Delft University of Technology
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Eigenschappen van de Racah polynomen met betrekking tot de Lie algebra representatie van $sl(2,\mathbb{C})$ (Engelse titel: Properties of the Racah polynomials with regard to the Lie algebra representation of $sl(2,\mathbb{C})$)

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"Eigenschappen van de Racah polynomen met betrekking tot de Lie algebra representatie van $sl(2,\mathbb{C})$ "

(Engelse titel: "Properties of the Racah polynomials with regard to the Lie algebra representation of $sl(2,\mathbb{C})$")

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Abstract

The Racah polynomial $R_n(\lambda(x))$ is a polynomial of degree n and is variable in $\lambda(x)$. In this thesis two properties of this polynomial will be studied. One is the orthogonal property of the Racah polynomial. And the other is that the Racah polynomial can also be described as a polynomial of degree x and variable over $\lambda(n)$.

The Racah polynomials will be studied with the use of a representation of the Lie algebra of $sl(2, \mathbb{C})$ and hypergeometric series. To do this, this Lie algebra will first be defined and then we will work towards defining the tensor product of three representations of the Lie algebra $sl(2, \mathbb{C})$.

From the tensor product, a series representation for the Racah polynomials will be found, which can be rewritten to a hypergeometric series.

Then, the orthogonal property of $sl(2, \mathbb{C})$ will be used to study the orthogonal property of the Racah polynomials. And the polynomial will be rewritten as a polynomial of degree x with the use of some identities of the hypergeometric series.

Het Racah polynoom $R_n(\lambda(x))$ is een polynoom van graad n en heeft als variabele $\lambda(x)$. In dit verslag zullen twee eigenschappen van het Racah polynoom worden bekeken. Dit zijn de orthogonaliteits eigenschap van het Racah polynoom, en de eigenschap dat het polynoom te schrijven is als een $x-$de graads polynoom variabel in $\lambda(n)$.

De Racah polynomen zullen worden bestudeerd met behulp van een representatie van de Lie algebra $sl(2, \mathbb{C})$ en de hypergeometrische functies. Om dit te doen, zal eerst de definie van een Lie algebra worden gegeven. Dat zal worden uitgebreid tot een definitie van een tensorproduct van drie representaties van de Lie algebra $sl(2, \mathbb{C})$.

Dit tensorproduct levert uiteindelijk een reeks op waarmee de Racah polynoom gedefinieerd kan worden. Deze reeks kan dan worden herschreven tot een hypergeometrische functie.

De orthogonliteits eigenschap van het Racah polynoom zal worden bewezen met behulp van de eigenschappen van de Lie algebra $sl(2, \mathbb{C})$. En het herschrijven van het Racah polynoom naar een polynoom van graad x, zal worden gedaan met behulp van eigenschappen van de hypergeometrische functies.

Contents

1 Introduction

Note, this thesis uses a lot from the work of Joris Van de Jeugt, [1]. Most of the given proofs, theorems and definitions come from this book. And this thesis follows the same guidelines to show the properties of the Racah polynomials as in the work of Joris Van de Jeugt, [1].

The Racah polynomials are polynomials with a few special properties. The properties of the Racah polynomials have been studied before, see [1], so the properties that we will look at are already well known.

The Racah polynomials $R_n(\lambda(x))$ are polynomials of degree n and variable in $\lambda(x)$. The two properties that we will look at in this thesis are the orthogonality property of the Racah polynomials, and that we can rewrite $R_n(\lambda(x))$ to a polynomial of degree x and variable over n.

When one would start with the definition of the Racah polynomial, it isn't immediately clear if the polynomial has the described properties. But, one can construct the polynomials with the use of the tensor product of three representations of $sl(2, \mathbb{C})$.

When we describe the Racah polynomials with the use of representations of the Lie algebra $sl(2, \mathbb{C})$. It becomes clear that we can use the orthonormality of the basis of this representation, to check the orthogonality property of the Racah polynomials.

For the other property we need to use hypergeometric series. Because the definition of the Racah polynomials that will be constructed, can be written as one of these series. And there are several identities for the hypergeometric series, which we can use to rewrite the Racah polynomial.

To do all of this, we will start by defining what a Lie algebra is, and what a Lie algebra representation is. In Chapter 2 we will take a brief look at the hypergeometric series, and some of their identities. And then we will only need to look at the tensor product of representations, and construct the definition of the Racah polynomial.

In this thesis, we will define the Lie algebra and Lie algebra representations over a field K. This field will be either the field of real numbers, $\mathbb{R}$, or the field of complex number, $\mathbb{C}$.

2 Lie algebra and representations

In this chapter we will give the definition of a Lie algebra. With the help of some examples, the definition of $sl(2,\mathbb{C})$ will be given. After that we will define a $\mathbf{g}-$module and a representation and look at some properties.

After that we will define a special $\mathbf{g}-$module, the Verma module of $sl(2,\mathbb{C})$. From this $\mathbf{g}-$module we can construct the representation that we will use to study the Racah polynomials. This is the $\star-$representation of D_j.

At the end of this chapter some extra details on the Verma module will be given. This is a more abstract approach and won't be used later on.

2.1 Lie algebra

Definition 2.1. Let $\mathbf{g}$ be a vector space over a field K endowed with a bilinear operation $[\,\cdot\,,\,\cdot\,]:\mathbf{g}\times\mathbf{g}\to\mathbf{g}$. Then, $\mathbf{g}$ is called a **Lie algebra** over K if the following properties are satisfied for $x,y,z\in\mathbf{g}$:

i. $[x,y]=-[y,x]$ (anti-symmetry)

ii. $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ (Jacobi-identity),

The operation $[\,\cdot\,,\,\cdot\,]$ is also referred to as the bracket, the Lie bracket or the commutator of $\mathbf{g}$. That the operation $[\,\cdot\,,\,\cdot\,]$ is sometimes called the commutator is quite obvious. We only have to look at how we can turn an algebra $\mathbf{g}$ over a field K into a Lie algebra. Note, an algebra $\mathbf{g}$ is just a vector space endowed with a bilinear operation, which is called the multiplication of $\mathbf{g}$.

Theorem 2.2. *Let $\mathbf{g}$ be an algebra over a field K, with the product of $\mathbf{g}$ denoted as $x\cdot y\equiv xy$, $x,y\in\mathbf{g}$. Then, $\mathbf{g}$ can be turned into a Lie algebra by defining the commutator as,*

$$[\,x\,,\,y\,]=xy-yx.$$

Proof. Take $x,y,z\in\mathbf{g}$ arbitrary. And define the commutator $[\,x\,,\,y\,]=xy-yx$ as above. Now it is easy to see that the commutator is bilinear, because the product of $\mathbf{g}$ is bilinear.
We also have $xy-yx=-(yx-xy)$, so the commutator is also anti-symmetric.
And we see that,

i. $[\,x\,,\,[\,y\,,\,z\,]\,]=xyz-xzy-yzx+zyx,$

ii. $[\,y\,,\,[\,z\,,\,x\,]\,]=yzx-yxz-zxy+xzy,$

iii. $[\,z\,,\,[\,x\,,\,y\,]\,]=zxy-zyx-xyz+yxz.$

So, the commutator also satisfies the Jacobi-identity. Which shows that $\mathbf{g}$ is indeed a Lie algebra over the field K. □

Now, note that a Lie algebra is only a vector space endowed with a bilinear operation. And because we know when a subset of a vector space is a subspace, it is also interesting to know when a subset of a Lie algebra $\mathbf{g}$ is a subalgebra of $\mathbf{g}$.

Definition 2.3. Let $\mathbf{g}$ be a Lie algebra over a field K and let $\mathbf{h}\subseteq\mathbf{g}$ be a subspace of $\mathbf{g}$. Then, $\mathbf{h}$ is a **subalgebra** of $\mathbf{g}$ if,

$$[\,x\,,\,y\,]\in\mathbf{h}\qquad\forall x,y\in\mathbf{h}.$$

As with subspaces, a subalgebra is also a Lie algebra on its own. Because a subalgebra is a vector space, and the properties for the Lie bracket are also satisfied when we look at $[\,x\,,\,y\,]$, for $x,y\in\mathbf{h}$. Now, a subspace $\mathbf{I}$ of a Lie algebra $\mathbf{g}$ could also satisfy a stronger property than the one given above.[2] Which leads to the definition of an ideal $\mathbf{I}$ of $\mathbf{g}$.

Definition 2.4. Let $\mathbf{g}$ be a Lie algebra over a field K, and let $\mathbf{I} \subseteq \mathbf{g}$ be a subspace of $\mathbf{g}$. Then, $\mathbf{I}$ is an **ideal** of $\mathbf{g}$ if,

$$[\,x\,,\,y\,] \in \mathbf{I} \qquad \forall x \in \mathbf{g} \text{ and } \forall y \in \mathbf{I}.$$

Notice that this means that an ideal of $\mathbf{g}$ is also a Lie algebra on its own. This follows from letting $x \in \mathbf{I}$ in the above definition. Also, a Lie algebra $\mathbf{g}$ always has $\{0\}$ and $\mathbf{g}$ as its ideals. Which is why these are called the **trivial ideals**.[2]

There are several easy and well known examples of Lie algebras. So, to get more familiar with Lie algebras and the bracket, some examples will be given. Some are well known examples, such as $\mathbb{R}^3$ endowed with the cross product. And one example will be given, because it will be used further on.

Example 2.5. Let's start with the well known example, $\mathbb{R}^3$ endowed with the cross product $\cdot \times \cdot : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. For $a, b \in \mathbb{R}^3$, with $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$, we have,

$$a \times b = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1).$$

Note that this is a bilinear operation and that $a \times b = -b \times a$. Because, for $a, b \in \mathbb{R}^3$ we have that,

i. $a_2 b_3 - a_3 b_2 = -(a_3 b_2 - a_2 b_3)$

ii. $a_3 b_1 - a_1 b_3 = -(a_1 b_3 - a_3 b_1)$

iii. $a_1 b_2 - a_2 b_1 = -(a_2 b_1 - a_1 b_2)$.

So we only have to check the Jacobi-identity. This is not really difficult, but it does take more time to check than the other properties. Now, take $x, y, z \in \mathbb{R}^3$ and $x \times y \equiv [a, b]$, then we have,

$$\begin{aligned}[x\,,\,[y\,,\,z]] &= [x\,,\,(y_2 z_3 - y_3 z_2\,,\,y_3 z_1 - y_1 z_3\,,\,y_1 z_2 - y_2 z_1)] \\ &= (x_2(y_1 z_2 - y_2 z_1) - x_3(y_3 z_1 - y_1 z_3)\,, x_3(y_2 z_3 - y_3 z_2) - x_1(y_1 z_2 - y_2 z_1) \\ &\quad , x_1(y_3 z_1 - y_1 z_3) - x_2(y_2 z_3 - y_3 z_2)).\end{aligned}$$

So, for a vector $p = (p_1, p_2, p_3) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \in \mathbb{R}^3$ we have,

$$\begin{aligned}p_1 = {} & x_2 y_1 z_2 - x_2 y_2 z_1 - x_3 y_3 z_1 + x_3 y_1 z_3 + y_2 z_1 x_2 - y_2 z_2 x_1 - y_3 z_3 x_1 + y_3 z_1 x_3 + z_2 x_1 y_2 - z_2 x_2 y_1 \\ & - z_3 x_3 y_1 + z_3 x_1 y_3 = 0\end{aligned}$$

$$\begin{aligned}p_2 = {} & x_3 y_2 z_3 - x_3 y_3 z_2 - x_1 y_1 z_2 + x_1 y_2 z_1 + y_3 z_2 x_3 - y_3 z_3 x_2 - y_1 z_1 x_2 + y_1 z_2 x_1 + z_3 x_2 y_3 - z_3 x_3 y_2 \\ & - z_1 x_1 y_2 + z_1 x_2 y_1 = 0\end{aligned}$$

$$\begin{aligned}p_3 = {} & x_1 y_3 z_1 - x_1 y_1 z_3 - x_2 y_2 z_3 + x_2 y_3 z_2 + y_1 z_3 x_1 - y_1 z_1 x_3 - y_2 z_2 x_3 + y_2 z_3 x_2 + z_1 x_3 y_1 - z_1 x_1 y_3 \\ & - z_2 x_2 y_3 + z_2 x_3 y_2 = 0.\end{aligned}$$

Which shows that $\mathbb{R}^3$ endowed with the cross product satisfies both properties of the Lie algebra. So, $\mathbb{R}^3$ endowed with the cross product is indeed a Lie algebra.

Example 2.6. A more general and abstract example is the ring End($\mathcal{V}$), the set of linear transformations from $\mathcal{V}$ to $\mathcal{V}$. To show that End($\mathcal{V}$) is indeed a Lie algebra, let's start with an arbitrary vector space $\mathcal{V}$ over a field K.
Then, the ring End($\mathcal{V}$) can be constructed with the use of all the endomorphisms $f\colon \mathcal{V} \to \mathcal{V}$. This is done by using the addition of $\mathcal{V}$, which is commutative. Also, we will define an addition on End($\mathcal{V}$), such that End($\mathcal{V}$) will be a group under addition. So, we define,

$$\begin{aligned} f(x+y) &= f(x)+f(y) = f(y)+f(x) = f(y+x), \qquad \text{for } x,y \in \mathcal{V}, f \in \text{End}(\mathcal{V}), \\ (f+g)(x) &= f(x)+g(x) \qquad \text{for } x \in \mathcal{V}, f,g \in \text{End}(\mathcal{V}), \end{aligned}$$

where the first equation is the definition of an endomorphism.[3]
Because the addition of $\mathcal{V}$ is commutative, End($\mathcal{V}$) is a group under addition. This is easily provable, because we only need to check that $f+g$, f,g $\in$End($\mathcal{V}$), is indeed an endomorphism. Because the identity is trivial, $id_{\mathcal{V}}(v) = 0, \forall v \in \mathcal{V}$, and the inverse of f $\in$End($\mathcal{V}$) is $f^{-1}(v) = -f(v), \forall v \in \mathcal{V}$. Now, take f,g $\in$End($\mathcal{V}$) and $x,y \in \mathcal{V}$ arbitrary. Then we have,

$$(f+g)(x+y) = (f+g)(v) = f(v)+g(v) = f(x)+f(y)+g(x)+g(y) \quad v = x+y \in \mathcal{V},$$

because of f and g are endomorphisms, and if $f+g$ $\in$End($\mathcal{V}$) we must satisfy,

$$(f+g)(x+y) = (f+g)(x)+(f+g)(y) = f(x)+g(x)+f(y)+g(y).$$

Which shows that the addition of $\mathcal{V}$ has to be commutative to satisfy both equations. In other words, if the addition of $\mathcal{V}$ is not commutative, End($\mathcal{V}$) can't be a group under addition, because $f+g$ won't be an endomorphism.
And, the multiplication of End($\mathcal{V}$) will be the composition of two endomorphisms, which is a bilinear operation. So,

$$(f \cdot g)(x) = f(g(x)), \qquad \text{for two endomorphisms } f,g \in \text{End}(\mathcal{V}), \text{ and } x \in \mathcal{V}. \text{ [4]}$$

Take f,g,h $\in$End($\mathcal{V}$), $a,b \in K$ and $x \in \mathcal{V}$. Then, the bilinear property of the multiplication can be proven with,

$$\begin{aligned} (f \circ (g+h))(x) &= f(g(x)+h(x)) = f(g(x))+f(h(x)), \\ ((g+h) \circ f)(x) &= (g+h)(f(x)) = g(f(x))+h(f(x)), \\ (a \cdot f) \circ (b \cdot g)(x) &= a \cdot f(b \cdot g(x)) = a \cdot \underbrace{f(g(x))+\cdots+f(g(x))}_{b \text{ times}} = a \cdot b \cdot f(g(x)). \end{aligned}$$

So, End($\mathcal{V}$) is indeed a ring. With this definition of End($\mathcal{V}$), we can use Theorem 2.2 to turn End($\mathcal{V}$) into a Lie algebra. This theorem can be used, because this ring is actually an algebra. This statement follows from the fact that, End($\mathcal{V}$) is a well-defined vector space under addition, and the multiplication of End($\mathcal{V}$) is bilinear.
So, from this we see that End($\mathcal{V}$), with the bracket defined as in Theorem 2.2, is a Lie algebra. And this Lie algebra is often denoted as $gl(\mathcal{V})$.

Example 2.7. The Lie algebra that will be used more often, is the Lie algebra $sl(2,\mathbb{C})$. As a vector space, $sl(2,\mathbb{C})$ consists of all the complex traceless $(2\times 2)-$matrices. And one can turn $sl(2,\mathbb{C})$ into a Lie algebra with the use of Theorem 2.2, because $sl(2,\mathbb{C})$ is an algebra over **C**.
Before looking at $sl(2,\mathbb{C})$, we will first look at a more general example. Namely, $gl(n,\mathbb{C})$, the vector space that consists of all the complex $(n\times n)-$matrices. So, take $\mathbf{g}=gl(n,\mathbb{C})$. Now, a basis for $\mathbf{g}$ is the set containing the n^2 unit matrices e_{ij}, $(i,j=1,\dots,n)$. In other words, the set that contains the n^2 matrices, which have only 0's as its values except for one 1. Where the place of this 1 is different for all non-equal matrices in the basis.
Now, we can define the commutator of $\mathbf{g}$ as,

$$[\,e_{ij}\,,\,e_{kl}\,]=\delta_{jk}e_{il}-\delta_{il}e_{kj}.$$

And we see that, with the use of Theorem 2.2, that $\mathbf{g}$ is a Lie algebra with this commutator.
One property of $(n\times n)-$matrices is that $tr(xy)=tr(yx)$, for $x,y\in\mathbf{g}$. So, for $x,y\in\mathbf{g}$ we have that $tr([\,x\,,\,y\,])=0$. So, for all traceless matrices $x,y\in\mathbf{g}$, we see that $[\,x\,,\,y\,]$ is again a traceless matrix. Which shows that $sl(n,\mathbb{C})$ is a subalgebra of $\mathbf{g}$.
A basis for $sl(n,\mathbb{C})$ can be made by first taking all the matrices from the basis of $\mathbf{g}$ with only zeros on the diagonal, which are n^2-n matrices. And then by taking all the matrices with a 1 in the first diagonal entry and a -1 somewhere else on the diagonal, and everything else 0. Hence, the basis of $sl(n,\mathbb{C})$ consists of n^2-1 elements.
Since $sl(n,\mathbb{C})$ is a subalgebra of $\mathbf{g}$, it is indeed a Lie algebra. And one could construct the following basis for $sl(2,\mathbb{C})$:

$$J_0=\begin{pmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{pmatrix},\qquad J_+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\qquad J_-=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{2.8}$$

With respect to this basis, the Lie algebra $sl(2,\mathbb{C})$ has the following commutator properties:

$$[\,J_0\,,\,J_\pm\,]=\pm J_\pm,\qquad [\,J_+\,,\,J_-\,]=2J_0.$$

Do remember that this Lie algebra will be used more often further on. And, this basis will be used as the standard basis for $sl(2,\mathbb{C})$.

2.2 Representations and modules

Eventually we want to use representations to study the properties of the Racah polynomials. But to be able to do that, we need to know what a Lie algebra representation is.

A Lie algebra representation is nothing more than a special Lie algebra homomorphism. So let's start with the definition of a Lie algebra homomorphism.

Definition 2.9. Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be Lie algebras over a field K. Then, a linear transformation $\varphi\colon \mathbf{g}_1\to\mathbf{g}_2$ is called **a Lie algebra homomorphism**, or **homomorphism**, if $\varphi([\,x\,,\,y\,])=[\,\varphi(x)\,,\,\varphi(y)\,]$.
Note, this implies that φ is a homomorphism between the vector spaces $\mathbf{g}_1$ and $\mathbf{g}_2$.

So a homomorphism between two Lie algebras is pretty much what we would expect. So, now that we know what a homomorphism is, lets finally look at what a representation is.

Definition 2.10. Let $\mathbf{g}$ be a Lie algebra over a field K and let $\mathcal{V}$ be a vector space over a field K. Define $gl(\mathcal{V})$ as in Example 2.6.
Then, a homomorphism $\varphi\colon\mathbf{g}\to gl(\mathcal{V})$ is called a **representation of g in** $\mathcal{V}$. And $\mathcal{V}$ is called the **representation space**.

When using representations it is often convenient to also use the language of modules. For the reason that they are equivalent, but it is sometimes easier to refer to a module than it is to refer to a representation, which is only a function.

Definition 2.11. Let $\mathbf{g}$ be a Lie algebra and $\mathcal{V}$ be a vector space, both over a field K. Let $\mathcal{V}$ be endowed with an operation $\mathbf{g} \times \mathcal{V} \to \mathcal{V}$, denoted as $(x, v) \mapsto \pi(x)(v)$.
Then, $\mathcal{V}$ is called a $\mathbf{g}-$**module** if the following conditions are satisfied:

i. $\pi(ax + by)(v) = a\pi(x)(v) + b\pi(y)(v)$,

ii. $\pi(x)(av + bw) = a\pi(x)(v) + b\pi(x)(w)$,

iii. $\pi([\,x\,,\,y\,])(v) = \pi(x)(\pi(y)(v)) - \pi(y)(\pi(x)(v))$,

with $x, y \in \mathbf{g}$, $v, w \in \mathcal{V}$ and $a, b \in K$.

The reason why the language of modules and representations is equivalent, is because the relation between modules and representations is a natural one. So, φ is a representation of $\mathbf{g}$ in the vector space $\mathcal{V}$ if and only if, $\mathcal{V}$ is a $\mathbf{g}-$module under the action $\pi(x)(v) = \varphi(x)(v)$.
Now, the $\mathbf{g}-$modules that will be studied won't be irreducible $\mathbf{g}-$modules. In other words, the $\mathbf{g}-$modules that will be studied have subsets, that are again $\mathbf{g}-$modules. That does sound inconvenient, but we will be lucky enough that the $\mathbf{g}-$modules are also completely reducible. Which means that they will be fully decomposable into smaller $\mathbf{g}-$modules. So, the larger $\mathbf{g}-$modules that will be studied, can be studied with the use of the smaller $\mathbf{g}-$submodules.

Let's first write these terms down properly. So that we can actually use them and also be able to properly check if a $\mathbf{g}-$module is irreducible.

Definition 2.12. Let $\mathcal{V}$ be a $\mathbf{g}-$module. Then, a subset $\mathcal{W} \subseteq \mathcal{V}$ is a $\mathbf{g}-$**submodule** if, $\pi(x)(w) \in \mathcal{W}$ $\forall x \in \mathbf{g}$ and $\forall w \in \mathcal{W}$.

Definition 2.13. Let $\mathcal{V}$ be a $\mathbf{g}-$module over a field K. Then, $\mathcal{V}$ is **irreducible** if it only has trivial $\mathbf{g}-$submodules. The trivial $\mathbf{g}-$submodules are $\mathcal{V}$ and the empty set. If $\mathcal{V}$ is irreducible, it is often referred to as **simple**.

Definition 2.14. Let $\mathcal{V}$ be a $\mathbf{g}-$module over a field K. Then, $\mathcal{V}$ is **called completely reducible** if, $\mathcal{V}$ is a direct sum of $\mathbf{g}-$submodules. In other words, $\mathcal{V}$ is completely reducible if every $\mathbf{g}-$submodule $\mathcal{W}$ of $\mathcal{V}$, has a compliment $\mathcal{W}'$, which is also a $\mathbf{g}-$submodule of $\mathcal{V}$. So, $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}'$, with $\mathcal{W}$ and $\mathcal{W}'$ $\mathbf{g}-$submodules of $\mathcal{V}$.

2.3 The Verma module and $\star-$representation of $sl(2, \mathbb{C})$

Now we know what a $\mathbf{g}-$module is, and when a $\mathbf{g}-$module is irreducible. We can now look at a special $\mathbf{g}-$module, the Verma module. For this part we will define the Verma module in a concrete way. Later on, we will look at it shortly in a more abstract way. To be more precise, we will first focus on the Verma module of $sl(2, \mathbb{C})$, because that is what we need later on. And later we can check if the given definition is also coherent with a more abstract definition.
However, before we can introduce the Verma module, we need to know what the enveloping algebra is. After defining those two, we will look at two new operations, a $\star-$operation and a Hermitian form. And finally, all of the information will be used to form a special representation of $sl(2, \mathbb{C})$, which will have an inner product defined on it.

In short, the enveloping algebra is the associative algebra generated by the basis elements of a Lie algebra $\mathbf{g}$. Let's look at this in a little bit more detail. Before we already saw that the Lie bracket is an operation on a vector space $\mathbf{g}$, which returns an element of $\mathbf{g}$.
Since a Lie algebra $\mathbf{g}$ is still a vector space under addition, this gives us that $[\,x\,,\,y\,]$, for $x, y \in \mathbf{g}$, has to be a linear combination of the basis elements of $\mathbf{g}$.

