitude of the heave displacement is about equal to the wave amplitude and the displacement at the bow due to the pitch motions, which is approximately in phase with the heave, is about three times the wave amplitude. Hence, the amplitude of the total bow displacement is almost four times the wave amplitude. Since the bow displacement is approximately out of phase with the wave, the relative displacement between the bow and the water surface may be almost five times the wave amplitude. For a draft-to-length ratio of the Mariner of 0.06, it follows that if the wave-height-to-wavelength ratio, \( H/\lambda = 0.020 \) the bottom of the bow will be just at the surface of the water. In the case of a typical destroyer hull, for example the Friesland, the relative bow displacement is about four times the wave amplitude at \( F = 0.35 \) [5] but since the hull draft-to-length ratio is only about 0.03 the bow will exit the surface when \( H/\lambda = 0.013 \). Figure 1 illustrates the bow motions of the Friesland for this particular wave condition.

![Figure 1 - Bow Motion of Destroyer Hull in Sinusoidal Wave with \( \lambda = 1.20L \) and \( H/\lambda = 0.013 \) (Froude Number, \( F = 0.35 \))]
Since it can be expected that a large percentage of the waves are much steeper than $H/\lambda = 0.020$ the assumption of small displacements at the bow will often be violated. Note that for non-breaking waves the maximum theoretical values of $H/\lambda$ is 0.14. It should be recognized that the large changes in the wetted surface at the bow will result in changes in the horizontal forces which again will introduce a nonlinear coupling between horizontal and vertical motions. This may have a considerable effect on the yaw motions in bow waves.

**Nonlinear Theories**

We shall here ignore the viscous effects and the free-surface tension and consider only the ship-motion problem which can be formulated within potential-flow theory. It should be recognized, however, that viscous damping can be an important part of the damping mechanism, as important as the nonlinear motion and free-surface effects. For example, in predicting the roll motions for most ship forms the viscous damping must be included in order to obtain reasonable accuracy. For low-water-plane catamarans viscous damping plays an important role even for pitch and heave motions.

The nonlinear potential flow formulation of the ship-motion problem consists of the nonlinear free-surface conditions to be satisfied on the unknown free surface and the body boundary condition to be satisfied at the instantaneous wetted surface of the ship hull. In the fluid domain the Laplace equation must be satisfied and some nonlinear incident-wave system must be specified. The representation of the nonlinear incident-wave system shall be discussed in more detail in the next section. Here we shall first consider the nonlinear free-surface and body-displacement effects and their relative importances.

Let us assume that the incident-wave system is some kind of periodic wave with fixed wave length. If the frequency is sufficiently high (i.e. wave length is short relative to ship dimensions) the ship motions will be small and the most important nonlinear effects will be the free-surface effects. In the limit as the frequency increases this will become a pure diffraction problem. In some intermediate frequency range it may be expected that body-displacement and free-surface effects are equally important, whereas in the low frequency range (i.e. long waves relative to the ship dimensions) the body generated free-surface disturbances become small even for large body motions. As the frequency decreases the ship-motion problem will in the limit be governed by purely hydrostatic-restoring and Froude-Krilló forces.

In the study of large ship motions only the intermediate and low frequency ranges are of any interest. If a general body shape is considered, both the nonlinear ship-hull condition and the nonlinear free-surface conditions must be satisfied in the intermediate frequency range, whereas in the low frequency range the linearized free-surface condition may be used with the body-condition satisfied at the exact location. If it is assumed that the hull is sufficiently slender, the free-surface conditions may also be linearized in the intermediate frequency range since the body generated waves can then be considered small. Therefore, a nonlinear numerical time-domain solution of the motions for a slender ship, valid both in the low and intermediate frequency range, can be obtained by a source distribution technique where the sources satisfy the linearized free-surface condition and are distributed on the instantaneous wetted surface of the hull. This would be similar to the method of Chang [4] except that the source strength would have to be computed for each time step, which would result in large computer times.

J.R. Pauling has worked for several years on the nonlinear problem of large-amplitude ship motions in following and quartering waves. With the assistance of some of his students, he has developed a time-domain numerical simulation technique [6] which has been used to predict even the very nonlinear phenomena of capsizing. In this method the forces due to body-generated waves (i.e. added mass, damping, and diffraction) are assumed to be small due to the low encounter frequency and therefore, are estimated very crudely; the hydrostatic forces are assumed to dominate the problem and are computed to a high-order of accuracy for the actual instantaneous submerged hull shape. The good agreement between computational and experimental results shown in [6] seems to indicate that this time-domain numerical method may not only be a useful tool for predicting capsizing but it may also be useful for the general dynamical problem of ship motions and course keeping at low encounter frequencies. Since strip theory is not applicable in the very low frequency range, I find it surprising that this method has not gained a wider recognition.

In this paper we shall present a frequency-domain nonlinear method which is assumed to be applicable for low-encounter frequencies and possibly also in the intermediate frequency range if the ship hull is sufficiently slender. It is assumed that the ship motions are dominated by the hydrostatic-restoring and Froude-Krilló forces and that the forces due to body-generated waves are small and only need to be determined to the first-order of accuracy.

**2. INCIDENT-WAVE REPRESENTATION**

In the linear theory the ship-motion problem is solved in the frequency domain, i.e. the motions are predicted for sinusoidal waves over a range of frequencies. The statistical responses of a
ship in irregular seas are then obtained by linear superposition, representing the seaway by an energy spectrum. For the nonlinear problem it would be very complicated to work with irregular waves with all frequency components present due to the nonlinear interactions between the different frequency components.

A ship's motions are very sensitive to the wave length. For example, the heave and pitch motions are only of critical magnitude when the wave length is approximately equal to the ship length. Therefore, ship responses in some respect can be considered as a filtering system which makes consideration of a complete wave spectrum unnecessary. Another important aspect of the problem is that observations seem to indicate that even in irregular waves, waves have a tendency to come in groups of up to three or four waves with approximately the same height and length. It is generally believed by ship operators that the maximum motions are experienced in such wave groups. In addition, it has been found for some ships, for example the AGOR-16 catamaran [7], that the most critical large amplitude motions occur in swell conditions. Thus, because large ship motions seem to occur in conditions which are closely related to periodic waves, and for the sake of simplicity, we shall represent the incident-wave system by a periodic nonlinear Stokes wave.

According to linear gravity-wave theory for infinite depth the complex potential for a periodic sinusoidal wave traveling in the negative x direction is

$$
\phi_1 + i \psi_1 = i \frac{\beta}{\omega} k_1 z \cdot e^{i k_1 z} e^{i \omega t}
$$

(2.1)

where the z axis is vertical and positive upward with z = 0 at the undisturbed free surface. Here \( \phi_1 \) and \( \psi_1 \) are the first-order velocity potential and stream functions, respectively, \( k_1 \) is the linear-theory wave number which is related to the frequency, \( \omega \) by

$$
k_1 = \frac{\omega^2}{g}
$$

(2.2)

and \( g \) is the gravitational acceleration. The first-order wave elevation is

$$
\xi_1 (x, t) = \beta \cos (k_1 x + \omega t)
$$

(2.3)

where \( \beta \) is the first-order wave amplitude. Stokes [8] was the first to show that the complex potential (2.1) is not only the linearized wave solution, but that it also satisfies the free-surface condition to second-order in \( \beta \). It can easily be shown from the free-surface boundary conditions that the second-order free-surface elevation is given by

$$
\xi_2 = \beta \cos (k_1 x + \omega t) + \frac{1}{2} k^2 \beta^2 \cos (2(k_1 x + \omega t))
$$

(2.4)

Rayleigh [9] pointed out that a solution of the form (2.1) can also be shown to be correct to third-order in \( \beta \) if the third-order wave number, \( k_3 \) is given as

$$
k_3 = k_1 (1 + k_1^2 \beta^2)
$$

(2.5)

Following Rayleigh we shall write the third-order complex potential as

$$
\phi_3 + i \psi_3 = i \frac{k_3 \beta}{k_1} \cdot e^{i k_1 x} e^{i \omega t}.
$$

(2.6)

It can be shown from Bernoulli's equation that the third-order free-surface elevation is

$$
\xi_3 = \left[ \beta + \frac{5}{3} k_3 \beta^2 \right] \cos (k_3 x + \omega t)
$$

$$
= \frac{1}{2} k_3 \beta^2 \cos (2(k_3 x + \omega t))
$$

$$
+ \frac{3}{5} k_3^2 \beta^3 \cos (3(k_3 x + \omega t)).
$$

(2.7)

It also follows that the third-order wave length, celerity, and wave height are given respectively by

$$
\lambda_3 = \lambda_1 \left( 1 - \frac{1}{2} k_1^2 \beta^2 \right) \text{ with } \lambda_1 = 2\pi/k_1
$$

$$
c_3 = c_1 \left( 1 - \frac{1}{2} k_1^2 \beta^2 \right) \text{ with } c_1 = \omega/k_1
$$

$$
H_3 = 2\beta \left( 1 + k_1^2 \beta^2 \right)
$$

(2.8)

For a steep wave with \( H/\lambda = 0.10 \) we have that \( k_1^2 \beta^2 \approx 0.10 \) which means that there will be approximately a 10 percent correction to the wave length, celerity, and wave height for a wave with this steepness.

If we apply the Bernoulli equation,

$$
\frac{p}{\rho} = -\frac{\partial \phi}{\partial t} - \frac{1}{2} \nabla \phi^2 - gz,
$$

(2.9)

to the potential for the Stokes wave (2.1), we find that the pressure field, correct to the third order, is given by

$$
\frac{p_1}{\rho g} = \frac{k_1 \beta}{k_1} e^{i k_1 x} \cdot \frac{\beta}{2} e^{i \omega t} e^{i k_1 x} e^{i \omega t} = \frac{1}{2} k_3 \beta^2 \cos (k_3 x + \omega t) - \frac{1}{2} k_1^2 \beta^2 e^{i k_1 x} e^{i \omega t} - z
$$

(2.10)

Close to the free surface we shall consider \( z \) small and of order \( \beta \); by expanding the exponentials in (2.10) we find that close to the surface the pressure correct to the first order is

$$
\frac{p_1}{\rho g} = \beta \cos (k_1 x + \omega t) - z
$$

(2.11)

which clearly satisfies the condition that \( p = 0 \) on \( z = \xi_1 \). To second order the pressure close to the surface is

$$
\frac{p_2}{\rho g} = \beta (1 + k_1 x) \cos (k_1 x + \omega t) - \frac{1}{2} k_1^2 \beta^2 - z,
$$

(2.12)

which satisfies the condition \( p = 0 \) on the second-order free
surface (2.4). Note that both (2.11) and (2.12) are valid extensions for \( z > 0 \).

We shall now consider a very small body, for example a cork, which is in the surface of a Stokes wave. If the dimensions of the body are sufficiently small relative to the wave length (low frequency), we can assume that the body's vertical motions are governed by the pressure field (2.11) to the first order and by (2.12) to the second order. If we assume further that the body has a cylindrical shape with its axis vertical, we have from Newton's second law, ignoring the inertia force that the weight of the body must be equal to the hydrodynamic pressure force and hence, correct to the first order,

\[
\rho g d A = \rho g A (\beta \cos \omega t - z_B)
\]  

(2.13)

where \( d \) is the draft of the body in calm water and \( A \) is the area of any horizontal cross-section. In writing (2.13) it is assumed that the body is at \( x = 0 \). Here \( z_B \) is the \( z \) coordinate of the bottom of the body. From (2.13) it follows that the vertical displacement of the body, \( z_v \), is given by

\[
z_v = d + z_B = \beta \cos \omega t,
\]

(2.14)

which states that the body moves with the wave and that there is no relative motion between the body and the free surface.

Similarly, to second order we have by (2.11) that

\[
\rho g d A = \rho g A \left[ \beta (1 + k z_B) \cos \omega t - \frac{1}{2} k \beta^2 - z_B \right]
\]

(2.15)

Hence, the vertical displacement of the body is

\[
z_v = d + z_B = \beta (1 - k d) \cos \omega t + \frac{1}{2} k \beta^2 \cos 2 \omega t
\]

(2.16)

The relative displacement between the body and the free surface correct to second order is then

\[
z_R = z_v - z_2 = - k d \cos \omega t,
\]

(2.17)

which states that the body will increase its draft by \( k d \) at the crest of the wave and decrease its draft by the same amount at the trough. For a steep wave with \( H/\lambda = 0.10 \) we have that \( \beta k = 0.31 \) so in this case the change in the draft is \( \pm 31 \) percent. This large draft change is often considered in the safety of smaller boats, and can also result in a significant change in the transverse stability of ships, as was first described qualitatively by W. Froude [10]. Numerical values for the change in the transverse stability of actual ship forms are given by J.R. Fauling [11], who used Trochoidal-wave theory in obtaining his results. The fact that is important to note here is that if linear-wave theory is used in computing the pressure forces on the body, in for example the time domain, this nonlinear effect will not be included. On the other hand, if in the linear theory the expansion \( \beta e^{kz} = \beta (1 + k z + \ldots) \) is used and the second-order term \( \beta k z \) is included in the pressure, even when \( z = 0 \) (note this is inconsistent) the same relative motion as given by (2.14) is obtained. Such an approach is often used in ship-theory work. However, if this nonlinear term is included in the first-order pressure expression (2.11), the pressure is no longer equal to zero at the free surface, \( z = \xi_1 \).

For most practical applications it may be sufficiently accurate to use a second-order Stokes wave. As we have seen in the previous section ship motion can become very large even for waves with small slopes, say \( H/\lambda = 0.05 \), when \( \lambda \) is close to the ship length. However, much steeper waves can occur when waves receive energy from currents or reflections. For example, R.S. Smith [12] points out that off the south-east coast of South Africa the rapid Aguthas Current will result in about a threefold amplification of the wave height and that this can result in waves with a wave length of 200 meters and a wave height of up to 20 meters \( (H/\lambda = 0.10) \). Furthermore, Smith points out that these large waves are steeper on the front part of the crest than at the back of the crest and that this effect can easily be represented by a Stokes wave by introducing a small phase shift, \( \delta \) in the second-order term. We may then write the wave profile as

\[
\xi_2 = \beta \cos k x + \frac{1}{2} k \beta^2 \cos 2 (k x + \delta)
\]

(2.18)

where we here have used a coordinate system fixed with respect to the wave profile. Figure 2 shows the difference between wave profile given by (2.18) and a sinusoidal wave. For the purpose of illustration, this is an unrealistically steep wave with \( H/\lambda = 0.167 \).

In summary, it is believed that the second- and third-order Stokes waves with their unique simplicity, are useful representations of incident-wave systems for many nonlinear ship-motion problems.

3. EQUATIONS FOR THE SHIP MOTIONS

J.N. Newman and E.O. Tuck [13] have derived a motion theory for slender ships by assuming the ship to be long compared to its beam and draft and that the wave lengths are of the same order or longer than the ship. To the first order in \( \epsilon \), where \( \epsilon \) is the beam/length ratio of the ship, they show that the equations which govern the heave and pitch motions are

\[
F_{FK} + F_{HS} = 0
\]

(3.1)

where \( F_{FK} \) is the hydrodynamic force (or moment) due to the pressure field of the undisturbed incident wave system (the "Froude-Krilov" force) and \( F_{HS} \) is the hydrostatic restoring force (or moment). In their formulation it is shown that the inertia forces are of \( O(\epsilon^2) \) and that the forces due both to
diffraction and to body motions are of \( O(e^2 \log e) \). Figures 3 and 4 which are both taken from their paper [14] show that this simple first-order slender-body theory gives a very good prediction of pitch and heave at zero forward speed. Note that this simple theory will be less accurate as the speed increases. These good results of Newman and Tuck have been a major motivating factor in deriving a higher-order theory where the Froude-Krilov and the hydrostatic forces are of leading order.

