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Theory for 1D GPR data inversion for a dissipative layered medium

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Abstract—**We present a data driven method of full waveform inversion in one dimension. This means that the inversion is carried out as a sequence of processing steps. The first step is known as Marchenko redatuming. In this step we retrieve focusing functions from the measured data. In the second step we isolate the last event in the focusing function to obtain the local reflection coefficient of a particular reflecting boundary. This is done for the dissipative and equivalent effectual model. An effectual medium amplifies a propagating wave in the same way as a dissipative medium attenuates it. From these two models the reflection coefficient of the corresponding lossless medium can be computed. This is then inverted for the electric permittivity. Once the permittivity is found, the individual layer thicknesses are obtained from the travel times. The ratio of the reflection coefficient in the physical and effectual medium provides an estimate of the attenuation in each layer from which the conductivity in each layer can be found. We show that in this case the full waveform inversion is a linear problem. We need reflection and transmission data measured at two sides of the medium. We use an unconditionally convergent iterative technique to compute the focusing functions. The method only needs the up- and downgoing parts of the electric field at the receiver levels. A 1D numerical example with a lossy model shows that the proposed GPR inversion method is effective on modeled data.**

Keywords— Ground Penetrating Radar (GPR); Marchenko inversion; focusing function; separated wavefield; lossy model;

I. INTRODUCTION

Ground penetrating radar (GPR) is increasingly being used for subsurface and civil engineering applications [1][2]. Geophysical inversion is a technique that aims at providing quantitative information for subsurface or material characterization [3]. This is done by defining a cost function that expresses this difference in computed and measured data by a single number. On measured data, inversion is often difficult due to lack of source control. Source amplitude and source repeatability are necessary information for full waveform inversion techniques. Because the exploration targets are always complicated leading to complex geophysical profiles. Inversion becomes very important in all geophysical methods [4]. At present, most of the works on GPR inversion are based on model driven methods. There are some weaknesses of model driven inversion [5][6][7]. For instance, a good starting model is mandatory because the problem is ill-posed and suffers from non-uniqueness in the

model solution. In addition, at every iteration a new data set has to be computed by forward modelling and an estimate of the inverse Hessian has to be computed, which is time consuming. A crucial parameter in the forward modeling steps is the source and receiver antenna effects, the source time function or its amplitude and phase at each frequency, which are particularly difficult to obtain or model when antennas are deployed at or close to the surface. Finally, once the data misfit is reduced to user specification there is no guarantee that the correct model is approximately found.

It has been shown that the focusing function can be computed from the reflection response by the Marchenko method [8]. It can be used to eliminate the effect of internal multiples in a seismic or GPR image [9][10][11]. A GPR inversion method for lossless medium based on Marchenko method has been developed [12]. This is a data-driven method, but it can deal only with lossless media and still needs to know the source signature. A 3D Marchenko redatuming method for dissipative acoustic media has also been derived [13].To circumvent the need to know the source time function, a new implementation was derived for marine seismic data [10].

Here we show how a Marchenko-type scheme can be used for a dissipative GPR model. We assume the up- and downgoing components of the electric field can be obtained from the GPR data. Then we define the actual dissipative medium and the equivalent effectual medium. The effectual medium is characterized by negative conductivity and is a non-physical medium. Its response can be computed from reflection and transmission responses at two side of the physical dissipative medium. We use an unconditionally convergent iterative technique (e.g., LSQR) to solve the resulting Marchenko equations. Finally, we formulate full waveform inversion using the focusing function and show that it is a stepwise linear problem. We end with a numerical example showing the inversion results with accompanying errors of the retrieved parameters.

II. REFLECTION RESPONSE OF AN EFFECTUAL MEDIUM AND GENERALIZED LOCAL REFLECTION COEFFICIENT

Consider a *N*-layered model with isotropic homogeneous layers. Free space magnetic permeability, μ_0 , is used in each layer. We define two media, one dissipative and the other

effectual with negative dissipation. The wavefields in the dissipative medium are denoted with an over hat ($\sim$), e.g. $\tilde{E}(z;\omega)$ denotes the electric field in the dissipative medium. Each layer has a constant permittivity ε [F/m] and conductivity σ [S/m]. The wavefields in the corresponding effectual medium are denoted with an overbar (-). We define a generalized local reflection coefficient:

$$\breve{r}_n = r_n \prod_{k=1}^{n} e^{-2\alpha_k d_k} \qquad (1)$$

Where α_k denotes the attenuation of the kth layer, d_k denotes the thickness of the kth layer, r_n denotes the local reflection coefficient of the nth boundary in the lossless medium. For a lossless medium $\alpha_k = 0$. For a dissipative medium $\alpha_k > 0$. For effectual medium $\alpha_k < 0$.

