CHARACTERIZING THE EVOLVING INTERNAL LENGTH SCALE IN STRAIN LOCALIZATION FOR COSSERAT MEDIA

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SUMMARY

The mechanical behaviour of granular materials under high plastic deformation is quite complex due to the fact that the granular assembly consists of discrete or discontinuous non-uniform particles. Each of the particles has its own mechanical behaviour, its micro-properties thus the soil mass behaviour is complex and in some cases the classical continuum theory fails to describe the real behaviour because granular materials behave as a continuum until failure where deformations begin to localize into a small but finite shear zone, which is called shear band.

In order to deal with the strain non-uniformities and strain softening a proper solution is to use additional degrees of freedom or high order gradient terms. Cosserat theory is one of the best classical micro-polar theories: it is quite realistic and it has a strong physical background due to the fact that is able to separate the micro-rotation of a material point from the overall rotation of the continuum. The micro-polar continuum is considered as a continuous group of particles behaving like rigid bodies. The theory combines two kinds of deformations at two different levels, micro-rotation at the particle level and macro-deformation at the structural level. If the particle can be considered as a Cosserat point we can include in the model internal length scale which is a parameter that characterizes the particle micro-properties.

A widely used length scale parameter is the mean particle size which links the micro-curvature of rotation to the couple stress. Parameters such as shape, surface roughness, and gradation of particles influence a lot strength and deformation properties of granular materials. The shape indices, $I_R$ (roundness index) and $I_{ sph}$ (sphericity index), have been used to account for the shape non-uniformity. The surface roughness and shape indices of granular materials, to some extent, are difficult to quantify. Visual comparison charts consisting of sets of grain images with known roundness are often used to make rapid visual estimation of grain roundness. Based on these parameters the initial length scales are calculated.

A cubic 3D specimen had been selected for the numerical simulations. The specimen consisting of dry dense sand (one phase material). The finite element mesh consists of 20-noded brick elements. A finite element mesh of 15x30x1 elements is taken in consideration.

The simulation consists in a biaxial test. First an isotropic loading is applied in the specimen and afterwards the deviatoric load is applied in the top of the specimen. The constitutive soil model that had been used in the numerical simulation is the Desai yield surface.

Three different type of particles had been chosen (A, B and C) and the length scale parametric study had been based in the variation of the main grain size $d_{50}$. 
Some numerical simulations were carried out to investigate the Cosserat shear modulus effect in the shear band formation. The grain parameters will be kept constant and the shear modulus will evolve from a small value which means that the Cosserat rotations will be not considerable till to the value $c=0.5G$ which is recommended in the literature.

Strong localized material dilatancy due to grain rearrangement and grain rotation are the dominant features of shear banding. It is questionable if a fixed value of the material length-scale is able to fully describe this phenomenon and to give results comparable with the experimental results. For these reasons different mathematical expressions were used to have an evolving length scale during the numerical simulation.

Some of the tests were performed with a constant length scale and afterwards the results were compared with the corresponding results obtained by an evolution formulation.

Some tests were performed with three different meshes in order to have some insights about the mesh dependency of the problem.
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1.1 PROBLEM DESCRIPTION.

The mechanical behavior of granular materials under high plastic deformation is quite complex due to the fact that the granular assembly consists of discrete or discontinuous non-uniform particles. Each of the material components has its own mechanical behavior its micro-properties thus the mixture behavior is complex and in some cases the classical continuum theory is not able to describe the real behavior such as micro-rotation in granular materials. During the hardening regime, granular materials behave as a continuum until the failure or the instability point where deformations begin to localize into a small but finite shear zone, which is termed as shear band. Standard soil models used in numerical simulations show a severe mesh dependency when they are used in problems involving strain softening. The non-uniformity of the particles in the shape and surface roughness at the micro or even at the nano level affects the material behavior. The shear band properties are dependent on some micro-length scale thus it is important to carefully incorporate this parameter. Several material parameters can be considered to express this internal length scales for granular materials such as the particle size, surface roughness, shape indices, etc.

For different materials analytical predictions have been done regarding the characteristic length scales. For fibrous composite materials the characteristic length is taken...
approximately as the spacing between fibbers (Hlavacek, 1975) for laminates it is close to the laminate thickness (Herrmann and Achenbach, 1967) and for cellular solids it can be taken as average cell size (Adomeit, 1967).
For materials with elliptic or spherical particles the characteristic length scales are predicted to be zero (Hlavacek, 1976; Berglund, 1982).

1.2 COSSERAT THEORY

Granular materials experience high deformations at failure (rotational and translational). The classical strain tensor (i.e. Cauchy-Green) can not describe the real kinematics such as micro-rotation in granular materials and other alternative tensors need to be introduced because the rotational kinematics are significant as the translational kinematics.

In order to deal with the strain non-uniformities and strain softening a proper solution is to use additional degrees of freedom or high order gradient terms.

A proper solution to those problems is the Cosserat theory which can separates the grain rotation from its translation adding three additional degrees of freedom to any point in the 3D continuum.

Thus in the Cosserat continuum apart of the linear inertia term also the spin inertia term is taken in account which is determined by the size and the shape of the micro elements.

The Cosserat continuum (or micropolar continuum) enhances the kinematic description of deformation by an additional field of local rotations, which can depend on the rotations corresponding to the displacement field, i.e. on the skew-symmetric part of the displacement gradient for the small displacement theory, or on the rotational part of the polar decomposition in the large-displacement theory.

Micro-polar theory originally had been used for metals thus in geomechanics until now the theory had been used mainly for sandy soils because it assumes that the behavior of sand aggregate is very similar to that of poly-crystals because the individual grain packing within the sand could be considered in a rough approximation to behave like randomly oriented crystals (Voyiadjis et al., 1995, 1992).

The main difference between sand particles and randomly ordered crystals consist in the pressure dependency of the sands, and the amount of slip affecting each of these packed grains, depends on the mean stress in contrast to the polycrystalline aggregate. Sand may experience dilation under shear which does not occur in polycrystalline aggregates. Another big difference is the fact that soils exhibit nonlinear inelastic stress strain behavior even for very small strains.

Yielding in granular media is a result of the nonlinear force-deformation behavior at the inter-particle contacts which causes nonlinear inelastic stress-strain behavior at very small strain levels. Thus cohesionless aggregates, unlike metals, do not have a clear “linear elastic region” defined by an initial yield surface.
CHAPTER II

CONTINUUM MECHANICS DEFINITION

2.1 CLASSICAL CONTINUUM MECHANICS FORMULATION

2.1.1 STRAIN

The deformation of the body consists in it being moved from its original reference configuration to a new deformed configuration (Fig. 2.1). All deformations can be de-composed in two different types; rigid motions and straining. Rigid motions are deformations for which the shape of the body does not change. Rigid motions consist in; rigid translation and rigid rotation (Fig. 2.2b, Fig. 2.2c). During rigid translation the body changes his location in space without changing its attitude in relation to the coordinate directions and during rigid rotation the attitude of the body changes, but not its position. The second type of deformation involves all the changes of shape of the body. It may be stretched, twisted, inflated or compressed. These types of deformations result in straining (Davis and Selvadurai, 2002). The deformation in every point in the continuum is given by the displacement vector which expresses the position of a point in the new deformed configuration as a function of its reference configuration.

\[ u = u(x, t) \]
Where \( x \) is the position of a point within the body and \( t \) is time. In a Cartesian coordinate system \( (x, y, z) \) the displacement vector \( u \) has components in \( x \), \( y \), and \( z \) directions \( u_x \), \( u_y \) and \( u_z \). Each component is a function of both position \((x, y, z)\) and time \( t \).

If the displacement vector field is known, then the deformation is also known. The displacement gradient matrix, \( \nabla u \) contains the spatial derivatives of the components of the displacement vector.

\[
\nabla u = \begin{bmatrix}
\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\
\frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\
\frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z}
\end{bmatrix}
\]

One popular deformation measure which is easy to compute is the right Cauchy-Green (Lagrangian) strain tensor. This tensor is symmetric for small strains and linear classical expression by neglecting the quadratic terms is used. The strain tensor \( \varepsilon \) is defined as:

\[
\varepsilon = \frac{1}{2} [\nabla u + (\nabla u)^T]
\]

Thus, the strain tensor is given:

\[
\varepsilon = \begin{bmatrix}
\varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\
\varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\
\varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz}
\end{bmatrix}
\]

\( \varepsilon_{xy} = \varepsilon_{yx} \), \( \varepsilon_{xz} = \varepsilon_{zx} \), \( \varepsilon_{yz} = \varepsilon_{zy} \)

Where:

\[ \varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \]

Fig. 2.2 - Different micro-mechanisms which gives the same macro-deformation.
CONTINUUM MECHANIC DEFINITIONS

\[ \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right), \quad \varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right), \quad \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \]

Where:

\[ \gamma_{xy} = \frac{\partial u_x}{\partial y}, \quad \gamma_{yx} = \frac{\partial u_y}{\partial x}, \quad \gamma_{xz} = \frac{\partial u_x}{\partial z}, \quad \gamma_{zx} = \frac{\partial u_z}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z}, \quad \gamma_{zy} = \frac{\partial u_z}{\partial y} \]

are the engineering shear strains.

### 2.1.2 STRESSES.

There are two types of forces in continuum: **body forces** and **contact forces**.

**Body forces** are caused by outside influences such as gravity or magnetism which are associated with the volume or mass of the body and they are fully specified at the outset of any problem.

**Contact forces** are associated with surfaces, either surfaces inside the body or segments of the exterior bounding surface of the body. Contact forces result from the action of the body on itself, such as an applied load on the upper surface of a beam (Davis and Selvadurai, 2002).

If we consider a continuum material volume \( \mathcal{V} \) enclosed by boundary surface \( \mathcal{A} \). The force vector \( \Delta f \), acting on a boundary area \( \Delta A \) with unit normal vector \( n \), can be expressed per unit area, giving the traction vector

\[ t^{(n)} = \frac{\Delta f}{\Delta A} \]  \hspace{1cm} (9)

The notion of traction can also be applied to a plane through the continuum material. If we choose a different surface element at the same point in the body, we will generally find a different traction vector. The orientation of the surface element is important because there are infinite possible orientations for a surface at a given point, thus we can define infinite traction vectors. The French mathematician A. Cauchy (1823) had solved the problem related to determination of the traction vector for a given surface.

\[ t^{(n)} = \sigma_{ij}n \]  \hspace{1cm} (10)

Where \( n \) is a unit vector normal to the surface and \( \sigma_{ij} \) is the Cauchy symmetric stress tensor. The Cauchy stress tensor is written:

\[ [\sigma] = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad \sigma_{xy} = \sigma_{yx}, \quad \sigma_{xz} = \sigma_{zx}, \quad \sigma_{yz} = \sigma_{zy} \]  \hspace{1cm} (11)

The three diagonal components represent the normal (compression or extension) stresses and the non-diagonal elements sharing (tangential) stress components.

### 2.2 MICRO-POLAR CONTINUUM FORMULATION.

The Cosserat theory (Cosserat, 1909) relates the local rotation of points and the translation assumed in classical continuum mechanics and the couple stress (a torque per unit area) with the force stress (force per unit area) which is referred simply as 'stress' in classical continuum theory due to the fact that is the only type of stress.
The idea of a couple stresses was introduced first in the pioneering work of Voigt (1887, 1894) during the formative period of the asymmetric theory of elasticity and was developed further by the Cosserat brothers (E. Cosserat and F. Cosserat 1909). They separated the displacement gradients from the rotations which is possibly the best way to deal with individual particles (Voyiadjis and Song 2006). The linearized or the constrained Cosserat theory developed by many researchers uses one independent displacement field from which the particle rotation can be easily computed. Fifteen years latter researchers started to show interest again in the work of Cosserat brothers and developing further the Cosserat theory by means of the capabilities of modern continuum mechanics (Ericksen and Truesdell, 1958; Mindlin, 1965; Eringen, 1868; Nowacki, 1970). Kunin (1982, 1983) presented in his work the relation between generalized continuum analysis and material defects, dislocations and other inhomogeneities. Eringen (1968) renamed Cosserat theory as the micro-polar theory and incorporated the micro-inertia (Voyiadjis and Song, 2006).

The polar decomposition of a positive non-symmetric matrix consists in a unique product.

\[ F_{ij} = R_{ik} U_{kj} \]

Where:
- \( R_{ik} \): orthogonal matrix (\( R^* R = R^T \cdot R = I \))
- \( V_{ik} U_{kj} \): positive definite symmetric matrix.

Where \( F \) is the deformation gradient tensor which is defined as:

\[ F = \frac{\partial X_i}{\partial x_j} = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} \]

For small strains the strain tensor (Molenkamp 2008) can be approximated by Taylor series:

\[ \varepsilon \approx U - I \] (14a)

The rotation tensor is also approximated:

\[ R \approx I + \Omega \] (14b)

Where \( \Omega \) is the skew-symmetric small rotation tensor.

If we express the deformation gradient tensor by using both approximations:

\[ F = R^* U \approx (I + \Omega)^*(I + \varepsilon) \approx I + \Omega + \varepsilon + \Omega^* \varepsilon \approx I + \Omega + \varepsilon \] (15)

The quadratic term is small compared to the linear terms \( \Omega \) and \( \varepsilon \) and therefore can be neglected.

\[ F = \frac{\partial x_i}{\partial x_j} = I + \frac{\partial u_i}{\partial x_j} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} = I + u \otimes \nabla \]

(16)

The small displacement gradient tensor is:

\[ A = F - I = u \otimes \nabla \] (17)

The small displacement gradient tensor can be split into the symmetric small strain tensor \( \varepsilon \) and a skew-symmetric small rotation tensor \( \Omega \).
The micro-polar continuum is a continuous assemblage of particles, which are considered as rigid bodies (Tejchman 2008). The micro-polar continuum combines two kinds of deformations at two different levels (Fig. 2.3):

- Micro-rotation at the particle level
- Macro-deformation at the structural level.

The micro-polar model is able to describe quite reasonably the behavior of granular material because it takes into account rotations and couple stresses, which have significant value during shearing tests, but they are negligible during homogeneous deformation without shear zones (Uesugi 1987, Tejchman 1989, Oda 1993, Voyiadjis and Song 2006).

Another advantage of the micro-polar model is the characteristic length scale, which is parameter related to the mean grain diameter and it is a realistic boundary condition at the interface of grains. The additional rotational degree of freedom of a Cosserat continuum originates by the mathematical homogenization of a discrete system of spherical grains with contact forces and moments (Pasternak and Mühlhaus 2001).

The particle ensemble has the character of a micro-polar Cosserat continuum and the couple stresses are generated from the eccentricities of normal contact forces thus Cosserat model is suitable for shear dominated problems but not for tension dominated applications (Ehlers et al. 2003).

The micro-polar (Cosserat) continuum in difference from the classical continuum introduces the additional independent rotations (Cosserat and Cosserat 1909, Mühlhaus...
1987, 1990). To each material point in 3D continuum are assigned 6 degrees of freedom
three translations and three rotations.

\[ u = \begin{bmatrix} u_x & u_y & u_z & \omega_x & \omega_y & \omega_z \end{bmatrix} \]  

(19)

Where:

\( \omega_x, \omega_y, \) and \( \omega_z \) are the micro-rotations in \( x, y \) and \( z \) direction respectively.

The stress tensor is similar to the one in classical continuum but it is not necessarily symmetric:

\[ \sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \]

(20)

The additional terms introduced by the micro-polar theory are the coupled momentum components:

\[ m = \begin{bmatrix} m_{xx} & m_{xy} & m_{xz} \\ m_{yx} & m_{yy} & m_{yz} \\ m_{zx} & m_{zy} & m_{zz} \end{bmatrix} \]

(21)

Where \( m_{ij} \) are the coupled stress components (\( m_{ii} \) are the torsion one and \( m_{ij} \) are the bending moments).

The Micro-polar (Cosserat) theory requires two independent kinematical fields; the first is the Cosserat objective strain tensor and the second is the curvature or the rotation gradient vector.

In difference from the classical continuum theory the shear strains are defined as the relative deformation between the macro-polar displacement gradients and the micro-rotation thus (Fig.2.7a):

\[ \varepsilon = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \]

(22)

\[ \varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{xy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{xz} = \frac{\partial u_z}{\partial z}, \quad \varepsilon_{yx} = \frac{\partial u_x}{\partial y} - \omega_z, \quad \varepsilon_{zx} = \frac{\partial u_x}{\partial z} + \omega_y \]

(22a)

Where:

\begin{align*}
\varepsilon_{xy} & = \frac{\partial u_y}{\partial x} - \omega_z & \varepsilon_{yx} & = \frac{\partial u_x}{\partial y} + \omega_z \\
\varepsilon_{yz} & = \frac{\partial u_y}{\partial y} & \varepsilon_{zy} & = \frac{\partial u_z}{\partial x} + \omega_y \\
\varepsilon_{xz} & = \frac{\partial u_z}{\partial z} & \varepsilon_{zx} & = \frac{\partial u_x}{\partial z}
\end{align*}

The Cosserat theory in additional to the normal and shear strains introduces the nine micro-curvatures

\[ k = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \]

(23)
CONTINUUM MECHANIC DEFINITIONS

\[ \kappa_{xx} = \frac{\partial \omega_z}{\partial x} \quad \kappa_{yy} = \frac{\partial \omega_z}{\partial y} \quad \kappa_{zz} = \frac{\partial \omega_z}{\partial z} \]

\[ \kappa_{xy} = \frac{\partial \omega_x}{\partial y} \quad \kappa_{yx} = \frac{\partial \omega_y}{\partial x} \quad \kappa_{xz} = \frac{\partial \omega_x}{\partial z} \quad \kappa_{zy} = \frac{\partial \omega_y}{\partial z} \]

\[ \kappa_{yz} = \frac{\partial \omega_z}{\partial x} \quad \kappa_{zx} = \frac{\partial \omega_x}{\partial y} \quad \kappa_{xy} = \frac{\partial \omega_y}{\partial z} \]

Where:

For plane strain or axial-symmetry the problem is simplified, to each material point three degrees of freedom are assigned: two translations \( u_x \) and \( u_y \), and one rotation \( \omega_z \) (Fig. 2.4). The rotation \( \omega_z \) is related to the micro-rotation of the micro-elements and is not determined from displacements as in a non-polar continuum.

\[ \mathbf{u} = \begin{bmatrix} u_x & u_y & \omega_z \end{bmatrix} \]

(24)

Stresses and strain quantities have two additional terms (Fig. 2.5 and Fig. 2.7b) compared to the classical continuum mechanics theory (Tejchman 2008). The two coupled moment’s \( m_{yz} \) and \( m_{xz} \) and the two curvatures \( \kappa_{xZ} \) and \( \kappa_{yZ} \).

\[ \mathbf{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & m_{xz} \\ \sigma_{yxy} & \sigma_{yy} & m_{yz} \\ m_{xz} & m_{yz} & \kappa_{xZ} & \kappa_{yZ} \end{bmatrix} \]

\[ \mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \gamma_{xy} \\ \varepsilon_{yxy} & \varepsilon_{yy} & \gamma_{yx} \\ \gamma_{xy} & \gamma_{yx} & \kappa_{xZ} & \kappa_{yZ} \end{bmatrix} \]

(25)

In difference from the classical continuum theory the shear strains are defined as the relative deformation between the macro-polar displacement gradients \( \partial u_y / \partial x \) or \( \partial u_x / \partial y \) and the micro-rotation \( \omega_z \).
The curvatures $\kappa_i$ (Fig. 2.7b) describe the gradients of the micro-rotation. $\varepsilon_{ij}$ and are invariant with respect to rigid body motions (Günther 1958, Mühlhaus 1989). According to Muhlhaus and Vardoulakis (1987) the continuum has an overall rotation ($\omega_{ij}$) which is different from that of the grain (Cosserat) rotation ($\omega_{ij}^c$). The deviation in the rotation would cause non-symmetry in the strain and stress tensors. Thus the strain

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u_x}{\partial x} \\
\varepsilon_{yy} &= \frac{\partial u_y}{\partial y} \\
\varepsilon_{xy} &= \frac{\partial u_y}{\partial x} - \omega_z \\
\varepsilon_{yx} &= \frac{\partial u_x}{\partial y} + \omega_z \\
\kappa_{x} &= \frac{\partial \omega_x}{\partial y} \\
\kappa_{y} &= \frac{\partial \omega_y}{\partial x}
\end{align*}
\]

(25a)
CONTINUUM MECHANIC DEFINITIONS

The tensor $\varepsilon_{ij}$ for plane strain conditions can be decomposed into a symmetric part $E_{ij}$ and a skew symmetric part $W_{ij}$ (Tejchman 2008, Voyiadjis and Song 2006):

$$
\varepsilon_{ij} = E_{ij} + W_{ij} - W_{ij}^c
$$

$$
E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
$$

$$
W_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)
$$

For the 2D (plain strain) case:

$$
W_{12}^c = -\omega_z
$$

$$
W_{21}^c = \omega_z
$$

(26a)

where:

$E_{ij}$ and $W_{ij}$ are the symmetric and skew symmetric part of the displacement gradient, respectively, and $W_{ij}^c$ is the skew symmetric tensor corresponding to the Cosserat rotation $\omega_z$. $E_{ij}$ is the deformation and $W_{ij}$ is the rotation tensor, which are characteristic of non-

---

a) Shear strain $\varepsilon_{12}$ and $\varepsilon_{21}$ in 2D Cosserat continuum

b) Micro-curvature $\kappa_1$ and $\kappa_2$ in 2D Cosserat continuum

Fig.2-7-
polar continuum. The skew symmetric part $W_{ij} - W_{ij}^c$ describes the difference between the macro- and micro-rotation (Tejchman 2008). If $W_{ij} = W_{ij}^c$, $\varepsilon_{ij}$ reduces to $E_{ij}$ and the kinematics of a non-polar continuum is retrieved. If $W_{ij} < W_{ij}^c$, an overall negative (clockwise) Cosserat rotation emerges. The anti-symmetry arises from the deviation between the micro and macro rotations “local spinning due to displacement and particle rotation” (Tejchman 2008). The conditions for compatibility of deformations and curvatures are (Günther 1958):

\begin{align*}
\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y} dy &= \sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy \\
\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial x} dx &= \sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx \\
\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx &= \sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y} dy
\end{align*}

Fig. 2.8- Equilibrium condition for Cosserat continuum
\[ \frac{\partial \varepsilon_{xx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial x} + \kappa_{xz} = 0 \]
\[ \frac{\partial \varepsilon_{yy}}{\partial x} - \frac{\partial \varepsilon_{yx}}{\partial y} + \kappa_{yz} = 0 \]
\[ \frac{\partial \kappa_{xz}}{\partial y} - \frac{\partial \kappa_{yz}}{\partial x} = 0 \]  
(27)

The deformation quantities are energy-conjugate with the stress quantities. The strain components \( \varepsilon_{ij} \) are associated with four components of the stress tensor \( \sigma_{ij} \) (plane strain) which is in general non-symmetric and the curvatures \( \kappa_{ij} \) are associated with couple stresses \( m_{ij} \) (Voyiadjis and Song 2006).

