Notes on the Unsteady Rectilinear Motion of a Perfect Gas

X. On the Influence of Gravity upon the Expansion of an Ideal Gas with $\gamma = 3$

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Summary

A vertical column of an ideal homentropic gas is at rest in a gravitational field, with constant acceleration of gravity $g$. At the upper end the column is bounded by a piston, while downwards it extends indefinitely. The gas pressure $p$ at the piston has the value $p_0$, while above the piston there is vacuum extending to infinity. At time $t = 0$ the piston is suddenly removed, permitting the gas to expand and to move into the vacuum. It is asked to study the motion of the gas for $t > 0$.

Burgers [1] considered this problem for $\gamma = \frac{5}{3}$. He found that the gas starts to move upwards, but after some time, due to the action of the gravity the velocity changes direction and the gas begins to fall back. In this phase of the motion the solution shows some singularities, which indicate that physically a shock-wave, moving in downwards direction, makes its appearance. The flow beyond the shock-wave can only be analyzed tentatively and the analysis retains a preliminary character. Also a final state of rest is only approximately obtained for large $t$.

In this Report Burgers' problem is considered once more but for a fictitious gas with $\gamma = 3$. This assumption simplifies the analysis and the singularities found for $\gamma = \frac{5}{3}$ are drastically reduced. Also since the answers are simpler it is possible to a large extent to invert the solutions.

Distinct from the case with $\gamma = \frac{5}{3}$ there is no overshoot, followed by a falling back, accompanied by a shock-wave. For $\gamma = 3$ the gas begins at $t = 0$ to move upwards but slows down and reaches a maximum height with speed zero, and density zero.
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1. Introduction

In 1948 Prof. J.M. Burgers, then at Delft, studied the following problem, and I quote from ref. [1]: 'We consider a vertical column of gas, which before the instant \( t = 0 \) is limited by a horizontal plane wall at its upper end, whereas downwards the column extends indefinitely. Lateral motion of the gas is prevented (it may be assumed that the gas is enclosed in an infinite vertical cylinder with perfectly smooth walls, or that the lateral dimensions of the column are infinite). The gas originally is everywhere at rest. The pressure of the gas has a certain finite value \( p_0 \) at the level where it is in contact with the boundary plane; downwards the pressure rises in consequence of the weight of the gas, according to the law valid for an atmosphere in adiabatic (isentropic) equilibrium. Above the boundary plane is vacuum extending towards infinity.

At the instant \( t = 0 \) the boundary plane is suddenly taken away, so that the gas can expand. It is asked to find the motion of the gas, taking account of its weight'. End of quote.

To solve this problem the equations of motion, taking account of a gravity field with constant acceleration \( g \), were put in the characteristic form with \( \theta \) and \( n \) denoting the Riemann-invariants. Taking the ratio of the specific heats, \( \gamma = \frac{c_p}{c_v} \), equal to 5/3, corresponding to a monatomic gas, and employing the transformation- and integration-method developed by B. Riemann [2], Prof. Burgers was able to construct the solution for the time \( t(\theta, n) \) valid during the expansion. Returning to the characteristic equations he obtained the position coordinate as function of \( \theta, n \) and later also a Lagrangian particle coordinate. Difficulties arose when the solutions in the \( \theta, n \)-plane were mapped into the 'physical plane' of the position coordinate and the time. Without going into details of the critical points, where \( \frac{\partial t}{\partial \theta}, \frac{\partial t}{\partial n} \) etc pass through zero, it was found that upon removal of the boundary plane the gas expands into the vacuum above. However after some time and due to the gravity the velocity reverses direction and the gas begins to fall back. From the appearance of critical points in the integration domain of the \( \theta, n \)-plane it is shown that in the 'physical-plane' a downwards moving shock-wave makes its appearance. Since the flow was assumed to be homentropic, the shock-wave, with discontinuities in entropy, puts the physical significance of the solution \( t(\theta, n) \) in jeopardy.
Burgers then explicitly inserts a shock-wave into the flow and attempts to continue the solution beyond it. However this attempt retains a preliminary character and could not be completed in detail. In particular a final state of rest of the gas in the field of gravity, with a pressure $p = 0$ at the upper end of the gas column, which seems physically most likely to appear for large $t$, was not obtained with certainty.

It is the purpose of the present investigation to consider the problem of Burgers anew, but for a (fictitious) gas where $\gamma$ has the value 3. This assumption simplifies the analysis as we shall see, and largely eliminates the singular points, except at one vertex of the rectangular domain in the $r,s$-plane, where the solution applies.

In particular we are able for $\gamma = 3$ to invert the solutions for the time and the position coordinate to a large extent and can proceed somewhat further with the analysis of the problem.

The Report is divided into 12 Sections.

In Section 2 the equations of motion are introduced in the Eulerian form, including a constant acceleration of gravity $g$ and they are put in a characteristic form. With these equations of motion the initial state of rest for $t < 0$ is easily dealt with in Section 3. Also the characteristics are considered and it is pointed out that the initial state is a 'general wave' in terms of the hierarchy 'constant domains', 'simple waves', 'general waves', which is often useful in flows where Riemann-invariants are present.

In Section 4 the equations of motion are made 'reducible' by the introduction of a 'freely-falling' reference-frame where the gravity is eliminated. Again the characteristics are considered and the transformation of Riemann. This leads to linear equations for $t$ and $x^*$ in terms of the independent variables $r$ and $s$, the Riemann-invariants. Some remarks about the Euler-Poisson-Darboux equation for $t(r,s)$ are also presented.

In Section 5 the equations of motion are introduced in the Lagrangian variables $t$, the time and $h$ the Lagrangian mass coordinate. Again characteristic equations
are considered and the Euler-Poisson-Darboux equations for $t(r,s)$ and $h(r,s)$. The initial state of the gas is also considered in the Lagrangian variables.

In Section 6 the expansion problem is formulated leading to a characteristic boundary value problem. The boundary-conditions are taken from Burgers' analysis (Ref. [1]). Then in Section 7 the fictitious ideal gas with $\gamma = 3$ is introduced. The simplifications which appear are discussed and some formulae and equations are collected.

In Section 8 the solution for the expansion problem is constructed, starting from the general solution for $t(r,s)$. This yields the three expressions $t(r,s)$, $x^*(r,s)$ and $h(r,s)$.

In Section 9 the solutions obtained in the previous Section are discussed as to their regularity. It is shown that almost the entire domain in the $r,s$-plane, where the solution is valid is composed of ordinary points. The only exceptions are the segment $-a_0 \leq s \leq +a_0$ of the characteristic $r = a_0$ and the point $r = a_0$, $s = a_0$. Also the Jacobian $J$ is considered and some curves $t = \text{const}$ and $h = \text{const}$ are presented in dimensionless form.

In Section 10 the expansion flow is considered in the $t,x^*$-plane. Since the $r$- and $s$-characteristics for $\gamma = 3$ are all straight lines this practically results in the inversion of $t(r,s)$ etc.

Finally in Section 11 the results of Section 10 are transferred to the physical $t,x$-plane.
In Section 12 some concluding remarks follow.
2. The Equations of Motion

The gas is assumed to be the ordinary ideal gas with the equation of state

\[ p = \rho RT, \quad (2.1) \]

where \( p \) denotes the pressure, \( \rho \) the density, \( T \) the temperature and \( R \) the gas constant per unit mass. The specific heats \( c_p \) and \( c_v \) are assumed constant.

Denoting the Cartesian coordinate along the gas column by \( x \), the velocity by \( u \), the time by \( t \) and the constant acceleration of gravity, acting in the negative \( x \)-direction, by \( g \), the equations of mass- and momentum conservation take the familiar form

\[ \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho \frac{\partial u}{\partial x} = 0, \quad (2.2) \]

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = -g. \quad (2.3) \]

Viscous forces have been neglected, and omitting also heat conduction and other forms of heat transfer, the equation for the conservation of energy, together with (2.1) and the constant ratio \( \gamma = c_p/c_v \), yields the Poisson relation

\[ p = B_0 \rho^\gamma = b_0^\gamma \rho^\gamma, \quad (2.4) \]

with \( B_0 = b_0^\gamma \) constants assumed to have the same value throughout the gas. The entropy \( S \) per unit mass has then the same value for all the gas elements and the gas is called homentropic (* = homogeneous entropy)*.

We also introduce the adiabatic speed of sound \( a \) and find

\[ a^2 = \frac{(dp)}{\rho \, dp} \bigg|_{S=\text{const}} = \gamma B_0^\gamma - 1 = \frac{\gamma P}{\rho} = \frac{\gamma - 1}{b_0 \rho}, \quad (2.5) \]

*) The term homentropic was coined by Prof. L. Howarth (see p. 3 of ref. [3]). The term is not generally known as ref. [4] may show.
Logarithmic differentiation of (2.5) yields the useful relations

\[
\frac{2}{a} \frac{da}{dt} = (\gamma - 1) \frac{1}{\rho} \frac{d\rho}{dp} = \frac{\gamma - 1}{\gamma} \frac{1}{\rho} \frac{dp}{d\rho} .
\]  

(2.6)

Expressing \( p \) and \( \rho \) from (2.2) and (2.3) in the speed of sound \( a \), upon employing (2.5) and (2.6), one obtains

\[
\frac{d}{dt} \left( \frac{2a}{\gamma - 1} \right) + u \frac{d}{dx} \left( \frac{2a}{\gamma - 1} \right) + a \frac{da}{dx} = 0 .
\]  

(2.7)

\[
\frac{du}{dt} + u \frac{du}{dx} + a \frac{d}{dx} \left( \frac{2a}{\gamma - 1} \right) = -g .
\]  

(2.8)

The reference frame with the coordinate \( x \) will be called the 'g-frame' with \( g \) for gravity, since an observer fixed in this frame will note the gravity field as eq. (2.8) shows.

The constant non-homogeneous term on the right hand side of eq. (2.8) can be removed by introducing \( u^* \) with

\[
u^* = u + gt , \quad \frac{du^*}{dx} = \frac{du}{dx} , \quad \frac{du^*}{dt} = \frac{du}{dt} + g .
\]  

(2.9)

Inserting (2.9) into (2.7) and (2.8) followed by addition and subtraction then yields the equations in the characteristic form

\[
\left\{ \frac{d}{dt} + (u + a) \frac{d}{dx} \right\} \left( u^* + \frac{2}{\gamma - 1} a \right) = 0 ,
\]  

(2.10)

\[
\left\{ \frac{d}{dt} + (u - a) \frac{d}{dx} \right\} \left( u^* - \frac{2}{\gamma - 1} a \right) = 0 ,
\]  

(2.11)

with

\[
u^* + \frac{2}{\gamma - 1} a = r , \quad u^* - \frac{2}{\gamma - 1} a = s ,
\]  

(2.12)

denoting the Riemann-invariants. From (2.10) and (2.11) it follows that along an \( r \)-characteristic with

\[
\frac{dx}{dt} = u + a = u^* + a - gt .
\]  

(2.13)
one has
\[ r = u^* + \frac{2}{\gamma - 1} a = u + gt + \frac{2}{\gamma - 1} a = \text{const.} \quad (2.14) \]

while along an s-characteristic with
\[ \frac{dx}{dt} = u - a = u^* - a - gt, \quad (2.15) \]

one has
\[ s = u^* - \frac{2}{\gamma - 1} a = u + gt - \frac{2}{\gamma - 1} a = \text{const.} \quad (2.16) \]

In Burgers' paper, ref. [1], the Riemann-invariants \( r, s \) are denoted by \( \theta, n \), while \( \gamma \) was taken 5/3, the value for the monatomic gas, yielding \( \frac{2}{\gamma - 1} = 3 \). Here \( \gamma \) is left free initially but later, from Section 7 onwards, \( \gamma \) is taken 3, yielding \( \frac{2}{\gamma - 1} = 1 \).
3. The Initial State of rest

Following the statement of Burgers' problem, the gas column is initially at rest with respect to the 'g-frame' and extends downwards from \( x = 0 \) to \( x = -\infty \). The speed of sound, the pressure etc. at \( x = 0 \), where a plane separates the gas from the vacuum above \( (x > 0) \) is denoted by \( a_o, p_o \) etc. In the initial state we have

(i) the situation is steady and \( \frac{\partial}{\partial t} = 0 \),

(ii) everywhere in the column \(- \infty \leq x \leq 0\), we have \( u = 0 \).