Definition 2.15. Let $\mathbf{g}$ be a Lie algebra over a field K, with basis elements x_i, $i = 1, 2, 3, \ldots$. Then, we have that

$$[\, x_i\, ,\, x_j\,] = \sum_k c_{ij}^k x_k.$$

And the constants c_{ij}^k are called **the structure constants of g**.

With this definition, we can find a way to create an associative algebra from a given Lie algebra $\mathbf{g}$ over a field K.

Definition 2.16. For a given Lie algebra $\mathbf{g}$ with basis elements x_i, $i = 1, 2, \ldots$. **The enveloping algebra $\mathcal{U}(\mathbf{g})$** is the associative algebra with unit, generated by the elements x_i, and subject to the relations

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k.$$

This relation gives us the multiplication for the associative algebra $\mathcal{U}(\mathbf{g})$. Do note that, the multiplication is not allowed to satisfy other relations.

This last statement is rather important. Because, if we look at the Lie algebra $\mathbf{g} = gl(n, \mathbb{C})$. Then, $\mathbf{g}$ already has a certain multiplication, the standard matrix multiplication. But with that multiplication, certain elements satisfy more relations than just the one given in Definition 2.16. For example, $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, satisfies the relation $M^2 = 0$. Which shows that we can't just make the Lie algebra $gl(n, \mathbb{C})$ into an enveloping algebra, if we don't change anything.
So, the multiplication for the enveloping algebra is a formal multiplication. For this reason, the enveloping algebra is sometimes also written with different elements or with the use of a linear map. But we will come back to these details at a later point.

For now, we will focus on some details of the enveloping algebra $\mathcal{U}(sl(2, \mathbb{C}))$. The following properties will be used to find some properties of the Verma module of $sl(2, \mathbb{C})$.

Proposition 2.17. Take $\mathbf{g} = sl(2, \mathbb{C})$. Then, for $\mathcal{U}(g)$ we have the following properties:

i. $J_0 J_\pm^n = J_\pm^n (J_0 \pm n)$

ii. $[\, J_-\, ,\, J_+^n\,] = -n(n-1) J_+^{n-1} - 2n J_+^{n-1} J_0$

iii. $[\, C\, ,\, x\,] = 0 \quad \forall x \in \mathcal{U}(sl(2, \mathbb{C}))$, with

$$C = J_+ J_- + J_0^2 - J_0 = J_- J_+ + J_0^2 + J_0 \quad \text{, the Casmir operator.}$$

With $X^n = \underbrace{X \cdots X}_{n-\text{times}}$, for some $X \in sl(2, \mathbb{C})$.

Note that $[\, X\, ,\, Y\,] = XY - YX$, for $X, Y \in \mathcal{U}(\mathbf{g})$. So, one should see the bracket as this operation. Because the enveloping algebra is an algebra, hence it has no Lie bracket.

Proof. The first property will be proven with the use of full induction to n. For $n = 1$,

$$[\, J_0\, ,\, J_\pm\,] = \pm J_\pm,$$

so we get $J_0 J_\pm = \pm J_\pm + J_\pm J_0 = J_\pm(J_0 \pm 1)$. By the induction hypothesis we get, for $n \in \mathbb{N}$,

$$J_0 J_\pm^n = J_\pm^n (J_0 \pm n).$$

And for $n = n+1$ we have,

$$\begin{aligned} J_0 J_\pm^{n+1} &= (J_0 J_\pm^n) J_\pm = (J_\pm^n (J_0 \pm n)) J_\pm = J_\pm^n J_0 J_\pm \pm n J_\pm^n J_\pm = J_\pm^n (J_\pm (J_0 + 1)) + n J_\pm^{n+1} \\ &= J_\pm^{n+1} J_0 \pm (n+1) J_\pm^{n+1}. \end{aligned}$$

Now, for the second property we will also use full induction. When $n = 1$, we have

$$[\, J_-\, ,\, J_+\,] = -2J_0,$$

which satisfies the commutator identity. By the induction hypothesis we get, for $n \in \mathbb{N}$,

$$[\, J_-\, ,\, J_+^n\,] = -n(n-1)J_+^{n-1} - 2nJ_+^{n-1}J_0.$$

For this part we will use the general commutator identity $[\, x\, ,\, yz\,] = y[\, x\, ,\, z\,] + [\, x\, ,\, y\,]z$. Then we get,

$$\begin{aligned} [\, J_-\, ,\, J_+^{n+1}\,] &= J_+^n [\, J_-\, ,\, J_+\,] + [\, J_-\, ,\, J_+^n\,] J_+ = J_+^n(-2J_0) + (-n(n-1)J_+^{n-1} - 2nJ_+^{n-1}J_0)J_+ \\ &= -2J_+^n J_0 + (-n(n-1)J_+^{n-1} - 2nJ_+^{n-1}J_0 J_+) = -2J_+^n J_0 + (-n(n-1)J_+^n - 2nJ_+^n(J_0+1)) \\ &= -2(n+1)J_+^n J_0 - (n^2 - n + 2n)J_+^n = -2(n+1)J_+^n J_0 - n(n+1)J_+^n. \end{aligned}$$

For the third property it is sufficient to prove it for $x = J_0, J_+, J_-$, because of the bilinear property of the Lie bracket and the previously given commutator identity. We will show the proof for $x = J_+$ only, because the proofs for the other cases are basically identical. So,

$$\begin{aligned} [\, C\, ,\, J_+\,] &= [\, J_+J_- + J_0^2 - J_0\, ,\, J_+\,] = J_+[\, J_-\, ,\, J_+\,] + J_0[\, J_0\, ,\, J_+\,] + [\, J_0\, ,\, J_+\,]J_0 - [\, J_0\, ,\, J_+\,] \\ &= -2J_+J_0 + J_0J_+ + J_+J_0 - J_+ = -J_+J_0 + J_+J_0 + J_+ - J_+ = 0. \end{aligned}$$

□

As the name already suggests, a Verma module is a $\mathbf{g}-$module. But to be able to use to previous properties, we need to have a $\mathcal{U}(\mathbf{g})-$module. However, with the given definitions of the enveloping algebra, we find the following result.

Corollary 2.18. Let $\mathbf{g}$ be a Lie algebra and let $\mathcal{V}$ be a vector space, both over the same field K. Let $\pi\colon \mathbf{g} \to \mathrm{End}(\mathcal{V})$ be a representation of $\mathbf{g}$ in $\mathcal{V}$. Then $\overline{\pi}\colon \mathcal{U}(\mathbf{g}) \to \mathrm{End}(\mathcal{V})$ defined as,

$$\overline{\pi}(X) = \pi(X), \qquad \text{for } X \in \mathbf{g},$$

is a representation of $\mathcal{U}(\mathbf{g})$. The converse is also true. So given a representation of $\mathcal{U}(\mathbf{g})$, one can define a representation of $\mathbf{g}$, in the same manner.

Proof. So for $X \in \mathbf{g}$, we have $\overline{\pi}(X) = \pi(X)$. Now, we have to show that $\overline{\pi}$ is a representation, when defined like this. Note, for a homomorphism f we have the following properties:

$$f(x+y) = f(x) + f(y),$$
$$f(x \cdot y) = f(x) \cdot f(y),$$

where the operations on the left- and right-hand side are allowed to be different.

Now, take $X_1, \ldots, X_n, Y_1, \ldots, Y_m \in \mathbf{g}$, arbitrary basis elements of $\mathbf{g}$. Then,

$$\pi(X_1 + \cdots + X_n) = \pi(X_1) + \cdots + \pi(X_n) = \overline{\pi}(X_1) + \cdots + \overline{\pi}(X_n),$$

which shows that $\overline{\pi}$ is properly defined as a homomorphism under addition. Because,

$$\overline{\pi}(X_1 + \cdots + X_n) = \overline{\pi}(X_1) + \cdots + \overline{\pi}(X_n) = \pi(X_1 + \cdots X_n)$$

is precisely what followed from how we defined $\overline{\pi}$.

Now, because π and $\overline{\pi}$ are endomorphisms of $\mathcal{V}$. We have,

$$\pi(X_1) \cdot \pi(X_2) \cdot \cdots \cdot \pi(X_n) = \pi(X_1) \circ \pi(X_2) \circ \cdots \circ \pi(X_n) = \overline{\pi}(X_1) \circ \cdots \circ \overline{\pi}(X_n),$$

which shows that $\overline{\pi}$ is also a properly defined homomorphism under multiplication. Because,

$$\overline{\pi}(X_1 X_2 \cdots X_n) = \overline{\pi}(X_1) \circ \cdots \circ \overline{\pi}(X_n) = \pi(X_1) \circ \pi(X_2) \circ \cdots \circ \pi(X_n)$$

is also exactly what followed from the definition of $\overline{\pi}$, with $X_1 X_2 \cdots X_n \in \mathcal{U}(\mathbf{g})$.

Now, the only thing left to do, is to show that the commutator property of $\overline{\pi}$ is also well defined. So we want to show that $\overline{\pi}([X_1, X_2]) = [\overline{\pi}(X_1), \overline{\pi}(X_2)]$ holds. But, $\mathcal{U}(\mathbf{g})$ has no Lie bracket, because it was an algebra. However, it does satisfy,

$$[X_1, X_2] = \sum_k c_{ij}^k x_k = X_1 X_2 - X_2 X_1,$$

where $\sum_k c_{ij}^k x_k$ is defined as in Definition 2.15. From this we get that,

$$\pi([X_1, X_2]) = [\pi(X_1), \pi(X_2)] = \pi(X_1)\pi(X_2) - \pi(X_2)\pi(X_1) = \overline{\pi}(X_1)\overline{\pi}(X_2) - \overline{\pi}(X_2)\overline{\pi}(X_1) = \overline{\pi}(X_1X_2 - X_2X_1),$$

is properly defined. Because the first and last part of the equation are π and $\overline{\pi}$ applied to the same sum, which is an element of $\mathbf{g}$. And the third equality follows from the definition of the commutator of $gl(\mathcal{V})$. So, $\overline{\pi}$ at least satisfies the commutator property when we only use elements of $\mathbf{g}$. Let's see if it is also true when we try to use elements of $\mathcal{U}(\mathbf{g})$.

For this part, denote $xy - yx \equiv \overline{[x, y]}$, for $x, y \in \mathcal{U}(\mathbf{g})$. Now, for $x, y, z, q \in \mathbf{g}$, arbitrary basis elements, we have

$$\overline{[x, yz]} = xyz - yzx = yxz - yzx + xyz - yxz = y[x, z] + [x, y]z,$$
$$\overline{[xy, z]} = xyz - xzy + xzy - zxy = x[y, z] + [x, z]y.$$

From this we get,

$$\overline{[xy, zq]} = z\overline{[xy, q]} + \overline{[xy, z]}q = z(x[y, q] + [x, q]y) + (x[y, z] + [x, z]y)q = zx[y, q] + z[x, q]y + x[y, z]q + [x, z]yq.$$

So, $\overline{[X_1 \cdots X_n, Y_1 \cdots Y_m]}$ can be rewritten to a sum consisting of products of X_i and a Lie bracket $[X_j, Y_k]$. Now, if we start with $[\overline{\pi}(X_1 \cdots X_n), \overline{\pi}(Y_1 \cdots Y_m)]$, the commutator of $gl(\mathcal{V})$, we can use the same property to rewrite it to a sum consisting of $\overline{\pi}$ applied to products of $\mathcal{U}(\mathbf{g})$ elements and a commutator of X_j and Y_k. This will look like $\overline{\pi}$ applied to a longer version of the above equation. This can be rewritten into terms of π applied to basis elements of $\mathbf{g}$ and π applied to the Lie bracket of two basis elements of $\mathbf{g}$, because $\overline{\pi}$ was a homomorphism under multiplication and addition. Which shows that it is properly defined in terms of π.

Now, because $\overline{\pi}$ was also properly defined for the commutator of x and y, for $x, y \in \mathbf{g}$. The described sum could also be rewritten, such that $\overline{\pi}$ is applied once to the full sum, and then we get $\overline{\pi}(\overline{[X_1 \cdots X_n, Y_1 \cdots Y_m]}) = [\overline{\pi}(X_1 \cdots X_n), \overline{\pi}(Y_1 \cdots Y_m)]$, with the second term the commutator of $gl(\mathcal{V})$. Which shows that $\overline{\pi}$ satisfies the commutator property when defined in terms of π.

Now, because π is a homomorphism, this is enough to proof that $\overline{\pi}$ is a representation. Because every element of $\mathbf{g}$ can be written as a linear combination of basis elements. But π applied to such a sum, is nothing more than then sum of π applied to the basis elements. Hence, $\overline{\pi}$ is a representation of $\mathcal{U}(g)$.

The converse proof is almost the same, except that the Lie bracket only needs to be checked for elements of $\mathbf{g}$. Because the addition,multiplication and the described Lie bracket are shown to be equal in both directions, the converse is also true. □

Now we will define the Verma module. More specifically, we will define the Verma module of $sl(2, \mathbb{C})$. There are several ways to get the definition of the Verma module. One way is to start with an easy example, the Verma module of $sl(2, \mathbb{C})$, and then expend that to a general definition. The other is to start with the general definition and then construct concrete examples. Because we only need to Verma module of $sl(2, \mathbb{C})$, the first option will be used. So we will start with a definition of the Verma module of $sl(2, \mathbb{C})$. Later, a slightly more general definition will be given, but that won't be necessary for the further results.

Now, the Verma module of $sl(2, \mathbb{C})$, will be a $\mathbf{g}-$module that is generated by a single vector $v \in \mathcal{V}$. Such that v has a $\pi(X)$ eigenvalue for one $X \in sl(2, \mathbb{C})$, and such that v gets annihilated by some other $\pi(Y)$ for $Y \in sl(2, \mathbb{C})$.

Definition 2.19. Let $\mathbf{g}$ be the Lie algebra $sl(2, \mathbb{C})$ and $\mathcal{V}$ be a vector space. Let V be the $\mathbf{g}-$module generated by a single vector $v \in \mathcal{V}$. Then, we call V **the Verma module of** $sl(2, \mathbb{C})$.
In particular, the vector v satisfies

$$\pi(J_0)(v) = \lambda v,$$
$$\pi(J_-)(v) = 0,$$

with J_0 and J_- defined as in Example 2.8. And we say that $\pi(J_-)$ annihilates v.

If we define a Verma module V of $sl(2, \mathbb{C})$ as above, it is clear that v and $\pi(J_+)(v)$ are elements of V. Because of Corollary 2.18 we know that we can construct an equivalent $\mathcal{U}(\mathbf{g})-$module. In other words, a representation of $\mathbf{g}$ has the same properties as a representation of $\mathcal{U}(\mathbf{g})$. So, we will be able to use the properties of $\mathcal{U}(sl(2, \mathbb{C}))$, from Proposition 2.17, to find some properties of the Verma module. Because of this, the following part will have many products of functions. So, for some convince, denote,

$$\underbrace{\pi(X) \circ \pi(X) \circ \cdots \circ \pi(X)}_{n-\text{times}} = \pi(X)^n.$$

Proposition 2.20. Let $V = V_\lambda$ be the Verma module of $sl(2,\mathbb{C})$ defined as in Definition 2.19 with $v = v_\lambda$, and $\pi(J_0)(v) = \lambda v$. Then, a basis for V can be given by $v_\lambda, \pi(J_+)(v_\lambda), \pi(J_+)^2(v_\lambda), \ldots$, and define $\pi(J_+)^n(v_\lambda) = v_{\lambda+n}$, for $n \in \mathbb{N}\bigcup\{0\}$. Then the actions of $\pi(J_0), \pi(J_+)$ and $\pi(J_-)$ on $v_{\lambda+n}$ are given by,

$$\begin{aligned}\pi(J_0)(v_{\lambda+n}) &= (\lambda+n)v_{\lambda+n},\\ \pi(J_+)(v_{\lambda+n}) &= v_{\lambda+n+1},\\ \pi(J_-)(v_{\lambda+n}) &= -n(2\lambda+n-1)v_{\lambda+n-1}.\end{aligned}$$

And,

$$\pi(C)(v_{\lambda+n}) = \pi(C)\pi(J_+^n)(v_\lambda) = \pi(J_+^n)\pi(C)(v_\lambda) = (\lambda^2-\lambda)v_{\lambda+n}.$$

Proof. First, the properties of the Verma module following from Proposition 2.17 and by using the definition of a homomorphism. Now, we'll show that the vectors $v_\lambda, v_{\lambda+1}, v_{\lambda+2}, \ldots$ form a basis for V_λ, and that the properties are correct. Note,

$$\begin{aligned}\pi(J_0)(v_{\lambda+n}) &= \pi(J_0J_+^n)(v_\lambda) = \pi(J_+^n)(\pi(J_0+n)(v_\lambda)) = \pi(J_+)^n(\pi(J_0)(v_\lambda)+\pi(n)(v_\lambda))\\ &= (\lambda+n)\pi(J_+)^n(v_\lambda) = (\lambda+n)(v_{\lambda+n}),\\ \pi(J_+)(v_{\lambda+n}) &= v_{\lambda+n+1},\\ \pi(J_-)(v_{\lambda+n}) &= \pi(J_-J_+^n)(v_\lambda) = \pi(J_+^nJ_- - n(n-1)J_+^{n-1} - 2nJ_+^{n-1}J_0)(v_\lambda)\\ &= \pi(J_+^n)(\pi(J_-)(v_\lambda)) - n(n-1)\pi(J_+^{n-1})(v_\lambda) - 2n\pi(J_+^{n-1})(\pi(J_0)(v_\lambda))\\ &= -n(n-1)v_{\lambda+n-1} - 2n\pi(J_+^{n-1})(\lambda v_\lambda) = -n(n-1)v_{\lambda+n-1} - 2n\pi(J_+^{n-1})(\lambda v_\lambda)\\ &= -n(2\lambda+n-1)v_{\lambda+n-1} \quad (n>0).\end{aligned}$$

So, we see that the vectors $v_{\lambda+n}$ are non-equal, because their $\pi(J_0)$ eigenvalues are non-equal. It is also clear that the vectors $v_{\lambda+n}$ are linear independent. Because if they aren't, you could write $v_{\lambda+n}$ as a sum of $v_{\lambda+i}$, with $i<n$, and use $\pi(J_-)$ to conclude that some $v_{\lambda+i}=0$ with $i \in \mathbb{N}\bigcup\{0\}$. The elements of V_λ are per definition of the form $\pi(X)(v_\lambda)$ with $X \in \mathcal{U}(\mathbf{g})$, and from the definition of a homomorphism, it follows that $\pi(X)$ is nothing more than applying $\pi(J_0), \pi(J_+)$ or $\pi(J_-)$ several times in a specific order on v_λ. Which shows, that all vectors of V_λ are linear combinations of the vectors $v_{\lambda+n}$, for $n \in \mathbb{N}\bigcup\{0\}$.
And the shown equations are also the properties that we needed to proof. □

Because of the actions of $\pi(J_0), \pi(J_+)$ and $\pi(J_-)$, we can easily see that V_λ is irreducible if $\pi(J_-)(v_{\lambda+n}) \neq 0\ \forall n \in \mathbb{N}$. In this case, we can get all the basis vectors $v_{\lambda+n}$ by letting $\pi(J_+)$ and $\pi(J_-)$ act on an arbitrary vector $v_{\lambda+i}$, $i \in \mathbb{N}\bigcup\{0\}$.
Now, if $-2\lambda \in \mathbb{N}\bigcup\{0\}$, then we have that $-2\lambda+1 \in \mathbb{N}$, so $\exists n \in \mathbb{N}$ such that $2\lambda - 1 + n = 0$. So, we have that if $\lambda \in -\frac{1}{2}\mathbb{N}\bigcup\{0\}$, then

$$\pi(J_-)(v_{\lambda+n}) = -n(2\lambda-1+n)v_{\lambda+n-1} = 0, \qquad \text{for some } n \in \mathbb{N}.$$

Which shows that V_λ is not always an irreducible $\mathbf{g}-$module.

Corollary 2.21. Let V_λ be the Verma module of $sl(2,\mathbb{C})$ defined in Definition 2.19. If $\lambda = -j$, with $j \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$, then V_λ has a submodule M_λ.
This submodule consists of all the vectors of the form v_{j+n}, for $n \in \mathbb{N}$.

Proof. Denote $v_\lambda \equiv v_{-j}$. Before we already saw that

$$\pi(J_-)(v_{j+1}) = \pi(J_-)(v_{-j+1+2j}) = (2j+1)(-2j+2j+1-1)v_j = 0.$$

So, if we start with v_{j+1} then this vector is an element of M_λ. And letting $\pi(J_0), \pi(J_+)$ or $\pi(J_-)$ act on this vector an arbitrary number of times, only gives 0 or $v_{j+n+1}, n \in \mathbb{N}$, except for some scalar multiplication. And all of those vectors are elements of M_λ, hence M_λ is a submodule of V_λ. □

So, now that we know that V_λ is reducible for some values for λ, we can finally construct the $\mathbf{g}-$module that we want to use. Namely, the quotient module $L_\lambda \equiv V_\lambda / M_\lambda$.

Corollary 2.22. Let V_λ be the Verma module of $sl(2,\mathbb{C})$ and M_λ the submodule of V_λ from Corollary 2.21, such that $\lambda \equiv -j \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$. Then, **the quotient module** $L_\lambda \equiv V_\lambda/M_\lambda$ is a finite dimensional vector space, and is spanned by $[v_m]$, the representatives of the $(2j+1)$ vectors v_m, with $m = -j, -j+1, \ldots, j$. And the actions of J_0, J_+ and J_- are given by,

$$\pi(J_0)([v_m]) = m[v_m],\ m = -j, -j+1, \ldots, j$$
$$\pi(J_+)([v_m]) = [v_{m+1}],\ m = -j, -j+1 \ldots -j-1, \qquad \pi(J_+)([v_j]) = 0$$
$$\pi(J_-)([v_m]) = (j+m)(j-m+1)[v_{m-1}],\ m = -j, -j+1, \ldots, j.$$

Proof. For $v_m, v_n \in V_\lambda$, recall that $[v_m] = \{v_m + v \colon v \in M_\lambda\}$. This shows that two representatives $[v_m]$ and $[v_n]$ are equal if,

$$\{v_m + v \colon v \in M_\lambda\} = \{v_n + v \colon v \in M_\lambda\} \iff v_m - v_n \in M_\lambda.$$

Let's now construct a basis for V_λ. First, define $\lambda \equiv -j \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$. Then, a basis for V_λ is given by $v_{-j}, v_{-j+1}, v_{-j+2}, \ldots$, with $v_{-j+n} = \pi(J_+)^n(v_{-j})$, $n \in \mathbb{N}\bigcup\{0\}$.
It is obvious that $[0] = [v_{j+1}] = [v_{j+2}] = \ldots$, because M_λ contains all of those vectors, so we see that $\pi(J_+)([v_j]) = 0$. For two different basis vectors v, w of V_λ, such that $v, w \notin M_\lambda$, we have

$$[v] = \{v + m \colon m \in M_\lambda\} \neq \{w + m \colon m \in M_\lambda\} = [w],$$

because $v - w \notin M_\lambda$. Because v and w are linearly independent, and $v - w$ would only be contained in M_λ if both v and w were elements of M_λ.
That gives $(2j+1)$ different representatives of the vectors $v_{-j}, v_{-j+1}, \ldots, v_j$. Now, note that all elements of V_λ are either contained in one representative $[v_m]$, $m = -j, -j+1, \ldots, j$, or contained in a sum of different representatives. Which shows that the representatives $[v_{-j+i}]$, $i = 0, \ldots, 2j$, form a basis for L_λ.
Now that we have a basis, the actions of J_0 and J_+ can easily be deduced from their actions on the basis vectors of V_λ. Just use $-\lambda = j$, and $\lambda + n = m$ with $m \in \{-j, -j+1, \ldots, j\}$. Where m is only defined this way to have the actions of J_0, J_+ and J_- on v_m, be the same action as on $[v_m]$. ☐

This $\mathbf{g}-$module is often denoted as D_j, with $j \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$. With the use of D_j, it is possible to define a representation of $sl(2,\mathbb{C})$ with an extra operation and an inner product. To do this, we will first define a new type of algebra, the $\star-$algebra.

Definition 2.23. Let $\mathbf{g}$ be a Lie algebra over a field K. Then we call $\mathbf{g}$ a $\star-$**algebra** if

$$\exists \star \colon \mathbf{g} \to \mathbf{g}, \quad \text{an operation denoted as } x \mapsto x^*,$$

with $\star$ a conjugate-linear anti-automorphic involution. Which means that the operation $\star$ has to satisfy the following properties:

i. $(x^*)^* = x$,

ii. $(ax + by)^* = \overline{a}x^* + \overline{b}y^*$,

iii. $[x, y]^* = [y^*, x^*]$,

with $x, y \in \mathbf{g}$ and $a, b \in K$. So, if $K = \mathbb{R}$, then $\star$ is a linear anti-automorphic involution.