Let \((x, y, z)\) be a right-handed orthogonal coordinate system fixed with respect of the mean position of the ship with \( z \) vertical upward through the center of gravity of the ship, \( x \) in the direction of the mean forward motion and the origin in the plane of the undisturbed free surface. Furthermore, let \((x_o, y_o, z_o)\) be a coordinate system fixed in the ship which coincides with \((x, y, z)\) when the ship is at rest. Viscous effects and surface tension will be disregarded and the problem will be formulated in terms of potential flow theory.

The total velocity potential, \( \Phi \) is assumed to have the following expansion

\[
\Phi(x, y, z; t) = \phi^{(1)}(x, y, z) e^{i \omega t} + \phi^{(2)}(x, y, z) e^{i 2 \omega t} + \ldots
\]

(3.2)

where the complex amplitudes \( \phi^{(j)} \) are of \( O(\alpha) \). Here \( \alpha \) is a parameter proportional to the wave slope which is assumed to be small. It is understood that the real part is to be taken in expressions involving \( e^{i \omega t} \) or \( e^{i 2 \omega t} \). The total potential may be written as

\[
\Phi = \Phi_o + \Phi_b
\]

(3.3)

where \( \Phi_o \) is the incident-wave potential and \( \Phi_b \) is the potential due to the body disturbance, including diffraction effects. We shall assume that

\[
\Phi_b \ll \Phi_o
\]

(3.4)

Figure 2 – Second-Order Stokes Wave with a Small Phase Shift

The total velocity potential, \( \Phi \) is assumed to have the following expansion

\[
\Phi(x, y, z; t) = \phi^{(1)}(x, y, z) e^{i \omega t} + \phi^{(2)}(x, y, z) e^{i 2 \omega t} + \phi^{(3)}(x, y, z) e^{i 3 \omega t} + \ldots
\]

(3.2)

where the complex amplitudes \( \phi^{(j)} \) are of \( O(\alpha) \). Here \( \alpha \) is a parameter proportional to the wave slope which is assumed to be small. It is understood that the real part is to be taken in expressions involving \( e^{i \omega t} \) or \( e^{i 2 \omega t} \). The total potential may be written as

\[
\Phi = \Phi_o + \Phi_b
\]

(3.3)

where \( \Phi_o \) is the incident-wave potential and \( \Phi_b \) is the potential due to the body disturbance, including diffraction effects. We shall assume that

\[
\Phi_b \ll \Phi_o
\]

(3.4)
This assumption is justified for ship hulls which are slender in the sense that both beam and draft are much smaller than their length and the length is of the same order as the wave length. In fact we shall assume that the incident-wave potential has the expansion
\[ \phi_o = \phi_o^{(1)} + \phi_o^{(2)} + \phi_o^{(3)} + \cdots, \tag{3.5} \]
where \( \phi_o^{(1)} \) and \( \phi_o^{(2)} \) are of \( O(\alpha^1) \), and where DC is used to indicate that a term is non-oscillatory and time independent. We shall assume that the body potential has the expansion
\[ \phi_B = \phi_B^{(1)} e^{i\omega t} + \phi_B^{(2)} n_3 + \cdots, \tag{3.6} \]
where \( \phi_B^{(1)} \) is of \( O(\alpha\epsilon) \). Here \( \epsilon \) is a small parameter related to the slenderness of the body and/or the frequency.* We shall in this derivation include only terms of \( O(\alpha), O(\alpha^2) \), and \( O(\alpha\epsilon) \).

If we let the incident wave be a second-order Stokes wave,
\[ \phi_o = \frac{1}{2} \beta \rho g \epsilon x_k k x e^{i\omega t}, \tag{3.7} \]
then
\[ \phi_B^{(2)} = 0 \quad \text{and} \quad \phi_B^{(1)} \text{DC} = 0, \tag{3.8} \]
since (3.7) is correct to second order in \( \beta \). Note that \( \phi_B^{(1)} \) shall here be considered as a “second-order” term since it is of \( O(\alpha\epsilon) \); however, it only has to satisfy the linearized free-surface condition since it is only of order one in \( \alpha \).

Uncoupled Heave Motions

For simplicity we shall consider first only pure heave motions. We have by Newton’s second law that
\[ M \ddot{\eta} = -\rho g V_o + F_H, \tag{3.9} \]
where \( F_H \) is the hydrodynamic force, \( \rho g V_o \) is the weight and \( M \) is the mass of the body. The vertical displacement of the body is
\[ \eta = \eta^{(1)} + \eta^{(2)} + \cdots = \eta^{(1)} e^{i\omega t} + \eta^{(2)} e^{2i\omega t} + \eta^{(3)} e^{3i\omega t} + \cdots \tag{3.10} \]
Here \( \eta^{(1)} \) and \( \eta^{(2)} \) will include terms both of \( O(\alpha) \) and \( O(\alpha\epsilon) \), whereas \( \eta^{(3)} \) and \( \eta^{(4)} \) will include terms only of \( O(\alpha^2) \).

The vertical component of the hydrodynamic force is
\[ F_H = \int_{S_t} p n_3 \, ds \tag{3.11} \]
where \( p \) is the pressure and \( n_3 \) is the vertical component of the outward unit normal vector. Here \( S_t \) is the instantaneous wetted surface of the body.

By Bernoulli’s equation, the pressure is
\[ p = -\frac{\partial \Phi}{\partial t} - \rho \frac{1}{2} \left| \nabla \Phi \right|^2 - \rho g z \tag{3.12} \]
which by equations (3.3), (3.5), (3.6) and (3.8) can be written as
\[ p = -i\omega \rho \left( \phi_o^{(1)} + \phi_B^{(1)} \right) e^{i\omega t} - \rho \frac{1}{2} \left| \nabla \phi_o^{(1)} \right|^2 - \rho g z \tag{3.13} \]
Introducing (3.13) into (3.11), the vertical force is given by
\[ F_H = -\int_{S_t} \left( i\omega \rho \phi_o^{(1)} e^{i\omega t} + \rho g \epsilon \right) n_3 \, ds \tag{3.14} \]
\[ -\int_{S_o} \left( i\omega \rho \phi_o^{(1)} e^{i\omega t} + \frac{1}{2} \right) \left| \nabla \phi_o^{(1)} \right|^2 n_3 \, ds \]
where \( S_o \), the undisturbed or mean position of the hull, replaces \( S_t \) in the second integral because both terms in the second integrand are of the second order; i.e., the first term in the second integral is of \( O(\alpha\epsilon) \) and the second term is of \( O(\alpha^2) \).

The first term in (3.14) is
\[ f_1 = -i\omega \rho \int_{C_t} e^{ikz} n_3 \, ds \tag{3.15} \]
which by (3.7) becomes
\[ f_1 = \rho g \beta \int_L e^{ikz} \int_{C_t} e^{kz} n_3 \, d\xi \, dx \, e^{i\omega t} \tag{3.16} \]
where \( \beta \) designates integration over the length of the ship and \( d\xi \) is an element of arc along the instantaneous wetted cross section \( C_t \). Now consider only the integral over the cross section in (3.16) which is only a function of \( x \) and \( t \)
\[ f_1(x, t) = \int_{C_t} e^{kz} n_3 \, d\xi \tag{3.17} \]
and let
\[ z = z_o + \eta \tag{3.18} \]
where \( z_o \) is the \( z \) coordinate of the undisturbed hull surface, \( S_o \), and \( \eta \) is the vertical displacement of the hull. We can rewrite (3.17) as
\[ f_1(x, t) = \int_{C_o} e^{kz_o} n_3 \, d\xi + \int_{\Delta C_t} e^{kz_o} n_3 \, d\xi \tag{3.19} \]
where \( C_o \) is the portion of the cross section which is wetted at the rest position and \( \Delta C_t \) is the additional wetted portion of the cross section (see Figure 5). Then by (3.18) it follows that
\[ f_1(x, t) = (1 + k \eta^{(1)}) \int_{C_o} e^{kz_o} n_3 \, d\xi + \Delta b_1 \tag{3.20} \]
where
\[ \Delta b_1 = \int_{\Delta C_t} e^{kz_o} n_3 \, d\xi \]
Here $\Delta b_t$ may be interpreted as the additional beam due to the body motion. If the angle between the $z$ axis and the hull surface is $\gamma$ (see Figure 5) we have

$$ f_1(x, t) = (1 + k \eta^{(1)}) \int_C e^{ikx} \tan \gamma \cdot n_3 d\xi + \eta^{(1)} \cdot \tan \gamma $$

Using the result (3.21), we can rewrite equation (3.16) as

$$ I_1 = (1 + k \eta^{(1)}) \rho g \beta \int_L e^{ikx} \int_C e^{kz_0} n_3 d\xi \cdot d\eta \cdot e^{i\omega t} $$

$$ + \eta^{(1)} \rho g \beta \int_L e^{ikx} \tan \gamma \cdot n_3 d\eta \cdot e^{i\omega t} $$

which we may write as

$$ I_1 = (1 + k \eta^{(1)}) F_0^{(1)} + \eta^{(1)} \rho g \beta \int_L e^{ikx} \tan \gamma \cdot n_3 d\eta \cdot e^{i\omega t} $$

Here $F_0^{(1)}$ is the first-order Froude-Krill0v force

$$ F_0^{(1)} = \rho g \beta \int_L e^{ikx} \int_C e^{kz_0} n_3 d\xi \cdot d\eta \cdot e^{i\omega t} $$

For simplicity we will write (3.23) as

$$ I_1 = F_0^{(1)} + F_0^{(2)} $$

where the second-order force $F_0^{(2)}$ is defined as

$$ F_0^{(2)} = \eta^{(1)} \left( k \cdot F_0^{(1)} + \rho g \beta \int_L e^{ikx} \tan \gamma \cdot dx \cdot e^{i\omega t} \right) $$

Now let us consider the second term in the first integral in (3.14)

$$ I_2 = - \int_{S_t} \rho g \eta n_3 d\xi $$

which by (3.18) becomes

$$ I_2 = - \int_{S_0} \rho g (z_0 + \eta) n_3 d\xi - \int_{\Delta S_t} \rho g (z_0 + \eta) n_3 d\xi $$

where $\Delta S_t$ is the change in the wetted surface relative to the undisturbed wetted surface $S_0$. It can easily be shown that (3.28) can be written as

$$ I_2 = \rho g \eta C \left[ \eta^{(1)} + \eta^{(2)} \right] + F_Y^{(2)} $$

where $C$ is the hydrostatic restoring coefficient which is equal to the area of the water plane at the rest position and where the last term is defined as

$$ F_Y^{(2)} = - \rho g \int_{C_t} (z_0 + \eta) n_3 d\xi $$

The first term in the second integral in (3.14) is

$$ I_3 = - i\omega \rho \int_{S_0} \phi^{(1)} n_3 \cdot e^{i\omega t} $$

We shall write the potential due to the body disturbance as

$$ \phi^{(1)} = \phi_7^{(1)} + \xi^{(1)} \phi $$

where $\phi_7^{(1)}$ is the diffraction potential and $\phi$ is the velocity potential for unit heave motion in calm water. Note that both $\phi_7^{(1)}$ and $\xi^{(1)} \phi$ are of $O(\epsilon^2)$. By (3.32) we have that

$$ I_3 = - i\omega \rho \int_{S_0} (\phi_7^{(1)} + \xi^{(1)} \phi) n_3 d\xi \cdot e^{i\omega t} $$

Now following conventional linearized ship-motion theory (see reference 4) we have

$$ I_3 = F_Y^{(1)} + (\omega^2 A - i\omega B) \xi^{(1)} \phi $$

where $F_Y^{(1)}$ is the diffraction force and $A$ and $B$ are the added-mass and damping coefficients.

Finally, the last term in (3.14) is

$$ I_4 = - \frac{1}{2} \rho \int_{S_0} \nabla \phi^{(1)} \cdot \nabla \phi^{(1)} n_3 d\xi $$
which by introducing (3.7) can be reduced to the form

\[ I_4 = F_{DC}^{(1)} = -\frac{1}{2} \rho g k \beta^2 \int_{S_0} e^{2kz_o} n_3 \, ds \]  
(3.36)

Note that \( I_4 \) is time independent.

Now, returning to Newton's second law (3.9), which states that

\[ M\ddot{\eta} = -\rho g V + F_H \]
(3.37)

Here \( F_H \) which is given by (3.14), is the sum of the four terms \( I_1, I_2, I_3, \) and \( I_4 \) given by equations (3.25), (3.29), (3.34), and (3.36) respectively. Introducing these equations in (3.37) we have that

\[ M\ddot{\eta} + A\dot{\eta} + B\eta + C\eta = F^{(1)} + F^{(2)} + F_{DC}^{(1)} \]

(3.38)

Since it is assumed that the ship is slender and that the wave length is of the order of the ship length the acceleration term, \( M\ddot{\eta} \) is of \( O(\epsilon^2) \) so that we only need to include \( M\ddot{\eta} \). We may write (3.38) as two equations, one equation including only single frequency \( (\omega) \) terms and the other equations including double frequency \( (\omega^2) \) and frequency independent \( (DC) \) terms as shown here

\[ (M + A)\ddot{\eta}^{(1)} + B\dot{\eta}^{(1)} + C\eta^{(1)} = F_{o}^{(1)} + F_{\gamma}^{(1)} \]
(3.39)

and

\[ C\eta^{(2)} = F_{DC}^{(2)} + F_{V}^{(2)} + F_{DC}^{(2)} \]
(3.40)

Note that in (3.39) the acceleration term, \( (M + A)\ddot{\eta}^{(1)} \), the damping velocity term, \( B\dot{\eta}^{(1)} \), and the diffraction force \( F_{\gamma}^{(1)} \) are all of \( O(\epsilon^2) \) whereas the hydrostatic restoring term \( C\eta^{(1)} \) and the Froude-Krilov exciting force are of \( O(\epsilon) \). Even though these terms are of different order it is most convenient to include them all in one equation since they are all single frequency \( (\omega) \) terms and since equation (3.39) is the conventional formulation of the linearized ship-motion strip theory. Equation (3.40) includes all the second-order term of \( O(\epsilon^2) \).