It follows that eqs. (2.2) and (2.7) are identically satisfied, while (2.8) reduces to the form

\[
\frac{\partial}{\partial x} (a^2) + (\gamma - 1) g = 0 .
\]  

(3.1)

Upon integration and using \( a = a_o \) for \( x = 0 \), this yields

\[
a^2 = a_o^2 - (\gamma - 1) gx .
\]  

(3.2)

Employing the expression \( a^2 = \gamma RT \) for the speed of sound, eq. (3.2) can be converted into

\[
c_p T + gx = c_p T_o .
\]  

(3.3)

which states that the total enthalpy, defined as the sum of the local enthalphy of the gas and the potential energy in the field of gravity, is constant along the gas column \((- \infty < x \leq 0)\).

In particular we note again (cf ref. [1]) that the temperature increases, when proceeding downwards into the gas column, according to the rule valid for a isentropic atmosphere at rest in a constant field of gravity.

This completes the solution for the initial state. In connection with things to follow it is useful to consider some further points.
The characteristics in the initial state are obtained from eqs. (2.13) - (2.16). Since \( u = 0 \) throughout, these equations read

\[
\frac{dx}{dt} = a, \quad r = gt + \frac{2}{\gamma - 1} a = \text{const.},
\]

\[
\frac{dx}{dt} = -a, \quad s = gt - \frac{2}{\gamma - 1} a = \text{const.}.
\]

Using the expression for \( a \) in (3.2) the equations \( \frac{dx}{dt} = \pm a \) are easily integrated and yield the expressions for \( r \) and \( s \) in (3.4) respectively in (3.5). Expressed in \( x \) and \( t \) the formulae take the form

\[
r = gt + \frac{2}{\gamma - 1} \sqrt{\frac{a^2}{g} - (\gamma - 1) gx},
\]

\[
s = gt - \frac{2}{\gamma - 1} \sqrt{\frac{a^2}{g} - (\gamma - 1) gx}.
\]

Removing the square root in (3.6) and (3.7), they may be rewritten

\[
\frac{a^2}{(\gamma - 1)g} - x = \frac{(\gamma - 1)g}{4} \left( \frac{r}{g} - t \right)^2 = \frac{(\gamma - 1)g}{4} \left( t - \frac{S}{g} \right)^2,
\]

yielding a pencil of parabolas in the \((t,x)\) plane with parameter \( r \) or \( s \). For a fixed value of \( r \) or \( s \) the eq. (3.8) represents a parabola with vertex at

\[
x = \frac{a^2}{(\gamma - 1)g}, \quad t = \frac{r}{g} \quad \text{or} \quad t = \frac{S}{g}.
\]

The symmetry axes of the parabolas are at \( t = \frac{r}{g} \) or \( t = \frac{S}{g} \), while the two branches extend from the vertex into the negative \( x \)-direction. Since the speed of sound \( a \) is always positive, the slope of the \( r \)-characteristic in (3.4) is always positive, while for the \( s \)-characteristic in (3.5) the slope is always negative. It follows that the left branches of the parabolas in (3.8), for values \( x \leq 0 \), represent the \( r \)-characteristics with \( t - \frac{r}{g} > 0 \), and the right branches, again for \( x \leq 0 \), the \( s \)-characteristics with \( t - \frac{S}{g} > 0 \). The characteristics are shown in Fig. 1, where also some of the \( r \)- and \( s \)-values are indicated.
In connection with the expansion flow to be studied we need to know the characteristics through \( t = 0, x = 0 \). Inspection of (3.6) and (3.7) shows that these characteristics have \( r = \frac{2}{\gamma - 1} a_0 \) and \( s = -\frac{2}{\gamma - 1} a_0 \), while (3.9) shows that they are segments of the left branch of the parabola with symmetry axis \( t = \frac{2}{\gamma - 1} a_0 \), respectively the right-branch of the parabola with symmetry axis \( t = -\frac{2}{\gamma - 1} \frac{a_0}{g} \). The \( s \)-characteristic through 0 is the so-called 'first sound wave', which will initiate the expansion at \( t = 0, x = 0 \) and proceeds from \( x = 0 \) in the negative \( x \)-direction. It also represents the curve in the \( t,x \)-plane along which the initial state of rest and the expansion flow are joined.

The \( r \)-characteristic intersecting the 'first sound wave' in \( x = 0, t = 0 \) has the value \( r = \frac{2}{\gamma - 1} a_0 \), while \( r \)-characteristics intersecting this sound wave at points with \( x < 0 \) have larger \( r \)-values, since they are segments of parabolae with symmetry axes further to the right in the \( t,x \)-plane and so \( t > \frac{2}{\gamma - 1} \frac{a_0}{g} \).

It follows that the \( r \)-characteristics intersecting the 'first sound wave' and proceeding into the expansion flow have \( r \)-values in the range

\[
\frac{2}{\gamma - 1} a_0 \leq r \leq +\infty .
\]

(3.10)

Addition and subtraction of \( r \) and \( s \) in (3.6) and (3.7) allows the expression of \( t \) and \( x \) in terms of \( r \) and \( s \).

One finds

\[
t = \frac{1}{2g} (r + s) ,
\]

(3.11)

\[
x = \frac{a_0^2}{(\gamma - 1)g} - \frac{\gamma - 1}{16g} (r - s)^2 ,
\]

(3.12)

and also

\[
x^* = x + \frac{1}{2} gt^2 = \frac{a_0^2}{(\gamma - 1)g} - \frac{\gamma - 1}{16g} (r - s)^2 + \frac{1}{8g} (r + s)^2 .
\]

(3.13)

The coordinate \( x^* \) will be discussed in the next section.
Since the values of $r$ and $s$ in the 'initial state' vary from one characteristic to another, the 'initial state' is not a 'constant domain' or 'simple wave', as obtained in the simplest flows where Riemann-invariants occur, but a 'general wave' with $r$ and $s$ varying both.

From the formulae (3.11) - (3.13) one notes that along the 'first sound wave' with $s = -\frac{2a_0}{\gamma-1}$ one obtains

$$t = \frac{1}{2g} \left[ r - \frac{2}{\gamma-1} a_0 \right], \quad (3.14)$$

$$x = \frac{a_0^2}{(\gamma-1)g} - \frac{\gamma-1}{16g} \left[ r + \frac{2}{\gamma-1} a_0 \right]^2, \quad (3.15)$$

$$x^* = \frac{a_0^2}{(\gamma-1)g} - \frac{\gamma-1}{16g} \left[ r + \frac{2}{\gamma-1} a_0 \right]^2 + \frac{1}{8g} \left[ r - \frac{2}{\gamma-1} a_0 \right]^2. \quad (3.16)$$

These relations will serve as boundary conditions along the 'first sound wave' for the expansion flow to be discussed.

The 'initial state' has now been discussed sufficiently to indicate with precision the domain where it applies.

In the $t,x$-plane the range of $x$ is $-\infty < x \leq 0$ for all $t \leq 0$. For $t > 0$ the values of $x$ range over the interval with underlimit $-\infty$, and the upperlimit formed by the 'first sound wave', i.e. the $s$-characteristic with $s = -\frac{2a_0}{\gamma-1}$ and passing through $t = 0$, $x = 0$. In detail the upperlimit of $x$, for $t > 0$ is

$$x = \frac{a_0^2}{(\gamma-1)g} - \frac{\gamma-1}{4g} \left[ t + \frac{2}{\gamma-1} a_0 \right], \quad (3.17)$$

obtained from (3.8) with $s = -\frac{2}{\gamma-1} a_0$ or from (3.14) and (3.15).

To consider the initial state in the $r,s$-plane, one observes from (3.11) and (3.12) that lines $t = \text{const}$ correspond with $r + s = \text{const}$, while lines $x = \text{const}.$ are $r - s = \text{const}.$ In particular $t = 0$ corresponds with $r + s = 0$ and $x = 0$ with $r - s = \frac{4a_0}{\gamma-1}$.
Since the gas column is bounded at $x = 0$ for $t \leq 0$, the straight line $r - s = \frac{4a_0}{\gamma - 1}$ is the boundary in the $r,s$-plane for $t \leq 0$. The point $t = 0, x = 0$ is reached for $r = \frac{2}{\gamma - 1} a_0$, $s = -\frac{2}{\gamma - 1} a_0$, and from then on the boundary is formed by the 'first sound wave', that is the line $s = -\frac{2}{\gamma - 1} a_0$ in the $r,s$-plane. The $r,s$-plane is shown in Fig. 2.
4. Further remarks on the equations of motion

The equations of motion were reduced to the characteristic forms (2.10) and (2.11) where the dependent variables are the Riemann-invariants \( r \) and \( s \). Since the coefficients of the partial derivatives in (2.10) and (2.11) do not only depend upon \( r \) and \( s \), but contain also the independent variable \( t \) in the term \( gt \), the equations are not reducible in the ordinary sense (See p. 79 of ref. [5], p. 39 of ref. [6]).

In this case however this deficiency can simply be removed. The expansion will be started at \( t = 0 \) and following Burgers [1], in addition to the speed \( u^* \) introduced in (2.9) a new coordinate \( x^* \) is introduced defined by \( \text{cf eq. (3.13)} \),

\[
x^* = x + \frac{1}{2} gt^2.
\]

(4.1)

The reference frame with the Cartesian coordinate \( x^* \) will be called the 'ff-frame' with 'ff' for 'falling-freely' under the influence of gravity. This nomenclature is easily verified by inspection of \( u \) and \( u^* \) in (2.9) and \( x \) and \( x^* \) in (4.1). The 'ff-frame' can be imagined to have started a very long time ago, with its origin at a very large negative \( x \), and with a very large positive velocity. Ever since that moment it has been falling freely. The retardation due to the continuously acting gravity is such that at \( t = 0 \) the velocity has just reached the value zero, and reverses its direction. Then at the instant \( t = 0 \) the 'g-frame' and the 'ff-frame' coincide, the 'ff-frame' is instantaneously at rest with respect to the physical 'g-frame' and the gas velocity \( u = u^* = 0 \) everywhere.

Changing the independent variables \( (t,x) \) in (2.10) and (2.11) to \( (t,x^*) \), while noting that

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x^*}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t} + gt \frac{\partial}{\partial x^*},
\]

(4.2)

one obtains
\[
\left(\frac{\partial}{\partial t} + (u^* + a) \frac{\partial}{\partial x^*}\right)\left[ u^* + \frac{2}{\gamma - 1} a \right] = 0 ,
\]
(4.3)
\[
\left(\frac{\partial}{\partial t} + (u^* - a) \frac{\partial}{\partial x^*}\right)\left[ u^* - \frac{2}{\gamma - 1} a \right] = 0 ,
\]
(4.4)
and observes that in the \(t,x^*\)-plane along an \(r\)-characteristic
\[
\frac{dx^*}{dt} = u^* + a , \quad r = u^* + \frac{2}{\gamma - 1} a = \text{const.} ,
\]
(4.5)
while along an \(s\)-characteristic
\[
\frac{dx^*}{dt} = u^* - a , \quad s = u^* - \frac{2}{\gamma - 1} a = \text{const.}
\]
(4.6)
The equations (4.3) and (4.4) are reducible in the ordinary sense, and the dependent and independent variables can, following Riemann [2], be interchanged in the usual way. Since
\[
u^* + a = \frac{\gamma - 1}{4} r + \frac{3 - \gamma}{4} s , \quad u^* - a = \frac{3 - \gamma}{4} r + \frac{\gamma + 1}{4} s ,
\]
(4.7)
and \(r\) is constant along an \(r\)-characteristic, we find upon taking \(r\) and \(s\) as independent variables, that the first equation in (4.5) can be written
\[
\frac{\partial x^*}{\partial s} - \left[\frac{\gamma + 1}{4} r + \frac{3 - \gamma}{4} s\right] \frac{\partial t}{\partial s} = 0 .
\]
(4.8)
In the same way along an \(s\)-characteristic, from (4.6)
\[
\frac{\partial x^*}{\partial r} - \left[\frac{3 - \gamma}{4} r + \frac{\gamma + 1}{4} s\right] \frac{\partial t}{\partial r} = 0 .
\]
(4.9)
Eliminating \(x^*\) and \(t\) in turn from the system (4.8) and (4.9) by taking the mixed second order derivatives \(x^*_{rs}\) and \(t_{rs}\) (subscripts denoting partial derivatives) and applying some simple algebra, one obtains for \(t(r,s)\) and \(x^*(r,s)\) the linear partial differential equations of second order
\[
(r - s) t_{rs} - m (t_r - t_s) = 0 ,
\]
(4.10)
\[
(r - s) x^*_{rs} - m \left[\frac{(\gamma + 1)r + (3 - \gamma)s}{(3 - \gamma)r + (\gamma + 1)s} x^* - \frac{(3 - \gamma)r + (\gamma + 1)s}{(\gamma + 1)r + (3 - \gamma)s} x^*\right] = 0 ,
\]
(4.11)
with

\[ m = \frac{\gamma + 1}{2(\gamma - 1)} \]  \hspace{1cm} (4.12)

The equation (4.10) is of the so-called Euler-Poisson-Darboux- or EPD-type. One may check that for \( \gamma = 3 \), for the monatomic gas with \( \gamma = \frac{5}{3} \), and for the diatomic gas with \( \gamma = \frac{7}{5} \), the coefficient \( m \) takes the values \( m = 1, 2 \) respectively 3.

