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In this paper the concept of invariance factors and invariance vectors to obtain invariants (or first integrals) for difference equations will be presented. This concept turns out to be more or less similar to the concept of integrating factors and integrating vectors for ordinary differential equations.

Keywords: invariance factor, invariance vector, invariant, first integral, functional equation, difference equation.

AMS Subject Classifications: 39A10, 39A12, 39B12.

1 Introduction

The fundamental concept of how to make a simple first order ordinary differential equation (ODE) exact by means of integrating factors was discovered by Euler in the period 1732-1734. Euler showed that for a first order ODE all integrating factors have to satisfy a single, first order, linear partial differential equation. Finding an integrating factor for a given first order ODE was and still is a difficult and usually impossible task. Euler, however, used special types of integrating factors obtaining (and so solving) classes of first order ODEs. In our papers [3, 7, 8] the fundamental concept of how to make second and higher order ODEs as well as systems of first order ODEs exact by means of integrating factors and integrating vectors has been presented. Like most methods for differential equations there is a more or less similar method for difference equations (see for instance [2, 3]). Recently first integrals and invariants for difference equations obtained a lot of attention in the literature (see for instance [1, 5, 6, 7, 10]). However, for difference equations a method like the method of integrating vectors for ODEs seems to be not available to construct invariants or first integrals.
Based upon the definitions (as presented in [1, 5, 6, 7, 10]) for invariants and first integrals for difference equations we propose the following definition for an invariant for a system of $k$ first order difference equations.
Definition:
Consider the system of difference equations

\[
\begin{align*}
    x_{n+1} &= f_1(x_n, y_n, ..., z_n, n), \\
    y_{n+1} &= f_2(x_n, y_n, ..., z_n, n), \\
    &\vdots \\
    z_{n+1} &= f_k(x_n, y_n, ..., z_n, n),
\end{align*}
\]

for \( n = 0, 1, 2, ... \), and assume that there exists a nontrivial function
\( I(x_n, y_n, ..., z_n, n) \) such that for every solution \( \{x_n, y_n, ..., z_n\} \) of (1.1) and for every \( n = 0, 1, 2, ... \)

\[
I(x_{n+1}, y_{n+1}, ..., z_{n+1}, n + 1) = I(x_n, y_n, ..., z_n, n) = \text{constant} \iff \quad (1.2)
\]

\[
\Delta I(x_n, y_n, ..., z_n, n) = 0. \quad (1.3)
\]

Then \( I(x_n, y_n, ..., z_n, n) \) is an invariant for system (1.1), and (1.3) is an exact difference equation.

In this paper the concept of invariance factors and invariance vectors to obtain invariants for (systems of) difference equations will be presented. This paper is organized as follows. In section 2 the concept of invariance factors for a single first order difference equation will be given, and in section 3 of this paper the concept of invariance vectors for a system of first order difference equations will be presented. Finally in section 4 of this paper some conclusions will be drawn and some future directions for research will be indicated.

2 Invariance factors for a single first order difference equation

Before formulating the concept of invariance factors for a single first order difference equation we will first treat some elementary examples to show how invariants can be obtained or equivalently how difference equations can be made exact by using invariance factors.

Example 2.1:
Consider the difference equation

\[
x_{n+1} = 2x_n. \quad (2.1)
\]

An invariance factor for (2.1) is \( \frac{1}{2^{n+1}} \), since if we multiply (2.1) with this factor we obtain
\[ \frac{x_{n+1}}{2^{n+1}} = \frac{2x_n}{2^n} \Leftrightarrow \frac{x_{n+1}}{2^{n+1}} - \frac{x_n}{2^n} = 0 \Leftrightarrow \]
\[ \Delta \left( \frac{x_n}{2^n} \right) = 0 \Leftrightarrow \Delta I_1(x_n, n) = 0, \quad (2.2) \]

where \( I_1(x_n, n) = \frac{x_n}{2^n} \) = constant. Obviously (2.2) is an exact equation, and \( I_1(x_n, n) \) is an invariant of (2.1).