**Coupled Pitch and Heave Motions**

For the more general case of coupled heave and pitch motions it follows from conventional strip theory [3] and from the results derived here (equations 3.39 and 3.40) that if we apply the assumptions used in strip theory in addition to the assumptions introduced here, there will be two sets of coupled equations governing the heave \( \eta_3 \) and pitch \( \eta_5 \). The first set of coupled equations is identical to equations used in strip theory

\[ (M + A_{33})\ddot{\eta}_3^{(1)} + B_{33}\dot{\eta}_3^{(1)} + C_{33}\eta_3^{(1)} + A_{35}\ddot{\eta}_5^{(1)} + B_{35}\dot{\eta}_5^{(1)} + C_{35}\eta_5^{(1)} = F_{3} e^{i\omega t}, \]
(3.41)

\[ (I_5 + A_{33})\ddot{\eta}_5^{(1)} + B_{33}\dot{\eta}_5^{(1)} + C_{33}\eta_5^{(1)} + A_{35}\ddot{\eta}_3^{(1)} + B_{35}\dot{\eta}_3^{(1)} + C_{35}\eta_3^{(1)} = F_{5} e^{i\omega t}, \]
(3.42)

where \( M \) is the mass of the ship and \( I_5 \) is the pitch mass moment of inertia. Here \( A_{jk} \) and \( B_{jk} \) are the added mass and damping coefficients, \( C_{jk} \) are the hydrostatic restoring coefficients, and \( F_j \) are the exciting force and moment. This set of equations can be solved by conventional strip-theory computer programs [3] and results in the first-order solution of the form

\[ \eta^{(1)}_j = \tilde{\eta}^{(1)}_j e^{i\omega t} \]  with \( j = 3 \) and 5
(3.43)

Note that all of the terms of \( O(\beta) \) and \( O(\epsilon \beta) \) are included in (3.41) and (3.42). The second set of equations for the second-order heave \( \eta_3^{(2)} \) and pitch \( \eta_5^{(2)} \) are

\[ C_{33}\ddot{\eta}_3^{(2)} + C_{35}\ddot{\eta}_5^{(2)} = F_{o}^{(2)} + F_{V}^{(2)} + F_{DC}^{(2)} \]
(3.44)

\[ C_{35}\ddot{\eta}_5^{(2)} + C_{53}\ddot{\eta}_3^{(2)} = M_{o}^{(2)} + M_{V}^{(2)} + M_{DC}^{(2)} \]
(3.45)

where it follows from (3.26) that

\[ F_{o}^{(2)} = \rho g \beta \int_{L_1}^{L_2} (f_1 + f_2) \, dx \, e^{i\omega t} \]
(3.46)

and

\[ M_{o}^{(2)} = -\rho g \beta \int_{L} \int_{L_1}^{L_2} (f_1 + f_2) \, dx \, e^{i\omega t} \]
(3.47)

with

\[ f_1 = k (\eta_3^{(1)} - x \eta_5^{(1)}) e^{ikx} \int_{C_o} e^{2kz_o} n_3 \, d\xi \]
(3.48)

and

\[ f_2 = (\eta_5^{(1)} - x \eta_3^{(1)}) e^{ikx} \tan \gamma \]
(3.49)

From (3.30) we have that

\[ F_{V}^{(2)} = -\rho g \frac{1}{2} \int_{L_1}^{L_2} (\eta_3^{(1)} - x \eta_5^{(1)})^2 \tan \gamma \, dx \]
(3.50)

and

\[ M_{V}^{(2)} = \rho g \frac{1}{2} \int_{L} \int_{L_1}^{L_2} (\eta_3^{(1)} - x \eta_5^{(1)})^2 \tan \gamma \, dx \]
(3.51)

and finally from (3.36) it follows that

\[ F_{DC}^{(2)} = -\frac{1}{2} \rho g k \beta^2 \int_{C_o} e^{2kz_o} n_3 \, d\xi \, dx \]
(3.52)

and
\[ M_{BC}^{(2)} = \frac{1}{2} \rho g k \beta^2 \int x \int e^{2kz_0} n_3 \, d^2 \, dx. \]  

(3.53)

The solution to equations (3.44) and (3.45) will be in the form

\[ \eta_y^{(2)} = \eta_y^{(1)} e^{j2\omega t} + M_{DC} \]  

(3.54)

The relative motion at any cross section \( x \) is

\[ z_x = (\eta_y^{(1)} - x \eta_y^{(1)}) + (\eta_y^{(1)} - x \eta_y^{(1)}) - \xi_2 \]  

(3.55)

where \( \xi_2 \) is the second-order free-surface elevation (2.4).

Hence, the derived equations for the heave and pitch motions consist of two sets or coupled equations. The one set of equations (3.41) and (3.42) which is identical to the conventional strip-theory equations [3] and the other set of equations (3.44) and (3.45) which contains the additional terms of \( O(\alpha^2) \).

4. NUMERICAL RESULTS

We shall present here numerical results of the first- and second-order exciting forces for three typical ship sections. For simplicity, we shall select the coordinate system such that \( x = 0 \) at the section in question. The sectional Froude-Krilov force* is then by (3.24)

\[ f_0^{(1)} = \rho g \beta \int e^{kz_0} n_3 \, d^2 \cdot \cos \omega t \]  

(4.1)

We shall here assume that in the integral in (4.1) we may replace \( \exp (kz) \) by \( \exp (-kds) \) 

(4.2)

where \( d \) is the sectional draft and \( s \) is the sectional area coefficient. This assumption is often used in strip-theory computations (see for example reference 4) and has been shown to give accurate results for conventional ship-hull forms. Making the replacement (4.2) in (4.1) we have that

\[ f_0^{(1)} = \rho g \beta b \, e^{-kds} \cdot \cos \omega t \]  

(4.3)

It will be convenient here to normalize the sectional forces by \( \rho g \beta b \) and denote them by an asterisk so that the normalized sectional Froude-Krilov force is

\[ *f_0^{(1)} = e^{-kds} \cos \omega t \]  

(4.4)

It is shown in equations (18) and (32) of [3] that by applying (4.2) the normalized sectional diffraction force is

\[ *f_7^{(1)} = -e^{-kds} \left( \frac{\omega^2}{\rho g b} \, a_{33} - i \, \frac{\omega}{\rho g b} \, b_{33} \right) e^{j\omega t} \]  

(4.5)

where \( a_{33} \) and \( b_{33} \) are the sectional added-mass and damping coefficients. The sectional forces of \( O(\alpha^2) \) are by (3.46), (3.50), and (3.52)

\[ *f_0^{(2)} = (\eta_x^{(1)} - x \eta_x^{(1)}) \left( k \, e^{-kds} + \frac{\tan \gamma \omega}{b} \right) \cos \omega t \]  

(4.6)

\[ *f_x^{(2)} = - \frac{1}{2} \eta_x^{(1)} \eta_x^{(1)} \left( k \, e^{-kds} \right) \frac{\tan \gamma \omega}{b} \]  

(4.7)

\[ *f_{DC}^{(2)} = - \frac{1}{2} \beta \, e^{-2kds} \]  

(4.8)

For the purpose of computing the sectional exciting forces let us consider a ship with beam-to-length and draft-to-length ratios

\[ B/L = 0.15 \quad \text{and} \quad D/L = 0.06 \]  

(4.9)

which are typical for moderate-speed merchant ships. Since the pitch and heave motions in head seas are maximum when the wave lengths are approximately equal to the ship length, we shall assume that

\[ \lambda = L \]  

(4.10)

and a moderately steep wave with

\[ H/\lambda = 0.06 \]  

(4.11)

It follows from (4.9) and (4.11) that the draft is equal to the wave height \( D = H \), which indicates large bow motions but we shall here assume that the sections considered are sufficiently far aft so that the sections do not exit the water.

First we shall consider a rectangular stern section of a transom stern ship with

\[ d/b = 0.10 \quad \text{and} \quad b = B \]  

(4.12)

Here \( d \) and \( b \) are sectional draft and beam, respectively. It is realistic to assume that to the first-order of accuracy, the stern sections move in phase with the wave and with the same amplitude as the wave amplitude so that

\[ \eta_x^{(1)} = \beta \cdot \cos \omega t \]  

(4.13)

We shall select two bow sections both with

\[ d/b = 1.50 \quad \text{and} \quad d = D \]  

(4.14)

One section is rectangular and the other is triangular. For the bow sections it is realistic to assume that to the first order of accuracy the displacement is out of phase with the wave and that the amplitude of the displacement is equal to the draft.

\[ \eta_x^{(1)} - x \eta_x^{(1)} = -d \cos \omega t \]  

(4.15)

The sectional forces (4.4) through (4.8) are given in Table 1 for these three ship sections for the wave condition stated.

*Note that lower case \( f \) will be used for sectional forces, whereas capital \( F \) is used for the total three-dimensional force.
The results given in Table 1 show that for the stern section the additional second-order forces $f^{(2)}_0$ and $f^{(2)}_{DC}$ are approximately 20 percent and 9 percent of the first-order Froude-Krilov force, $f^{(1)}_0$, and for the rectangular bow section they are about 38 percent and 6 percent of the Froude-Krilov force, whereas for the V-shaped bow section the second-order forces $f^{(2)}_0$, $f^{(2)}_V$, and $f^{(2)}_{DC}$ are almost 100 percent, 16 percent, and 7 percent, respectively, of the Froude-Krilov force. The assumed incident wave is probably too large ($H/\lambda = 0.06$) for this perturbation scheme to be applicable; however, the trend is believed to be correct. Namely, the second-order effects will be most important for bow sections where the relative motions are large and particularly for V-shaped bow sections where the hydrostatic restoring will change drastically with large bow displacements.

5. CONCLUDING REMARKS

A second-order theory for predicting ship heave and pitch motions has been derived by assuming that the ship is slender and/or that the frequency is low. Under these assumptions it is shown that the equations governing the motions consist of two sets of coupled equations. One set is identical to the existing conventional strip theory [3] and another set which consists of second-order terms which are only products of the first-order motions and hydrostatic-restoring and Froude-Krilov forces. It has been shown that the hydrodynamic problems associated with added mass, damping, and diffraction must only be solved to the first-order of accuracy with respect to wave slope.

The numerical results for the second-order sectional exciting forces seem to indicate that these second-order forces will be most significant for the bow sections where the motions are usually large and particularly, for V-shaped bow sections. It is anticipated that consideration of these second-order effects will improve on the relative bow motions computed by conventional strip theory. Good prediction accuracy in the relative bow motions is important in the prediction of deck wetness and slamming.

A time-domain method which is similar to this second-order frequency-domain method could also be developed. If a strip-theory approach was used in the time domain, then not only the hydrostatic-restoring and Froude-Krilov forces could be determined for the instantaneous wetted hull shape, but the sectional hydrodynamic forces could also be determined for each time step using the actual wetted surface by adopting the method of R.B. Chapman [14]. Chapman has developed a numerical solution for large-amplitude two-dimensional motions. The body-boundary condition is satisfied at the exact location and the free-surface conditions are linearized by assuming low-frequency motions.

ACKNOWLEDGMENT

This work was supported by the Numerical Naval Hydrodynamics Program at the David W. Taylor Naval Ship Research and Development Center. This program is jointly sponsored by the DTNSRDC and Office of Naval Research.
REFERENCES


ON THE MECHANICS OF COSserAT SURFACES

by

JUSTIN H. McCARTHY
DAVID W. TAYLOR NAVAL SHIP R&D CENTER
BETHESDA, MARYLAND, U.S.A.

ABSTRACT

The paper reviews the continuum mechanics of a material Cosserat surface, $\mathcal{S}$, whose motion at every point is specified by a position vector and a director vector. The kinematics, conservation equations, and constitutive relations are treated. Tensor fields are defined on $\mathcal{S}$ and changes in the reference coordinate frame and changes in the reference material configuration of $\mathcal{S}$ are discussed. Integral forms of the conservation equations, which account for motion of the director vectors, are motivated and postulated for balance of energy, moment of momentum, linear momentum, mass and entropy. Alternate local expressions for the balance equations are derived, and very general forms of the constitutive relations are assumed for all of the constitutive response functions such as stress, internal energy and entropy. The entropy balance equation and the principle of material-frame-indifference are applied to determine restrictions on the functional form of the constitutive response relations. Stress relations are derived for isotropic elastic and fluid Cosserat surfaces with and without incompressibility constraints.

1. KINEMATICS OF SURFACES

Surface Geometry

Let $\mathcal{S}$ be a deforming material surface contained in a three-dimensional Euclidean space, $E_3$, and let $\mathbf{e}_k$ ($k = 1, 2, 3$) be a fixed, right-handed, rectangular Cartesian coordinate system in $E_3$, with unit base vectors $\mathbf{e}_k$. Let $\mathcal{S}_R$ be some reference configuration of the surface in $E_3$, and choose a curvilinear coordinate system $x^\alpha$ ($\alpha = 1, 2, 3$) such that $x^1$ and $x^2$, denoted $x^\alpha$ ($\alpha = 1, 2$), are drawn in $\mathcal{S}_R$ and $x^3$ is directed normal to $\mathcal{S}_R$. The $x^\alpha$ coordinates identify, once and for all, the material particles comprising $\mathcal{S}$ and its surroundings. If attention is now restricted to the case when there is no interchange of particles between $\mathcal{S}$ and its surroundings, $\mathcal{S}$ is a material surface defined...
by \( x^3 = 0 \); the \( x^\alpha (\alpha = 1, 2) \) are material surface coordinates and define a convected coordinate system for a given motion of \( J \).

It is assumed that the coordinates of a given particle in \( E_3' \), relative to the fixed Cartesian coordinate system, may be specified for all time, \( t \), by an invertible mapping

\[
2^k = 2^k (x^\alpha, t), \det \left| \frac{\partial 2^k}{\partial x^\alpha} \right| > 0.
\]

The material particles comprising \( E_3' \), together with all coordinate systems related by proper coordinate transformations \((1.1)_2\) form an oriented simple coordinate manifold \([1]\). The Cartesian position vector \( r = (x^1, x^2, x^3) \) of particles on \( J \) is given by,

\[
r = i_k z^k (x^\alpha, t) \mid_{x^3 = 0} = i_k z^k (x^\alpha, t) = r (x^\alpha, t)
\]

and the surface deformation gradient by,

\[
r;\alpha = \frac{\partial r}{\partial x^\alpha} = i_k z^k (x^\alpha, t)
\]

As indicated in \((1.3)\) a semicolon will always be used to denote partial differentiation with respect to \( x^\alpha \).

It is easily seen that \( r;\alpha \), considered as a space vector, is tangent to \( J \), and that \( z^\alpha \) forms a double tensor field \([2]\). The first fundamental tensor (metric) of \( J \), \( a_{\alpha\beta} \) is defined by,

\[
a_{\alpha\beta} = a_{\alpha\beta} (x^\alpha, t) = r;\alpha \cdot r;\beta = z^1_{\alpha\beta} z^2_{\alpha\beta}
\]

Unless otherwise stated, repeated lower case Latin indices, \( i \), will always imply summation over \( i = 1, 2, 3 \). For Greek indices, \( \alpha, \beta \), summation will be implied over \( \alpha = 1, 2 \) only for diagonally repeated indices, e.g. \( a_{\alpha\beta} b_{\alpha\beta} = a_{11} b_{11} + a_{22} b_{22} \). Since \( \det |\partial 2^k/\partial x^\alpha| \neq 0 \) on \( J \subset E_3' \), it is may be shown that there exists a conjugate metric tensor \( a_{\alpha\beta} \) such that \( a_{\alpha\beta} a_{\gamma\delta} = \delta_{\alpha\gamma} \delta_{\beta\delta} \) where \( \delta_{\alpha\gamma} \) is the Kronecker delta in a two-dimensional space and takes the values \( \delta_{11} = \delta_{22} = 1, \delta_{21} = \delta_{12} = 0 \). The components of the conjugate metric are given by,

\[
a_{11} = \frac{a_{22}}{a}, a_{22} = \frac{a_{11}}{a}, a_{12} = a_{21} = -\frac{a_{12}}{a}
\]

or,

\[
a_{\alpha\beta} = \varepsilon_{\alpha\lambda\mu} \varepsilon_{\beta\mu\nu} a_{\lambda\nu},
\]

where \( a = \det || a_{\alpha\beta} || \) and \( \varepsilon_{\alpha\beta} \) is the contravariant alternating (axial) tensor for a two-dimensional space and takes the values \( \varepsilon_{11} = \varepsilon_{22} = 0, \varepsilon_{12} = -\varepsilon_{21} = a^{1/2} \). It may be shown that \([1]\)

\[
a_{\alpha\beta, \gamma} = a_{\alpha\beta, \gamma} = a_{\alpha\gamma} = \varepsilon_{\alpha\beta, \gamma} = \varepsilon_{\alpha\gamma} = 0,
\]

where a comma will always be used to denote covariant differentiation with respect to \( a_{\alpha\beta} \). For any covariant surface tensor, say \( t_{\alpha\beta, \gamma} \), an associated covariant-contravariant tensor, \( t_{\alpha, \beta, \gamma} \) is defined by \( t_{\alpha, \beta, \gamma} = a_{\alpha, \beta, \gamma} \), which implies

\[
t_{\alpha, \beta, \gamma} = a_{\alpha\beta} t_{\alpha\gamma},
\]

This type of definition may obviously be extended to all surface tensors and will be taken for granted in the future.