There exist two non-equivalent $\star-$operations of $sl(2,\mathbb{C})$. These are given by,

$$J_0^* = J_0, \qquad J_\pm^* = J_\mp, \tag{2.24}$$

and

$$J_0^* = J_0, \qquad J_\pm^* = -J_\mp. \tag{2.25}$$

We will only need the first $\star-$operation. This $\star-$operation is associated with $su(2)$. The reason for this, and a description of $su(2)$, will be explained later on, because the details aren't needed for the later parts, but may be useful to know.

We will eventually use $\star-$representations to study the Racah polynomials. So, we also need to define an inner product for them. Because one of the properties we will look at is the orhtonormality of the Racah polynomials. To define this inner product, we will first define a Hermitian form.

Definition 2.26. Let $\mathbf{g}$ be a Lie algebra and $\mathcal{V}$ a $\mathbf{g}-$module, both over the same field K. Then the operation $\langle\, ,\, \rangle\colon \mathcal{V}\times\mathcal{V}\to\mathbb{C}$, with

$$\langle u,\, v\rangle = \overline{\langle v,\, u\rangle},$$

$$\langle u,\, av+bw\rangle = a\langle u,\, v\rangle + b\langle u,\, w\rangle,$$

for $u, v, w \in \mathcal{V}$ and $a, b\in\mathbb{C}$, is called a **Hermitian form**.

Now that we know what a Hermitian form is and what a $\star-$algebra is, we can finally define a $\star-$representation.

Definition 2.27. Let $\mathbf{g}$ be a Lie algebra and $\mathcal{V}$ a $\mathbf{g}-$module. Induce $\mathbf{g}$ with a $\star-$operation, and let $\langle\, ,\, \rangle\colon \mathcal{V}\times\mathcal{V}\to\mathbb{C}$ be a Hermitian form on $\mathcal{V}$. Then, $\mathcal{V}$ is **a $\star-$representation** if,

$$\langle \pi(x)(v),\, w\rangle = \langle v,\, \pi(x^*)(w)\rangle, \qquad \forall x\in\mathbf{g},\ \text{and } \forall v,w\in\mathcal{V}.$$

We would love to extend the definition of the $\star-$representation, to a $\star-$representation with an inner product. For that, the Hermitian form needs to be an inner product. Luckily enough, the definition of a Hermitian form is basically the definition of an inner product, except for the positive definite property

$$\langle v, v\rangle > 0, \qquad \text{for } v\in\mathcal{V},\ v\neq 0;\ \ \text{with } \langle\,\cdot,\cdot\,\rangle\colon \mathcal{V}\times\mathcal{V}\to\mathbb{C}.$$

So, to define an inner product on a $\star-$representation, we only need to make sure the above property also holds. Next, we will look at the Verma module V_λ, from Definition 2.19, and the quotient module D_j, from Corollary 2.22, and check if one of them can have an inner product defined on them. This will be done by using the $\star-$operation associated with $su(2)$, equation 2.24, and a positive definite Hermitian form.

Keep in mind, that the $\star-$operation that will be used in this part is the $\star-$operation from equation 2.24. Let's start with the $\star-$representation of the Verma module V_λ. And, because we want to study the orthogonality of the Racah polynomials, define a Hermitian form on V_λ such that,

$$\langle v_\lambda, v_\lambda\rangle = 1,\ \text{ and } \langle\pi(x)(v), w\rangle = \langle v, \pi(x^*)(w)\rangle, \qquad \text{for } v, w, v_\lambda\in V_\lambda, x\in sl(2,\mathbb{C}).$$

We are looking at a $\star-$representation, so we only need to define our Hermitian form to just satisfy the first statement. Because all elements in the $\star-$representation of V_λ are linear combinations of $\pi(J_+)^n(v_\lambda)$, $n\in\mathbb{N}\bigcup\{0\}$. The second statement follows from the definition of a $\star-$representation of V_λ.

From the action of J_0, we get,

$$\overline{\lambda}\langle v_\lambda, v_\lambda\rangle = \langle\pi(J_0)(v_\lambda), v_\lambda\rangle = \langle v_\lambda, \pi(J_0)(v_\lambda)\rangle = \lambda\langle v_\lambda, v_\lambda\rangle,$$

which implies that λ has to be real, else we won't have $\lambda = \overline{\lambda}$. The actions of J_+ and J_- gives us,

$$\begin{aligned}\langle v_{\lambda+n}, v_{\lambda+n}\rangle &= \langle \pi(J_+)(v_{\lambda+n-1}), v_{\lambda+n}\rangle = \langle v_{\lambda+n-1}, \pi(J_-)(v_{\lambda+n})\rangle = -n(2\lambda+n-1)\langle v_{\lambda+n-1}, v_{\lambda+n-1}\rangle \\ &= -n(2\lambda+n-1)\langle \pi(J_+)(v_{\lambda+n-2}), v_{\lambda+n-1}\rangle = \cdots = (-1)^n n!(2\lambda+n-1)(2\lambda+n-2)\cdots(2\lambda) \\ &= n!(-2\lambda)(-2\lambda-1)\cdots(-2\lambda-n+1),\end{aligned}$$

for $n \in \mathbb{N}\bigcup\{0\}$. From this, it follows that if $\lambda > 0$, then $\langle v_{\lambda+n}, v_{\lambda+n}\rangle$ is negative if n is odd and positive if n is even. Which means that it certainly isn't an inner product. If λ is negative we get two cases. One is $-\lambda \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$, when V_λ is reducible, the other is when $-\lambda \notin \frac{1}{2}\mathbb{N}\bigcup\{0\}$, when V_λ is irreducible. If V_λ is irreducible we have, for $n \in \mathbb{N}\bigcup\{0\}$, $-2\lambda > n$ or $-2\lambda < n$, so we have $2\lambda + n < 0$ or $-2\lambda - n < 0$. For a fixed λ, we have for some n that $-2\lambda - n > 0$. So, we have that $\langle v_{\lambda+n}, v_{\lambda+n}\rangle > 0$. But when n gets large enough, we have $-2\lambda - n < 0$, and terms of the form $-2\lambda - n + i$, $i = 1, 2, \ldots, n-1$, will be negative for some i. Which shows that the Hermitian form is also no inner product in this case, because $\langle v_{\lambda+n}, v_{\lambda+n}\rangle$ is negative for some n.

So we are left with the case that $-\lambda \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$. Let's denote $\lambda \equiv -j$ with $j \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$, as we did before. If we would try to make the Hermitian form an inner product on V_λ, we get the same problem as before. Because for some $n \in \mathbb{N}\bigcup\{0\}$, we have $-2\lambda = n$. So, for large enough n, $\langle v_{\lambda+n}, v_{\lambda+n}\rangle$ is either positive or negative. But, on D_j we don't have that problem, with D_j the quotient module from Corollary 2.22. Let $m \in \{-j, -j+1, \ldots, j\}$, then we have

$$\begin{aligned}\langle [v_m], [v_m]\rangle &= (j+m)!(2j)(2j-1)\cdots(2j-m+1)\langle [v_{-j}], [v_{-j}]\rangle \\ &= (j+m)!(2j)(2j-1)\cdots(j-m+1)\langle [v_{-j}], [v_{-j}]\rangle \\ &= \frac{(j+m)!(2j)!}{(j-m)!},\end{aligned}$$

because $[v_m] = [v_{\lambda+n}]$, hence $n = j+m$ in the previous formula for this Hermitian form. And we see that, for all $m \in \{-j, -j+1, \ldots, j\}$, $\langle v_m, v_m\rangle > 0$, so we can define an inner product on the $\star-$representation of D_j.

With this inner product, we can change our old basis to an orthonormal basis. This gives us the following theorem.

Theorem 2.28. *Let $j \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$ and define D_j as in Corollary 2.22. Let $\star$ be the $\star-$operation associated with $su(2)$. Then D_j is an irreducible $\star-$representation of $sl(2,\mathbb{C})$ with an inner product $\langle e_m^{(j)}, e_{m'}^{(j)}\rangle = \delta_{m,m'}$, and orthonormal basis spanned by the elements $e_m^{(j)} = \sqrt{\frac{(j-m)!}{(j+m)!(2j)!}} \cdot [v_m]$ of D_j, $m \in \{-j, -j+1, \ldots, j\}$. And the actions of J_0, J_+ and J_- are given by,*

$$\begin{aligned}\pi(J_0)(e_m^{(j)}) &= m e_m^{(j)}, \\ \pi(J_+)(e_m^{(j)}) &= \sqrt{(j-m)(j+m+1)}e_{m+1}^{(j)}, \\ \pi(J_-)(e_M^{(j)}) &= \sqrt{(j+m)(j-m+1)}e_{m-1}^{(j)}.\end{aligned}$$

And for the Casmir operator we have $C^* = C$, and $\pi(C)e_m^{(j)} = j(j+1)e_m^{(j)}$.

Proof. If we define $\langle v_m, v_m\rangle$, for $m \in \{-j, -j+1, \ldots, j\}$ as before, it is trivial that the vectors $e_m^{(j)}$ are the normalised form of v_m. The actions follow directly from Corollary 2.22, and the Casmir operator follows from the fact that $J_0^* = J_0$ and $J_\pm^* = J_\mp$. That it is an orthonormal basis follows from,

$$\begin{aligned}\langle e_m^{(j)}, e_{m'}^{(j)}\rangle &= \langle \pi(J_+)(e_{m-1}^{(j)}), e_{m'}^{(j)}\rangle = \langle e_{m-1}^{(j)}, \pi(J_-)(e_{m'}^{(j)})\rangle = \sqrt{(j+m')(j-m+1)}\langle e_{m-1}^{(j)}, e_{m'-1}^{(j)}\rangle = \cdots \\ &= \sqrt{(j+m')!(j-m'+1)(j-m'+2)\cdots(j-m'+m'+j)}\langle e_{m-m'-j}^{(j)}, e_{m'-m'-j}\rangle \\ &= \sqrt{(j+m')!(j-m'+1)\cdots(-2j)}\langle e_{m-m'-j-1}^{(j)}, \pi(J_-)e_{-j}\rangle \\ &= \sqrt{(j+m')!(j-m'+1)\cdots(-2j)\cdot((j-j)(j-(-j-1)))}\langle e_{m-m'-j-1}, e_{-j-1}\rangle = 0,\end{aligned}$$

with $m' < m$. If $m > m'$, then we can use $\langle e_m^{(j)}, e_{m'}^{(j)} \rangle = \overline{\langle e_{m'}^{(j)}, e_m^{(j)} \rangle}$, which shows that it is still 0, if $m \neq m'$. □

The value m is called the weight of the vector, which will return in the extra details. Now, if we look at the actions of J_+ and J_-, we see that we can get all the vectors $e_m^{(j)}$, multiplied with some scalar, by letting J_+ act on $e_{-j}^{(j)}$ or by letting J_- act on $e_j^{(j)}$. This leads to the following formula,

$$e_m^{(j)} = \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}} \pi(J_-)^{j-m}(e_j^{(j)}) = \sqrt{\frac{(j-m)!}{(2j)!(j+m)!}} \pi(J_+)^{j+m}(e_{-j}^{(j)}).$$

Now that we have a $\star-$representation with an orthonormal basis, we finally have something that we can use to study the Racah polynomials. But before we do that, we will first look at hypergeomtric series, and the Clebsch-Gordan coefficients, because those will also be used when we look at some properties of the Racah polynomials.

2.4 Extra details on the Verma module of $sl(2, \mathbb{C})$

This section will be about the Verma module of $sl(2, \mathbb{C})$. But, this time we will look at it in a more abstract way. Everything that was done before is defined correctly, but one could say that some minor details were not given. However, these details aren't needed for the later parts, but are given to show that the Verma module is correctly defined.

The goal is to give a more abstract definition of the Verma module that will be sufficient enough to define the Verma module of $sl(2, \mathbb{C})$ that was given before. To do this, we will need some more definitions. So we will first start with a Lie algebra $\mathbf{g}$, and then try to extend that to another definition for the Verma module.

To be more precise. We start with some extra definitions of a real Lie algebra. Then extend that to a complex Lie algebra. After that we will define a special subalgebra and look back at the $\star-$operation from equation 2.24. After that we will be able to define an inner product that satisfies some extra properties. And with the use of that inner product and special subalgebra, the roots of a Lie algebra and the weight of a representation can be defined. At that point we have everything that we need, to be able to define a so called highest weight cyclic representation. And we'll show that, that satisfies the definition of a Verma module for $sl(2, \mathbb{C})$.

We will start with the definition of a commutative, a simple and semisimple Lie algebra.

Definition 2.29. Let $\mathbf{g}$ be a Lie algebra over the field K. Then $\mathbf{g}$ is a **commutative** Lie algebra if,

$$[\, x \,,\, y \,] = 0, \qquad \forall x, y \in \mathbf{g}.$$

If $\mathbf{g}$ is commutative it is often referred to as an **abelian** Lie algebra

Definition 2.30. Let $\mathbf{g}$ be a Lie algebra over the field K. Then we call $\mathbf{g}$ **simple**, if the dimension of $\mathbf{g}$ is more than 1, and the only ideals of $\mathbf{g}$ are $\{0\}$ and $\mathbf{g}$ itself.
We call $\mathbf{g}$ **semisimple** if $\{0\}$ is the only abelian ideal of $\mathbf{g}$.[5]

First, notice that this shows that $sl(2, \mathbb{C})$ is an example of a complex semisimple Lie algebra. Because from the commutator properties,

$$[\, J_0 \,,\, J_\pm \,] = \pm J_\pm, \qquad [\, J_+ \,,\, J_- \,] = 2J_0,$$

it follows that if an ideal of $sl(2, \mathbb{C})$ contains at least 1 basis vector, it contains all three basis vectors. Hence, it would be equal to $sl(2, \mathbb{C})$.

For the next part we will be looking at a complexification of a Lie algebra. Which can be described as starting with a real Lie algebra, and then extending it to a complex Lie algebra. Let's first give an example for this in term of $sl(2, \mathbb{C})$ and $su(2)$.

Example 2.31. The real Lie algebra $su(2)$ is the vector space over R, containing all $2 \times 2-$matrices X such that the $X^* = -X$. Where X^* is the conjugate transpose of X. So, $X^* = \overline{X^T}$. Because this is a vector space of $2 \times 2-$matrices, it is an algebra. So, we can define the Lie bracket on $su(2)$ as $[\, x \,,\, y \,] = xy - yx, x, y \in su(2)$. Which is the same Lie bracket as $sl(2, \mathbb{C})$, but defined on a different domain.

A basis for $su(2)$ can be given by the elements[2],

$$X_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad X_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad X_3 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For these matrices it is obvious that $X^* = -X$. And this basis has the following commutator properties:

$$[\, X_1 \,,\, X_2 \,] = X_3, \qquad [\, X_3 \,,\, X_1 \,] = X_2, \qquad [\, X_2 \,,\, X_3 \,] = X_1.$$

Which are different commutator properties than that of $sl(2,\mathbb{C})$. But with the help of this basis, we can define $sl(2,\mathbb{C})$ in terms of $su(2)$. For the basis elements of $sl(2,\mathbb{C})$ we have,

$$J_0 = iX_1, \qquad J_+ = 2(-X_3 - iX_2), \qquad J_- = 2(X_3 - iX_2).$$

And this shows that we can write all the elements of $sl(2,\mathbb{C})$ as $aX + (b \cdot i)Y$, for $X, Y \in sl(2)$. Note, this only works for $sl(2,\mathbb{C})$ and $su(2)$ as Lie algebras over $\mathbb{R}$. Because the bracket of $su(2)$ isn't defined for complex scalars. However, there does exists an extension of the Lie bracket of $su(2)$ to a Lie bracket of $sl(2,\mathbb{C})$ over $\mathbb{C}$, so the Lie algebra $sl(2,\mathbb{C})$ as we know it from example 2.7. We will show this in the following proposition.

Definition 2.32. Let V be a vector space over the field $\mathbb{R}$. The **complexification** of V is the vector space consisting of the linear combinations

$$a \cdot v_1 + b \cdot i \cdot v_2, \qquad v_1, v_2 \in V, \text{ and } a, b \in \mathbb{R},$$

which is denoted by $V_\mathbb{C}$. And $V_\mathbb{C}$ becomes a complex vector space if we define,

$$i \cdot (a \cdot v_1 + b \cdot i \cdot v_2) = a \cdot i \cdot v_1 - b \cdot v_2.$$

We will assume that a complexification of a vector space is a complex vector space unless it is said to be a real vector space. That will make some things a little more convenient, because most complexifiactions that will be used in this section are complex vector spaces.

Proposition 2.33. Let $\mathbf{g}$ be a real Lie algebra, and $\mathbf{g}_\mathbb{C}$ the complexification of $\mathbf{g}$. Then, there exists an extension of the Lie bracket of $\mathbf{g}$ to $\mathbf{g}_\mathbb{C}$ such that $\mathbf{g}_\mathbb{C}$ becomes a complex Lie algebra.

Proof. Let $X_1, X_2, Y_1, Y_2 \in \mathbf{g}$ and $a, b \in \mathbb{R}$. If a Lie brackets exists for $\mathbf{g}_\mathbb{C}$, then it has to be bilinear. So, the existence of the Lie bracket, as an extension of the Lie bracket of $\mathbf{g}$, is shown by taking,

$$[\,X_1{+}iY_1\,,\,X_2{+}iY_2\,] = [\,X_1\,,\,X_2{+}iY_2\,]{+}i[\,Y_1\,,\,X_2{+}iY_2\,] = ([\,X_1\,,\,X_2\,]{-}[\,Y_1\,,\,Y_2\,]){+}i([\,X_1\,,\,Y_2\,]{+}[\,Y_1\,,\,X_2\,]),$$

and proving that this bracket is bilinear over $\mathbb{C}$, skew-symmetric and also satisfies the Jacobi identity.

Because the bracket is defined in terms of the bracket of $\mathbf{g}$, it is clear that it is at least bilinear over $\mathbb{R}$. So, we only have to check that it is also bilinear when multiplied with i. Before we do that, we will first show that the given bracket is skew-symmetric. Because then we only need to proof that it is linear in one argument, and it will follow that it is bilinear.

Note that,

$$\begin{aligned}[\,X_2 + iY_2\,,\,X_1 + iY_1\,] &= ([\,X_2\,,\,X_1\,] - [\,Y_2\,,\,Y_1\,]) + i([\,Y_2\,,\,X_1\,] + [\,X_2\,,\,Y_1\,]) \\ &= -([\,X_1\,,\,X_2\,] - [\,Y_1\,,\,Y_2\,]) - i([\,X_1\,,\,Y_2\,] + [\,Y_1\,,\,X_2\,]).\end{aligned}$$

So the given bracket is indeed skew-symmetric. So, we only need to show that $[\,i(X_1 + iY_1)\,,\,X_2 + iY_2\,]$ is linear, to proof that it is bilinear over $\mathbb{C}$.

$$\begin{aligned}[\,i(X_1 + iY_1)\,,\,X_2 + iY_2\,] &= [\,-Y_1 + iX_1\,,\,X_2 + iY_2\,] = ([\,-Y_1\,,\,X_2\,] - [\,X_1\,,\,Y_2\,]) + i([\,-Y_1\,,\,Y_2\,] + [\,X_1\,,\,X_2\,]) \\ &= i[\,X_1\,,\,X_2\,] - i[\,Y_1\,,\,Y_2\,] - [\,X_1\,,\,Y_2\,] - [\,Y_1\,,\,X_2\,] \\ &= i\Big(([\,X_1\,,\,X_2\,] - [\,Y_1\,,\,Y_2\,]) + i([\,X_1\,,\,Y_2\,] + [\,Y_1\,,\,X_2\,])\Big) = i[\,(X_1 + iY_1)\,,\,X_2 + iY_2\,].\end{aligned}$$

So the only thing left to do is showing that the given bracket also satisfies the Jacobi identity. But, the Jacobi identity,

$$[\,X\,,\,[\,Y\,,\,Z\,]\,] + [\,Y\,,\,[\,Z\,,\,X\,]\,] + [\,Z\,,\,[\,X\,,\,Y\,]\,] = 0,$$

holds when $X, Y, Z \in \mathbf{g}$. And if we take $X \in \mathbf{g}_\mathbb{C}$ and $Y, Z \in \mathbf{g}$. Then we see that it sill hold, because the bracket was also bilinear over $\mathbb{C}$. For the same reasoning, we find that it also holds when $X, Y \in \mathbf{g}_\mathbb{C}$ and also when $X, Y, Z \in \mathbf{g}_\mathbb{C}$. □

If we now apply this proposition to extend the Lie bracket of $su(2)$, we see that the basis of $su(2)$ is also a basis for $sl(2,\mathbb{C})$. This follows from applying the theorem on the Lie bracket from Example 2.31. With this we can construct a special subalgebra of a semi simple Lie algebra $\mathbf{g}_{\mathbb{C}}$.

Definition 2.34. Let $\mathbf{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra. Let $\mathbf{h} \subset \mathbf{g}_{\mathbb{C}}$ be a subspace of $\mathbf{g}_{\mathbb{C}}$ such that,

i. For all $H_1, H_2 \in \mathbf{h}$, $[\, H_1 \,,\, H_2 \,] = 0$,

ii. If $X \in \mathbf{g}_{\mathbb{C}}$, such that $[H, X] = 0$ for all $H \in \mathbf{h}$, then $X \in \mathbf{h}$,

iii. For all $H \in \mathbf{h}$, ad_H is diagonalizable, where $\mathrm{ad}_H(X) = [\,\mathrm{H}\,,\mathrm{X}\,]$.

Then we call $\mathbf{h}$ the **Cartan subalgebra** of $\mathbf{g}_{\mathbb{C}}$.

Normally, it should first be proven that a Cartan subalgebra exists. But we will be focusing on the Verma module of $sl(2,\mathbb{C})$, so we won't be bothered with those details, the same for the third property. The first two properties give us that the Cartan subalgebra is a maximum abelian subalgebra of $\mathbf{g}_{\mathbb{C}}$. So, there is no other abelian subalgebra of $\mathbf{g}_{\mathbb{C}}$, such that the Cartan subalgebra is a subalgebra of that abelian subalgebra. Where this other subalgebra is not equal to the Cartan subalgebra. Also, we will only need those two properties to be able to define a Cartan subalgebra of $sl(2,\mathbb{C})$.[2]

Example 2.35. One example of a Cartan subalgebra of $sl(2,\mathbb{C})$ is the subalgebra containing all traceless diagonal matrices. First, notice that none of the commutator properties are zero, and that $sl(2,\mathbb{C})$ is a semisimple Lie algebra.

So, we can never have two basis elements of $sl(2,\mathbb{C})$ in the Cartan subalgebra. Because two basis elements of $sl(2,\mathbb{C})$ don't commute. Now, $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so a basis for the subalgebra that contains all traceless diagonal matrices is given by J_0. And because we have,

$$[\, aJ_0 \,,\, bJ_0 \,] = 0, \qquad \text{for } a, b \in \mathbb{C},$$

it shows that this is indeed an abelian subalgebra of $sl(2,\mathbb{C})$. And we can't add any other basis element or linear combination of other basis elements of $sl(2,\mathbb{C})$, because those don't commute. Hence, this subalgebra is a Cartan subalgebra of $sl(2,\mathbb{C})$.

When we look at this example, it is obvious that we can create other Cartan subalgebras of $sl(2,\mathbb{C})$, by just taking the subalgebra generated by exactly 1 basis element of $sl(2,\mathbb{C})$.

Next we will be looking at a special inner product, but that definition will make use of an extra operation. Which will also be denoted by a $\star$. The reason for this, will be that in the case of the Lie algebra $sl(2,\mathbb{C})$, the $\star$ will be a $\star-$operation of Definition 2.23.

Now that we know this, we could first look at why the $\star-$operation 2.24, from Definition 2.23, is associated with $su(2)$. The reason for this, follows from the earlier defined $\star-$operation of $su(2)$.

Let $\mathbf{g} = su(2)$ and $\mathbf{g}_{\mathbb{C}} = sl(2,\mathbb{C})$. Earlier we defined $X^* = -X$, for $X \in su(2)$. Now, this $\star-$operation is only defined on $\mathbf{g}$, but can also be extended to $\mathbf{g}_{\mathbb{C}}$, just like the Lie bracket. First, notice that the complex conjugate satisfies the definition of the $\star-$operation from Definition 2.23. Because, for $X, Y \in \mathbf{g}$, $a, b \in \mathbb{R}$,

$$\begin{aligned} (X^*)^* &= -(-X) = X, \\ (aX + bY)^* &= \overline{a}X^* + \overline{b}Y^* = -aX - bY, \\ [\, X \,,\, Y \,]^* &= -[\, X \,,\, Y \,] = [\, Y \,,\, X \,] = [\, -Y \,,\, -X \,] = [\, Y^* \,,\, X^* \,]. \end{aligned}$$

Where the second equation follows from the fact that the complex conjugate is a linear operation for real scalars. And the last equation follows form the fact that $[\, X \,,\, Y \,] \in \mathbf{g}$. So it is indeed a

$\star-$operation. If we take $Z \in \mathbf{g}_{\mathbb{C}}$, then we can write $Z = aX + biY$, for some $X, Y \in \mathbf{g}$ and $a, b \in \mathbb{R}$. Then we see that an extension for this $\star-$operation can be given by,

$$(aX + biY)^* = -aX + biY = \overline{a}X^* + \overline{bi}Y^*.$$

Then it is at least a $\star-$operation over $\mathbb{R}$. Because it is basically written as the second property, so the first is also satisfied, and the third one follows from the bilinearity of the bracket.