In a Cosserat point, the balance laws are nonlinear and the material is arbitrary, thus it can be applied to granular material.

Force and moment equilibrium required for plane strain are (Fig.2.8):
\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = f_x \]
\[ \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{yx}}{\partial x} = f_y \]
\[ \frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + \sigma_{yx} - \sigma_{xy} = m \]  
(28)

Where \( f_x, f_y \) and \( m \) are the volume body forces and volume body moment, respectively.

2.3- STRESS AND STRAIN INVARIANTS.

2.3.1-STRESS INVARIANTS.

At any point in the body exists at least three surfaces where the shear components are zero. These surfaces are called principal planes and the corresponding stresses principal stresses. To determine the principal stresses (eigenvalues) and the corresponding directions (eigenvectors) the homogeneous algebraic equations of the following form is used:
\[ \sigma x = \lambda \ x \ (\sigma - \lambda I)x = 0 \]  
(29)

Where \( \sigma \) is a given square real matrix, \( x \) an unknown column matrix, often indicated as vector and \( \lambda \) an unknown scalar. The condition to have non-trivial solutions for \( x \) is:
\[ \det(\sigma - \lambda I) = 0 \]  
(30)

For a matrix \( \sigma \ 3x3 \) the equation can be written:
\[ \det(\sigma - \lambda I) = \begin{vmatrix} \sigma_{xx} - \lambda & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - \lambda & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - \lambda \end{vmatrix} = 0 \]  
(31)

After some elaboration the final equation is the third order polynomial equation:
\[ \lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0 \]  

(32)

Where:

\[ I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{ii} = \text{tr}(\sigma) \]  

is the first stress invariant

\[ I_2 = \sigma_{xx} \sigma_{yy} + \sigma_{yy} \sigma_{zz} + \sigma_{zz} \sigma_{xx} - \sigma_{xy} \sigma_{yx} - \sigma_{yz} \sigma_{zy} - \sigma_{zx} \sigma_{xz} \]  

is the second stress invariant

\[ I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} - \sigma_{xy}\sigma_{yx} - \sigma_{yz}\sigma_{zy} - \sigma_{zx}\sigma_{xz} \]  

is the third stress invariant

The solution of the cubic equation (Abramovitz, Stegun 1970) for a symmetric matrix will always give real quantities:

Often the stress tensor is decomposed into an isotropic and a deviatoric: \( \sigma^{(iso)} \) and \( \sigma^\prime \) (or \( s \)).

\[ \sigma^{(iso)} = \sigma_m \delta_{ij} e_i \otimes e_j \quad \text{and} \quad \sigma' = \sigma - \sigma^{(iso)} = (\sigma_{ij} - \sigma_m \delta_{ij}) e_i \otimes e_j \]  

(33)

\[ \left[ \sigma^{(iso)} \right] = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} \quad \left[ s \right] = \begin{bmatrix} \sigma_{xx} - p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - p & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - p \end{bmatrix} \]  

(33a)

Where:

\[ \sigma_m = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = p \]  

is the hydrostatic pressure.

The second deviatoric invariant is defined in similar way as the second stress invariant \( I_2 \):

\[ J_2 = \frac{1}{6} \left[ (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6\sigma_{xy}^2 + 6\sigma_{yz}^2 + 6\sigma_{xz}^2 \right] \]  

(34)

The third deviatoric invariant is defined as:

\[ J_3 = \left| \sigma' \right|^2 = (\sigma_{xx} - p)(\sigma_{yy} - p)(\sigma_{zz} - p) - (\sigma_{xx} - p)\sigma_{yz}^2 - (\sigma_{yy} - p)\sigma_{zx}^2 - (\sigma_{zz} - p)\sigma_{xy}^2 \]  

(35)

If we consider the principal directions as the coordinate axes, we can draw a plane which normal vector makes equal angles with each of the principal axes, i.e. having direction cosines equal to \( \frac{1}{\sqrt{3}} \). This plane is called the \( \Pi \)-plane (octahedral or deviatoric plane).

Consequently the isotropic principal stress vector \( p \) is directed along unit vector \( n \) and the deviatoric principal stress vector \( s \) is normal to \( n \), thus located in the \( \Pi \) -plane (Fig.2.9).

To define the direction of the deviatoric component in the \( \Pi \)-plane the Lode angle is used (Fig.2.9). The Lode angle is defined as a function of the second and third deviatoric invariant:

\[ \sin 3\theta = \frac{3\sqrt{3}}{2} \left( \frac{-J_3}{\sqrt{J_2}} \right) \quad \text{→} \quad \theta = \frac{1}{3} \arcsin \left( \frac{3\sqrt{3}}{2} \left( \frac{-J_3}{\sqrt{J_2}} \right) \right) \]  

(36)
2.3.2-STRAIN INVARIANTS.

In the same way the strain invariants are calculated. The volumetric strain is expressed as:
\[ \varepsilon_v = \text{tr}(\varepsilon) = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \nabla \cdot u \]  
(37)

The isotropic strain is expressed:
\[ \varepsilon_{\text{iso}} = \frac{\varepsilon_v}{3} \delta_{ij} \]  
(38)

The deviatoric strain tensor is given:
\[ \text{dev}(\varepsilon) = (\varepsilon_{ij} - \frac{\varepsilon_{\text{vol}}}{3} \delta_{ij}) \]  
(38)

The deviatoric strain is given:
\[ \varepsilon_d = \sqrt{\frac{2}{3} |\text{dev}(\varepsilon)|} \]  
(39)

2.3.3- MICROPOLAR FORMULATION

The stress invariants for a micro-polar continuum are determined in similar way as for the classical formulation. Thus the first stress invariant is equal with the one determined in the classical continuum formulation.

The second and the third invariant are determined by involving also the effect of the couple moments. In order to take in account only the symmetric part of the stress tensor a correction matrix \( P \) is introduced (Liu and Scarpas 2007).

\[ \begin{pmatrix} \bar{\mathbf{s}} \\ \bar{m} \end{pmatrix} = P \begin{pmatrix} \mathbf{s} \\ m \end{pmatrix} \]  
(40)

Where \( P \) is the correction matrix which has dependency on the length scales. The matrix \( P \) is given in appendix 1.
CHAPTER II

For a micro-polar continuum, the second and the third invariant of the deviatoric stresses can be written as:

\[ J_2 = \frac{1}{2}(\overline{s}_{ij} * \overline{s}_{ij} + \overline{m}_{ij} * \overline{m}_{ij}) = \]

\[ \frac{1}{2}\left[ \sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 + \left(\sigma_{xy} + \sigma_{yx}\right)^2 + \left(\sigma_{xz} + \sigma_{zx}\right)^2 + \left(\sigma_{yz} + \sigma_{zy}\right)^2 \right] \]  

\[ + \frac{1}{2l_c^2}(m_{xx}^2 + m_{yy}^2 + m_{zz}^2) + \frac{1}{2l_c^2}(m_{xy}^2 + m_{yx}^2 + m_{xz}^2 + m_{zx}^2 + m_{yz}^2 + m_{zy}^2) \]

\[ J_3 = \frac{1}{3}(\overline{s}_{ij} * \overline{s}_{jk} * \overline{s}_{ik} + \overline{m}_{ij} * \overline{m}_{jk} * \overline{m}_{ik}) = \frac{1}{3}\left(\sigma_{xx}^3 + \sigma_{yy}^3 + \sigma_{zz}^3 \right) \]

\[ + \frac{1}{8}\left((\sigma_{xy} + \sigma_{yx})^2(\sigma_{xz} + \sigma_{zx})^2(\sigma_{yz} + \sigma_{zy})^2 \right) \]

\[ + \frac{1}{2}\left((\sigma_{xy} + \sigma_{yx})^2 + (\sigma_{xz} + \sigma_{zx})^2 + (\sigma_{yz} + \sigma_{zy})^2 \right) \]

\[ + \frac{1}{2}\left((\sigma_{yz} + \sigma_{zy})^2 + (\sigma_{xz} + \sigma_{zx})^2 + (\sigma_{xy} + \sigma_{yx})^2 \right) \]

\[ + \frac{1}{2}\left((\sigma_{yz} + \sigma_{zy})^2 + (\sigma_{xz} + \sigma_{zx})^2 + (\sigma_{xy} + \sigma_{yx})^2 \right) \]

\[ + \frac{m_{xx}}{3l_c^2}(m_{xy}^2 + m_{yx}^2 + m_{xz}^2 + m_{yz}^2 + m_{x}m_{y} + m_{x}m_{z}) \]

\[ + \frac{m_{yy}}{3l_c^2}(m_{xy}^2 + m_{yx}^2 + m_{yz}^2 + m_{yz}^2 + m_{y}m_{z} + m_{y}m_{z}) \]

\[ + \frac{m_{zz}}{3l_c^2}(m_{xz}^2 + m_{yz}^2 + m_{xz}^2 + m_{yz}^2 + m_{z}m_{x} + m_{z}m_{y}) \]

\[ + \frac{m_{xy}m_{yz}m_{xz} + m_{xz}m_{yz}m_{xy} + m_{yz}m_{xz}m_{xy}}{3l_c^2} \]

With the help of the formulation in eq. (41) and eq. (41a) the J2 or J3 plasticity models can be formulated in the micro-polar continuum (Liu and Scarpas 2007).

2.4 ISOTROPIC LINEAR ELASTICITY

2.4.1 CONSTITUTIVE TENSOR

The relation between stress and strain for linear elasticity is defined by Hooke’s law:

\[ \sigma = D^e : \varepsilon \] (42)

Where \( D^e \) is the elastic material tensor, which for the simple case of a homogenous isotropic material is depending in two parameters. Several alternative sets of 2 parameters occur, depending on the type of application. First the mathematically based set of the Lamé’s parameters \( \lambda \) and \( \mu \), then the alternative set of bulk modulus \( K \) and shear modulus \( G \), rather popular in geomechanics, and finally the relation between these sets and the third parameter set of Young’s modulus \( E \) and Poisson’s ratio \( \nu \), commonly applied in structural mechanics (Molenkamp 2008):

The fourth order elasticity tensor as a function of the former parameters is:

\[ D_{ijkl} = \frac{E}{1 + \nu} \left( \frac{\nu}{1 - 2\nu} \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} \right) \] (43)
Thus the relation between stress and strain is:

\[ \sigma_{kl} = \lambda \varepsilon_{rr} \delta_{kl} + 2 \mu \varepsilon_{kl} \]  

(44)

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{13}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 
1-\nu & \nu & 0 & 0 & 0 \\
\nu & 1-\nu & \nu & 0 & 0 \\
\nu & \nu & 1-\nu & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2-\nu} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2-\nu} \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
2\varepsilon_{12} \\
2\varepsilon_{13} \\
2\varepsilon_{23}
\end{bmatrix} 
\]  

(44a)

The bulk modulus and the shear modules are expressed in terms of the Young modulus \( E \) and Poisson ratio \( \nu \):

\[ K = \frac{E}{3(1-2\nu)} \]

\[ G = \frac{E}{2(1+\nu)} \]  

(45)

And the Lame constants:

\[ \mu = G \]

\[ \lambda = K - \frac{2}{3} G \]  

(46)

2.4.2- MICROPOLAR FORMULATION

According to De Borst (1992) the elastic stiffness matrix for 2D micro-polar space (plane strain) is given:

\[
D^e = \begin{bmatrix}
2\mu c_1 & 2\mu c_2 & 2\mu c_2 & 0 & 0 & 0 \\
2\mu c_2 & 2\mu c_1 & 2\mu c_2 & 0 & 0 & 0 \\
2\mu c_2 & 2\mu c_2 & 2\mu c_1 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu + \mu_c & \mu - \mu_c & 0 \\
0 & 0 & 0 & \mu - \mu_c & \mu + \mu_c & 0 \\
0 & 0 & 0 & 0 & 0 & 2\mu l^2 \\
0 & 0 & 0 & 0 & 0 & 2\mu l^2
\end{bmatrix} 
\]  

(47)

Where:

\[ c_1 = \frac{1-\nu}{1-2\nu} \]

\[ c_2 = \frac{\nu}{1-2\nu} \]

\[ \mu = \frac{E}{2(1+\nu)} \]  

(48)

\( l \)- is the length scale parameter

\( \mu_c \)-is the Cosserat \( \mu \)
In this matrix two new parameters are introduced the length scale parameter \( l \) and \( \mu_c \) which are equal to zero if the coupled stress are not present. The elasticity stiffness tensor extended for the 3D case (Liu and Scarpas 2007) is given in appendix 1. The new parameters introduced here are three: \( \mu_c, l_c, l_t \).

Where:

- \( l_c \) is the length scale parameter related to the bending coupled stress
- \( l_t \) is related to the torsion couple stress


\[
E = M\lambda P_a \left[ \left( \frac{I_1}{P_a} \right)^2 + R_\lambda \frac{J_2}{P_a^2} \right]^{\frac{\lambda}{\lambda - 1}}
\]

(49)

\[
R_\lambda = 6 \frac{1+\nu}{1-2\nu}
\]

(49a)

Where:

- \( P_a \) = is the atmospheric pressure
- \( I_1 \) = is the first stress invariant
- \( J_2 \) = is the second deviatoric stress invariant enhanced to account for the couple stress
- \( \nu \) = is the Poisson ratio
- \( M \lambda \) = and \( \lambda = \) are dimensionless material parameters determined from experiments that have loading-unloading-reloading cycles.

The couple stresses in Cosserat continuum represent spatial averages of distributed moments per unit area, like the stress represents a spatial average of force per unit area. The moments occur because the interatomic forces propagate further than one atomic spacing (Kröner, 1963) which occur in all solids, but the characteristic lengths would be of atomic range and not amenable to macroscopic mechanical experiment. Moments can be transmitted on a larger scale through fibbers in fibber-reinforced materials, or in the cell ribs or walls in cellular solids. In this case the characteristic lengths would then be associated with the physical size scales in the microstructure, and be sufficiently large to observe experimentally.
CHAPTER III

DESAI HIERARCHICAL SINGLE-SURFACE PLASTICITY MODEL

3.1-PLASTICITY
In the elastic region the deformations are reversible, but once that the plasticity is reached the deformations become irreversible. The condition that defines the limit of elasticity and the beginning of plastic deformation under any possible combination of stresses is known as yield criterion.

3.1.1- Yield criterion.
For 1D loading the yield criterion is a point beyond which the plasticity will occur. For 2D loading the yield criterion is a curve and the plasticity will occur when the combination of stresses applied in both directions will touch the curve. Similarly for the 3D case the yield criterion is a surface and the plasticity occur when a combination of stresses applied in three directions will touch the surface.

The yield criterion can be expressed as a function of stress tensor or the three stress invariants:

$$f(I_1, J_2, J_3) = 0$$

(1)

The basic principle of elasto-plasticity denotes that strains and strain rates can be decomposed into the elastic and plastic part.
\[ \varepsilon = \varepsilon^e + \varepsilon^p \quad d\varepsilon = d\varepsilon^e + d\varepsilon^p \]  

(2)

From the Hook’s law the relation between elastic stress and strain is defined:

\[ \sigma = D^e \varepsilon \rightarrow \varepsilon^e = D^e (\varepsilon - \varepsilon^p) \]  

(3)

To determine the plastic strains the plastic potential is used, which can be defined:

\[ g(\sigma_{ij}) = g(I_1, J_2, J_3) = 0 \]  

(4)

If the plastic potential and the yield surface coincide, then the plastic flow rule is called associated flow otherwise it is called non associated flow rule. The associated flow rule follows from considerations of the polycrystalline aggregates in which individual crystals deform by slipping over preferred planes (Bishop and Hills, 1951).

The most popular expression of the plastic strains rates is given as the product of a positive scalar \( \lambda \) and the vector \( m(\partial g / \partial \sigma_{ij}) \) which gives the direction of the plastic flow (von Mises, 1928; Melan, 1938; Hill, 1950).

\[ d\varepsilon_{ij}^p = \lambda \frac{\partial g}{\partial \sigma_{ij}} \]  

(5)

If the unit vector normal to the plastic potential approaches a finite number of linearly independent limiting values as the stress point approaches the singular point in question, Koiter (1953) proposes the following generalized flow rule:

\[ d\varepsilon_{ij}^p = \sum_{i=1}^{n} d\lambda_i \frac{\partial g_i}{\partial \sigma_{ij}} \]  

(6)

Where \( d\lambda_i \) are nonnegative and \( \partial g_i / \partial \sigma_{ij} \) are the linearly independent gradients.

### 3.1.2 - Drucker postulate

The maximum plastic work theorem (von Mises, 1928; Hill, 1950) is based in two main statements:

a) The yield surface is convex.

b) The plastic strain rate is normal to the yield surface.

But both definitions are purely mathematical so Drucker (1952, 1958) introduced an additional fundamental stability postulate.

\[ (\sigma_{ij} - \sigma_{ij}^*) d\varepsilon_{ij}^p > 0 \]  

(7)

\[ d\sigma_{ij} d\varepsilon_{ij}^p \geq 0 \]

So for an additional increment of the stress, both the total and plastic strain should also show and additional increment. This material behaviour is called stable. So in general a stable material should have the following properties:

a) The yield surface is convex

b) The plastic strain rate is normal to the yield surface (with an associated flow rule).

c) The rate of strain hardening must be positive or zero (work “softening” must not occur).

Plastic deformations often lead to hardening of the material and the increase of its elastic region so the yield surface in general is not fixed in stress space thus if the yield contour is expanding in size the material is hardening and if the yield surface is contracting in size then the material is softening.
The change of size of the yield surface is related to the plastic strain rates. The most common measure involve the total plastic work per unit volume, the accumulated plastic strain (Hill, 1950), the volumetric plastic strain rate (Schofield and Worth, 1968) or a combination of volumetric and shear plastic strains rates (Wilde, 1977; Yu at al.2005).

Hardening in the theory of plasticity means that the yield surface changes, in size, locations or even in shape, with the loading history and the rule of hardening defines the initial yield modification during the process of plastic flow.

Most plasticity models assume that the shape of the yield surface remains unchanged, although it may change in size or location. This restriction is more a mathematical convection, rather than based in any physical principle or experimental evidence.

The two most widely used rule of hardening are:

- Isotropic hardening where the yield surface can grow (or shrink), and it may change dimensions but not in shape.
- Kinematic (or anisotropic) hardening where the yield function neither translates nor rotates in the stress space changing neither shape nor dimensions.

The combination of both the rules is called mixed hardening, which is used in the advanced soil models.

The yield surface for a strain-hardening or softening material is also called the loading surface. The loading surface, which changes with plastic deformation, is expressed:

\[ f(\mathbf{\sigma}_y, \mathbf{\varepsilon}_y^p) = 0 \]  

(8)

where \( \mathbf{\varepsilon}_y^p \) is the plastic strain tensor.

If the yield surface is fixed in the stress strain space the material is called perfectly plastic and the behaviour is elastic as long as the stress state lies inside the yield surface. Plastic strains will occur only when the stress state lies on or travels along the yield surface. The stress condition for elastic and plastic behaviour may be stated as:

Elastic:
\[ f(\mathbf{\sigma}_y) < 0 \text{ or } df = \frac{\partial f}{\partial \mathbf{\sigma}_y} d\mathbf{\sigma}_y < 0 \]

Plastic:
\[ f(\mathbf{\sigma}_y) = 0 \text{ or } df = \frac{\partial f}{\partial \mathbf{\sigma}_y} d\mathbf{\sigma}_y = 0 \]  

(9)

Unloading from plastic state:
\[ f(\mathbf{\sigma}_y) = 0 \text{ or } df = \frac{\partial f}{\partial \mathbf{\sigma}_y} d\mathbf{\sigma}_y < 0 \]

The stress conditions for elastic and plastic behaviour for a strain hardening material are:

Elastic:
\[ f(\mathbf{\sigma}_y, \mathbf{\varepsilon}_y^p) < 0 \text{ or } df = \frac{\partial f}{\partial \mathbf{\sigma}_y} d\mathbf{\sigma}_y \leq 0 \]

Plastic:
\[ f(\mathbf{\sigma}_y, \mathbf{\varepsilon}_y^p) = 0 \text{ or } df = \frac{\partial f}{\partial \mathbf{\sigma}_y} d\mathbf{\sigma}_y = 0 \]  

(10)

3.2-PLASTICITY MODELS

Tresca (1864) work on the yield criterion of metal is considered as the birth of the classical theory of plasticity, but research on the failure or yielding of soils had been carried out much earlier by Coulomb (1773) and applied by Rankine (1853) to solve earth pressure problems in retaining walls.
3.2.1- PERFECT PLASTICITY MODELS.

