Streamline curvature and bed resistance in shallow water flow

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Internal report no. 1-79

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SUMMARY

The relationship between streamline curvature and bed resistance in shallow water flow with little side constraint, as derived in 1970 by H.J. Schoemaker, is reconsidered. Schoemaker concluded that the bed resistance causes the curvature of a free streamline to grow exponentially with the distance along this streamline, thus giving rise to a destabilizing tendency. The present analysis shows the bed shear stress to act in a stabilizing way and, as far as it is possible to isolate the influence of the bed resistance on the development of streamline curvature, it is shown to be a damping one.
In addition, the applicability of the shallow water equations to the scaling of curved alluvial river models is discussed. It is suggested to introduce additional terms into the stream-wise momentum equation, accounting for the advective influence of the secondary flow.

LIST OF SYMBOLS

a	scalar
A	cross-sectional area of the stream tube considered
B	channel width
C	Chezy's factor
f	scalar function
F	stream function of the secondary flow
f_1	damping factor in the vorticity transport equation
f_2	source term in the vorticity transport equation
g	acceleration due to gravity
h	depth of flow
I_n	transverse slope of the piezometric head
I_s	streamwise slope of the piezometric head
l_1	streamwise length function in the streamline coordinate system
l_2	normal length function in the streamline coordinate system
L_ω	characteristic length of vorticity variations
n	distance along a normal line in the streamline coordinate system
p	pressure
Q	total discharge
r	local radius of curvature of the streamlines (according to H.J. Schoemaker)
R_n	local radius of curvature of the normal lines
R_s	local radius of curvature of the streamlines
R_ξ, R_η	radius of curvature of a generalized curvilinear coordinate axis
s	distance along a streamline
$\bar{v}$	total depth-averaged velocity
v_n	normal velocity-component
v_s	streamwise velocity-component
v_x, v_y	horizontal velocity-components in a cartesian coordinate system
v_z	vertical velocity component
v_ξ , v_η	horizontal velocity-components in a generalized curvilinear coordinate system
x,y	horizontal coordinates in a cartesian system

z	vertical coordinate
z_b	bed elevation
z_s	free surface elevation
$\alpha = \overline{v^2}/\bar{v}^2$	
α_2	advection factor
α_3	bed shear stress factor
η	distance along an axis of a generalized curvilinear coordinate system
κ	Von Karman's constant
ξ	distance along an axis of a generalized curvilinear coordinate system
ρ	mass density of the fluid
τ	horizontal turbulent shear stress
τ_{xy}, τ_{yz}	component of the horizontal turbulent shear stress
ϕ	normal coordinate in the streamline coordinate system
ψ	streamwise coordinate in the streamline coordinate system
ω	vorticity of the depth-averaged flow

CONTENTS

1. Introduction

An important class of problems in hydraulic engineering concerns steady or gradually varying, nearly horizontal shear flow with a horizontal length-scale that is much larger than the depth of flow (shallow rivers, lakes, estuaries and seas).
Physical scalemodels are often used to investigate this kind of problems and in such models a correct representation of the horizontal flow field is very important. This implies i.a. that the modelling of the bed resistance, which influences the streamline curvature (SCHOEMAKER 1970), must be adequate, i.e. some relationship between the roughness-scale and the length-scales must be satisfied.
SCHOEMAKER (1970) derived a relationship between streamline curvature and bed resistance in shallow water flow with little side constraint. This relationship forms the basis for the scale-law that has to be satisfied in order to attain a correct modelling of the flow in this respect.
It follows directly from SCHOEMAKER's relationship that, going along a streamline of the horizontal flow field, the bed shear stress causes an increase of the streamline curvature, i.e. the bed shear stress makes the streamlines curl in the horizontal plane, so it has a destabilizing effect on the flow. As this is in contrast with the usually damping and stabilizing character of the bed shear stress in similar situations (cf. the damping of curvature effects beyond a shallow river bend (ROZOVSKII, 1961; DE VRIEND, 1978b)), SCHOEMAKER's relationship was subject to a closer investigation.
Most of the study underlying this report was carried out in 1976. The results were not published, however, until they appeared to be used for the derivation of scale laws for alluvial river models (DELFT HYDRAULICS LABORATORY, 1978). This gave rise to an additional discussion on the applicability of the present results for this purpose, especially if bends are included (see chapter 4).

2. Formulation of the problem

As the influence of the bed resistance on streamline curvature in shallow water flow is considered, it will be attempted to formulate the problem in such a way, that all other phenomena influencing streamline curvature (adjacent lateral boundaries or bottom discontinuities, secondary flow induced by curvature, et cetera) are absent or of negligible importance. Therefore considerations are limited to steady shallow shear flow of mild curvature over a horizontal or gently sloping bottom, far from obstacles or lateral boundaries, i.e. with little side constraint.
In a cartesian coordinate system (x,y,z) with vertical z-axis, this type of flow can be described by the following set of differential equations, representing the conservation of mass and momentum:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \qquad (1)$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z} \qquad (2)$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial \tau_{yz}}{\partial z} \qquad (3)$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \qquad (4)$$

, where v_x, v_y, v_z = turbulence-averaged velocity components in x-, y- and z-direction, respectively,

τ_{xz}, τ_{yz} = shear stress components (including the Reynolds stress)

p = pressure

ρ = mass density of the fluid

g = acceleration due to gravity

Only the boundary conditions at the bottom and at the free surface are relevant to the present problem. These conditions read

$$v_x,\ v_y,\ v_z = 0 \quad \text{at the bottom } (z = z_b) \qquad (5)$$

$$\tau_{xz},\ \tau_{yz},\ p = 0 \text{ and } v_z = v_x \frac{\partial z_s}{\partial x} + v_y \frac{\partial z_s}{\partial y} \quad \text{at the surface } (z = z_s) \qquad (6)$$

Equation (4) and the pressure condition at the surface yield the hydrostatic pressure distribution

$$p = \rho g(z_s - z) \tag{7}$$

The differential equations (1) through (3) can be integrated over the depth of flow. Making use of (5) and (6), this leads to :

$$\frac{\partial}{\partial x} (\bar{v}_x h) + \frac{\partial}{\partial y} (\bar{v}_y h) = 0 \tag{8}$$

$$\frac{\partial}{\partial x} (\overline{v_x^2} h) + \frac{\partial}{\partial y} (\overline{v_x v_y} h) = -gh \frac{\partial z_s}{\partial x} - \left.\frac{\tau_{xz}}{\rho}\right|_{z_b} \tag{9}$$

$$\frac{\partial}{\partial x} (\overline{v_x v_y} h) + \frac{\partial}{\partial y} (\overline{v_y^2} h) = -gh \frac{\partial z_s}{\partial y} - \left.\frac{\tau_{yz}}{\rho}\right|_{z_b} \tag{10}$$

in which the overbar denotes the depth-averaged value and $h = z_s - z_b$ is the depth of flow.

If the vertical distribution of the velocity is assumed to be similar throughout the flow field, the depth-averaged velocity products can be written as

$$\overline{v_x^2} = \alpha \bar{v}_x^2 \; ; \quad \overline{v_x v_y} = \alpha \bar{v}_x \bar{v}_y ; \quad \overline{v_y^2} = \alpha \bar{v}_y^2 \tag{11}$$

in which α is a constant. If the velocity distribution is logarithmic,

$$\alpha = 1 + \frac{g}{\kappa^2 C^2} \tag{12}$$

in which κ = Von Karman's constant

C = Chezy's factor

Usually, the second term is small with respect to unity, so that it can be neglected. This is consistent with neglecting the effects of the vertical redistribution of the velocity due to streamwise accelerations and of secondary flow induced by curvature. If these phenomena are included, the factor α becomes (DE VRIEND, 1976)

$$\alpha = 1 + 3 \frac{g}{\kappa^2 C^2} - 2 \frac{g\sqrt{g}}{\kappa^3 C^3} \tag{13}$$

The components of the bed shear stress are assumed to correspond with Chezy's law for uniform shear flow, so that the depth-integrated momentum equations become

$$\frac{\partial}{\partial x}(\bar{v}_x^2 h) + \frac{\partial}{\partial y}(\bar{v}_x \bar{v}_y h) = -gh\frac{\partial z_s}{\partial x} - \frac{g}{C^2}\bar{v}_x(\bar{v}_x^2 + \bar{v}_y^2)^{1/2} \qquad (14)$$

$$\frac{\partial}{\partial x}(\bar{v}_x \bar{v}_y h) + \frac{\partial}{\partial y}(\bar{v}_y^2 h) = -gh\frac{\partial z_s}{\partial y} - \frac{g}{C^2}\bar{v}_y(\bar{v}_x^2 + \bar{v}_y^2)^{1/2} \qquad (15)$$