The unit normal vector to \( J \), \( n = i_k n^k (x^\alpha, t) \), is defined by

\[
n = \frac{1}{2} \varepsilon_{\alpha\beta} r;\alpha \wedge r;\beta
\]

and satisfies the equations,

\[
n \wedge r;\alpha = \varepsilon_{\alpha\gamma} n^\beta r;\gamma, n \wedge n = 0.
\]

As defined in \((1.7)\), \( n \) is an axial vector which changes sign under improper transformations of either the spatial or surface coordinates, i.e. transformations with negative Jacobian. With \( b_{\alpha\beta} \) denoting the second fundamental form of \( J \), the following formulae may be quoted from classical differential geometry:

\[
r;\alpha \beta = b_{\alpha\beta} n, \quad n;\alpha = -b_{\alpha\beta} r;\beta, \quad b_{\alpha\beta, \gamma} = b_{\alpha\beta, \gamma}.
\]

It is easily seen that \( b_{\alpha\beta} \) is a symmetric, covariant, axial, surface tensor. The mean curvature, \( H \), and total curvature, \( K \), of \( J \) are defined by

\[
H = \frac{2}{a} b_{\alpha\beta} b_{\alpha\beta} \quad K = \frac{b}{a},
\]

where \( b = \det || b_{\alpha\beta} || \). \( H \) is an axial scalar and \( K \) an absolute scalar.

For future reference Green's Theorem for surface integrals will also be recorded here. If \( J \) is a 'regular' portion of surface with boundary curve \( \partial \mathcal{A} \), and \( \nu = \nu_{\alpha} r;\alpha \) is the unit outer normal to \( \partial \mathcal{A} \) tangent to \( J \), then for every contravariant surface vector, \( V^\alpha \), which is continuously differentiable on \( \mathcal{A} \),

\[
\int_{\partial \mathcal{A}} V^\alpha \cdot d\mathcal{A} = \int_{\mathcal{A}} V^\alpha \cdot \nu_{\alpha} \cdot ds.
\]

Explicit formulae for \( V^\alpha \cdot \nu_{\alpha} \) and \( d\mathcal{A} \) are given by,

\[
V^\alpha \cdot \nu_{\alpha} = a^{1/2} (a^{1/2} V^\alpha)_{\alpha},
\]

\[
d\mathcal{A} = a^{1/2} dx^1 dx^2.
\]

Tensor Fields Defined on \( J \)

In the following we consider a Cosserat surface, \( J \), whose motion at every point is specified by a position vector \( r = i_k z^k (x^\alpha, t) \) and a single absolute 'director' vector \( d = i_k d^k (x^\alpha, t) \) associated with each particle of \( J \), having dimensions of length. Heuristically, a mono-molecular film might be regarded as a Cosserat surface, where one thinks of \( d \) as characterizing the orientation of the molecules composing \( J \).
In subsequent formulations of conservation and constitutive equations, mixed surface gradients and material derivatives of \( r \) and \( d \) will occur. For later convenience, then, we now introduce the components of such space vectors with respect to the basis \( \{ r_\alpha, n \} \). That the triple \( \{ r_\alpha, n \} \) forms a basis for all space vectors defined on \( J \) follows from (1.1).\(^2\)

With reference to the basis \( \{ r_\alpha, n \} \), decomposition of the director vector into components tangent and normal to \( J \) yields

\[
d = d^\alpha r_\alpha + d_3 n
\]  
(1.13)

where, by (1.4) and (1.8),

\[
d_\alpha = a_\beta^\alpha d^\beta = r_\alpha \cdot d, \quad d_3 = d = n \cdot d. \]
(1.14)

Equations (1.14) may be taken as definitions of \( d_\alpha \) and \( d_3 \). They imply that \( d_\alpha \) are the components of an absolute covariant surface vector, and that \( d_3 \) is an axial scalar. These sorts of results apply to all space vector fields defined on \( J \), so that the director surface gradient, \( d_\alpha \), may be written,

\[
d_\alpha = \lambda_\beta^\alpha r_\beta + \lambda_3^\alpha n, \]
(1.15)

Explicit expressions for the surface tensors \( \lambda_\beta^\alpha \) and the axial surface vectors \( \lambda_3^\alpha \), in terms of \( d_\alpha \) and \( d_3 \), may be obtained by direct covariant differentiation of (1.13), making use of equations (1.9)\(^1\),\(^2\). The results are,

\[
\lambda_\beta^\alpha = d_\beta - b_\beta^\alpha d_3, \quad \lambda_3^\alpha = d_3 + d_\beta^\alpha b_\beta^\alpha. \]
(1.16)

The surface velocity, \( \mathbf{r} \), and director velocity, \( \mathbf{d} \), associated with a given particle on \( J \), are defined respectively as the material derivatives of \( r \) and \( d \), e.g., \( \mathbf{d} = \frac{d}{dt}(d) = \partial/dt(d)\big|_{x^\alpha \text{ const.}} \). As before we may write the decompositions,

\[
\mathbf{r} = v^\alpha r_\alpha + v^3 n, \quad v_\alpha = a_\beta^\alpha v^\beta = r_\alpha \cdot \mathbf{r}, \quad v_3 = v^3 = n \cdot \mathbf{r}, \]
(1.17)

and,

\[
\mathbf{d} = w^\alpha r_\alpha + w^3 n, \quad w_\alpha = a_\beta^\alpha w^\beta = r_\alpha \cdot \mathbf{d}, \quad w_3 = w^3 = n \cdot \mathbf{d}. \]
(1.18)

Similarly, for the surface velocity gradient, \( \mathbf{r}_\alpha \), and the surface director velocity gradient, \( \mathbf{d}_\alpha \),

\[
\mathbf{r}_\alpha = V^\beta_\alpha r_\beta + V^3_\alpha n, \quad \mathbf{d}_\alpha = W^\beta_\alpha r_\beta + W^3_\alpha n. \]
(1.19)

With reference to the basis \( \{ r_\alpha, n \} \), decomposition of the director vector into components tangent and normal to \( J \) yields

\[
\mathbf{d} = d^\alpha r_\alpha + d_3 n
\]  
(1.13)

where, by (1.4) and (1.8),

\[
d_\alpha = a_\beta^\alpha d^\beta = r_\alpha \cdot d, \quad d_3 = d = n \cdot d. \]
(1.14)

Equations (1.14) may be taken as definitions of \( d_\alpha \) and \( d_3 \).

They imply that \( d_\alpha \) are the components of an absolute covariant surface vector, and that \( d_3 \) is an axial scalar. These sorts of results apply to all space vector fields defined on \( J \), so that the director surface gradient, \( d_\alpha \), may be written,

\[
d_\alpha = \lambda_\beta^\alpha r_\beta + \lambda_3^\alpha n, \]
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Explicit expressions for the surface tensors \( \lambda_\beta^\alpha \) and the axial surface vectors \( \lambda_3^\alpha \), in terms of \( d_\alpha \) and \( d_3 \), may be obtained by direct covariant differentiation of (1.13), making use of equations (1.9)\(^1\),\(^2\). The results are,

\[
\lambda_\beta^\alpha = d_\beta - b_\beta^\alpha d_3, \quad \lambda_3^\alpha = d_3 + d_\beta^\alpha b_\beta^\alpha. \]
(1.16)

The surface velocity, \( \mathbf{r} \), and director velocity, \( \mathbf{d} \), associated with a given particle on \( J \), are defined respectively as the material derivatives of \( r \) and \( d \), e.g., \( \mathbf{d} = \frac{d}{dt}(d) = \partial/dt(d)\big|_{x^\alpha \text{ const.}} \).

As before we may write the decompositions,

\[
\mathbf{r} = v^\alpha r_\alpha + v^3 n, \quad v_\alpha = a_\beta^\alpha v^\beta = r_\alpha \cdot \mathbf{r}, \quad v_3 = v^3 = n \cdot \mathbf{r}, \]
(1.17)

and,

\[
\mathbf{d} = w^\alpha r_\alpha + w^3 n, \quad w_\alpha = a_\beta^\alpha w^\beta = r_\alpha \cdot \mathbf{d}, \quad w_3 = w^3 = n \cdot \mathbf{d}. \]
(1.18)

Similarly, for the surface velocity gradient, \( \mathbf{r}_\alpha \), and the surface director velocity gradient, \( \mathbf{d}_\alpha \),

\[
\mathbf{r}_\alpha = V^\beta_\alpha r_\beta + V^3_\alpha n, \quad \mathbf{d}_\alpha = W^\beta_\alpha r_\beta + W^3_\alpha n. \]
(1.19)

Alternatively, by direct material differentiation of (1.13) and (1.15), making use of (1.4), (1.8)\(^1\) and (1.19), it follows that (1.18)\(^3\) and (1.20)\(^3\) are given by the formulae,

\[
w_\beta = d_\beta - d^\alpha v^\gamma_\alpha - d^3 v^3_\beta, \quad w_3 = d_3 + d^\alpha v^3_\alpha. \]
(1.21)

\[
W_\beta = \lambda_\beta^\alpha r_\alpha r^\gamma_\alpha - \lambda^3_\alpha v^3_\beta, \quad W_3 = \lambda_3^\alpha + \lambda^3_\alpha v^3_\beta. \]

For future reference, the following useful formulae are also obtained. First, we note by comparison with (1.19)\(^2\) that the material derivative of the surface metric, defined in (1.4), is given by,

\[
\dot{a}_\beta^\alpha = 2 a_\beta^\alpha V^\gamma_\alpha - a_\beta^\alpha (c^\alpha_\beta + c^\beta_\alpha). \]
(1.22)

where \( V^\alpha_\beta \) denotes the symmetric part of \( V^\alpha_\beta \). It follows from the definition of \( a \), making use of (1.5) and (1.22), that

\[
\dot{a}_\beta^\alpha = 2 a_\beta^\alpha V^\gamma_\alpha - a_\beta^\gamma (c^\alpha_\gamma + c^\gamma_\alpha). \]
(1.23)

Hence, by (1.12)\(^2\), for a continuously differentiable function \( \xi \), defined on a regular material area \( \mathcal{A} \subset J \),

\[
\frac{d}{dt} \int_{\mathcal{A}} \xi d\mathcal{A} = \int_{\mathcal{A}} \left( \dot{\xi} + \dot{\mathbf{v}}_\alpha \frac{\partial}{\partial x^\alpha} \xi \right) d\mathcal{A} \]
(1.24)

\[= \int_{\mathcal{A}} \left( \dot{\xi} + \frac{\partial}{\partial t} \hat{a}_\alpha^\beta \xi \right) d\mathcal{A}. \]

Change of Reference Frame and Objectivity

Following Truesdell and Noll [3], a 'change of (reference) frame' is defined as a one-to-one mapping of space-time onto itself, such that distances, time intervals, and temporal order are preserved. Under a change of frame, the Cartesian components of a given motion of \( J \) in each frame are related most generally by transformations of the type,

\[
z^* = (x^a, t^*) = c^i (t) + Q_{ij} (t) z^j (x^a, t), \]
\[
\dot{z}^* = (x^a, t^*) = Q_{ij} (t) \dot{z}^j (x^a, t), \]
(1.25)

where \( c^i \) is some constant scalar, \( c^i \) is some time-dependent space vector, and \( Q_{ij} \) is some time-dependent orthogonal space tensor, i.e., \( Q_{ik} Q_{jk} = \delta_{ij} \). In a change of frame the same motion of \( J \)
is being described; only the frame of reference of the observer is changed. To illustrate, suppose \( t = t^* \) and that \( \mathbf{z} \) and \( \mathbf{d} \) describe the motion of \( \mathbf{J} \) relative to the spatially fixed reference coordinate system originally selected at the outset of this work.

Then, the asterisked terms in (1.25)_1,2 would describe the motion of \( \mathbf{J} \) relative to some rotating and translating Cartesian Coordinate system, not necessarily right-handed, in terms of the motion originally described in the spatially fixed reference coordinate system. By direct differentiation of (1.25)_1,2, the various derivatives of \( \mathbf{z}^* \) and \( \mathbf{d}^* \) are given by,

\[
\begin{align*}
\dot{\mathbf{z}}^* &= \mathbf{Q}_{\mathbf{ij}} \mathbf{z}_i^* \mathbf{\Omega}_{\mathbf{jk}} \mathbf{k}_j, \\
\dot{\mathbf{d}}^* &= \mathbf{Q}_{\mathbf{ij}} \mathbf{d}_i^* \mathbf{\Omega}_{\mathbf{jk}} \mathbf{k}_j,
\end{align*}
\]

(1.26)

where \( \mathbf{\Omega}_{\mathbf{ij}} \) is a skew-symmetric space tensor, as follows from \( \mathbf{Q}_{\mathbf{ij}} \) being orthogonal.