We use these coefficients to invert for the subsurface parameters: layer thickness, permittivity and conductivity. Reflection and transmission responses measured at the actual dissipative medium can be used to compute the reflection response of the corresponding effectual medium [13].

III. ELECTROMAGNETIC MARCHENKO EQUATIONS FOR A DISSIPATIVE MEDIUM

To derive one-way convolution type Marhchenko equation, we define two non-identical electromagnetic wave fields ("state A" and "state B", see Fig.1). The investigation area in state A is identical to the actual dissipative medium, and outside of this area it is reflection-free. For state B, we choose the actual dissipative medium.
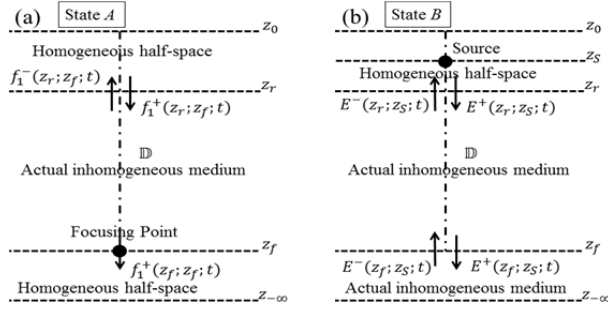

Fig. 1 (a) The configurations for the fields in the medium with the focusing functions of state A; (b) The configurations for the fields in the medium with the reflection data of state B

In our 1D example (see Fig. 1), the convolution type reciprocity relations of one-way electromagnetic wave is given by [14]

$$\left[\hat{E}^{+B}\hat{H}^{-A} + \hat{E}^{-B}\hat{H}^{+A} \right]\big|_{z_r} = -\left[\hat{E}^{+A}\hat{H}^{-B} + \hat{E}^{-A}\hat{H}^{+B} \right]\big|_{z_f} \qquad (2)$$

Where $\hat{E}^-$ and $\hat{E}^+$ are the up- and downgoing components of the electric wavefield in the frequency domain respectively; $\hat{H}^-$ and $\hat{H}^+$ are the up- and downgoing components of magnetic wavefield in the frequency domain respectively.

In a similar way, the correlation type reciprocity relationship of one-way electromagnetic wave is given by

$$\left[\left(\hat{H}^{+A}\right)^* \hat{E}^{+B} + \left(\hat{H}^{-A}\right)^* \hat{E}^{-B} \right]\big|_{z_r} = \left[\left(\hat{E}^{+A}\right)^* \hat{H}^{+B} + \left(\hat{E}^{-A}\right)^* \hat{H}^{-B} \right]\big|_{z_f} \qquad (3)$$

Equation (3) requires one media is dissipative and the other one is effectual. If we define magnetic field of state A (See Fig. 1a) as a focusing wavefield $f_1(z;\omega)$, then in the dissipative medium, we can express the upgoing part of the magnetic field for a receiver at z_f and a source at z_S in terms of the up- and downgoing parts of the measured electric field and the focusing function at the receiver level z_r and with its focus level at z_f. All fields are in the dissipative medium. On the other hand, the downgoing part of the magnetic field in the dissipative medium for a receiver at z_f and a source at z_s can be expressed in terms of the up- and downgoing parts of the computed electric field in the effectual medium and the focusing function in the dissipative medium at the receiver level z_r and with its focus level at z_f. We can interchange the dissipative and effectual media to obtain similar expression for the up- and downgoing components of the magnetic field in the effectual medium[13]. They are given by

$$\tilde{H}^-(z_f;z_S;\omega) = -2\left[\tilde{E}^+(z_r;z_S;\omega)\tilde{f}_1^-(z_r;z_f;\omega) + \tilde{E}^-(z_r;z_S;\omega)\tilde{f}_1^+(z_r;z_f;\omega) \right] \qquad (4)$$

$$\bar{H}^+(z_f;z_S;\omega) = 2\left[\left(\tilde{f}_1^+(z_r;z_f;\omega)\right)^* \bar{E}^+(z_r;z_S;\omega) + \left(\tilde{f}_1^-(z_r;z_f;\omega)\right)^* \bar{E}^-(z_r;z_S;\omega) \right] \qquad (5)$$

$$\bar{H}^-(z_f;z_S;\omega) = -2\left[\bar{E}^+(z_r;z_S;\omega)\bar{f}_1^-(z_r;z_f;\omega) + \bar{E}^-(z_r;z_S;\omega)\bar{f}_1^+(z_r;z_f;\omega) \right] \qquad (6)$$

$$\tilde{H}^+(z_f;z_S;\omega) = 2\left[\left(\bar{f}_1^+(z_r;z_f;\omega)\right)^* \tilde{E}^+(z_r;z_S;\omega) + \left(\bar{f}_1^-(z_r;z_f;\omega)\right)^* \tilde{E}^-(z_r;z_S;\omega) \right] \qquad (7)$$