The earliest and simplest plastic models are the perfect plasticity models. These models are divided in two groups for cohesive soil models (undrained behaviour) and friction soil models.

a) Cohesive soils.

For cohesive soils the two most used plasticity models are Tresca (1864) and Von Mises (1913). The behaviour of fully saturated cohesive soils, under undrained conditions can be modelled accurately by Tresca or von Mises plasticity theory.

**Tresca model**

Tresca yield criterion is:

\[ f = \sigma_1 - \sigma_3 - 2c_u = 0 \]  \hspace{1cm} (10)

where \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \) and \( c_u \) is the undrained shear strength.

By using the 6 stress permutation six different surfaces are obtained. Tresca yield surface is a regular hexagon on the \( \Pi \)-plane and in principal stress space is an infinite six-faceted cylinder in both directions.

![Tresca model](image)

**Von Mises**

Von Mises yield criterion is expressed as follow:

\[ f = \sqrt{J_2} - k = 0 \]  \hspace{1cm} (11)

Where \( k \) is the undrained shear strength of the soil. Von Mises yield contour is represented from a circle in the \( \Pi \)-plane.

Both Tresca and von Mises are stress independent. By choosing the suitable value of \( k \) von Mises circle can pass through the corners of Tresca hexagon.
b) Frictional-cohesive soils

For frictional-cohesive materials both Tresca and Von Mises are not suitable because of their stress independence. The main feature of yielding of frictional materials is their mean pressure dependence.

Mohr-Coulomb

The oldest and the most used yield criterion for frictional material is the empirical formulation proposed by Coulomb (1773). It states that the yield starts when the shear stresses and the normal stresses satisfy the following equation:

\[ |r| = c + \sigma_n \tan \phi \]  \hspace{2cm} (12)

In terms of principal stresses the Mohr-Coulomb yield contour is:

\[ f = \sigma_1 - \sigma_3 - (\sigma_1 - \sigma_3) \sin \phi - 2c \cos \phi = 0 \]  \hspace{2cm} (13)

Tresca model might be considered a special case of Mohr- Coulomb yield criterion with friction angle equal to zero. In the principal stress space the yield contour is infinite cone and in the \( \pi \)-plane is a hexagon.
It is important to mention that Mohr-Coulomb model has a non-associated flow rule based on the dilatancy angle. One of the most successful stress-dilatancy models is the one developed by Rowe (1962), which has been further simplified by Bolton (1986) as follows:

\[ \psi = 1.25(\phi - \phi_{cv}) \]  

(14)

where \( \phi_{cv} \) is the angle of friction at the critical state.

**Drucker-Prager**

Von Mises is not suitable to model the yielding of frictional materials as it does not include the effect of the isotropic stress. To overcome this limitation Drucker and Prager (1952) proposed the revised function for frictional soils.

![Drucker-Prager model](image)

Fig. 3.5-Drucker-Prager model

\[ f = \sqrt{J_2} - aI_1 - k = 0 \]  

(15)

Where \( a \) and \( k \) are material constants. On deviatoric plane the equation is plotted again as a circle as for the von Mises yield surface. In principal stress space the Drucker-Prager yield surface is a cone whilst the von Mises is an infinitely long cylinder. To select the material constants \( a \) and \( k \) the Drucker-Prager yield surface is matched with Mohr-Coulomb using a certain criterion.

\[
\alpha = \frac{2 \sin \phi}{\sqrt{3}(3 - \sin \phi)} \quad k = \frac{2c \sin \phi}{\sqrt{3}(3 - \sin \phi)} \\
\alpha = \frac{2 \sin \phi}{\sqrt{3}(3 + \sin \phi)} \quad k = \frac{2c \sin \phi}{\sqrt{3}(3 + \sin \phi)}
\]

(Compression)

(Tension)

**Lade- Duncan and Matsuoka - Nakai**

The major disadvantage of Mohr-Coulomb (Tresca) model are the corners which require special numerical techniques because the gradient to the yield surface when two surfaces
become active in the same time is not uniquely defined. From a computational point of view, a smooth line is the best yield surface. For this reason the yield surfaces proposed by Lade and Duncan (1975) and Matsuoka and Nakai (1974, 1982) which had tried to overcome this problem had been also used in geotechnical analysis. The yield contour proposed by Lade and Duncan can be written in terms of the first and the third invariant:

\[ f = \frac{I_3}{I_1} - k_1 = 0 \]  

(16)

Where \( k_1 \) is a soil constant depending on density.

The yield contour proposed by Matsuoka and Nakai is written in terms of the three stress invariants:

\[ f = \frac{I_1 I_2}{I_3} - (9 + 8 \tan^2 \phi) = 0 \]  

(17)

Where \( I_1, I_2 \) and \( I_3 \) are the first the second and the third invariants of effective stress tensor.

3.2.2- ADVANCED SOIL MODELS

Cap hardening.

Drucker, Gibson and Henkel (1957) in analogy with metal plasticity modelled soil as an elasto-plastic strain material. They added a spherical end –cap to the Drucker-Prager model to control the plastic volumetric change of soil or dilatancy. When the soil strain hardens, the cone and cap expand. The model contains two innovations. First the spherical cap fitted to the cone, second it use the current soil density as strain hardening parameter to determine the successive loading.

Fig.3.6- Lade- Duncan and Matsuoka – Nakai models

\[ f = \frac{I_1 I_2}{I_3} - (9 + 8 \tan^2 \phi) = 0 \]  

(17)

Where \( I_1, I_2 \) and \( I_3 \) are the first the second and the third invariants of effective stress tensor.

3.2.2- ADVANCED SOIL MODELS

Cap hardening.

Drucker, Gibson and Henkel (1957) in analogy with metal plasticity modelled soil as an elasto-plastic strain material. They added a spherical end –cap to the Drucker-Prager model to control the plastic volumetric change of soil or dilatancy. When the soil strain hardens, the cone and cap expand. The model contains two innovations. First the spherical cap fitted to the cone, second it use the current soil density as strain hardening parameter to determine the successive loading.

Fig.3.7 Drucker cap model
surface. The authors had found good agreement with the behaviour of virgin soils in drained triaxial tests.

The Cap models from Di Maggio and Sandler (1971) can be considered as critical state models with modified supercritical yield surface. The cap yield surface moves according to the changes in plastic volumetric strain, but the failure surface is fixed. No softening behaviour is, therefore predicted.

The model is mathematically advantageous and it has also good agreement with experimental data. The model is composed by two lines; one line for the over-consolidated state and another line for the normally consolidated state which meet smoothly at the connection point. At the connection point, the slopes of two lines are identical (the two lines are tangential to each other at the intersection point) thus the normality of plastic strains with respect to the yield surface does not cause mathematical problems.

This Cap model achieves a better agreement with experimental data than do the modified Cam Clay model even for flatter lines for the over-consolidated soils.

3.2.3- CRITICAL STATE MODELS (ISOTROPIC HARDENING)

The critical state concept was proposed independently by Roscoe et al. (1958) and Parry (1956, 1958). The critical state depend on the mean effective stress $p$, shear stress $q$ and soil specific volume $\nu = 1 + e$ ($e$-void ratio). At the critical state, soil behaves as a frictional fluid so yielding occurs at constant volume and stresses thus the plastic volumetric strain increment will be zero, since elastic strain increments will be zero. The critical state lines are unique for a given soil and independent from the stress path or initial condition. The two equations that determine the critical state line are:

\[ q = M_p \]
\[ \Gamma = \nu + \lambda \ln p \]

(18)

Where $M, \Gamma$ and $\lambda$ are soil properties and $p$ and $q$ the isotropic and deviatoric stress. The critical state can be considered till certain extent as the ultimate state by Drucker at al. (1957) or as the concept of steady state proposed latter by Poulos (1981). Desai and Toth (1996) and Desai (2001) proposed the concept of disturbed state. The critical state can be considered as the Desai’s fully disturbed state.

Cam-Clay and Modified Cam-Clay

The classical critical state models are Cam-Clay (named after the river Cam which flows behind the Cambridge Engineering laboratories) and Modified Cam-Clay. The first Cam Clay was developed by Roscoe and Schofield (1963) and Schofield and Wroth (1968) and afterwards Roscoe and Burland (1968) proposed the modified Cam clay model. The yield function formula is based on the simple analysis of Taylor (1948) on shear box tests results. Taylor assumed that the plastic work is dissipated entirely in friction

\[ dW_{dis} = Mpd\varepsilon_q^p \]

(19)

Where $M = q/p$ at the critical state.
The plastic work per unit volume of a triaxial sample is given as a function of the isotropic and deviatoric stress:

\[ dW_{\text{pl}} = pd\varepsilon_p^p + qd\varepsilon_q^p \quad (20) \]

where \( d\varepsilon_p^p \) and \( d\varepsilon_q^p \) are the volumetric and shear plastic strains. From energy conservation criterion we can write:

\[ dW_{\text{in}} = dW_{\text{dis}} \quad (21) \]

The volumetric and shear components of the strain can be written as a function of plastic multiplier \( \lambda \) and the components of plastic flow vector:

\[ d\varepsilon_p^p = d\lambda \frac{\partial g}{\partial p} \quad d\varepsilon_q^p = d\lambda \frac{\partial g}{\partial q} \quad (22) \]

Finally the equation is written:

\[ \frac{q}{p} - \frac{dq}{dp} = M \quad (23) \]

The critical stress ratio in triaxial compression \( M \) can be determined as:

\[ M = \frac{6\sin \phi_c}{3 - \sin \phi_c} \quad (23a) \]

Where \( \phi_c \) is the critical friction angle.

For the associated flow rule the Cam clay model can be written:

\[ \text{Fig.3.7 - Critical state line} \]

The critical state line in triaxial compression can be determined as:

\[ \text{For the associated flow rule the Cam clay model can be written:} \]
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\[ f = f(p, q, p_0) = \frac{q}{pM} + \ln\left(\frac{P}{p_0}\right) = 0 \]  
(24)

The size of the yield contour is represented by the pre-consolidation pressure \( p_0 \) which changes with the following low:

\[ dp_0 = \frac{vP_0}{\lambda - \kappa} \frac{d\varepsilon_p^o}{\lambda - \kappa} \]  
(25)

Where \( \lambda, \kappa \) and \( N \) are characteristic properties of a particular soil. \( \lambda \) is the slope of the normal compression (virgin consolidation) line or the critical state line in \( v-ln'p \) space, \( \kappa' \) is the slope of swelling line in \( v-ln p \) space and \( N \) is the specific volume of normal compression line at unit pressure and it depend on the units.

To avoid the discontinuity of the original model at \( p=0 \) the modified Cam-clay adopted an ellipse for the shape of the yield surface.

\[ f = f(p, q, p_0) = \left(\frac{q}{pM}\right)^2 - \left(\frac{p}{p_0} - 1\right) = 0 \]  
(26)

Several other modifications had been applied to the original Cam clay, but the model still shows limitations such:

- The yield surface significantly overestimate failure stresses in the “dry side”
- The model with an associated flow rule is not able to predict an important feature observed on lose sands or normally consolidated clays, the peak in deviatoric stress before reaching that the critical state (Bishop, 1972; Sladen et al, 1985) and bifurcation is not possible in the hardening regime (Vermeer, 1982).
- Cam Clay had been not able to model sand (Schofield and Wroth, 1968) because the model failed to predict observed softening and dilatancy of dense sands and the undrained behaviour of very lose sands.

Thus the critical state models are mostly limited to saturated clays and silts. Stiff and over consolidated clays are not modelled with critical state models due to the poor performance of critical state models in the “dry side”.

Fig.3.8- Cam-clay model
3.2.4-DOUBLE HARDENING MODELS

Critical state models had been not so successful to model granular materials thus a lot of “double hardening” sand models like the model developed at Imperial College by Nyaoro(1989) or Lade (1977), or Nova and Wood(1979) and Vermeer(1978) had been proposed, but they weren’t very successful in numerical analysis, because two separate yield surfaces are used to model hardening and softening, which causes numerical difficulties. The double hardening models involve two work hardening softening yield surfaces called the conical and cap yield surfaces.

Lade's double hardening model

Lade's double hardening model (Lade 1977) is an example of a constitutive model that involves two yield surfaces. The model is based on nonlinear elasticity and isotropic work hardening softening plasticity theories and is suitable for simulating the behavior of granular soils can be applied for three dimensional stress conditions, even that the input parameters can all be derived entirely from the results of standard laboratory tests.

The model involves two work hardening (the conical) softening (the cap) yield surfaces. The behavior depends on the direction of the stress path. If the stress path reaches the cap than the cap is activated and expands and the other one remains fixed or vice versa the cone is activated and the cap remains fixed. If the stress path reaches their intersection then both yield contours became active simultaneously. The conical yield function is:

\[ f = 27 \left( \frac{p_3^3}{I_3} - 1 \right) \left( \frac{3p_y}{p_a} \right)^m - H_1 = 0 \]  

where \( H_1 \) is hardening softening parameter which defines the size of the surface. At failure \( H_1 = \eta_1 \)

\[ \eta_1 = 27 \left( \frac{p_3^3}{I_3} - 1 \right) \left( \frac{3p_y}{p_a} \right)^m \]  

where \( m \) and \( \eta_1 \) are material parameters. They determine the apex angle and the curvature of the yield contour. The flow rule for this part of the model is non-associated thus the plastic potential has the form:
\[ g = 27p^* \left[ 27 + \eta_2 \left( \frac{p^*}{3p^*} \right)^m \right] I_3 \]  
\[ \text{(29)} \]

Where: \( \eta_2 = \rho H_1 + R \left( \frac{\sigma_y}{p_u} \right)^{\frac{1}{2}} + t \) and \( p, R \) and \( t \) are dimensionless parameters.

The cap yield surface has a spherical shape, with its centre at the origin of principal stress space. The yield function equation is:

\[ f = 9p^2 + 2I_2 - H_2 = 0 \]  
\[ \text{Where } H_2 \text{ is a work hardening parameter which defines the size of the surface.} \]

**Nova and Wood double hardening model (sand)**

The yield contour is given from the formula:

\[ g = q - \frac{Mp^*}{1-\mu} \left[ 1 - \mu \left( \frac{p^*}{p_{ug}} \right)^{(1-\mu)/\mu} \right] \]  
\[ \text{where } p_{ug} \text{ is the value of the isotropic pressure for } \mu=M. \text{ For } \mu=1 \text{ the equation is reduced to the Cam-Clay model.} \]

The yield locus for a given pre-consolidation pressure will be:

\[ p_u = \frac{p_c}{\sqrt{1+\mu}} e^{-M/2m} \]  
\[ \text{where } p_c, M \text{ and } m \text{ are material properties.} \]  
\[ \text{where } \eta_2 \text{ is the work hardening parameter.} \]

![Graph showing yield contours](image)

3.10- Nova and Wood double hardening yield contour

**Hardening soil model (Plaxis)**

Hardening soil model can be considered as an advanced soil model. It is an isotropic soil model not capable to model cyclic loading or hysteresis behaviour. Softening not included in the model, due to the dilatancy and debonding effect.
3.2.6- SINGLE HARDENING MODELS

Single hardening constitutive model Kim and Lade

Based on experimental data, Kim and Lade (1988) proposed a single surface hardening constitutive model which has the following plastic potential function:


\[ f = \left[ \psi_1 \frac{I_1}{I_3} - \psi_2 \frac{I_2}{I_3} \right] \frac{I_1}{p_e} \]

(37)

where:

- \( \psi_1 \) is a weighting factor between the triangular and circular shape in the octahedral plane;
- \( \psi_2 \) determines where this curve will intercept the hydrostatic axis;
- \( \mu \) controls the curvature of meridians.

This model is similar to the HISS model proposed by Desai, but there are still some differences like the number of parameters required for the Desai model is 8 for non-associated behavior and the Kim and Lade model requires 11 for non-associated flow. Also, the softening behavior is considered as contraction of the yield surface in difference from Desai original model where softening occurs due to the damage coming from micro-cracking and discontinuous nature of the material.

3.3-DESAI HIERARCHICAL SINGLE SURFACE PLASTICITY MODEL

Double hardening soil models use two separate yield surfaces to model hardening and softening, which may cause significant numerical difficulties and use a large number of constants (some of them doesn’t have a clear physical meaning) which need to be determined before these models can be applied.

The hierarchical single-surface (HISS) plasticity models provide a unified approach, which includes most of the other plasticity models as special cases. It eliminates or reduces the deficiencies of...
two separate yield surfaces and usually requires fewer parameters, compared to those in
the previously available models of comparable capacity.

The hierarchical single-surface (HISS) plasticity models provide a general formulation
for the elasto-plastic characterization of the material behavior. These models, which can
allow for isotropic and anisotropic hardening, and associated and non-associated
plasticity characterizations, can be used to represent material response based on the
continuum plasticity theory.

There are several version models of HISS like the basic one $\delta_0$ based on associated
plasticity, $\delta_1$ for non associated plasticity, $\delta_2$-models for kinematic and anisotropic
hardening and $\delta_{vp}$ for viscoplasticity, but in this study we will limit our self in the $\delta_0$-
model.

### 3.3.1 Desai yield contour formulation.

The single surface plasticity model proposed by Desai (1980) in general form has the
following yield contour:

$$
F = \frac{J_2}{p_2^a} - F_b * F_p = 0
$$

(38)

For the non associated flow $\delta_1$ the plastic potential is:

$$
Q = \frac{J_2}{p_2^a} - \left[ -\alpha \left( \frac{I_1 + R}{p_a} \right)^n + \gamma \left( \frac{I_1 + R}{p_a} \right)^e \right] \left( 1 - \beta * \frac{3 \sqrt{3}}{2} \frac{J_3}{\sqrt{J_2}} \right)^m = 0
$$

(39)

For the associated flow $\delta_0$ the yield contour is:

$$
F = \frac{J_2}{p_2^a} - \left[ -\alpha \left( \frac{I_1 + R}{p_a} \right)^n + \gamma \left( \frac{I_1 + R}{p_a} \right)^e \right] \left( 1 - \beta * \frac{3 \sqrt{3}}{2} \frac{J_3}{\sqrt{J_2}} \right)^m = 0
$$

(40)

where:

![Desai yield contour](image-url)
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$$F_b = \left[ -\alpha \left( \frac{I_1 + R}{p_a} \right)^p + \gamma \left( \frac{I_1 + R}{p_a} \right)^g \right]$$

(41)

Is the basic function describing the shape in $I_1$-$J_2$ space

$$F_s = \left( 1 - \beta \cdot \frac{3\sqrt{3}}{2} \frac{J_3}{(\sqrt{J_2})^3} \right)^m$$

(42)

Is the shape function describing the shape in the $\Pi$-plane.

$$S_r = \frac{3\sqrt{3}}{2} \frac{J_3}{(\sqrt{J_2})^3} = \cos 3\theta$$

(43)

Is the stress ratio (which is a function of the Lode angle).

The main difference between associated model and the non-associated one consist in the hardening parameter which for non-associated flow is extended:

$$\alpha_0 = \alpha + \kappa_c (\alpha_0 - \alpha)(1 - \chi_v)$$

(44)

Where:

$\alpha_0$ is the value at the initiation of non-associativeness.

$\kappa_c$ is the non-associative material (hardening) parameter.

$\chi_v$ controls the contribution of volumetric plastic deformation to the expansion of the potential surface ($\chi_v = \xi_v / \xi$).

A simple form of the hardening rule is:

$$\alpha = \frac{a_1}{\xi^{n_1}}$$

(45)

Where $\xi$ is the trajectory of plastic strains and is defined:

$$\xi = \int (de^p_{ij}de^p_{ij})^{1/2}$$

(46)

The plastic strain is composed by the isotropic ($\xi_v$) and the deviatoric ($\xi_D$) components.

$$\xi = \xi_v + \xi_D = \int \frac{1}{\sqrt{3}} |de^p_{ii}| + \int (de^p_{ij}de^p_{ij})^{1/2}$$

(47)

Where $E^p_{ij} = e^p_{ij} - \frac{1}{3} e^p_{ii}\delta_{ij}$

When $\alpha=0$ than the ultimate state had been reached and the equation is reduced:

$$F = \frac{J_2}{p_a} - \gamma \left( \frac{I_1 + R}{p_a} \right)^g \left[ 1 - \beta \cdot \frac{3\sqrt{3}}{2} \frac{J_3}{(\sqrt{J_2})^3} \right]^m = 0$$

(48)

The parameter $\beta$ defines the shape of the yield contour in the principal stress space. In order to have a convex surface this parameter should vary between $0 \leq \beta \leq 0.76$. For $\beta=0.77$ the yield surface starts to be concave and $\beta=0$ the yield contour in $\Pi$-plane has a circular shape. The value of the parameters $m$ and $g$ usually are $m=-0.5$ and $g=2$. 