These are rather complicated equations in which the streamline curvature does not figure explicitly, so that it will be a fairly complicated task to derive a relationship between this curvature and the bed resistance from these equations. Therefore the depth-averaged conservation laws (8), (14) and (15) will be transformed to a stream-oriented coordinate system with one vertical axis and two horizontal axes tangent to the streamlines and the normal lines in every point of the flow.
If s denotes the distance along a streamline and n is the distance along a normal line, this (curvilinear) coordinate system can be indicated by (s,n,z)*). Transformation of equations (8), (14) and (15) to this coordinate system yields (see Appendix I)

$$\frac{\partial(\bar{v}h)}{\partial s} + \frac{\bar{v}h}{R_n} = 0 \qquad (16)$$

$$\bar{v}h\frac{\partial\bar{v}}{\partial s} = -gh\frac{\partial z_s}{\partial s} - \frac{g}{C^2}\bar{v}^2 \qquad (17)$$

$$-\frac{\bar{v}^2 h}{R_s} = -gh\frac{\partial z_s}{\partial n} \qquad (18)$$

where $\bar{v}$ = total depth-averaged velocity

$1/R_s$ = local streamline curvature (positive when the normal lines diverge)

$1/R_n$ = local curvature of the normal lines (positive when the streamlines diverge)

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* This is a symbolic notation, s and n not being the actual coordinates of the system (s is not necessarily constant along a normal line and n may vary along a streamline).

3. Variation of flow curvature along a streamline

In order to derive a relationship between streamline curvature and bed resistance, the water surface z_s must be eliminated from the momentum equations (17) and (18). This can be done using the following rule, that is derived mathematically in Appendix B and physically in Appendix C:

$$\frac{\partial}{\partial n}\left(\frac{\partial z_s}{\partial s}\right) + \frac{1}{R_s}\frac{\partial z_s}{\partial s} - \frac{\partial}{\partial s}\left(\frac{\partial z_s}{\partial n}\right) - \frac{1}{R_n}\frac{\partial z_s}{\partial n} = 0 \tag{19}$$

This rule is a representation of the vector identity

$$\vec{\text{curl}}\ (\vec{\text{grad}}\ a) \equiv 0 \tag{20}$$

holding for any scalar a (KUIPERS EN TIMMAN, 1966). When applied to z_s, the physical interpretation of this identity is that, following a closed path over the water surface and observing the local water level, this level will not have changed when returning in the point of departure, whatever path has been followed.
Equations (17) and (18) can be rewritten as expressions for the components of the free surface slope

$$\frac{\partial z_s}{\partial s} = -\frac{\bar{v}}{g}\frac{\partial \bar{v}}{\partial s} - \frac{\bar{v}^2}{C^2 h} \tag{21}$$

$$\frac{\partial z_s}{\partial n} = \frac{\bar{v}^2}{gR_s} \tag{22}$$

Substituting these expressions into (19) yields

$$\frac{\partial}{\partial s}\left(\frac{1}{R_s}\right) = -\frac{1}{\bar{v}^2}\frac{\partial}{\partial n}\left(\bar{v}\frac{\partial \bar{v}}{\partial s}\right) - \frac{3}{R_s}\frac{1}{\bar{v}}\frac{\partial \bar{v}}{\partial s} - \frac{1}{R_s R_n} - \frac{g}{C^2 h}\left\{\frac{1}{R_s} + \frac{2}{\bar{v}}\frac{\partial \bar{v}}{\partial n} - \frac{1}{h}\frac{\partial h}{\partial n}\right\} \tag{23}$$

Adopting the same simplifying assumptions as SCHOEMAKER (1970), viz.

. all velocity gradients are negligible
. the divergence of the streamlines (and hence the curvature of the normal lines) are negligible
. the normal gradient of h in (23) is negligible

, equation (23) reduces to

$$\frac{\partial}{\partial s}\left(\frac{1}{R_s}\right) = -\frac{g}{C^2 h}\frac{1}{R_s} \qquad (24)$$

(see also Appendix C), whereas SCHOEMAKER, probably as a consequence of a sign error, found

$$\frac{\partial R_s}{\partial s} = -\frac{g}{C^2 h} R_s \qquad (25)$$

, which is equivalent to

$$\frac{\partial}{\partial s}\left(\frac{1}{R_s}\right) = +\frac{g}{C^2 h}\frac{1}{R_s} \qquad (26)$$

Expressions (24) and (26) are contradictory, the former stating that the bed resistance straightens the streamlines and the latter stating that the bed resistance makes the streamlines curl.
Equation (26) may be expected to be wrong on the basis of the physical argument, that the bed shear stress uses to stabilize the depth-averaged flow field. In a straight section after a river bend, for example, the asymmetric distribution of the depth-averaged velocity caused by the bend is damped by the bed shear stress (ROZOVISKII, 1961); DE VRIEND, 1978b).
Finally it should be noted that the scale-laws following from equations (24) and (26) are identical, so that the incompatibility of these two equations has no repercussions for the physical modelling of shallow water flow with little side constraint.

4. Discussion

4.1. On the damping effect of bed resistance

The physical consideration that friction usually has a damping and stabilizing effect corroborates the results of the present mathematical analysis. As an illustration of this damping influence, an example concerning shallow water flow is given here.
Consider a long straight section of a shallow rectangular channel, with a depth of flow h (say 1.00 m) at the downstream end and a constant channel width B (say 20.00 m), with slipping sidewalls and a longitudinal slope given by

$$\frac{\partial z_b}{\partial x} = - \frac{Q^2}{C^2 B^2 h^3} \tag{27}$$

, Q denoting the total discharge. If the slope is chosen this way, no backwater effects will occur in the fully developed stage of steady turbulent flow satisfying Chezy's law. Now the transverse distribution of the inflow velocity is taken asymmetric with respect to the channel axis and the streamwise variation of the flow is investigated using a mathematical model based on equations (8), (14) and (15), (see DE VRIEND, 1976 and 1977)[*].
The results of the computations, carried out both for a smooth bed ($C = 70\ m^{\frac{1}{2}}/s$) and for a rough bed ($C = 30\ m^{\frac{1}{2}}/s$), are shown in figure 1. The main velocity distribution turns out to adapt smoothly to its fully developed configuration and the streamline curvature appears to damp out exponentially, without any tendency to become unstable. In addition, the adaptation to fully developed flow and the damping of the streamline curvature occur over a longer distance in case of a smooth bed, as was to be expected on the basis of equation (24).

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*) This implies that secondary flow, either due to streamline curvature or due to turbulence in the sidewall regions, is neglected and that the vertical distribution of the main velocity is assumed to be similar (for instance: logarithmic) throughout the flow field.