Example 2.2:
Consider the difference equation
\[ x_{n+1} = \frac{x_n}{x_n - 1}. \quad (2.3) \]

An invariance factor for (2.3) is \( \frac{x_{n+1}}{x_n} \), since if we multiply both sides of (2.3) with this factor we obtain (using (2.3))
\[ \frac{x_{n+1}^2}{x_n - 1} = \frac{x_{n+1}}{x_n} - 1 \frac{x_n}{x_n - 1} = \frac{x_n}{x_n - 1} - \frac{x_n}{x_n - 1} = \frac{x_n^2}{x_n} \Leftrightarrow \]
\[ \Delta \left( \frac{x_n^2}{x_n - 1} \right) = 0 \Leftrightarrow \Delta I_2(x_n, n) = 0, \quad (2.4) \]

where \( I_2(x_n, n) = \frac{x_n^2}{x_n - 1} \) = constant. Obviously (2.4) is an exact equation, and \( I_2(x_n, n) \) is an invariant of (2.3).

Example 2.3:
Consider the difference equation
\[ x_{n+1} = 2x_n(1 - x_n). \quad (2.5) \]

An invariance factor for (2.5) is \( \frac{1}{x_{n+1}^2} (1 - 2x_{n+1})^{\frac{1}{2}} \), since if we multiply both sides of (2.5) with this factor we obtain (using (2.5))
\[ (1 - 2x_{n+1})^{\frac{1}{2}} = \frac{1}{x_{n+1}^2} (1 - 2x_{n+1})^{\frac{1}{2}} 2x_n(1 - x_n) = \]
\[ \frac{1}{2x_n(1 - x_n)} (1 - 4x_n(1 - x_n))^{\frac{1}{2}} 2x_n(1 - x_n) = (1 - 2x_n)^{\frac{1}{2}} \Leftrightarrow \]
\[ \Delta \left( (1 - 2x_n)^{\frac{1}{2}} \right) = 0 \Leftrightarrow \Delta I_3(x_n, n) = 0, \quad (2.6) \]
where \( I_3(x_n, n) = (1 - 2x_n)^{\frac{1}{2}} = \) constant. Obviously (2.6) is an exact equation, and \( I_3(x_n, n) \) is an invariant of (2.5).

When we consider a single first order difference equation

\[
x_{n+1} = f(x_n, n),
\]

(2.7)

where \( f \) is a sufficiently smooth function, an invariant can be represented by

\[
I(x_{n+1}, n + 1) = I(x_n, n) = \text{constant} \Leftrightarrow \Delta I(x_n, n) = 0.
\]

(2.8)

Keeping in mind the examples 2.1, 2.2, and 2.3 we now make (2.7) exact by multiplying both sides of (2.7) with an invariance factor \( \mu(x_{n+1}, n + 1) = \mu(f(x_n, n), n + 1) \), yielding

\[
\mu(x_{n+1}, n + 1)x_{n+1} = \mu(f(x_n, n), n + 1)f(x_n, n).
\]

(2.9)

Since \( \mu \) is an invariance factor we have (by definition) an exact equation (2.9). The relationship between \( I \) and \( \mu \) follows from the equivalence of (2.8) and (2.9), yielding

\[
\begin{aligned}
I(x_{n+1}, n + 1) &= \mu(x_{n+1}, n + 1)x_{n+1}, \\
I(x_n, n) &= \mu(f(x_n, n), n + 1)f(x_n, n).
\end{aligned}
\]

(2.10)

Elementarily \( I \) can be eliminated from (2.10), and then it follows that all invariance factors for the difference equation (2.7) have to satisfy

\[
\mu(x_n, n)x_n = \mu(f(x_n, n), n + 1)f(x_n, n).
\]

(2.11)

When an invariance factor has been determined from the functional equation (2.11) an invariant for (2.7) easily follows from (2.10), yielding

\[
I(x_n, n) = \mu(x_n, n)x_n.
\]

(2.12)