-34-
If we consider the case when $\beta=0$ the yield contour can be written:

$$F = J_2 - \gamma (I_1 + R)^3 = 0$$

(49)

Where the parameter $\gamma$ is related to the ultimate strength of the material. It denotes the slope of the ultimate stress response surface.

The parameter $R$ takes into account the material cohesion and approximately is expressed:

$$3R = \frac{c}{\sqrt{\gamma}}$$

(50)

So for non-cohesive soils the equation is further reduced:

$$F = J_2 - \gamma I_1^2 = 0$$

(51)

### 3.3.2 Material parameters

The formulation of the yield contour which had been used in this thesis is the associated flow rule which is given by the HISS basic model $\delta_0$. The parameters involved in the $\delta_0$ model are:

**The ultimate state parameters $\gamma$ and $\beta$:**

The ultimate yield envelope often is a curved line, but it can be approximated as an average straight line. This line can be different for different stress paths (compression, extension or simple shear).

Both parameters are determined when $\alpha=0$ which correspond to the ultimate state so the yield contour is given with the simplified form:

$$F = \frac{J_2}{p_a} - \gamma \left( \frac{I_1 + R}{p_a} \right)^{\varepsilon} \left( 1 - \beta \frac{3\sqrt{3}}{2} \frac{J_3}{(\sqrt{J_2})^3} \right)^{\mu} = J_{2D} - \gamma J_{1D} \cdot F_s = 0$$

(52)

Thus $\gamma$ is determined:

$$\gamma = \frac{J_{2D}}{J_{1D} \cdot F_s}$$

(53)
The slope of the ultimate envelope is given:

\[ \tan^2 \theta = \frac{J_{20}}{J_{10}} = \gamma \cdot F_s = \gamma (1 - \beta S_r)^{-0.5} \]  \hspace{1cm} (54)

For a simple shear stress path \( S_r = 0 \) thus:

\[ \tan^2 \theta = 1 \]  \hspace{1cm} (55)

For the other cases the stress path is \( S_r = \pm 1 \) for compression and extension.

Compression:

\[ \tan^2 \theta = \gamma (1 - \beta)^{-0.5} \]  \hspace{1cm} (55a)

Extension:

\[ \tan^2 \theta = \gamma (1 + \beta)^{-0.5} \]

For \( \beta = 0 \):

\[ \tan^2 \theta = 1 \]

If \( \beta = 1 \) then \( \tan^2 \theta = 0 \) which is not possible and if \( \beta > 1 \) \( \tan^2 \theta < 0 \) which is again impossible. Thus: \( 0 \leq \beta < 1 \).

**The phase change parameter \( n \)**

The phase change parameter \( n \) is related to the state of stress at which the material response changes from compaction to dilation (\( n \) is related to the stress state when the volume change is equal to zero similar to the critical state definition). For plastic hardening parameters such a state can occur during the hardening process. For dense compacted soil this state is reached close to the peak stress when the material state changes from compaction to dilation and for loose materials is reached at high deformations state. The zero plastic volume change occurs when the increment of volumetric plastic strain is equal to zero.

Mathematically this statement is expressed:

\[ \frac{\partial f}{\partial I_1} = 0 \]  \hspace{1cm} (56)

After some elaborations we have:

\[ \frac{\partial F}{\partial I_1} = \frac{1}{p_a} \left( n \alpha \left( \frac{I_1 + R}{p_a} \right)^{n-1} - 2 \gamma \left( \frac{I_1 + R}{p_a} \right) \right) \left( 1 - \beta \cdot \frac{3 \sqrt{3}}{2} \frac{J_3}{(\sqrt{J_2})^3} \right)^m = 0 \]  \hspace{1cm} (57)

\( (n \alpha T_1^{n-1} - 2 \gamma T_1) \cdot F_s = 0 \)

The shape function \( F_s \neq 0 \) thus \( I_1 \) at the transition point is:

\[ (I_1)_t = \left( - \frac{2 \gamma}{\alpha n} \right)^{\frac{1}{n-2}} \]  \hspace{1cm} (58)

The yield contour at the transition point now is expressed:

\[ (\bar{J}_{2D})_t = [-\alpha_t (\bar{J}_1)_{t}^{y} + \gamma (\bar{J}_1)_{t}^{z}] \cdot F_{st} \]  \hspace{1cm} (59)

This leads to:

\[ \left( \frac{J_{2D}}{J_{10}} \right)_t = \left[ \gamma - \frac{2 \gamma}{n} \right] F_{st} \]  \hspace{1cm} (60)

So finally \( n \) is expressed:
Bonding stress $R$.

The parameter $R$ is related to the tensile strength of the geological materials. The value of $R$ can be obtained by assuming the ultimate envelope as a straight line so the bounding stress is proportional with the cohesion of the material.

$$R = \frac{c}{3\sqrt{F}}$$  \hspace{1cm} (62)

To evaluate $R$ the uniaxial tensile strength is used.

**Hardening parameters.**
The hardening rule in HISS basic model $\delta_0$ can has different formulation. The simplest one is:

$$\alpha = \frac{a_1}{\xi^\eta_1}$$  \hspace{1cm} (63)

Where $a_1$ and $\eta_1$ are material parameters and $\xi$ the plastic strain.

In a more extended formulation of hardening rule the deviatoric and the isotropic plastic strains effect on the hardening response is taken separately. A hardening rule formulation can be:

$$\alpha = \frac{h_1}{\left[\xi_v + h_3 \xi_d \right]^{h_2}}$$  \hspace{1cm} (64)

Where $h_i$ are material parameters.

The third type is an exponential hardening rule:

$$\alpha = \overline{h}_i \exp\left(-\overline{h}_b \xi \left(1 - \frac{\xi_d}{\overline{h}_3 + \overline{h}_4 \xi_d}\right)\right)$$  \hspace{1cm} (65)

Where $\overline{h}_i$ are material parameters.

In order to calculate the permanent deformations under cyclic loading a special procedure had been developed based on the bounding surface approach Mroz et al. during cyclic loading the yield surface expands with the number of cycles $N$ which causes cyclic hardening. The accumulated plastic strain at the end of each cycle is given:

$$\xi_i = \xi_0 + \left(1 - \frac{1}{N^h}\right) (\xi_b - \xi_0)$$  \hspace{1cm} (66)

Where:
- $\xi_0$-initial strain
- $\xi_b$-correspond to the bounding surface strain for the maximum load
- $h_c$-cyclic hardening parameter.

Fig.4.5-Desai bounding surface
In our research we will limit ourself only in monotonic loading. In this thesis for simulation of the hardening softening response of the material, parameter $\alpha$ is expressed as a function of both volumetric and deviatoric hardening components, $\alpha_v$ and $\alpha_d$:

**a) Hardening phase.**

For the hardening response the parameter $\alpha$ which will be used in this study (Liu and Scarpas 2007) can be expressed:

$$\alpha = \eta_h \alpha_v + (1 - \eta_h) \alpha_d$$

(67)

where:

$$\alpha = \eta_h \alpha_v + (1 - \eta_h) \alpha_d$$

$$\alpha_v = a_1 \cdot e^{b_1 \varepsilon}$$

$$\alpha_d = c_1 \cdot \left[ 1 - \left( \frac{\xi_d}{d_1 + \xi_d} \right)^2 \right]$$

(68)

$$\eta_h = \frac{\varepsilon_v}{\xi_d + \xi_v}$$

where the effective plastic strain is defined as:

$$\xi = \int \left( \frac{d\varepsilon^p}{d\varepsilon^p} \right)^{1/2} = \int \left[ N^T N \right]^{1/2} d\lambda$$

(69)

And the deviatoric and volumetric effective plastic strains are

$$\xi_d = \int \left( \frac{d\varepsilon^p}{d\varepsilon^p} \right)^{1/2} = \int \left[ N^T N \right]^{1/2} d\lambda = \left[ \frac{\partial Q^T}{\partial q} \frac{\partial Q}{\partial q} \right]^{1/2} d\lambda$$

(69a)

$$\xi_v = \int \frac{1}{\sqrt{3}} \left( d\varepsilon^p_{kk} d\varepsilon^p_{kk} \right)^{1/2} = \int \frac{1}{\sqrt{3}} \frac{\partial Q}{\partial p} d\lambda = \int [N] d\lambda$$

(69b)

With:

$$\varepsilon^p = \varepsilon_{kk} - \frac{1}{3} \varepsilon_{kk}$$

$$\xi = \sqrt{\xi_v^2 + \xi_d^2}$$

**b) Softening phase.**

For the softening phase the hardening parameter is expressed:

$$\alpha = \alpha_h + \eta_s (\alpha_u - \alpha_h)$$

(70)

Where:

$$\eta_s = e^{-\kappa_1 \xi_v}$$

(70a)

The post fracture plastic strain is determined:

$$\xi_{pf} = \int \left( \frac{d\varepsilon^p}{d\varepsilon^p} \right)^{1/2} = \int \left[ N^T N \right]^{1/2} d\lambda = \left[ \frac{\partial Q^T}{\partial q} \frac{\partial Q}{\partial q} \right]^{1/2} d\lambda$$

(70b)

Where:

$\kappa_1$—is the material degradation rate.
CHAPTER IV

LENGTH SCALE FORMULATIONS

4.1 STRAIN LOCALIZATION PHENOMENON.

The Cosserat micro-polar theory was first developed by the Cosserat brothers (1909) but the real rebirth of micromechanics was in the 60s when researchers started to show again interest and republishing Cosserat work. Cosserat continuum started to be used in geomechanics, in mid 1970’s to investigate the strain localization phenomenon. The work from Mühlhaus and Vardoulakis (1987) is one of the best examples where the Cosserat continuum had been used to investigate the strain localization in granular materials. The Cosserat theory was used to investigate the thickness of shear bands in granular materials. Roscoe (1970) was one of the first to investigate the localization phenomenon in different granular materials. One of the conclusions of his experimental work regarding the failure surface was the dilatancy (kinematic basis) effect on the inclination of the failure surface. A lot of experimental work has been done to investigate the strain localisation phenomena. From the geotechnical experience it is well known the fact that loose sand will dilate under low confining pressure and dense sand might contract under high confining pressure. The shear band thickness can be linked to the amount of the localized dilatation (for a higher dilatancy we get a thicker shear band).

The recent experimental work on strain localization has shown that the shear band inclination and thickness depend on the micro-structural properties (size, shape and roughness) of the materials, and the boundary and loading conditions. Thus we can say that we can control the shear band to develop in a certain direction. Han and Drescher (1993) conducted a series of bi-axial tests on dry coarse sand and they observed that the shear strain inside the shear band tends to increase and the inclination of the shear band tends to decrease as the confining pressure increases. The inclination of the shear band at high confining pressure had deviated from the Mohr-Coulomb solution. Mokni and Desrues (1998) found that the inclination of shear band in sand with respect to the minimum principal stress decreases when the mean stress level and the specimen density increase. The Mohr-Coulomb solution was close to their result for the inclination of the shear band in the loose sand, (no dilatancy).
Alshibli and Sture, (2000) performed a series of biaxial testing on three different sands; they observed that the shear band inclination increased with density, and decreased when the confining pressure and mean grain size increased. Mühlhaus and Vardoulakis (1987), reported shear band inclination of 62.5° and 60.1° for fine and medium sands respectively. Alsaleh (2004) supported the argument that loose sand will show thicker shear bands (high localized dilation) under low confining pressure which had been shown also in his numerical model.

Shear bands can develop in both drained and undrained conditions. Their thickness decreases as the confining pressure increases. Shear band boundaries experience high jumps in strains and excess pore water pressure, thus the phenomenon can give some indications related to the liquefaction phenomenon in saturated loose sands. Mokni and Desrues, (1998) Finno et al. (1997) had tested the behaviour of the saturated sand, testing under drained and undrained condition to study the possible localization phenomenon in the saturated soils. Desrues (1998) and Desrues et al. (1996) had studied the strain localization in sands and clays by means of the stereophotogrammetry and the computed tomography techniques. They observed, not only the shear banding, but also the presence of the cracks during shearing in ductile materials such as clays. Desrues and Hammad (1989) measurement of the shear band inclination angle had a fair agreement with the Mohr-Coulomb solution for the loose samples and had a significant deviation from the Mohr-Coulomb solution for the dense specimens. This can be explained by the effect of the dilation on the inclination of the failure surface for dense specimens which had been observed by Roscoe and are missing in the Mohr-Coulomb solution.

4.2 COSSEERAT THEORY LENGTH SCALES

During the hardening regime, granular materials behave as a continuum until the failure or the instability point where deformations begin to localize into a small but finite shear zone. The material stops behaving as one block and is divided in to several independent fragments that split by the shear bands failure zones (Voyiadjis and Song 2006). Softening behaviour occurs when the shape of the body and the boundary conditions induce a non-homogeneous state of deformation. As a consequence the microstructure reaches a heterogeneous state of deformation. This heterogeneity can cause shear bands in different materials such as metals, polymers, and granular materials. The possible causes of these failures can be the material heterogeneity or the defects present within the material such as cracks voids or dislocations (Voyiadjis and Song 2006). The localisation can be generated from the coupling of those defects with the plastic flow and the fracture.

From the analytical solution for the 1D case the shear band width can be written for isotropic hardening plasticity:

\[
w = 2\pi \sqrt{c_R} = 2\pi l
\]

where \(l\) is the internal length scale.
Cosserat theory is one of the best classical micro-polar theories: it is quite realistic and it has a strong physical background due to the fact that it is able to separate the micro-rotation of a material point from the overall rotation of the continuum. The micro-polar continuum is considered as a continuous group of particles behaving like rigid bodies. The theory combines two kinds of deformations at two different levels, micro-rotation at the particle level and macro-deformation at the structural level (Fig.2.3).

The Cosserat theory separates the grain rotation from its translation, by adding three other independent degrees of freedom to any point in the 3D continua. Thus in a Cosserat point we have not only translational but, also spinning (or rotational) deformation.

Kanatani (1979) studied the flow of granular materials by means of the micro-polar theory. He assumed that the grains were homogenous rigid spheres with the same size. The velocity $v_i$ and the rotation $\omega_{ij}$ of the grains were considered independently from each other as two variables that describe the deformation of the continuum by using independently the conservation laws of linear momentum and angular momentum. Kanatani equations of motion for the granular materials assume uniform contact distribution because the grains have spherical shape. Thus the non-uniformity of the particles that is faced in real granular material it’s not taken in account.

Oda et al. (1996) suggested that for granular materials the rotations of the grains are dominant at failure and must be taken into consideration with their couple stresses.

The grain rotations affect the behaviour of the granular media at failure, thus the dilatancy of the granular materials is highly affected by the grain rotations.

Oda and Iwashita (1999), emphasized that rotations and couple stresses are important quantities and they should be considered in the modelling of granular materials. Similarly Adhikary and Dyskin, (1997) found that the couple (moment) stresses were found of high importance on the behavior of granular materials. They have developed a model for layered geomaterials based on the Cosserat theory to incorporate the particle rotations and the couple stresses that cause those rotations.

Adhikary et al. (1999) emphasized that the couple stresses used in the Cosserat continuum are able to overcome the internal instabilities and interfaces interaction.

Ehlers and Volk (1997) had used the micro-polar kinematics to develop coupled equations for a saturated cohesive-frictional porous media where they distinguished between the continuum and the point rotation.

The couple stress can be smaller than the stress tensor but still it has an important effect on triggering the shear bands.

If the particle can be considered as a Cosserat point we can include in the model internal length scale which is a parameter that characterizes the particle micro-properties. A widely used length scale parameter is the mean particle size ($d_{50}$) which is one of the major factors that may govern the length scale magnitude. When the mean particle size increases/decreases also the magnitude of the length scale will increase/decrease. The length scale defined by Vardoulakis and Sulem (1995) is:

$$l = \frac{M}{G}$$

Where:

$G$- is the shear modulus

$M$- is the bending modulus that takes the unit of a force.
l- is the material bending length which may be considered as an internal length scale (i.e. the mean particle size can be chosen to represent this length). In addition, we can express the shear modulus as a function of the bulk modulus (K) and the Poisson ratio (v):

$$\frac{K}{G} = \frac{1}{1-2v}$$  \hspace{1cm} (3)

Another parameter is introduced as a function of the shear modulus (G) and Cosserat shear modulus ($\mu_c$), which relates the anti-symmetric part of the relative deformation to the anti-symmetric part of the resulted shear stress component. The constant $\alpha$ is called the coupling number (Vardoulakis and Sulem, 1995).

$$\alpha = \frac{1}{\sqrt{1 + \frac{G}{G^c}}}$$  \hspace{1cm} (4)

The parameter $\alpha$ can vary between $0 \leq \alpha \leq 1$. When $\alpha = 0$ the Cosserat effect vanishes ($G_c = 0$) and when $\alpha = 1$, this means $G_c$ is infinitely large (Alsaleh 2004). Cosserat theory uses two types of length scale (Fig. 4.1): the length of the contact surface ($l_s$) and the arm of rotation ($l_a$). The arm of particle rotation is not a unique value but it depends on the particle shape and the surface roughness (Fig. 4.1b). The actual geometrical dimensions $L$ of the bulk material must be significantly larger than length scale $l$. For a large $L$ the influence of the rotations will be small and vice versa for larger $l$, the Cosserat effects will become bigger.

![Fig.4-1- Two random adjacent particles](image1)

![Fig.4.2-Different length scales based in the particle shape](image2)
When we consider contact length scale by means of contact area then the rolling resistance terms parameters should be considered (Oda et al. 1982). Walsh and Tordesillas (2004) experimental and numerical results have shown that the particle rotations are large and unrealistic without a rolling resistance or friction (Arslan and Sture 2008). The contact moment resistance is defined in similar to the Coulomb friction criterion:

\[ M^k = P^N \tan \phi_c \]  

(5)

Where:
- \( M^k \) is the contact moment,
- \( P^N \) is the maximum contact force,
- \( \phi_c \) is the contact rolling resistance.

The maximum contact force for equal spheres according to H. Hertz (1982) is:

\[ P = \frac{2}{3} \sigma_N (\pi a^2) \]  

(6)

Where \( A = 4R^2 \) is the contact area of particles for equal and unequal spheres. In order to express the equation, in terms of average stress we can express the average stress as a function of the maximum force:

The average stress can be calculated as:

\[ \sigma_{avg} = \frac{P}{A} = \frac{P}{4R^2} \]  

(6a)

Thus, the contact moment of particles becomes:

\[ M^u - M^k = K' \Delta \theta \]

\[ K' \Delta \theta = l^2 G - \frac{2}{3} \sigma_N (\pi a^2) \tan \phi_c \]

\[ l^2 = \frac{1}{G} \left( K' \Delta \theta + \frac{2}{3} \sigma_N (\pi a^2) \tan \phi_c \right) \]  

(7)

Where:
- \( K' \) is the rotational stiffness modulus
- \( \Delta \theta \) is the relative rotation at contacts.
If the average stress is used the equation becomes:

\[ l^2 = \frac{1}{G} \left( K' \Delta \theta + \frac{2}{3} \sigma_{\text{avg}} 4 R \tan \phi \right) \]  

(8)

The relative rotation at contact (Walsh and Tordesillas 2004) is defined as:

\[ \Delta \theta = 2 R \omega \]  

(9)

where \( \omega \) is the curvature of the rotation.

If we substitute Eq. (7) into Eq. (6), the length scale equations will be:

\[ l^2 = \frac{1}{G} \left( K' 2 R \omega + \frac{2}{3} \sigma'_N (\pi a^2) \tan \phi^c \right) \]  

\[ l^2 = \frac{1}{G} \left( K' 2 R \omega + \frac{2}{3} \sigma_{\text{avg}} 4 R \tan \phi^c \right) \]  

(10)

The above length scale equations include the effect of plastic micro-rotation the effect of normal stress and contact area for spherical particles. Such a relation will simplify the length scale expression.

Walsh and Tordesillas (2004) had recommended a formulation that involves the rolling resistance and internal friction angle. Similar relations in which the rotational stiffness modulus and shear modulus is involved are not available yet (Arslan and Sture 2008).

4.3-GRAIN PARAMETERS.

A granular assembly consists of discrete non-uniform particles that have different shapes and sizes. Each particle has its own micro-properties providing in this way a very heterogeneous complex mixture.

The heterogeneity in the material properties is mainly represented by the spatial distribution of the particle size, surface roughness and shape indices.

Rowe (1962) was the first to show the effect of the inter-particle friction on the strength of granular material by performing tests by using different materials. His experimental and theoretical work was on ideal assemblies of rods and spheres and has show that the shear strength and dilatancy depend on the surface friction, the packing of the particles and on the energy losses. His results support the fact that the strain localization will be affected by the shape of the particles ‘packing’ and by the surface roughness which controls the friction.

During the simple shear tests and triaxial tests, he observed a narrow failure zone within the specimen which was distorted at failure. This zone is the so called shear band. This shear band which represents the failure zone can be considered as a failure surface in a slope or soil mass under certain boundary conditions. The shear band is of interest in the geotechnical practice because it is the cause of major failures in many real geotechnical structures such as retaining walls, dams, foundations, highways, earth-fill embankments, slopes etc.