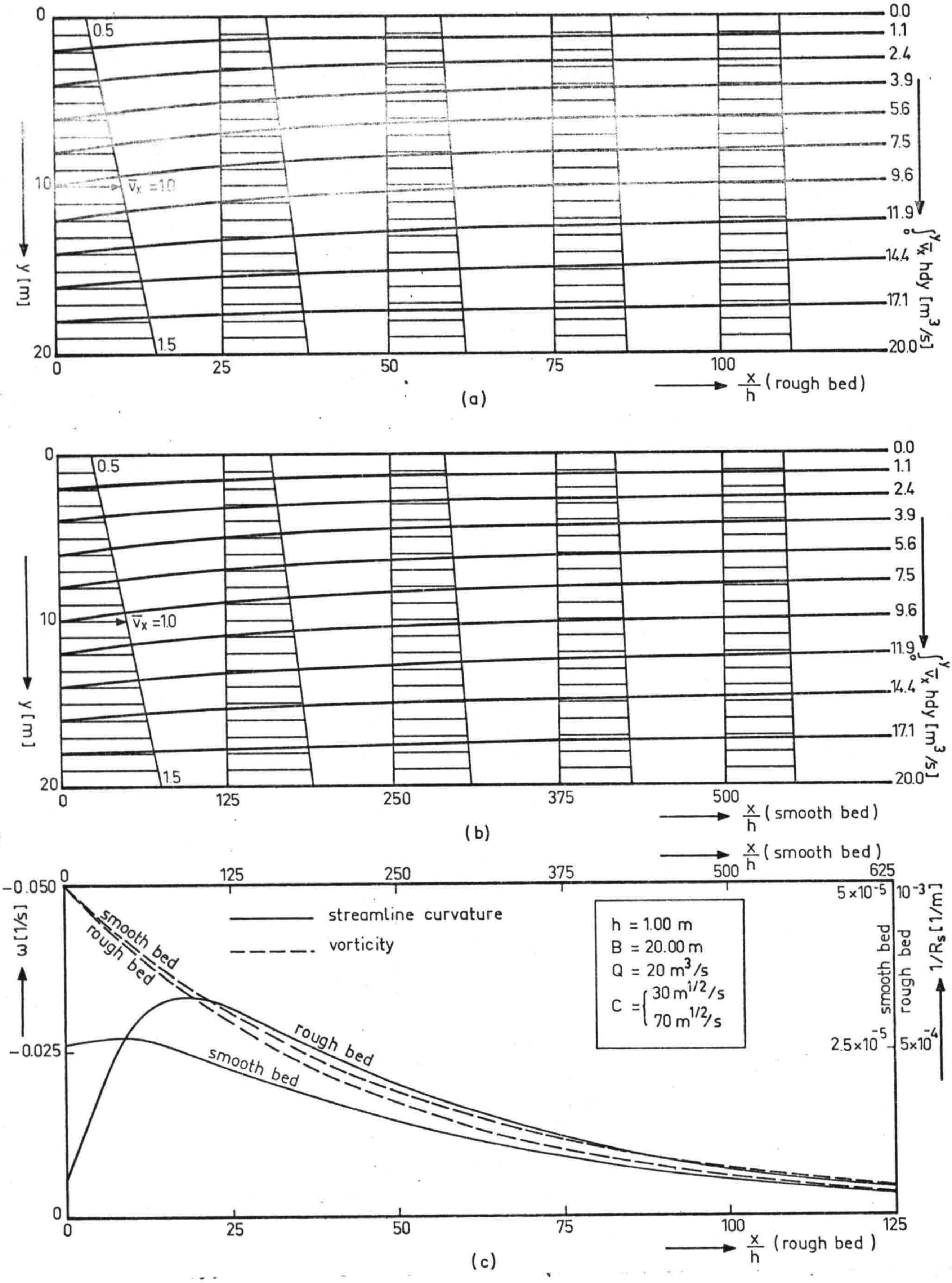

Figure 1. Damping influence of bed resistance in a straight channel with non-uniform inflow.

(a) Velocity distributions and streamline pattern (rough bed)

(b) Velocity distributions and streamline pattern (smooth bed)

(c) Damping of vorticity and streamline curvature

4.2. Admissibility of the simplifications

In order to arrive at equation (24), two important groups of simplifications had to be made, viz.:

- those leading to the depth-averaged equations for shallow water flow (equations (8), (14) and (15) or equation (16) through (18)),
- the simplifications needed to reduce equation (23) to equation (24).

The most important simplifications belonging to the first group are the direct correlation between the bed shear stress and the depth-averaged velocity*) through Chezy's law and the neglect of the influence of the secondary flow due to streamline curvature.
DE VRIEND (1978 a, 1978 b) showed that if the streamline curvature remains over a sufficiently long distance, the secondary flow influences the vertical and the horizontal distribution of the main flow. The effect on the vertical distribution establishes over a rather short distance after the beginning of curvature and it works out in an increase of the bed shear stress with respect to the values according to Chezy's law. The effect on the horizontal distribution of the main flow needs a much longer distance to establish, especially if the convex boundary is far away. It works out in a tendency of the streamlines to move away from the centre of curvature. The importance of both effects depends on the rate of curvature, i.e. the ratio h/R_s.
Even if the distance over which curvature exists is too short for the direct effect of the secondary flow on the streamline configuration to become important, the effect on the bed shear stress may be considerable. In that case the bed shear stress depends, with a certain retardation, on the streamline curvature and the relation between this curvature and the bed resistance is greatly complicated.
On the other hand, if the rate of curvature is small enough to have a negligible effect of the secondary flow on the bed shear stress, but curvature exists over a long distance, the effect of the secondary flow on the streamline pattern may grow important. In that case there is no direct correlation between streamline curvature and bed resistance.
Hence it must be concluded that the present shallow water equations can only be used for the derivation of a relationship between curvature and bed resistance if the curvature is small and exists over not too long distances.

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*) The assumption that such a correlation exists is related to the similarity hypothesis for the vertical distribution of the main flow.

The second group of simplifications, leading from equation (23) to equation (24), included the neglect of the spatial variations of the depth-averaged velocity and the depth of flow as well as the curvature of the normal lines. In general, however, a streamwise variation of the streamline curvature will involve a variation of the depth-averaged velocity, so that it is not correct to maintain the variation of the curvature and neglect the variation of the velocity. This can be made clear by considering the vorticity of the depth-averaged flow

$$\omega = - \frac{\partial \bar{v}}{\partial s} - \frac{\bar{v}}{R_s} \tag{28}$$

This is a flow property that is produced, transported and dissipated as other properties, such as momentum, and hence it satisfies a transport equation.
Adopting the shallow water approximation represented by equations (16) through (18), this vorticity transport equation reads (see Appendix D)

$$\bar{v}\frac{\partial \omega}{\partial s} - \frac{\bar{v}}{h}\frac{\partial h}{\partial s}\,\omega = - \frac{g}{C^2}\frac{\bar{v}}{h}\left\{2\omega + \frac{\bar{v}}{R_s} + \frac{\bar{v}}{h}\frac{\partial h}{\partial n}\right\} \tag{29}$$

This can be considered as a rewritten version of equation (23) and if the spatial variations of $\bar{v}$ and h are neglected this equation reduces to equation (24).
According to equation (29), the vorticity has a certain inertia, i.e. if the flow encounters a change in the production terms, the vorticity shows a retarded adaptation to this change (see Appendix D). The characteristic length L_ω of this adaptation is given by

$$\frac{L_\omega}{h} = \frac{C^2}{2g} \tag{30}$$

i.e. 50-250 times the depth of flow, according as the bed is rough or smooth. This does not imply, however, that both the curvature and the transverse velocity gradient shall vary with a characteristic length L_ω, but only that steeper variations of either quantity are compensated by opposite variations of the other quantity.
Therefore equation (24) represents only a very specific case of shallow water flow, occurring when the transverse velocity gradient varies with

a much larger length scale than the vorticity. In general, equation (24) should be replaced by an equation concerning the vorticity rather than the streamline curvature. If the Froude number is small and the bottom is nearly horizontal, the variations of h are negligible and the equation, to be derived from the vorticity transport equation (29), reads

$$\frac{\partial \omega}{\partial s} = - \frac{g}{C^2 h} \left(2\omega + \frac{\bar{v}}{R_s} \right) \tag{31}$$

It may become evident from these arguments, that only in very specific cases of shallow water flow that will not often occur in nature, the relationship between the streamline curvature and the bed resistance is as simple as equation (24). Therefore it will be a rather hard, if not impossible task to verify this equation experimentally.

4.3. Applicability of the results for scaling alluvial river models

Some of the results presented in this report, viz. equation (23) and the vorticity transport equation (29), were used to derive scale laws for alluvial river models (DELFT HYDRAULICS LABORATORY, 1978). Regarding the considerations given in section 4.2., the use of these equations is at least preferable to the use of equation (24), but especially in river bends the applicability of the shallow water approximation underlying these equations must be doubted (DE VRIEND, 1976; DE VRIEND AND KOCH, 1977 and 1978).

The main reason why the shallow water approximation fails in this case is the neglect of the advective influence of the secondary flow on the main flow, as became evident from a thorough analysis of the main velocity redistribution in a channel bend (DE VRIEND, 1978a and b). It is this advective influence that causes the effect of the secondary flow on the vertical and the horizontal distribution of the main flow as mentioned in section 4.2.