Finding an invariance factor from the functional equation (2.11) is a difficult and mostly impossible task. However, we can use special types of invariance factors obtaining and solving classes of first order difference equations. For instance if we take as an invariance factor

\[
\mu(x_n, n) = \frac{a_n + b_n}{x_n},
\]

(2.13)

where \( a_n, b_n, c_n, \) and \( d_n \) are arbitrary, it follows easily from (2.11) that
\[
\frac{a_n x_n + b_n}{c_n x_n + d_n} = \frac{a_{n+1} f(x_n, n) + b_{n+1}}{c_{n+1} f(x_n, n) + d_{n+1}} \iff \\
\frac{f(x_n, n) = \frac{(b_{n+1} c_n - d_{n+1} a_n)x_n + b_{n+1} d_n - d_{n+1} b_n}{(c_{n+1} a_n - a_{n+1} c_n)x_n + c_{n+1} b_n - a_{n+1} d_n}}{(2.14)}
\]

So all difference equations \(x_{n+1} = f(x_n, n)\), where \(f\) is given by (2.14), can be made exact by using the invariance factor (2.13). An invariant in this case is (see also (2.12))

\[
I(x_n, n) = \mu(x_n, n)x_n = \frac{a_n x_n + b_n}{c_n x_n + d_n} = \text{constant}
\]

By just experimentally taking other types of invariance factors it is quite obvious that invariants (and/or solutions) for first order difference equations can be found, which are not known in the literature at the moment.

3 Invariance vectors for a system of first order difference equations

Before formulating the concept of invariance vectors for a system of \(k\) first order difference equations we will first treat an elementary example to show how invariants can be obtained, or equivalently how an exact equation can be obtained from the system of difference equations by using invariance vectors.

Example 3.1:
Consider the system of difference equations

\[
x_{n+1} = \cos(\omega)x_n + \sin(\omega)y_n, \quad (3.1)
\]

\[
y_{n+1} = -\sin(\omega)x_n + \cos(\omega)y_n, \quad (3.2)
\]

where \(\omega\) is an arbitrary fixed constant. By multiplying (3.1) with \(x_{n+1}\), and (3.2) with \(y_{n+1}\), and by adding the so-obtained equations we get (using (3.1) and (3.2))

\[
x_{n+1}^2 + y_{n+1}^2 = x_{n+1}(\cos(\omega)x_n + \sin(\omega)y_n) + y_{n+1}(-\sin(\omega)x_n + \cos(\omega)y_n) \\
= (\cos(\omega)x_n + \sin(\omega)y_n)^2 + (-\sin(\omega)x_n + \cos(\omega)y_n)^2 \\
= x_n^2 + y_n^2 \\
\iff \Delta(x_n^2 + y_n^2) = 0 \iff \Delta I_4(x_n, y_n, n) = 0, \quad (3.3)
\]

where \(I_4(x_n, y_n, n) = x_n^2 + y_n^2 = \text{constant}\). Obviously (3.3) is an exact equation, and \(I_4(x_n, y_n, n)\) is an invariant of (3.1)-(3.2). By multiplying (3.1) with \(\sin((n + 1)\omega)\), and
(3.2) with \( \cos((n + 1)\omega) \), and by adding the so-obtained equations we similarly obtain (using (3.1) and (3.2))

\[
\sin((n + 1)\omega)x_{n+1} + \cos((n + 1)\omega)y_{n+1} = \sin((n + 1)\omega)(\cos(\omega)x_n + \sin(\omega)y_n) + \cos((n + 1)\omega)(-\sin(\omega)x_n + \cos(\omega)y_n) = \sin(n\omega)x_n + \cos(n\omega)y_n \\
\]

\[
\Delta(\sin(n\omega)x_n + \cos(n\omega)y_n) = 0 \iff \Delta I_5(x_n, y_n) = 0, \quad (3.4)
\]

where \( I_5(x_n, y_n) = \sin(n\omega)x_n + \cos(n\omega)y_n \) = constant. Obviously (3.4) is an exact equation, and \( I_5(x_n, y_n) \) is an invariant of (3.1)-(3.2). Observe that

\[
\begin{pmatrix}
  x_{n+1} \\
  y_{n+1}
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
  \sin((n + 1)\omega) \\
  \cos((n + 1)\omega)
\end{pmatrix}
\]

are invariance vectors, that is, using these vectors we obtain exact equations, and so invariants for system (3.1)-(3.2).