For positive integer values of \( m \) the EPD-equation (4.10) is closely related to the standard wave equation

\[ w_{rs} = 0 \]  \hspace{1cm} (4.13)

with the general solution, usually called after d'Alembert,

\[ w(r,s) = R(r) + S(s) \]  \hspace{1cm} (4.14)

where \( R(r) \) and \( S(s) \) are arbitrary functions of the variable \( r \), respectively \( s \).

The general solution of (4.10), for positive integer \( m \) can be written

\[ t = \frac{2(m-1)}{\partial_r^{m-1}\partial_s^{m-1}} \left( \frac{R(r) + S(s)}{r-s} \right) \]  \hspace{1cm} (4.15)

as observed by G. Darboux (Ref. [7] pp. 62-65), where \( R(r) \) and \( S(s) \) are again arbitrary functions of \( r \), respectively of \( s \). Clearly the simplest form of (4.15) is obtained when \( m = 1 \) and \( \gamma = 3 \). The next simplest is the monatomic gas with \( \gamma = \frac{5}{3} \) and \( m = 2 \), considered by Burgers in ref. [1].

For greater values of \( m \), the differentiations required in (4.15) quickly make the expressions prohibitively complex. To integrate (4.10) for higher integer values of \( m \), and for non-integer \( m \) the integration method of Riemann [2] can be used. Burgers [1] used this method also for \( m = 2 \).

The equation (4.11) for \( x^*(r,s) \) is clearly more complex, than eq. (4.10) for \( t(r,s) \) and a general solution will not be attempted. The procedure to be followed is like this: we shall consider the general solution \( t(r,s) \) for \( \gamma = 3 \),
m = 1, and select the forms \( R(r) \) and \( S(s) \) in such a way, that the boundary conditions, still to be specified, are satisfied. Returning then to the characteristic equations (4.8) and (4.9) the solution \( x^*(r,s) \) is obtained by a quadrature. Finally it is checked that \( x^*(r,s) \) satisfies the equation (4.11).

Returning for a moment to the solutions of the initial state in Section 3 it is easily checked that the 'general waves' \( t(r,s) \) in (3.11) and \( x^*(r,s) \) in (3.13) are solutions of eq. (4.10) resp. of eq. (4.11).
5. The Equations of Motion in Lagrangian variables

Before turning to the expansion problem another point will be raised. So far the equations of motion were written in the Eulerian form with \( t \) and \( x \), or \( t \) and \( x^* \) as independent variables. It is often attractive, although not often done, to consider also the equations of motion in the Lagrangian variables \( t \) and \( h \), with \( h \) denoting the Lagrangian mass coordinate determined by

\[
dh = \rho \, dx \ , \tag{5.1}
\]

at a fixed time \( t = t_0 \), with \( t_0 \) suitably chosen.

The equations of motion in the Lagrangian variables then take the form

\[
\frac{\partial V}{\partial t} - \frac{\partial u}{\partial h} = 0 \ , \tag{5.2}
\]

\[
\frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = -g \ , \tag{5.3}
\]

\[
p V' = B_0 = b_0' \ , \tag{5.4}
\]

with \( V = \rho^{-1} \) denoting the specific volume, and \( u \) denoting the velocity with respect to the 'g-frame'. The first equation (5.2) is identically satisfied if a function \( E(t,h) \) can be found such that

\[
dE = ud\tau + Vdh \ , \quad E_t = u \ , \quad E_h = V \ , \tag{5.5}
\]

with subscripts denoting partial derivatives. It is easily verified that \( E(t,h) \) represents the Cartesian coordinate \( x \), indicating the position in the 'g-frame' of the fluid element \( h \) at time \( t \). In particular for fixed \( t = t_0 \) and \( dt = 0 \) eq. (5.5) reduces to (5.1).

To remove the non-homogeneous g-term in eq. (5.3) \( u \) is replaced by \( u^* \) defined in (2.9) and leading to the alternative homogeneous equations

\[
\frac{\partial V}{\partial t} - \frac{\partial u^*}{\partial h} = 0 \ , \tag{5.6}
\]

\[
\frac{\partial u^*}{\partial t} + \frac{\partial p}{\partial h} = 0 \ , \tag{5.7}
\]
Employing also eq. (4.1) the mass conservation equation (5.6) is identically satisfied by

\[ dx^*(t,h) = d\{x + \frac{1}{2} gt^2\} = u^*dt + Vdh , \]

\[ x^*_t = u^* , \quad x^*_h = V . \]  \hspace{1cm} (5.8)

The equations of motion (5.6) and (5.7) can be put in characteristic form by using the specific acoustic impedance \( \frac{a}{V} \) defined by

\[ \left( \frac{a}{V} \right)^2 = \frac{\gamma p}{V} = - \left[ \frac{\partial p}{\partial V} \right]_{S=\text{const.}} . \]  \hspace{1cm} (5.9)

Equations (5.6) and (5.7) then take the form

\[ \left( \frac{V}{a} \right)^2 \frac{\partial p}{\partial t} + \frac{\partial u^*}{\partial h} = 0 , \]  \hspace{1cm} (5.10)

\[ \frac{\partial u^*}{\partial t} + \frac{\partial p}{\partial h} = 0 , \]  \hspace{1cm} (5.11)

and the equations for the characteristics can be easily deduced.

Along an r-characteristic one has

\[ dh - \frac{a}{V} dt = 0 , \quad dp + \frac{a}{V} du^* = 0 , \]  \hspace{1cm} (5.12)

and along an s-characteristic

\[ dh + \frac{a}{V} dt = 0 , \quad dp - \frac{a}{V} du^* = 0 . \]  \hspace{1cm} (5.13)

One may check that the second equation in (5.12) can be written

\[ d\{u^* + \frac{2}{\gamma - 1} a\} = 0 , \]  \hspace{1cm} (5.14)

and the second equation in (5.13)

\[ d\{u^* - \frac{2}{\gamma - 1} a\} = 0 . \]  \hspace{1cm} (5.15)
It shows that along an r-characteristic the Riemann-invariant \( r \) in eqs. (4.5) is constant and similarly that \( s \) in eq. (4.6) is constant along an s-characteristic.

To obtain linear equations with independent variables \( r \) and \( s \), we note in analogy with the discussion in Section 4 that the first equation in (5.12) can be written in the forms

\[
h_s - \frac{a}{V} t_s = 0 , \quad \frac{V}{a} h_s - t_s = 0 . \tag{5.16}
\]

Also along the s-characteristics, from (5.13)

\[
h_r + \frac{a}{V} t_r = 0 , \quad \frac{V}{a} h_r + t_r = 0 . \tag{5.17}
\]

By cross-differentiation, addition and subtraction, the variables \( h \) and \( t \) can be eliminated in turn leading to

\[
t_{rs} + \frac{1}{2} \frac{a}{s} \left[ \ln \frac{a}{V} \right] t_r + \frac{1}{2} \frac{a}{sr} \left[ \ln \frac{a}{V} \right] t_s = 0 , \tag{5.18}
\]

\[
h_{rs} + \frac{1}{2} \frac{a}{s} \left[ \ln \frac{V}{a} \right] h_r + \frac{1}{2} \frac{a}{sr} \left[ \ln \frac{V}{a} \right] h_s = 0 . \tag{5.19}
\]

Employing the Riemann-invariants \( r \) and \( s \) one may deduce

\[
\frac{a}{V} = \left( \gamma B_0 \right) \frac{-1}{Y-1} \left( \frac{Y-1}{4} (r-s) \right)^{\frac{Y+1}{Y-1}} , \tag{5.20}
\]

and

\[
\ln \frac{a}{V} = \ln \text{(const)} + \frac{Y+1}{Y-1} \ln (r-s) . \tag{5.21}
\]

Substitution of (5.21) into (5.18) and (5.19) then finally yields

\[
(r-s) t_{rs} - m \left[ t_r - t_s \right] = 0 , \tag{5.22}
\]

\[
(r-s) h_{rs} + m \left[ h_r - h_s \right] = 0 , \tag{5.23}
\]

with \( m \) given by eq. (4.12).
Considering also the second equations in (5.12) and (5.13) with \( dp \) and \( du^* \) one verifies that the equation for \( u^* \) is identical to eq. (5.22) and the equation for \( p \) identical to (5.23). Since

\[
\begin{align*}
u^* &= \frac{1}{2} (r + s) \quad , \\
a &= \frac{\gamma - 1}{4} (r - s) = \sqrt{\frac{\gamma}{b_o}} \frac{\gamma - 1}{p} \quad ,
\end{align*}
\]

(5.24) (5.25)

it is easily verified that \( u^* \) in (5.24) is a solution of eq. (5.22) and \( p \) in (5.25) a solution of (5.23).

Inspection shows that eqs. (4.10) and (5.22) are identical while eq. (5.23) for \( h \) is simpler than eq. (4.11) for \( x^* \). For positive integer \( m \) the general solution of \( h \) is

\[
h = (r - s) \frac{2m+1}{\delta r^{m+1} \delta s^{m+1}} \left[ \frac{R(r) + S(s)}{r-s} \right] ,
\]

(5.26)

where \( R(r) \) and \( S(s) \) are arbitrary functions of \( r \) respectively of \( s \). Other relations between the solutions of \( t(r,s) \) and \( h(r,s) \), which also satisfy the characteristic equations in (5.16) and (5.17) can be found in ref. [7], in the Appendix to ref. [8] and in ref. [9].

Once \( t(r,s) \) has been found the Lagrangian \( h(r,s) \) will be obtained by a quadrature from (5.12) and (5.13) or (5.16) and (5.17), similar to the procedure described in Section 4 for the calculation of \( x^*(r,s) \).

We conclude this Section with a discussion of the initial state in terms of \( t \) and \( h \) as companion to the discussion in Section 3.