In numerical simulations the shear band is identified by a finite zone with high volumetric strains, Cosserat rotations and concentration of void ratio. Thicker shear bands indicate less shear strength and the inclination angle of such bands with respect to the minimum principal axis gives an idea about the stability of the soil mass.
Several researchers have investigated the effects of the grain size, density and micro-properties on the thickness and evolution of the shear bands but, the equations that have been developed until now to quantify the effect of the surface roughness and grain shape on the thickness or/and the inclination of the shear band are still not well formulated. Mokni and Desrues (1998) found that the thickness of the shear band was in the range of $(7.5 \div 9.6) \cdot d_{50}$ (a parameter which is called the mean grain size) and it has the tendency to increase with decreasing density. Alshibli and Sture, (1999) found a shear band thickness which ranges from $(10.63 \div 13.86) \cdot d_{50}$ and increases when the mean grain size increases and decreases with density. Their results are consistent with the results obtained from Vardoulakis (1978) who reported values of $10 \div 15$ times the mean grain size. Mühlhaus and Vardoulakis (1987) reported shear band thickness in the range of $13 \div 18.5 \cdot d_{50}$ (mean grain size).

De Jong and Frost (2002) stated that the increase of the grain angularity will not affect the thickness, but the thickness will decrease when the confining stress will increase. Oda et al. (1997) showed that the shear band doesn’t have straight boundaries and the thickness varies along the failure surface. Lade and Wang (2001) found that the shear band develops during the hardening regime and the localization will affect the peak stress value; its inclination increased slightly with the density for sand tested on triaxial apparatus.

The angularity of the grains seems to controls to some extent the polar friction which will affect the rotational resistance of the grains (Oda and Iwashita 1999; Gudehus and Nübel 2002) and the surface roughness affects the inter-particle slipping in granular materials. (Vardoulakis and Sulem 1995).

The shear band thickness should increase when the surface roughness increases because the shear band thickness is dependent on the internal length scale which a function of the roughness. This statement is physically justified because the shear band thickness is depending on the dilation which will increase when the roughness of the surface of contact between the particles will increase.

Assuming a uniform surface roughness with an average value measured at the micro-scale allows incorporating this parameter in the formulations at the finite element level. Huang et al. (2002) attempted to bring the surface roughness and shape of the grains into a hypoplastic model to study their effect on the strain localization in granular materials but still this theory suffer from linking the theoretical parameters to real physical quantities.

The size is the most striking property of a particle. This property serves to classify particles as gravels, sands, silts and clays. Several scales had been proposed to divide granular materials in size classes. The Wentworth (1922) grade scale is the most used one by geologists (Fig. 4.4).
Fig. 4.4 - Wentworth (1922) grade scale

Fig. 4.5 - Chart indicating different soil types and their properties.
The charts in fig.4.5 and fig.4.6 give an overview of the different types of granular materials which are found in nature, their sizes, shapes, sorting and different soil matrices.

A well sorted material is composed mainly from grains which fall in the same class on Wentworth scale. A visual estimation of the sorting can be done from the graphs of fig.4.6 or can be calculated from the grain size distribution curve.

Parameters such as shape, surface roughness, and gradation of particles influence a lot the strength and deformation properties of granular materials. Thus angular sands have a
greater friction angle than rounds one (Koerner 1968) and fine sands have a bigger friction angle then lose ones.

The shape of the particles affects the behaviour of the material thus particles with big angularity have more polar friction and as a consequence the dilation will be higher or the surface of contact between adjacent particles is highly affected by the shape of those particles( Fig.4.7).

The particle shape in sedimentology is expressed in terms of the surface texture, roundness and sphericity. 

The surface texture describes the surface of particles (e.g. polished, greasy, dull, frosted, etc.) That is characteristics that are too small to affect the overall shape. Roundness describes aspects of grain surface like the sharpness of corners and edges larger than the surface texture, but smaller than the dimensions of the grain. Sphericity describes the form of the particle irrespective of the sharpness of edges and corners by comparing it with the shape of the sphere. 

The surface texture of sedimentary particles has been studied widely and several attempts had been done to relate the texture to the depositional processes. The surface can be marked by a variety of small-scale low-relief features such as pits, scratches, fractures and ridges. On the geological records is said that windblown sand grains has opaque frosted surfaces and water laid sands have clear translucent surfaces. Electronic microscopic studies show that there are several types of surface texture on sand grains produced by glacial eolian and aqueous processes (Krinsley, 1998). Water deposited sands have V-shaped percussions, pits and groves. Glacial sands have chonchoidal fractures and irregular angled micro-topography (Fig.4.8).

A lot of attempts had been done to define the shape of the particles and the controlling factors of grain shape. The shape refers to the gross overall configuration of the particles and reflects variations on their proportions. Same particles can be close to the shape of a sphere others one can be flat or they can have rod shape. Zingg (1935) had defined a
scheme to classify pebbles. He used a different approach by proposing the use of two different shape indices $D_1/D_L$ and $D_S/D_1$ to define the shape fields (Fig.4.9).

According to him based on the ratios between length, breadth and thickness four classes can be divided:
- Spherical (equant)
- Oblate (disk or tabular)
- Bladed
- Prolate (roller)

Wadell (1932) defined that shape and roundness are not the same geometrical concepts and has chosen the sphere as a standard (Fig.4.10 shows that the sphericity and the roundness are two different parameters). Thus roundness is the measure of the sharpness of the grain corners and is determined as the ratio of the average of radii of corners of the grain image and the maximum radius of the inscribed circle. Well rounded grains have smooth corners and edges, poorly rounded grains have sharp or angular corners and edges.

The concept of sphericity was introduced by Wadell (1932). According to Wadell (1932) the sphericity is an expression of the extent to which the form of the particle approaches the shape of the sphere. If all the three axes have about the same length the particle has high sphericity but, if the axes differ significantly in length the particle has low sphericity. Thus we can write:

$$Sphericity = \frac{\text{surface area of the particle}}{\text{surface area of the sphere of equal volume}}$$  \hspace{1cm} (11)

Krumbein (1941) modified the sphericity concept of Wadell’s and his expression is:
\[ \psi = \sqrt[3]{\frac{\text{volume of the particle}}{\text{volume of the circumscribing sphere}}} \]  

(11a)

Considering the volume of the sphere is equal to: \( V = \frac{\pi}{6} D^3 \) and if we assume that the particle is a triaxial ellipsoid with three diameters \( D_L, D_I \) and \( D_S \) where \( L, I \) and \( S \) are the length of the long, intermediate and short axes of the ellipsoid. By substituting this formula can be expressed:

\[ \psi = \sqrt[3]{\frac{\pi}{6} D_L D_I D_S} = \sqrt[3]{\frac{D_S D_I}{D_L^2}} \]  

(11b)

The sphericity calculated with this formula is called the intercept sphericity and can be calculated by measuring the long, the intermediate and the short axes of the ellipsoid. Sneed and Folk proposed a different sphericity measure called the maximum projection sphericity which is the ratio between the maximum projection area of a sphere with the same volume as the particle and the maximum projection of the particle.

\[ \psi_p = \sqrt[3]{\frac{D^2}{D_L D_I}} \]  

(11c)

![Diagram showing particle shapes and their descriptors](Fig.4.10- Schematic representations of particle shape.)
The first definition of roundness was defined by Wentworth (1919) as the ratio of the average radius of the sharpest corner to the radius of the largest inscribed circle. This statement was modified by Wadell (1932). According to Wadell (1932) the roundness is defined as the ratio of the average radius of all the corners and edges to the radius of the large inscribed circle:

$$Roundness = \frac{\text{average radius of corners and edges}}{\text{radius of maximum inscribed circle}}$$

$$R_w = \frac{\sum (r / R)}{RN} = \sum \frac{r}{N}$$

Other authors have given other definitions for the roundness (e.g., Russel and Taylor 1937; Pettijohn 1949; Powers 1953, 1982). Sames (1966) proposed criteria to distinguish between fluvial and littoral pebble samples by using a combination of shape and roundness. Sames samples were restricted to isotropic rocks like chert and quartzite. For smaller particles it is more feasible to analyze the samples statistically. Sand grains can be analyzed by means of microscopic techniques. The sphericity is defined as the ratio of the surface area of a sphere of the same volume as the fragment to the surface area of the grain. The definition of particle shape in terms of sphericity and roundness is widely used and their analysis is often made visually.

The definition of Wadell is more statistically appropriate but, for practical purposes Wentworth’s definition is simpler and faster to use. Additional definitions have been proposed by Russell and Taylor (1937) and Powers (1953). According to Powers (1953) the roundness does not depend on particle shape but it depends on the sharpness of edges. He described the shape and the roundness index as a function of sphericity. In 1982 he enriched his chart with more classes for sphericity and
assigned index numbers for the different roundness and sphericity classes. Visual comparison charts consisting of sets of grain images with known roundness are often used to make rapid visual estimation of grain roundness. The visual charts by Krumbein (1941) and Powers (1953) are the mostly used for this comparison purposes.

Alshibli and Alsaleh 2004 proposed new roundness and sphericity indices which are shown in Figure 4.13 in comparison with the roundness and sphericity indices defined by Powers (1982).

Tables 4.1 and 4.2 are showing the roundness classifications according to Russel and Taylor (1953), Pettijohn and Powers (1953).

<table>
<thead>
<tr>
<th>CLASSIFICATION</th>
<th>RUSSEL AND TAYLOR</th>
<th>PETTIJOHN</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Class limit</td>
<td>Arithmetic midpoint</td>
</tr>
<tr>
<td>Angular</td>
<td>0-0.15</td>
<td>0.075</td>
</tr>
<tr>
<td>Sub-angular</td>
<td>0.15-0.3</td>
<td>0.225</td>
</tr>
<tr>
<td>Sub-rounded</td>
<td>0.3-0.5</td>
<td>0.4</td>
</tr>
<tr>
<td>rounded</td>
<td>0.5-0.7</td>
<td>0.6</td>
</tr>
<tr>
<td>well rounded</td>
<td>0.7-1.0</td>
<td>0.85</td>
</tr>
</tbody>
</table>

Table 4.1-Roundness grades according to Russel&Taylor and Pettijohn classifications (Powers 1953)
Surface texture can change without significantly changing the shape or the roundness but, a change in shape or roundness will change the surface texture because new surfaces will be exposed. If we think in a hierarchical way it can be said that the shape is a first order property, the roundness a second order property superimposed in shape and the surface texture is a third order property superimposed on both the corners of the grain and the surfaces between corners (Barret 1980).

<table>
<thead>
<tr>
<th>Roundness</th>
<th>Very Angular</th>
<th>Angular</th>
<th>Sub Angular</th>
<th>Sub Rounded</th>
<th>Rounded</th>
<th>Well Rounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_R$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>&gt;1.5</td>
<td>1.4-1.5</td>
<td>1.3-1.4</td>
<td>1.2-1.3</td>
<td>1.1-1.2</td>
<td>1.0-1.1</td>
</tr>
<tr>
<td></td>
<td>0.5</td>
<td>1.5</td>
<td>2.5</td>
<td>3.5</td>
<td>4.5</td>
<td>5.5</td>
</tr>
</tbody>
</table>

Fig. 4.13-Powers (1982) and Alshibli and Alsaleh 2004: roundness and sphericity indices
The arm of particle rotation, $l_a$, seems to vary depending on the surface roughness, $R$, and the shape indices ($R_r$ and $S_{SPH}$). If we assume a uniform surface roughness an average value can be used to represent the variation like a sinusoidal oscillation. The average value measured at the micro-scale can be used to incorporate this parameter in the formulations at the finite element level.

The shape of the particles affects the behaviour of the material too. Particle with high angularity result to have more polar friction and as a result higher dilation. The surface of contact between adjacent particles is strongly related to their shape. Granular particles have the tendency to lay on their long direction for more stability. The couple stress depends on the size of the contact surface and it can be called the bending length. The shape indices, $R_r$(roundness index) and $S_{SPH}$ (sphericity index), will be used to account for the shape non-uniformity (eq.15 and eq.16).

If we look at the particles at the micro-scale, we can measure infinite number of radii from the centre of gravity to the surface of the particles. We can adjust the radius in such a way that can take in account the surface roughness. This radius is used together with the shape indices to determine the arm of rotation and the length of the contact surface (eq.15 and eq.16).

The surface roughness and shape indices of granular materials, to some extent, are difficult to quantify. Alsaleh (2004) did surface measurements using optical interferometry performed on two glass sizes of beads (Fig.4.14) and three silica sands (Fig.4.15). For each of the materials 120 particles which were arbitrarily selected were scanned. His new roundness and sphericity indices are given in Fig.4.13.

The surface roughnesses were measured using an optical surface profiler (Fig.4.16). To avoid errors due to particle’s surface curvature and edges only the center part of particles were used in roughness calculations (Fig.4.16). The height of the surface at a specific coordination is measured based on the phase data (i.e., pixel value) and the wavelength of the source light.

In his work he determined; the average roughness $R_a$ as the arithmetic mean of the absolute values of the surface departure from the mean plane.

$$R_a = \frac{1}{MN} \sum_{i=1}^{M} \sum_{j=1}^{N} |Z_{ij}|$$

(13)

Where $M$ and $N$ are the number of pixels in X and Y direction, $Z$ is the surface height at a specific pixel relative to the reference mean plane and $Z_0$ is defined as:

$$Z_{ij} = \lambda L_{ij}$$

(14)
Where $\lambda$ is the wavelength used in the scan and $L$ is the wave value for specific coordinates at the particle surface. $R_a$ is usually used to describe the roughness of a finished surface, so it can be used to describe the roughness of the sand particle surface.

The main disadvantage of the average roughness, is that the effect of a single spurious, peak or valley will be averaged out and have only small influence on the overall roughness. Thus the average roughness doesn’t give information about the shape of the irregularities or the surface of the particle. For granular materials $R_a$ represents, to some extent, the overall roughness used for friction calculations.

![Fig.4-14 Glass beads with different sizes](image)

- a) small glass beads with a size range of 0.75 – 1.0 mm
- b) large glass beads with range of 3.30 – 3.6 mm

Fig.4-14 Glass beads with different sizes
a) fine-grained quartz sand (Ottawa sand) with mean particle size $d_{50} = 0.22$ mm.

b) medium-grained white quartz sand (industrial) with mean particle size $d_{50} = 0.55$ mm

c) crushed silica sand (Connecticut) with mean particle size of $d_{50} = 1.6$ mm

Fig. 4.15- Particle size, shape and surface roughness
In the table 6.3 some statistical results for the five different materials taken in account (Fig. 4.16) from Alsaleh (2004) are shown:

<table>
<thead>
<tr>
<th>Roughness ($\mu m$)</th>
<th>Ottawa sand</th>
<th>White quartz sand</th>
<th>Crushed silica sand</th>
<th>Small glass beads</th>
<th>Large glass beads</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>0.262</td>
<td>0.3997</td>
<td>0.6515</td>
<td>0.0949</td>
<td>0.0581</td>
</tr>
<tr>
<td>Maximum</td>
<td>2.916</td>
<td>3.1205</td>
<td>2.6958</td>
<td>1.66</td>
<td>0.569</td>
</tr>
<tr>
<td>Mean</td>
<td>0.8487</td>
<td>1.0822</td>
<td>1.1169</td>
<td>0.33649</td>
<td>0.213536</td>
</tr>
<tr>
<td>Median</td>
<td>0.7075</td>
<td>0.813</td>
<td>0.9639</td>
<td>0.258</td>
<td>0.193</td>
</tr>
<tr>
<td>RMS</td>
<td>1.0105</td>
<td>1.2685</td>
<td>1.2227</td>
<td>0.40831</td>
<td>0.236783</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.551</td>
<td>0.6657</td>
<td>0.5033</td>
<td>0.23244</td>
<td>0.10283</td>
</tr>
<tr>
<td>Std. err.</td>
<td>0.0538</td>
<td>0.071</td>
<td>0.0768</td>
<td>0.02324</td>
<td>0.010283</td>
</tr>
</tbody>
</table>

Table 4.3-Roughness results for three different types of sand and two types of glass beads according to Alsaleh (2004)
4.4-LENGTH SCALE FORMULATIONS.

During numerical simulations it is easier to use a constant value for the length scale. Thus the material stiffness matrix \((D_e)\) remains the same throughout the whole simulation. Unfortunately this approach is unable to simulate the variation in the material strength which depends not only in the grain size but also in the plastic strains. Strong localized material dilatancy due to grain rearrangement and grain rotation are the dominant features of shear banding (Arslan and Sture 2008). It is questionable if a fixed value of the material length-scale is able to fully describe this phenomenon and to give results comparable with the experimental results.

Aifantis (1999) and Tsagarakis and Aifantis (2002) have indicated that a fixed value of the material length-scale is not always realistic and that different problems could require different values.

Abu Al-Rub and Voyiadjis (2004) and Nix and Gao (1998) suggest that the material length scale is not a fixed material parameter but is proportional with the mean free path of the dislocations \(L\) (eq.18) and changes with the deformation microstructure because of the variation of the mean free path with dislocations evolution. The change in length-scale magnitude is physically based because of the continuous modification of material characteristics with deformation.

Begley and Hutchinson (1998) showed that \(l\) has different values for different hardening exponents \(m\) (eq.18 a , eq.18b). Stolken and Evans (1998) also showed that \(l\) does not change if \(m\) is constant.

Abu Al-Rub and Voyiadjis (2004) had shown that \(l\) depend on the plastic strain level, and on the hardening level. Tsagarakis and Aifantis (2002) and Zbib and Aifantis (2003) have used two different values of the length scale parameter to fit the Fleck et al. (1994) micro-torsion test for copper material and the Stolken and Evans (1998) micro-bend test for nickel. Haque and Saif (2003) found that the length scale parameter doesn’t have a fixed value but it depends on the grain size.

Some other authors have also pointed out the necessity of a length-scale parameter that change with accumulated plastic strain in gradient theories in order to achieve an efficient computational convergence while conducting multi-scale simulations (e.g. Pamin, 1994). Thus we can say that length scales are dependent on the degree of accumulated plastic strain in the material. An evolution law for the length scale parameter is needed in order to have the proper modification required to fully utilize the theories which had been developed to solve the size effect problem. The consideration of the evolution of the material intrinsic length parameter with time, in dynamic problems becomes more necessary.

In micro-polar (Cosserat) continuum two length scales are needed (Fig.4.1):
- The length of contact surface \(l_s\) which is an internal length scale that represents the length of the contact surface between two adjacent particles.
- The rotational arm \(l_a\) which represent the distance from the centre of the particle to the point of the contact under consideration.

A widely used length scale is \(d_{50}\) (the mean particle size) which is not so accurate and most of the time will provide inaccuracy in modelling the behaviour of granular material. The couple stress depends on the size of the contact surface (the bending length). If we take in account the effect of shape and size of the particles in the length scale we can
observe that more the particle is elongated the longer is the contact surface and less the arm of rotation. 

The length of surface contact increases as the sphericity index increases and decreases as the roundness index increases (fig.4.13). Based in this the following formula is proposed for the length of the contact surface (Voyiadjis et al 2004):

\[ I_s = \frac{I_{sp}}{I_R} l_{ave} \]  \hspace{1cm} (15)

Where:
- \( I_s \) - the length of the surface of contact
- \( I_{sp} \) - sphericity index
- \( I_R \) - roundness index

The arm of particle rotation, \( l_a \), seems to oscillate depending on the surface roughness, \( R \), and the shape indices. If the surface roughness is assumed uniform with an average value we can assume an oscillation to represent it (sinusoidal oscillation had been assumed in our case). Thus the length of arm of rotation is determined:

\[ l_a = \frac{I_R}{I_{sp}} l_{ave} + R_a \]  \hspace{1cm} (16)

Where:
- \( l_a \) - the length of the arm-of-rotation
- \( R_a \) - the mean surface roughness

Where the \( I_{sp} \) (sphericity index) and \( I_R \) (roundness index) are determined respectively:

\[ I_{sp} = \frac{D_{equ} - d_L}{d_L} \] 

\[ I_R = \frac{P_{act}}{\pi \left( \frac{d_s + d_L}{2} \right)} \]  \hspace{1cm} (17)

where:
- \( d_L \) - the longest particle dimension,
- \( d_I \) - the intermediate particle dimension,
- \( d_s \) - the shortest particle dimension
- \( D_{equ} \) - is the equivalent particle diameter (perimeter/ \( \pi \))
- \( P_{act} \) - is the actual perimeter of the particle

The particles are considered as rigid bodies and the particle size itself is not assumed to change (no particle damage is taken in account). The evolution equations will take in account for the changes in the length scales due to the translational and rotational deformations of the non-uniform-shaped particles.

4.4.1- Gracio (1994) length scale formulation.

The aim of his work was to check the presence of statistical and geometrical dislocations in the grains of polycrystalline copper, and to show that in the early stage of plastic deformation the mean free path of dislocations is of the order of the grain size, which leads to a double effect of the grain size on the work hardening behaviour:

- Limitation of the mean free path of dislocations due to the presence of grain boundaries;
Influence on the complexity of the intergranular accommodation.
In metallic crystals, plastic deformations are carried out on a microscopic scale by defects called dislocations, which are created by fluctuations in local stress fields within the material culminating in a lattice rearrangement as the dislocations propagate through the lattice. At normal temperatures the dislocations are not annihilated but they accumulate. Material deformation in metals enhances the dislocation formation, the dislocation motion, and the dislocation storage. The dislocation storage causes material hardening (Stelmashenko et al., 1993; De Guzman et al., 1993; Fleck et al., 1994 Gao and Huang, 2003).

