Seismic Modeling 3: Computational/Theoretical Aspects

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Fast Laplace-Hankel Transform for Forward Modeling

Marianne Vissinga, Jacob T. Fokkema, and Peter M. van den Berg, Delft University of Technology, Netherlands

SUMMARY

A new method to compute the Hankel transform is presented. In this transform the original integral with a Bessel function is replaced by a spatial Fourier transform followed by a square-root filter. The Fourier transform is done by using standard FFT routines. The square-root filtering is performed by introducing a new interpolation procedure which improves the accuracy of the final results. For a simple scalar problem, the method is illustrated and the results are compared with the Cagniard-de Hoop method.

INTRODUCTION

The Hankel transform plays an important role in modelling of horizontally rotational acoustic-wave problems and in the so-called plane-wave decomposition (r-p transform) of seismic data (Treitel et al., 1982). The numerical implementation of the transform is time consuming and the accuracy is difficult to control due to evaluation and oscillatory behaviour of the Bessel function. Brysk and McCowan (1986) modified the transform such that the original integral with the Bessel function is replaced by a conventional slant-stack procedure followed by a square-root filter. Although their principal aim was the r-p transform, the method could also in principal be used for inverse transformation from r-p to t-x. The latter case is relevant in modelling. Then, the input is created in the omega-p domain followed by a Fourier transform to r-p. The main difficulty, when using their approach, is that the square-root singularities of the vertical wave number cannot be controlled in omega-p and further it appears that the slant-stack procedure in r-p is to coarse to show the details needed in modelling. In the present paper we replace the original integral by a spatial Fourier transform with the same square-root filter as Brysk and McCowan (1986) to transform from s-k to s-x, followed by an inverse Laplace transform to go back to t-x. For the latter we use a Bromwich integral along a path in the right-half of the complex s-plane, where the integrand is analytic. When the path runs parallel to the imaginary axis in this plane, the numerical evaluation of the Laplace inversion formula reduces to a standard FFT procedure with taper. In order to facilitate the computation of the square-root filter, a new interpolation of a convolution type is introduced with an appropriate interpolation function. This function is chosen such that the square-root filter operations to these functions can be determined analytically. It will be shown that this procedure performs some filtering with the inverse of the spatial Fourier transform of our interpolation function. To illustrate the method we consider in this paper the simple case of scalar wave motion. The method however can easily be extended to more complicated problems as, for example, full elastic-wave-propagation problems.

THE CONFIGURATION

The Cartesian coordinates (x, y, z) locate a point in a horizontally stratified medium. The impulsive wave motion \( u = u(x, y, z, t) \) is generated by a point source that starts to act at \( t = 0 \). The wave motion is a function of the vertical z-coordinate and independent of the horizontal coordinates x, y, and the time coordinate t. Since we are interested in the behaviour of the wave motion in a time-invariant configuration and since the source starts to act at \( t = 0 \), one can take advantage of this situation mathematically by carrying out an one-sided Laplace transformation with respect to time.

TIME LAPLACE TRANSFORMATION

The Laplace transform of our causal wave motion is then given by

\[
\hat{u}(x, y, z, s) = \int_0^\infty e^{-st} u(x, y, z, t) \, dt ,
\]

where \( \text{Re}(s) > 0 \). For each causal wave function \( u \), we have the property that \( \hat{u} \) is an analytic function of \( s \) in the right half of the complex s-plane. The inverse Laplace transform is obtained as a Bromwich integral (Widder, 1946)

\[
u(x, y, z, t) = \frac{1}{2\pi i} \int_{L-i\infty}^{L+i\infty} e^{st} \hat{u}(x, y, z, s) \, ds ,
\]

where \( \delta > 0 \). Introducing the frequency \( f \) in relation to the Laplace transformation parameter as

\[
s = \delta - i2\pi f ,
\]

we obtain the inverse Laplace transform as

\[
u(x, y, z, t) = e^{\delta t} \int_{-\infty}^{\infty} e^{-i2\pi ft} \hat{u}(x, y, z, s) \, df .
\]

How \( \hat{u} \) will be computed, is discussed in the next sections. As soon as \( \hat{u} \) has been calculated, the integral in the right-hand side of Eq. (4) can, for various time instants, be
computed efficiently using a Fast-Fourier-Transform (FFT) routine, provided a proper positive value of $\delta$ has been chosen. The function $e^\delta$ is the so-called taper.