A set of depth-averaged conservation laws equivalent to (16) through (18), but accounting for this advective influence, would read (DE VRIEND, 1979)

$$\frac{\partial(\bar{v}h)}{\partial s} + \frac{\bar{v}h}{R_n} = 0 \tag{16}$$

$$\bar{v}h\,\frac{\partial\bar{v}}{\partial s} + \alpha_2\,\{\bar{F}h(\frac{\partial v}{\partial n} + \frac{v}{R_s}) + \bar{v}h\,(\frac{\partial\bar{F}}{\partial n} + \frac{\bar{F}}{R_s}) + \bar{v}\,\bar{F}\,\frac{\partial h}{\partial n}\} =$$

$$- gh\,\frac{\partial z_s}{\partial s} - \alpha_3\bar{v}^2 \tag{32}$$

$$- \frac{\bar{v}^2 h}{R_s} = - gh\,\frac{\partial z_s}{\partial n} \tag{18}$$

in which $\bar{F}$ = the depth-averaged stream function of the secondary flow*),
$\alpha_2 = - \overline{v\,\frac{\partial F}{\partial z}}\,/(\bar{v}\bar{F})$ = the advection factor,
α_3 = bed shear stress factor.
As the vertical distribution of the main flow tends to become more uniform under the influence of secondary flow advection, the factor $\overline{v^2}/\bar{v}^2$ will lie even closer to unity than in case of a logarithmic distribution. Hence it is consistent with equations (17) and (18) to set this factor equal to unity. On the other hand, the bed shear stress factor α_3 may become much larger than the "undisturbed" value g/C^2.
The extended vorticity transport equation to be derived from equations (32) and (18) reads

$$\bar{v}\,\frac{\partial\omega}{\partial s} - \frac{\bar{v}}{h}\,\frac{\partial h}{\partial s}\,\omega = - \alpha_2\left[\bar{F}(\frac{\partial\omega}{\partial n} + \frac{\omega}{R_s}) + \omega(\frac{\partial\bar{F}}{\partial n} + \frac{\bar{F}}{R_s} + \frac{\bar{F}}{h}\frac{\partial h}{\partial n}) + \right.$$

$$\left. - \bar{v}\,\{\frac{\partial^2\bar{F}}{\partial n^2} + \frac{\partial\bar{F}}{\partial n}\,(\frac{1}{R_s} + \frac{1}{h}\,\frac{\partial h}{\partial n}) + \bar{F}\,\frac{\partial}{\partial n}\,(\frac{1}{R_s} + \frac{1}{h}\,\frac{\partial h}{\partial n})\}\right]$$

$$- \alpha_3\,\frac{\bar{v}}{h}\,(2\omega + \frac{\bar{v}}{R_s} + \frac{v}{h}\,\frac{\partial h}{\partial n}) \tag{33}$$

For $\alpha_2 = 0$ and $\alpha_3 = g/C^2$, this equation reduced to equation (29). In order to analyse the differences between these two equations they are rewritten as

*) Here F is defined by $v_n = - \frac{\partial F}{\partial z}$ and $v_z = \frac{\partial F}{\partial n} + \frac{F}{R_s}$

$$\frac{\partial\omega}{\partial s} + \left\{-\frac{1}{h}\frac{\partial h}{\partial s} + 2\frac{g}{C^2 h}\right\}\omega = -\frac{g}{C^2}\frac{\bar{v}}{h}\left(\frac{1}{R_s} + \frac{1}{h}\frac{\partial h}{\partial n}\right) \qquad (34)$$

and

$$\frac{\partial\omega}{\partial s} + \frac{\alpha_2\bar{F}}{\bar{v}}\frac{\partial\omega}{\partial n} + \left[-\frac{1}{h}\frac{\partial h}{\partial s} + \frac{\alpha_2}{\bar{v}}\left(\frac{\partial\bar{F}}{\partial n} + 2\frac{\bar{F}}{R_s} + \frac{\bar{F}}{h}\frac{\partial h}{\partial n}\right) + 2\frac{\alpha_3}{h}\right]\omega =$$

$$\alpha_2\left[\frac{\partial^2\bar{F}}{\partial n^2} + \frac{\partial\bar{F}}{\partial n}\left(\frac{1}{R_s} + \frac{1}{h}\frac{\partial h}{\partial n}\right) + \bar{F}\frac{\partial}{\partial n}\left(\frac{1}{R_s} + \frac{1}{h}\frac{\partial h}{\partial n}\right)\right] - \frac{\alpha_3}{h}\left(\frac{\bar{v}}{R_s} + \frac{\bar{v}}{h}\frac{\partial h}{\partial n}\right) \qquad (35)$$

Equation (34) has one real characteristic, coincident with the streamline. Equation (35) has also one real characteristic, but it is given by

$$\frac{dn}{ds} = \frac{\alpha_2\bar{F}}{\bar{v}} \qquad (36)$$

, i.e. it does not coincide with the streamline, but tends to deviate outwards. In practical cases the deviation angle will be small, but if, starting from a certain point, the secondary flow acts over a sufficiently long distance, the characteristic through this point finally deviates considerably from the streamline.

Not only the characteristic direction, but also the damping factor and the source term are influenced by the secondary flow advection. The damping factor is increased near the inner bank and decreased near the outer bank and the source term is influenced such, that the depth-averaged velocity tends to decrease near the inner bank and increase near the outer bank (DE VRIEND, 1978a and 1978b).

It may be evident from the foregoing that it is important to adequately represent secondary flow advection when modelling rivers of curvilinear alignment.

5. Conclusions

The following conclusions can be drawn from the present analysis regarding the influence of the bed resistance on the curvature of streamlines in shallow water flow with little side constraint:

- the bed resistance has a damping influence on the flow and hence on the curvature of a free streamline,
- the damping is stronger as the resistance is greater,
- the vorticity transport equation for the depth-averaged flow provides information on the damping influence of the bed resistance; only in very specific cases this equation reduces to a direct relationship between streamline curvature and bed resistance,
- experimental verification of the relationship between streamline curvature and bed resistance will be difficult, as in most flow cases the curvature is also influenced by other effects that can hardly be eliminated (velocity gradients, secondary flow).

In addition, the analysis gives rise to the following more general conclusion as to shallow water flow:

- if the rate of curvature or the distance over which the streamlines are curved is not small, the original shallow water equations (16) through (18) must be corrected for the advective influence of the secondary flow on the main flow; when modelling this type of flow by a physical scale model, the scale laws to be satisfied must ensure an adequate representation of this advective influence.

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*) Any other handbook on mathematics dealing with vector analysis and generalized curvilinear coordinates can be used instead.

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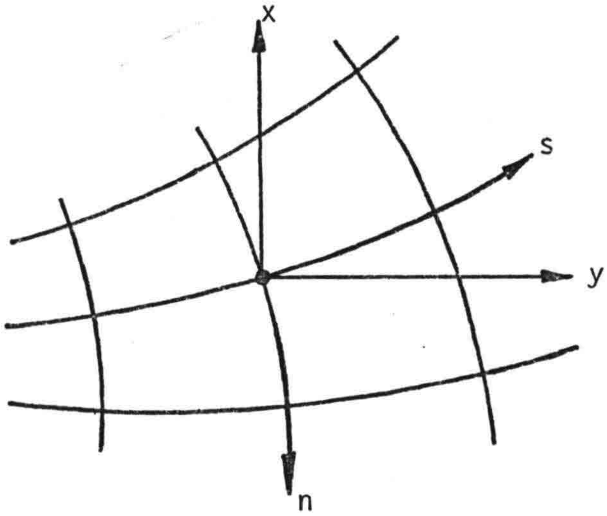
APPENDICES

Appendix A. Conservation laws in streamline coordinates

One of the possibilities to derive depth-averaged conservation laws equivalent to equations (8), (14) and (15) is to carry out a transformation from the cartesian coordinate system (x,y,z) to the streamline-coordinate system (s,n,z).



In general, the transformation rules can be written as

$$\frac{\partial}{\partial s} = \frac{\partial}{\partial x}\frac{dx}{ds} + \frac{\partial}{\partial y}\frac{dy}{ds} \tag{A.1}$$

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial x}\frac{dx}{dn} + \frac{\partial}{\partial y}\frac{dy}{dn} \tag{A.2}$$

Since s and n are defined along streamlines and normal lines, the transformation factors are

$$\frac{dx}{ds} = \frac{\bar{v}_x}{\bar{v}} \; ; \; \frac{dy}{ds} = \frac{\bar{v}_y}{\bar{v}} \; ; \; \frac{dx}{dn} = -\frac{\bar{v}_y}{\bar{v}} \; ; \; \frac{dy}{dn} = \frac{\bar{v}_x}{\bar{v}} \tag{A.3}$$

Making use of these transformation laws, equations (8), (14) and (15) can be reduced to

$$\frac{\partial(\bar{v}h)}{\partial s} = \frac{h}{\bar{v}^2}\left\{-\bar{v}_y^2\frac{\partial\bar{v}_x}{\partial x} - \bar{v}_x^2\frac{\partial\bar{v}_y}{\partial y} + \bar{v}_x\bar{v}_y\left(\frac{\partial\bar{v}_x}{\partial y} + \frac{\partial\bar{v}_y}{\partial x}\right)\right\} \tag{A.4}$$

$$gh\frac{\partial z_s}{\partial s} = -\frac{g}{C^2}\bar{v}^2 - \bar{v}h\frac{\partial\bar{v}}{\partial s} \tag{A.5}$$

$$gh\frac{\partial z_s}{\partial n} = \frac{h}{\bar{v}}\left\{\bar{v}_x\bar{v}_y\left(\frac{\partial\bar{v}_x}{\partial x} - \frac{\partial\bar{v}_y}{\partial y}\right) + \bar{v}_y^2\frac{\partial\bar{v}_x}{\partial y} - \bar{v}_x^2\frac{\partial\bar{v}_y}{\partial x}\right\} \tag{A.6}$$

Physical considerations lead to the conclusion that the terms in braces in equations (A.4) and (A.6) must be related to the divergence and the curvature of the streamlines, respectively. Therefore it will be attempted to express the radii of curvature of the streamlines and the normal lines in terms of $\bar{v}_x$ and $\bar{v}_y$ (see also DE VRIEND, 1976).