The theory of dislocations was introduced by Orowan, Taylor and Polanyi (1934) to explain the lower values of yield strength obtained experimentally compared to the theoretical one. This type of mechanisms allows deformation within the material without slipping the entire plane which means the simultaneous braking of all the bonds. Dislocations exist in the material but they start to influence the behaviour of the material only after that the yield had been reached.

There are two different types of dislocations.

- Statistically stored dislocations (SSD) which are dislocations stored by trapping each other in random way.
- Geometrically necessary dislocations (GND) are dislocations that relieve deformation incompatibilities.

In agreement with previous work on poly-crystalline copper the effect of grain size on the work hardening behaviour must be due either to statistical and/or to geometrical dislocations and at the initial stage of plastic deformation the mean free path of dislocations (L) is of the order of the grain size.

The strain compatibility between adjacent grains of a poly-crystal generates geometrical dislocations which participate in the strengthening mechanism in conjunction with statistically stored dislocations which are related to the single-crystal behaviour. Other types of dislocations are indistinguishable, but they may contribute to cell formation.

The dislocation structure of poly-crystals is a function of the major or minor intergranular readjustments. At intermediate strain values the readjustments are distributed over the cells leading to a linear relationship between the tensile stress and the inverse of the cell size, whatever the grain size of the tested samples.

Garcio original formulation of internal length scale related to poly-crystals is given below:

\[ L = \frac{\delta D}{\delta + D\xi} \]

Where:
- L - is the free path of dislocations
- D - is the grain size
- \( \delta \) - is a material constant
- \( \xi \) - is the effective plastic strain.

Begley and Hutchinson (1998) had shown that for different values of hardening exponents’ different internal length scale are obtained. In similar way Stolken and Evans (1998) showed that the internal length scale does not change for constant hardening exponents. Based on these results Aifants et al (1999, 2002) proposed a new relation for
the mean free path of dislocations as a function of the plastic strain and the hardening parameter:

\[
L = \frac{\delta D}{\delta + D \xi^m}
\]  

(18)

Gracio’s (1994) evolution equation to update the average length scale for granular materials by taking as the representative diameter the mean particle size \( d_{50} \) can be written:

\[
l_{\text{Ave}} = \frac{f_0 d_{50}}{\delta + d_{50} \xi^m}
\]  

(19)

Where:
- \( d_{50} \) - is the mean particle size,
- \( m \) - is a material constant,
- \( f \) - is a material constant,
- \( \delta \) - is a constant taken as mean surface roughness of particles
- \( \xi \) - is the effective plastic strain.

4.4.2- Abu Al-Rub (2004), length scale formulations.

The evolution equation proposed by Abu Al-Rub, 2004 assumes that the internal length scale starts with an initial value and then decreased exponentially to a final value of \( l \to 0 \) at saturation (corresponds to the plasticity limit) as a function of the amount of the effective plastic strain and a material constant coefficient \( k_0 \). The asymptotic variation of the length scale with the effective plastic strain is physically more appropriate than a constant value of \( l \). The evolution equation is given:

\[
l_{\text{Ave}} = l_0 e^{-k_0 \xi}
\]  

(20)

Where:
- \( l_{\text{Ave}} \) - is average length scale
- \( k_0 \) - is a constant coefficient taken here as unity
- \( l_0 \) - is the initial length scale
- \( \xi \) - effective plastic strain

This exponential two-parameter function gives enough freedom for the evolution of the material intrinsic length-scale, and is consistent with the experimental results. The coefficient \( k_0 \) is characterized by the hardening exponent \( m \). Thus \( k_0 \) determines the rate at which the size effect starts to diminish toward the conventional plasticity limit (Abu Al-Rub 2004).

The case of a constant length-scale \( l_0 = l \) at yield (\( \xi = 0 \)) indicates that size effect is present even in the elastic domain, where \( l_0 = l \) corresponds to the effect of prior dislocation density, which can be a combination of \( \rho^S \) (density of the statistically stored dislocations) and \( \rho^G \) (density of the geometrically necessary dislocations which is a function of the strains) which nucleated during the specimen preparation (Alsahleh 2004).

When the mean grain diameter \( d_{50} \) is used as the primary parameter for the initial length scale \( l_0 \), and it is assumed that the effective plastic strain is the primary factor in
controlling the length scale, according to Voyadjis(2005) eq.20 can be written in a more extended way in order to take in account the particle micro-properties:

\[ l_{\text{ave}} = l_0 e^{-k_0 \xi} = a_1 d_{50} \left[ a_2 R_a + a_3 A_s \right] e^{-b_0 \xi} \]  

(20a)

Where:
- \( a_1 \) - material parameter
- \( a_2 \) - material parameter
- \( a_3 \) - material parameter
- \( R_a \) - Surface roughness
- \( d_{50} \) - mean particle size.
- \( \xi \) - effective plastic strain
- \( A_s \) - aspect ratio

Equation (20a) is similar in form to (20), but the additional parameters \( a_1 \), \( a_2 \) and \( a_3 \) are added in order to taken in account the physical properties of soils. Equation (20a) is an expression for one direction; in similar way the expressions for other directions may be obtained.

Another initial length scale (Alsaleh and Voyiadjis 2004) that we will use in this research is assumed to be as follows:

\[ I_{\text{SPH}} R s 5 0 R i n I d 2 \]  

(21)

Where:
- \( I_{\text{SPH}} \) - sphericity index
- \( I_R \) - roundness index
- \( R \) - surface roughness
- \( d_{50} \) - mean particle size.

The length scales can decrease or increase based on the shape of particles and the deformation in each loading step. This formulation does not have real physical basis and is only used for comparison purposes and to check the model response.

4.4.3- Voyiadjis at al. length scale formulations.

Another formulation from Voyiadjis et al. 2005 is expressed in terms of lengths of contact surface and arms rotation. This formulation is based on the behaviour of granular materials which exhibit large translational and rotational deformations due to the discreteness of the materials. The rotation of the particles will cause changes of the length scales due to the non uniformity of the particle shape. The length scales oscillations are assumed to develop according to the Cosserat rotation. Thus the length scale oscillation can increase or decrease and has a physical meaning.

Length of surface contact:

\[ l_s = \frac{I_{\text{SPH}}}{I_R} d_{50} \sin \left( \omega \xi \right) \]  

(22)

Length of arm of rotation:
The average length
\[ l_{\text{ave}} = l_a + l_s \tag{24} \]

Where:
- \( \omega_c \) - Equivalent rotation.
- ISPH - Sphericity index
- IR - Roundness index
- \( d_{50} \) - The mean particle size

In the Fig. 4.17 a) two random particles and their respective contact length scale and the arm of rotation are shown. In the Fig. 4.17 b) a perfect spherical particle which rotates in 2D with \( \omega_c \) is shown and the trigonometrically relation used in the length scale formulation is shown.

Taking in account that for real particles, the length of surface contact increases as the sphericity index increases and decreases as the roundness index increases, the formulation is corrected by the ratio ISPH/IR (eq. 22).

The same way of reasoning is used for the arm of rotation. The initial length of the arm rotation increases when the sphericity index decreases and increases when the roundness index and the surface roughness increase (eq. 23). Thus the length of the arm rotation for real particles is corrected by the ratio IR/ISPH and the surface roughness is added.

4.4.4 – Liu length scale formulations.

Two other length scale formulations which will be used in these research are from Liu (2009).

The first formulation is an exponential law where the evolution of the length scale depends on the deviatoric and the rotational effective plastic strains. As the plastic effective strains increase, the length scale decreases asymptotically to zero. In this
formulation the effect of the rotation and shear is not taken separately but in the formulation it had been taken their summation. The first formulation which involves the exponential low is given as below:

\[ l_{\text{Ave}} = l_0 e^{-k_0 (\xi_D + \xi_R)} \] (25)

Where

- \( \xi_D \) - the deviatoric effective plastic strain
- \( \xi_R \) - the rotational effective plastic strain
- \( l_0 \) - the initial length scale determined in eq.(21)
- \( k_0 \) - a material parameter

The second formulation involves two different length scales; the deviatoric length scale and the rotational length scale. The first one the deviatoric length scale \( l_D \) is depending only on the deviatoric plastic strains and has an exponential evolution low which in similar way as the previous formulation tends to go asymptotically to a length scale equal to zero when the effective plastic strains increase. This formulation is similar to the evolution equation proposed by and Abu Al-Rub,(2004) which assumes that the internal length scale start with an initial value and then decreased exponentially to a final value of \( l \rightarrow 0 \). the difference consist that in the previous formulation the total plastic strains are taken in account.

The second length scales the rotational one and it is dependent on the rotational effective plastic strains. The evolution low in difference from the previous formulation is a second order power low depending only on the rotational effective plastic strains. As the rotational effective plastic strains increase the length scale tend to the reach the asymptotic value of zero.

The total length scale is the result of the effect of both length scales by taking in account the weight effect that those two different types of plastic strains have on the overall behavior of the sample. This is expressed through the parameter \( \eta \) which is the ratio between the deviatoric strains and the total strains.

The second length scale formulation is expressed:

\[ l_{\text{Ave}} = \eta l_D + (1 - \eta) l_R \] (26)

\[ l_D = l_0 e^{-k_0 \delta} \]

\[ l_R = l_0 \left[ 1 - \left( \frac{\xi_R}{\xi_D + \xi_R} \right)^2 \right] \] (26a)

\[ \eta = \frac{\xi_D}{\xi_D + \xi_R} \]

Where:

- \( k_0 \) - is a material parameter
- \( l_0 \) - is the initial length scale determined in eq(19)
- \( \delta \) - is a material parameter
CHAPTER V

NUMERICAL IMPLEMENTATION

5.1 CONSISTENT TANGENT OPERATOR

Numerical simulation in solid mechanics usually leads to highly (geometrically and materially) nonlinear problems. Two kinds of methods can be applied to solve them; implicit and explicit methods. Implicit methods are usually preferred, because they are unconditionally stable but, for large scale problems explicit methods are widely used, because the computational effort is lower.

Various nonlinear solvers can be used to solve this type of problems but, a very common choice is the Newton-Raphson method which ensures the quadratic convergence which is obtained from the consistent tangent operators. The concept of consistent tangent operator was introduced for simple elasto-plastic models by Simo and Taylor (1985).

The term Consistent means consistent with the numerical time–integration scheme used to solve the local problems (Gauss–point level), which is typically the backward Euler (or the midpoint rule). Consistent tangent matrix formulations for several material models can be found in the literature.

Consistent tangent matrices had been efficient to solve through implicit methods complex problems in nonlinear computational mechanics (Simo and Taylor 1985) of the global problem with quadratic convergence, through Newton-Raphson linearization but the quadratic convergence should be obtained also for the local problem.

The Newton-Raphson method it had been not applicable for some non-trivial constitutive laws due to difficulties to define the analytical expression and the computation of consistent operators, thus other nonlinear solvers were used to integrate the constitutive model but without obtaining quadratic convergence because these methods, aren’t based on a consistent linearization of all the equations with respect to all the unknowns (Pérez Foguet 2000).

For advanced constitutive models (like the nonlinear coupling between hardening–softening parameters), the convergence at Gauss point’s level can be troublesome, but the same problem can occur also with simple models for the stress states in zones of high
curvature of the yield function. In both cases the plastic corrector has problems with the return mapping to the yield surface (Bićanić and Pearce 1996). Another solution can be smaller steps but then the most restrictive Gauss point will control the global problem. Thus for a better initial approximation of the local problem line search techniques can be used to enlarge the convergence region of the Newton-Raphson method.

Elasto-plastic models like Tresca, Mohr Coulomb or the cone–cap models have yield functions which are not a single line, leading to a non-smooth transition of the flow equations at the intersection. The problem is solved, by means of corner return mapping algorithms based on the Koiter’s rule, (Simo et al. 1988).

To obtain the consistent tangent matrix we should compute the derivatives of the constitutive equation with respect to stresses and internal variables which, are needed to solve the local problems by means of the Newton-Raphson method.

5.2- NEWTON-RAPHSON SCHEME FORMULATION

In general form for an elasto-plastic model we can write (Ortiz and Popov 1985):

\[ \varepsilon = \varepsilon^e + \varepsilon^p \]
\[ \sigma = D^e \varepsilon^e \]
\[ \varepsilon^p = \lambda \cdot m(\sigma, \alpha) \]
\[ \dot{k} = \dot{\lambda} h(\sigma, \alpha) \]  

(1)

Where \( \varepsilon^e \) and \( \varepsilon^p \) the elastic and plastic strain tensor, \( \sigma \) the stress tensor, \( D^e \) the elastic stiffness tensor, \( m \) is the flow vector, \( \alpha \) is the hardening parameter and \( h \) the hardening modulus. The plastic multiplier can be determined from the following conditions:

\[ f(\sigma, \alpha) \leq 0 \]
\[ \dot{\lambda} \geq 0 \]
\[ f(\sigma, \alpha) \dot{\lambda} = 0 \]  

(2)

For a backward Euler scheme the non-linear equations are used (Ortiz and Popov 1985, Simo and Hughes 1998)

\[ \sigma^{n+1} + \lambda D^e m(\sigma^{n+1}, a^{n+1}) = \sigma^n + D^e \Delta \varepsilon \]
\[ a^{n+1} + \lambda h(\sigma^{n+1}, a^{n+1}) = a^n \]
\[ f(\sigma^{n+1}, a^{n+1}) = 0 \]  

(3)

Where the quantities \( \sigma^n, a^n \) and the increment of total strains \( \Delta \varepsilon \) from the n step to n+1 step is known. The unknowns are \( \sigma^{n+1}, a^{n+1} \) and the plastic multiplier \( \lambda \). To solve the local and the global problem by using the Newton-Raphson method, the Jacobian of the residuals is needed.
In order to solve the problems an initial value for the unknowns is needed (Pérez Foguet, 2000). The most common choice given in the literature for those initial values is the elastic trial state:

\[ \sigma_{n+1}^0 = \sigma_n + D^e \Delta \varepsilon \]

\[ a_{n+1}^0 = a_n \]

\[ \lambda^0 = 0 \]

When the Newton-Raphson iterative procedure is used the iterative update of the primary variables is given:

\[ \begin{bmatrix}
\Delta \sigma_{n+1}^k \\
\Delta a_{n+1}^k \\
\Delta \lambda_{n+1}^k
\end{bmatrix} = - \left[ J^{-1} \right] \begin{bmatrix}
F(\sigma_{n+1}^k, a_{n+1}^k, \lambda_{n+1}^k) \\
S(\sigma_{n+1}^k, a_{n+1}^k, \lambda_{n+1}^k) \\
G(\sigma_{n+1}^k, a_{n+1}^k, \lambda_{n+1}^k)
\end{bmatrix} \]

Where:

\[ F(\sigma_{n+1}, a_{n+1}, \lambda_{n+1}) = 0 \]

\[ G(\sigma_{n+1}, a_{n+1}, \lambda_{n+1}) = \sigma_n - \sigma_n + \Delta \lambda D^e m_n = 0 \]

\[ S(\sigma_{n+1}, a_{n+1}, \lambda_{n+1}) = a_n - a_n - \Delta \lambda h(\sigma_{n+1}, \alpha_{n+1}) = 0 \]

To achieve the quadratic convergence for the global problem the consistent tangent matrix is needed. To compute this matrix the consistent tangent modulus \( d^{n+1} \sigma / d^{n+1} \varepsilon \) at each Gauss point is calculated by linearizing the eq. (3).

### 5.3-DESAI YIELD SURFACE DERIVATIVES

If we use an implicit method to integrate a constitutive law the local problem will be nonlinear. The Newton-Raphson method, which ensures a quadratic convergence, is widely used to solve the local problem (Dennis and Schnabel 1983), thus the Jacobian of the residual at the Gauss–point level is needed. Also the global problem is solved by means of an incremental/iterative approach (Crisfield 1991) thus at each load increment, a nonlinear system of equations should be solved. For this purpose the consistent tangent matrix computed with the consistent elastoplastic modules is needed (Simo and Taylor 1985, Runesson et al. 1986). The derivatives of the yield surface are needed to compute both the Jacobean of the residual for the local problem and the consistent elastoplastic modules for the global one (Pérez Foguet A. 2000).

For simple plasticity models, analytical derivatives are easily computed, which gives closed form return mapping algorithms in the Gauss point’s level and compact, explicit
expressions of the consistent elastoplastic modules for the global problem (Simo and Taylor 1985, Simo and Hughes 1998).

The analytical differentiation is difficult for the advanced constitutive models and algebraic manipulators like Maple can be used to obtaining those analytical derivatives. The Jacobian for the k iteration is given:

\[
\begin{bmatrix}
\frac{\partial F}{\partial \Delta \lambda} & \frac{\partial F}{\partial \sigma} & \frac{\partial F}{\partial \alpha} \\
\frac{\partial G}{\partial \Delta \lambda} & \frac{\partial G}{\partial \sigma} & \frac{\partial G}{\partial \alpha} \\
\frac{\partial S}{\partial \Delta \lambda} & \frac{\partial S}{\partial \sigma} & \frac{\partial S}{\partial \alpha}
\end{bmatrix}
\begin{bmatrix}
\delta \Delta \lambda^k \\
\delta \sigma^k \\
\delta \alpha^k
\end{bmatrix} = - \begin{bmatrix}
F(\sigma^k, \alpha^k, \Delta \lambda^k) \\
G(\sigma^k, \alpha^k, \Delta \lambda^k) \\
S(\sigma^k, \alpha^k, \Delta \lambda^k)
\end{bmatrix}
\] (8)

The three equations in incremental form are:

\[
\begin{align*}
F(\sigma^k, \alpha^k, \Delta \lambda^k) &= \frac{\partial F}{\partial \Delta \lambda} \delta \Delta \lambda^k + \frac{\partial F}{\partial \sigma} \delta \sigma^k + \frac{\partial F}{\partial \alpha} \delta \alpha^k \\
G(\sigma^k, \alpha^k, \Delta \lambda^k) &= \frac{\partial G}{\partial \Delta \lambda} \delta \Delta \lambda^k + \frac{\partial G}{\partial \sigma} \delta \sigma^k + \frac{\partial G}{\partial \alpha} \delta \alpha^k \\
S(\sigma^k, \alpha^k, \Delta \lambda^k) &= \frac{\partial S}{\partial \Delta \lambda} \delta \Delta \lambda^k + \frac{\partial S}{\partial \sigma} \delta \sigma^k + \frac{\partial S}{\partial \alpha} \delta \alpha^k
\end{align*}
\] (9)

Thus to solve the problem 9 derivatives are needed. The first three one are related to the yield contour. The formulation of the yield contour which had been used in this thesis is the HISS basic model \(\delta_0\) for the associated flow (Liu and Scarpas 2007). The yield contour is given:

\[
F = \frac{J_2}{p_o} \left[ -\alpha \left( \frac{I_1 + R}{p_o} \right)^n + \beta \left( \frac{I_1 + R}{p_o} \right)^m \right] \left( 1 - \beta * \frac{3\sqrt{3}}{2} \frac{J_3}{(\sqrt{J_2})^3} \right)^m = 0
\] (10)

The yield contour is expressed as a function of the three stress invariants \(I_1, J_2, J_3\) thus the derivatives related to the stresses are obtained by applying the chain rule:

\[
\frac{\partial F}{\partial \sigma_y} = \frac{\partial F}{\partial I_1} \frac{\partial I_1}{\partial \sigma_y} + \frac{\partial F}{\partial J_2} \frac{\partial J_2}{\partial \sigma_y} + \frac{\partial F}{\partial J_3} \frac{\partial J_3}{\partial \sigma_y}
\] (11)

The derivatives related to first invariant \(I_1\) are:

\[
\frac{\partial F}{\partial I_1} = \frac{1}{p_o} \left( \frac{I_1 + R}{p_o} \right)^{-a-1} - g \gamma \left( \frac{I_1 + R}{p_o} \right)^{g-1} \left( 1 - \beta * \frac{3\sqrt{3}}{2} \frac{J_3}{(\sqrt{J_2})^3} \right)^m
\] (12)

\[
\frac{\partial I_1}{\partial \sigma_y} = \delta_y
\] (13)

In similar way for the second deviatoric invariant \(J_2\):
\[
\frac{\partial F}{\partial J_2} = \frac{1}{p_a} + m \left[ \alpha \left( \frac{I_1 + R}{p_a} \right)^n - \gamma \left( \frac{I_1 + R}{p_a} \right)^g \right] \left( 1 - \beta * \frac{3\sqrt{3}}{2} \left( \frac{J_3}{\sqrt{J_2}} \right)^3 \right)^{m-1} 
\]

* \( \beta \left( \frac{3\sqrt{3}}{2} \frac{J_1}{\sqrt{J_2}} \right) \)

\[
\frac{\partial J_2}{\partial \sigma_y} = S_y = \sigma_y - \frac{1}{3} \delta_y 
\]

(15)

The derivatives related to the third deviatoric invariant are \( J_3 \):

\[
\frac{\partial F}{\partial J_3} = - \left[ -\alpha \left( \frac{I_1 + R}{p_a} \right)^n + \gamma \left( \frac{I_1 + R}{p_a} \right)^g \right] * m \left( 1 - \beta * \frac{3\sqrt{3}}{2} \left( \frac{J_3}{\sqrt{J_2}} \right)^3 \right)^{m-1} 
\]

* \( \beta \left( \frac{3\sqrt{3}}{2} \frac{1}{\sqrt{J_2}} \right) \)

\[
\frac{\partial J_3}{\partial \sigma_y} = S_{i4}S_{4y} - \frac{2}{3} J_2 \delta_y 
\]

(17)

The plastic multiplier \( \Delta \lambda \) is a function of the hardening parameter \( \alpha \), Eq.(7) thus the derivative of the yield surface related to the hardening parameter is determined:

\[
\frac{\partial F}{\partial \Delta \lambda} = \frac{\partial F}{\partial \alpha} \frac{\partial \alpha}{\partial \Delta \lambda} 
\]

\[
\frac{\partial F}{\partial \alpha} = \left( \frac{I_1 + R}{p_a} \right)^n * \left( 1 - \beta * \frac{3\sqrt{3}}{2} \left( \frac{J_3}{\sqrt{J_2}} \right)^3 \right)^{m} 
\]

(18a)

\[
\frac{\partial \alpha}{\partial \Delta \lambda} = g(\sigma_{n+1}, \alpha_{n+1}) 
\]

The stress function derivatives are:

\[
\frac{\partial G}{\partial \Delta \lambda} = \frac{\partial F}{\partial \sigma} N = \frac{\partial F}{\partial \sigma} \frac{\partial F}{\partial \sigma} 
\]

\[
\frac{\partial G}{\partial \sigma} = I + \Delta \lambda \frac{\partial}{\partial \sigma} (D^e : N) = I + \Delta \lambda \frac{\partial}{\partial \sigma} (D^e : \frac{\partial F}{\partial \sigma}) 
\]

\[
\frac{\partial G}{\partial \alpha} = \Delta \lambda \frac{\partial}{\partial \alpha} (D^e : N) = \Delta \lambda \frac{\partial}{\partial \alpha} (D^e : \frac{\partial F}{\partial \sigma}) 
\]

Where \( N = \frac{\partial F}{\partial \sigma} \) is the vector that gives the direction of the plastic flow.