Now, for future reference when formulating constitutive equations, suppose that \( \mathbf{X} \) and \( \mathbf{X}^* \) are sets of quantities defined by,

\[
\begin{align*}
\mathbf{X} &= (t, \mathbf{z}, \mathbf{\dot{z}}, \mathbf{z}_i^*, \mathbf{d}, \mathbf{\dot{d}}, \mathbf{d}_i^*) \\
\mathbf{X}^* &= (t^*, \mathbf{z}^*, \mathbf{\dot{z}}^*, \mathbf{z}_i^{*,*}, \mathbf{d}^*, \mathbf{\dot{d}}^*, \mathbf{d}_i^{*,*})
\end{align*}
\]

and that \( s = s(\mathbf{X}) \) and \( v^i = v^i(\mathbf{X}) \) are respectively scalar and vector valued functions under a change of frame. Then, \( s \) and \( v^i \) are said to be 'material-frame-indifferent' or 'objective' if

\[
\begin{align*}
s &= s(\mathbf{X}) = s(\mathbf{X}^*), \\
v^i &= v^i(\mathbf{X}) = v^i(\mathbf{X}^*),
\end{align*}
\]

(1.27)

under all changes of frame. The requirement of objectivity imposes restrictions on the admissible functional forms of \( s \) and \( v^i \). For example, if we set \( \mathbf{Q}_{\mathbf{ij}} = \delta_{\mathbf{ij}} \) and successively let \( \mathbf{c}^i = 0, \mathbf{c}^i = \text{const} \) and \( \mathbf{c}^i = \text{const} \), it follows from (1.25) and (1.26) that \( s \) and \( v^i \) can not have explicit dependence on \( t \) nor depend on \( \mathbf{z}^i \) or \( \mathbf{d}^i \). Thus, for objectivity we must have,

\[
\begin{align*}
s &= s(\mathbf{Y}) = s(\mathbf{Y}^*), \\
v^i &= v^i(\mathbf{Y}) = Q_{\mathbf{ij}} v^j(\mathbf{Y}^*),
\end{align*}
\]

(1.27)

where,

\[
\mathbf{Y} = (d^i, \mathbf{\dot{z}}^i, \mathbf{\dot{d}}^i, \mathbf{d}_i^*),
\]

If the independent variables, \( \mathbf{Y} \), in (1.27) are considered as space vectors, to say that \( s \) and \( v^i \) are objective is equivalent to requiring that \( s \) and \( v^i \) be isotropic functions of these vectors, independent of \( \mathbf{Q}_{\mathbf{ij}} \) and \( \mathbf{\Omega}_{\mathbf{ij}} \). From the representation formula for isotropic vector functions [4] it follows that,

\[
v^i = d^i A^i(\mathbf{Y}) + \mathbf{d}^i B(\mathbf{Y}) + d^i_\mathbf{\alpha} B^\mathbf{\alpha}(\mathbf{Y}),
\]

(1.28)

where \( A^i, B \) and \( B^\mathbf{\alpha} \) are to be objective scalar functions with respect to changes of frame. By Cauchy's representation formula [4] \( s, A^i, B \) and \( B^\mathbf{\alpha} \) can be expressed as a function of the inner products of the vectors which comprise \( \mathbf{Y} \). With reference to (1.25) and (1.26), when one writes out all the inner products, the following maximal set of independent objective variables may be constructed:

\[
\begin{align*}
s_{\mathbf{\alpha} \mathbf{\beta}} &= \mathbf{z}^i \mathbf{z}^j, \\
d_{\mathbf{\alpha} \mathbf{\beta}} &= \mathbf{z}^i \mathbf{d}^j, \\
\lambda_{\mathbf{\alpha} \mathbf{\beta}} &= \mathbf{d}^i \mathbf{z}^j, \\
\lambda_{\mathbf{\alpha} \mathbf{\beta}} &= \mathbf{d}^i \mathbf{d}^j, \\
\mathbf{D} &= \frac{1}{2} \mathbf{d}^i \mathbf{d}^j = \frac{1}{2} (\mathbf{d}^i \mathbf{d}^j + \mathbf{d}^j \mathbf{d}^i), \\
\mathbf{D}_{\mathbf{\alpha} \mathbf{\beta}} &= \mathbf{d}^i \mathbf{d}^j - \mathbf{d}^j \mathbf{d}^i, \\
\dot{\mathbf{D}} &= \mathbf{d}^i \mathbf{d}^j = \mathbf{d}^j \mathbf{d}^i, \\
\dot{\mathbf{D}}_{\mathbf{\alpha} \mathbf{\beta}} &= \mathbf{d}^i \mathbf{d}^j - \mathbf{d}^j \mathbf{d}^i = \mathbf{d}^i \mathbf{d}^j - \mathbf{d}^j \mathbf{d}^i, \\
\dot{\mathbf{D}}_{\mathbf{\alpha} \mathbf{\beta}} &= \mathbf{d}^i \mathbf{d}^j - \mathbf{d}^j \mathbf{d}^i
\end{align*}
\]

(1.30)

In writing (1.30), use has been made of the definitions given in equations (1.13) to (1.22). Finally, then, for any objective scalar, \( s = s(\mathbf{Y}) \), we have,

\[
\begin{align*}
s &= s(\mathbf{Y}) = s(\mathbf{Y}^*), \\
\dot{s} &= \frac{\partial s}{\partial t} = \sum_{\mathbf{\alpha} \mathbf{\beta} \mathbf{\gamma}} \frac{\partial s}{\partial d^i} \frac{\partial d^i}{\partial t} \mathbf{d}^i, \\
\dot{\mathbf{Y}} &= \frac{\partial s}{\partial \mathbf{Y}} = \sum_{\mathbf{\alpha} \mathbf{\beta} \mathbf{\gamma}} \frac{\partial s}{\partial \mathbf{d}^i} \frac{\partial \mathbf{d}^i}{\partial \mathbf{Y}} \mathbf{d}^i
\end{align*}
\]

(1.31)

In the special case when the director vector is constrained to be of constant length, \( D = \text{const} \) and \( D_{\mathbf{\alpha} \mathbf{\beta}} = \dot{D} = \dot{D}_{\mathbf{\alpha} \mathbf{\beta}} = 0 \), so that,

\[
s|_{\mathbf{D} = \text{const}} = \sum_{\mathbf{\alpha} \mathbf{\beta} \mathbf{\gamma}} \frac{\partial s}{\partial \mathbf{d}^i} \frac{\partial \mathbf{d}^i}{\partial \mathbf{Y}} \mathbf{d}^i
\]

(1.32)

Change of Reference Configuration

At the outset we chose a time-independent reference configuration for our material surface. While such a fixed configuration will be conceptually preferable when formulating balance and constitutive equations, for the solution of many problems it is often more convenient to take the (time-dependent) current configuration of the surface for reference. This will be true, for example, when calculating the flow inside a fluid surface of fixed geometry. For completeness, then, we now establish the transformation laws which relate surface tensor fields referred to time-independent and time-dependent reference configurations.
In the current configuration the surface coordinates, denoted $x^\Delta$, $\Delta = 1, 2$, may be related to the fixed reference configuration coordinates, $x^\alpha$, $\alpha = 1, 2$, by a time-dependent invertible mapping:

$$x^\Delta = x^\Delta(x^\alpha, t).$$

(1.33)

Thus, for the surface position vector, and director vector, $\mathbf{d}$, one may write

$$\mathbf{r} = \mathbf{r}(x^\alpha, t) = \hat{\mathbf{r}}(x^\Delta, t)$$

$$\mathbf{d} = \mathbf{d}(x^\alpha, t) = \hat{\mathbf{d}}(x^\Delta, t)$$

(1.34)

where the hat will be used here to denote functional dependence on $x^\Delta$ rather than $x^\alpha$. We also may write,

$$\mathbf{n} = \mathbf{n}(x^\alpha, t) = \pm \hat{\mathbf{n}}(x^\Delta, t),$$

(1.35)

where, as mentioned earlier, the sign of $\hat{\mathbf{n}}$ is positive or negative according as transformation (1.33) is proper or improper.

Relative to the current configuration coordinates, the first and second fundamental tensors, $\hat{\mathbf{a}}_{\Delta \Gamma}$ and $\hat{\mathbf{b}}_{\Delta \Gamma}$, of the surface are defined by,

$$\hat{\mathbf{a}}_{\Delta \Gamma} = \hat{\mathbf{r}}_{\alpha\beta} \mathbf{a}_{\alpha\beta}$$

(1.36)

It follows from (1.34) by the chain rule for differentiation that the surface velocity, $\dot{\mathbf{r}}$, and director velocity, $\dot{\mathbf{d}}$, associated with the surface particle at $x^\Delta$, may be written,

$$\dot{\mathbf{r}} = \frac{\partial}{\partial t} \hat{\mathbf{r}} + u^\Delta \hat{\mathbf{r}}_{\Delta}$$

$$\dot{\mathbf{d}} = \frac{\partial}{\partial t} \hat{\mathbf{d}} + u^\Delta \hat{\mathbf{d}}_{\Delta},$$

(1.37)

where,

$$u^\Delta \equiv u^\Delta(x^\Gamma, t) \equiv \dot{x^\Delta}$$

(1.38)

is the particle's surface velocity relative to the current configuration coordinates, and $\partial/\partial t = \partial/\partial t |_{x^\alpha = \text{const}}$. Another differentiation and use of (1.36), gives the corresponding accelerations of surface particles at fixed $x^\Delta$:

$$\ddot{\mathbf{r}} = \frac{\partial^2}{\partial t^2} \hat{\mathbf{r}} + 2\frac{\partial}{\partial t} \frac{\partial}{\partial t} \hat{\mathbf{r}}_{\Delta} + \left(\frac{\partial u^\Delta}{\partial t} + u^\Gamma \hat{\mathbf{r}}_{\Gamma, \Delta}\right) \hat{\mathbf{r}}_{\Delta} + u^\Delta u^\Gamma \hat{\mathbf{b}}_{\Delta \Gamma} \hat{\mathbf{n}},$$

(1.39)

$$\ddot{\mathbf{d}} = \frac{\partial^2}{\partial t^2} \hat{\mathbf{d}} + 2\frac{\partial}{\partial t} \frac{\partial}{\partial t} \hat{\mathbf{d}}_{\Delta} + \left(\frac{\partial u^\Delta}{\partial t} + u^\Gamma \hat{\mathbf{d}}_{\Gamma, \Delta}\right) \hat{\mathbf{d}}_{\Delta} + u^\Delta u^\Gamma \hat{\mathbf{b}}_{\Delta \Gamma} \hat{\mathbf{n}}.$$

For steady motion inside surfaces of fixed geometry, terms involving $\partial/\partial t$ are zero, so that (1.37) and (1.39) simplify considerably.

We now obtain the transformation laws for the objective variables listed in (1.30). First, it follows by direct chain rule differentiation of (1.34), making use of (1.33), that,

$$\dot{\mathbf{r}}_\alpha = \frac{\partial x^\Gamma}{\partial x^\alpha} \dot{\mathbf{r}}_\Gamma, \dot{\mathbf{d}}_\alpha = \frac{\partial x^\Gamma}{\partial x^\alpha} \dot{\mathbf{d}}_\Gamma.$$  

(1.40)

In terms of the current configuration coordinates the triple $\{\hat{\mathbf{r}}, \hat{\mathbf{d}}, \hat{\mathbf{n}}\}$ forms a basis for all surface vectors, so that corresponding to (1.13) to (1.15) one may write the decompositions,

$$\hat{\mathbf{d}} = \hat{\mathbf{d}}_\Delta \hat{\mathbf{r}}_\Delta + \hat{\mathbf{d}}^\Gamma \hat{\mathbf{n}},$$

$$\hat{\mathbf{d}}_{\Delta \Gamma} = \hat{\mathbf{d}}_\Delta \hat{\mathbf{r}}_{\Delta \Gamma} + \hat{\mathbf{d}}^\Gamma \hat{\mathbf{n}}.$$  

(1.41)

where

$$\hat{\mathbf{d}}_\Delta = \hat{\mathbf{d}}_{\Delta \Sigma} \hat{\mathbf{d}}^\Sigma = \hat{\mathbf{r}}_{\Delta \Gamma} \cdot \hat{\mathbf{d}}_{\Gamma}, \hat{\mathbf{d}}^\Gamma = \hat{\mathbf{n}} \cdot \hat{\mathbf{d}}.$$  

(1.42)

We also define,

$$\hat{\mathbf{\lambda}}_{\Delta \Gamma} = \hat{\mathbf{d}}_{\Delta \Sigma} \hat{\mathbf{\lambda}}^\Sigma = \hat{\mathbf{r}}_{\Delta \Gamma} \cdot \hat{\mathbf{\lambda}}_{\Gamma}, \hat{\mathbf{\lambda}}_{\Delta \Gamma} = \hat{\mathbf{\lambda}}_\Delta \cdot \hat{\mathbf{\lambda}}_{\Gamma}.$$  

(1.43)

Substituting (1.34), (1.40) and (1.41) into (1.30), and making use of (1.42) and (1.43), it is easily verified that

$$a_{\alpha\beta} = \frac{\partial x^\Delta}{\partial x^\alpha} \frac{\partial x^\Gamma}{\partial x^\beta} b_{\Delta \Gamma} + \hat{\mathbf{\lambda}}_{\Delta \Gamma},$$

$$\lambda_{\alpha\beta} = \frac{\partial x^\Delta}{\partial x^\alpha} \frac{\partial x^\Gamma}{\partial x^\beta} b_{\Delta \Gamma} + \hat{\mathbf{\lambda}}_{\Delta \Gamma},$$

(1.44)

$$\hat{\mathbf{D}} = \hat{\mathbf{D}}_\Delta + \hat{\mathbf{D}}_{\Delta \Gamma},$$

indicating the absolute tensor character of the above set of objective variables under time-dependent transformations of the reference configuration. By direct differentiation of (1.44) it follows that,

$$\dot{a}_{\alpha\beta} = \frac{\partial x^\Delta}{\partial x^\alpha} \frac{\partial x^\Gamma}{\partial x^\beta} b_{\Delta \Gamma} + \hat{\mathbf{\lambda}}_{\Delta \Gamma},$$

$$\dot{\lambda}_{\alpha\beta} = \frac{\partial x^\Delta}{\partial x^\alpha} \frac{\partial x^\Gamma}{\partial x^\beta} b_{\Delta \Gamma} + \hat{\mathbf{\lambda}}_{\Delta \Gamma},$$

$$\hat{\mathbf{D}} = \frac{\partial}{\partial t} \hat{\mathbf{D}} + \mathbf{u} \cdot \hat{\mathbf{D}}_\Gamma.$$  

(1.45)
2. SURFACE BALANCE EQUATIONS

Basic Postulates

In order to describe the mechanical-thermodynamical behavior of Cosserat surfaces, five surface balance equations are postulated. They are energy, moment of momentum, linear momentum, mass and entropy balance equations. As will be seen, the first four balance equations postulated lead to forms previously proposed by other authors, and the entropy balance equation may be viewed as a generalization of the Clausius-Duhem inequality. Rather than trying to motivate the balance equations at this point, we first state them in integral form and then motivate and interpret them in some detail. Subsequently, the balance equations are reduced to local form, and independent variables identified.

In the following, all motion is referred to our spatially fixed Cartesian coordinate system. Let \( \mathcal{A} \subset \mathbb{R}^3 \) be a material area and \( \partial \mathcal{A} \) its boundary curve with unit outer normal \( \nu \equiv \nu^\alpha \mathbf{r}_\alpha \) tangent to \( \mathcal{A} \). The following notation is used: \( \rho \) is the surface mass per unit area, \( \varepsilon \) internal energy per unit mass, \( F \) external force per unit area, \( G \) external 'director force' per unit area, \( r \) external heat supply rate per unit area, \( \mathbf{t} \) intrinsic boundary traction force per unit length, \( \mathbf{s} \) intrinsic boundary 'director traction' force per unit length, and \( h \) the heat flux rate per unit length out of \( \mathcal{A} \) through \( \partial \mathcal{A} \). The quantities \( \rho, \varepsilon, r \) and \( h \) are assumed to be absolute scalars and \( \mathbf{t} \) and \( \mathbf{s} \) are assumed to be absolute spatial vectors. The assumed surface balance equations are:

**POSTULATE I (Energy Balance)**

\[
\frac{d}{dt} \int_{\mathcal{A}} \left( e + \frac{1}{2} \mathbf{r} \cdot \mathbf{r} + \frac{1}{2} \mathbf{d} \cdot \mathbf{d} \right) d\mathcal{A} = \int_{\mathcal{A}} (F \cdot \mathbf{r} + G \cdot \mathbf{d} + \mathbf{t}) d\mathcal{A} + \oint_{\partial \mathcal{A}} (\mathbf{r} \cdot \mathbf{r} + \mathbf{d} \cdot \mathbf{d} - h) ds.
\]

**POSTULATE II (Moment of Momentum Balance)**

\[
\frac{d}{dt} \int_{\mathcal{A}} \rho (\mathbf{r} \times \mathbf{d}) \cdot \mathbf{d} d\mathcal{A} = \int_{\mathcal{A}} (\mathbf{r} \times F + \mathbf{d} \times G) d\mathcal{A} + \oint_{\partial \mathcal{A}} (\mathbf{r} \times \mathbf{t} + \mathbf{d} \times \mathbf{s}) ds.
\]

**POSTULATE III (Linear Momentum Balance)**

\[
\frac{d}{dt} \int_{\mathcal{A}} \rho \mathbf{r} d\mathcal{A} = \int_{\mathcal{A}} F d\mathcal{A} + \oint_{\partial \mathcal{A}} \mathbf{t} ds.
\]

**POSTULATE IV (Mass Balance)**

\[
\frac{d}{dt} \int_{\mathcal{A}} \rho d\mathcal{A} = 0.
\]

**POSTULATE V (Entropy Balance)**

\[
\frac{d}{dt} \int_{\mathcal{A}} \rho \eta d\mathcal{A} = \int_{\mathcal{A}} \left( \frac{\eta}{\mathbf{r}} + \varepsilon \right) d\mathcal{A} - \oint_{\partial \mathcal{A}} k ds.
\]

where \( \eta \) denotes surface entropy per unit mass, \( \theta \) surface temperature, \( e \) intrinsic entropy production rate per unit area, and \( k \) intrinsic entropy flux rate per unit length out of \( \mathcal{A} \) through \( \partial \mathcal{A} \). It is assumed that \( \theta > 0 \). The quantities \( \eta, \theta, e \) and \( k \) are assumed to be absolute scalars.