The streamlines of the depth-averaged flow field are defined by

$$\frac{dy}{dx} = \frac{\bar{v}_y}{\bar{v}_x} \tag{A.7}$$

and if the streamline curvature is defined as in chapter 2, i.e. the curvature is taken positive when the normal lines diverge, it follows from

$$\frac{1}{R_s} = \frac{-\dfrac{d^2y}{dx^2}}{\{1 + (\dfrac{dy}{dx})^2\}^{3/2}} \tag{A.8}$$

The numerator can be elaborated using (A.8) to yield

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\bar{v}_y}{\bar{v}_x}\right) = \frac{1}{\bar{v}_x^2}\left(\bar{v}_x \frac{d\bar{v}_y}{dx} - \bar{v}_y \frac{d\bar{v}_x}{dx}\right) \tag{A.9}$$

Since

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{\bar{v}_y}{\bar{v}_x}\frac{\partial}{\partial y} \tag{A.10}$$

expression (A.9) can be written

$$\frac{d^2y}{dx^2} = \frac{1}{\bar{v}_x^2}\left(\bar{v}_x \frac{\partial\bar{v}_y}{\partial x} + \bar{v}_y \frac{\partial\bar{v}_y}{\partial y} - \bar{v}_y \frac{\partial\bar{v}_x}{\partial x} - \frac{\bar{v}_y^2}{\bar{v}_x}\frac{\partial\bar{v}_x}{\partial y}\right) \tag{A.11}$$

Substitution of (A.7) and (A.11) leads to

$$\frac{1}{R_s} = -\frac{1}{\bar{v}^3}\left(\bar{v}_x^2 \frac{\partial\bar{v}_y}{\partial x} + \bar{v}_x\bar{v}_y \frac{\partial\bar{v}_y}{\partial y} - \bar{v}_y^2 \frac{\partial\bar{v}_x}{\partial y} - \bar{v}_x\bar{v}_y \frac{\partial\bar{v}_x}{\partial x}\right) \tag{A.12}$$

The normal lines of the depth-averaged flow field are defined by

$$\frac{dy}{dx} = -\frac{\bar{v}_x}{\bar{v}_y} \tag{A.13}$$

and their curvature follows from

$$\frac{1}{R_n} = \frac{-\dfrac{d^2y}{dx^2}}{\{1 + (\frac{dy}{dx})^2\}^{3/2}} \tag{A.14}$$

In a similar way as described above for the streamline curvature, the following relation between R_n and $\bar{v}_x$ and $\bar{v}_y$ can be derived

$$\frac{1}{R_n} = -\frac{1}{\bar{v}^3}\left(-\bar{v}_x^2 \frac{\partial \bar{v}_y}{\partial y} - \bar{v}_y^2 \frac{\partial \bar{v}_x}{\partial x} + \bar{v}_x \bar{v}_y \frac{\partial \bar{v}_x}{\partial y} + \bar{v}_x \bar{v}_y \frac{\partial \bar{v}_y}{\partial x}\right) \tag{A.15}$$

Then the transformed conservation laws can be rewritten as

$$\frac{\partial(\bar{v}h)}{\partial s} + \frac{\bar{v}h}{R_n} = 0 \tag{A.16}$$

$$\frac{\partial z_s}{\partial s} = -\frac{\bar{v}^2}{C^2 h} - \frac{\bar{v}}{g}\frac{\partial \bar{v}}{\partial s} \tag{A.17}$$

$$\frac{\partial z_s}{\partial n} = \frac{\bar{v}^2}{g R_s} \tag{A.18}$$

According to the first equation $\bar{v}h$ tends to decrease along a streamline when R_n is positive (i.e. when the streamlines diverge), which is physically correct. The last equation states that the transverse slope is positive when R_s is positive, i.e. when the normal lines diverge. This is physically correct, as well. So the signs of R_n and R_s in the above equations are correct.

An alternative way to arrive at these equations is to start from the conservation laws in an orthogonal coordinate system with vertical z-axis and general curvilinear horizontal coordinates. If ξ and η denote the distances along the horizontal coordinate lines, this system can be indicated by (ξ,η,z) and the depth-averaged continuity and momentum equations read (cf. ROUSE, 1959)

$$\frac{\partial}{\partial \xi}(\bar{v}_\xi h) + \frac{\partial}{\partial \eta}(\bar{v}_\eta h) + \frac{\bar{v}_\xi h}{R_\eta} + \frac{\bar{v}_\eta h}{R_\xi} = 0 \tag{A.19}$$

$$\bar{v}_\xi h \frac{\partial \bar{v}_\xi}{\partial \xi} + \bar{v}_\eta h \frac{\partial \bar{v}_\xi}{\partial \eta} + \frac{\bar{v}_\xi \bar{v}_\eta}{R_\xi} - \frac{\bar{v}_\eta^2}{R_\eta} = -gh \frac{\partial z_s}{\partial \xi} - \frac{g}{C^2} \bar{v}_\xi \bar{v} \tag{A.20}$$

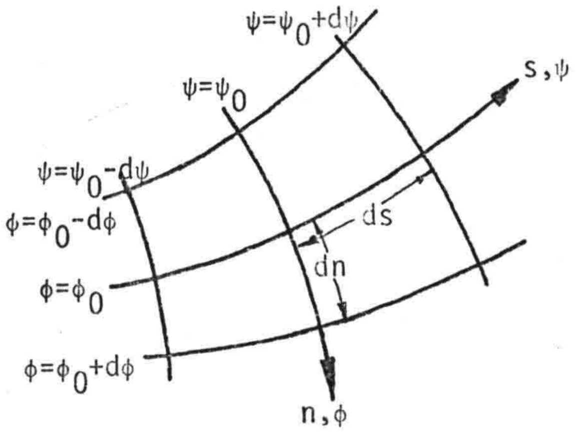
$$\bar{v}_\xi h \frac{\partial \bar{v}_\eta}{\partial \xi} + \bar{v}_\eta h \frac{\partial v_\eta}{\partial \eta} + \frac{\bar{v}_\xi \bar{v}_\eta}{R_\eta} - \frac{\bar{v}_\xi^2}{R_\xi} = -gh \frac{\partial z_s}{\partial \eta} - \frac{g}{C^2} \bar{v} \bar{v}_\eta \tag{A.21}$$

If the horizontal coordinate lines coincide with the streamlines and the normal lines of the depth-averaged flow field,

$$\bar{v}_\xi \equiv \bar{v} \quad \text{and} \quad \bar{v}_\eta \equiv 0 \tag{A.22}$$

by definition. Equations (A.19) through (A.21) then reduce to equations (A.16) through (A.18).

Appendix B. Exchange of the sequence of differentiation in a two-dimensional curvilinear coordinate system



If ψ and ϕ denote the actual coordinates in the streamline coordinate system (s,n)*, two functions $l_1(\psi,\phi)$ and $l_2(\psi,\phi)$ can be defined such that

$$ds = l_1(\psi,\phi)d\psi \qquad \text{and} \qquad dn = l_2(\psi,\phi)d\phi \tag{B.1}$$

(see, for instance, KUIPERS EN TIMMAN, 1966). Then the first derivatives of a smooth scalar function $f(\psi,\phi)$, denoted symbolically by $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial n}$, can be written as

$$\frac{\partial f}{\partial s} = \frac{1}{l_1}\frac{\partial f}{\partial \psi} \qquad \text{and} \qquad \frac{\partial f}{\partial n} = \frac{1}{l_2}\frac{\partial f}{\partial \phi} \tag{B.2}$$

In a polar coordinate system (ϕ,r), for instance, $l_1 = r$ and $l_2 = 1$, so that $ds = rd\phi$ and $dn = dr$. Then the first derivatives of a smooth function $f(\phi,r)$ are

$$\frac{\partial f}{\partial s} = \frac{1}{r}\frac{\partial f}{\partial \phi} \qquad \text{and} \qquad \frac{\partial f}{\partial n} = \frac{\partial f}{\partial r} \tag{B.3}$$

As $f(\psi,\phi)$ is a scalar, its second cross-derivative is independent of the sequence of differentiation, so

$$\frac{\partial^2 f}{\partial\psi\partial\phi} = \frac{\partial^2 f}{\partial\phi\partial\psi} \tag{B.4}$$

* i.e. streamlines are lines of constant ϕ and normal lines are lines of constant ψ.