Due to the fact that \( D^e \) is independent from the hardening parameter \( \alpha \) the derivatives are not influenced thus it is possible to write:

\[
\frac{\partial}{\partial \alpha} (D^e : \frac{\partial F}{\partial \sigma}) = D^e : \frac{\partial}{\partial \alpha} \left( \frac{\partial F}{\partial I_1} \frac{\partial I_1}{\partial \sigma_y} + \frac{\partial F}{\partial J_1} \frac{\partial J_1}{\partial \sigma_y} + \frac{\partial F}{\partial J_2} \frac{\partial J_2}{\partial \sigma_y} + \frac{\partial F}{\partial J_3} \frac{\partial J_3}{\partial \sigma_y} \right) 
\]

(19a)
\[
\frac{\partial F}{\partial \alpha} \left( \frac{\partial F}{\partial I_i} \right) = I_i^{n+1} \delta_y \left( 1 - \beta \ast \frac{3\sqrt{3}}{2} \left( \frac{J_3}{\sqrt{J_2}} \right)^m \right)
\]

\[
\frac{\partial F}{\partial \alpha} \left( \frac{\partial F}{\partial J_2} \right) = \left( \frac{I_i + R}{p_a} \right)^n \left( 1 - \beta \ast \frac{3\sqrt{3}}{2} \left( \frac{J_3}{\sqrt{J_2}} \right)^m \right) \left( \sigma_y - \frac{I_i}{3} \delta_y \right)
\]

\[
\frac{\partial F}{\partial \alpha} \left( \frac{\partial F}{\partial J_3} \right) = \left( \frac{I_i + R}{p_a} \right)^n \left( 1 - \beta \ast \frac{3\sqrt{3}}{2} \left( \frac{J_3}{\sqrt{J_2}} \right)^m \right) \left( S_{ij} S_{ij} - \frac{2}{3} J_2 \delta_y \right)
\]

The derivatives of the last function \( S(\sigma_{n+1}, \alpha_{n+1}, \Delta \lambda_{n+1}) = 0 \) are:

\[
\frac{\partial S}{\partial \Delta \lambda} = -g
\]

\[
\frac{\partial S}{\partial \sigma} = -\Delta \lambda \frac{\partial g}{\partial \sigma}
\]

\[
\frac{\partial S}{\partial \alpha} = 1 - \Delta \lambda \frac{\partial g}{\partial \alpha}
\]

From Eq.(7) the increment of hardening parameter is determined as a function of \( \Delta \lambda \) the incremental magnitude of plastic strains and \( g(\sigma_{n+1}, \alpha_{n+1}) \) a function of both stress and hardening/softening parameter (Liu 2003, Liu and Scarpas 2007).

\[
\Delta \alpha = \alpha_{n+1} - \alpha_n = \Delta \lambda g(\sigma_{n+1}, \alpha_{n+1})
\]

Thus the function \( g(\sigma_{n+1}, \alpha_{n+1}) \) has a different definition for each material phase.

**a)Hardening phase.**

For the hardening response the parameter \( \alpha \) has the following rule:

\[
\alpha = \eta_h \alpha_v + (1 - \eta_h) \alpha_d
\]

Where:

\[
\alpha_v = a_v \cdot e^{b_v \xi_v}
\]

\[
\alpha_d = c_d \left[ 1 - \left( \frac{d_1 + \xi_d}{d_1} \right)^2 \right]
\]

\[
\eta_h = \frac{\xi_v}{\xi_d + \xi_v}
\]
In which the effective plastic strain is defined as
\[
\xi = \int \left[ \left( \frac{d\varepsilon^p}{d\lambda} \right)^T R d\varepsilon^p \right]^{1/2} = \int \left[ N^T R N \right]^{1/2} d\lambda
\]
(23)

Where \( R \) is a correction matrix which takes in account the micro-polar effect (Liu and Scarpas 2007). The matrix \( R \) is given in appendix 1. The deviatoric and volumetric effective plastic strains are:
\[
\xi_d = \int \left[ \left( \frac{d\varepsilon^p}{d\lambda} \right) R d\varepsilon^p \right]^{1/2} = \int \left[ N^T R N \right]^{1/2} d\lambda = \left[ \frac{\partial F^T}{\partial q} R \frac{\partial F}{\partial q} \right]^{1/2} d\lambda
\]
\[
\xi_v = \int \frac{1}{\sqrt{3}} \left( d\varepsilon^p_{kk} \right)^{1/2} = \int \frac{1}{\sqrt{3}} \frac{\partial F}{\partial q} d\lambda = \int \left[ N \right]_v d\lambda
\]
(23a)

With
\[
d\varepsilon^p_{kk} = d\varepsilon^p_k - \frac{1}{3} d\varepsilon^p_{kk}
\]
\[
\xi = \sqrt{\xi_v^2 + \xi_d^2}
\]
(24)

In incremental form the volumetric and deviatoric components are:
\[
\Delta \xi_d = \left[ N^T R N \right]^{1/2} \Delta \lambda
\]
\[
\Delta \xi_v = \left[ N \right]_v \Delta \lambda
\]
(23b)

The components of \( N \) represent the volumetric and deviatoric part thus:
\[
p = \frac{1}{3} \rightarrow I_1 = 3p \quad \quad q = \sqrt{2J_2} \rightarrow J_2 = \frac{q^2}{2}
\]
(25)

\[
\left[ N \right]_v = \frac{1}{\sqrt{3}} \frac{\partial F}{\partial q} = \frac{3}{\sqrt{3}} \left[ \alpha \left( \frac{I_1 + R}{p_a} \right)^{\eta_1} - \beta \left( \frac{I_1 + R}{p_a} \right)^{\eta_1} \right] \left[ 1 - \beta \frac{3\sqrt{3}}{2} \frac{J_3}{(J_2)^{3}} \right]^{m-1}
\]
\[
\left[ N \right]_d = \frac{\partial F}{\partial p} = \frac{q}{p_a} - \mu \left[ \alpha \left( \frac{I_1 + R}{p_a} \right)^{\eta_1} - \gamma \left( \frac{I_1 + R}{p_a} \right)^{\eta_1} \right] \left[ 1 - \beta \frac{3\sqrt{3}}{2} \frac{J_3}{(J_2)^{3}} \right]^{m-1}
\]
\[
\left( \frac{9\beta\sqrt{6}}{(q)^2} \right)^2
\]
(26)

The hardening parameter in incremental form can be written:
\[
\Delta \alpha = \Delta \lambda g(\sigma_{\alpha_1}, \alpha_{\alpha_1}) = \frac{\partial \alpha}{\partial \alpha_v} \frac{\partial \alpha}{\partial \alpha_v} \Delta \xi_v + \frac{\partial \alpha}{\partial \alpha_d} \frac{\partial \alpha}{\partial \Delta \xi_d}
\]
\[
\frac{\partial \alpha}{\partial \alpha_v} = \eta_1 \quad \quad \frac{\partial \alpha}{\partial \alpha_d} = 1 - \eta_1
\]
\[
\frac{\partial \alpha_v}{\partial \xi_v} = b_m \cdot \alpha_v \quad \quad \frac{\partial \alpha_\xi_d}{\partial \xi_d} = b_m \cdot \alpha_v
\]
\[
\frac{\partial \alpha_d}{\partial \xi_d} = -2c_1 \cdot \frac{d_1 \xi_d}{(d_1 + \xi_d)^3}
\]
(27a)
NUMERICAL IMPLEMENTATION

The hardening parameter can be expressed in incremental form by eq.23b and eq.27a:

$$\Delta \alpha = \Delta \lambda g(\sigma_{n+1}, \alpha_{n+1}) = \eta_h \alpha, b_h \Delta \xi_v + 2c_1 \frac{d\xi_v (\eta_h - 1)}{(d_1 + \xi_d) \Delta \xi_d}$$

(28)

$$\Delta \lambda \eta_h \alpha, b_h [N]^n + \Delta \lambda 2c_1 \frac{d\xi_v (\eta_h - 1)}{(d_1 + \xi_d)^2} \left[ N^T R N \right]^{1/2}$$

Finally the stress function \( g(\sigma_{n+1}, \alpha_{n+1}) \) will be:

$$g(\sigma_{n+1}, \alpha_{n+1}) = \eta_h \alpha, b_h [N]^n + 2c_1 \frac{d\xi_v (\eta_h - 1)}{(d_1 + \xi_d) \Delta \xi_d} \left[ N^T R N \right]^{1/2}$$

(29)

The derivatives of this function related to its variables will be respectively:

$$\frac{\partial g}{\partial \alpha} = b_1 \eta_h \alpha, \frac{\partial N}{\partial \alpha} + \frac{\partial \left[ N^T R N \right]^{1/2}}{2c_1 \frac{d\xi_v (\eta_h - 1)}{(d_1 + \xi_d)^2}} \left[ N^T R N \right]^{1/2}$$

(30)

$$\frac{\partial g}{\partial \sigma} = b_1 \eta_h \alpha, \frac{\partial N}{\partial \sigma} + \frac{\partial \left[ N^T R N \right]^{1/2}}{2c_1 \frac{d\xi_v (\eta_h - 1)}{(d_1 + \xi_d)^2}} \left[ N^T R N \right]^{1/2}$$

(31)

b) Softening phase.

For the softening phase the hardening parameter (Liu and Scarpas 2007) has the following expression:

$$\alpha = \alpha_R + \eta_s (\alpha_u - \alpha_R)$$

(31)

Where:

$$\eta_s = e^{-\kappa_1 \xi_v}$$

(32)

The post fracture plastic strain is determined:

$$\xi_{pf} = \int \left( d e^p R d e^p \right) d d = \int \left[ N^T R N \right]^{1/2} d \lambda = \left[ \frac{\partial F^T}{\partial q} \right] \frac{\partial F}{\partial q} \left[ N^T R N \right]^{1/2} d \lambda$$

(33)

\( \kappa_1 \) is the material degradation rate.

In similar way like the for the hardening phase \( \alpha \) can be expressed in incremental form.

$$\Delta \alpha = \alpha_{n+1} - \alpha_n = \Delta \lambda g(\sigma_{n+1}, \alpha_{n+1}) = \Delta \lambda \eta_i (\alpha_R - \alpha_{n+1}) \left[ N^T R N \right]^{1/2}$$

(34)

The stress \( g(\sigma_{n+1}, \alpha_{n+1}) \) function for the softening phase is defined:

$$g(\sigma_{n+1}, \alpha_{n+1}) = \eta_i (\alpha_R - \alpha_{n+1}) \left[ N^T R N \right]^{1/2}$$

(35)

The derivatives of this function related to its variables will be respectively:

$$\frac{\partial g}{\partial \alpha} = -\eta_i \left[ N^T R N \right]^{1/2} + (\alpha_R - \alpha_{n+1}) \frac{\partial \left[ N^T R N \right]^{1/2}}{\partial \alpha}$$

(36)

$$\frac{\partial g}{\partial \sigma} = \eta_i (\alpha_R - \alpha_{n+1}) \frac{\partial \left[ N^T R N \right]^{1/2}}{\partial \sigma}$$

Then the overall material continua can be studied numerically based on the formulations that have the Cosserat point continuum as the theoretical foundation. In this aspect, the nonlinear finite elements formulations can be built and solved for the whole continuum. The Cosserat point can be defined as a zero dimensional point surrounded by small but finite region of material (Rubin, 2000).
CHAPTER VI

NUMERICAL EXAMPLE

6.1- GEOMETRY AND BOUNDARY CONDITIONS.

A cubic 3D specimen had been selected for the numerical simulations. The specimen consisting of dry dense sand (one phase material). The finite element mesh consists of 20-noded brick elements. A finite element mesh of 15x30x1 elements is taken in consideration. The dimensions of the specimen are shown in table 6.1.

![Finite element geometry and boundary conditions](image)

Fig.6.1- The finite element geometry and boundary conditions.
RESULTS AND CONCLUSIONS

Table 6.1 Specimen dimensions

| Height (L) | 100 mm |
| Width (B) | 50 mm  |
| Thickness (W) | 10 mm |

A confining pressure of 150 kPa is applied to all boundaries of the specimen and kept constant throughout the analysis. Incremental displacements are applied on a rigid plate at the top of the specimen. The left, right and front planes of the specimen can move freely in the normal direction of each plane. The bottom plane of the specimen is constrained in the y-direction. In order to simulate the real test conditions, interface elements are introduced at the top and the bottom of the sand specimen between the rigid platens and the specimen. By adjusting the bond stiffness $D_{tt}$ of the interfaces, the influence of the roughness of the platen on strain localization within the specimen can be simulated.

The simulation consists in a biaxial test. First an isotropic loading is applied in the specimen and afterwards the deviatoric load is applied in the top of the specimen. The first loading phase is divided in 200 steps and least 0.2 sec (200x0.001) and the second one is dived in 7.8sec (7800x0.001) steps. The duration of the whole test is 8 sec.

6.2-MATERIAL PARAMETERS

During the numerical simulation parameters such as the Young modulus, the Poisson ratio and the constitutive soil model (Desai yield surface) parameters had been kept constant in order to investigate only the effect of the internal length scale (based in the four different length scale formulations which are given in chapter V) and the Cosserat shear modulus $\mu_c$ in the shear band formation. The numerical simulations were performed only in the dry material thus parameters like the permeability or the pore pressure had been neglected.

The constitutive soil model that had been used in the numerical simulation is the Desai yield surface which had been discussed in chapter III and the necessary parameters are given in table 6.2.

Table 6.2-Material properties:

| Young modulus | $E=79.0$ Mpa |
| Poisson ratio | $\nu=0.29$   |
| Constitutive soil model parameters (Desai model) | $n=4.36$  |
| | $a_1=3.28e-03$  |
| | $b_1=-2403.15$  |
| | $c_1=3.28e-03$  |
| | $d_1=0.19e-03$  |
| | $\alpha=4.8e-04$  |
| | $\kappa_c=0.035$  |
| | $P_a=-0.1$Mpa  |
| | $\kappa_1=10.8$  |
| | $\gamma=0.155$  |
| | $R=-0.00001$Mpa  |
6.3 TEST DESCRIPTION.

As it had been discussed before in chapter IV the length scale is a function of several parameters such as the mean grain size the sphericity index ($I_{SPH}$) the roundness index ($I_R$) and the roughness ($R_a$). Based on the fig. 6.3 three different type of particles had been chosen (A, B and C) and the length scale parametric study had been based in the main grain size $d_{50}$. The roughness values will be taken from table 4.3 which is based on the results obtained by Alsaleh (2004).

Some numerical simulations were carried out to investigate the Cosserat shear modulus effect in the shear band formation. The grain parameters will be kept constant and the shear modulus will evolve form a small value which mean that the Cosserat rotations will be not considerable till to the value $\mu_C=0.5G$ which is recommended in the literature (De Borst 1991).

The list of the performed is shown in the table 6.4.

![Fig. 6.3- Particle type selection for the numerical simulations.](image)

In this research four different evolving lows will be used named option 1, option 2, and option 3 and option 4 in the finite element program. The four different evolving formulations are:


-76-
The initial length scale (Alsaleh and Voyiadjis 2004) for this formulation is determined as:

\[
I_o = \frac{I_{SPH} d_{50} + I_R d_{50} + R}{2}
\]  

(1)

The evolution law is a function of the total accumulated effective plastic strain and a constant (material parameter) which had been taken equal to the unit:

\[
I_{Ave} = I_o e^{-k_0 \zeta}
\]  

(2)

**OPTION 2 Liu (2009) length scale formulations.**

The initial length scale is the same as the one used in the previous formulation:

\[
I_o = \frac{I_{SPH} d_{50} + I_R d_{50} + R}{2}
\]  

(1)

The length scale evolution depends only on the deviatoric and rotational effective plastic strain:

\[
I_{Ave} = I_o e^{-k_0 (\zeta_d + \zeta_r)}
\]  

(3)

**OPTION 3 Liu (2009) length scale formulations.**

The initial length scale for the option 3 is calculated in the same way as for the other two options:

\[
I_o = \frac{I_{SPH} d_{50} + I_R d_{50} + R}{2}
\]  

(1)

The length scale has two different components the deviatoric and the rotational length scale:

\[
I_{Ave} = \eta I_d + (1 - \eta) I_R
\]  

(4)

The evolution law for the deviatoric length scale is exponential law as function of deviatoric effective plastic strain:

\[
I_d = I_o e^{-k_0 \tilde{\zeta}_{50}}
\]  

(5)

The evolution law for the rotational length scale is a power law depending on the rotational plastic strains

\[
I_R = I_o \left[ 1 - \left( \frac{\varepsilon_R}{\delta + \tilde{\zeta}_R} \right)^2 \right]
\]  

(6)

Where the constant (material parameter) \( \delta \) is a parameter that controls the evolution rate of the length scale components.

The ratio between the deviatoric strain and the sum of the rotation strains is determining the weight that each of the length scales has in the total one.
\[ \eta = \frac{\varepsilon_D}{\varepsilon_D + \varepsilon_R} \]  

**OPTION 4 Voyiadjis et al. (2005) length scale formulations.**

The option 4 in difference from the previous assumptions calculates two different length scales the contact length scale and the arm of rotation. Both the evolution laws are functions of the rotational displacements.

The initial value for the length of the contact surface is determined as the ratio between the sphericity index \( I_{SPH} \) and the roundness index \( I_R \) multiplied by the mean particle size \( d_{50} \):

\[ l_s = \frac{I_{SPH}}{I_R} d_{50} \sin \left( \alpha^C \right) \]  

(8)

The initial value for the length of rotation arm is the inverse of the contact surface length and is determined as the ratio between the roundness index \( I_R \) and the sphericity index \( I_{SPH} \) multiplied by the mean particle size \( d_{50} \):

\[ l_a = \frac{I_R}{I_{SPH}} d_{50} \cos \left( \alpha^C \right) + R_a \]  

(9)

The total length scale is the summation of both components:

\[ l_{Ave} = l_a + l_s \]  

(10)

The initial length scales for the three different grain types and for the four evolving laws are summarized in table 6.3. The mean particle size \( d_{50} \) varies from \( d_{50}=0.05 \) mm to \( d_{50}=0.72 \) mm.

**Table 6.3- Initial length scale values.**

<table>
<thead>
<tr>
<th>( d_{50} ) (mm)</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.377417</td>
<td>0.09332</td>
<td>0.05473</td>
<td>0.7515</td>
<td>0.171284</td>
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<td>0.10473</td>
<td>1.5015</td>
<td>0.334082</td>
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<tr>
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<td>0.271474</td>
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<td>2.2515</td>
<td>0.496879</td>
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<td>3.0015</td>
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<tr>
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<td>0.25946</td>
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<tr>
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<td>0.538705</td>
<td>0.30473</td>
<td>4.5015</td>
<td>0.985271</td>
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<tr>
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<td>10.8015</td>
<td>2.352768</td>
<td>0.72946</td>
</tr>
</tbody>
</table>

Some of the tests were performed with a constant length scale and afterwards the results were compared with the corresponding results obtained by an evolution formulation. For this test the type A (very angular discoidal) grain particle was selected.

The summary of the simulations performed during this research is given in table 6.4. Some tests were performed with the 20x40 mesh in order to have some insights about the
e mesh dependency of the problem. For this purpose the grain type A and option 1 were selected.

Table 6.4- Numerical simulations summary.