In the initial state \( u = 0 \), \( u^* = gt \) and assuming that \( x = 0 \), with \( p = p_o \) etc, corresponds with \( h = 0 \), the equations of motion (5.6) and (5.7) reduce to

\[
\begin{align*}
\delta u^* &= 0 \quad , \\
g + \delta p &= 0 \quad ,
\end{align*}
\]

(5.27)

leading to \( p = p_o - gh \). Via the homentropic relation (5.4) this leads to
\[ V = b_0 \frac{1}{\gamma} = b_0 \left( p_0 - gh \right)^{\frac{1}{\gamma}} = \left( \frac{\gamma}{\gamma-1} \right)^{\frac{1}{\gamma}} \frac{1}{\gamma} = v_0 \left( \frac{\gamma}{\gamma-1} \right)^{\frac{1}{\gamma}}. \]

\[
\left( \frac{a}{V} \right)^2 = \gamma \frac{p}{v} = \frac{\gamma}{b_0} \left( p_0 - gh \right)^{\frac{y+1}{\gamma}} = \left( \frac{a_0}{v_0} \right)^2 \left( \frac{p_0}{v_0} \right)^{\frac{y+1}{\gamma}}. \tag{5.28}
\]

\[ a^2 = a_0^2 \left( \frac{p_0}{v_0} \right)^{\frac{y-1}{\gamma}}. \]

With the expressions \( u^* = gt, \ p = p_0 - gh \) it follows that the two equations for the \( r \)-characteristics in (5.12) coincide. The same applies for the equations in (5.13) for the \( s \)-characteristics. Since integration of (5.12) leads to the Riemann-invariant \( r \), we have along the \( r \)-characteristics

\[
r = u^* + \frac{2}{\gamma-1} a = gt + \frac{2}{\gamma-1} a_0 \left( \frac{p_0}{v_0} \right)^{\frac{y-1}{2\gamma}} = \]

\[ = gt + \frac{2}{\gamma-1} a_0 \left( 1 - \frac{gh}{p_0} \right)^{\frac{y-1}{2\gamma}} = \text{const.} \tag{5.29}
\]

In the same way one deduces along the \( s \)-characteristics

\[
s = gt - \frac{2}{\gamma-1} a_0 \left( 1 - \frac{gh}{p_0} \right)^{\frac{y-1}{2\gamma}} = \text{const.} \tag{5.30}
\]

Equations (5.29) and (5.30) are the analogues in the \( t,h \)-plane of (3.6) and (3.7) in the \( t,x \)-plane.

From (5.29) and (5.30) \( t \) and \( h \) can be expressed in terms of \( r \) and \( s \). One obtains

\[
t = \frac{1}{2g} \left( r + s \right), \tag{5.31}
\]

\[
\frac{g}{p_0} h = 1 - \left( \frac{\gamma-1}{4a_0} (r - s) \right)^{\frac{2\gamma}{\gamma-1}},
\]

which may be compared to (3.11) - (3.13).
The characteristics through \( t = 0, \ h = 0 \) (coinciding with \( x = 0 \)) are \( r = \frac{2}{\gamma-1} a_0 \),
\( s = - \frac{2}{\gamma-1} a_0 \), the latter representing the 'first sound wave'.
Clearly along the 'first sound wave' one has

\[
t = \frac{1}{2g} \left( r - \frac{2}{\gamma-1} a_0 \right),
\]

\[
\frac{gh}{p_0} = 1 - \left( \frac{\gamma-1}{4a_0} \right) \left( r + \frac{2}{\gamma-1} a_0 \right)^{\frac{2\gamma}{\gamma-1}},
\]

which may be compared to (3.14) - (3.16).

Finally one may verify that the expressions for \( t \) and \( h \) in (5.31) are solutions of the equations (5.22) respectively (5.23).

Comparison of (3.6), (3.7) and (5.29), (5.30) also shows that \( h \) in the initial state, independent of \( t \), is simply a stretched version of the Cartesian \( x \), leading to

\[
1 - (\gamma - 1) \frac{\frac{\gamma-1}{a_0} \frac{gh}{p_0}}{\gamma} = \left( 1 - \frac{gh}{p_0} \right)^{\frac{\gamma-1}{\gamma}},
\]

which is easily manipulated to express \( h \) in \( x \), or \( x \) in \( h \).
6. The Formulation of the Expansion problem

The expansion of the gas is initiated, when at \( t = 0 \), the bounding plane at \( x = 0 \) is suddenly removed and the gas begins to move upwards into the vacuum above \( x = 0 \). The 'first sound wave', which is the s-characteristic through \( x = 0 \), \( t = 0 \) then proceeds down the gas column. Along this s-characteristic \( t \) and \( x^* \) are given by (3.14) and (3.16), \( t \) and \( h \) by (5.32), with \( s = -\frac{2a_0}{\gamma - 1} \) and \( \frac{2a_0}{\gamma - 1} \leq r \leq +\infty \) as discussed in Section 3. The above relations yield the first boundary condition for \( t[r, -\frac{2a_0}{\gamma - 1}], x^*[r, -\frac{2a_0}{\gamma - 1}] \) and \( h[r, -\frac{2a_0}{\gamma - 1}] \) during the expansion in the \( r,s \)-plane.

To obtain a second boundary condition for the expansion problem Burgers suggests, that locally in the immediate vicinity of \( x = 0 \), \( t = 0 \), and immediately after the sudden removal of the boundary plane at \( x = 0 \), the flow to be calculated will differ little from a complete centered simple wave with \( r = \frac{2a_0}{\gamma - 1} \). Such a flow is obtained when a semi-infinite amount of uniform gas at rest, with speed of sound \( a = a_0 \) and gravity absent, which occupies the domain \( -\infty < x \leq 0 \) for \( t < 0 \), is released at \( t = 0 \), by the sudden removal of a bounding plane at \( x = 0 \), and allowed to move into the vacuum extending along \( 0 < x \leq +\infty \).

In the homentropic simple wave \( r = \text{const.} \) and so we take

\[
r = u + \frac{2}{\gamma - 1} a = \frac{2}{\gamma - 1} a_0 .
\]

(6.1)

Since the s-characteristics are straight and centered we obtain for them

\[
\frac{dx}{dt} = u - a = \frac{x}{t} .
\]

(6.2)

From (6.1) and (6.2) one finds the well-known answers for the complete, homentropic, centered simple wave

\[
a = \frac{2}{\gamma + 1} a_0 - \frac{\gamma - 1}{\gamma + 1} \frac{x}{t} ,
\]

(6.3)

\[
u = \frac{2}{\gamma + 1} \left( a_0 + \frac{x}{t} \right) .
\]

(6.4)
Along the 'first sound wave', the first s-characteristic moving into the uniform gas, the velocity \( u \) is zero and \( \frac{x}{t} = -a_0 \). Along this s-characteristic also \( a = a_0 \) and \( s = -\frac{2a_0}{\gamma - 1} \).

The last s-characteristic in the simple wave is the vacuum characteristic, with \( a = 0 \) and \( \frac{x}{t} = \frac{2}{\gamma - 1} a_0 \), yielding also \( u = \frac{2}{\gamma - 1} a_0 = s \). It follows that in the homentropic complete centered simple wave \( r = \frac{2}{\gamma - 1} a_0 \), while the range of \( s \) is \(-\frac{2}{\gamma - 1} a_0 \leq s \leq \frac{2}{\gamma - 1} a_0\).

Assuming that the s-characteristics near \( x = 0, t = 0 \), in the expansion with gravity, coincide with or more precisely are tangent to the straight s-characteristics of the complete centered simple wave, just presented, we obtain, following Burgers, a second boundary condition for \( t(r,s) \) and \( r^*(r,s) \) in the \( r,s \)-plane which can be stated in the form: for \( r = \frac{2}{\gamma - 1} a_0 \),
\[-\frac{2}{\gamma - 1} a_0 \leq s \leq \frac{2}{\gamma - 1} a_0 \] we have \( t = 0, x^* = 0 \) and also \( x = 0 \) and \( h = 0 \).

Considering that the two boundary conditions, one along the 'first sound wave', the s-characteristic with \( s = -\frac{2a_0}{\gamma - 1} \), the other along the r-characteristic \( r = \frac{2}{\gamma - 1} a_0 \), are now given along segments of two characteristics, we have a characteristic boundary value problem or Goursat-problem (See ref. [10]), for the equations (4.10) and (4.11) and the equations (5.22) and (5.23).

From the theory of partial differential equations it is known that the solution to be constructed will apply in the rectangular domain of the \( r,s \)-plane bounded by \( \frac{2}{\gamma - 1} a_0 \leq r \leq +\infty, \frac{2}{\gamma - 1} a_0 \leq s \leq \frac{2}{\gamma - 1} a_0 \).

Finally it may be asked whether, once the motion has started, it will proceed indefinitely, or whether a state of rest will return. Burgers suggests that such a final state of rest will differ little from the initial state, except that the pressure and the speed of sound will be zero at the top of the column.

Assuming that further down in the column little has changed a reasonable estimate is that the top of the column will be at most at
\[
x = \frac{2a_0}{(\gamma - 1)g},
\]
which follows from eq. (3.2).
7. The fictitious ideal gas with $\gamma = 3$

It has been mentioned that Burgers worked out the expansion problem for the monatomic gas with $\gamma = \frac{5}{3}$, leading to $m = 2$. A simpler case is obtained by taking $\gamma = 3$ and $m = 1$. The general solution of the EPD-equation (4.10) is then

$$ t = \frac{R(r) + S(s)}{r-s} , \quad (7.1) $$

and differentiations according to (4.15) of the expression (7.1) are not required.

Most expressions obtained so far simplify when $\gamma = 3$ and these will be discussed in a moment. First some arguments are presented to clarify the physics of a gas with $\gamma = 3$. There are at least two distinct ways to give some background.

It is a classical result that the macroscopic ideal gas law (2.1) agrees with the molecular picture of a gas composed of (smooth spherical) molecules, which is in thermal equilibrium and so diluted or rarefied that the volume of the molecules is negligible compared to the volume occupied by the gas as a whole. In that situation each degree of freedom of a molecule has the energy $\frac{1}{2} kT$, with $k$ denoting the Boltzmann-constant and $T$ the equilibrium temperature in degrees-Kelvin.

In a monatomic gas composed of spherical molecules with three translational degrees of freedom, and molecular mass $m$, we obtain for the internal energy $U$ per unit mass

$$ U = c_v T = \frac{3}{2} \frac{k}{m} T = \frac{3}{2} R T = \frac{3}{2} \frac{R_M}{M} T . \quad (7.2) $$

with $R$ the gas constant per unit mass, $R_M$ the universal molar gas constant, $M$ the mass of a mole of the gas, while the Boltzmann-constant $k$ is the gas constant per molecule.

Since for the ideal gas also

$$ c_p - c_v = R = \frac{k}{m} , \quad (7.3) $$
one easily finds $\gamma = \frac{5}{3}$ for the monatomic gas.

For the gas composed of diatomic molecules with three translational- and 2 rotational-degrees of freedom (dumbbell-molecules) one finds in the same way $\gamma = \frac{7}{5}$.

Taking a gas composed of molecules with only one degree of freedom (one-dimensional on a molecular scale) we obtain by the same reasoning

$$c_v = \frac{1}{2} \frac{k}{m} = \frac{1}{2} R, \quad c_p = \frac{3}{2} \frac{k}{m} = \frac{3}{2} R,$$  \hspace{1cm} (7.4)

and find $\gamma = 3$. This is probably the best way to consider the gas with $\gamma = 3$.

Clearly at a given temperature $T$ and given $R$ the internal energy of a gas with $\gamma = 3$ is $\frac{1}{3}$ of the internal energy of the monatomic gas with the same $T$ and $R$.

A second method to consider the case $\gamma = 3$ is to note that the appearance of $\gamma$ in most gasdynamic relations originates from the isentropic relation (2.4). If the isentropic relation were replaced by

$$p = B_0 \rho^n,$$  \hspace{1cm} (7.5)

everywhere $\gamma$ would be replaced by $n$. Relations of the kind (7.5) are obtained for polytropic processes when every change of temperature in the gas is accompanied by heat addition or extraction according to the rule

$$dQ = T dS = c dT,$$  \hspace{1cm} (7.6)

with $c$ a constant specific heat and $S$ the entropy per unit mass. Clearly such a process is not adiabatic and isentropic. The second law of thermodynamics for the gas then takes the form

$$c dT = c_v dT + p d \left( \frac{1}{\rho} \right).$$  \hspace{1cm} (7.7)

Using the equation of state (2.1) and its logarithmic derivative, the relation (7.7) can be written in the form
\[ 0 = \left( c_v - c \right) \frac{dp}{p} - \left( c_p - c \right) \frac{dp}{\rho}, \quad (7.8) \]

and yields upon integration the polytropic relation (7.5) with \( n = \frac{c_p - c}{c_v - c} \).