Motivation

**POSTULATE I:**

The surface energy balance equation (2.1) has a form similar to that used by Green, Nagdhi and Wainwright [6], and is fully analogous to that proposed by Ericksen [7] in his theory of bulk liquid crystals. To motivate the form of his balance equation (see also, Truesdell and Noll [31, § 127] considers a three-dimensional medium composed of rod-like molecules. For a Cosserat surface, consider the analogous model: a monomolecular film, each molecule of which is represented by a uniformly extensible line vector \( \mathbf{d} \) along which mass is non-uniformly distributed. Let \( \mathbf{r'} \) be the position vector of any point on a given molecule and \( \mathbf{r} \) the position vector of the molecule's center of mass relative to our fixed Cartesian coordinate system. Then, if \( s \in [-s_0, 1-s_0] \) is the fractional distance measured along the molecule from its center of mass, \( \gamma(s) \) the mass per unit \( s \), \( m \) the molecule's mass and \( I \) the molecule's mass moment of inertia about its centroid, we may write:

\[
\mathbf{r'} = \mathbf{r} + s \sqrt{m} \mathbf{d}, \quad m = \int_{-s_0}^{1-s_0} \gamma(s) ds
\]

\[
1 = \int_{-s_0}^{1-s_0} \gamma(s) s^2 ds, \quad \mathbf{r} = \int_{-s_0}^{1-s_0} \gamma(s) s \mathbf{d} ds.
\]

The first equation may be taken as the definition of the director vector \( \mathbf{d} \). It follows, then, that the kinetic energy of each molecule is given by,

\[
\int_{-s_0}^{1-s_0} \gamma(s) \mathbf{r'} \cdot \mathbf{r'} ds = \frac{m}{2} (\mathbf{r} \cdot \mathbf{r} + \mathbf{d} \cdot \mathbf{d}),
\]

which corresponds to the kinetic energy terms shown on the left-hand side of (2.1).
Similarly, if \( \tau(s) \) is an intrinsic force per unit \( s \), exerted on a surface molecule by the other molecules comprising \( J \), then the rate of working of a molecule by the intrinsic force is given by,

\[
\int_{-s_0}^{1-s_0} \tau(s) \cdot i' \, ds = t \cdot \dot{r} + s \cdot \dot{d},
\]

(2.1.3)

Thus, the rate of working by external forces shown in the line integral on the right-hand side of (2.1). In this model we note that when \( \tau(s) = c \), independent of \( s \), and the molecule’s center of mass is at its geometric center, i.e. \( s_0 = 1/2 \), then \( t = c \) and \( s = 0 \).

In addition to the intrinsic forces there will also be external forces acting on a surface molecule, which may be considered in two parts. First, there may be long-range body forces, \( f_B(s) \), per unit mass acting on each molecule of \( J \), and secondly there may be contact forces exerted by material on each side of \( J \). The latter, denoted \( f_c(s) \) and \( f_r(s) \), which are forces per unit \( s \), correspond to forces exerted by material on, say, the upper and lower sides of \( J \). Thus, the rate of working by external forces on each molecule is given by:

\[
\int_{-s_0}^{1-s_0} [\gamma(s) f_B(s) + f_c(s) + f_r(s)] \cdot i' \, ds = F \cdot \dot{r} + G \cdot \dot{d},
\]

(2.1.4)

\[
F = m F_B + F_c + F_r, \quad G = m G_B + G_c + G_r,
\]

\[
F_B = \frac{1}{m} \int_{-s_0}^{1-s_0} \gamma(s) f_B(s) \, ds, \quad G_B = \frac{1}{m} \int_{-s_0}^{1-s_0} \gamma(s) f_B(s) \, ds,
\]

\[
F_c \equiv \int_{-s_0}^{1-s_0} f_c(s) \, ds, \quad G_c \equiv \int_{-s_0}^{1-s_0} f_c(s) \, ds,
\]

The above corresponds to the rate of working by external forces in the surface integral on the right-hand side of (2.1). We note that if \( f_B(s) = C_B \) independent of \( s \), then \( F_B = C_B \) and \( G_B = 0 \). This would be the case if gravity were the only source of body forces. Also, if \( f_c(s) = C_c \) independent of \( s \), and the molecule’s center of mass is at its geometric center, then \( F_c = C_c \) and \( G_c = 0 \). In subsequent developments the external body force density, \( f_c(s) \), will be regarded as arbitrary, so that \( F \) and \( G \) may be considered arbitrary.

As in the preceding, the external heat supply rate to each molecule of \( J \) will be considered in two parts. First, there may be an external body heat supply rate, \( r_B(s) \), per unit mass of a given molecule, and, secondly, heat supply rates, \( r_c(s) \) and \( r_r(s) \), per unit \( s \), from material in contact with each side of \( J \). Thus, the total external heat supply rate to a molecule is given by

\[
\int_{-s_0}^{1-s_0} \left[ \gamma(s) r_B'(s) + r_c'(s) + r_r'(s) \right] \, ds = m r_B + r_c + r_r = r,
\]

(2.1.5)

This corresponds to the external heat supply rate term shown in the surface integral on the right-hand side of (2.1). The external body heat supply rate, \( r_B(s) \), will be regarded as arbitrary, so that \( r \) may be considered arbitrary.

The remaining terms, \( e \) and \( h \), appearing in (2.1), have obvious analogs in the monomolecular film model adopted here. If we assume that all surface torques are moments of forces and that intrinsic inter-molecular forces are equal and oppositely directed, it follows from the above analog that (2.1) is equivalent to the usual assertion of energy balance: the rate of change of total energy of every material part, \( A \subset J \), is equal to the rate of working by \( A \subset J \), plus the net rate of supply of heat to \( A \subset J \).

For the external forces, \( F \) and \( G \), and external heat supply rate \( r \), we have the decompositions,

\[
F = \rho F_B + F_c + F_r, \quad G = \rho G_B + G_c + G_r, \quad r = \rho r_B + r_c + r_r,
\]

(2.1.6)

which, for a surface continuum, have meanings analogous to those already given above.

**POSTULATES II, III and IV**

The same model used to motivate the form of the surface energy balance equation, along with the same definitions and assumptions, may be easily shown to result in the surface moment of momentum, linear momentum, and mass balance equations given by (2.2), (2.3) and (2.4). Equations (2.1) through (2.4) may of course be useful in describing surfaces whose molecules are more complex than those considered in the above analog.

**POSTULATE V**

The surface entropy balance equation, (2.5), is given as a postulate, no attempt being made to give a detailed molecular motivation. It may be taken as a 'general' balance equation (see Truesdell and Toupin [8], § 157) for something called total entropy, whose 'external' entropy supply rate per unit area is assumed equal to \( r/t \). The existence of an entropy density, \( \eta \), follows by assuming that the total entropy of every material part \( A \subset J \) is an additive set function, absolutely continuous with respect to surface mass. The surface temperature, \( T \), should be taken as some 'average' temperature of each material element composing \( J \).
If the 'intrinsic' entropy production rate, $\dot{e}$, is assumed to be non-negative then (2.5) reduces to a form of the generalized entropy production inequality proposed by Müller [9]. If, in addition, the entropy flux rate is assumed to be equal to the heat flux rate divided by temperature, i.e., $k = h/\theta$, then (2.5) reduces to the form of the Clausius-Duhem inequality used by Green, Nagdhi and Wainwright [6] for Cosserat surfaces. Here, we make neither of these assumptions.

Local Forms

MASS BALANCE

By (1.24), the mass balance equation (2.4) may be written in the form,

$$\int_A \left( \dot{\rho} + \frac{1}{2} \rho \mathbf{a} \mathbf{a} \mathbf{a} \mathbf{a} \mathbf{a} \right) \, dA = 0,$$

or, since $A$ is arbitrary the local form is given by:

$$\dot{\rho} + \frac{1}{2} \rho \mathbf{a} \mathbf{a} \mathbf{a} \mathbf{a} \mathbf{a} = 0. \tag{2.6}$$

Alternatively, (2.4) could have been integrated immediately to yield

$$\int_A \rho \, dA = \int_A \rho_R \, dA_R,$$

where subscript R refers to quantities evaluated in the fixed reference configuration. Use of (1.12) leads to the local form,

$$\rho = J \rho_R, \quad J = \left( \frac{\rho_R}{\rho} \right)^{1/2}. \tag{2.7}$$

Equations (2.6) and (2.7) are equivalent statements of mass balance. For incompressible material surfaces $J = 1$ and $\rho = \rho_R$, $\dot{a} = 0$, and $\mathbf{a} \mathbf{a} \mathbf{a} \mathbf{a} \mathbf{a} = 0$.

LINEAR MOMENTUM BALANCE

If the linear momentum balance equation (2.3) is applied to an elementary curvilinear triangle drawn on $A$, having two sides along the surface coordinate lines $x^a = \text{const}$ and the third side arbitrary, it follows by allowing the area of the triangle to shrink to zero that the boundary traction force, $t$, may be represented by,

$$t = T^a \nu_\alpha$$

or $d = T^a \nu_\alpha,$ \tag{2.8} \end{align*}

where $T^a$ is a double tensor called the stress tensor. The physical components of traction force along the lines $x^a = \text{const}$ are given by $T^a (a^{a a} g_{1/2})$. Substitution of (2.8) into (2.3), use of (1.24) and (2.6), and application of Green's Theorem (1.11) leads to the local form of the linear momentum balance equation given by:

$$\rho \mathbf{t} - \mathbf{F} = -T^a \nu_\alpha = 0 \tag{2.9}$$

MOMENT OF MOMENTUM BALANCE

Substitution of (2.8) and (2.9) into (2.2), and use of (1.24) and (1.26), leads to the following form of the moment of momentum balance equation:

$$\int_A d_A \left( \rho \ddot{d} - \mathbf{G} - S^a \nu_\alpha \right) = m,$$ \tag{2.10}

where $S^a (a^{a a} g_{1/2})$ are the physical components of the director traction force along the lines $x^a = \text{const}$. It will be assumed that $S^a$ is a double tensor. Substitution of (2.10) into the above equation and application of Green's Theorem (1.11) leads to the following local form of the moment of momentum balance equation:

$$d_A \left( \rho \ddot{d} - \mathbf{G} - S^a \nu_\alpha \right) = m,$$ \tag{2.11}

where $m = (m \cdot d) d + d_A (m \cdot d) (d \cdot d)^{-1}$.

It follows that,

$$m = d_A \ddot{g}, \quad m = (m \cdot d) d + d_A (m \cdot d) (d \cdot d)^{-1} \tag{2.12}$$

Substitution of (2.12) into (2.11), gives,

$$d_A \left( \rho \ddot{d} - \mathbf{G} + \ddot{g} - S^a \nu_\alpha \right) = 0,$$

which implies that

$$\rho \ddot{d} - \mathbf{G} + \ddot{g} - S^a \nu_\alpha = \Gamma d,$$

where $\Gamma$ is some arbitrary scalar function defined on $A$.

Alternatively, one may write,

$$\rho \ddot{d} - \mathbf{G} + \ddot{g} - S^a \nu_\alpha = 0,$$ \tag{2.13}

$$g = \ddot{g} + \Gamma d,$$ \tag{2.14}

where, by (2.11), $g$ satisfies the identity,

$$T^a A_A (d_A \ddot{d} - d_A S^a \nu_\alpha) + d_A g = 0,$$ \tag{2.15}
and \( g \) is assumed to be an intrinsic surface director force specified by a constitutive equation. Identity (2.15) then represents a symmetry condition to be satisfied by the stresses and intrinsic director force. Following these authors, we will subsequently assume that \( \Gamma \), and hence \( g \), is not arbitrary, but given by a constitutive equation.

**ENERGY BALANCE**

Substitution of (2.8), (2.9) and (2.13) into (2.1), and use of (1.24) and (2.6), leads to the following form of the energy balance equation:

\[
\int_{A} \left[ \rho \frac{\partial \dot{e}}{\partial t} - \dot{r} \cdot \mathbf{T} \mathbf{a} + \dot{d} \cdot (g + \mathbf{S} \alpha \gamma) \right] \, dA - \int_{\partial A} (\dot{d} \cdot s - h) \, ds = 0.
\]

Application to an elementary curvilinear surface triangle yields,

\[
\dot{d} \cdot s - h = (\dot{d} \cdot \mathbf{S} \alpha \gamma - \dot{q} \alpha \gamma) \nu_{\alpha},
\]

where \( \dot{q} \alpha \gamma = (\alpha \nu \gamma)^{1/2} \) are the physical components of heat flux across the lines \( x \alpha = \text{const} \). Since \( \mathbf{S} \alpha \gamma \) is assumed to be a double tensor, \( \dot{q} \alpha \gamma \) is a contravariant surface vector. Substitution of (2.16) into the above equation and use of Green’s Theorem (1.11) leads to a reduced local form of the energy balance equation given by,

\[
\rho \frac{\partial \dot{e}}{\partial t} + \dot{q} \alpha \gamma - \dot{r} \cdot \mathbf{T} \mathbf{a} - \dot{d} \cdot (g + \mathbf{S} \alpha \gamma + \dot{d} \cdot g) = 0 \tag{2.17}
\]

**ENTROPY BALANCE**

Application of (2.5) to an elementary curvilinear surface triangle gives,

\[
k = p^\alpha \nu_{\alpha}, \tag{2.18}
\]

where \( p^\alpha \) is a contravariant surface vector. The physical components of entropy flux rate across the lines \( x \alpha = \text{const} \) are given by \( p^\alpha = (p^\alpha \gamma)^{1/2} \). Substitution of (2.18) into (2.5) following the usual procedure, yields the local form of entropy balance equation given by,

\[
\rho \frac{\partial \dot{e}}{\partial t} + \dot{q} \alpha \gamma - \dot{r} \cdot \mathbf{T} \mathbf{a} - \dot{d} \cdot (g + \mathbf{S} \alpha \gamma + \dot{d} \cdot g) = 0 \tag{2.19}
\]

We select \( \rho, r, d \) and \( \theta \) as the independent variables which characterize the mechanical-thermodynamical behavior of \( \mathbf{J} \). The tensors \( \mathbf{T} \alpha \gamma, \mathbf{S} \alpha \gamma, g, e, q \alpha \gamma, \theta, r \alpha \gamma, \) appearing in the above equations are assumed to be intrinsic surface quantities whose functional dependence will be specified by constitutive equations for the particular material surface under consideration. If the constitutive equations do not involve dependence on quantities other than the independent variables \( \rho, r, d, \theta \), and their surface gradients and material derivatives, then we expect that there are well-posed boundary value problems for which equations (2.6), (2.9), (2.13) and (2.17) are sufficient to determine unique fields \( \rho, r, d \) and \( \theta \). Here, \( \mathbf{F}, \mathbf{G} \) and \( r \) are regarded as having been arbitrarily prescribed in advance. In this view the reduced entropy balance equation (2.19) must then be satisfied identically by the constitutive quantities appearing in them, for all solutions \( \rho, r, d \) and \( \theta \) which satisfy (2.6), (2.9), (2.13) and (2.17). Consequently, (2.19) will impose restrictions on the forms of the constitutive equations in much the same way that restrictions arise when the usual Clausius-Duhem inequality is used [11].