The two terms in this equation can be elaborated to

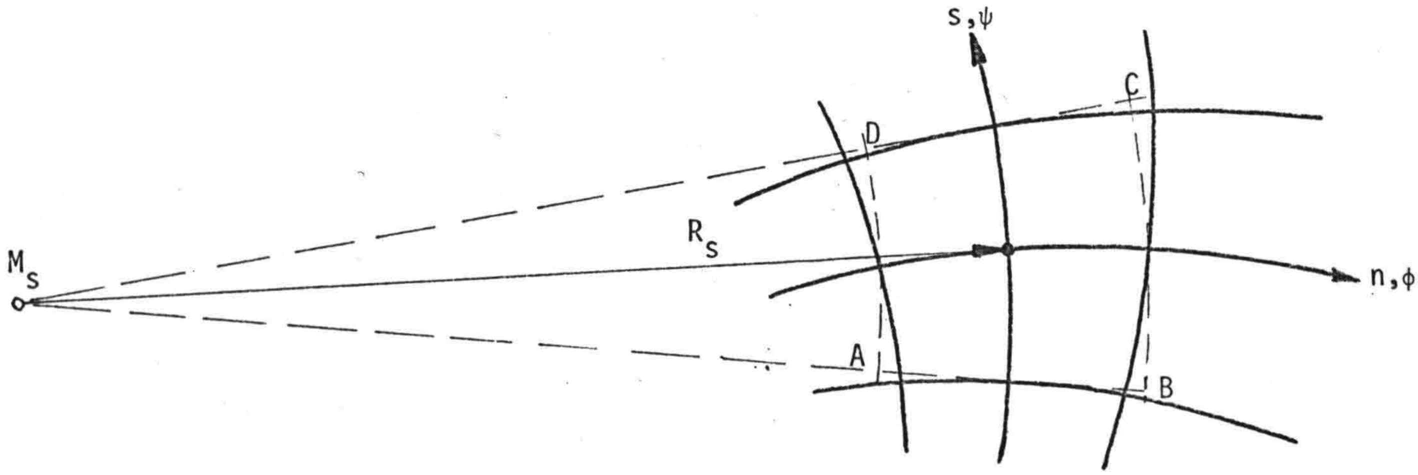
$$\frac{\partial^2 f}{\partial\psi\partial\phi} = \frac{\partial}{\partial\psi}\left(\frac{\partial f}{\partial\phi}\right) = \frac{\partial}{\partial\psi}\left(l_2 \frac{1}{l_2}\frac{\partial f}{\partial\phi}\right) = l_2 \frac{\partial}{\partial\psi}\left(\frac{1}{l_1}\frac{\partial f}{\partial\phi}\right) + \frac{1}{l_2}\frac{\partial f}{\partial\phi}\frac{\partial l_2}{\partial\psi} \tag{B.5}$$

$$\frac{\partial^2 f}{\partial\psi\partial\phi} = \frac{\partial}{\partial\phi}\left(\frac{\partial f}{\partial\psi}\right) = \frac{\partial}{\partial\phi}\left(l_1 \frac{1}{l_1}\frac{\partial f}{\partial\psi}\right) = l_1 \frac{\partial}{\partial\phi}\left(\frac{1}{l_1}\frac{\partial f}{\partial\psi}\right) + \frac{1}{l_1}\frac{\partial f}{\partial\psi}\frac{\partial l_1}{\partial\phi} \tag{B.6}$$

Then equation (B.4) becomes

$$\frac{1}{l_1}\frac{\partial}{\partial\psi}\left(\frac{1}{l_2}\frac{\partial f}{\partial\phi}\right) + \frac{1}{l_2}\frac{\partial f}{\partial\phi}\frac{1}{l_1 l_2}\frac{\partial l_2}{\partial\psi} = \frac{1}{l_2}\frac{\partial}{\partial\phi}\left(\frac{1}{l_1}\frac{\partial f}{\partial\psi}\right) + \frac{1}{l_1}\frac{\partial f}{\partial\psi}\frac{1}{l_1 l_2}\frac{\partial l_1}{\partial\phi} \tag{B.7}$$

The distance between two adjacent streamlines, measured along a normal line, is proportional to l_2. Hence the streamwise variation of l_2 indicates the rate of divergence of the streamlines, which is related to the curvature of the normal lines. Similarly, the streamline curvature is related to the divergence of the normal lines, i.e. to the normal variation of l_1. These relationships can be derived as follows (see also: ROUSE, 1959).



The similarity of the sectors M_sAD and M_sBC leads to

$$\overline{BC} : \overline{AD} = \overline{M_sC} : \overline{M_sD} \tag{B.8}$$

or

$$(l_1 d\psi)_{BC} : (l_1 d\psi)_{AD} = (R_s + \frac{1}{2} l_2 \, d\phi) : (R_s - \frac{1}{2} l_2 \, d\phi) \tag{B.9}$$

$$(l_1 + \frac{1}{2} \frac{\partial l_1}{\partial \phi} d\phi) : (l_1 - \frac{1}{2} \frac{\partial l_1}{\partial \phi} d\phi) = (R_s + \frac{1}{2} l_2 \, d\phi) : (R_s - \frac{1}{2} l_2 \, d\phi) \tag{B.10}$$

Then series expansions of the quotients yield

$$1 + \frac{1}{l_1} \frac{\partial l_1}{\partial \phi} d\phi = 1 + \frac{1}{R_s} l_2 \, d\phi + O(d\phi^2) \tag{B.11}$$

so that for $d\phi \downarrow 0$

$$\frac{1}{R_s} = \frac{1}{l_1 l_2} \frac{\partial l_1}{\partial \phi} \tag{B.12}$$

This result is consistent with the sign convention for the streamline curvature: the curvature is positive when the normal lines diverge. A similar expression for the curvature of the normal lines can be derived in the same way. It reads

$$\frac{1}{R_n} = \frac{1}{l_1 l_2} \frac{\partial l_2}{\partial \psi} \tag{B.13}$$

Now equation (B.7) can be rewritten as

$$\frac{1}{l_1} \frac{\partial}{\partial \psi} (\frac{1}{l_2} \frac{\partial f}{\partial \phi}) + \frac{1}{l_2} \frac{\partial f}{\partial \phi} \frac{1}{R_n} = \frac{1}{l_2} \frac{\partial}{\partial \phi} (\frac{1}{l_1} \frac{\partial f}{\partial \psi}) + \frac{1}{l_1} \frac{\partial f}{\partial \psi} \frac{1}{R_s} \tag{B.14}$$

or, using the symbolic notation of the derivatives,

$$\frac{\partial}{\partial s} (\frac{\partial f}{\partial n}) + \frac{1}{R_n} \frac{\partial f}{\partial n} = \frac{\partial}{\partial n} (\frac{\partial f}{\partial s}) + \frac{1}{R_s} \frac{\partial f}{\partial s} \tag{B.15}$$

The above rule can also be derived from the vector identity

$$\vec{curl} \, (\vec{grad} \, f) \equiv 0 \tag{B.16}$$

holding for any scalar f (KUIPERS EN TIMMAN, 1966). The curl of a vector $\vec{v}$ with components v_s and v_n is defined as

$$\vec{\text{curl}}\ (\vec{v}) = \frac{1}{l_1 l_2}\frac{\partial}{\partial\psi}(v_n l_2) - \frac{1}{l_1 l_2}\frac{\partial}{\partial\phi}(v_s l_1) \tag{B.17}$$

or, making use of (B.12) and (B.13),

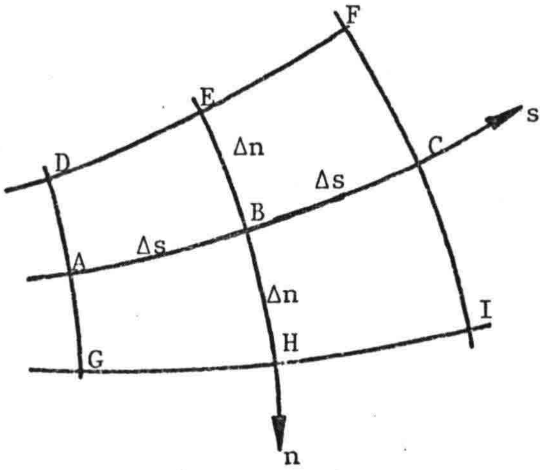
$$\vec{\text{curl}}\ (\vec{v}) = \frac{\partial v_n}{\partial s} + \frac{v_n}{R_n} - \frac{\partial v_s}{\partial n} - \frac{v_s}{R_s} \tag{B.18}$$

If the components of the vector $\vec{\text{grad}}$ f are substituted for v_s and v_n, this expression becomes

$$\vec{\text{curl}}\ (\vec{\text{grad}}\ f) = \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial n}\right) + \frac{1}{R_n}\frac{\partial f}{\partial n} - \frac{\partial}{\partial n}\left(\frac{\partial f}{\partial s}\right) - \frac{1}{R_s}\frac{\partial f}{\partial s} \tag{B.19}$$

When combining this expression with the identity (B.16), equation (B.15) is found.