Taking a monatomic gas with \( \gamma = \frac{5}{3} \), the value \( n = 3 \) is obtained if \( c = \frac{2}{3} c_v \).
Considering eq. (7.7) in that case, a change of \( c_v dT \) in the right-hand side, is for two thirds eliminated or compensated by \( cdT \) in the left-hand side, and effectively from the three degrees of freedom only one remains.

In the same way for \( \gamma = \frac{7}{5} \), the value \( n = 3 \) yields \( c = \frac{4}{5} c_v \) and again effectively one degree of freedom remains.

Especially R von Mises in ref. [11] has considered equations of the form (7.5) and more general one's, where \( n \) is not necessarily the ratio of the specific heats. He calls them 'specifying relations'. Formally it is not difficult to replace the isentropic- or homentropic-relation (2.4) by the polytropic relation (7.5), but conceptually and physically problems remain. Is there an adiabatic speed of sound in a gas where the polytropic relation (7.5) applies, and how is the continuous supply and extraction of heat to be realized, which changes of \( p \) and \( \rho \) require. It is not unlikely that problems of this nature have reduced the interest in the specifying relations, so that mainly the isentropic- or Poisson-relation (2.4) has remained.

In view of the preceding remarks we consider the gas with \( \gamma = 3 \) as a fictitious gas, where each molecule has only one degree of freedom, and which is allowed to pass through adiabatic changes of state.

Comparing the Riemann invariants \( r \) for gases with \( \gamma = 3, \frac{5}{3} \) and \( \frac{7}{5} \) one obtains
\[ r = u^* + a, \quad r = u^* + 3a, \quad r = u^* + 5a. \quad (7.9) \]

If an \( r \)-characteristic originates in a region with \( u^* = 0 \) and \( a = a_0 \) and proceeds to a domain where vacuum reigns, with \( a = 0 \) while \( u^* \) reaches its maximum value, it is clear that for the sequence of \( \gamma 's \) one has
\[ u_{max}^* = a_0, \quad u_{max}^* = 3a_0, \quad u_{max}^* = 5a_0. \quad (7.10) \]
Clearly molecules with more degrees of freedom can store more internal energy to be converted into kinetic energy during the expansion leading to larger $u_{\text{max}}$.

Finally we turn to the changes and simplifications appearing when $\gamma$ takes the value 3.

The isentropic relation (2.4) takes the form

$$p = B_0 \rho^3 = b_0^3 \rho^3,$$  \hspace{1cm} \text{(7.11)}

and several relations in Section 2 are obtained by substituting $\gamma = 3$.

In the initial state of rest we have from (3.2) that

$$u = 0, \quad a = \sqrt{a_0^2 - 2gx}.$$  \hspace{1cm} \text{(7.12)}

The $r$- and $s$-characteristics in (3.6) - (3.8) take the form

$$r = gt + \sqrt{a_0^2 - 2gx},$$  \hspace{1cm} \text{(7.13)}

$$s = gt - \sqrt{a_0^2 - 2gx},$$  \hspace{1cm} \text{(7.14)}

$$\frac{a_0^2}{2g} - x = \frac{g}{2} \left( \frac{E}{g} - t \right)^2 = \frac{g}{2} \left( t - \frac{s}{g} \right)^2,$$  \hspace{1cm} \text{(7.15)}

which represent two pencils of parabola in the $t,x$-plane (with parameter $r$, respectively $s$) with vertex at

$$x = \frac{a_0^2}{2g}, \quad t = \frac{r}{g} \quad \text{and} \quad t = \frac{s}{g}.$$  \hspace{1cm} \text{(7.16)}

The left-branches of the parabola, where $\frac{E}{g} - t > 0$, are the $r$-characteristics, the right-branches where $t - \frac{s}{g} > 0$ are the $s$-characteristics as discussed in Section 3.
Turning to Section 4 one notes that the Riemann-invariants \( r \) and \( s \) in (4.5) and (4.6) now take the form

\[
    r = u^* + a, \quad s = u^* - a. \tag{7.17}
\]

The equation (4.3) and (4.4) then read

\[
    \frac{\partial r}{\partial t} + r \frac{\partial r}{\partial x^*} = 0, \quad \frac{\partial s}{\partial t} + s \frac{\partial s}{\partial x^*} = 0. \tag{7.18}
\]

From (7.18) one concludes that along an \( r \)-characteristic

\[
    \frac{dx^*}{dt} = r, \quad r = \text{const}, \tag{7.19}
\]

and along an \( s \)-characteristic

\[
    \frac{dx^*}{dt} = s, \quad s = \text{const}. \tag{7.20}
\]

It follows that all the characteristics in the \( t,x^* \)-plane, both \( r \)- and \( s \)-characteristics, are straight lines. This is the most interesting simplification for the gas with \( \gamma = 3 \).

Inspection of (4.8) - (4.11) yields for \( \gamma = 3 \)

\[
    \frac{\partial x^*}{\partial s} - r \frac{\partial t}{\partial s} = 0, \quad \frac{\partial x^*}{\partial r} - s \frac{\partial t}{\partial r} = 0, \tag{7.21}
\]

\[
    (r - s) \frac{t_{rs}}{t_s} - (t_r - t_s) = 0, \tag{7.22}
\]

\[
    (r - s) \frac{x_{rs}}{x_s} - (\frac{t_s}{r} x_r - \frac{s}{r} x_s) = 0. \tag{7.23}
\]

*) From (4.8) and (4.9) one observes that in addition to \( \gamma = 3 \) also \( \gamma = -1 \) is a special case, which leaves (4.8) and (4.9) in the form

\[
    \frac{\partial x^*}{\partial s} - s \frac{\partial t}{\partial s} = 0, \quad \frac{\partial x^*}{\partial r} - r \frac{\partial t}{\partial r} = 0 \tag{7.24}
\]

The special value \( \gamma = -1 \) became well-known in the study of 2-dimensional steady gas flows by the work of S.A. Chaplygin [12], von Karman-Tsien [13] and others.
Next the characteristics (7.13) - (7.15) in the initial state in the t,x-plane are transformed to the t,x*-plane.

This yields

\[
\frac{a_0^2}{2g} - x^* + \frac{1}{2} gt^2 = \frac{g}{2} \left( \frac{r}{g} - t \right)^2 = \frac{g}{2} \left( t - \frac{s}{g} \right)^2 ,
\]

which is easily converted into the pencils of straight lines

\[
x^* - \frac{a_0^2}{2g} = r \left( t - \frac{r}{2g} \right) = s \left( t - \frac{s}{2g} \right) .
\]

The r-characteristics have slope r in the t,x*-plane and pass through

\[
x^* = \frac{a_0^2}{2g}, \quad t = \frac{r}{2g} ,
\]

while the s-characteristics have slope s and pass through

\[
x^* = \frac{a_0^2}{2g}, \quad t = \frac{s}{2g} .
\]

The expressions (7.26) can also be written

\[
x^* = rt - \frac{r^2 - a_0^2}{2g} , \quad x^* = st + \frac{a_0^2 - s^2}{2g} .
\]

The r- and s-characteristics through \( x = 0 , \ t = 0 , \ x^* = 0 \) have \( r = a_0 \), respectively \( s = -a_0 \) for \( \gamma = 3 \) and so they are

\[
x^* = a_0 t , \quad x^* = -a_0 t .
\]

The t,x*-plane with the characteristics for the steady initial state of the gas is shown in Fig. 3.

Considering finally the relations in Section 5 for \( \gamma = 3 \) we note that

\[
\frac{a}{V} = \left( 3 \ B_0 \right)^{-\frac{1}{2}} \left[ \frac{1}{2} \ (r - s) \right]^2 ,
\]

from (5.20). Also (5.29) and (5.30) take the form
\[ r = gt + a_0 \left( 1 - \frac{gh}{p_0} \right)^{\frac{1}{3}} = \text{const}, \]  
\[ s = gt - a_0 \left( 1 - \frac{gh}{p_0} \right)^{\frac{1}{3}} = \text{const}, \]

and so on. Finally we note that (5.33) transforms into

\[ 1 - \frac{2gx}{a_0} = \left( 1 - \frac{gh}{p_0} \right)^{\frac{2}{3}}, \]

which again allows us to express \( x \) in \( h \), or conversely \( h \) in \( x \).
8. The expansion problem for \( \gamma = 3 \), and its solution

In Section 6 the expansion problem was formulated for arbitrary \( \gamma \). When \( \gamma = 3 \) it takes the following form:

expressions \( t(r,s) \) and \( x^*(r,s) \) have to be found which satisfy eq. (7.22), respectively eq. (7.23) and moreover the following boundary-conditions

(i) along the characteristic \( s = -a_0 \), representing the 'first sound wave', with \( a_0 < r \leq \infty \), \( t \) respectively \( x^* \) have the form

\[
t = \frac{1}{2g} (r - a_0) , \quad x^* = -\frac{a_0}{2g} (r - a_0) .
\]

which follows from (3.14), respectively (3.16) when \( \gamma \) is taken 3.

(ii) The second boundary-condition follows from Section 6 by taking \( \gamma = 3 \) and reads: for \( r = a_0 \), \( -a_0 \leq s \leq a_0 \), one has

\[
t(a_0, s) = 0 , \quad x^*(a_0, s) = 0 .
\]

The solution to be constructed is valid in the domain of the \( r,s \)-plane bounded by

\[
a_0 \leq r \leq \infty , \quad -a_0 \leq s \leq a_0 .
\]

The \( r,s \)-plane is shown in Fig. 4.

The general solution for \( t(r,s) \) which satisfies eq. (7.22) has the form (7.1). Substituting the boundary-condition for \( t \) in (8.1) yields

\[
\frac{1}{2g} (r - a_0) = \frac{R(r) + S(-a_0)}{r + a_0} ,
\]

and determines \( R(r) \), leading to \( t(r,s) \) in the form

\[
t(r,s) = \frac{1}{2g} \frac{r^2 - a_0^2}{r-s} + \frac{S(s) - S(-a_0)}{r-s} .
\]
The second boundary-condition for \( t \) in (8.2) yields

\[
0 = \frac{S(s) - S[-a_0]}{a_0 - s}.
\]

leading to \( S(s) = S[-a_0] = \text{const.} \), provided \( a_0 - s \neq 0 \). In this way one arrives at

\[
t = \frac{1}{2g} \frac{r^2 - a_0^2}{r - s},
\]

valid for \( a_0 \leq r \leq +\infty \), \( -a_0 \leq s \leq a_0 \).

It may be checked that the same answer is obtained if the boundary-conditions are applied in the reverse order. Direct substitution of (8.7) in eq. (7.22) shows that this equation is satisfied.

The expression \( x^*(r,s) \) is obtained from the characteristic equations (7.21). Differentiation of (8.7) yields

\[
t_r = \frac{1}{2g} \left[ 1 + \frac{a_0^2 - s^2}{(r-s)^2} \right], \quad t_s = \frac{1}{2g} \frac{r^2 - a_0^2}{(r-s)^2},
\]

and with (7.21)

\[
x_r^* = \frac{1}{2g} s \left[ 1 + \frac{a_0^2 - s^2}{(r-s)^2} \right], \quad x_s^* = \frac{1}{2g} \frac{r(r^2 - a_0^2)}{(r-s)^2}.
\]

Integration of \( x_s^* \) yields

\[
x^* = \frac{1}{2g} \left[ \frac{r(r^2 - a_0^2)}{r - s} + f(r) \right],
\]

with \( f(r) \) an 'integration constant', which may be a function of \( r \). Differentiation of \( x^* \) in (8.10) with respect to \( r \) should result in \( x_r^* \) in (8.9). From (8.10) one obtains
\[ x_r^* = \frac{1}{2g} \left( 2r + s \frac{s(a_0^2 - s^2)}{(r-s)^2} + f'(r) \right), \quad (8.11) \]

and comparison with (8.9) yields

\[ f'(r) + 2r = 0, \quad f(r) = C - r^2 \quad (8.12) \]

From the second boundary condition in (8.2) one obtains \( C = a_0^2 \) and so from (8.10)

\[ x^*(r,s) = \frac{1}{2g} \left( \frac{r[r^2 - a_0^2]}{r-s} - \left(r^2 - a_0^2\right) \right) = \frac{1}{2g} \frac{s[r^2 - a_0^2]}{r-s}. \quad (8.13) \]

One may check that \( x^* \) satisfies the boundary-condition for \( x^* \) in (8.1) and is a proper solution of eq. (7.23).