Equations (4) and (5) form one set of equations that can be solved in the time domain for the up- and downgoing parts of the focusing wave field in the dissipative medium if the up- and downgoing components of the electric field at surface are known in the dissipative and in the effectual medium. Equations (6) and (7) form a similar set that can be solved for the focusing wavefield in the effectual medium using the same data. Here, we present how to solve (4)-(5). Transforming (4)-(5) to time domain gives

$$\tilde{H}^-(z_f;z_S;t) = -2\left[\int_{t'=-\infty}^{t} \tilde{E}^+(z_r;z_S;t-t')\tilde{f}_1^-(z_r;z_f;t')dt' \right.$$
$$\left. + \int_{t'=-\infty}^{t} \tilde{E}^-(z_r;z_S;t-t')\tilde{f}_1^+(z_r;z_f;t')dt' \right] \qquad (8)$$

$$\bar{H}^+(z_f;z_S;t) = 2\left[\int_{t'=-\infty}^{t} \tilde{f}_1^+(z_r;z_f;-t')\bar{E}^+(z_r;z_S;t-t')dt' \right.$$
$$\left. + \int_{t'=-\infty}^{t} \tilde{f}_1^-(z_r;z_f;-t')\bar{E}^-(z_r;z_S;t-t')dt' \right] \qquad (9)$$

In state B, we let the source be exited at $t=0$ and the first arrival is recorded in $z=z_f$ at $t = t_d(z_f;z_S)$. where $t_d(z_f;z_S)$ is the direct arrival time from z_S to z_f. Therefore, before $t_d(z_f;z_S)$, the magnetic wavefield is zero, i.e. both left-hand sides of (8)-(9) are zero. In state A, we let the focusing point be located in $z=z_f$ and focus the wavefield at $t=0$, so it is a non-causal wavefield. Moreover, in state A, below z_f there is a non-reflecting half-space, and the focusing wavefields do not exist for $|t| > t_d(z_f;z_r)$. So equations (8)-(9) can be simplified as

$$\int_{t'=t_d}^{t_d} \left[\tilde{E}^+(z_r;z_S;t-t')\tilde{f}_1^-(z_r;z_f;t') \right]dt' = -\int_{-t_d}^{t_d} \left[\tilde{E}^-(z_r;z_S;t-t')\tilde{f}_1^+(z_r;z_f;t') \right]dt' \qquad (10)$$

$$\int_{t'=t_d}^{t} \left[\bar{E}_x^+(z_r;z_S;t-t')\tilde{f}_1^+(z_r;z_f;-t') \right]dt' = -\int_{t'=-t_d}^{t} \left[\bar{E}_x^-(z_r;z_S;t-t')\tilde{f}_1^-(z_r;z_f;-t') \right]dt' \qquad (11)$$

At the instant $t = \mp t_d(z_f;z_r)$, $\tilde{f}_1^-(z_r;z_f;t) = 0$ but $\tilde{f}_1^+(z_r;z_f;t) \neq 0$. To focus the field at depth, we can write the downgoing part of the focusing wavefield as

$$\tilde{f}_1^+(z_r; z_f; t) = \tilde{f}_{1d}^+(z_r; z_f; t) + \tilde{f}_{1m}^+(z_r; z_f; t) \tag{12}$$

Where $\tilde{f}_{1d}^+(z_r; z_f; t)$ denotes the first arrival of the inverse of the transmission response between z_r and z_f; $\tilde{f}_{1m}^+(z_r; z_f; t)$ represents the coda following the first arrival.