Now, if we take $a, b \in \mathbb{C}$, then we still have

$$((aX + biY)^*)^* = (\overline{a}X^* + \overline{bi}Y^*)^* \overline{\overline{a}}X + \overline{\overline{bi}}Y = aX + biY.$$

The second property follows from the definition of our extended $\star-$operation. And the third property still follows from the bilinearity of the bracket. Because the bracket was a bilinear for scalars in $\mathbb{C}$.

So we see that if we first start with the $\star-$operation of $su(2)$, defined as taking the complex conjugate, then we can extend that to a $\star-$operation for $sl(2, \mathbb{C})$. And for that $\star-$operation we have,

$$\begin{aligned} J_0^* &= (iX_1)^* = (-i)(-X_1) = iX_1 = J_0, \\ J_+^* &= (-X_3 - iX_2)^* = -(-X_3) - iX_2 = X_3 - iX_2 = J_-, \qquad \text{so, } J_-^* = (J_+^*)^* = J_+. \end{aligned} \tag{2.36}$$

Which is the $\star-$operation of $sl(2, \mathbb{C})$ given by equation 2.24, which was the $\star-$operation associated with $su(2)$. So the reason why it is associated with $su(2)$, is because the $\star-$operation from equation 2.24 is the $\star-$operation of $su(2)$ extended to be defined on $sl(2, \mathbb{C})$.

Next we will look at a special inner product and subalgebra of a Lie algebra. Before we didn't need an inner product to define a Verma module, but this time we do. It will be used to determine the weight of a vector, and then we can use that to define the Verma module as a highest weight representation.

Theorem 2.37. *Let $\mathbf{g}_{\mathbb{C}}$ be a semi simple Lie algebra. Then there exists an inner product on $\mathbf{g}_{\mathbb{C}}$ that is real valued on $\mathbf{g}$ and,*

$$\langle [\, X \,,\, Y \,], Z \rangle = -\langle Y, [\, X \,,\, Z \,] \rangle,$$

for all $X \in \mathbf{g}$ and $Y, Z \in \mathbf{g}_{\mathbb{C}}$. If we define an operation $\star\colon \mathbf{g}_{\mathbb{C}} \to \mathbf{g}_{\mathbb{C}}$,

$$(X_1 + iX_2)^* = -X_1 + iX_2, \qquad \textit{for } X_1, X_2 \in \mathbf{g}.$$

Then, the inner product also satisfies

$$\langle [\, X \,,\, Y \,], Z \rangle = \langle Y, [\, X^* \,,\, Z \,] \rangle,$$

for all $X, Y, Z \in \mathbf{g}_{\mathbb{C}}$.[2]

So, if we would have $\mathbf{g}_{\mathbb{C}} = sl(2, \mathbb{C})$, then the operation $\star$ would be the same $\star-$operation from Definition 2.23. In other words, the $\star-$operation that we just extended from $su(2)$ to $sl(2, \mathbb{C})$.

The proof of this theorem won't be given, because we need some extra definitions and other properties to proof it properly. And those would only be needed to proof this theorem. So, we won't be focusing on those details.

Now that we know what a Cartan subalgebra is and that we have this inner product we can define what a root is.

Definition 2.38. Let $\mathbf{g}_{\mathbb{C}}$ be a semisimple Lie algebra and $\mathbf{h}$ a Cartan subalgebra of $\mathbf{g}_{\mathbb{C}}$. Define an inner product as in Theorem 2.37. Then we call $\alpha \in \mathbf{h}$ a **root** for $\mathbf{g}_{\mathbb{C}}$ relative to $\mathbf{h}$, or a **root**, if $\exists X \in \mathbf{g}_{\mathbb{C}}$, with $X \neq 0$, such that

$$[\, H \,,\, X \,] = \langle \alpha, H \rangle X,$$

for all $H \in \mathbf{h}$. The sets of all roots is denoted by R.

Notice that if $\alpha \in \mathbf{h}$ is a root, then $-\alpha$ is also a root. Because

$$[\,H\,,\,-X\,] = -[\,H\,,\,X\,] = -\langle \alpha, H\rangle X = \langle -\alpha, H\rangle X.$$

The same holds for scalar multiples of α.

Definition 2.39. Let R be the set of all roots for a semisimple Lie algebra $\mathbf{g}_{\mathbb{C}}$. Then we can construct a base Δ for R. Then we denote the set of positive roots, with respect to Δ, by R^+. And we denote the set of negative roots, with respect to Δ, by R^-.

Here we split up R into two sets, the positive and negative roots. And Δ is needed to know which are positive and which aren't. So, if Δ contains α and then we say that α is a positive root, and positive scalar multiples of α will be positive. This works, because the roots α are pure imaginary.[2]

Definition 2.40. Let $\mathbf{g}_{\mathbb{C}}$ be semisimple Lie algebra and $\mathbf{h}$ a Cartan subalgebra of $\mathbf{g}_{\mathbb{C}}$. And define an inner product on $\mathbf{g}_{\mathbb{C}}$ defined as in Theorem 2.37. And let α be a root. Then the space of all X in $\mathbf{g}_{\mathbb{C}}$ such that, $[\,H\,,\,X\,] = \langle \alpha, H\rangle X$ for all $H \in \mathbf{h}$, is called the **root space**. This is denoted by $\mathbf{g}_\alpha$.

Now we have basically everything that we need to construct another definition for the Verma module, if we only want to define it for $\mathbf{g}_{\mathbb{C}} = sl(2,\mathbb{C})$. So for this part, let $\mathbf{g}_{\mathbb{C}}$ be a **complex semisimple Lie algebra**, let $\mathbf{h} \subset \mathbf{g}_{\mathbb{C}}$ be a **fixed Cartan subalgebra** of $\mathbf{g}_{\mathbb{C}}$. Then define an **inner product** on $\mathbf{g}_{\mathbb{C}}$ defined as in Theorem 2.37, and let R be **the set of roots** of $\mathbf{g}_{\mathbb{C}}$ relative to $\mathbf{h}$ with a **base** Δ.

Definition 2.41. Let $\pi : \mathbf{g}_{\mathbb{C}} \to \mathrm{End}(V)$ be a representation of $\mathbf{g}_{\mathbb{C}}$. Then, $\lambda \in \mathbf{h}$ is called a **weight** of π if

$$\exists v \in V : \pi(H)(v) = \langle \lambda, H\rangle v, \qquad \forall H \in h \text{ and } v \neq 0.$$

The set of all $v \in V$ satisfying the above equation, is called the **weight space of** λ and the **multiplicity** of λ is the dimension of the weight space.

This definition of a weight of π is actually the same as the one we saw in the proof of Theorem 2.28. There we called it the weight of the vector, and here the weight of the representation. But that difference doesn't really matter, because in both cases, the weight described the same thing. Only this time, it is more abstractly defined.

This now leads to the following definition.

Definition 2.42. A representation $\pi\colon \mathbf{g}_{\mathbb{C}} \to \mathrm{End}(V)$ of $\mathbf{g}$ is **highest weight cyclic** with highest weight $\mu \in \mathbf{h}$ if there exists a non-zero vector $v \in V$ such that,

i. $\pi(H)(v) = \langle \mu, H\rangle v$ for all $H \in \mathbf{h}$,

ii. $\pi(X)(v) = 0$ for all $X \in \mathbf{g}_\alpha$, with $\alpha \in R^+$,

iii. the smallest invariant subspace containing v is V.

And we can define a Verma module as a **"maximal" highest weight cyclic** representation, with a specific weight.[2]

Now, if we look at the earlier given definition of the Verma module, Definition 2.19, then we see that it also corresponds to a highest weight cyclic representation. First, do note that we won't be able to give a concrete inner product on $sl(2,\mathbb{C})$, but that won't be a problem. Because we only need to check the values of the inner product for the basis values of $sl(2,\mathbb{C})$, because of the linearity in the second variable. See the definition of a Hermitian form, Definition 2.26.

First, notice that we defined the Verma module V_λ as the vector space generated by v. So, V_λ is per definition the smallest invariant subspace that contains v. Next, define $\mathbf{h}$ as the submodule generated by J_0. In other words, the Cartan subalgebra $\mathbf{h}$, is the Lie algebra with basis $J\_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which is the same Cartan subalgebra as the one that was given in Example 2.35. Then we see that,

$$\pi(J_0)(v_\lambda) = \lambda v_\lambda = \langle \mu, J_0 \rangle v_\lambda, \text{ for some } \mu \in \mathbf{g}_{\mathbb{C}}.$$

Where $\mu \in \mathbf{h}$ is the highest weight of π.
Now, note that we have $H \in \mathbf{h}$ arbitrary, then $H = aJ_0$ for some $a \in \mathbb{C}$. So,

$$[\, H \,,\, J_- \,] = a[\, J_0 \,,\, J_- \,] = -aJ_- = \langle \alpha, H \rangle J_-, \qquad \text{for some } \alpha \in \mathbf{h}.$$

And then we have that $\langle \alpha, J_0 \rangle = -1$, so $\langle -\alpha, J_0 \rangle = 1$. Which gives us that,

$$[\, H \,,\, J_+ \,] = \langle -\alpha, H \rangle J_+.$$

So, we have that $J_- \in \mathbf{g}_\alpha$ for $\alpha \in R^+$. And we also have found all our roots. So we will have $\pi(J_-)(v_\lambda) = 0$. Which is exactly what we wanted. So, we saw that,

$$\pi(J_0)(v_\lambda) = \lambda v_\lambda = \langle \mu, J_0 \rangle v_\lambda,$$
$$\pi(J_-)(v_\lambda) = 0,$$

and that V_λ was generated by v_λ. And the given Verma module of $sl(2, \mathbb{C})$, is indeed a Verma module according to both definitions.

3 Hypergeometric series and transformation formulas

In this chapter the hypergeometric series and the Pochhammer symbols will be discussed. First, the Pochhammer symbols will be defined, followed by some identities. Then, the definition of the hypergeometric series will be given followed by some properties of some special cases. Also, some formulas will be discussed. These formulas will show how to get another hypergeometric series with more parameters from one with less parameters.
The purpose of this chapter is to get the reader accustomed to the notation and some properties of the hypergeometric series, as well as some properties of combinatorial operations and the Pochhammer symbol.

3.1 Pochhammer symbols and ${}_2F_1$eries

Let's start with the definition of the Pochhammer symbol.

Definition 3.1. Let $n \in \mathbb{N}$ and $\alpha \in K$, with K a field. Then **the Pochhammer symbol** is given by,

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$$

The Pochhammer symbol gives rise to some simple, yet useful equations such as:

$$(\alpha)_{n-k} = \frac{(\alpha)_n}{(-1)^k(1-\alpha-n)_k}, \tag{3.2}$$

$$(n+k)! = n!(n+1)_k, \tag{3.3}$$

$$\frac{(-n)_k}{n!} = \frac{(-1)^k}{(n-k)!}, \tag{3.4}$$

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = (-1)^k\frac{(-n)_k}{k!}, \tag{3.5}$$

$$(\alpha-\beta-n+1)_n = (-1)^n(\beta-\alpha)_n, \tag{3.6}$$

$$(a-k)! = \frac{a!}{(-1)^k(-a)_k}. \tag{3.7}$$

These kinds of equations will be used to rewrite hypergeometric series and later on to rewrite equations that contain hypergeometric series.

Definition 3.8. The **hypergeometric series** is defined as,

$$F(a,\, b;\, c\, z) ={}_2 F_1(a,\, b;\, c;\, z) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k.$$

For the hypergeometric series it is known that it converges absolutely when $|z| < 1$. Also, when one of the numerators a or b is a negative integer $-n$, the series terminates and consists of only $n+1$ terms. This is something what will be used in this chapter, and almost all the given series will be terminating series. Now, in case of a terminating series, c is allowed to be a negative integer, but it must be smaller than the numerator that causes the termination.
The given hypergeometric series are also a solution of a second order differential equation. This second order differential equation is given by,

$$z(1-z)\frac{d^2y}{dz^2} + [c-(a+b+z)z]\frac{dy}{dz} - aby = 0.$$

The most general solution to the above equation, for $|z| < 1$, is given by,

$$y = A_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) + Bz^{1-c}{}_2F_1\left(\begin{matrix} a+1-c, b+1-c \\ 2-c \end{matrix}; z\right). \tag{3.9}$$

Another solution for this second order differential equation, for $|z| < 1$, is given by,

$$(1-z)^{c-a-b}{}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right) \tag{3.10}$$

So, equation 3.10 can be written in terms of equation 3.9. This can be done with the use of Euler's transformation formula. Now, we will assume the next equality to be correct without proof. By comparing coefficients, one can get the following equality:

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = (1-z)^{c-a-b}{}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}; z\right). \quad [1] \tag{3.11}$$

The first thing we want to do is to rewrite the terminating hypergeometric series into terms of the Pochhammer symbol. So, because the hypergeometric series that we will be looking at is terminating, we can write it as,

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{k=0}^{n} \frac{(a)_k(b)_k}{k!(c)_k} z^k.$$

So this will be the series that we want to write in terms of the Pochhammer symbol. To do this, we will look at some sums and binomial coefficients that may look random at first, but they will help rewriting the hypergeometric series.

First, let u and v be two positive integers. Then, we can get the number of combinations of n elements from a set of $u+v$ elements in two different ways. The first would be by using $\binom{u+v}{n}$, the second option is by first taking k elements from the set of u elements and combining those with $n-k$ elements of the set of v elements, so we also have $v \geq n$. Which gives us,

$$\sum_{k=0}^{n} \binom{u}{k}\binom{v}{n-k} = \binom{u+v}{n}.$$

This is a polynomial expression in u and v, so this is true in general, if $n \leq u+v$ and $v \geq n$ or $u \geq n$. And with the use of equation 3.5, this can be rewritten to

$$\sum_{k=0}^{n} \binom{u}{k}\binom{v}{n-k} = \sum_{k=0}^{n}(-1)^k \frac{(-u)_k}{k!}(-1)^{n-k}\frac{(-v)_{n-k}}{(n-k)!} = (-1)^n \sum_{k=0}^{n} \frac{(-u)_k(-v)_{n-k}}{k!(n-k)!}$$
$$= (-1)^n \frac{(-u-v)_n}{n!} = \binom{u+v}{n},$$

which gives us the following equation:

$$\sum_{k=0}^{n} \frac{n!(-u)_k(-v)_{n-k}}{k!(n-k)!} = \sum_{k=0}^{n} \binom{n}{k}(-u)_k(-v)_{n-k} = (-u-v)_n. \tag{3.12}$$

And this gives us:

$$\sum_{k=0}^{n} \binom{n}{k}(-u)_k(-v)_{n-k} \overset{(3.5)}{=} \sum_{k=0}^{n}(-1)^k \frac{(-n)_k(-u)_k(-v)_{n-k}}{k!} \overset{(3.2)}{=} \sum_{k=0}^{n}(-1)^k \frac{(-n)_k(-u)_k}{k!}\frac{(-v)_n}{(-1)^k(1+v-n)_k}$$
$$= \sum_{k=0}^{n} \frac{(-n)_k(-u)_k(-v)_n}{k!(1+v-n)_k} \overset{(3.8)}{=} {}_2F_1\left(\begin{matrix} -n, -u \\ 1+v-n \end{matrix}; 1\right) \cdot (-v)_n \overset{(3.12)}{=} (-u-v)_n. \tag{3.13}$$

And the last part of the above equation gives us the following theorem.

Theorem 3.14. *Let n be a non-negative integer and $c > 0$ or $c \leq n$, then*

$$ {}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix};1\right) = \frac{(c-b)_n}{(c)_n}. \tag{3.15}$$

And this equation is known as Vandermonde's summation formula for a terminating ${}_2F_1$ of unit argument.

Proof. If we take $b = -u$, $1 + v - n = c$ in Equation 3.13, we get

$$ {}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix};1\right) = \frac{(b-c-n+1)_n}{(-c-n+1)_n} \overset{(3.6)}{=} \frac{(-1)^n}{(-1)^n}\frac{(c-b)_n}{(c)_n}$$

□

3.2 General hypergeometric series

Until now, the hypergeometric series, ${}_2F_1$, that we have seen has two numerator parameters, and one denominator parameter. But, a more general hypergeometric series can also be defined.

Definition 3.16. The **generalised hypergeometric series** is

$$ {}_pF_q\left(\begin{matrix}a_1, \dots, a_p\\ b_1, \dots, b_q\end{matrix};z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k}.$$

We will only be interested in the case when $p = q + 1$. For this case it is also known that the series converges absolutely for $|z| < 1$, but that won't really concern us. Because, the series we will study will, again, be terminating series.
To construct a formula for the ${}_3F_2$ series of unit argument, Euler's transformation formula 3.11 will be used. This can be done by taking the expansion of $(1-z)^{c-a-b}$. This gives us,

$$(1-z)^{c-a-b}{}_2F_1\left(\begin{matrix}c-a, c-b\\ c\end{matrix};z\right) = \sum_k \binom{c-a-b}{k}(-1)^k z^k \sum_l \frac{c-a)_l(c-b)_l}{l!(c)_l} z^l,$$

because we had that $|z| < 1$. Do note that this series representation of $(1-z)^{c-a-b}$ also works when $z = 1$.

If we now compare the coefficients of z^n, we get the following equation:

$$\sum_l \frac{(c-a)_l(c-b)_l(-n)_l(a+b-c)_n}{l!(c)_l(1-a-b+c-n)_l n!} = \frac{(a)_n(b)_n}{(c)_n n!}.$$

Which can be rewritten to,

$$ {}_3F_2\left(\begin{matrix}c-a, c-b, -n\\ c, 1-a-b+c-n\end{matrix};1\right) = \frac{(a)_n(b)_n}{(c)_n(a+b-c)_n} \tag{3.17}$$

gives us the following theorem.

Theorem 3.18. *Let n be a non-negative integer, and $a + b - n + 1 = c + d$. Then,*

$$ {}_3F_2\left(\begin{matrix}a, b, -n\\ c, d\end{matrix};1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$$

and this equation is known as the Pfaff-Saalschütz summation formula.

Notice that the proof of this theorem was already done before it was given. By just rewriting equation 3.17, the equation of the theorem can easily be made.

For the general ${}_4F_3$ hypergeometric series a same kind of formula can be given.

Theorem 3.19. *Let n be a non-negative integer, and $a+b+c-n+1 = d+e+f$, which is the balance condition. Then,*

$$ {}_4F_3\left(\begin{matrix}-n, a, b, c\\ d, e, f\end{matrix}; 1\right) = \frac{(e-c)_n(f-c)_n}{(e)_n(f)_n} {}_4F_3\left(\begin{matrix}-n, d-a, d-b, c\\ d, d+e-a-b, d+f-a-b\end{matrix}; 1\right) $$

Proof. Let a, b, c, d, e, f be arbitrary parameters. The coefficient of z^n in,

$$ {}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; z\right) {}_2F_1\left(\begin{matrix}d, e\\ f\end{matrix}; z\right) $$

is given by,

$$ \begin{aligned} \sum_k \frac{(a)_k(b)_k}{k!(c)_k}\frac{(d)_{n-k}(e)_{n-k}}{(n-k)!(f)_{n-k}} &\overset{(3.2)}{=} \sum_k \frac{(a)_k(b)_k}{k!(c)_k}\frac{(d)_n}{(1-d-n)_k}\frac{(e)_n}{(1-e-n)_k}\frac{(1-f-n)_k}{(f)_n}\frac{(-n)_k}{n!} \\ &= \frac{(d)_n(e)_n}{n!(f)_n} {}_4F_3\left(\begin{matrix}a, b, 1-f-n, -n\\ c, 1-d-n, 1-e-n\end{matrix}; 1\right) \end{aligned} \tag{3.20} $$

If we use Euler's transformation formula 3.11 on both ${}_2F_1$ series, we get

$$ (1-z)^{c-a-b+f-d-e} {}_2F_1\left(\begin{matrix}c-a, c-b\\ c\end{matrix}; z\right) {}_2F_1\left(\begin{matrix}f-d, f-e\\ f\end{matrix}; z\right). $$

Now, to satisfy the balance condition we take $c-a-b+f-d-e = 0$. This works, because we have $a+b+1-f-n-n+1 = c+1-d-n+1-e-n$ as our balance condition in equation 3.20. So the coefficient of z^n in the above equation becomes:

$$ \frac{(f-d)_n(f-e)_n}{n!(f)_n} {}_4F_3\left(\begin{matrix}c-a, c-b, 1-f-n, -n\\ c, 1-f+d-n, 1-f+e-n\end{matrix}; 1\right). \tag{3.21} $$

To finish the proof, we only need to write

$$ \frac{(d)_n(e)_n}{n!(f)_n} {}_4F_3\left(\begin{matrix}a, b, 1-f-n, -n\\ c, 1-d-n, 1-e-n\end{matrix}; 1\right) = \frac{(f-d)_n(f-e)_n}{n!(f)_n} {}_4F_3\left(\begin{matrix}c-a, c-b, 1-f-n, -n\\ c, 1-f+d-n, 1-f+e-n\end{matrix}; 1\right). $$

And then use, $a = a, b = b, 1-f-n = c, -n = -n, c = d, 1-d-n = e, 1-e-n = f$ and the balance condition. Then we see that $f-d = e-c$ and $f-e = f-c$. So we already get,

$$ {}_4F_3\left(\begin{matrix}a, b, 1-f-n, -n\\ c, 1-d-n, 1-e-n\end{matrix}; 1\right) = {}_4F_3\left(\begin{matrix}a, b, c, -n\\ d, e, f\end{matrix}; 1\right) = {}_4F_3\left(\begin{matrix}-n, a, b, c\\ d, e, f\end{matrix}; 1\right) $$

And because we have have $(a-b-n+1)_n = (-1)^n(b-a)$, we can rewrite the last two denominator terms in the left hand side of the last equation.

So, that gives use that the last terms, $1-f+d-n$ and $1-f+e-n$, become $f-d$ and $f-e$ respectively. Which are then rewritten to $e-c$ and $f-c$ respectively, and can be rewritten to $d+f-a-b$ and

$d+e-a-b$ respectively, with the use of the balance condition given in the theorem. So,

$$\begin{aligned}
&\frac{(f-d)_n(f-e)_n}{(d)_n(e)_n}{}_4F_3\left(\begin{matrix}c-a, c-b, 1-f-n, -n\\ c, 1-f+d-n, 1-f+e-n\end{matrix};1\right)\\
&=\frac{(f-d)_n(f-e)_n}{(1-d-n)_n(1-e-n)_n}{}_4F_3\left(\begin{matrix}c-a, c-b, 1-f-n, -n\\ c, f-d, f-e\end{matrix};1\right)\\
&=\frac{(e-c)_n(f-c)_n}{(e)_n(f)_n}{}_4F_3\left(\begin{matrix}d-a, d-b, c, -n\\ d, e-c, f-c\end{matrix};1\right)\\
&=\frac{(e-c)_n(f-c)_n}{(e)_n(f)_n}{}_4F_3\left(\begin{matrix}d-a, d-b, c, -n\\ d, d+f-a-b, d+e-a-b\end{matrix};1\right)\\
&=\frac{(e-c)_n(f-c)_n}{(e)_n(f)_n}{}_4F_3\left(\begin{matrix}-n, d-a, d-b, c\\ d, d+e-a-b, d+f-a-b\end{matrix};1\right)
\end{aligned}$$

And because we can swap those two terms without changing the ${}_4F_3$ series, we get the equation as given in the theorem. □

With this transformation formula of ${}_4F_3$ another transformation formula for the ${}_3F_2$ series can be made.

Corollary 3.22. Let n be a non-negative integer. Then we have,

$${}_3F_2\left(\begin{matrix}-n, b, c\\ d, e\end{matrix};1\right)=\frac{(e-c)_n}{(e)_n}{}_3F_2\left(\begin{matrix}-n, d-b, c\\ d, 1+c-e-n\end{matrix};1\right).$$

Proof. In Theorem 3.19 replace a by $f-a$. Then, the left hand side will be equal to

$${}_4F_3\left(\begin{matrix}-n, f-a, b, c\\ d, e, f\end{matrix};1\right).$$

Now, by taking the limit $f\to\infty$, we get,

$${}_3F_2\left(\begin{matrix}-n, b, c\\ d, e\end{matrix};1\right).$$

The right hand side becomes,

$$\frac{(e-c)_n(f-c)_n}{(e)_n(f)_n}{}_4F_3\left(\begin{matrix}-n, d-f+a, d-b, c\\ d, 1+c-f-n, 1+c-e-n\end{matrix};1\right).$$

Again, taking the same limit $f\to\infty$ yields,

$$\frac{(e-c)_n}{(e)_n}{}_3F_2\left(\begin{matrix}-n, d-b, c\\ d, 1+c-e-n\end{matrix};1\right).$$

□

There are more transformation functions, and one could even study them in more detail. But, only the transformation formulas and Pochhammer symbol identities given in this chapter will be used further on.