Appendix C. Variation of the transverse surface slope along a streamline



A most essential tool in the derivation of the relationship between streamline curvature and bed resistance (see chapter 3) is equation (19), reading

$$\frac{\partial}{\partial n}\left(\frac{\partial z_s}{\partial s}\right) + \frac{1}{R_s}\frac{\partial z_s}{\partial s} - \frac{\partial}{\partial s}\left(\frac{\partial z_s}{\partial n}\right) - \frac{1}{R_n}\frac{\partial z_s}{\partial n} = 0 \qquad \text{(C.1)}$$

A strictly mathematical derivation of this rule is given in Appendix B. In the present appendix a more physical approach is used to arrive at the same equation.
According to equation (18), the streamline curvature is related to the transverse surface slope: if $\bar{v}$ and h are almost constant, the curvature is proportional to $\partial z_s/\partial n$. So the streamwise variation of this curvature is determined by the streamwise variation of the transverse surface slope $\partial/\partial s\ (\partial z_s/\partial n)$. Therefore this quantity is subject to a closer investigation.
Consider an elementary domain DFIG (see figure), bounded by two streamlines and two normal lines. The water surface elevation in this domain, and so along the line FI, is determined uniquely by the elevation along the line DG and the longitudinal slope in every point. Hence the transverse surface slopes on DG and FI are related through the longitudinal slope.
In order to establish this relationship, the following approximations, holding for small Δs and Δn, are introduced:

$$z_s(D) \simeq z_s(A) - L_{DA} \left.\frac{\partial z_s}{\partial n}\right|_A \tag{C.2}$$

$$z_s(G) \simeq z_s(A) + L_{AG} \left.\frac{\partial z_s}{\partial n}\right|_A \tag{C.3}$$

$$z_s(F) \simeq z_s(D) + L_{DF} \left.\frac{\partial z_s}{\partial s}\right|_E \tag{C.4}$$

$$z_s(I) \simeq z_s(G) + L_{GI} \left.\frac{\partial z_s}{\partial s}\right|_H \tag{C.5}$$

$$L_{FI} \left.\frac{\partial z_s}{\partial n}\right|_C \simeq z_s(I) - z_s(F) \tag{C.6}$$

$$L_{DG} \simeq (R_n - \Delta s) \frac{2\Delta n}{R_n} \tag{C.7}$$

$$L_{FI} \simeq (R_n + \Delta s) \frac{2\Delta n}{R_n} \tag{C.8}$$

$$L_{DF} \simeq (R_s - \Delta n) \frac{2\Delta s}{R_s} \tag{C.9}$$

$$L_{GI} \simeq (R_s + \Delta n) \frac{2\Delta s}{R_s} \tag{C.10}$$

in which L_{PQ} denotes the distance between the points P and Q measured along the streamline or the normal line these point have in common.

Making use of these approximations, the relationship between the transverse surface slopes in A and C can be shown to read

$$(1 + \frac{\Delta s}{R_n})\, 2\Delta n \left.\frac{\partial z_s}{\partial n}\right|_C \simeq (1 - \frac{\Delta s}{R_n})\, 2\Delta n \left.\frac{\partial z_s}{\partial n}\right|_A + (1 + \frac{\Delta n}{R_s})\, 2\Delta s \left.\frac{\partial z_s}{\partial s}\right|_H +$$

$$- (1 - \frac{\Delta n}{R_s})\, 2\Delta s \left.\frac{\partial z_s}{\partial s}\right|_E \tag{C.11}$$

As for small values of Δs and Δn

$$\left.\frac{\partial z_s}{\partial n}\right|_C + \left.\frac{\partial z_s}{\partial n}\right|_A \simeq 2\left.\frac{\partial z_s}{\partial n}\right|_B \tag{C.12}$$

$$\left.\frac{\partial z_s}{\partial n}\right|_C - \left.\frac{\partial z_s}{\partial n}\right|_A \simeq 2\Delta s \left.\left\{\frac{\partial}{\partial s}\left(\frac{\partial z_s}{\partial n}\right)\right\}\right|_B \tag{C.13}$$

$$\left.\frac{\partial z_s}{\partial s}\right|_H + \left.\frac{\partial z_s}{\partial s}\right|_E \simeq 2\left.\frac{\partial z_s}{\partial s}\right|_B \tag{C.14}$$

$$\left.\frac{\partial z_s}{\partial s}\right|_H - \left.\frac{\partial z_s}{\partial s}\right|_E \simeq 2\Delta n \left.\left\{\frac{\partial}{\partial n}\left(\frac{\partial z_s}{\partial s}\right)\right\}\right|_B \tag{C.15}$$

, equation (C.11) leads to

$$\frac{\partial}{\partial s}\left(\frac{\partial z_s}{\partial n}\right) + \frac{1}{R_n}\frac{\partial z_s}{\partial n} = \frac{\partial}{\partial n}\left(\frac{\partial z_s}{\partial s}\right) + \frac{1}{R_s}\frac{\partial z_s}{\partial s} \tag{C.16}$$

, which is equivalent to (C.1) and equation (19). In fact, the present derivation is nothing but a finite area representation of the derivation in Appendix B. Integrating identity (B.16) over the domain DFIG yields

$$\iint_A \vec{\text{curl}}\,(\vec{\text{grad}}\; z_s)\; dA = 0 \tag{C.17}$$

, where A is the area of DFIG. Then Stokes' theorem leads to

$$\oint_{\mathcal{S}} (\vec{\text{grad}}\; z_s)\cdot\vec{s}\; ds = 0 \tag{C.18}$$

, $\mathcal{S}$ denoting the path along the boundary of the domain.

Making use of approximations (C.7) through (C.10) and

$$\{(\vec{\text{grad}}\ z_s)\ .\ \vec{s}\}\Big|_{DG} \simeq -\ L_{DG}\ \frac{\partial z_s}{\partial n}\Big|_A \tag{C.19}$$

$$\{(\vec{\text{grad}}\ z_s)\ .\ \vec{s}\}\Big|_{FI} \simeq +\ L_{FI}\ \frac{\partial z_s}{\partial n}\Big|_C \tag{C.20}$$

$$\{(\vec{\text{grad}}\ z_s)\ .\ \vec{s}\}\Big|_{DF} \simeq +\ L_{DF}\ \frac{\partial z_s}{\partial s}\Big|_E \tag{C.21}$$

$$\{(\vec{\text{grad}}\ z_s)\ .\ \vec{s}\}\Big|_{GI} \simeq -\ L_{GI}\ \frac{\partial z_s}{\partial s}\Big|_H \tag{C.22}$$