Substituting (12) in (10)-(11), we obtain the Marchenko equations:

$$\wp_{11} f_{1m}^+ + \wp_{12} f_1^- = d_1 \tag{13}$$

$$\wp_{21} f_{1m}^+ + \wp_{22} f_1^- = d_2 \tag{14}$$

Where

$$d_1 = -\int_{t'=-t_d}^{t} \left[\tilde{E}^-(z_r; z_S; t-t') \tilde{f}_{1d}^+(z_r; z_f; t') \right] dt' \tag{15}$$

$$\wp_{11} f_{1m}^+ = \int_{t'=-t_d}^{t} \left[\tilde{E}_x^-(z_r; z_S; t-t') \tilde{f}_{1m}^+(z_r; z_f; t') \right] dt' \tag{16}$$

$$\wp_{12} f_1^- = \int_{t'=-t_d}^{t} \left[\tilde{E}^+(z_r; z_S; t-t') \tilde{f}_1^-(z_r; z_f; t') \right] dt' \tag{17}$$

$$d_2 = -\int_{t'=-t}^{t_d} \left[\overline{E}_x^+(z_r; z_S; t+t') \tilde{f}_{1d}^+(z_r; z_f; t') \right] dt' \tag{18}$$

$$\wp_{21} f_{1m}^+ = \int_{t'=-t}^{t_d} \left[\overline{E}_x^+(z_r; z_S; t+t') \tilde{f}_{1m}^+(z_r; z_f; t') \right] dt' \tag{19}$$

$$\wp_{22} f_1^- = \int_{t'=-t}^{t_d} \left[\overline{E}_x^-(z_r; z_S; t+t') \tilde{f}_1^-(z_r; z_f; t') \right] dt' \tag{20}$$

When the source signature is not a single delta-like event in the measured data, equations (13)-(14) cannot be solved by the Neumann-type iterative scheme. However, these equations can be solved by direct matrix inversion, or unconditionally convergent iterative methods such as LSQR. Iterative methods are preferred because we do not need to compute the system matrix but only its action on the fields. Analysis has shown that LSQR method is more accurate and stable than other methods [15][16]. We choose LSQR to solve (13)-(14).

IV. DATA-DRIVEN INVERSION

The advantage of the present formulation is that the measured data does not need to be deconvolved by the source wavelet. Once (13)-(14) are solved, we have obtained $\tilde{f}_1^-(z_r; z_f; t)$. We assume $\tilde{f}_{1d}^+(z_r; z_f; t)$ is a delta-function in time, we observe that the last event of $\tilde{f}_1^-(z_r; z_f; t)$ is the local reflection coefficient $\tilde{r}_f$ in the dissipative medium. In a similar way, the local reflection coefficient $\overline{r}_f$ in the effectual medium is obtained as the last event in $\overline{f}_1^-(z_u; z_f; t)$. From (1), the local reflection coefficient r_n in the lossless medium can be approximated by [13]

$$r_n = sign(\overline{r}_n)\sqrt{\overline{r}_n \bullet \tilde{r}_n} \tag{21}$$

Once the local reflection coefficients in the equivalent lossless medium are obtained, we start in the air with $n = 0$ where the receivers are present and work our way down into the layered medium. Starting by assuming we know the medium parameters in air, $\varepsilon_r = 1$, $\sigma_0 = 0$, we find the solution recursively for the layer below an interface. Once we have the electric permittivity, we know the propagation velocities inside each layer and can transform the one-way vertical travel time to depth. Using the electromagnetic parameters and the depth information, we can find the solution for the conductivity of each layer.

We start at the first reflector and obtain an estimate for the electric permittivity below the reflector from the general expression, which can be made explicit as

$$\left(\frac{1-r_i}{1+r_i}\right)^2 \varepsilon_{r,i} = \varepsilon_{r,i+1} \tag{22}$$

Where $\varepsilon_{r,i}$ is the relative electric permittivity of the ith layer. Once we obtain the electric permittivity ε_{i+1} of the $(i+1)^{th}$ layer, estimate the thickness of the $(i+1)^{th}$ layer as

$$d_{i+1} = \frac{(t_{i+1} - t_i)\sqrt{\varepsilon_{i+1}\mu_0}}{2} \tag{23}$$

Where $\mu_0 \approx 4\pi \times 10^{-7} [H/m]$. t_i we can pick from the imaging times that correspond to the time of the maximum amplitude. To invert for the electric conductivity σ, we take

$$P_n = \ln(\frac{\overline{r}_n}{\tilde{r}_n}) \tag{24}$$

The attenuation coefficient α of the $(n+1)^{th}$ layer can be solved by:

$$\alpha_{n+1} = \frac{P_{n+1} - P_n}{4d_{n+1}} \tag{25}$$

As we know, the attenuation coefficient α of the $(n+1)^{th}$ layer is given by

$$\alpha_{n+1} = \omega \sqrt{\frac{\mu_{n+1}\varepsilon_{n+1}}{2}\left[\sqrt{1+\left(\frac{\sigma_{n+1}}{\omega\varepsilon_{n+1}}\right)^2} - 1\right]} \tag{26}$$