4 The Clebsch-Gordan coefficients of $su(2)$

In this following chapter, the $\star-$representation of D_j will be studied. More specifically, a realisation of the $\star-$representation of D_j, from Theorem 2.28, will be given. This realisation will be studied shortly. The study of the realisation will consist of the correctness of it, and to give some concrete values to the elements of the representation. The given values of the realisation will be really useful in the later parts of the chapter, compared to using the abstract definition.
After that, a tensor product of two D_j $\star-$representations will be defined and then decomposed into a direct sum of irreducible $\star-$representations of D_j. After decomposing them, and forming orthonormal bases for these representations, a formula for the Clebsch-Gordan coefficients can be constructed. These coefficients will then be studied, and some relations, symmetries and recurrence relations for them will be given. The study of these coefficients is rather important, because they will be used to construct the Racah polynomials.

4.1 A realisation of the D_j $\star-$representation

In this part, we will give a realisation of the irreducible D_j $\star-$representation of Theorem 2.28 and the Lie algebra $su(2)$. A realisation of $su(2)$ will be given to be able to use the same methods and values as in the work of Joris Van der Jeugt ([1]), but to stay consistent, we will define our realisation as a vector field over $\mathbb{C}$. So, let $j \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$ and let our representation and Lie algebra be defined over the field $\mathbb{C}$. Then we can define a realisation of D_j, by defining the vectors $e_m^{(j)}$, $m \in \{-j, -j+1, \ldots, j\}$, as homogeneous polynomials of degree $2j$ in the variables x and y. So define,

$$e_m^{(j)} = \frac{x^{j+m}y^{j-m}}{\sqrt{(j+m)!(j-m)!}}, \qquad \text{for } m \in \{-j, -j+1, \ldots, j\}.$$

By defining the vectors as these polynomials, the basis of the $\star-$representation will still be orthonormal. This can be achieved by defining the Hermitian form as,

$$\langle x^a y^b, x^{a'} y^{b'} \rangle = a!b!\delta_{a,a'}\delta_{b,b'}.$$

This is a well defined Hermitian form and can be defined as,

$$\langle P(x,y), Q(x,y) \rangle = \overline{P(\overline{\partial_x}, \overline{\partial_y})} \cdot Q(x,y)|_{x=y=0},$$

for two arbitrary polynomials $P(x,y)$ and $Q(x,y)$ of D_j. This means that we change the polynomial $P(x,y)$ to a polynomial consisting of differential operators for x and y, and the inner product will be the constant part of $P(\partial_x, \partial_y) \cdot \overline{Q(\overline{x}, \overline{y})}$. This is a well defined Hermitian form on D_j. Let's check this for three arbitrary homogeneous polynomials of degree n. So for $P(x,y) = \sum_{k=0}^{n} a_k x^k y^{n-k}$, $Q(x,y) = \sum_{k=0}^{n} b_k x^k y^{n-k}$ and $W(x,y) = \sum_{k=0}^{n} c_k x^k y^{n-k}$, $u, v, a_k, b_k, c_k \in \mathbb{C}$, we have,

$$\begin{aligned}
\langle P(x,y), Q(x,y)\rangle &= \overline{P(\overline{\partial_x}, \overline{\partial_y})}Q(x,y)|_{x=y=0} = (\overline{\sum_{k=0}^{n} \overline{a_k}\overline{\partial_x^k}\overline{\partial_y^{n-k}}})(\sum_{k=0}^{n} b_k x^k y^{n-k})|_{x=y=0} \\
&= \sum_{k'=0}^{n}\sum_{k=0}^{n}(\overline{a_k} b_{k'} \partial_x^k \partial_y^{n-k} x^{k'} y^{n-k'})|_{x=y=0} = \sum_{k'=0}^{n}\sum_{k=0}^{n}(\overline{a_k} b_{k'} k'!(n-k')!\delta_{k,k'}\delta_{k,k'}) \\
&= \sum_{k=0}^{n} \overline{a_k} b_k k!(n-k)!, \\
\langle Q(x,y), P(x,y)\rangle &= \overline{Q(\overline{\partial_x}, \overline{\partial_y})}P(x,y)|_{x=y=0} = (\overline{\sum_{k=0}^{n} \overline{b_k}\overline{\partial_x^k}\overline{\partial_y^{n-k}}})(\sum_{k=0}^{n} a_k x^k y^{n-k})|_{x=y=0} \\
&= \sum_{k=0}^{n} a_k \overline{b_k} k!(n-k)! = \overline{\langle P(x,y), Q(x,y)\rangle}, \\
\langle P(x,y), u\cdot Q(x,y) + v\cdot W(x,y)\rangle &= \overline{P(\overline{\partial_x}, \overline{\partial_y})}(u\cdot Q(x,y) + v\cdot W(x,y))|_{x=y=0} \\
&= (\overline{\sum_{k=0}^{n} \overline{a_k}\overline{\partial_x^k}\overline{\partial_y^{n-k}}})(u\cdot\sum_{k=0}^{n} b_k x^k y^{n-k} + v\cdot\sum_{k=0}^{n} c_k x^k y^{n-k})|_{x=y=0} \\
&= u\cdot\sum_{k'=0}^{n}\sum_{k=0}^{n}(\overline{a_k} b_{k'} \partial_x^k \partial_y^{n-k} x^{k'} y^{n-k'})|_{x=y=0} \\
&+ v\cdot\sum_{k'=0}^{n}\sum_{k=0}^{n}(\overline{a_k} c_{k'} \partial_x^k \partial_y^{n-k} x^{k'} y^{n-k'})|_{x=y=0} \\
&= u\cdot\sum_{k=0}^{n} \overline{a_k} b_k k!(n-k)! + v\cdot\sum_{k=0}^{n} \overline{a_k} c_k k!(n-k)! \\
&= u\cdot\langle P(x,y), Q(x,y)\rangle + v\cdot\langle P(x,y), W(x,y)\rangle.
\end{aligned}$$

The complex conjugate can easily be rewritten, because of the fact that it is linear in multiplication and addition. And the three equations show that this is indeed a Hermitian form. It is even an inner product, because of the earlier given property. Now, this proof can easily be extended to arbitrary polynomials, but that won't be needed for this realisation.

Now, one can represent the actions of the basis elements of $su(2)$ on the $\star-$representation D_j in the following way,

$$\pi(J_0) = \frac{1}{2}(x\partial_x - y\partial_y), \qquad \pi(J_+) = x\partial_y, \qquad \pi(J_-) = y\partial_x.$$

These actions are well defined, because when they act on a vector $e_m^{(j)}$, we get the same equations as in Theorem 2.28.

For these elements, we have the following commutator relations,

$$[\pi(J_0), \pi(J_+)] = \pi(J_+), \qquad [\pi(J_0), \pi(J_-)] = \pi(J_-), \qquad [\pi(J_+), \pi(J_-)] = 2\pi(J_0),$$

which are coherent with the commutator relations of $su(2)$. Do note that these are other relations than the ones of the basis elements of $sl(2,\mathbb{C})$, but this won't be an issue for the later parts. These elements also form a basis for $sl(2,\mathbb{C})$, as was seen in Example 2.31.

Now that we have some concrete vectors for D_j and differential operators for the actions of the basis elements of $su(2)$, we can look at the tensor product of two D_j representations. We will do this with these concrete vectors and operators to keep it easier and to get faster results.

4.2 Tensor product of D_j and the Clebsch-Gordan coefficients

Now, we will use the concrete values of the $\star-$representation D_j to decompose the tensor product of two $\star-$representations D_j, into irreducible $\star-$representations. We will use the decomposition to construct an orthonormal basis for these irreducible $\star-$representations. And at the same time, we will find a definition for the Clebsch-Gordan coefficients.

First we need to define how the tensor product of two $\mathbf{g}-$modules, becomes a $\mathbf{g}-$module on its own. First, for two vector spaces $\mathcal{V}$ and $\mathcal{W}$, with respective bases $v_1, v_2, \dots$ and $w_1, w_2, \dots$. The tensor product $\mathcal{V} \bigotimes \mathcal{W}$ has a basis consisting of the vectors $v_i \otimes v_j$.

Definition 4.1. Let $\mathbf{g}$ be a Lie algebra and let $\mathcal{V}$ and $\mathcal{W}$ be two $\mathbf{g}-$modules all over the same field K. Then, $\mathcal{V} \bigotimes \mathcal{W}$ is a $\mathbf{g}-$module on its own when we define,

$$\pi(x)(v \otimes w) = \pi_{\mathcal{V}}(x)(v) \otimes w + v \otimes \pi_{\mathcal{W}}(x)(w), \qquad \text{for } x \in \mathbf{g}, v \in \mathcal{V}, w \in \mathcal{W} \text{ and } v \otimes w \in \mathcal{V} \bigotimes \mathcal{W},$$

with $\pi_{\mathcal{V}}$ and $\pi_{\mathcal{W}}$ the representations of $\mathbf{g}$ in $\mathcal{V}$ and $\mathcal{W}$ respectively.
This is well defined. Because when we define the operation $\pi(x)$ in this way, we get,

$$\begin{aligned}
\pi(ax+by)(v \otimes w) &= \pi_{\mathcal{V}}(ax+by)(v) \otimes w + v \otimes \pi_{\mathcal{W}}(ax+by)(w) \\
&= (a \cdot \pi_{\mathcal{V}}(x)(v) + b \cdot \pi_{\mathcal{V}}(y)(v)) \otimes w + v \otimes (a \cdot \pi_{\mathcal{W}}(y)(w) + b \cdot \pi_{\mathcal{W}}(x)(w)) \\
&= a \cdot (\pi_{\mathcal{V}}(x)(v) \otimes w + v \otimes \pi_{\mathcal{W}}(x)(w)) + b \cdot (\pi_{\mathcal{V}}(y)(v) \otimes w + v \otimes \pi_{\mathcal{W}}(y)(y)) \\
&= a \cdot \pi(x)(v \otimes w) + b \cdot \pi(y)(v \otimes w) \\
\pi(x)(a(v \otimes w) + b(v' \otimes w')) &= \pi(x)((av+bv') \otimes (aw+bw')) \\
&= \pi_{\mathcal{V}}(x)(av+bv') \otimes (aw+bw') + (av+bv') \otimes \pi_{\mathcal{W}}(x)(aw+bw') \\
&= (a \cdot \pi_{\mathcal{V}}(x)(v) + b \cdot \pi_{\mathcal{V}}(x)(v')) \otimes (aw+bw') \\
&+ (av+bv') \otimes (a \cdot \pi_{\mathcal{W}}(x)(w) + b \cdot \pi_{\mathcal{W}}(x)(w')) \\
&= a(\pi_{\mathcal{V}}(x)(v) \otimes w) + b(\pi_{\mathcal{V}}(x)(v') \otimes w') + a(v \otimes \pi_{\mathcal{W}}(x)(w)) + b(v' \otimes \pi_{\mathcal{W}}(x)(w')) \\
&= a \cdot \pi(x)(v \otimes w) + b \cdot \pi(x)(v' \otimes w') \\
\pi[x\,,\,y](v \otimes w) &= \pi_{\mathcal{V}}[x\,,\,y](v) \otimes w + v \otimes \pi_{\mathcal{W}}[x\,,\,y](w) \\
&= (\pi_{\mathcal{V}}(x)(\pi_{\mathcal{V}}(y)(v)) - \pi_{\mathcal{V}}(y)(\pi_{\mathcal{V}}(x)(v))) \otimes w + v \otimes (\pi_{\mathcal{W}}(x)(\pi_{\mathcal{W}}(y)(w)) \\
&- \pi_{\mathcal{W}}(y)(\pi_{\mathcal{W}}(x)(w))) \\
&= \pi_{\mathcal{V}}(x)(\pi_{\mathcal{V}}(y)(v)) \otimes w - \pi_{\mathcal{V}}(y)(\pi_{\mathcal{V}}(x)(v)) \otimes w + v \otimes \pi_{\mathcal{W}}(x)(\pi_{\mathcal{W}}(y)(w)) \\
&- v \otimes \pi_{\mathcal{W}}(y)(\pi_{\mathcal{W}}(x)(w)) \\
&= \pi_{\mathcal{V}}(x)(\pi_{\mathcal{V}}(y)(v)) \otimes w + \pi_{\mathcal{V}}(y)(v) \otimes \pi_{\mathcal{W}}(x)(w) + \pi_{\mathcal{V}}(x)(v) \otimes \pi_{\mathcal{W}}(y)(w) \\
&+ v \otimes \pi_{\mathcal{W}}(x)(\pi_{\mathcal{W}}(y)(w)) - (\pi_{\mathcal{V}}(y)(\pi_{\mathcal{V}}(x)(v)) \otimes w + \pi_{\mathcal{V}}(x)(v) \otimes \pi_{\mathcal{W}}(y)(w)) \\
&- (\pi_{\mathcal{V}}(y)(v) \otimes \pi_{\mathcal{W}}(x)(w) + v \otimes \pi_{\mathcal{W}}(y)(\pi_{\mathcal{W}}(x)(w)) \\
&= \pi(x)(\pi_{\mathcal{V}}(y)(v) \otimes w + v \otimes \pi_{\mathcal{W}}(y)(w)) - \pi(y)(\pi_{\mathcal{V}}(x)(v) \otimes w + v \otimes \pi_{\mathcal{W}}(x)(w)) \\
&= \pi(x)(\pi(y)(v \otimes w)) - \pi(y)(\pi(x)(v \otimes w)),
\end{aligned}$$

$$\text{for } x, y \in \mathbf{g}; \qquad v \otimes w, v' \otimes w' \in \mathcal{V} \bigotimes \mathcal{W}; \qquad a, b \in K.$$

This shows that $\mathcal{V} \bigotimes \mathcal{W}$ is $\mathbf{g}-$module. Because the operation π satisfies the 3 conditions of Definition 2.11.

Let's now construct the tensor product of two $\star-$representations of D_j. So, let $j_1, j_2 \in \frac{1}{2}\mathbb{N} \bigcup \{0\}$ and consider the tensor product $D_{j_1} \bigotimes D_{j_2}$. As we saw before, a basis for $D_{j_1} \bigotimes D_{j_2}$ consists of the vectors $e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2}$, $m_1 \in \{-j_1, -j_1+1, \dots, j_1\}, m_2 \in \{-j_2, -j_2+1, \dots, j_2\}$. This shows that $D_{j_1} \otimes D_{j_2}$ is a $(2j_1+1)(2j_2+1)$ dimensional vector space. Now, we can define an inner product on this vector space by defining,

$$\langle v \otimes w, v' \otimes w' \rangle = \langle v, v' \rangle \langle w, w' \rangle, \text{ for} v, v' \in D_{j_1}, w, w' \in D_{j_2}.$$

And with the use of the realisation and the previous definition, Definition 4.1, we get,

$$\pi(J_0)(e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2}) = \pi_{D_{j_1}}(J_0)(e^{(j_1)}_{m_1}) \otimes e^{(j_2)}_{m_2} + e^{(j_1)}_{m_1} \otimes \pi_{D_{j_2}}(J_0)(e^{(j_2)}_{m_2}) = (m_1 + m_2)e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2}.$$

This shows that the maximal $\pi(J_0)$ eigenvalue of $e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2}$ is $j_1 + j_2$, with $e^{(j_1)}_{j_1} \otimes e^{(j_2)}_{j_2}$ as the corresponding eigenvector.

With this vector, we can construct a submodule of $D_{j_1} \bigotimes D_{j_2}$. This can be done by acting with $\pi(J_-)$ on $e^{(j_1)}_{j_1} \otimes e^{(j_2)}_{j_2}$ until we get $0 \otimes 0$, then we get $2(j_1+j_2)+1$ vectors. And all of those are contained, and form a basis for a submodule of $D_{j_1} \bigotimes D_{j_2}$. Because, letting $\pi(J_0)$ act on one of those vectors just gives the same vector multiplied with a scalar, and letting $\pi(J_+)$ act on one of those vectors gives a vector that is in the set, but multiplied with a scalar. The last thing follows from that fact we started with the vector with the highest weight and $\pi(J_+)(e^{(j_1)}_{j_1} \otimes e^{(j_2)}_{j_2}) = 0$, and if we let $\pi(J_+)$ act on a vector $e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2}$, on which we first applied $\pi(J_-)$, we get the old vector, on which we applied $\pi(J_-)$, multiplied with a scalar.

To give an example of such a set, let $j_1 = j_2 = 1$ then,

$$\begin{aligned}
\pi(J_-)(e^{(1)}_1 \otimes e^{(1)}_1) &= e^{(1)}_0 \otimes e^{(1)}_1 + e^{(1)}_1 \otimes e^{(1)}_0, \\
\pi(J_-)(e^{(1)}_0 \otimes e^{(1)}_1 + e^{(1)}_1 \otimes e^{(1)}_0) &= e^{(1)}_{-1} \otimes e^{(1)}_1 + e^{(1)}_0 \otimes e^{(1)}_0 + e^{(1)}_1 \otimes e^{(1)}_{-1}, \\
\pi(J_-)(e^{(1)}_{-1} \otimes e^{(1)}_1 + e^{(1)}_0 \otimes e^{(1)}_0 + e^{(1)}_1 \otimes e^{(1)}_{-1}) &= e^{(1)}_{-1} \otimes e^{(1)}_0 + e^{(1)}_0 \otimes e^{(1)}_{-1} \\
\pi(J_-)(e^{(1)}_{-1} \otimes e^{(1)}_0 + e^{(1)}_0 \otimes e^{(1)}_{-1}) &= e^{(1)}_{-1} \otimes e^{(1)}_{-1}.
\end{aligned}$$

In this example we let all scalar multiples be 1, which can always be achieved by multiplying the constructed vectors with a specific scalar. Also, duplicates were removed. So in the cases were two vectors would have the same subscript, they were turned into 1 vector.

This shows that $D_{j_1} \bigotimes D_{j_2}$ isn't irreducible, because the submodule that we constructed earlier is not equal to $D_{j_1} \bigotimes D_{j_2}$ itself. In fact, it is only equal to $D_{j_1} \bigotimes D_{j_2}$ when $j_1 = j_2 = 0$. Because only in that case, we have that the dimension of $D_{j_1} \bigotimes D_{j_2}$ is equal to $2(j_1 + j_2) + 1$.

Let's now decompose $D_{j_1} \bigotimes D_{j_2}$ into irreducible submodules. First, note that the $\pi(J_0)$ eigenvalues of all the basis vectors are in the set $\{|j_1-j_2|, |j_1-j_2+1|, \ldots, j_1+j_2\}$. Then we see that one possibility could be

$$D_{j_1} \bigotimes D_{j_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} D_j,$$

but this could only work if the irreducible components are again $\star-$representations of $sl(2, \mathbb{C})$, with the usual $\star-$operation. To check this, we will give a realisation of $D_{j_1} \bigotimes D_{j_2}$, for this we will use the realisation of the $\star-$representation D_j that was given before. So, we define the actions of J_0, J_+ and J_- as,

$$\pi(J_0) = \frac{1}{2}(x_1\partial_{x_1} - y_1\partial_{y_1}) + \frac{1}{2}(x_2\partial_{x_2} - y_2\partial_{y_2}), \quad \pi(J_+) = x_1\partial_{y_1} + x_2\partial_{y_2}, \quad \pi(J_-) = y_1\partial_{x_1} + y_2\partial_{x_2},$$

which is well defined according to Definition 4.1. And define,

$$e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2} = \frac{x_1^{j_1+m_1} y_1^{j_1-m_1}}{\sqrt{(j_1+m_1)!(j_1-m_1)!}} \frac{x_2^{j_2+m_2} y_2^{j_2-m_2}}{\sqrt{(j_2+m_2)!(j_2-m_2)!}}.$$

Then, the inner product will be given by,

$$\langle x_1^{u_1} y_1^{v_1} x_2^{u_2} y_2^{v_2}, x_1^{u_1'} y_1^{v_1'} x_2^{u_2'} y_2^{v_2'} \rangle = u_1! v_1! u_2! v_2! \delta_{u_1,u_1'} \delta_{v_1,v_1'} \delta_{u_2,u_2'} \delta_{v_2,v_2'}.$$

Now we want to find vectors of $D_{j_1} \bigotimes D_{j_2}$ that get annihilated by $\pi(J_+)$. Because if we find such a vector, and it isn't $e^{(j_1)}_{j_1} \otimes e^{(j_2)}_{j_2}$, we could construct a submodule of $D_{j_1} \bigotimes D_{j_2}$, which will be a

$\star-$representation. We also want to get all the $\pi(J_0)$ eigenvalues of $D_{j_1} \bigotimes D_{j_2}$, by letting $\pi(J_0)$ act on the basis vectors of the decomposed irreducible submodules. For the reason that, if the given possible decomposition is indeed a decomposition. The submodules will be D_j $\star-$representations, hence we will then be able to get all the basis vectors of a submodule by letting $\pi(J_-)$ act on the highest weight vector of D_j. And the highest weight should be j, which is an eigenvalue of $D_{j_1} \bigotimes D_{j_2}$.

Let's start with finding vectors that get annihilated by $\pi(J_+)$. Note that for the vectors,

$$x_1^{2j_1-k} x_2^{2j_2-k} (x_1y_2 - x_2y_1)^k,$$

it is easy to see that they get annihilated by $\pi(J_+)$, because

$$\pi(J_+)((x_1y_2 - x_2y_1)^k) = k(x_2x_1 - x_1x_2)(x_1y_2 - x_2y_1)^{k-1} = 0.$$

And, the given vector is also an element of $D_{j_1} \bigotimes D_{j_2}$, for $k \in \{0, 1, \ldots, \min(2j_1, 2j_2)\}$, because it is a homogeneous polynomial of degree $2(j_1 + j_2)$, for these values of k. However, this is easier to see after we rewrite the vector.

Now, we eventually want to construct an orthonormal basis for the D_j submodules. So we want to normalise the vector that was given earlier. Define,

$$e_j^{(j_1j_2)j} = c\sum_{l=0}^{k}(-1)^l \binom{k}{l} x_1^{2j_1-l} y_1^l x_2^{2j_2-k+l} y_2^{k-l} = c(x_1^{2j_1-k} x_2^{2j_2-k}(x_1y_2 - x_2y_1)^k),$$

the normalised version of the constructed vector in $D_{j_1} \bigotimes D_{j_2}$, with $j = j_1 + j_2 - k$ and $j \in \{j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|\}$. Now it is obvious that this is indeed a vector of $D_{j_1} \bigotimes D_{j_2}$, and it has a $\pi(J_0)$ eigenvalue of $j_1 + j_2 - k = j$.

Now, the c in $e_j^{(j_1j_2)j}$ is some constant that we have to determine. Because the coefficient c will make sure that $e_j^{(j_1j_2)j}$ will be an orthonormal vector in $D_{j_1} \bigotimes D_{j_2}$.

Hence, we want those vectors to have norm 1. So, we need to solve the following equation,

$$1 = \langle e_j^{(j_1j_2)j}, e_j^{(j_1j_2)j} \rangle = c^2 \sum_{l=0}^{k} \binom{k}{l}^2 (2j_1 - l)!l!(2j)2 - k + l)!(k-l)!.$$

Now, we will solve this with the use of the identities 3.7 and 3.3, and some other previously given identities and theorems.

$$\begin{aligned}
\sum_{l=0}^{k} \binom{k}{l}^2 (2j_1 - l)!l!(2j)2 - k + l)!(k-l)! &= \sum_{l=0}^{k} k!^2 \frac{(2j_1 - l)!l!(2j)2 - k + l)!(k-l)!}{l!(k-l)!} \\
&\overset{(3.7),(3.3)}{=} k!^2 \sum_{l=0}^{k} \frac{(2j_1)!(-k)_l(2j_2-k)!(2j_2-k+1)_l}{l!(-2j_1)_l k!} \\
&= k!(2j_1)!(2j_2-k)!{}_2F_1\left(\begin{matrix}-k, 2j_2-k+1 \\ -2j_1\end{matrix}; 1\right) \\
&\overset{(3.15)}{=} k!(2j_1)!(2j_2-k)! \frac{(-2j_1-2j_2+k-1)_k}{(-2j_1)_k} \\
&\overset{(3.4)}{=} \frac{k!(2j_1-k)!(2j_2-k)!(2j_1+2j_2-k+1)!}{(2j_1+2j_2-2k+1)!}.
\end{aligned}$$

So this gives us a formula for c up to a sign, because the given equation was equal to $\frac{1}{c^2}$. Now, c is usually chosen to be positive, so we will also do that. That gives us,

$$c = \sqrt{\frac{(2j_1+2j_2-2k+1)!}{k!(2j_1-k)!(2j_2-k)!(2j_1+2j_2-k+1)!}}, \quad \text{with } k = j_1 + j_2 - j. \tag{4.2}$$

Now we saw that the vectors $e_j^{(j_1j_2)j}$, $j = j_1 + j_2 - k$, give us all the $\pi(J_0)$ eigenvalues of $D_{j_1} \bigotimes D_{j_2}$ and they were annihilated by $\pi(J_+)$.