**Alternate Local Forms**

For many problems it will be convenient to work with local balance equations which have been reduced to surface component form. The mass-balance equation (2.6) is of course already in surface-component form, and in the following we obtain corresponding surface component forms for the linear momentum, moment of momentum and energy balance equations.

The scalar \( \psi \) may be identified as a surface Helmholtz free energy per unit mass and \( \partial^2// \partial \) corresponds to an 'extra' intrinsic entropy flux rate.

**Summary**

From the preceding development, the following set of local balance equations is adopted,

\[
\rho \frac{\partial \dot{e}}{\partial t} + \frac{1}{2} \rho a^{\alpha \beta} \dot{a}_{\alpha \beta} = 0, \tag{2.6}
\]

\[
\rho \dot{r} - \mathbf{F} - \mathbf{T} \alpha \gamma = 0, \tag{2.9}
\]

\[
\rho \dot{d} - (G + g) - \mathbf{S} \alpha \gamma = 0, \tag{2.13}
\]

\[
\rho \dot{e} - r + \dot{q} \alpha \gamma - \dot{r} \cdot \mathbf{T} \mathbf{a} - \dot{d} \cdot \mathbf{S} \alpha \gamma + \dot{d} \cdot g = 0, \tag{2.17}
\]

\[
\theta \dot{e} + \rho (\dot{\psi} + \dot{\theta}) - (\phi \alpha \gamma - p^\alpha \gamma \dot{\theta}_{\alpha \gamma}) = 0, \tag{2.19}
\]

where \( g, \mathbf{T} \alpha \gamma \) and \( \mathbf{S} \alpha \gamma \) satisfy the identity,

\[
\mathbf{T} \alpha \gamma \cdot r_{\alpha \gamma} - \mathbf{d}_{\alpha \gamma} \cdot \mathbf{S} \alpha \gamma + \mathbf{d} \cdot g = 0. \tag{2.15}
\]
Following the decomposition method used earlier to obtain (1.13) to (1.20), $T^\alpha$, $S^\alpha$, and $g$ may be resolved into components relative to the basis $\{r, o_\alpha, n\}$, so that we may write

\[
T^\alpha = \sigma^\alpha \beta r_\beta + \sigma^3 \alpha n, \quad S^\alpha = \pi^\alpha \beta r_\beta + \pi^3 \alpha n, \quad g = g^\beta r_\beta + g^3 n. \tag{2.21}
\]

Since $T^\alpha$, $S^\alpha$, and $g$ are absolute tensors it follows that $\sigma^\alpha \beta$, $\pi^\alpha \beta$ and $g^\beta$ are absolute contravariant surface tensors, and $\sigma^3 \alpha$, $\pi^3 \alpha$ and $g^3$ are respectively axial contravariant surface vectors and an axial scalar. Similarly, for the accelerations $\ddot{r}$ and $\ddot{u}$ and the external forces $F$ and $G$, we may write the decompositions,

\[
\ddot{r} = c^\beta r_\beta + c^3 n, \quad \ddot{u} = g^\beta r_\beta + g^3 n, \tag{2.22}
\]

Substitution of (2.21) and (2.22) into (2.9) and (2.13), use of (1.19), and taking the scalar products of the resulting expressions with $r_\gamma$ and $n$, leads to the surface component forms of the linear momentum equations given by,

\[
p_c^\gamma - F^\gamma = \sigma^\alpha \beta r_\beta - b^\gamma \sigma^3 \alpha, \tag{2.23}
\]

\[
p_c^3 - F^3 = \sigma^3 \alpha - b^\gamma \sigma^\alpha \beta \pi^\gamma, \tag{2.24}
\]

Identify (2.15), which is equivalent to the symmetry condition,

\[
P_{ij}^\gamma = P_{ji}^\gamma = T^\alpha j^\alpha i - d^\gamma i^\alpha S^\alpha + d^i d^j, \tag{3.1}
\]

may also be reduced to surface component forms. Defining $P_{\alpha \beta} = z^\gamma \alpha j^\beta P_{ji}$ and $P_{3 \beta} = n^\gamma \beta j^\gamma P_{ji}$, we have from definitions,

\[
P_{\alpha \beta} = P_{\alpha \beta}^\gamma = \sigma_{\alpha \beta} - \lambda_\gamma \pi_{\alpha \gamma} + d_\alpha d_\beta, \tag{3.2}
\]

and from $P_{3 \beta} = P_{3 \beta}$,

\[
\sigma_{3 \beta} = (\sigma_{\gamma \alpha} \lambda_{3 \gamma} \pi_{\gamma \alpha} + (e_3 d_\beta - g_3 d_3). \tag{3.3}
\]

In the special case when the director-associated stress and force vanish, i.e. $\sigma = 0$, (2.25) and (2.26) reduce to $\sigma_{\alpha \beta}^\gamma = \sigma_{\alpha \beta}$ and $\sigma_{3 \alpha} = 0$, which assert the symmetry of the surface stress tensor and the impossibility of shear stresses directed normal to the surface.

Finally, substituting (1.18) to (1.21) and (2.21), and making use of (1.22), (2.25) and (2.26), the energy balance equation (2.17) may be reduced to the surface component form,
but in the above form, not objectivity. In order that objectivity be satisfied, it follows from the arguments leading to (1.28) and (1.31) that

\[ \begin{align*}
T^{\alpha} &= r_{\beta} A^{\beta \alpha} + d_{\beta} B^{\beta \alpha}, \\
S^{\alpha} &= r_{\beta} \tilde{A}^{\beta \alpha} + d_{\beta} \tilde{B}^{\beta \alpha}, \\
g &= r_{\beta} A_{\beta}^{\alpha} + d_{\beta} \tilde{B}_{\beta}^{\alpha},
\end{align*} \]  
(3.3)

where the various A's and B's appearing in (3.3) are absolute surface tensors, each a function of the objective variables

\[ \theta, \theta^{\alpha}, a_{\alpha \beta}, \lambda_{\alpha \beta}, d_{\alpha}, \lambda_{\alpha}, \alpha_{\alpha}, \lambda_{\alpha \beta}, D, D_{\alpha \beta}, D_{\alpha}, D_{\alpha \beta}. \]  
(3.4)

The remaining constitutive response functions, namely \( \epsilon, q_{\alpha \beta}, \psi, \psi^{\alpha}, \psi^{\beta}, \) are also absolute surface scalars and vectors, each a function of the objective variables listed in (3.4). It follows from (2.21) and (3.3) that the surface components of \( T^{\alpha}, S^{\alpha} \) and \( g \) are given by

\[ \begin{align*}
o^{\beta \alpha} &= o^{\beta \gamma} r_{\gamma} \times T^{\alpha} = A^{\beta \gamma} + d_{\beta} B^{\beta \gamma}, \\
o^{\beta \alpha} &= o^{\beta \gamma} r_{\gamma} \times S^{\alpha} = \tilde{A}^{\beta \gamma} + d_{\beta} \tilde{B}^{\beta \gamma}, \\
o^{\beta \alpha} &= o^{\beta \gamma} r_{\gamma} \times g = \tilde{A}_{\beta}^{\alpha} + d_{\beta} \tilde{B}_{\beta}^{\alpha}, \\
o^{\beta \alpha} &= o^{\beta \gamma} r_{\gamma} \cdot \tilde{A}^{\beta \gamma} + d_{\beta} \tilde{B}^{\beta \gamma}, \\
o^{\beta \alpha} &= o^{\beta \gamma} r_{\gamma} \cdot \tilde{B}_{\beta}^{\alpha}.
\end{align*} \]  
(3.5)

As noted earlier, \( o^{\beta \alpha}, \pi^{\beta \alpha} \) and \( o^{\beta 3} \) are axial surface vectors and an axial scalar, their axial character indicated by the presence of \( d^{3} \) and \( \lambda^{3} \gamma \) in the above representations.

In the special case of curvilinearly aeototropic material surfaces whose constitutive response functions (3.1) do not depend on the time rates of change of \( r \) and \( d \), the list of objective variables (3.4) reduces to:

\[ \theta, \theta^{\alpha}, a_{\alpha \beta}, d_{\alpha}, \lambda_{\alpha \beta}, \lambda_{\alpha}, \alpha_{\alpha}, D, D_{\alpha \beta}. \]  
(3.6)

This list is similar to, but not the same as, that assumed at the outset by Green, Nagdhi and Wainwright [6], § 5) as their definition of an elastic Cosserat surface. In their thermo-elastic theory constitutive equations for \( \mathbf{s}, \mathbf{g}, \mathbf{T}^{\alpha} \) and \( S^{\alpha} \) follow from a surface form of the Clausius-Duhem inequality as derivatives of the surface free energy function. In the present notation, their free energy function, \( \psi \), is assumed to depend on

\[ \theta, a_{\alpha \beta}, d_{\alpha}, \lambda_{\alpha \beta}, \lambda_{\alpha}, \lambda_{3 \beta}, \]  

multiplied by the reference configuration (or initial) values of \( a_{\alpha \beta}, d_{\alpha}, \lambda_{\alpha \beta} \) and \( \lambda_{3 \beta} \). In the present theory the free energy function depends on \( d_{3} \) and \( \lambda_{3 \beta} \) (i.e., \( d_{3}, \beta \)) only through \( A_{\alpha \beta}, D \) and \( D_{\alpha \beta} \) (cf. (3.6)). This is a result of the form of objectivity assumed at the outset of the present work, which does not distinguish between proper and improper orthogonal reference frames. It should also be noted that only the heat flux, \( q^{\alpha} \), was assumed by Green, Nagdhi and Wainwright [6] to depend on the temperature gradient \( \theta^{\alpha} \).

For subsequent use, we now define hemitropic and isotropic surface materials and derive corresponding representation formulae for scalar-valued and vector-valued functions of surface vectors. The definitions are analogous to those used for bulk materials and proceed from material symmetries with respect to a three-dimensional Cartesian coordinate frame which is locally tangent to the material surface in its reference configuration, \( J_{R} \). A surface material will be called hemitropic (isotropic) if at every point of \( J_{R} \) the constitutive functions for the material are "objective" with respect to rigid rotations (rigid rotations and reflections) of the tangent-plane coordinates about the local surface normal.

At a given material point on \( J_{R} \) we choose a right-handed rectangular Cartesian coordinate system with unit vectors \( \{ a_{\alpha}, a^{3} \} \), where the \( a_{\alpha}, \alpha = 1, 2, \) are tangent to \( J_{R} \) and \( a^{3} \) is normal to \( J_{R} \). Let \( \Phi \) be a scalar-valued function defined on \( J \) which is a function of the surface vectors \( \mathbf{u} \) and \( v^{p} \), \( P = 1, 2 \ldots N \), i.e.

\[ \Phi = \mathbf{u} = \mathbf{v}^{p}. \]  
(3.7)

At the given material point we have the decompositions:

\[ u = u_{\alpha} a^{\alpha} + v^{p} = v^{p} a^{\alpha} a^{3}. \]  
If \( \Phi \) is a hemitropic function of \( \mathbf{u} \) and \( v^{p} \) at each material point, it follows from the basic representation theorem for hemitropic scalar functions of vectors (see e.g., Truesdell and Noll [3], § 11) that \( \Phi \) can depend only on the inner products and determinant products of \( u \) and \( v^{p} \), i.e.

\[ \Phi = \Phi(\mathbf{u} \cdot \mathbf{v}^{p}, \mathbf{u} \times \mathbf{v}^{p}), \]  
(3.7)\(_{1}\)

If \( \Phi \) is an isotropic function, it depends only on the inner products, i.e.

\[ \Phi = \Phi(\mathbf{u} \cdot \mathbf{v}^{p}, \mathbf{u}, \mathbf{v}^{p}), \]  
(3.7)\(_{2}\)

Now consider a surface vector \( \phi \) which is a function of the surface vectors \( v^{p} \), \( P = 1, 2 \ldots N \). For an arbitrary surface vector \( u \), we may define a scalar \( \Phi \) by:

\[ \Phi = \phi \cdot \mathbf{u} + \Phi(\mathbf{v}^{p}) \cdot \mathbf{u} = \Phi(\mathbf{u}, \mathbf{v}^{p}). \]  
(3.7)\(_{2}\)
where, \( A_p = A_p (v^K \cdot v^P, \xi_{\alpha\beta} v^\alpha v^\beta) \)
\[ B_p = B_p (v^K \cdot v^P, \xi_{\alpha\beta} v^\alpha v^\beta) \]

But, \( u \) is an arbitrary vector, so that
\[ \phi_\alpha = A_p v^\alpha + B_p \xi_{\alpha\beta} v^\beta \]  
(3.8) \]

For isotropic vector functions, \( \phi_\alpha \) reduces to the form:
\[ \phi_\alpha = A_p v^\alpha \]  
(3.8) \]

where, \( A_p = A_p (v^K \cdot v^P) \), \( K, P = 1, 2 \ldots N \).