**SPATIAL FOURIER TRANSFORMATION**

In view of the spatial invariance of the configuration in the horizontal directions, we introduce a two-dimensional Fourier transform of the wave motion $\tilde{u}$ as

$$\tilde{u}(\alpha, \beta, z, s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i2\pi \alpha\psi + i2\pi \beta \phi} \delta(x, y, z, s) \, dx \, dy.$$  \hspace{1cm} (5)

In practice, for each value of $\alpha$ and $\beta$, the spectral quantity $\tilde{u} = \tilde{u}(\alpha, \beta, z, s)$ can be obtained from a solution of either an integral equation or a differential equation in the vertical $z$-direction; in the case of a piecewise homogeneous configuration, a proper transfer-matrix or a scattering-matrix formulation can be employed. As soon as the spectral quantity $\tilde{u}$ has been arrived at, the wave motion $u$ is obtained from the inverse Fourier transform

$$\hat{u}(x, y, z, s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i2\pi \alpha \psi - i2\pi \beta \phi} \tilde{u}(\alpha, \beta, z, s) \, d\alpha \, d\beta.$$  \hspace{1cm} (6)

As next step, we take advantage of the rotational symmetry in the horizontal directions, viz.

$$x = d \cos \theta, \quad \alpha = 2\pi \lambda \cos \psi,$$

$$y = d \sin \theta, \quad \beta = 2\pi \lambda \sin \psi.$$  \hspace{1cm} (7)

where $0 \leq \theta < \infty, 0 \leq \psi < 2\pi.$ Further, using the property that $\tilde{u}(\alpha, \beta, z, s) = \tilde{u}(\lambda, z, s)$ is an even function of $\lambda$, the wave motion $\hat{u} = \hat{u}(x, y, z, s) = \hat{u}(d, z, s)$ is obtained as

$$\hat{u}(d, z, s) = 2 \int_{-\infty}^{\infty} \int_{0}^{2\pi} \hat{u}(\lambda, z, s) |\lambda| \, d\lambda \, \frac{d\theta}{2\pi} e^{-i2\pi \lambda \cos \theta} d\phi$$

$$= 2 \pi J_0(2\pi \lambda d) \lambda \, d\lambda,$$  \hspace{1cm} (8)

in which $J_0$ is the Bessel function of first kind and zero order. The second expression of Eq. (8) is known as the Hankel transform. A direct numerical evaluation of the Hankel transform is comparatively inefficient. For this reason, we return our attention to the first expression of Eq. (8). In this expression we have extended the definition of $\lambda$ to the range $-\infty < \lambda < \infty.$ In the remainder of this paper we employ this extended definition. Introducing a new variable of integration $r = d \cos \theta$ in the second integral of Eq. (8) and changing the order of integrations, we arrive at

$$\hat{u}(d, z, s) = \int_{0}^{\infty} \int_{0}^{\infty} \hat{u}(r, z, s) \, \frac{d\phi}{(d^2 - r^2)^{1/2}} \, dr,$$  \hspace{1cm} (9)

where $\delta = \delta(r, z, s)$ is obtained from the one-dimensional inverse Fourier transform with $\lambda$ as transformation parameter, viz.

$$\delta = \mathbf{F}^{-1}\{\delta\} = \int_{-\infty}^{\infty} e^{-i2\pi \lambda \phi} \hat{u}(\lambda, z, s) \, d\lambda,$$  \hspace{1cm} (10)

in which

$$\hat{u}(\lambda, z, s) = 2 |\lambda| \tilde{u}(\lambda, z, s).$$  \hspace{1cm} (11)

Eq. (9) represents the square-root filtering introduced by Brysk and McCowan (1986). In the definition of Eq. (10) and in the remainder of this paper, we extend the definition of $r$ to the range $-\infty < r < \infty.$ Note that in the special case $d = 0,$ Eq. (8) reduces to

$$\hat{u}(0, z, s) = \frac{\pi}{2} \delta(0, z, s),$$  \hspace{1cm} (12)

where $\delta$ is defined in Eqs. (10) - (11).

For various values of $r$, the inverse Fourier transform of Eq. (10) can be computed efficiently using a FFT routine. In order to compute the integral of the right-hand side of Eq. (9), we need the values of $\hat{u}(r, z, s)$ for all $r$ in the range of integration $0 < r < d$. The discrete Fourier transform of the FFT routine, applied to Eq. (10), yields these values at discrete $r$-points only. A direct numerical approximation of Eq. (9) leads to significant inaccuracies. In the next section, we therefore discuss a proper interpolation technique.

**INTERPOLATION PROCEDURE**

In order to have interpolated values of $\hat{u}$ at various points $r$ and to take full advantage of the properties of Fourier transforms, we now define an interpolation formula of the convolution type

$$\hat{u}(r, z, s) = \int_{-\infty}^{\infty} \hat{v}(r', z, s) \hat{\varphi}(r - r') \, dr'.$$  \hspace{1cm} (13)

This expression represents the continuous counterpart of a discrete expansion of the type (see Harrington, 1968, p. 13)

$$\hat{u}(r, z, s) = \sum_{n=-\infty}^{\infty} \hat{v}_n(z, s) \hat{\varphi}(r - r_n).$$  \hspace{1cm} (14)