Combining (26) with (25), the conductivity is estimated as

$$\sigma_{n+1} = \frac{P_{n+1} - P_n}{2d_{n+1}} \bullet \sqrt{\frac{\varepsilon_{n+1}}{\mu_0}} \tag{27}$$

To quantify the accuracy of the retrieved parameters in terms of data prediction, we use the inversion parameters to compute the reflection data e_i and compare that data with the reflection data E_i to estimate the errors. The global normalized root mean square error is given by

$$\rho = \sqrt{\frac{\sum_{i=1}^{nt}(e_i - E_i)^2}{\sum_{i=1}^{nt} E_i^2}} \tag{28}$$

where nt denotes the number of data points in the data. This error will be used to evaluate inversion quality.

V. NUMERICAL RESULTS

To illustrate the modified Marchenko inversion method in a dissipative medium, we give a numerical example. The transmitter emits a Ricker wavelet with 250 MHz center frequency. The model consists of a layered medium with 7 reflecting interfaces below the source and receiver. The layered model is given in Table I. The source and receiver depths are $z_s = -0.84$ m and $z_r = 0$ m. In the dissipative medium, the up- and downgoing components of the data are shown in Figure 2a). In the effectual medium, they are shown in Figure 2b). This data are used to find the upgoing

focusing function for all possible times available in the data from (13)-(14). In a similar way, we find the upgoing focusing function in the effectual medium.

TABLE I. MODEL PARAMETERS.

d (m)	0.84	0.82	0.30	0.33	0.37	0.31	0.32	∞
ε_r	1.0	4.3	10.1	15.9	11.4	13.8	12.2	13.5
σ (mS/m)	0	3.9	11.1	20.7	12.9	8.7	9.9	7.5

Finally, ε_r , σ and the thickness of the layers are computed from (22)-(27) and the results are shown in Table 2. We compare the retrieved parameters with the model values. The errors shown in the table are computed according to following generalized formula *Err.* = *$100*(|U_{mod} - U_{inv}|)/U_{mod}$*, where U can be ε_r , σ or d.
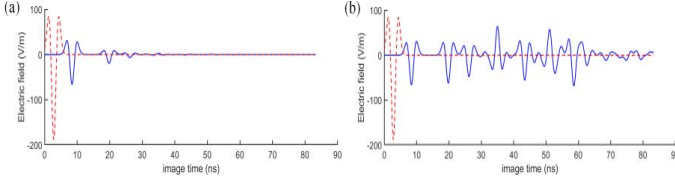

Fig. 2 The up- (solid blue line) and downgoing (dash red line) component of reflection data: a) The electric field data in the dissipative medium; b) The electric field data in the effectual medium

TABLE II VALUES FOR INVERTED MODEL PARAMETERS AND THEIR ERRORS

layer	ε_r	Err.(%)	σ(mS/m)	Err.(%)	d (m)	Err.(%)
1	1.00	-	0.00	-	0.84	-
2	4.31	+ 0.34	3.91	+ 0.22	0.82	-0.17
3	10.13	+ 0.29	11.47	+ 3.37	0.30	-0.15
4	15.89	-0.07	20.39	-1.48	0.33	+0.03
5	11.42	+ 0.22	12.13	-5.98	0.37	-0.11
6	13.79	-0.07	8.67	-0.38	0.31	+0.03
7	12.23	+ 0.27	10.15	+ 2.56	0.32	-0.14
8	13.50	+0.025	-	-	-	-

From Table II, it can be seen that the largest error occurs in the permittivity of layer 2 where ε is overestimated by 0.34%. The largest error occurs in the conductivity σ of layer 5 where σ is underestimated by 5.98%. The largest error occurs in the thickness estimate of layer 2 is underestimated by -0.17%. In practice, we do not know the true parameter values and use the global normalized root mean square error, ρ, between the data modelled with the medium parameters obtained by inversion and the original reflection data. The global normalized root mean square error for this example is 0.36%.

VI.   CONCLUSION

We have presented a Marchenko inversion scheme for 1D GPR data. This constitutes a full waveform inversion method that obtains all model parameters from the data. It does not require any model information. We have reformulated Marchenko redatuming such that we need only to separate the up- and downgoing parts from measurements.

A disadvantage for field applications of this method is its requirement to have two-sided access to the medium to measure the reflection response and transmission responses. This is necessary to compute the reflection response in the effectual medium. How to implement Marchenko inversion for lossy medium which do not need to put the source inside the medium remains to be investigated. The second challenge

is to generalize the current method to 2D and 3D models. Finally, the behaviour with noisy data remains to be investigated.

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