The main question is now how can we make a system of \( k \) first order difference equation exact. Consider

\[
x_{n+1} = f(x_n, n) \quad (3.5)
\]

for \( n = 0, 1, 2, \ldots \), and where \( x_n = (x_{1,n}, x_{2,n}, \ldots, x_{k,n})^T \), \( f = (f_1, f_2, \ldots, f_k)^T \) in which the superscript indicates the transposed, and where \( f_i = f_i(x_n, n) = f_i(x_{1,n}, x_{2,n}, \ldots, x_{k,n}, n) \)

are sufficiently smooth functions (for \( i = 1, 2, \ldots, k \)). We also assume that an invariant for (3.5) can be represented by

\[
I(x_{n+1}, n + 1) = I(x_n, n) = \text{constant} \iff \Delta I(x_n, n) = 0. \quad (3.6)
\]

Keeping in mind example 3.1 we now try to find an invariant for (3.5). By multiplying each \( i \)-th equation in (3.5) with an invariance factor \( \mu_i(x_{n+1}, n + 1) = \mu_i(f(x_n, n), n + 1) \) for \( i = 1, 2, \ldots, k \), and by adding the so-obtained equations we obtain

\[
\mu_i(x_{n+1}, n + 1) \cdot x_{n+1} = \mu_i(f(x_n, n), n + 1) \cdot f(x_n, n), \quad (3.7)
\]

where \( \mu = (\mu_1, \mu_2, \ldots, \mu_k)^T \), and \( \mu_i = \mu_i(x_{n, n}) = \mu_i(x_{1,n}, x_{2,n}, \ldots, x_{k,n}, n) \) for \( i = 1, 2, \ldots, k \).

In fact, \( \mu \) can be considered an invariance vector. An exact equation (3.7) has now been obtained. The relationship between \( I \) and \( \mu \) follows from the equivalence of (3.6) and (3.7), yielding

\[
\begin{cases}
  I(x_{n+1}, n + 1) = \mu(x_{n+1}, n + 1) \cdot x_{n+1}, \\
  I(x_n, n) = \mu(f(x_n, n), n + 1) \cdot f(x_n, n).
\end{cases} \quad (3.8)
\]

Elementarily \( I \) can be eliminated from (3.8), and then it follows that all invariance vectors for the system of difference equations (3.5) have to satisfy
\[ \mu(x_n, n) \cdot x_n = \mu(f(x_n, n), n + 1) \cdot f(x_n, n) \ . \]  

(3.9)

When an invariance vector has been determined from the functional equation (3.9) an invariant for (3.5) easily follows from (3.8), yielding

\[ I(x_n, n) = \mu(x_n, n) \cdot x_n . \]  

(3.10)

Finding an invariance vector for a given system of \( k \) first order difference equations is a difficult and usually impossible task. On the other hand we can use invariance vectors of some special form, and so we can obtain invariants for special classes of systems of \( k \) first order difference equations.

4 Conclusions and future directions

In this paper the concept of invariance factors and invariance vectors to obtain invariants (or first integrals) for difference equations has been presented. This concept turns out to be more or less similar to the concept of integrating factors and integrating vectors for ordinary differential equations (see for instance [4, 8, 9]). Finding an invariance factor (or vector) for a given (system of) first order difference equation(s) is usually a difficult and mostly impossible task. However, we can use special types of invariance factors or vectors obtaining and so solving classes of difference equations. Another interesting application of the concept of invariance vectors is to use this concept to construct approximations of invariants for weakly perturbed difference equations. In [8, 9] the equivalent concept of integrating vectors for ordinary differential equations has been applied successfully to classes of weakly perturbed ordinary differential equations to construct approximations of first integrals on long time-scales.

References