To obtain the Lagrangian mass-coordinate \( h(r,s) \) we note that the equations for the characteristics in the \( t,h \)-plane, and for \( \gamma = 3 \), take the form, obtained from (5.16) and (5.17), with \( \frac{a}{v} \) given by (7.29). They are

\[ h_s = \frac{1}{4a_0 v_0} (r - s)^2 t_s, \quad h_r = -\frac{1}{4a_0 v_0} (r - s)^2 t_r. \quad (8.14) \]

Substitution of \( t(r,s) \) from (8.7) and (8.8) yields

\[ h_s = \frac{1}{8a_0 v_0 g} \left( r^2 - s_0^2 \right), \quad (8.15) \]

\[ h_r = -\frac{1}{8a_0 v_0 g} \left( 2rs - r^2 - s_0^2 \right). \quad (8.16) \]

Integration of (8.15) yields

\[ h = \frac{1}{8a_0 v_0 g} \left( \left[r^2 - s_0^2\right] s + F(r) \right). \quad (8.17) \]

Differentiation of (8.17) to \( r \) and comparison with (8.16) yields

\[ F''(r) = -\left(r^2 + a_0^2\right), \quad (8.18) \]
and upon integration

\[ F(r) = -\frac{1}{3} r^3 - a_0^2 r + C. \quad (8.19) \]

Imposing the condition that \( \eta = 0 \) coincides with \( x = 0 \), and so for \( r = a_0 \),
\(- a_0 \leq s \leq a_0 \) one has \( h = 0 \), then leads to \( C = \frac{4}{3} a_0^3 \) yielding for \( h \) the
alternative forms

\[
\begin{align*}
 h &= \frac{1}{8a_0 V_0 G} \left[ \left( r^2 - a_0^2 \right) s - \frac{1}{3} r^3 - a_0^2 r + \frac{4}{3} a_0^3 \right] = \\
 &= \frac{1}{8a_0 V_0 G} \left[ - \left( r - a_0 \right) a_0^2 + \left( r^2 - a_0^2 \right) s - \frac{1}{3} \left( r^3 - a_0^3 \right) \right] = \\
 &= \frac{1}{8a_0 V_0 G} \left[ \left( r - a_0 \right) \left( rs - \frac{1}{3} r^2 + a_0 s - \frac{1}{3} a_0^2 r - \frac{4}{3} a_0^2 \right) \right].
\end{align*}
\quad (8.20)
\]

It may be verified by direct substitution that \( h(r,s) \) in (8.20) is a solution of
eq. (5.23) for \( m = 1 \) and that along the 'first sound wave' \( s = -a_0 \) the eq.
(5.32) for \( \gamma = 3 \) is satisfied.

This completes the construction of \( t(r,s), x^*(r,s) \) and \( h(r,s) \) in the expansion
phase when \( \gamma = 3 \). Comparison with the answers obtained by Burgers in Ref. [1]
for \( \gamma = \frac{5}{3} \), shows that the answers for \( \gamma = 3 \) are simpler. This simplicity
enables us to invert the expressions (8.7) and (8.13) to a considerable degree.
It will be considered in the following Sections. In Section 9 we begin with the
discussion in the \( r,s \)-plane.
9. The solutions in the r,s-plane

In general the expressions (8.7), (8.13) and (8.20) obtained for \( t(r,s) \), \( x^*(r,s) \) and \( h(r,s) \) yield single values for any pair \( r,s \) of the independent variables in the domain \( a_0 \leq r \leq +\infty, -a_0 \leq s \leq +a_0 \).

Taking \( r = a_0 \) in (8.7), (8.13) and (8.20), while leaving \( s \) free yields \( t[a_0,s] = 0 \), \( x^*[a_0,s] = 0 \) and \( h[a_0,s] = 0 \), in agreement with the boundary condition (8.2) and the relation imposed upon \( h \) to obtain (8.20).

Taking \( s = -a_0 \), the boundary conditions (8.1) valid for \( t \) and \( x^* \) along the 'first sound wave' are retrieved, while \( h \) takes the form (5.32) with \( \gamma = 3 \).

Next we take \( s = +a_0 \) and find along this last \( s \)-characteristic of the 'expansion fan',

\[
t = \frac{1}{2g} (r + a_0), \quad x^* = \frac{a_0}{2g} (r + a_0), \quad h = -\frac{p_0}{g} \left( \frac{r-a_0}{2a_0} \right)^3.
\]  

(9.1)

Approaching the point \( [a_0, a_0] \) along \( s = a_0 \) by taking the limit \( r \to a_0^+ \) one obtains

\[
t[a_0, a_0] = \frac{a_0}{g}, \quad x^*[a_0, a_0] = \frac{a_0^2}{g}, \quad h = 0.
\]

(9.2)

However, approaching the point \( [a_0, a_0] \) along \( r = a_0 \) we just found

\[
t[a_0, a_0] = 0, \quad x^*[a_0, a_0] = 0, \quad h[a_0, a_0] = 0,
\]

(9.3)

and so \( t \) and \( x^* \) are multivalued in the point \( [a_0, a_0] \) of the r,s-plane.

It will now be shown that \( t \) and \( x^* \) at \( [a_0, a_0] \) can assume all the values in the intervals

\[
0 \leq t \leq \frac{a_0}{g}, \quad 0 \leq x^* \leq \frac{a_0^2}{g}.
\]

(9.4)
depending upon the direction from which the point \([a_0, a_0]\) is approached.

Consider therefore the straight line

\[ s - a_0 = -m \left( r - a_0 \right), \]  \hspace{1cm} (9.5)

with slope \(-m\) \((0 \leq m \leq +\infty)\), passing through \([a_0, a_0]\).

For \(m = 0\) eq. (9.5) represents the characteristic \(s = a_0\), for \(m = \infty\) it yields \(r = a_0\), when approaching \(a_0\) from below.

Substituting (9.5) into (8.7) and (8.13) yields

\[ t = \frac{1}{2g} \frac{r + a_0}{1 + m}, \quad x^* = \frac{1}{2g} \frac{r + a_0}{1 + m} [a_0 - m \left(r - a_0\right)]. \]  \hspace{1cm} (9.6)

Keeping \(m\) fixed and allowing \(r\) to approach \(a_0\) from above, this leads to

\[ t = \left[ \frac{a_0}{g} \right] \frac{1}{1 + m}, \quad x^* = \frac{a_0}{g} \frac{1}{1 + m}. \]  \hspace{1cm} (9.7)

One observes that \(t\) and \(x^*\) in (9.7) pass through the intervals in (9.4) when \(m\) passes through \(0 \leq m \leq +\infty\).

The above discussion shows that \([a_0, a_0]\) is a singular point of the solution where \(t\) and \(x^*\) are multivalued.

In order to study the inversion of \(t(r, s)\) and \(x^*(r, s)\), one notes that in the neighbourhood of an ordinary point \((r, s)\) one has

\[ dt = t_r \, dr + t_s \, ds, \]  \hspace{1cm} (9.8)

\[ dx^* = r^* \, dr + x^*_s \, ds. \]

The increments \(dr\) and \(ds\), and therefore also increments in \(u^*\) and the speed of sound \(a\), can be uniquely expressed in \(dt\) and \(dx^*\) provided the Jacobian \(J\) of the system (9.8), defined by
\[ J = t_r x^*_s - x_r^* t_s, \quad (9.9) \]

is different from zero. The Jacobian is easily found from (8.8) and (8.9) and
yields
\[ J = \frac{1}{4g^2} \frac{[r^2 - a_0^2][r^2 - 2rs + a_0^2]}{(r-s)^3}. \quad (9.10) \]

Inspection shows that \( J = 0 \) for \( r = a_0 \) and for
\[ r^2 - 2rs + a_0^2 = 0, \quad (9.11) \]
provided \( r - s \neq 0 \).

The only point in the domain \( a_0 \leq r \leq +\infty, -a_0 \leq s \leq +a_0 \) where \( r - s = 0 \) is the point \([a_0, a_0]\) discussed in connection with the multivalued \( t[a_0, a_0] \) and \( x^*[a_0, a_0] \).

Clearly in this singular point also inversion breaks down.

Along the characteristic \( r = a_0, -a_0 \leq s \leq +a_0 \) the boundary-condition (8.2) required \( t = 0, x^* = 0 \), while (8.8) and (8.9) yield \( t_s = x_s^* = 0 \) along it. Clearly along \( r = a_0, -a_0 \leq s \leq +a_0 \) the Jacobian vanishes making the inversion of (9.8) impossible there, as the inspection of (9.10) already indicated.

Finally the curve in (9.11), along which \( J = 0 \), represents a hyperbola, which is outside the domain \( a_0 \leq r \leq +\infty, -a_0 \leq s \leq +a_0 \), with the exception of the point \([a_0, a_0]\).

In this point the hyperbola is tangent to \( s = +a_0 \). Differentiation of (9.11) yields
\[ 2r - 2s - 2r \frac{ds}{dr} = 0, \quad (9.12) \]
and \( \frac{ds}{dr} = 0 \) for \( r = s = a_0 \). Also for \( r = s = a_0 \) eq. (9.11) is satisfied indicating that \([a_0, a_0]\) is a point of the hyperbola.
It is not difficult to reduce (9.11) to standard form and to verify the points mentioned. Since there are no terms linear in $r$ and $s$ in (9.11) the centre of the conic is at $(0,0)$. By a rotation of axes, defined by

$$r = r' \cos \alpha - s' \sin \alpha,$$

$$s = r' \sin \alpha + s' \cos \alpha,$$

the expression (9.11) takes the form

$$\left[ \frac{1}{2} (1 + \cos 2\alpha) - \sin 2\alpha \right] (r')^2 - \left[ \sin 2\alpha + 2 \cos 2\alpha \right] r's' + \left[ \frac{1}{2} (1 - \cos 2\alpha) + \sin 2\alpha \right] (s')^2 + a_0^2 = 0.$$  \hspace{1cm} (9.14)

The $r's'$- term in (9.14) can be removed by selecting

$$\tan 2\alpha = -2, \quad \sin 2\alpha = -\frac{2}{\sqrt{5}}, \quad \cos 2\alpha = \frac{1}{\sqrt{5}},$$ \hspace{1cm} (9.15)

leading to the standard form

$$\frac{1}{2} (\sqrt{5} - 1) (s')^2 - \frac{1}{2} (\sqrt{5} + 1) (r')^2 = a_0^2.$$ \hspace{1cm} (9.16)

From (9.15) one easily finds

$$\alpha = -31.717^\circ = -32^\circ.$$ \hspace{1cm} (9.17)

A picture of the $r,s$-plane with the hyperbola (9.11) or (9.16) is shown in Fig. 5.

Writing (9.16) also in the standard form

$$\frac{(s')^2}{A^2} - \frac{(r')^2}{B^2} = 1,$$ \hspace{1cm} (9.18)

with $A$ and $B$ the semi-major, and semi-minor axis one easily checks
\[ A^2 = \frac{\sqrt{5} + 1}{2} a_0^2, \quad B^2 = \frac{\sqrt{5} - 1}{2} a_0^2, \]  
(9.19)

and

\[ A = 1.2720 \ a_0, \quad B = 0.7861 \ a_0. \]  
(9.20)

This concludes to a large extent the discussion of \( t(r,s), x^*(r,s) \) and \( h(r,s) \) in the \( r,s \)-plane. The main result is that, apart from 2 exceptions, the entire domain \( a_0 \leq r \leq + \infty, \ -a_0 \leq s \leq a_0 \) represent ordinary points for the solutions \( t(r,s), x^*(r,s) \) and \( h(r,s) \), yielding single values for these parameters, and permitting the local inversion.