In Eq. (13), $\hat{\varphi}(r)$ is a suitably chosen interpolation function such that the integral

$$I(d, r') = \int_{0}^{r} \frac{\hat{\varphi}(r - r')}{{(d^2 - r'^2)}^{1/2}} \, dr', \hspace{0.5cm} d > 0,$$  \hspace{1cm} (15)

can be computed analytically in closed form for all $r$. The values of $\hat{V}(r, z, s)$ can be determined as follows. Since, we only know exactly the quantities of $\hat{u}(\lambda, z, s)$ in the $\lambda$-domain, we transform Eq. (13) to this Fourier domain. Using the one-dimensional Fourier transform in relation to the inverse Fourier transform defined in Eq. (10), viz.,
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\[ \tilde{u} = F(\tilde{v}) = \int_{-\infty}^{\infty} e^{ix\lambda} \tilde{v}(r, z, s) \, dr, \quad (16) \]

we obtain

\[ \tilde{\phi}(\lambda, z, s) = \tilde{\phi}(\lambda) \tilde{V}(\lambda, z, s), \quad (17) \]

where

\[ \tilde{V}(\lambda, z, s) = F\{\tilde{V}(r, z, s)\}, \]
\[ \tilde{\phi}(\lambda) = F\{\tilde{\phi}(r)\}, \quad (18) \]

From Eq. (17) it directly follows that

\[ \tilde{V}(\lambda, z, s) = \frac{\tilde{V}(\lambda, z, s)}{\tilde{\phi}(\lambda)}. \quad (19) \]

Now, the final result can be written as

\[ \tilde{u}(d, z, s) = \int_{-\infty}^{\infty} I(d, r) \tilde{V}(r, z, s) \, dr, \quad (20) \]

with

\[ \tilde{V}(r, z, s) = F^{-1}\{\frac{\tilde{V}(\lambda, z, s)}{\tilde{\phi}(\lambda)}\}, \quad (21) \]

where the inverse Fourier transformation is defined in Eq. (10). As soon as we have chosen our interpolation function \( \phi(r) \), we can calculate analytically the Fourier transform of \( \phi(r) \) and the integrals \( I(d, r) \). For example, a suitable interpolation function is the triangle function with support 2L, defined as (Harrington, 1968, p. 12)

\[ \phi(r) = \begin{cases} \frac{1}{L} & |r| \leq L, \\ 0 & |r| > L. \end{cases} \quad (22) \]

The Fourier transform of this interpolation function is given by

\[ \tilde{\phi}(\lambda) = \frac{\sin^2(\pi \lambda L)}{\pi^2 \lambda^2 L}. \quad (23) \]

This interpolation function is used in our numerical program. Nevertheless higher-order interpolation functions, e.g. a cubic spline, can be used as well. With the choice of the triangle function as interpolation function the maximum range of the integration interval in Eq. (15) is 2L. This means \( I(d, r) = 0 \) when \( |d - r| > 2L \); consequently, the infinite range of integration in Eq. (20) also reduces to a finite interval 2L. We remark that the only part of the computations to be evaluated numerically is the integration of Eq. (20) over a finite interval 2L and the inverse Fourier transform of Eq. (21). The latter can be computed efficiently using a FFT routine, while the former can be replaced by a simple finite summation.

NUMERICAL RESULTS

In our numerical example we have used a FFT routine with \( N = 512 \) sample points. The FFT period in the spatial domain is taken in such a way that the field values beyond are negligible. In the \( \lambda \)-domain, we have chosen the sample interval \( \Delta \lambda \) to be \( s \)-dependent such that for increasing \( |s| \) we take increasing values of \( \Delta \lambda \). Since in the FFT routine the spatial sampling interval is related to \( \Delta \lambda \) as \( \Delta \lambda = \frac{\pi}{\Delta r} \), it means that in the spatial domain the sampling interval decreases for increasing values of \( |s| \). This is equivalent to take a finer mesh in the spatial domain for higher frequencies \( f \).

We shall present some numerical results for the nine-layer configuration for which Drijkoningen and Fokkema (1989, Table 3) have computed the space-time response by using the exact Cagniard-De Hoop method with primary reflections only. For the source-wavelet we have taken the Blackman-Harris window function. The Cagniard-De Hoop results are presented in Fig. 1, while Fig. 2 shows our present results. No noticeable differences can be observed. Our computations of Fig. 2 have also been carried out with primary reflections only. However, with a small increase of computation time, the computation of all internal multiples can be included. This is in contrast with the Cagniard-De Hoop technique where an inclusion of an extra generalised ray leads to extra computational effort. Also loss mechanisms can now be taken into account; the presence of losses leads even to a more accurate result of our inverse Laplace transform.

The computation time on a Gould PN 6000 amounts to 75 minutes CPU time. Within this time we have computed 80 traces of our example of Fig. 2.

CONCLUSIONS

We have developed a computationally efficient procedure to compute the space-time response of a point source in a horizontally stratified medium. The technique of the spatial Fourier transform in this paper can also be used advantageously for a plane-wave decomposition \((r-p)\) transform\) of seismic data. For this application, the present scheme avoids the inherent interpolation problems of the slant-stack procedure of Brysk and McCowan (1986).

REFERENCES


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**Fig. 1.** Space-time results for the nine-layer configuration: Cagniard-De Hoop method.

**Fig. 2.** Space-time results for the nine-layer configuration: present method.