If we let $\pi(J_-)$ act on $e_j^{(j_1j_2)j}$ several times, with $j = j_1 + j_2 - k \in \{j_1 + j_2, j_1 + j_2 - 1 \ldots, |j_1 - j_2|\}$, then we know from Theorem 2.28 that,

$$e_m^{(j_1j_2)j} = \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}}\pi(J_-)^{j-m}e_j^{(j_1j_2)j}. \tag{4.3}$$

This is true, because the vector $e_j^{(j_1j_2)j}$ will always be the highest weight vector of a submodule D_j of $D_{j_1} \bigotimes D_{j_2}$. Later we will also check if the vectors are also independent. Now, the given expression can be explicitly written as,

$$e_m^{(j_1j_2)j} = c'(y_1\partial_{x_1} + y_2\partial_{x_2})^{j-m}x_1^{2j_1-k}x_2^{2j_2-k}(x_1y_2 - x_2y_1)^k,$$

with,

$$\begin{aligned} c' &= \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}} \cdot c \\ &= \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}} \cdot \sqrt{\frac{(2j+1)!}{(j_1+j_2-j)!(j_1-j_2+j)!(-j_1+j_2+j)!(j_1+j_2+j+1)!}}. \end{aligned}$$

By working this out, we will be able to give the definition of the Clebsch-Gordan coefficients.

$$\begin{aligned} e_m^{(j_1j_2)j} &= c'(y_1\partial_{x_1} + y_2\partial_{x_2})^{j-m}x_1^{2j_1-k}x_2^{2j_2-k}(x_1y_2 - x_2y_1)^k \\ &= c'\sum_{i=0}^{j-m}\binom{j-m}{i}(y_1\partial_{x_1})^i(y_2\partial_{x_2})^{j-m-i} \times \sum_{l=0}^{j_1+j_2-j}(-1)^l\binom{j_1+j_2-j}{l}x_1^{2j_1-l}y_1^l x_2^{j_2-j_1+j+l}y_2^{j_1+j_2-j-l} \\ &= c'\sum_{i,l}(-1)^l\binom{j-m}{i}\binom{j_1+j_2-j}{l}\frac{(2j_1-l)!}{(2j_1-l-i)!}\frac{(j_2j_1+j+l)!}{(j_2-j_1+l+m+i)!} \\ &\times x_1^{2j_1-l-i}y_1^{l+i}x_2^{j_2-j_1+l+m+i}y_2^{j_1+j_2-l-i}, \end{aligned}$$

or, by taking $r = l + i$, one could rewrite the last equality to,

$$\begin{aligned} e_m^{(j_1j_2)j} &= c'(j-m)!\sum_r\left(\sum_l(-1)^l\binom{2j_1-l}{r-l}\binom{j_2-j_1+j+l}{j+l-m-r}\binom{j_1+j_2-j}{l}\right)x_1^{2j_1-r}y_1^r x_2^{j_2-j_1+m+r}y_2^{j_1+j_2-m-r} \\ &= c'(j-m)!\sum_r\left(\sum_l(-1)^l\binom{2j_1-l}{r-l}\binom{j_2-j_1+j+l}{j+l-m-r}\binom{j_1+j_2-j}{l}\right) \\ &\times \sqrt{(2j_1-r)!r!(j_2-j_1+m+r)!(j_1+j_2-m-r)!}e_{j_1-r}^{(j_1)} \otimes e_{m+r-j_1}^{(j_2)}. \end{aligned}$$

The first equality is achieved by rewriting the divisions, in the previous equation, to binomial coefficients. Now, by taking $m_1 = j_1 - r$, this can be rewritten to,

$$e_m^{(j_1j_2)j} = \sum_{m_1}C_{m_1,m-m_1,m}^{j_1,j_2,j}e_{m_1}^{(j_2)} \otimes e_{m-m_1}^{(j_2)},$$

where

$$\begin{aligned} &C_{m_1,m-m_1,m}^{j_1,j_2,j} \\ &= \sqrt{(j+m)!(j-m)!(j_1+m_1)!(j_1-m_1)!(j_2-m_1+m)!(j_2+m_1-m)!} \\ &\times \sqrt{\frac{2j+1}{(-j_1+j_2+j)!(j_1-j_2+j)!(j_1+j_2-j)!(j_1+j_2+j+1)!}} \\ &\times \sum_l(-1)^l\binom{2j_1-l}{j_1-m_1-l}\binom{j_2-j_1+j+l}{j-j_1-m+m_1+l}\binom{j_1+j_2-j}{l}, \end{aligned} \tag{4.4}$$

are the so called **Clebsch-Gordan coefficients of** $su(2)$. Notice that these coefficients are real-valued. This follows from the fact that $j_1, j_2 \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$.

Now that we have constructed some highest weight vectors of a submodule D_j of $D_{j_1} \bigotimes D_{j_2}$. We get the following theorem.

Theorem 4.5. *Let $j_1, j_2 \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$. Then the tensor product $D_{j_1} \bigotimes D_{j_2}$ decomposes into irreducible $\star-$representations D_j of $sl(2,\mathbb{C})$,*

$$D_{j_1} \bigotimes D_{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D_j.$$

An orthonormal basis of $D_{j_1} \bigotimes D_{j_2}$ is given by the vectors,

$$e_m^{(j_1 j_2)j} = \sum_{m_1} C^{j_1,j_2,j}_{m_1,m-m_1,m} e^{(j_2)}_{m_1} \otimes e^{(j_2)}_{m-m_1},$$

where the coefficients $C^{j_1,j_2,j}_{m_1,m-m_1,m}$ are given by equation 4.4. The action of J_0, J_+ and J_- on the basis vectors $e_m^{(j_1 j_2)j}$ is the standard action of the representation D_j defined in Theorem 2.28.

Proof. We constructed the vectors $e_j^{(j_1 j_2)j}$ to have norm 1. Then, by using equation 4.3, it follows that $e_m^{(j_1 j_2)j}$ has norm 1 too. We already saw that if the different D_j submodules are disjoint, and if the sum of the dimensions is equal to $(2j_1+1)(2j_2+1)$, that the decomposition is indeed correct.

First, we will show that the decompositions are indeed disjoint. Using the action of J_0,

$$\langle \pi(J_0)(e_m^{(j_1 j_2)j}), e_{m'}^{(j_1 j_2)j'} \rangle = \langle e_m^{(j_1 j_2)j}, \pi(J_0)(e_{m'}^{(j_1 j_2)j'}) \rangle,$$

we see that if $m \neq m'$, then the vectors $e_m^{(j_1 j_2)j}$ and $e_{m'}^{(j_1 j_2)j'}$ are orthogonal. If $m = m'$, and assume $j' > j$, then

$$\langle e_m^{(j_1 j_2)j}, e_m^{(j_1 j_2)j'} \rangle \sim \langle e_j^{(j_1 j_2)j}, \pi(J_-)^{j'-j}(e_{j'}^{(j_1 j_2)j'}) \rangle \sim \langle \pi(J_+)^{j'-j} e_j^{(j_1 j_2)j}, e_{j'}^{(j_1 j_2)j'} \rangle = 0.$$

So all vectors $e_m^{(j_1 j_2)j}$ are orthonormal vectors. Hence, they are all independent of each other. Which shows that all the D_j submodules are disjoint.

And the number of such vectors is,

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} (2j+1) = (j_1+j_2-|j_1-j_2|+1)(j_1+j_2+|j_1-j_2|+1) = (2j_1+1)(2j_2+1).$$

So the vectors also form an orthonormal basis for $D_{j_1} \bigotimes D_{j_2}$. □

Now that we have decomposed our tensor product $D_{j_1} \bigotimes D_{j_2}$ and defined the Clebsch-Gordan coefficients, we can look at some identities of these coefficients.

4.3 Identities of the Clebsch-Gordan coefficients

In this part the Clebsch-Gordan coefficients will be studied. Some of the identities that will be found will also be used to find identities for the Racah polynomials. Not all of the identities will be shown, but for some identities it will be explained how to find it. And with that information, one could find more identities and use those to get extra results.

To find these identities, we will first extend the definition of the Clebsch-Gordan coefficients. If we look at equation 4.4, note that the Clebsch-Gordan coefficients could be seen as a real function of six arguments. So let's redefine our definition.

Definition 4.6. The real function $C^{j_1,j_2,j}_{m_1,m_2,m}$, with the arguments $j_1, j_2, j, m_1, m_2, m \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$, is defined as in equation 4.4 if the arguments satisfy the following conditions,

i. (j_1, j_2, j) forms a **triad**, so $-j_1 + j_2 + j$, $j_1 - j_2 + j$ and $j_1 + j_2 - j$ are non-negative integers,

ii. m_1 is a **projection** of j_1, so $m_1 \in \{-j_1, -j_1 + 1, \dots, j_1\}$; m_2 is a projection of j_2, and m is a projection of j,

iii. $m = m_1 + m_2$,

else we have $C^{j_1,j_2,j}_{m_1,m_2,m} = 0$.

With this definition, we can now write,

$$e^{(j_1 j_2)j}_m = \sum_{m_1} C^{j_1,j_2,j}_{m_1,m-m_1,m} e^{(j_2)}_{m_1} \otimes e^{(j_2)}_{m-m_1} = \sum_{m_1,m_2} C^{j_1,j_2,j}_{m_1,m_2,m} e^{(j_1)}_{m_1} e^{(j_2)}_{m_2}$$

and,

$$C^{j_1,j_2,j}_{m_1,m_2,m} = C' \sum_l (-1)^l \binom{2j_1 - l}{j_1 - m_1 - l} \binom{j_2 - j_1 + j + l}{j - j_1 - m_2 + l} \binom{j_1 + j_2 - j}{l}, \tag{4.7}$$

where the summation is over all integers l for which the binomial coefficients aren't zero. So $\max(0, j_1 + m_2 - j) \le l \le \min(j_1 - m_1, j_1 + j_2 - j)$. So, if $j_1 + m_2 - j < 0$, equation 4.7 can be rewritten into terms of a ${}_3F_2$ series. For this, we will use some identities from Chapter 3, to rewrite factorials to Pochhammer symbols.

$$\begin{aligned}
C^{j_1,j_2,j}_{m_1,m_2,m} &= C' \sum_l (-1)^l \binom{2j_1 - l}{j_1 - m_1 - l} \binom{j_2 - j_1 + j + l}{j - j_1 - m_2 + l} \binom{j_1 + j_2 - j}{l} \\
&\overset{3.5}{=} C' \sum_l (-1)^l \frac{(2j_1 - l)!(j_2 - j + j + l)!}{(j_1 - m_1 - l)!(j_1 + m_1)!(j - j_1 - m_2 + l)!(j_2 + m_2)!} \cdot (-1)^l \frac{(-j_1 - j_2 + j)_l}{l!} \\
&\overset{3.7}{=} C' \sum_l \frac{(j_2 - j + j + l)!}{(j_1 + m_1)!(j - j_1 - m_2 + l)!(j_2 + m_2)!} \cdot \frac{(-j_1 - j_2 + j)_l}{l!} \cdot \frac{(2j_1)!}{(-2j_1)_l} \cdot \frac{(m_1 - j_1)_l}{(j_1 - m_1)!} \\
&\overset{3.3}{=} C' \sum_l \frac{(2j_1)!}{(j_1 + m_1)!(j_2 + m_2)!(j_1 - m_1)!} \cdot \frac{(-j_1 - j_2 + j)_l (m_1 - j_1)_l}{l!(-2j_1)_l} \cdot \frac{(j_2 - j_1 + j)!(j_2 - j_1 + j + 1)_l}{(j - j_1 - m_2)!(j - j_1 - m_2 + 1)_l} \\
&= C' \frac{(2j_1)!(j_2 - j_1 + j)!}{(j_1 + m_1)!(j_2 + m_2)!(j_1 - m_1)!(j - j_1 - m_2)!} \cdot {}_3F_2\left(\begin{matrix} -j_1 + m_1,\ -j_1 - j_2 + j,\ -j_1 + j_2 + j + 1 \\ -2j_1,\ j - j_1 - m_2 + 1 \end{matrix} ; 1 \right).
\end{aligned} \tag{4.8}$$

But this is off course not the only way to rewrite equation 4.7. One could also replace l with $j_1 - m_1 - l$ in equation 4.7, which leads to,

$$C^{j_1,j_2,j}_{m_1,m_2,m} = C' \sum_l (-1)^{j_1 - m_1 - l} \binom{j_1 + m_1 + l}{l} \binom{j_2 + j - m_1 - l}{j - m - l} \binom{j_1 + j_2 - j}{j_1 - m_1 - l}, \tag{4.9}$$

and this can be rewritten to a ${}_3F_2$ expression in the same way as we did in equation 4.8. Now, with the use of transformation formulas for ${}_3F_2$ series, one could find other formulas. For example, by using Corollary 3.22 on equation 4.8, assuming that $j - j_2 + m_1 \ge 0$, one finds,

$$C^{j_1,j_2,j}_{m_1,m_2,m} = C' \frac{(-j_1 + j_2 + j)!(j_1 - j_2 + j)!}{(j_2 + m_2)!(j_1 - m_1)!(j - j_1 - m_2)!(j - j_2 + m_1)!} {}_3F_2\left(\begin{matrix} -j_1 + m_1, j_1 - j_2 + j,\ -j_2 - m_2 \\ j - j_2 + m_1 + 1, j - j_1 - m_2 + 1 \end{matrix} ; 1 \right). \tag{4.10}$$

By rewriting the ${}_3F_2$ series, we can get a new explicit form of $C^{j_1,j_2,j}_{m_1,m_2,m}$. So, equation 4.10 becomes,

$$\begin{aligned} C^{j_1,j_2,j}_{m_1,m_2,m} &= \Delta(j_1,j_2,j)\sqrt{(2j+1)} \\ &\times \sqrt{(j-m)!(j+m)!(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!} \\ &\times \sum_l \frac{(-1)^l}{l!(j_1-m_1-l)!(j_1+j_2-j-l)!(j_2+m_2-l)!(j-j_2+m_1+l)!(j-j_1-m_2+l)!}, \end{aligned} \tag{4.11}$$

with

$$\Delta(j_1,j_2,j) = \sqrt{\frac{(-j_1+j_2+j)!(j_1-j_2+j)!(j_1+j_2-j)!}{(j_1+j_2+j+1)!}}. \tag{4.12}$$

The given expression has a rather symmetrical form due to Van der Waerden and Racah, and it is valid for all arguments as long as the conditions from Definition 4.6 are satisfied. Which means that the sum in equation 4.10 is over all integer values l, such that the factorials in the denominator aren't 0.

For some special cases, the sum in equation 4.11 becomes a single term. For example, if $m = j$ and we look at the sum over l. Then second term in the denominator is $(j_1 - m_1 - l)!$ and the last factor is $(j - j_1 - m_2 + l)! = (m_1 - j_1 + l)! = (-(j_1 - m_1 - l))!$. So the only value valid value for l is $j_1 - m_1$, and in this case we get,

$$C^{j_1,j_2,j}_{m_1,j-m_1,j} = (-1)^{j_1-m_1}\sqrt{\frac{(2j+1)!(j_1+j_2-j)!(j_1+m_1)!(j_2+j-m_1)!}{(j_1-m_1)!(j_2-j+m_1)!(-j_1+j_2+j)!(j_1-j_2+j)!(j_1+j_2+j+1)!}}, \tag{4.13}$$

which is a **closed form expression** for the Clebsch-Gordan coefficient.

One could off course express the Clebsch-Gordan coefficients in many other ways. Those can than also be used to study the Racah polynomials, but we will only be using the expressions given here.

But there are some identities of the Clebsch-Gordan coefficients that we still want to look at. These are the symmetry and orthogonality relations of the Clebsch-Gordan coefficients. We will first look at some symmetries of the coefficients.

Symmetries of the Clebsch-Gordan coefficients

Do note that not all of the possible symmetries will be shown in this part. Only some will be shown, and those will be used later on. But one could find other symmetries with the same methods.

With the use of equation 4.11 we can quickly find a symmetry relation for the Clebsch-Gordan coefficients. By simply replacing $j_1 = j_2, m_1 = -m_2, m = -m$, we get

$$C^{j_1,j_2,j}_{m_1,m_2,m} = C^{j_2,j_1,j}_{-m_2,-m_1,-m}. \tag{4.14}$$

By replacing l, in the summation of equation 4.11, by $j_1 + j_2 - j - l$ one finds,

$$C^{j_1,j_2,j}_{m_1,m_2,m} = (-1)^{j_1+j_2-j} C^{j_1,j_2,j}_{-m_1,-m_2,-m}. \tag{4.15}$$

This equation follows from comparing the coefficients of both Clebsch-Gordan coefficients. Now, by combining the previous two symmetries, we get,

$$C^{j_1,j_2,j}_{m_1,m_2,m} = (-1)^{j_1+j_2-j} C^{j_2,j_1,j}_{m_2,m_1,m}. \tag{4.16}$$

And, when one would replace l by $j - j_2 + m_1 + l$ in the summation of equation 4.11, one would get the following symmetry,

$$C^{j_1,j_2,j}_{m_1,m_2,m} = (-1)^{m_1-j_2+j}\sqrt{\frac{2j+1}{2j_2+1}} C^{j_1,j,j_2}_{-m_1,m,m_2}. \tag{4.17}$$

More symmetries could be found, by just changing arguments of $C^{j_1,j_2,j}_{m_1,m_2,m}$ and then comparing the two expressions. There are other methods, but we aren't really interested in those. Mostly, because the symmetries that we want to use can easily be found by the described methods above.

Orthogonality relations of the Clebsch-Gordan coefficients

To construct the Clebsch-Gordan coefficients, we started with an orthonormal basis for $D_{j_1} \bigotimes D_{j_2}$ and constructed an orthonormal basis for $\sum_{j=|j_1-j_2|}^{j_1+j_2} D_j$. These bases consisted of the elements $e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2}$, $m_1 \in \{-j_1, -j_1+1, \ldots, j_1\}, m_2 \in \{-j_2, -j_2+1, \ldots, j_2\}$ and $e^{(j_1 j_2)j}_m$, $j \in \{j_1+j_2, j_1+j_2-1, \ldots, |j_2-j_2|\}$, $m \in \{-j, -j+1, \ldots, j\}$, respectively. And, we defined

$$e^{(j_1 j_2)j}_m = \sum_{m_1,m_2} C^{j_1,j_2,j}_{m_1,m_2,m} e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2}. \tag{4.18}$$

Both of the bases can be ordered. Say,

$$B_1 = \{e^{(j_1)}_{-j_1} \otimes e^{(j_2)}_{-j_2}, e^{(j_1)}_{-j_1} \otimes e^{(j_2)}_{-j_2+1}, \ldots, e^{(j_1)}_{j_1} \otimes e^{(j_2)}_{j_2}\},$$

so first going over all the values of j_2 before changing the value of m_1, and

$$B_2 = \{e^{(j_1 j_2)j_1+j_2}_{-(j_1+j_2)}, e^{(j_1 j_2)j_1+j_2}_{-(j_1+j_2)+1}, \ldots, e^{(j_1 j_2)|j_1-j_2|}_{|j_1-j_2|}\},$$

so first going over all values of m before changing the value of j.

Because B_1 and B_2 are both orthonormal bases of $D_{j_1} \bigotimes D_{j_2}$. We know that the relation between them is defined by a transition matrix, say C. Because the bases are orthonormal, C is also an orthogonal matrix. With equation 4.18, we can easily define C as the transition matrix from B_2 to B_1. So, the i^{th} column of C will be given by,

$$(C^{j_1,j_2,j}_{-j_1,-j_2,m}, C^{j_1,j_2,j}_{-j_1,-j_2+1,m}, \ldots, C^{j_1,j_2,j}_{j_1,j_2,m}),$$

where $e^{(j_1 j_2)j}_m$ is the i^{th} basis vector of B_1. So, the values of m_1 and m_2 in the column first change in the m_2 direction, before changing in the m_1 direction. Which is the same order as in the basis B_2. Now, notice that equation 4.18 is nothing more than this column multiplied with $\overline{B_1}^T$. Where $\overline{B_1}$ is the row vector with its i^{th} value equal to the i^{th} basis element of B_1.

Now, C is actually orthonormal, so we have $C^{-1} = C^T$. This will be shown in the following corollary. Which gives us the relation,

$$e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2} = \sum_{j,m} C^{j_1,j_2,j}_{m_1,m_2,m} e^{(j_1 j_2)j}_m. \tag{4.19}$$

This leads to the following.

Corollary 4.20. The Clebsch-Gordan coefficients satisfy the following orthogonality relations:

$$\sum_{m_1,m_2} C^{j_1,j_2,j}_{m_1,m_2,m} C^{j_1,j_2,j'}_{m_1,m_2,m'} = \delta_{j,j'} \delta_{m,m'}, \tag{4.21}$$

$$\sum_{m,j} C^{j_1,j_2,j}_{m_1,m_2,m} C^{j_1,j_2,j}_{m'_1,m'_2,m} = \delta_{m_1,m'_1} \delta_{m_2,m'_2}. \tag{4.22}$$

Proof. The first equation follows from,

$$\langle e_m^{(j_1j_2)j}, e_{m'}^{(j_1j_2)j'}\rangle = \delta_{m,m'}\delta_{j,j'},$$

where we use equation 4.18 to replace $e_m^{(j_1j_2)j}$, and use the orthonormality property of the basis elements $e_{m_1}^{(j_1)} \otimes e_{m_2}^{(j_2)}$. And one should think of j_1, j_2, j, j', m and m' as fixed parameters. Then, the double sum can be seen as a single sum. And use the linearity of the Hermitian form, with the fact that the Clebsch-Gordan coefficients are real-valued.
The second equation can be proven in a similar manner. But, then we need to use equation 4.19 and

$$\langle e_{m_1}^{(j_1)} \otimes e_{m_2}^{(j_2)}, e_{m_1'}^{(j_1)} \otimes e_{m_2'}^{(j_2)}\rangle = \delta_{m_1,m_1'}\delta_{m_2,m_2'},$$

follows from the orthonormality of the basis elements $e_m^{(j_1j_2)j}$. □

So there are several ways how one can denote the Clebsch-Gordan coefficients. While some expressions do require some extra assumptions, they can sometimes be useful. We have also seen that one can construct several other expressions for the Clebsch-Gordan coefficients with the use of some symmetries. Which also showed that some coefficients are exactly the same, even if they have different arguments. Lastly, we saw that the Clebsch-Gordan coefficients satisfy two orthogonality relations. Which is helpful later on. Because we will look at a sum of a product of several Clebsch-Gordan coefficients. Which can be reduced with these relations.

5 Racah coefficients and the Racah polynomials

In this chapter we will first look at the tensor product of three $\star-$representations of D_j, with the same $\star-$operation as before. The tensor product will be decomposed into irreducible submodules. And from that decomposition, follows an expression for the Racah coefficients. Some symmetries and orthogonal properties for these Racah coefficients will be deduced. As well as some expressions. In other words, we will look at some different ways to denote these coefficients.

Eventually the Racah polynomials will be defined with the use of the Racah coefficients. After we define these polynomials, the previous expressions and symmetries will be used to look at some special properties of the Racah polynomials.

5.1 Tensor product decomposition of three $\star-$representations

We have seen how we can decompose the tensor product of two $\star-$representations of D_j, so let's do the same but this time for three. We will be using the same realisation as in Chapter 4, but we will extend it to three representations. So we will represent $D_{j_1} \bigotimes D_{j_2} \bigotimes D_{j_3}$ as the vector space consisting of homogeneous polynomials of degree $(2j_i)$ in x_i and y_i, $i \in \{1,2,3\}$. It is obvious that this is possible. Because, if we look at Definition 4.1 we could easily let $\mathcal{V}$ or $\mathcal{W}$ be a tensor product on its own. So, the definition is easy to extend to a tensor product of more representations.