The two exceptions are

(i) The point \( r = s = a_0 \), where \( t \) and \( x^* \) are multivalued and no inversion is possible,

(ii) the segment \( -a_0 \leq s \leq a_0 \) of \( r = a_0 \) where again the inversion is impossible.

Compared to the results obtained by Burgers for \( \gamma = \frac{5}{3} \) the solutions obtained for \( \gamma = 3 \) seem simple.

In the two figures 6 and 7 the \( r,s \)-plane is shown in a dimensionless form. In Fig. 6 lines \( t = \text{const} \) are shown and in Fig. 7 lines \( h = \text{const} \), which represent particle paths.

To make (8.7) dimensionless we put

\[ x = \frac{r}{a_0}, \quad y = \frac{s}{a_0}, \quad z = \frac{2s}{a_0} t, \]  
(9.21)

and \( t \) assumes the form

\[ z = \frac{x^2 - 1}{x - y}. \]  
(9.22)

To make \( h(r,s) \) in (8.20) dimensionless we put
\[ x = \frac{r}{a_0}, \quad y = \frac{s}{a_0}, \quad z = \frac{8v_0^2}{a_0^2} h = \frac{8g}{3p_0} h. \]  

(9.23)

and obtain from (8.20)

\[ z = \left[ x^2 - 1 \right] y - \left( \frac{1}{3} x^3 + x - \frac{4}{3} \right). \]  

(9.24)

Curves \( z = \text{const}, \) obtained from (9.22) and (9.24) have been drawn in the rectangular domain \( 1 \leq x \leq 11, \ -1 \leq y \leq 1, \) in Fig. 6 resp. Fig. 7.

Before finishing this Section it is of interest also to bring the decompositions of the Riemann-invariants \( r \) and \( s \) in the discussion. For \( \gamma = 3 \) we have from (7.17)

\[ r = u^* + a, \quad s = u^* - a. \]  

(7.17)

Along \( r = a_0, \ -a_0 \leq s \leq a_0, \) where \( t = 0 \) and \( x^* = 0 \) we obtain from (7.17)

\[ u^* = \frac{1}{2} \left[ a_0 + s \right], \quad a = \frac{1}{2} \left[ a_0 - s \right]. \]  

(9.25)

This indicates that \( a \) reduces from the value \( a = a_0 \) to \( a = 0 \) when \( s \) runs from \( -a_0 \) to \( +a_0 \) along \( r = a_0, \) while \( u^* \) increases from \( 0 \) to \( a_0. \) This all occurs at the instant \( t = 0 \) and location \( x^* = 0, \) and \( x = 0. \)

Considering also the multivalued \( t \) in \( [a_0, a_0] \) and the relation (2.9) between \( u^* \) and \( u, \) one has

\[ u^* [a_0, a_0] = u + gt = a_0, \quad a = 0. \]  

(9.26)

With the interval of \( t \) in (9.4) one then finds that in the \( g \)-frame \( u \) reduces from the value \( u = a_0, \) to \( u = 0 \) when \( t \) proceeds from \( t = 0 \) to \( t = \frac{a_0}{g} \) in the point \( [a_0, a_0] \) of the \( r,s \)-plane. Considering also the interval of \( x^* \) in (9.4) and the relation (4.1) between \( x \) and \( x^* \) in the '\( g \)-frame' and the 'ff-frame' one finds that \( x \) proceeds from the value \( 0 \) to \( \frac{a_0^2}{2g} \) in the point \( [a_0, a_0] \) of the \( r,s \)-plane.
10. The flow in the $t,x^*$-plane

Before considering the 'expansion flow' it is desirable to discuss the 'initial state' in the $t,x^*$-plane.

In the initial state the gas column is limited by the boundary plane at $x = 0$, which according to the formula (4.1) corresponds to the parabola

$$x^* = \frac{1}{2} gt^2,$$  \hspace{1cm} (10.1)

with vertex at $t = 0, x^* = 0$ in the $t,x^*$-plane.

Due to the choice $\gamma = 3$, it was found in Section 7 that all the characteristics in the $t,x^*$-plane are straight lines. For the initial state they are given by the equations (7.25) - (7.27). The $r$- and $s$-characteristics through $t = 0, x^* = 0$ are given by (7.28). The $s$-characteristic $x^* = - a_0 t$ is the 'first sound wave' and forms together with the left branch of the parabola (10.1) the boundary of the domain in the $t,x^*$-plane, where the expressions, valid in the initial state, apply.

If no expansion would take place, by keeping the lid on the gas column the relations of the initial state would persist also for $0 \leq t \leq +\infty$ and the complete parabola in (10.1) would form the boundary.

The answers for the expansion flow $t(r,s)$ and $x^*(r,s)$ were obtained in Section 8. Inspection of (8.7) and (8.13) shows that

$$x^* = st, \quad - a_0 \leq s \leq a_0,$$ \hspace{1cm} (10.2)

and substituting $s$ from (10.2) into (8.7) yields

$$x^* = rt - \frac{1}{2g} \left(r^2 - a_0^2\right), \quad a_0 \leq r \leq +\infty.$$ \hspace{1cm} (10.3)

In this way the $r$- and $s$-characteristics are shown to be two pencils of straight lines. The $s$-characteristics are centered at $x^* = 0, t = 0$. The pencil of $r$-characteristics in the expansion flow is identical with eq. (7.27) for the $r$-characteristics in the initial state. Since for $\gamma = 3$ all characteristics are
straight, and the flow suffers no discontinuities, when passing from the 'initial state' into the 'expansion flow' across the 'first sound-wave' (the s-characteristic \( s = -a_0 \)), the r-characteristics in the 'expansion flow' are the continuations of the straight r-characteristics from the 'initial state' and they retain their form. For \( \gamma \) different from 3 this is clearly not the case.

The s-characteristics represent the expansion fan due to the sudden removal of the bounding plane at \( t = 0, x^* = 0 \). The first s-characteristic is the 'first sound wave' with \( s = -a_0 \) and the last s-characteristic in the expansion fan has \( s = +a_0 \) and since there is vacuum above the bounding plane it seems reasonable to assume \( a = 0 \) along this 'vacuum-characteristic'. It follows then that along it

\[
a = 0, \quad s = u^* - a = u^* = a_0.
\]

(10.4)

Inspection also shows that the last s-characteristic \( x^* = a_0 t \), of the expansion fan can also be considered as the continuation of the r-characteristic \( r = a_0 \) of the initial state, passing through \( t = 0, x^* = 0 \) and shown in (7.28), or by putting \( r = a_0 \) in (10.3). With the characteristics \( r = a_0 \) and \( s = a_0 \) coinciding one necessarily has

\[
a = 0, \quad u^* = a_0,
\]

(10.5)

giving a rigorous argument for the assumption \( a = 0 \).

The characteristic with \( r = a_0 \) forms the limit of the r-characteristics in the 'initial state' and proceeding into the 'expansion-flow' across the 'first sound wave'. The coordinates \( t \) and \( x^* \) of the point where an r-characteristic intersects the 'first sound wave' are determined by solving simultaneously

\[
x^* = -a_0 t, \quad x^* = rt - \frac{1}{2g} \left( r^2 - a_0^2 \right),
\]

(10.6)

and yield

\[
t = \frac{1}{2g} \left( r - a_0 \right), \quad x^* = -\frac{a_0}{2g} \left( r - a_0 \right),
\]

(10.7)
on the 'first sound wave', while \( a_0 \leq r \leq a \), Eq. (10.7) clearly represents the boundary condition (8.1).

The greater values of \( r \) occur further down in the gas column and emerge later from the 'initial state' passing into the 'expansion domain'.

Since \( r \) also represents the slope of the \( r \)-characteristic, the greater \( r \)-values, penetrating later into the 'expansion domain', will intersect the \( r \)-characteristics which penetrated earlier into the 'expansion flow' and which have a smaller slope in the \( t,x^* \)-plane.

If characteristics of one family intersect, shock-waves are anticipated and the inviscid theory loses its physical significance due to the steep gradients occurring already before the actual intersection takes place. To see how this process develops here we construct the envelope of the \( r \)-characteristics.

Writing the family of \( r \)-characteristics in the form

\[
f(t,x;r) = x^* - rt + \frac{r^2 - a_0^2}{2g} = 0 ,
\]

(10.8)

the envelope is found upon eliminating the parameter \( r \) from

\[
f(t,x^*;r) = 0 , \quad \frac{\delta f}{\delta r} (t,x^*;r) = 0 .
\]

(10.9)

One finds

\[
\frac{\delta f}{\delta r} = -t + \frac{r}{g} = 0 ,
\]

(10.10)

and elimination of \( r \) results in

\[
(x^* - \frac{a_0^2}{2g}) - \frac{1}{2} gt^2 = 0 ,
\]

(10.11)

representing a parabola of the same form as (10.1) but with vertex
\[ t = 0, \quad x^* = \frac{a_0^2}{2g}. \]  

(10.12)

The intersection of the envelope (10.11) with the 'vacuum-characteristic' \( x^* = a_0 t \) results in the equation

\[ \left( t - \frac{a_0}{g} \right)^2 = 0, \]

(10.13)
yielding a double root \( t = \frac{a_0}{g}, \quad x^* = \frac{a_0^2}{g} \), indicating that the 'vacuum-characteristic' is tangent to the parabola (10.11) in the point obtained. The values \( t = \frac{a_0}{g}, \quad x^* = \frac{a_0^2}{g} \) appeared, together with \( t = 0, \quad x^* = 0 \), in Section 9, eq. (9.4) when the multiple values of \( t \) and \( x^* \) were discussed in the point \( \{a_0, a_0\} \) of the \( r,s \)-plane.

Considering the intersection of the parabola (10.11) with an \( r \)-characteristic, whose \( r \)-value is greater than \( a_0 \), yields a double root \( t = \frac{r}{g} \) and \( x^* = \frac{1}{2g} \left( r^2 + a_0^2 \right) \). Since \( s = a_0 \) is the last \( s \)-characteristic of the expansion fan given in the \( t,x^* \)-plane by

\[ x^* = a_0 t, \]

(10.14)
and the boundary of the expansion domain in the \( t,x^* \)-plane it seems that the intersection points

\[ t = \frac{r}{g}, \quad x^* = \frac{1}{2g} \left( r^2 + a_0^2 \right), \quad r > a_0, \]

(10.15)
on the parabola (10.11) have no physical significance since they are located beyond the boundary (10.14) of the expansion domain, where our solution applies.

Considering the intersection of the last \( s \)-characteristic \( s = a_0 \) in (10.14) with an \( r \)-characteristic, in (10.3), one easily obtains

\[ t = \frac{1}{2g} \left( r + a_0 \right), \quad x^* = \frac{a_0}{2g} \left( r + a_0 \right), \]

(10.16)
for the position of the intersection point, corresponding with \( t \) and \( x^* \) in (9.1).

The only point of the envelope (10.11), which lies in the domain where our solutions apply is then the point \( t = \frac{a_0}{g} \), \( x^* = \frac{a_0^2}{g} \), and even this is not entirely true, since the physical significance of the solutions vanishes already when the envelope is approached closely.

Since the singularity concerns only one point of the domain in the \( r,s \)-plane, and only one point in the \( t,x^* \)-plane, while we are not at the moment prepared for a more detailed analysis, we proceed by ignoring the difficulty which appeared and assume that the \( s \)-characteristic \( s = a_0 \), given by (10.14) will remain the boundary of the expansion domain, as it was in the \( r,s \)-plane. The intersection of (10.14) with an \( r \)-characteristic whose \( r \) value is greater than \( a_0 \) then yields the coordinates (10.16) for the intersection point and in addition

\[
r = u^* + a, \quad s = a_0 = u^* - a ,
\]

(10.17)

leading to

\[
u^* = \frac{1}{2} \left[ r + a_0 \right], \quad a = \frac{1}{2} \left[ r - a_0 \right].
\]

(10.18)

Considering also the relation (2.9) between \( u \) and \( u^* \), and the coordinate \( t \) of the intersection point in (10.16) we obtain

\[
u^* = u + gt = u + \frac{1}{2} \left[ r + a_0 \right].
\]

(10.19)

Comparison of (10.18) and (10.19) then yields \( u = 0 \), indicating that in the physical \( g \)-frame the state of rest, returns beyond the point \( t = \frac{a_0}{g} \), \( x^* = \frac{a_0^2}{g} \), in the \( t,x^* \)-plane along the last \( s \)-characteristic \( s = a_0 \).