<table>
<thead>
<tr>
<th>μc (MPA)</th>
<th>d_{50} (mm)</th>
<th>Option 1</th>
<th>Option 2</th>
<th>Option 3</th>
<th>Option 4</th>
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<td>0.72</td>
<td>x</td>
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</table>
6.4-RESULTS AND INTERPRETATIONS

Several researchers have investigate the effects of the grain size, density and micro-properties on the thickness and evolution of the shear bands (Alsaleh, 2004; Alshibli and Sture, 1999; Mühlhaus and Vardoulakis, 1987; Abu Al-Rub and Voyiadjis 2004) but, the equations that have been developed until now to quantify the effect of the surface roughness and grain shape on the thickness or/and the inclination of the shear band are still not well formulated. It is clear that the size (in our case the mean particle size $d_{50}$) which is a property that serves to classify particles as gravels, sands, silts and clays still remains the most striking property of a particle.

6.4.1-LENGTH SCALE INVESTIGATION

6.4.1.1 Influence of the initial length scale on the shear band formation and inclination

In this research the initial internal length scale had been calculated depending on the mean particle size ($d_{50}$). Three different types of soil particles were selected (fig.6.3) named as case A, B and C and the initial length scale has been calculated for four different length scale formulations named as option 1, option 2, option 3 and option 4. The initial length scales for grain mean size from 0.05-0.72 mm has been shown in table 6.2 (chapter VI).

The initial length scale for options 1, 2 and 3 were the same for the same grain type and size.

The initial length scales for option 4 for the grain type A and B were slightly higher than the length scales obtained for option 1, 2 and 3. The values are comparable only for the case C (Prismoidal well-rounded particle). In figure 6.4 the results obtained for case A for four different mean grain sizes and the four different evolving length formulations has been shown. The shear bands obtained for the smallest grain size (smallest length scale) have the highest inclination angle and for the highest grain size the inclination angle becomes zero. Thus the shear band inclination depends on the grain size and, while the grain size increases, the shear band angle decreases.

Another feature of the shear bands mentioned in the literature is the shear band thickness. The experimental results (Mokni and Desrues, 1998; Alshibli and Sture, 1999; Vardoulakis, 1978) had shown that this feature depends on the grain size. Several authors have tried to define quantitatively the shear band thickens as a function of the main grain size (Alshibli and Sture, 1999; Mühlhaus and Vardoulakis, 1987).

In accordance with the indications obtained from the literature, when we compare the results of fig.6.4 it is clear that the thickest shear bands has been obtained for the $d_{50}=0.3$ mm for all the evolving formulations and the thinnest were obtained for $d_{50}=0.05$ mm. Another important conclusion is related to the comparison of the shear bands with the same main grain size but with a different initial length scale. Thus the shear bands for the options 1, 2 and 3 which have the same initial length scale do not have a significant difference in their thickness. On the contrary the shear band obtained by option four is slightly thicker. This can be explained by the fact that the initial length scale used for option four is higher than the initial length scale used for the other options with the same mean grain size ($d_{50}$).
RESULTS AND CONCLUSIONS

Fig. 6.4 Shear band formation for case A (very angular discoidal particles)

1) \(d_{50}=0.05 \text{ mm}\)  
2) \(d_{50}=0.1 \text{ mm}\)  
3) \(d_{50}=0.2 \text{ mm}\)  
4) \(d_{50}=0.3 \text{ mm}\)

Options:

- a) option 1
- b) option 2
- c) option 3
- d) option 4
Another observation is the dependency of the specimen stiffness on the main grain size. During the shear band formation it has been observed that when the grain size increases the rotational stiffness of the material also increases which is a logical conclusion because the terms of the elastic stiffness matrix $D_e$ which relate the coupled stress and the micro-curvature are a function of the corresponding length scale. The rotations seem to develop in the same locations as the shear bands and the maximum location of the rotations correspond to the maximum total effective strain. Rotations had been detected as one of the factors causing the plastic strains initialisation which are necessary to develop the shear band (Fig. 6.14).

6.4.1.2 Shear band development by using constant length scale and evolving length scale

During numerical simulations it is easier to assign a constant value to the length scale. Thus the material stiffness remains the same throughout the whole simulation. Unfortunately this approach is unable to simulate the variation in the material strength which depends not only on the grain size but also in the plastic strains. Thus it is questionable if a fixed value of the material length-scale is able to describe the size effect for different type of problems.

Some numerical simulations were performed by using a constant length scale. In order to compare the results the initial length scales values which had been used in the numerical simulation belonging to the particle type A (fig. 6.3) and to option 1 have been used. The Cosserat shear modulus has been taken equal to the half of the shear modulus $\mu_c=15\text{MPa}=0.5G$

When we compare the results obtained by means of the evolving formulation and the results obtained for a constant length scale it can be observed that the sample shows a stiffer response in the case of the constant length scale (Fig. 6.5).

If we take in consideration the shear bands in fig. 6.5 a$_1$ and fig. 6.5 b$_1$ it is observed that the shear band shape is the same and the difference consists in the shear band thickness. The shear band obtained for the evolving length scale is thicker than the shear band obtained for a constant length scale. Thicker shear bands indicate less shear strength. Thus by using an evolution low it is possible to obtain a softer material response under the same loading conditions due to the variation in the length scale (contact length scale and the arm of the rotation). Physically the phenomena can be explained by the continuous rotations and the rearrangement of the adjacent particles toward each other within the sample. This type of behaviour is more close to the real soil behaviour.

6.4.1.3 The influence of evolving length scale on shear band thickness and shear band inclination

While comparing the shear bands in fig. 6.5 a$_4$ and fig. 6.5 b$_4$ two important conclusions have been found:

- The inclination angle by means of a constant length scale is smaller than the inclination angle when an evolving length scale formulation had been utilized.
- The shear band obtained for the constant length scale is thicker than the shear band obtained by means of an evolution low.
RESULTS AND CONCLUSIONS

The shear band inclination with a zero angle is unrealistic which emphasizes more why an evolving (decay) length scale should be utilized. The material shows an unrealistic stiffness and unrealistic shear band inclination. The results obtained by means of the evolving length scale are more reasonable.

6.4.1.4 The effects of different evolution equations on shear band formation.

Another important factor that is influencing the shear band formation and its shape is the type of the evolving formulation used in the numerical simulation. When we compare the results on fig.6.4 the conclusion is that also results obtained by using an evolving length scale can be unrealistic like in the fig 6.4 b1 or fig. 6.4 d3 that means that is important to incorporate the proper evolving length scale in the model. The evolution law determines the decay rate of the length scale. Mathematically an exponential law tends to reach asymptotically the zero value. The evolving formulations used in the options 1, 2 and 3 belong to this type of mathematical expressions. Thus the
length scale tends to decrease while the plastic strain increases. The rate of decay depends on the amount of the plastic strains.

From the results in fig.6.4 option 2 had length scale decay smaller than option 1 and 3. In this option only two components of the plastic strains appear in the evolution law the rotation and the deviatoric one, while in option 1 the total plastic strain has been considered. As a result the decay rate is smaller than the decay rate obtained by using option 1.

Option 3 combines two different length scales one decaying exponentially and the other one decays based on a power law. The decay obtained is higher than the one obtained with option 2 even through the same strain quantities are taken in account ($\xi_D$ and $\xi_R$).

Fig.6.6- Total plastic effective strain and rotations $\omega_z$ for different particles main size

From the results in fig.6.4 option 2 had length scale decay smaller than option 1 and 3. In this option only two components of the plastic strains appear in the evolution law the rotation and the deviatoric one, while in option 1 the total plastic strain has been considered. As a result the decay rate is smaller than the decay rate obtained by using option 1.

Option 3 combines two different length scales one decaying exponentially and the other one decays based on a power law. The decay obtained is higher than the one obtained with option 2 even through the same strain quantities are taken in account ($\xi_D$ and $\xi_R$).
RESULTS AND CONCLUSIONS

Thus for different evolution formulations, different decay rates can be obtained. It is important to find the proper formulation which brings the numerical results close to the experimental data.

A parameter that had been not discussed in this research is the material parameter $k_0$ (here it had been taken equal to the unit). This parameter can control the decay rate if other values than unit will be used.

6.4.1.5 The effect of mesh refinement.

Numerical models show a pathological mesh dependency when they are used in problems involving strain softening. In order to deal with the strain non-uniformities and strain softening a proper solution is to use additional degrees of freedom or high order gradient terms.

Three different geometries were selected initially for the numerical simulations the 20X40 elements and 15X30 elements and the 10x20 specimen. Some of the numerical simulations were carried out with all the geometries. The same displacements were applied at the top of the specimen for all three geometries. The load- displacement diagram (Fig. 6.7) obtained for the case when the grain type A, the evolving formulation Option 1 and the main particle size $d_{50}=0.05\text{mm}$ were used shows that the solution obtained is unique. The shear band (plastic effective strains) and the deformed mesh obtained from these tests are shown in Fig.6.8.

The results obtained with the 20X40 element mesh and 15X30 for the grain type A (fig.6.3) and evolution low 1 were compared. It was observed that the results for the 15X30 mesh was comparable with the finest mesh 20X40 (Fig.6.9) thus the whole simulations were carried out with that 15X30 geometry due to the fact that the simulation time needed was 4 times less than for the finest mesh 20X40.

The shear bands obtained by using the 15x30 mesh and 20x40 are given in fig.6.9.
6.4.1.6 Stress and strain investigation in the shear band.

From the investigation of the stress and strain path at the Gauss point level it had been observed that the shear band formation starts in the softening regime which is in full agreement with the observations obtained from the literature which are stating that: During the hardening regime, granular materials behave as a continuum until the failure or the instability point where deformations begin to localize into a small but finite shear zone, which is termed as shear band (Voyiadjis and Song, 2006).

In fig.6.10 the shear band and the deformed mesh together with some element locations are shown. We have taken in consideration three elements and only one of them is completely outside the shear band influence (element 111). From the $q$-$\varepsilon_{yy}$ graph (Fig.6.11a) we can see that the stress strain path shows almost an elastic unloading. From the $\varepsilon_{yy}$-$\varepsilon_{vol}$ graph (Fig.6.11b) we see that the element 111 shows no dilatancy and the plastic strain values are quite zero.
RESULTS AND CONCLUSIONS

The Gauss point in element 407 shows the highest value of vertical plastic strains along loading direction and very small value of volumetric strains which can be observed from the deformed mesh. Due to the effects of the boundary conditions the specimen doesn’t show any expansion in that location.

Element 250 lies on the main shear band show distinctly different behaviour. This element has experienced a strong dilative expansion which can be seen also from the deformed mesh. From the $e_{yy}$-$e_{vol}$ graph we can see that the element first is under compression and afterwards starts to dilate. The axial strain is smaller then the one in element 407 but the softening effect is higher in element 250 even through the vertical strains are smaller in that location. The decrease in strength of the material is higher in the element 250 than in element 407 which means that the material inside the shear band is much softer.

Fig. 6.9 – Shear bands obtained by means of two mesh refinement for different main grain size.

The Gauss point in element 407 shows the highest value of vertical plastic strains along loading direction and very small value of volumetric strains which can be observed from the deformed mesh. Due to the effects of the boundary conditions the specimen doesn’t show any expansion in that location.

Element 250 lies on the main shear band show distinctly different behaviour. This element has experienced a strong dilative expansion which can be seen also from the deformed mesh. From the $e_{yy}$-$e_{vol}$ graph we can see that the element first is under compression and afterwards starts to dilate. The axial strain is smaller then the one in element 407 but the softening effect is higher in element 250 even through the vertical strains are smaller in that location. The decrease in strength of the material is higher in the element 250 than in element 407 which means that the material inside the shear band is much softer.
Plastic strains and Cosserat rotations reach the highest value at the centre of shear band while the rotation curvature and the couple stresses at failure will vanish outside the shear band and they switch their direction at the centre of the shear band (Fig. 6.12). This phenomenon has a physical meaning in the sense that the centreline of the shear band can be considered as a slip surface where the particles will be sheared with high shearing loads and therefore their rotational gradient and couple stresses will flip from clockwise to counter.

It had been observed from the investigation on the shear band formation and its evolution in time that both the plastic effective strains and the rotations develop parallel to each other in time and space (Fig. 6.14). Rotations are one of the initializing causes for the shear band formation and they appear before the plastic strains.

It is observed in all the tests that the shear band shape and distribution coincide with the shape of the rotation band and the distribution. Thus the location of the highest strain value coincides with location of the highest rotation. This phenomenon is shown in fig. 6.41 where the evolution in time of the plastic effective strain, the rotations \( \omega_z \) and the specimen deformation had been plotted for different time sequences from the starting of the shear band formation until the end of the simulation.

Fig. 6.10- Effective plastic strains and deformed mesh visualisation for case A option 1 and \( d_{\text{so}}=0.05 \text{ mm} \)
6.4.2-COSSERAT SHEAR MODULUS INVESTIGATION.

The Cosserat shear modulus appears in the elastic stiffness tensor in the terms related to shear stress and strain components as a term which is added or subtracted to the normal shear modulus. The elastic stiffness matrix components in detailed way are given in

Fig.6.11-Stress and strain relation for 5 different elements for case A option 1 and d_{50}=0.05 mm

a) deviatoric stress versus axial strain

b) volumetric strain versus axial strain
appendix 1. In order to investigate the influence of the shear modulus on the shear band formation (effective plastic strains and rotations) a parametric study of $\mu_c$ had been carried out. For this reason different values were used for the $\mu_c$ starting from 1MPa till 15MPa which in our case is half of the shear modulus which is a value recommended from the literature (De Borst 1991).

It has been observed that for small mean grain size ($d_{50}$) that the variation of the Cosserat shear modulus values doesn’t influence significantly the behaviour of the material. The results in fig. 6.12 had been obtained for a small grain size and the plots of strains and rotations are the same for different Cosserat shear modulus. The same phenomena were observed for high values of the length scale. For an intermediate value of the length scale the effects of the shear modulus were visible (Fig.6.13).

The shear strains in Cosserat continua have two components the first part which is function of the translative displacements and the second component which is the rotation angle. If this rotation angle is very small (negligible) the strains assume the classical formulation and the Cosserat effect on these terms vanishes. The strain tensor becomes symmetric and the influence of the Cosserat shear modulus vanishes because we add and subtract the same quantity to determine the stress. This phenomenon (small rotations) has been observed for higher initial internal length scales.
RESULTS AND CONCLUSIONS

Fig. 6.13 - Plastic effective strains and rotations (grain type C and option 4).

Effective plastic strain:
- $\mu_c = 1$ MPa
- $\mu_c = 15$ MPa

Rotations $\omega_z$
- $\mu_c = 1$ MPa
- $\mu_c = 15$ MPa

a) $d_{50} = 0.05$ mm  b) $d_{50} = 0.1$ mm  c) $d_{50} = 0.2$ mm

$L_0 = 0.7515$ mm  $L_0 = 1.5015$ mm  $L_0 = 3.015$ mm

Fig. 6.13- Plastic effective strains and rotations (grain type C and option 4).

-91-
Similarly if the characteristic length \(d_{50}\) becomes infinitely small, the micro-polar model reduces to the non-polar one since the effect of the micro-polar quantities disappears. For the intermediate values, the shear stress components effect (\(\mu c\)) is visible. Thus the reduction in stiffness by decreasing the Cosserat shear modulus can be clearly observed in the numerical results. The material will become softer for a small Cosserat shear modulus and it will become stiffer for a higher value of this parameter. If the Cosserat shear modulus becomes infinitely small the stress tensor will become symmetric and the micropolar effect will disappear. Thus by increasing the Cosserat shear modulus the micro-polar effect (rotations) influence in the shear stress components will also increase. In this research it was not possible to fully cover the effect of the Cosserat shear modulus in the shear band formation thus further investigations should be done related to this effect in the future.
Fig. 6.14 - Shear band evolution in time a) total effective plastic strains b) rotations $\omega$, c) deformed mesh.
CHAPTER VII

EPILOGUE

7.1- CONCLUSION

The mechanical behaviour of granular materials under high plastic deformation is quite complex due to the fact that the granular assembly consists of discrete or discontinuous non-uniform particles.

Each of the particles has its own mechanical behaviour, its micro-properties thus the soil mass behaviour is complex and in some cases the classical continuum theory fails to describe the real behaviour because granular materials behave as a continuum until failure where deformations begin to localize into a small but finite shear zone, which is called shear band.

In order to deal with the strain non-uniformities and strain softening a proper solution is to use additional degrees of freedom or high order gradient terms.

Cosserat theory is one of the best classical micro-polar theories: it is quite realistic and it has a strong physical background due to the fact that it is able to separate the micro-rotation of a material point from the overall rotation of the continuum. The micro-polar continuum is considered as a continuous group of particles behaving like rigid bodies. The theory combines two kinds of deformations at two different levels, micro-rotation at the particle level and macro-deformation at the structural level.

If the particle can be considered as a Cosserat point we can include in the model internal length scale which is a parameter that characterizes the particle micro-properties.

A widely used length scale parameter is the mean particle size which links the micro-curvature of rotation to the couple stress.

Parameters such as shape, surface roughness, and gradation of particles influence a lot strength and deformation properties of granular materials. The shape indices, \( I_R \) (roundness index) and \( I_{sp} \) (sphericity index), have been used to account for the shape non-uniformity.

The surface roughness and shape indices of granular materials, to some extent, are difficult to quantify. Visual comparison charts consisting of sets of grain images with known roundness are often used to make rapid visual estimation of grain roundness. Based on these parameters the initial length scales are calculated.

During numerical simulations it is easier to assign a constant value to the length scale. Thus the material stiffness remains the same thorough the whole simulation. Unfortunately this approach is unable to simulate the variation in the material strength which depends not only in the grain size but also in the plastic strains.

Strong localized material dilatancy due to grain rearrangement and grain rotation are the dominant features of shear banding (Arslan and Sture 2008). It is questionable if a fixed value of the material length-scale is able to fully describe this phenomenon and to give results...
comparable with the experimental results. For these reason different mathematical expressions were used to have an evolving length scale during the numerical simulation. The conclusions of this research can be summarized:

- The initial length scale value is an important factor that controls the material behaviour. Thus the thickness and the inclination of the shear band are strongly depending on the value of the initial length scale. It is important to find the proper value of the initial length scale in order to obtain reasonable soil response.
- The evolving formulation of the length scale controls the decay of the length scale and is responsible for the material response and for shear band formation. By controlling the rate of decay it is possible to control the material stiffness and the shear band properties such the inclination or the thickness. Different rates of decay can be obtained by using different evolution formulations.
- When the length scale tends to zero the problem from the micro-polar model reduces to the non-polar one since the effect of micro-polar quantities disappears. If this rotation angles are very small (negligible) the strains assume the classical formulation and the Cosserat effect on these terms vanishes.
- The Cosserat shear modulus influences the behaviour of the material. The material will become softer for a small Cosserat shear modulus and it will become stiffer for a higher value of this parameter.
- The numerical simulations show that the problem becomes mesh independent when Cosserat formulation is used which ensures the uniqueness of the numerical solution.

7.2- RECOMMENDATIONS.

- In this research only one type of initial length scale was used assuming that the length of the arm rotation(torsion) and the contact length scale (bending) have the same numerical value which is not always true because those two parameters are strongly related to the particle shape and roundness thus often they have different values depending on the grain parameters which had been explained in details in chapter IV. Those two length scales also represent two different physical quantities because the arm rotation is associated to the torsion moments and the contact length scale is related to the bending moments. Due to the fact that the investigation of the length scale evaluation is a complex problem some simplifications have been done in this research in order to have results which could be interpreted. Thus one of the main recommendations for the future is to investigate the evolving internal length scale localisation not only by using one length scale but by using the two different one which have a different physical meaning.
- Another issue is the determination of the initial length scale based on the grain parameters such as the Sphericity index (I_{SPH}), Roughness (R) or roundness index (I_{R}). Further investigations are needed in order to obtain initial length scale values which will give results close to the real soil behaviour.
- Another weakness of this research is the fact that we assume homogenous grain size distribution and homogenous grain shape by assuming the same initial length scale which is not the case in practice where the soil fabric is quite complex and heterogeneous. This weakness is also related to the fact that the initially micro-polar theory which we are using have been developed for metals where the
materials have uniform grain size (crystals) distribution and the same poly-crystal shape.

During the literature study it was found out that related to one of the parameters of micro-polar theory the Cosserat shear modulus has been underestimated because there are not so clear explanations related to physical meaning and the values used for this parameter are taken empirically. Thus further investigation is needed to study the effect of this parameter in micro-polar continua.

During this research the specimen dimensions were kept constant thus it was not possible to investigate the influence of the specimen on the shear band formation. Thus this is an issue which has to be investigated in the future.
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APPENDICES
APPENDIX 1
Elasticity stiffness tensor for 3D micro-polar media

\[ D^* = \begin{bmatrix}
2\mu c_1 & 2\mu c_2 & 2\mu c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu c_2 & 2\mu c_1 & 2\mu c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu c_3 & 2\mu c_2 & 2\mu c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu + \mu_c & \mu - \mu_c & \mu - \mu_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu - \mu_c & \mu + \mu_c & \mu + \mu_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu + \mu_c & \mu - \mu_c & \mu - \mu_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu - \mu_c & \mu + \mu_c & \mu + \mu_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu d^2_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu d^2_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu d^2_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu d^2_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu d^2_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu d^2_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu d^2_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\mu d^2_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]
Correction matrix which takes in account only the symmetric part of the stress tensor and it has dependency on the length scales.

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Correction matrix which takes in account the micro-polar effect on the total effective strain tensor.

\[
R = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& & & & & & & & & & & & & & & & \\
\end{bmatrix}
\]