So, let $j_1, j_2, j_3 \in \frac{1}{2}\mathbb{N} \bigcup \{0\}$ and consider the tensor product $D_{j_1} \bigotimes D_{j_2} \bigotimes D_{j_3}$. Then we see that the vectors $e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2} \otimes e^{(j_3)}_{m_3}$, $m_1 \in \{-j_1, -j_1+1, \ldots, j_1\}, m_2 \in \{-j_2, -j_2+1, \ldots, j_2\}$ and $m_3 \in \{-j_3, -j_3+1, \ldots, j_3\}$ form a basis for $D_{j_1} \bigotimes D_{j_2} \bigotimes D_{j_3}$. Because that follows from the definition. So, this shows that we have a $(2j_1+1)(2j_2+1)(2j_3+1)$ dimensional vector space. Also, letting one D_{j_i} be a tensor product in Chapter 4, shows that we can define the inner product of $D_{j_1} \bigotimes D_{j_2} \bigotimes D_{j_3}$ as,

$$\langle u \otimes v \otimes w, u' \otimes v' \otimes w' \rangle = \langle u, u' \rangle \langle v, v' \rangle \langle w, w' \rangle, \quad \text{for } u \otimes v \otimes w, u' \otimes v' \otimes w' \in D_{j_1} \bigotimes D_{j_2} \bigotimes D_{j_3}.$$

And we can realise the actions of J_0, J_+ and J_- by,

$$\pi(J_0) = \frac{1}{2}(x_1\partial_{x_1} - y_1\partial_{y_1}) + \frac{1}{2}(x_2\partial_{x_2} - y_2\partial_{y_2}) + \frac{1}{2}(x_3\partial_{x_3} - y_3\partial_{y_3}),$$
$$\pi(J_+) = x_1\partial_{x_1} + x_2\partial_{y_2} + x_3\partial_{y_3}, \quad \pi(J_-) = y_1\partial_{x_1} + y_2\partial_{x_2} + y_3\partial_{x_3},$$

and

$$e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2} \otimes e^{(j_3)}_{m_3} = \frac{x_1^{j_1+m_1} y_1^{j_1-m_1}}{\sqrt{(j_1+m_1)!(j_1-m_1)!}} \frac{x_2^{j_2+m_2} y_2^{j_2-m_2}}{\sqrt{(j_2+m_2)!(j_2-m_2)!}} \frac{x_3^{j_3+m_3} y_3^{j_3-m_3}}{\sqrt{(j_3+m_3)!(j_3-m_3)!}}.$$

To decompose $D_{j_1} \bigotimes D_{j_2} \bigotimes D_{j_3}$ into irreducible submodules, we can off course decompose it one for one. In other words, because we already know how to decompose a tensor product of two $\star-$representations, we will first decompose a tensor product of two representations. And then we will construct a decomposition of the new tensor product of two $\star-$representations. So, we can choose to either decompose $D_{j_1} \bigotimes D_{j_2}$ or $D_{j_2} \bigotimes D_{j_3}$ first. Both will lead to a decomposition that we already know, namely the decomposition of Chapter 4. So from Theorem 4.5, it follows that,

$$D_{j_1} \bigotimes D_{j_2} = \sum_{j_{12}=|j_1-j_2|}^{j_1+j_2} D_{j_{12}},$$
$$D_{j_2} \bigotimes D_{j_3} = \sum_{j_{23}=|j_2-j_3|}^{j_2+j_3} D_{j_{23}}.$$

And both decompositions will lead to a decomposition of the tensor product of three representations. Which we get by applying Theorem 4.5 to $D_{j_{12}} \bigotimes D_{j_3}$ and $D_{j_{23}} \bigotimes D_{j_1}$. So, we get

$$D_{j_1} \bigotimes D_{j_2} \bigotimes D_{j_3} = \left(\sum_{j_{12}=|j_1-j_2|}^{j_1+j_2} D_{j_{12}} \right) \bigotimes D_{j_3} = \sum_{j_{12}=|j_1-j_2|}^{j_1+j_2} \sum_{j=|j_{12}-j_3|}^{j_{12}+j_3} D_j, \tag{5.1}$$

$$D_{j_1} \bigotimes D_{j_2} \bigotimes D_{j_3} = D_{j_1} \bigotimes \left(\sum_{j_{23}=|j_2-j_3|}^{j_2+j_3} D_{j_{23}} \right) = \sum_{j_{23}=|j_2-j_3|}^{j_2+j_3} \sum_{j=|j_1-j_{23}|}^{j_1+j_{23}} D_j. \tag{5.2}$$

And both are well defined decompositions, but they are not always the same decomposition. Another thing that is important is that these decompositions have some D_j several times. For example, if $j_1 = 1, j_2 = 2, j_3 = 2$, then,

$$\sum_{j_{12}=1}^{3} \sum_{j=|j_{12}-2|}^{j_{12}+2} D_j = (D_1 + D_2 + D_3) + (D_0 + D_1 + D_2 + D_3 + D_4) + (D_1 + D_2 + D_3 + D_4 + D_5)$$

$$= D_0 + 3D_1 + 3D_2 + 3D_3 + 2D_4 + D_5 = \sum_{j_{23}=0}^{4} \sum_{j=|1-j_{23}|}^{1+j_{23}} D_j.$$

So both decomposition lead to the same sum of representations, but we do need to do something to deal with the multiple submodules with the same label. Because we won't be able to construct an orthonormal basis like this. If we wouldn't do anything, it would become difficult to see if the given vectors are indeed linear independent. So to resolve the multiples, we add an extra label. These will be j_{12} or j_{23}, depending on the order in which the tensor product was decomposed. So we can construct two bases for the decomposition of the tensor product. These are build up by the following vectors respectively,

$$\begin{aligned} e_m^{((j_1j_2)j_{12}j_3)j} &= \sum_{m_{12},m_3} C^{j_{12},j_3,j}_{m_{12},m_3,m} e^{(j_1j_2)j_{12}}_{m_{12}} \otimes e^{(j_3)}_{m_3} \\ &= \sum_{\substack{m_1,m_2,m_3 \\ m_1+m_2+m_3=m}} C^{j_1,j_2,j_{12}}_{m_1,m_2,m_{12}} C^{j_{12},j_3,j}_{m_{12},m_3,m} e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2} \otimes e^{(j_3)}_{m_3}, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} e_m^{(j_1(j_2j_3)j_{23})j} &= \sum_{m_1,m_{23}} C^{j_1,j_{23},j}_{m_1,m_{23},m} e^{(j_1)}_{m_1} \otimes e^{(j_2j_3)j_{23}}_{m_{23}} \\ &= \sum_{\substack{m_1,m_2,m_3 \\ m_1+m_2+m_3=m}} C^{j_2,j_3,j_{23}}_{m_2,m_3,m_{23}} C^{j_1,j_{23},j}_{m_1,m_{23},m} e^{(j_1)}_{m_1} \otimes e^{(j_2)}_{m_2} \otimes e^{(j_3)}_{m_3}. \end{aligned} \tag{5.4}$$

And we see that only j_1, j_2 and j_3 are fixed. And, from Definition 4.6 we see that, in equation 5.3, (j_{12}, j_3, j) forms a triad, m_{12},m_3 and m are a projection of j_{12}, j_3 and j respectively. And $m_{12} + m_3 = m$. And, in equation 5.4, (j_1, j_{23}, j) forms a triad, m_1,m_{23} and m are a projection of j_1, j_{23} and j respectively. And $m_1 + m_{23} = m$.

We know that the transition between two bases of the same vector space yields a matrix. And by constructing this matrix, we will find the Racah coefficients. So, we want to construct the transition matrix U between the basis that contains the vectors from equation 5.3, and the basis that contains the vectors from equation 5.4. First note that both bases are orthonormal bases, hence U is an orthogonal matrix.

Now, the elements of U are given by,

$$\langle e_m^{(j_1(j_2j_3)j_{23})j}, e_{m'}^{((j_1j_2)j_{12}j_3)j'}\rangle,$$

and by using the general action of the Casmir operator and the action of J_0, from Theorem 2.28, we will find that it is easy to simplify the matrix. Because,

$$\langle \pi(C)(e_m^{(j_1(j_2j_3)j_{23})j}), e_{m'}^{((j_1j_2)j_{12}j_3)j'}\rangle = \langle e_m^{(j_1(j_2j_3)j_{23})j}, \pi(C)(e_{m'}^{((j_1j_2)j_{12}j_3)j'})\rangle,$$

and $\overline{j(j+1)} = j(j+1) = j'(j'+1) \iff j = j'$. So, if $j \neq j'$, then the element of U is equal to zero. And with the action of J_0, we get

$$\langle \pi(J_0)(e_m^{(j_1(j_2j_3)j_{23})j}), e_{m'}^{((j_1j_2)j_{12}j_3)j'}\rangle = \langle e_m^{(j_1(j_2j_3)j_{23})j}, \pi(J_0)(e_{m'}^{((j_1j_2)j_{12}j_3)j'})\rangle.$$

And $\overline{m} = m = m' \iff m = m'$, so the elements of U are also zero if $m' \neq m$. Which gives us that the only non-zero elements of U are the elements where $j = j'$ and $m = m'$.

Now we will use the action of J_+ to show that the elements of U are also independent of m. So,

$$\begin{aligned}
\langle e_m^{(j_1(j_2j_3)j_{23})j}, e_m^{((j_1j_2)j_{12}j_3)j}\rangle &= \langle \pi(J_+)(e_{m-1}^{j_1(j_2j_3)j_{23})j}, e_m^{((j_1j_2)j_{12}j_3)j}\rangle \\
&= \sqrt{(j-m+1)(j+m)}\langle e_{m-1}^{(j_1(j_2j_3)j_{23})j}, e_m^{((j_1j_2)j_{12}j_3)j}\rangle \\
&= \langle e_{m-1}^{(j_1(j_2j_3)j_{23})j}, \pi(J_-)(e_m^{((j_1j_2)j_{12}j_3)j})\rangle \\
&= \sqrt{(j+m)(j-m+1)}\langle e_{m-1}^{(j_1(j_2j_3)j_{23})j}, e_{m-1}^{((j_1j_2)j_{12}j_3)j}\rangle,
\end{aligned}$$

shows that the values of U are not dependent on m. Because it shows that,

$$\langle e_m^{(j_1(j_2j_3)j_{23})j}, e_m^{((j_1j_2)j_{12}j_3)j}\rangle = \langle e_{m-1}^{(j_1(j_2j_3)j_{23})j}, e_{m-1}^{((j_1j_2)j_{12}j_3)j}\rangle,$$

so we can define the elements of U as either 0 or dependent on j, j_{12} and j_{23}. Now, let's denote

$$\langle e_m^{(j_1(j_2j_3)j_{23})j}, e_{m'}^{((j_1j_2)j_{12}j_3)j'}\rangle = \delta_{j,j'}\delta_{m,m'}U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}}, \tag{5.5}$$

where $U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}}$ are the **Racah coefficients**. And from this, it follows that,

$$e_m^{((j_1j_2)j_{12}j_3)j} = \sum_{j_{23}} U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}} e_m^{(j_1(j_2j_2)j_{23})j}, \tag{5.6}$$

and

$$e_m^{(j_1(j_2j_2)j_{23})j} = \sum_{j_{12}} U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}} e_m^{((j_1j_2)j_{12}j_3)j}. \tag{5.7}$$

Where the second equation follows from the fact that U was an orthogonal matrix. And the first equation can be constructed by ordering the bases. And then using the fact that,

$$a_i = \sum_k U_{ki} b_k,$$

where a_i and b_k are the i^{th} and k^{th} vector of the basis build up by the vectors from equation 5.4 and equation 5.3 respectively. And U_{ki} is the value of U, in the k^{th} row and i^{th} column. So, the matrix can be constructed in the same way as we did in Chapter 4, with the matrix C.

From this, we get another equation for the orthogonality of U. Namely,

$$\sum_{j_{12}} U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}} U_{j_3,j,j'_{23}}^{j_1,j_2,j_{12}} = \delta_{j_{23},j'_{23}}, \tag{5.8}$$

$$\sum_{j_{23}} U_{j_3,j,j_{23}}^{j_1,j_2,j_{12}} U_{j_3,j,j_{23}}^{j_1,j_2,j_{12'}} = \delta_{j_{12},j'_{12}}. \tag{5.9}$$

Which we get by using equation 5.6, equation 5.7 and the norm of two vectors from the same basis.

Later we will define the Racah polynomials, and we will be able to define them with the help of the Racah coefficients. But before we will do that, we will first give a proper definition for them and look at some properties.

5.2 Symmetries of the Racah coefficient

One way to define the Racah coefficients follows from equations 5.3 and 5.5,

$$U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}} = \sum_{\substack{m_1,m_2,m_3\\ m_1+m_2+m_3=m}} C^{j_1,j_2,j_{12}}_{m_1,m_2,m_{12}} C^{j_{12},j_3,j}_{m_{12},m_3,m} C^{j_2,j_3,j_{23}}_{m_2,m_3,m_{23}} C^{j_1,j_{23},j}_{m_1,m_{23},m}. \tag{5.10}$$

In this equation, the sum is over m_1, m_2 and m_3, such that $m_1 + m_2 + m_3 = m$ with m a fixed and arbitrary projection of j, so $m \in \{-j, -j+1, \dots, j\}$. And m_{12} stands for $m_1 + m_2$, and m_{23} stands for $m_2 + m_3$. Which shows that this is a double sum over the values of m_{12} and m_3 or over m_1 and m_{23}. All of this follows from the definition of the Clebsch-Gordan coefficients given in Definition 4.6. Also, we are able to let j be an arbitrary but fixed projection of m, because

$$\langle e^{(j_1(j_2j_3)j_{23})j}_{m}, e^{((j_1j_2)j_{12}j_3)j'}_{m'} \rangle$$

was independent of m. And equal to $U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}}$ if we take $m = m'$ and $j = j'$.

Now, a double sum of a product of 4 Clebsch-Gordan coefficients is rather complicated. So we will try to simply the given definition. But, this form is rather helpful to get some symmetries. So, before simplifying the summation, we will first look at some symmetries.

First, we know that the value of the Racah coefficient is independent of the choice of m. In other words, the sum in equation 5.10 is the same no matter how we choose m, as long as the value is valid. This gives us,

$$(2j+1)U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}} = \sum_{\substack{m,m_1,m_2,m_3\\ m_1+m_2+m_3=m}} C^{j_1,j_2,j_{12}}_{m_1,m_2,m_{12}} C^{j_{12},j_3,j}_{m_{12},m_3,m} C^{j_2,j_3,j_{23}}_{m_2,m_3,m_{23}} C^{j_1,j_{23},j}_{m_1,m_{23},m}, \tag{5.11}$$

when we also sum over all values of m. And if we know replace j with j_{23}, so $j \leftrightarrow j_{23}$, then equation 5.11 becomes

$$(2j_{23}+1)U^{j_1,j_{12},j_2}_{j_3,j_{23},j} = \sum_{\substack{m,m_1,m_2,m_3\\ m_1+m_2+m_3=m}} C^{j_1,j_{12},j_2}_{m_1,m_{12},m_2} C^{j_2,j_3,j_{23}}_{m_2,m_3,m_{23}} C^{j_{12},j_3,j}_{m_{12},m_3,m} C^{j_1,j,j_{23}}_{m_1,m,m_{23}}. \tag{5.12}$$

We can now use equation 4.17 to change the first and last Clebsch-Gordan coefficient in the last equation. That leads to,

$$(2j_{23}+1)U^{j_1,j_{12},j_2}_{j_3,j_{23},j} = \sqrt{\frac{(2j_2+1)(2j_{23}+1)}{(2j_{12}+1)(2j+1)}}(-1)^{j_2-j_{12}+j-j_{23}} \sum_{\substack{m,m_1,m_2,m_3\\ m_1+m_2+m_3=m}} C^{j_1,j_2,j_{12}}_{m_1,m_2,m_{12}} C^{j_2,j_3,j_{23}}_{m_2,m_3,m_{23}} C^{j_{12},j_3,j}_{m_{12},m_3,m} C^{j_1,j_{23},j}_{m_1,m_{23},m}, \tag{5.13}$$

where we make use of the fact that we sum over all values of m_1. So, instead of changing it to $-m_1$, we can replace it again with m_1. And this equation gives us the following symmetry,

$$U^{j_1,j_{12},j_2}_{j_3,j_{23},j} = (-1)^{j_2-j_{12}+j-j_{23}} \sqrt{\frac{(2j_2+1)(2j_{23}+1)}{(2j_{12}+1)(2j+1)}} U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}}. \tag{5.14}$$

This is off course only one of many symmetries. Because with the use of the symmetries of the Clebsch-Gordan coefficients and equation 5.11, several other symmetries can be found.

Something that is often introduced when speaking about the symmetries of the Racah coefficients, is the $6j-$**coefficient**. This is given by,

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{a+b+d+e} \frac{U^{a,b,c}_{d,e,f}}{\sqrt{(2c+1)(2f+1)}}, \tag{5.15}$$

with $(a,b,c), (d,e,c), (d,b,f)$ and (a,e,f) triads. The $6j-$coefficient is invariant under permutation of columns and also when swapping the upper and lower value in each of any two columns.[1] This isn't different then what we found earlier in equation 5.14, but it could be easier to some symmetries with the use of the $6j-$coefficient.

5.3 Expressions for the Racah coefficient

Earlier it was already noted that the given expression for the Racah coefficient was complicated. So in this section we will try to simplify the expression given in equation 5.10.

The first person to simplify the expression for $U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}}$, given in equation 5.10, was Racah [1]. This lead to a single sum expression for the Racah coefficient. And that expression can even be written as a general hypergeometric series, which is something that looks a lot less complex than the expression we have now.

We will start equation 5.7 and use equation 5.4 and equation 5.3 to replace the $e^{j_1(j_2j_3)j_{23})j}_m$ term and the $e^{((j_1j_2)j_{12}j_3)j}_m$ term respectively. That gives us,

$$C^{j_2,j_3,j_{23}}_{m_2,m_3,m_2+m_3}C^{j_1,j_{23},j}_{m_1,m_2+m_3,m_1+m_2+m_3} = \sum_{j_{12}} U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}} C^{j_1,j_2,j_{12}}_{m_1,m_2,m_{12}} C^{j_{12},j_3,j}_{m_{12},m_3,m_{12}+m_3}, \tag{5.16}$$

for fixed values of m_1, m_2 and m_3, and $m_{12} = m_1 + m_2$. Now, keep m_{12} fixed and multiply both sided with $C^{j_1,j_2,j'_{12}}_{m_1,m_2,m_{12}}$, with j'_{12} fixed, and sum over m_1 and m_2. Then we get,

$$\begin{aligned}\sum_{m_1,m_2} C^{j_2,j_3,j_{23}}_{m_2,m_3,m_2+m_3}C^{j_1,j_{23},j}_{m_1,m_2+m_3,m_1+m_2+m_3}C^{j_1,j_2,j'_{12}}_{m_1,m_2,m_{12}} &= \sum_{m_1,m_2}\sum_{j_{12}} U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}} C^{j_1,j_2,j_{12}}_{m_1,m_2,m_{12}} C^{j_{12},j_3,j}_{m_{12},m_3,m_{12}+m_3} C^{j_1,j_2,j'_{12}}_{m_1,m_2,m_{12}} \\ &\overset{(4.21)}{=} \sum_{j_{12}} U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}} \delta_{j_{12},j'_{12}} C^{j_{12},j_3,j}_{m_{12},m_3,m_{12}+m_3} \\ &= U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}} C^{j_{12},j_3,j}_{m_{12},m_3,m_{12}+m_3}\end{aligned} \tag{5.17}$$

So, if we let j'_{12} be equal to j_{12}, we still get the same. Using the same variable for j_{12} and j'_{12} at the start of the previous equation would be confusing. Because we would have one j_{12} that is fixed and one that isn't. But, by doing this now, we get the following expression:

$$\sum_{m_1,m_2} C^{j_2,j_3,j_{23}}_{m_2,m_3,m_2+m_3}C^{j_1,j_{23},j}_{m_1,m_2+m_3,m_1+m_2+m_3}C^{j_1,j_2,j_{12}}_{m_1,m_2,m_{12}} = U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}} C^{j_{12},j_3,j}_{m_{12},m_3,m_{12}+m_3}. \tag{5.18}$$

And this gives us a new expression for the Racah coefficients, namely

$$U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}} = \sum_{\substack{m_1,m_2 \\ m_1+m_2=m_{12}}} C^{j_2,j_3,j_{23}}_{m_2,m_3,m_2+m_3}C^{j_1,j_{23},j}_{m_1,m_2+m_3,m_1+m_2+m_3}C^{j_1,j_2,j_{12}}_{m_1,m_2,m_{12}} / C^{j_{12},j_3,j}_{m_{12},m_3,m_{12}+m_3}. \tag{5.19}$$

And in this equation we have m_{12} and m_3 fixed, and the sum is over all the values of m_1 and m_2 such that $m_1 + m_2 = m_{12}$. Do note that m_{12} and m_3 are arbitrary values, but still projections of j_{12} and j_3.

Now, let $m_{12} = j_{12}$ and $m_3 = j - j_{12}$. Then we have, $m_2 = j_{12} - m_1$, and equation 5.19 becomes,

$$U^{j_1,j_2,j_{12}}_{j_3,j,j_{23}} = \sum_{m_1} C^{j_2,j_3,j_{23}}_{j_{12}-m_1,j-j_{12},j-m_1}C^{j_1,j_{23},j}_{m_1,j-m_1,j}C^{j_1,j_2,j_{12}}_{m_1,j_{12}-m_1,j_{12}} / C^{j_{12},j_3,j}_{j_{12},j-j_{12},j},$$

which can be rewritten to,

$$U^{a,b,c}_{d,e,f} = \sum_{\alpha} C^{b,d,f}_{x-\alpha,e-c,e-\alpha}C^{a,f,e}_{a,e-\alpha,e}C^{a,b,c}_{\alpha,c-\alpha,c} / C^{c,d,e}_{c,e-c,e}. \tag{5.20}$$

Now, note that three of the Clebsch-Gordan coefficients are of the form $C^{a,b,c}_{d,e,c}$. So for those, we can use the closed form expression of the Clebsch-Gordan coefficient, equation 4.13. With this, equation 5.20

becomes,

$$
\begin{aligned}
U^{a,b,c}_{d,e,f} &= \sqrt{\frac{(2c+1)(a+b-c)!(a+f-e)!(c-d-e)!(c+d+e+1)!}{(-a+b+c)!(a-b+c)!(a+b+c+1)!(-a+f+e)!(a-f+e)!(a+f+e+1)!}} \\
&\times \sum_{\alpha} \sqrt{\frac{(b+c-\alpha)!(f+e-\alpha)!}{(b-c+\alpha)!(f-e+\alpha)!}} \frac{(a+\alpha)!}{(a-\alpha)!} C^{b,d,f}_{c-\alpha,e-c,e-\alpha}
\end{aligned} \tag{5.21}
$$

For the last Clebsch-Gordan coefficient we can use equation 4.9. Then we get the following expression for the Racah coefficient,

$$
\begin{aligned}
U^{a,b,c}_{d,e,f} &= \sqrt{\frac{(2c+1)(a+b-c)!(a+f-e)!(c-d-e)!(c+d+e+1)!}{(-a+b+c)!(a-b+c)!(a+b+c+1)!(-a+f+e)!(a-f+e)!(a+f+e+1)!}} \\
&\times \sum_{\alpha} \left[\sqrt{\frac{(b+c-\alpha)!(f+e-\alpha)!}{(b-c+\alpha)!(f-e+\alpha)!}} \frac{(a+\alpha)!}{(a-\alpha)!} \right. \\
&\times \sqrt{\frac{(2f+1)(f+e-\alpha)!(f-e+\alpha)!(b+c-\alpha)!(b-c+\alpha)!(d+e-c)!(d-e+c)!}{(-b+d+f)!(b-d+f)!(b+d-f)!(b+d+f+1)!}} \\
&\times \left. \sum_{l} (-1)^{b-c+\alpha-l} \binom{b+c-\alpha+l}{l} \binom{d+f-c+\alpha-l}{f-e+\alpha-l} \binom{b+d-f}{b-c+\alpha-l} \right] \\
&= \sqrt{\frac{(2c+1)(2f+1)(a+b-c)!(a+f-e)!}{(-a+b+c)!(a-b+c)!(a+b+c+1)!(-a+f+e)!(a-f+e)!}} \\
&\times \sqrt{\frac{(c-d-e)!(c+d+e+1)!(d+e-c)!(d-e+c)!}{(a+f+e+1)!(-b+d+f)!(b-d+f)!(b+d-f)!(b+d+f+1)!}} \\
&\times \sum_{\alpha} \left[\sqrt{\frac{(b+c-\alpha)!(f+e-\alpha)!}{(b-c+\alpha)!(f-e+\alpha)!}} \frac{(a+\alpha)!}{(a-\alpha)!} \sqrt{(f+e-\alpha)!(f-e+\alpha)!(b+c-\alpha)!(b-c+\alpha)!} \right. \\
&\times \left. \sum_{l} (-1)^{b-c+\alpha-l} \frac{(b+c-\alpha+l)!(d+f-c+\alpha-l)!(b+d-f)!}{l!(b+c-\alpha)!(f-e+\alpha-l)!(d+e-c)!(b-c+\alpha-l)!(d+c-f-\alpha+l)!} \right] \\
&= \sqrt{\frac{(2c+1)(2f+1)(a+b-c)!(a+f-e)!}{(-a+b+c)!(a-b+c)!(a+b+c+1)!(-a+f+e)!(a-f+e)!}} \\
&\times \sqrt{\frac{(c-d-e)!(c+d+e+1)!(d-e+c)!(b+d-f)!}{(a+f+e+1)!(-b+d+f)!(b-d+f)!(b+d+f+1)!(d+e-c)!}} \\
&\times \sum_{\alpha,l} \frac{(-1)^{b-c+\alpha-l}(a+\alpha)!(f+e-\alpha)!(b+c-\alpha+l)!(d+f-c+\alpha-l)!}{l!(f-e+\alpha-l)!(b-c+\alpha-l)!(d+c-f-\alpha+l)!(a-\alpha)!}
\end{aligned} \tag{5.22}
$$

If we now rewrite $\alpha - l$ to k in the summation over α and l, in the last equation. That summation becomes,

$$
\sum_{\alpha,k} \frac{(-1)^{b-c+k}(a+\alpha)!(f+e-\alpha)!(b+c-k)!(d+f-c+k)!}{(\alpha-k)!(f-e+k)!(b-c+k)!(d+c-f-k)!(a-\alpha)!}.
$$

And from equation 3.15, it follows that,

$$
\sum_{\alpha} \frac{(a+\alpha)!(f+e-\alpha)!}{(\alpha-k)!(a-\alpha)!} = \frac{(a+k)!(f+e+a+1)!(f+e-a)!}{(a-k)!(f+e+k+1)!},
$$

so the previously given summation is equal to

$$\sum_k \frac{(-1)^{b-c+k}(b+c-k)!(d+f-c+k)!(a+k)!(f+e+a+1)!(f+e-a)!}{(f-e+k)!(b-c+k)!(d-f+c-k)!(a-k)!(f+e+k+1)!}.$$

With this, we can now rewrite equation 5.22 to only contain a single sum. So we get, when also replacing k with $a-k$,

$$\begin{aligned} U^{a,b,c}_{d,e,f} &= (-1)^{a+b-c}\sqrt{(2c+1)(2f+1)}\frac{\nabla(a,e,f)\nabla(d,e,c)}{\nabla(a,b,c)\nabla(b,d,f)} \\ &\times \sum_k \frac{(-1)^k(b+c-a+k)!(d+f-c+a-k)!(2a-k)!}{k!(f-e+a-k)!(b-c+a-k)!(d-f+c-a+k)!(1+f+e+a-k)!}, \end{aligned} \tag{5.23}$$

with,

$$\nabla(a,b,c) = \sqrt{\frac{(-a+b+c)!(a-b+c)!(a+b+c+1)!}{(a+b-c)!}}.$$

The summation given in the last equation is over all values of k, such that the factorials in the denominator aren't 0. So, the summation is from $\max(0, a-d+f-c))$, to $\min(a+b-c, a+f-e)$.