Beyond this point of the envelope there is however no vacuum-state since \( a \) is given by the expression in (10.18) and increases steadily with \( r \).
In Fig. 3 the tx*-plane was shown for $\gamma = 3$ in the state of rest when no sudden expansion at $t = 0$ would occur.

In Fig. 8 the tx*-plane is shown with the expansion fan centered at $t = 0$, $x^* = 0$ and the different straight characteristics ($\gamma = 3$).
11. The flow in the t,x-plane

The t,x-plane is the physical plane where the actual flow in the gravity-field takes place. Its relation with the flow in the t,x*-plane, discussed in Section 10, is easily obtained from the transformation formulae (2.9) and (4.1)

\[ u^* = u + gt, \quad (2.9) \]
\[ x^* = x + \frac{1}{2} gt^2. \quad (4.1) \]

The 'initial state' of the gas in the t,x-plane was discussed in Section 3.

From the solutions of t(r,s) and x*(r,s) it was found that the r-characteristics in the t,x*-plane were a set of straight lines, which passed without change from the domain of the 'initial state' into the 'expansion flow' upon crossing the 'first sound wave', the s-characteristic s = - a_0.

Applying the transformation (4.1) it follows that the r-characteristics in the 'expansion flow' are segments of the parabola, which were obtained already when discussing the 'initial state' in Section 3.

It is easily checked that also the straight and centered s-characteristics are parabola in the t,x-plane. They can be written

\[ x^* = x + \frac{1}{2} gt^2 = st, \quad (11.1) \]

which is easily converted to

\[ [x - \frac{s^2}{2g}] + \frac{1}{2} g \left[t - \frac{s}{g}\right]^2 = 0. \quad (11.2) \]

This represents a pencil of parabola, with parameter s and vertex at

\[ x = \frac{s^2}{2g}, \quad t = \frac{s}{g}. \quad (11.3) \]

Inspection shows that they all pass through t = 0, x = 0.
The values of $s$ in the expansion flow are in the interval $-a_0 \leq s \leq a_0$, with $s = -a_0$ the 'first sound wave' and $s = a_0$ the last $s$-characteristic. The values $s = -a_0$ and $s = a_0$ yield the parabola

$$\left(x - \frac{a_0^2}{2g}\right) + \frac{1}{2} g \left(t + \frac{a_0}{g}\right)^2 = 0 \, ,$$

(11.4)

with vertex $x = \frac{a_0^2}{2g}$, $t = -\frac{a_0}{g}$ and symmetry-axis $t = -\frac{a_0}{g}$, together with

$$\left(x - \frac{a_0^2}{2g}\right) + \frac{1}{2} g \left(t - \frac{a_0}{g}\right)^2 = 0 \, ,$$

(11.5)

which has the vertex $x = \frac{a_0^2}{2g}$, $t = \frac{a_0}{g}$ and symmetry-axis $t = \frac{a_0}{g}$.

The solution obtained applies in the domain bounded by these two parabolae.

While the vertices of the $s$-characteristics vary in both $t$- and $x$-direction when $s$-changes, the $r$-characteristics have vertices only changing in $t$-direction. The $r$-characteristics, given in the $t,x^*$-plane by the second relation in (10.6) assume upon applying formula (4.1) the form

$$\left(x - \frac{a_0^2}{2g}\right) + \frac{1}{2} g \left(t - \frac{r}{g}\right)^2 = 0 \, ,$$

(11.6)

representing a pencil of parabolae with vertices $t = \frac{r}{g}$, $x = \frac{a_0^2}{2g}$. With increasing $r$ the symmetry-axes shift to larger $t$-values. One also checks that the envelope of the $r$-characteristics in the $t,x^*$-plane given by eq. (10.11) is mapped into the straight line $x = \frac{a_0^2}{2g}$, which is the tangent to all the parabolae in (11.6) in their vertices. Only one point of this envelope, the point

$$x = \frac{a_0^2}{2g} \, , \quad t = \frac{a_0}{g} \, ,$$

(11.7)

belongs to the domain of the expansion flow indicated above. All the other points of the envelope fall outside the domain where our solutions apply. This
behaviour corresponds nicely with the features in the \( t, x^* \)-plane discussed in Section 10.

The \( r \)-characteristic \( r = a_0 \) and the left branch of the \( s \)-characteristic \( s = a_0 \) coincide and along this segment \( t \) and \( x \) pass through the interval

\[
0 \leq t \leq \frac{a_0}{g}, \quad 0 \leq x \leq \frac{a_0^2}{2g}, \tag{11.8}
\]

which may be compared with the multiple values of \( t \) and \( x^* \) in the point \( r = s = a_0 \) of the \( r, s \)-plane and given in formulae (9.4). Application of (4.1) then shows that \( t = \frac{a_0}{g}, \; x^* = \frac{a_0^2}{g} \) leads to \( x = \frac{1}{2} \frac{a_0^2}{g} \) shown in (11.7).

Repeating some of the discussion already presented in Sections 9 and 10, we note again that along the segment where the \( r \)-characteristic \( r = a_0 \) and the \( s \)-characteristic \( s = a_0 \) coincide one has

\[
\begin{align*}
    r &= u^* + a = a_0, \\
    s &= u^* - a = a_0,
\end{align*} \tag{11.9}
\]

yielding

\[
\begin{align*}
    a &= 0, \\
    u^* &= a_0, \\
    u &= a_0 - gt.
\end{align*} \tag{11.10}
\]

Hence the vacuum state is reached instantaneously at \( t = 0 \) and persists during \( 0 \leq t \leq \frac{a_0}{g} \).

The velocity \( u \) instantaneously assumes the value \( a_0 \) at \( t = 0 \), but then decreases to zero when \( t \) proceeds through \( 0 \leq t \leq \frac{a_0}{g} \). Clearly in the vertex of the parabola \( x = \frac{a_0^2}{2g} \) the gas reaches its highest point with \( a = 0 \) and \( u = 0 \).

Considering then intersections of the right-branch of the parabola \( s = a_0 \), with left branches of \( r \)-characteristics with \( r > a_0 \), the discussion in Section 10 with the relations (10.17) - (10.19) is essentially repeated.

From \( r = u^* + a, \; s = a_0 = u^* - a \) one has
\[ u^* = \frac{1}{2} (r + a_0) , \ u = \frac{1}{2} (r + a_0) - gt , \ a = \frac{1}{2} (r - a_0) . \quad (11.11) \]

Also upon intersecting the parabola (11.2) with \( s = a_0 \) and the \( r \)-characteristic (11.6) with \( a_0 \leq r \leq a_0 + \infty \) one obtains

\[ t = \frac{r + a_0}{2g} , \quad (11.12) \]

in the point of intersection, yielding \( u = 0 \) in (11.11) and indicating that the gas has returned to a state of rest while its speed of sound increases with \( r \) as shown in (11.11), or already earlier in (10.18).

In Fig. 9 the \( t, x \) plane for \( \gamma = 3 \), is shown with the expansion fan centered at \( t = 0, \ x = 0 \). Both families of characteristics are parabolae, with the \( s \)-characteristics forming the expansion fan.
12. Concluding Remarks

A problem considered by J.M. Burgers in Ref. [1] for $\gamma = \frac{5}{3}$, has been calculated in this Report for the value $\gamma = 3$. It is found, as surmised, that this simplifies the answers to some degree.

In particular it is found that the critical points in the solution-domain, found by Burgers for $\gamma = \frac{5}{3}$, do not appear, when the value $\gamma = 3$ is chosen. The only singular points found, when $\gamma = 3$, are along the edge of the solution-domain. To be precise they are
(i) the segment $a_0 \leq s \leq a_0$ of the characteristic $r = a_0$,
(ii) the point $(a_0, a_0)$ in the $r,s$-plane.

In the case $\gamma = \frac{5}{3}$ singularities also appear along curves inside the solution domain. It follows that the behaviour of the fictitious gas with $\gamma = 3$, is much simpler in this situation than the gas with $\gamma = \frac{5}{3}$. Physically speaking there is no overshoot of the velocity and no flow reversal accompanied by the appearance of a shock-wave.

Since the different values of $\gamma$ are related to the amount of internal energy stored in the molecules with different numbers of degrees of freedom it is of interest to note the relations between the internal energy stored in molecules with one, three, and possibly five degrees of freedom and the distinct macroscopic features of the flow patterns, which occur under kinematically speaking identical conditions.
References


10. I.G. Petrowski, 'Vorlesungen über partielle Differentialgleichungen'.


12. S.A. Chaplygin, 'On Gas Jets', Moscow, 1902, Sci Mem. Moscow Univ., Math-

13. H.S. Tsien, 'Two-dimensional subsonic flow - of compressible fluids'.
J. Aeron Sc. 6, 1939, pp. 399-407.
PARABOLAE

1. \( \frac{a_0^2}{(\gamma-1)g} - x = \frac{(\gamma-1)g}{4} \left( t + \frac{2a_0}{(\gamma-1)g} \right)^2 \)

2. \( \frac{a_0^2}{(\gamma-1)g} - x = \frac{(\gamma-1)g}{4} gt^2 \)

3. \( \frac{a_0^2}{(\gamma-1)g} - x = -\frac{(\gamma-1)g}{4} \left( \frac{2a_0}{(\gamma-1)g} - t \right)^2 \)

FIG. 1. THE STATE OF REST IN THE t,x-PLANE.
BA - $x=0$ for $t\leq 0$
AD - "FIRST SOUND WAVE", $s=-\frac{2a_0}{\gamma-1}$
Point A is $x=0, t=0$
Point 0 is $x=\frac{a_0^2}{(\gamma-1)g}, t=0$

FIG. 2: THE $r,s$-PLANE FOR THE INITIAL STATE.
FIG. 3: \( t, x^* \) - PLANE FOR \( \gamma = 3 \)
AD: "FIRST SOUND WAVE" $s = -a_0$
\[ t = \frac{1}{2g} (r - a_0), x^* = -\frac{a_0}{2g} (r - a_0) \]

AB: $r = a_0, -a_0 \leq s \leq a_0$
\[ t(a_0, s) = 0, x^*(a_0, s) = 0 \]

DABE: Solution domain for the expansion flow.

FIG. 4: THE $r,s$-PLANE FOR THE INITIAL STATE AND THE EXPANSION-FLOW.
FIG. 5: THE \( r,s \)-PLANE WITH THE HYPERBOLA WHERE \( J = 0 \).
Graph of $z = \frac{x^2 - 1}{x - y}$

$1 \leq x \leq +\infty$

$-1 \leq y \leq +1$

$x = \frac{r}{a_0}, y = \frac{s}{a_0}, z = \frac{2a}{a_0} t$

FIG. 6: DIMENSIONLESS FORM OF $t(r,s) = \text{CONST.}$
GRAPH OF $z = (x^2-1)y - \frac{1}{3}(x^3+3x-4)$

$1 \leq x \leq = \infty$

$-1 \leq y \leq 1$

$x = \frac{r}{a_0}$, $y = \frac{s}{a_0}$, $z = \frac{8V_0 g}{3a_0^2}$, $h = \frac{8g}{3p_0}$

1. $z = -0.70$
2. $z = -3.33$
3. $z = -10.67$
4. $z = -24$
5. $z = -45.3$
6. $z = -76.67$
7. $z = -120$
8. $z = -177.33$

FIG. 7: DIMENSIONLESS FORM OF $h(r,s) = \text{CONST.}$
FIG. 8: THE FLOW IN THE $t,x^*$-PLANE.
FIG. 9: THE FLOW IN THE t,x-PLANE.