Unconstrained Materials

In this section we determine the restrictions imposed on the various constitutive response functions (3.1) by the reduced local form of the entropy balance equation (2.19). No constraints, such as incompressibility, i.e. \( \dot{\varepsilon} = \dot{\varepsilon} = 0 \), are imposed initially on the material surface. The method used is an extension of one proposed by Coleman and Noll [11] in a mechanical-thermodynamical theory of simple materials. First, \( \dot{\psi} \) and \( \phi^\alpha_\alpha \) must be evaluated for substitution into (2.19). For reasons of simplicity, at the outset we will work with the constitutive response functions in form (3.2) and constitutive variables in form (3.4). Then, using Cartesian tensor notation, by the chain rule,
\[ \dot{\psi} = \frac{\partial \psi}{\partial \theta} \theta + \frac{\partial \psi}{\partial \phi_\alpha} \phi_\alpha \alpha + \frac{\partial \psi}{\partial \phi^i_\alpha} \phi^i_\alpha \alpha + \frac{\partial \psi}{\partial \phi^i_\alpha} \phi^i_\alpha \alpha \]
and, making use of (1.12)\(_1\),
\[ \phi^\alpha_\alpha = \left[ \frac{\partial \phi^\alpha_\alpha}{\partial \phi^\beta_\alpha} \theta^\beta + \frac{\partial \phi^\alpha_\alpha}{\partial \phi^\beta_\alpha} \phi^\beta_\alpha + \frac{\partial \phi^\alpha_\alpha}{\partial \phi^i_\alpha} \phi^i_\alpha \right] \frac{1}{2} \]
where, \( \phi^\alpha_\alpha = a^{1/2} \phi^\alpha_\alpha \). Substitution of these expressions into (2.19) and rearrangement of terms results in:
\[ \psi = e \theta + \left( \phi^\alpha_\alpha - \frac{\partial \phi^\alpha_\alpha}{\partial \theta} \right) \theta \alpha + \left( \frac{\partial \psi}{\partial \phi^i_\alpha} \phi^i_\alpha - \frac{1}{2} \phi^i_\alpha \right) \frac{1}{2} \alpha \]
\[ - \frac{\partial \phi^\alpha_\alpha}{\partial \phi^i_\alpha} \phi^i_\alpha \alpha + \phi \left( \frac{\partial \phi^\alpha_\alpha}{\partial \phi^i_\alpha} \phi^i_\alpha + \frac{1}{2} \phi^i_\alpha \right) \frac{1}{2} \alpha \]
\[ + \rho \left( \frac{\partial \psi}{\partial \theta} + \eta \right) \theta + \frac{\partial \psi}{\partial \phi^i_\alpha} \phi^i_\alpha \alpha + \frac{\partial \psi}{\partial \phi^i_\alpha} \phi^i_\alpha \alpha \]
which has the solution
\[ \chi^\alpha = e \alpha^\beta A_{\alpha} \xi_{\alpha\beta} + A^\alpha \]
where \( A^\alpha \) and \( \alpha^\alpha \) are independent of \( \xi_{\alpha\beta} \) and \( e \alpha^\beta = e^{1/2} \xi_{\alpha\beta} \). In general, \( A^\alpha \) and \( \alpha^\alpha \) can depend at most on only \( \theta, \phi^i_\alpha \) and \( \dot{\phi}^i_\alpha \). It follows that the most general expression for \( \phi^\alpha_\alpha \) is given by:
\[ \phi^\alpha_\alpha = e^{1/2} \left[ \xi_{\alpha\beta} A_{\alpha} \theta, + A_{\alpha} \xi_{\alpha\beta} + A_{\alpha} \xi_{\alpha\beta} + A_{\alpha} \xi_{\alpha\beta} + A_{\alpha} \xi_{\alpha\beta} \right] \]
where \( A_{\alpha} \), \( A_{\alpha} \) and \( \xi_{\alpha\beta} (K = 2, 3, 4, 5) \) are functions of \( \theta, \phi^i_\alpha \) and \( \dot{\phi}^i_\alpha \). Since \( \phi^\alpha_\alpha = e^{1/2} \phi^\alpha_\alpha \) is assumed to be an objective function, \( \phi^\alpha_\alpha \) can depend on the constitutive variables only through the objective list given by (3.4) and defined in (1.30). It then follows that \( \phi^\alpha_\alpha \) is given most generally by:
\[ \phi^a = \sqrt{\alpha} A^a + \alpha^\beta \left[ A_1 \theta_\beta + A_2 \dot{d}_\beta + A_3 \ddot{d}_\beta + A_4 D_{,\beta} + A_5 \dot{D}_{,\beta} \right] \]

where,
\[ A^a = A^a(\theta, D, \dot{D}) \] and \( A_\kappa = A_\kappa(\theta, D, \dot{D}), \kappa = 1, 2, 3, 4, 5. \)

The entropy balance equation \( (3.9) \) finally reduces to the identity:
\[ O = e \theta + \left( \frac{\partial \phi^a}{\partial \theta} - \theta \frac{\partial x}{\partial \theta} - \frac{\partial \theta}{\partial \theta} \right) z^i_{,\theta} + \left( \rho \frac{\partial \psi}{\partial \alpha} - \rho \frac{\partial \phi^a}{\partial \alpha} - \rho \frac{\partial \theta}{\partial \alpha} \right) \dot{d}^i_{,\alpha} \]

For hemitropic surface materials it follows from representation \( (3.8) \) that \( \theta^a = 0 \) in expression \( (3.14) \) for \( \phi^a \). In the case of isotropic materials, to which later attention will be restricted, it follows from \( (3.8) \) that \( \theta^a \equiv 0 \) so that by \( (2.20) \):
\[ \rho^a = \rho^a(\theta) \]

and \( (3.15) \) reduces to:
\[ O = e \theta + \left( \frac{\partial \phi^a}{\partial \theta} - \theta \frac{\partial x}{\partial \theta} - \frac{\partial \theta}{\partial \theta} \right) z^i_{,\theta} + \left( \rho \frac{\partial \psi}{\partial \alpha} - \rho \frac{\partial \phi^a}{\partial \alpha} - \rho \frac{\partial \theta}{\partial \alpha} \right) \dot{d}^i_{,\alpha} \]

Expression \( (3.16) \) asserts that for isotropic materials the entropy flux rate, \( \rho^a \), is equal to the heat flux rate, \( \phi^a \), divided by the temperature, \( \theta \). This equality is usually assumed at the outset of most theories, rather than being proved.

**Incompressible Materials**

If the surface density, \( \rho \), and the magnitude of the director vector, \( |d^i| \), for each material point of a Cosserat surface have the same values for all time the material is said to be incompressible. Mathematically, it follows from \( (2.6), (2.7) \) and definitions that the incompressibility condition is given by
\[ \dot{a} = 0 \quad \text{or} \quad a^\beta \dot{z}^i_{,\beta} z^i_{,\beta} = 0 \]
\[ \dot{D} = 0 \quad \text{or} \quad d^i \dot{d}^i = 0 \]

Differentiation of \( (3.16) \) with respect to \( x^a \) yields
\[ d^i_{,\alpha} z^i_{,\alpha} \dot{d}^i_{,\alpha} = 0 \]

Now, expressions \( (3.18) \) and \( (3.19) \) impose constraints on the values which may be taken by the constitutive variables listed in \( (3.2) \). Thus, in order for the entropy balance equation \( (3.9) \) to be satisfied identically it is necessary to multiply the constraint equations \( (3.18)_1, (3.18)_2 \) and \( (3.19) \) by Lagrange multipliers \( \lambda, \dot{P} \) and \( P^a \) and add the resulting expressions to the entropy balance equation.
If attention is restricted to unconstrained, isotropic elastic materials, expressions (3.12), (3.16) and (3.17) hold. Since \( t_{\alpha\beta} \) and \( \dot{d}_\alpha \) may be arbitrarily varied in the entropy balance equation, (3.17), it follows that:

\[
T^{\alpha\beta} = \rho \left( \frac{\partial \psi}{\partial \dot{\alpha}} - \frac{\partial \psi}{\partial d_\alpha} \right) \quad \text{and} \quad O = \phi \theta + \frac{\partial \alpha}{\partial \theta} \theta; \alpha \cdot \frac{\partial ^{\alpha}}{\partial \theta^\alpha} \cdot \frac{\partial ^{\beta}}{\partial \theta^\beta}.
\]

For isothermal processes, \( \theta_{\alpha\beta} = 0 \), so that by (3.23) the intrinsic entropy production rate \( e \) is identically zero. Now, the surface components of \( T^{\alpha\beta}, s^\alpha, \) and \( g^\beta \) relative to the basis \( \{ t_{\alpha\beta}, n \} \) are given by equations (3.5). Substitution of (3.22) into (3.5) and chain rule differentiation of \( \psi \), where \( \psi \) is given by (3.12) in terms of objective variables, leads to the following constitutive relations for isotropic, unconstrained elastic Cosserat surfaces:

\[
\begin{align*}
\sigma_{\beta\alpha} &= \rho \left[ 2 \frac{\partial \psi}{\partial a_{\alpha\beta}} d^2\alpha + \frac{\partial \psi}{\partial d_\alpha} \lambda^\beta \gamma \right], \\
\sigma_3 \alpha &= \rho \left[ 2 \frac{\partial \psi}{\partial a_\alpha} d^2\alpha + \frac{\partial \psi}{\partial d_\alpha} \lambda^\beta \gamma \right], \\
\pi_{\alpha\beta} &= \rho \left[ \frac{\partial \psi}{\partial \alpha_{\alpha\beta}} + \frac{\partial \psi}{\partial \alpha_\alpha} \lambda^\beta \gamma + \frac{\partial \psi}{\partial \alpha_\beta} - \frac{\partial \psi}{\partial \alpha_\gamma} d^2\alpha \right], \\
\pi_3 \alpha &= \rho \left[ 2 \frac{\partial \psi}{\partial \alpha_\alpha} d^2\alpha + \frac{\partial \psi}{\partial \alpha_\beta} \lambda^\beta \gamma + \frac{\partial \psi}{\partial \alpha_\gamma} d^2\alpha \right], \\
g^{\alpha \beta} &= \rho \left[ \frac{\partial \psi}{\partial d_\alpha} d^2\alpha + \frac{\partial \psi}{\partial d_\beta} d^2\beta \right], \\
g_3 &= \rho \left[ \frac{\partial \psi}{\partial d_\alpha} d^2\alpha + \frac{\partial \psi}{\partial d_\beta} d^2\beta \right].
\end{align*}
\]

These expressions are consistent with the general 'objective' expressions listed on the right-hand sides of equations (3.5).

In the case of incompressible, isotropic elastic materials, expressions (3.19), (2) and (3.21) hold. Repeating the same types of arguments which led to (3.24) it follows that, for incompressible, isotropic elastic Cosserat surfaces:

\[
\begin{align*}
\sigma_{\beta\alpha} &= P \alpha^2 \gamma \delta^\beta \gamma + \rho \left[ 2 \frac{\partial \psi}{\partial a_{\alpha\beta}} d^2\alpha + \frac{\partial \psi}{\partial d_\alpha} \lambda^\beta \gamma \right], \\
\sigma_3 \alpha &= \rho \left[ \frac{\partial \psi}{\partial a_\alpha} d^2\alpha + \frac{\partial \psi}{\partial d_\alpha} \lambda^\beta \gamma \right], \\
\pi_{\alpha\beta} &= \rho \left[ \frac{\partial \psi}{\partial \alpha_{\alpha\beta}} + \frac{\partial \psi}{\partial \alpha_\alpha} \lambda^\beta \gamma + \frac{\partial \psi}{\partial \alpha_\beta} - \frac{\partial \psi}{\partial \alpha_\gamma} d^2\alpha \right], \\
\pi_3 \alpha &= \rho \left[ 2 \frac{\partial \psi}{\partial \alpha_\alpha} d^2\alpha + \frac{\partial \psi}{\partial \alpha_\beta} \lambda^\beta \gamma + \frac{\partial \psi}{\partial \alpha_\gamma} d^2\alpha \right], \\
g^{\alpha \beta} &= \rho \left[ \frac{\partial \psi}{\partial d_\alpha} d^2\alpha + \frac{\partial \psi}{\partial d_\beta} d^2\beta \right], \\
g_3 &= \rho \left[ \frac{\partial \psi}{\partial d_\alpha} d^2\alpha + \frac{\partial \psi}{\partial d_\beta} d^2\beta \right] + \frac{\partial \psi}{\partial d_\alpha} \lambda^\beta \gamma.
\end{align*}
\]

For isothermal processes, \( \theta_{\alpha\beta} = 0 \), so that by (3.23) the intrinsic entropy production rate \( e \) is identically zero. Now, the surface components of \( T^{\alpha\beta}, s^\alpha, \) and \( g^\beta \) relative to the basis \( \{ t_{\alpha\beta}, n \} \) are given by equations (3.5). Substitution of (3.22) into (3.5) and chain rule differentiation of \( \psi \), where \( \psi \) is given by (3.12) in terms of objective variables, leads to the following constitutive relations for isotropic, unconstrained elastic Cosserat surfaces:

\[
\begin{align*}
\sigma_{\beta\alpha} &= P \alpha^2 \gamma \delta^\beta \gamma + \rho \left[ 2 \frac{\partial \psi}{\partial a_{\alpha\beta}} d^2\alpha + \frac{\partial \psi}{\partial d_\alpha} \lambda^\beta \gamma \right], \\
\sigma_3 \alpha &= \rho \left[ \frac{\partial \psi}{\partial a_\alpha} d^2\alpha + \frac{\partial \psi}{\partial d_\alpha} \lambda^\beta \gamma \right], \\
\pi_{\alpha\beta} &= \rho \left[ \frac{\partial \psi}{\partial \alpha_{\alpha\beta}} + \frac{\partial \psi}{\partial \alpha_\alpha} \lambda^\beta \gamma + \frac{\partial \psi}{\partial \alpha_\beta} - \frac{\partial \psi}{\partial \alpha_\gamma} d^2\alpha \right], \\
\pi_3 \alpha &= \rho \left[ 2 \frac{\partial \psi}{\partial \alpha_\alpha} d^2\alpha + \frac{\partial \psi}{\partial \alpha_\beta} \lambda^\beta \gamma + \frac{\partial \psi}{\partial \alpha_\gamma} d^2\alpha \right], \\
g^{\alpha \beta} &= \rho \left[ \frac{\partial \psi}{\partial d_\alpha} d^2\alpha + \frac{\partial \psi}{\partial d_\beta} d^2\beta \right], \\
g_3 &= \rho \left[ \frac{\partial \psi}{\partial d_\alpha} d^2\alpha + \frac{\partial \psi}{\partial d_\beta} d^2\beta \right] + \frac{\partial \psi}{\partial d_\alpha} \lambda^\beta \gamma.
\end{align*}
\]
from the entropy balance equation, (3.17), that in an equilibrium state, the intrinsic entropy production rate, \( e \), is identically zero. Successive differentiations of (3.17) with respect to \( \theta, \alpha \), \( \psi \), and \( \phi \) indicate that equations (3.22) for \( T^\alpha, S^\alpha \) and \( \phi^\alpha \) for an unconstrained, isotropic elastic surface also hold for an unconstrained, isotropic fluid surface when in a state of static, isothermal equilibrium. In this case the surface component expressions given by (3.24) also hold for unconstrained, isotropic fluids, except that \( \partial \psi / \partial \alpha \beta \) of (3.24) is replaced by
\[
-\rho \ a^{3/2} (\partial \psi / \partial \rho) \delta_{\alpha \beta},
\]
since \( \psi \) depends on \( \alpha \beta \) through \( \rho \) for fluid surfaces. Repeating the same arguments, it follows that incompressible, isotropic, fluid surfaces in a state of static, isothermal equilibrium will satisfy expressions (3.25) for an incompressible, isotropic elastic surface if \( \partial \psi / \partial \alpha \beta \) in (3.25) is replaced by
\[
-\rho \ a^{3/2} (\partial \psi / \partial \rho) \delta_{\alpha \beta}. \tag{3.29}
\]
In the special case when the theory is restricted to isotropic fluid surfaces for which director vectors are not specified, then \( \psi = \psi (\theta, \rho) \) which implies that \( \sigma^{\alpha} = \pi^{\alpha \beta} = \pi^{\alpha 3} = g^{\alpha} = g^{3} = 0 \), so that for unconstrained and incompressible surfaces, respectively,
\[
\sigma^{\alpha} = -\rho \ a^{3/2} \frac{\partial \psi}{\partial \rho} \delta^{\alpha} + \sigma^{\alpha} (\theta, \rho, \alpha \beta),
\]
or
\[
\sigma^{\alpha} = P \varepsilon^{\alpha} - \rho \ a^{3/2} \frac{\partial \psi}{\partial \rho} \delta^{\alpha} + \sigma^{\alpha} (\theta, \rho, \alpha \beta). \tag{3.29}
\]
The leading terms in the expressions are those occurring in static, isothermal processes and the \( \sigma^{\alpha} \) and \( \sigma^{\alpha} \) represent the additional functions required for general processes. Equations (3.29) are appropriate to conventional isotropic fluid surfaces of the type investigated by Scriven [13].

Acknowledgements

An early version of this paper benefitted from comments of Prof. J.L. Ericksen of Johns Hopkins University. The author also thanks Dr. M.S. Chang of DTNSRDC for her critical review of the paper. The work reported here was supported by the Center's Independent Research Program.

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