With the help of the last equation it is much easier to understand what the Racah coefficients are.

Corollary 5.24. Let $a,b,c,d,e,f \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$. Then, $U^{a,b,c}_{d,e,f}$ is a function of the 6 arguments a,b,c,d,e,f. If $(a,b,c),(d,e,c),(d,b,f)$ and (a,e,f) are all triads, then $U^{a,b,c}_{d,e,f}$ is given by equation 5.23. Else, $U^{a,b,c}_{d,e,f} = 0$.

And this expression is less complicated then the one given earlier, the expression given in equation 5.10. And we can even write it as an expression of the hypergeometric series ${}_4F_3$. But, to be able to do that, we need to make one extra assumption. We need to assume that $-a+d+c-f \geq 0$. If this isn't the case, it will still work. But, we need to take $k' = -a+d+c-f+k$ and rewrite equation 5.23 as a sum over k'. We can then use the Pochhammer identities 3.3 and 3.7 to rewrite the factorials in the summation to Pochhammer symbols of the form $(a)_k$. We can then rewrite it to a ${}_4F_3$ series, with the definition of a hypergeometric series. This gives us the following corollary.

Corollary 5.25. Let $a,b,c,d,e,f \in \frac{1}{2}\mathbb{N}\bigcup\{0\}$. Such that $(a,b,c),(d,e,c),(d,b,f)$ and (a,e,f) are triads, and $-a+d+c-f \geq 0$. Then,

$$\begin{aligned} U^{a,b,c}_{d,e,f} &= (-1)^{a+b-c}\sqrt{(2c+1)(2f+1)}\frac{\nabla(a,e,f)\nabla(d,e,c)}{\nabla(a,b,c)\nabla(b,d,f)}\frac{(b+c-a)!(d+f-c+a)!(2a)!}{(f-e+a)!(b-c+a)!(d-f+c-a)!(1+f+e+a)!} \\ &\times {}_4F_3\left(\begin{matrix} 1-a+b+c,\ -a+e-f,\ -a-b+c,\ -1-a-e-f \\ -a+c-d-f,\ 2a,\ 1+d-f+c-a \end{matrix};1\right). \end{aligned}$$

Just like the Clebsch-Gordan coefficients, there are also more expressions for the Racah-coefficients that we won't be looking at. But, one could find them in the same manner as we found the previous corollary. It is also possible to find expression with the use of this corollary. Because then one could change the ${}_4F_3$ series to find a different expression, for example.

5.4 Racah polynomials

We will now describe the Racah polynomials with the help of the Racah coefficients. In other words, we will give a relation between the Racah polynomials and the $\star-$representation D_j. After the definition is given, we will look at two special properties of the Racah polynomials.

To describe the Racah polynomials, we will be using Corollary 5.25. To do this, we need to introduce some new arguments. Take

$$\begin{aligned}&n = a - e + f, x = a + b - c, \alpha \equiv -N - 1 = -a + c - d - f - 1,\\&\beta = -a + d - c - f - 1, \gamma = -2a - 1, \delta = 2c + 1.\end{aligned}$$

Then, let a, c, d, f be fixed in $U^{a,b,c}_{d,e,f}$ such that

$$c - a \geq |d - f|, \qquad \text{and } c - d \geq |a - f|.$$

So, we have b and e as variables, such that b is a value varying from $c - a$ to $d + f$, and is e is a value varying from $c - d$ to $a + f$. Note, that this gives as extra condition that $c - a \leq d + f$. But, with the use of equation 5.14, or the $6j-$coefficient, it can be shown that, that condition can always be given by interchanging the arguments. Also, from these conditions, it follows that $(a, b, c), (d, e, f), (d, b, f)$ and (a, e, f) are triads. Hence, we can use equation 5.25.

For the new arguments, this means that N is a fixed non-negative integer argument, and x and n are non-negative integer variables. Where $0 \leq x \leq N$ and $0 \leq n \leq N$. And the ${}_4F_3$ series of equation 5.25 becomes,

$$\begin{aligned}{}_4F_3\left(\begin{matrix}x + \gamma + \delta + 1, -n, -x, n + \alpha + \beta + 1\\ \alpha + 1, \gamma + 1, 1 + \beta + \delta\end{matrix}; 1\right) &= {}_4F_3\left(\begin{matrix}-n, n + \alpha + \beta + 1, -x, x + \gamma + \delta + 1\\ \alpha + 1, 1 + \beta + \delta, \gamma + 1\end{matrix}; 1\right)\\ &= R_n(\lambda(x); \alpha, \beta, \gamma, \delta) \equiv R_n(\lambda(x)).\end{aligned} \tag{5.26}$$

And in the case of the given variables, we find that $1 + \beta + \delta$ is the only non-negative value in the denominator. Because $\alpha + 1 = N$ and $\gamma + 1 \leq -N$. And $\lambda(x) = x(x + \gamma + \delta + 1)$.

Corollary 5.27. Let $\alpha, \beta, \gamma, \delta$ be any parameters and n any integer, then we have that the ${}_4F_3$ series of equation 5.26 is a polynomial of degree n in the variable $\lambda(x) = x(x + \gamma + \delta + 1)$.

And when one of the denominator parameters $\alpha + 1, \beta + \delta + 1$ or $\gamma + 1$ is equal to a negative integer $-N$, then we call this the **Racah polynomial**. Do note that, the other two denominator parameters shouldn't belong to $\{0, -1, -2, \ldots, -N + 1\}$, for $0 \leq n \leq N$.

The remark in the Corollary is there to make sure that equation 5.26 is well defined for all integers n, given that $0 \leq n \leq N$. Which is due to Theorem 3.19. Because the balance condition is satisfied, and if two or more denominator arguments belong to that set. Then $(e)_N = 0$, where e is one of the denominator arguments. Which would lead to division by zero, which won't happen when only one argument is equal to $-N$.

Now that we know what the Racah polynomial is, we can look at two special properties of it. Namely, its orthogonality property and the fact that you can actually swap n and x. Let's first look at the orthogonality property.

Orthogonality relation of the Racah polynomials

With the previously introduced variables, equation from Corollary 5.25 was still valid. And it becomes,

$$\begin{aligned}U^{a,b,c}_{d,e,f} &= (-1)^x \frac{N!(-\gamma - 1)!}{(\beta + \delta)!} \sqrt{\frac{(N + \gamma - \beta - n)!(-\beta - n - 1)!}{n!(-\gamma - n - 1)!(N - \beta - n)!(N - n)!}}\\ &\quad \times \sqrt{\frac{\delta(N + 1 + \gamma - \beta)(\beta + \delta + n)!(N + \delta - n)!(\gamma + \delta + x)!(\beta + \delta + x)!}{x!(-\gamma - x - 1)!(\delta + x)!(N - x)!(\gamma - \beta + x)!(N + 1 + \gamma + \delta + x)!}} R_n(\lambda(x), \alpha, \beta, \gamma, \delta),\end{aligned} \tag{5.28}$$

which can be found by just changing the variables. From equations 5.14 and 5.8, we get the following orthogonality relation,

$$\sum_c U^{a,b,c}_{d,e,f} U^{a,b,c}_{d,e,f'} = \sum_c (-1)^{f+f'} \frac{(2c+1)\sqrt{(2f+1)(2f'+1)}}{(2b+1)(2e+1)} U^{a,c,b}_{d,f,e} U^{a,c,b}_{d,f',e} = \delta_{f,f'}. \tag{5.29}$$

In this equation we can take $b = c$, $f = e$, $f' = e'$, and we can change $(-1)^{f+f'}$ with $(-1)^{f-f'}$. So we get,

$$\sum_b (-1)^{e-e'} \frac{(2b+1)\sqrt{(2e+1)(2e'+1)}}{(2c+1)(2f+1)} U^{a,b,c}_{d,e,f} U^{a,b,c}_{d,e',f} = \delta_{e,e'}. \tag{5.30}$$

With the use of equation 5.25, we see that we can remove the $(2c+1)$ and $(2f+1)$ terms from the denominator. And this equation becomes,

$$\begin{aligned} &\sum_{x=0}^{N} (-1)^{n-n'} (2x+\gamma+\delta+1)\sqrt{(2n+\beta+\alpha+1)(2n'+\beta+\alpha+1)} \frac{(N!)^2((-\gamma-1)!)^2}{((\beta-\delta)!)^2} \\ &\times \sqrt{\frac{(N+\gamma-\beta-n)!(N+\gamma-\beta-n')!(-\beta-n-1)!(-\beta-n'-1)!}{n!n'!(-\gamma-n-1)!(-\gamma-n'-1)!(N-\beta-n)!(N-\beta-n')!(N-n)!(N-n')!}} \\ &\times \frac{(\gamma+\delta+x)!(\beta+\delta+x)!\sqrt{(\beta+\delta+n)!(\beta+\delta+n')!(N+\delta-n)!(N+\delta-n')!}}{x!(-\gamma-x-1)!(\delta+x)!(N-x)!(\gamma-\beta+x)!(N+1+\gamma+\delta+x)!} \\ &\times R_n(\lambda(x)) R_{n'}(\lambda(x)) = \delta_{n,n'} \end{aligned} \tag{5.31}$$

So,

$$\begin{aligned} &\sum_{x=0}^{N} \frac{(2x+\gamma+\delta+1)(\gamma+\delta+x)!(\beta+\delta+x)!(N!)^2((-\gamma-1)!)^2}{x!(-\gamma-x-1)!(\delta+x)!(N-x)!(\gamma-\beta+x)!(N+1+\gamma+\delta+x)!((\beta+\delta)!)^2} \\ &= \frac{n!(-\gamma-n-1)!(N-\beta-n)!(N-n)!}{(N+\gamma-\beta-n)!(-\beta-n-1)!(\beta+\delta+n)!(N+\delta-n)!(2n+\beta+\alpha+1)} \delta_{n,n'}. \end{aligned}$$

Then, after rewriting the factorials to Pochhammer symbols, we get

$$\begin{aligned} &\sum_{x=0}^{N} \frac{(\gamma+\delta+1+2x)(\gamma+\delta+1,\alpha+1,\beta+\delta+1,\gamma+1)_x}{(\gamma+\delta+1)x!(\gamma+\delta-\alpha+1,\gamma-\beta+1,\delta+1)_x} R_n(\lambda(x)) R_{n'}(\lambda(x)) \\ &= \frac{(\gamma+\delta+2,-\beta)_N}{(\gamma-\beta+1,\delta+1)_N} \frac{n!(n+\alpha+\beta+1,\beta+1,\alpha-\delta+1,\alpha+\beta-\gamma+1)_n}{(\alpha+\beta+2)_{2n}(\alpha+1,\beta+\delta+1,\gamma+1)_n} \delta_{n,n'}, \end{aligned} \tag{5.32}$$

where $(a_1, a_2, \ldots, a_n)_x = \prod_{i=1}^{n} (a_i)_x$.

This is the orthogonality relation for the Racah polynomials when $N = -\alpha - 1$. Since the left and right hand side of equation 5.31 are both rational functions, this is well defined for all values of α, β, δ and γ as long as both sides are well defined for the given values of α, β, δ and γ.

It is possible to check if a weight function is positive or negative when the 4 arguments are given. It is also possible to define a condition for when a weight function is positive and when it is negative, but we won't be doing that. Because defining the full condition is tedious[1], and doesn't really add anything to this property of the Racah polynomials.

For the general case, when one of the denominators is equal to $-N$ and the other two don't belong to the set $\{0, -1, \ldots, -N+2, -N+1$, we have

$$\sum_{x=0}^{N} \frac{(\gamma+\delta+1+2x)(\gamma+\delta+1, \alpha+1, \beta+\delta+1, \gamma+1)_x}{(\gamma+\delta+1)x!(\gamma+\delta-\alpha+1, \gamma-\beta+1, \delta+1)_x} R_n(\lambda(x))R_{n'}(\lambda(x))$$
$$= M\frac{n!(n+\alpha+\beta+1, \beta+1, \alpha-\delta+1, \alpha+\beta-\gamma+1)_n}{(\alpha+\beta+2)_{2n}(\alpha+1, \beta+\delta+1, \gamma+1)_n}\delta_{n,n'}$$

, with,

$$M = \begin{cases} \dfrac{(\gamma+\delta+2)_N(-\beta)_N}{(\gamma-\beta+1)_N(\delta+1)_N}, & \text{if } \alpha+1=-N, \\ \dfrac{(\gamma+\delta+2)_N(\delta-\alpha)_N}{(\gamma+\delta-\alpha+1)_N(\delta+1)_N}, & \text{if } \beta+\delta+1=-N, \\ \dfrac{(-\delta)_N(\alpha+\beta+2)_N}{(\alpha-\delta+1)_N(\beta+1)_N}, & \text{if } \gamma+1=-N. \end{cases}$$

Where the other weight functions can be found in the same way as the one was found for the case $\alpha+1=-N$.

As a polynomial of degree x

Another property of the Racah polynomial is that we can at interchange the x and n. In other words, we can represent the Racah polynomial as a polynomial variable in $\lambda(n)$, and with its degree equal to x. Which is a special property for a polynomial.

To show this, we will be using the $6j-$coefficient and Theorem 3.19. Because we first described the Racah polynomials as the ${}_4F_3$ series of the equation in Corollary 5.25, when exactly one of the denominator parameters was equal to $-N$. And, from the $6j-$coefficient and equation 5.14 we get,

$$U^{a,b,c}_{d,e,f} = U^{a,e,f}_{d,b,c} = (-1)^{f-e+c-b}\sqrt{\frac{(2f+1)(2c+1)}{(2e+1)(2b+1)}}U^{a,f,e}_{d,c,b}.$$

For this part we will be using the same values for the arguments of the Racah polynomial as before. So,

$$n = a-e+f, x = a+b-c, \alpha = -a+c-d-f-1,$$
$$\beta = -a+d-c-f-1, \gamma = -2a-1, \delta = 2c+1.$$

That gives us the following equality,

$$\begin{aligned}
&(-1)^{f-e+c-b}\sqrt{\frac{(2e+1)(2b+1)}{(2f+1)(2c+1)}}U^{a,b,c}_{d,e,f} \\
&= (-1)^{a+f-e}\sqrt{(2e+1)(2b+1)}\frac{\nabla(a,e,f)\nabla(d,e,c)}{\nabla(a,b,c)\nabla(b,d,f)}\frac{(b+c-a)!(d+f-c+a)!(2a)!}{(f-e+a)!(b-c+a)!(d-f+c-a)!(1+f+e+a)!} \\
&\times {}_4F_3\left(\begin{matrix}1-a+b+c,\ -a+e-f,\ -a-b+c,\ -1-a-e-f \\ -a+c-d-f,\ -2a, 1+d-f+c-a\end{matrix}; 1\right) \\
&= (-1)^{a+f-e}\sqrt{(2b+1)(2e+1)}\frac{\nabla(a,c,b)\nabla(d,c,e)}{\nabla(a,f,e)\nabla(f,d,b)}\frac{(f+e-a)!(d+b-e+a)!(2a)!}{(b-c+a)!(f-e+a)!(d-b+e-a)!(1+b+c+a)!} \\
&\times {}_4F_3\left(\begin{matrix}1-a+f+e,\ -a+c-b,\ -a-f+e,\ -1-a-c-b \\ -a+e-d-b,\ -2a, 1+d-b+e-a\end{matrix}; 1\right) \\
&= U^{a,f,e}_{d,c,b}.
\end{aligned} \tag{5.33}$$

Note that we have to assume that $|f-c| \geq 0$, to use the previous equation. Else we had to rewrite the sum in equation 5.23 which could lead to a different equality. But we will only use this equation to verify that the following hypergeometric series is the Racah polynomial, where we swapped n and x. Note that, when we take $b \leftrightarrow f$, $c \leftrightarrow e$, then we see that $x \leftrightarrow n$. Which is the exact change of variables we used in the Racah coefficients above. Now, we have

$$1+1-a+b+c+-a+e-f+-a-b+c+-1-a-e-f = -a+c-d-f+2a+1+d-f+c-a,$$

which is the balance condition of Theorem 3.19. So the given ${}_4F_3$ series, which was used to define the Racah polynomial, satisfies the balance condition. Hence, we can use this theorem to rewrite our ${}_4F_3$ series. Now, if we take,

$$\begin{aligned} &-n=-n,\ c=-x,\ a=n+\alpha+\beta+1,\ b=x+\gamma+\delta+1,\\ &d=\gamma+1,\ e=\alpha+1,\ f=1+\beta+\delta, \end{aligned}$$

for the arguments of the hypergeometric series in Theorem 3.19. Then it follows that,

$$\begin{aligned} &{}_4F_3\left(\begin{matrix}1-a+b+c,\ -a+e-f,\ -a-b+c,\ -1-a-e-f\\ -a+c-d-f,2a,1+d-f+c-a\end{matrix};1\right)\\ &={}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,\ -x,x+\gamma+\delta+1\\ \alpha+1,1+\beta+\delta,\gamma+1\end{matrix};1\right)\\ &=\frac{(\alpha+x+1)_n(1+\beta+\delta+x)_n}{(\alpha+1)_n(1+\beta+\delta)_n}{}_4F_3\left(\begin{matrix}-x,\ -n+\gamma-\alpha-\beta,\ -n,\ -x-\delta\\ -n-x-\beta-\delta,\ -n-\alpha-x,\gamma+1\end{matrix};1\right), \end{aligned} \tag{5.34}$$

where the last ${}_4F_3$ series is the last ${}_4F_3$ series of equation 5.33. Because the condition for the theorem is satisfied, we see that we can always use the replacements $b \leftrightarrow f$ and $c \leftrightarrow e$ in this hypergeometric series, given that we started with a Racah polynomial. This gives us a relation between the two different Racah polynomials. Because we have,

$$\begin{aligned} &R_n(\lambda(x);\alpha,\beta,\gamma,\delta)={}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,\ -x,x+\gamma+\delta+1\\ \alpha+1,1+\beta+\delta,\gamma+1\end{matrix};1\right)\\ &=\frac{(\alpha+x+1)_n(1+\beta+\delta+x)_n}{(\alpha+1)_n(1+\beta+\delta)_n}{}_4F_3\left(\begin{matrix}-x,\ -n+\gamma-\alpha-\beta,\ -n,\ -x-\delta\\ -n-x-\beta-\delta,\ -n-\alpha-x,\gamma+1\end{matrix};1\right)\\ &=\frac{(\alpha+x+1)_n(1+\beta+\delta+x)_n}{(\alpha+1)_n(1+\beta+\delta)_n}R_n(\overline{\lambda(n)};\overline{\alpha},\overline{\beta},\gamma,\overline{\delta}). \end{aligned}$$

Where $R_n(\overline{\lambda(n)};\overline{\alpha},\overline{\beta},\gamma,\overline{\delta})$ is a polynomial of degree x and variable in $\lambda(n)$. And $\overline{\alpha}=-a+e-d-b-1$, $\overline{\beta}=-a+d-e-b-1$, $\overline{\delta}=2e+1$ and $\overline{\lambda(n)}=n(n+\gamma+\overline{\delta}+1)$. So, the variables $\overline{p}$ are the same as the variable p, but with the change $e \leftrightarrow c$ and $b \leftrightarrow f$. With $p \in \{\alpha,\beta,\gamma\}$.

Which shows that the Racah polynomial is a polynomial of degree n and variable over $\lambda(x)$. And it can also be written as a polynomial of degree x and variable over $\overline{\lambda(n)}$.

6 Conclusion

So we saw that we could define the Racah polynomials with the use of the tensor product of three $\star-$representations of D_j. This tensor product gave us an orthogonality relation for the Racah coefficients, and different expressions for the Racah coefficients.

With the use of Corollary 5.25, we could define an expression for the Racah polynomials, in terms of a ${}_4F_3$ series. Namely,

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = {}_4F_3\left(\begin{matrix} x+\gamma+\delta+1,\ -n,\ -x, n+\alpha+\beta+1 \\ \alpha+1, \gamma+1, 1+\beta+\delta \end{matrix};1\right).$$

Then with the use of the orthonormality relation of the Racah coefficients we found a general expression for the orthogonality relation of the Racah polynomials. This was given by,

$$\begin{aligned} &\sum_{x=0}^{N} \frac{(\gamma+\delta+1+2x)(\gamma+\delta+1,\alpha+1,\beta+\delta+1,\gamma+1)_x}{(\gamma+\delta+1)x!(\gamma+\delta-\alpha+1,\gamma-\beta+1,\delta+1)_x} R_n(\lambda(x))R_{n'}(\lambda(x)) \\ &= M\frac{n!(n+\alpha+\beta+1,\beta+1,\alpha-\delta+1,\alpha+\beta-\gamma+1)_n}{(\alpha+\beta+2)_{2n}(\alpha+1,\beta+\delta+1,\gamma+1)_n}\delta_{n,n'} \end{aligned}$$

, with,

$$M = \begin{cases} \dfrac{(\gamma+\delta+2)_N(-\beta)_N}{(\gamma-\beta+1)_N(\delta+1)_N}, & \text{if } \alpha+1 = N, \\ \dfrac{(\gamma+\delta+2)_N(\delta-\alpha)_N}{(\gamma+\delta-\alpha+1)_N(\delta+1)_N}, & \text{if } \beta+\delta+1 = -N, \\ \dfrac{(-\delta)_N(\alpha+\beta+2)_N}{(\alpha-\delta+1)_N(\beta+1)_N}, & \text{if } \gamma+1 = -N. \end{cases}$$

Where M was a weight function, that could either be positive or negative.

And with the use of Theorem 3.19, which gave us a way to rewrite a ${}_4F_3$ series. And the fact that our definition for the Racah polynomial satisfied the balance equation, we found a relation between $R_n(\lambda(x))$ and $R_x(\overline{\lambda(n)})$. Where $R_i(j)$ is the Racah polynomial of degree i and variable in j. The relation that we found was given by,

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = \frac{(\alpha+x+1)_n(1+\beta+\delta+x)_n}{(\alpha+1)_n(1+\beta+\delta)_n} R_n(\overline{\lambda(n)};\overline{\alpha},\overline{\beta},\gamma,\overline{\delta}).$$

Where we defined $\overline{p}$ to be equal to p, when changing $c \leftrightarrow e$ and $b \leftrightarrow f$ for only p or $\overline{p}$. And $p \in \{\alpha,\beta,\delta\}$.

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