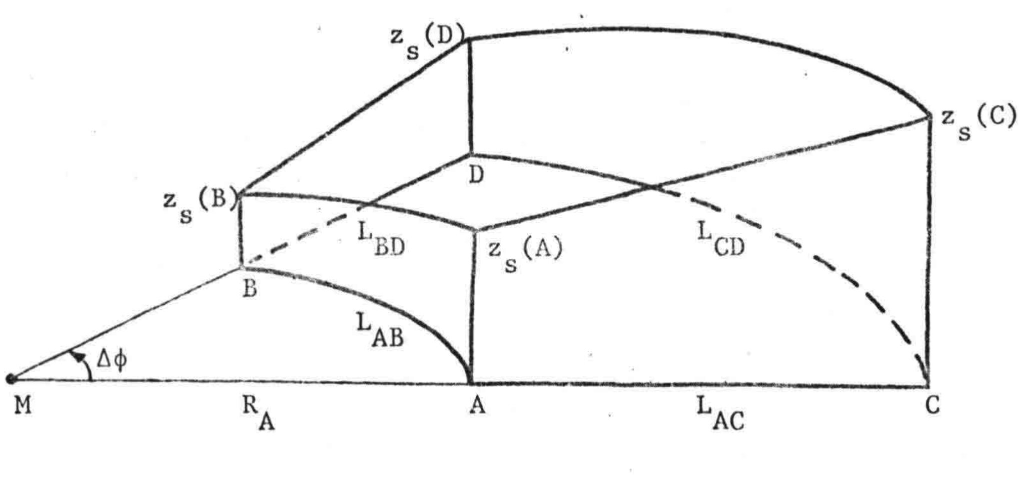
, identity (C.18) can be elaborated to

$$-\ (1 - \frac{\Delta s}{R_n})\ 2\Delta n\ \frac{\partial z_s}{\partial n}\Big|_A + (1 - \frac{\Delta n}{R_s})\ 2\Delta s\ \frac{\partial z_s}{\partial s}\Big|_E + (1 + \frac{\Delta s}{R_n})\ 2\Delta n\ \frac{\partial z_s}{\partial n}\Big|_C +$$

$$-\ (1 + \frac{\Delta n}{R_s})\ 2\Delta s\ \frac{\partial z_s}{\partial s}\Big|_H = 0 \tag{C.23}$$

This result corresponds with equation (C.11) and, for Δs and $\Delta n \downarrow 0$, with equation (C.16).

The decreasing tendency of the streamline curvature in case of an almost uniform velocity field with negligible gradients of the depth of flow (Schoemaker's assumptions; see chapter 3) is readily shown by the following simplified version of the present derivation.



Consider a domain ABDC, bounded by two streamline sections AB and CD being concentric circles and two straight normal line sections AC and BD. As a consequence of the assumptions made, the longitudinal slope is a constant throughout the domain and if the distance L_{AC} (= L_{BD}) is small, the transverse slope is approximately constant along AC and along BC. Be I_s the longitudinal slope of the water surface, defined by

$$I_s = - \frac{\partial z_s}{\partial s} \tag{C.24}$$

, then the surface elevations in B and D are related to those in A and C through

$$z_s(B) \simeq z_s(A) - L_{AB} I_s \tag{C.25}$$

$$z_s(D) \simeq z_s(C) - L_{CD} I_s \tag{C.26}$$

Hence

$$\left.\frac{\partial z_s}{\partial n}\right|_{BD} \simeq \frac{z_s(D) - z_s(B)}{L_{BD}} \simeq \frac{z_s(C) - z_s(A)}{L_{AC}} - I_s \frac{L_{CD} - L_{AB}}{L_{AC}} \simeq \left.\frac{\partial z_s}{\partial n}\right|_{AC} - I_s \Delta\phi \tag{C.27}$$

This implies that the transverse slope (and so the streamline curvature) tends to decrease along a streamline and that the rate of decrease is proportional to the longitudinal slope.

Appendix D. Vorticity transport equation

The vorticity $\vec{\omega}$ of a velocity field is defined as twice the rotation of the velocity vector (STREETER, 1971). Hence the vorticity of the depth-averaged shallow water flow considered here has only one non-zero component, viz. the vertical one, the magnitude of which is given by

$$\omega = \left| \vec{\nabla} \times \vec{\bar{v}} \right| \tag{D.1}$$

This quantity is a property of the flow, so it is independent of the horizontal configuration of the adopted coordinate system. In the cartesian system used in chapter 2, the vorticity follows from

$$\omega = \frac{\partial \bar{v}_y}{\partial x} - \frac{\partial \bar{v}_x}{\partial y} \tag{D.2}$$

and in the streamline coordinate system the expression reads (see also ROUSE, 1959)

$$\omega = -\frac{\partial \bar{v}}{\partial n} - \frac{\bar{v}}{R_s} \tag{D.3}$$

Starting from the conservation laws for mass and momentum, a transport equation for the vorticity can be derived by eliminating the pressure from the momentum equations.
On doing so in the cartesian coordinate system, equations (14) and (15) lead to (see also KUIPERS AND VREUGDENHIL, 1973)

$$\frac{\partial}{\partial x} (\bar{v}_x \omega) + \frac{\partial}{\partial y} (\bar{v}_y \omega) = -\frac{g}{C^2} \left\{ \frac{\partial}{\partial x} \left(\frac{\bar{v}_y \bar{v}}{h}\right) - \frac{\partial}{\partial y} \left(\frac{\bar{v}_x \bar{v}}{h}\right) \right\} \tag{D.4}$$

, which can be elaborated using the equation of continuity (8) and expression (A.12) for the streamline curvature. Then the following vorticity transport equation is found:

$$\bar{v}_x \frac{\partial \omega}{\partial x} + \bar{v}_y \frac{\partial \omega}{\partial y} - \frac{\omega}{h} \left(\bar{v}_x \frac{\partial h}{\partial x} + \bar{v}_y \frac{\partial h}{\partial y}\right) =$$

$$-\frac{g}{C^2} \frac{\bar{v}}{h} \left\{ 2\omega + \frac{\bar{v}}{R_s} - \frac{1}{h} \left(\bar{v}_y \frac{\partial h}{\partial x} - \bar{v}_x \frac{\partial h}{\partial y}\right) \right\} \tag{D.5}$$

The equivalent equation in the streamline coordinate system can be derived from (D.5) by transformation, using the transformation rules given in Appendix A.

$$\bar{v}\frac{\partial\omega}{\partial s} - \frac{\omega}{h}\bar{v}\frac{\partial h}{\partial s} = -\frac{g}{C^2}\frac{\bar{v}}{h}\{2\omega + \frac{\bar{v}}{R_s} + \frac{\bar{v}}{h}\frac{\partial h}{\partial n}\} \tag{D.6}$$

This equation can also be derived directly from equations (16) through (18), making use of the identity (B.15). Equations (17) and (18) can be rewritten as

$$g\frac{\partial}{\partial s}(z_s + \frac{\bar{v}^2}{2g}) = -\frac{g}{C^2}\frac{\bar{v}^2}{h} \tag{D.7}$$

$$g\frac{\partial}{\partial n}(z_s + \frac{\bar{v}^2}{2g}) = +\frac{\bar{v}^2}{R_s} + \frac{\partial\bar{v}}{\partial n} = -\bar{v}\omega \tag{D.8}$$

Substituting this into (B.15) applied to the energy head $z_s + \bar{v}^2/2g$ yields

$$\frac{\partial}{\partial s}(\bar{v}\omega) + \frac{\bar{v}\omega}{R_n} - \frac{g}{C^2}\frac{\partial}{\partial n}(\frac{\bar{v}^2}{h}) - \frac{g}{C^2}\frac{\bar{v}^2}{hR_s} = 0 \tag{D.9}$$

which can easily be elaborated to (D.6).
This equation can further be simplified by dividing it by $\bar{v}$, to yield

$$\frac{\partial\omega}{\partial s} + \omega(-\frac{1}{h}\frac{\partial h}{\partial s} + 2\frac{g}{C^2 h}) = -\frac{g}{C^2}\frac{\bar{v}}{h}(\frac{1}{R_s} + \frac{1}{h}\frac{\partial h}{\partial n}) \tag{D.10}$$

If $\omega = \omega_0$ is given in a point $s = s_0$ of a streamline and

$$f_1(s) = -\frac{1}{h}\frac{\partial h}{\partial s} + 2\frac{g}{C^2 h} \tag{D.11}$$

$$f_2(s) = -\frac{g}{C^2}\frac{\bar{v}}{h}(\frac{1}{R_s} + \frac{1}{h}\frac{\partial h}{\partial n}) \tag{D.12}$$

, the solution of equation (D.10) along this streamline has the form (see also DE VRIEND, 1976)

$$\omega = \exp\{-\int_{s_0}^{s} f_1(s')ds'\}\left[\omega_0 + \int_{s_0}^{s} f_2(s')\exp\{\int_{s_0}^{s'} f_1(s'')ds''\}\,ds'\right] \quad (D.13)$$

or, in another form

$$\omega = \omega_0 \exp\{-\int_{s_0}^{s} f_1(s')ds'\} + \int_{s_0}^{s} f_2(s')\exp\{-\int_{s'}^{s} f_1(s'')ds''\}ds' \quad (D.14)$$

As long as $f_1(s)$ is a positive quantity, as will be the case for the present class of problems, (D.14) is only a stable form if $s - s_o$ is positive, i.e. downstream of the point s_0. This implies that ω in a point s of a certain streamline is only influenced by what happens upstream along the streamline. In addition, expression (D.14) shows the influence of what happens in a point upstream to decrease as the distance (measured along the streamline) to the point considered increases.