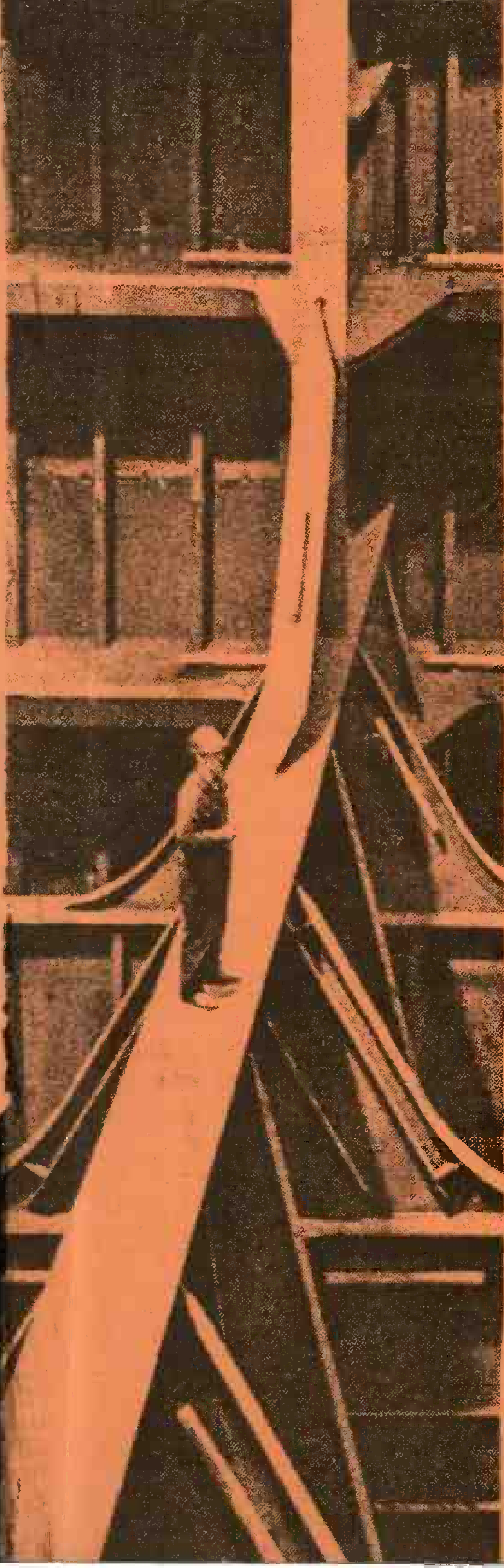

TOR VINJE

ON THE STATISTICAL DISTRIBUTION OF MAXIMA OF SLIGHTLY NON-LINEAR STOCHASTIC VARIABLES.

Report SK/M 27

TRONDHEIM
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MEDDELELSE SK/M 27

ON THE STATISTICAL DISTRIBUTION OF MAXIMA OF SLIGHTLY NON-LINEAR STOCHASTIC VARIABLES.

by

Tor Vinje.

June,1974

Trondheim

PREFACE.

I hope that this work not only will be a guide into the problem of calculation of the distribution of maxima of slightly non-linear variables, but also an introduction to the general theory of distribution of maxima. To make is so, I found that some **general statistical** theory had to be included (not the most elementary).

It can be discussed weather this theory should have been put into an appendix or not. In this work it is put in the beginning, as an introduction to the field.

I want to point out that in this report the following types have been used for vectors or matrices:

$\boldsymbol{\Lambda}, \boldsymbol{\lambda}, \mathbf{A}, \mathbf{v}$ and so on.

Einstein's summation convention is also used.

I want to thank Mrs. Ingrid Hansen for her help in typing the manuscript and correcting my mathematical calculations

Trondheim, June 27. 1974

Tor Vinje

CONTENTS

1. INTRODUCTION

Most problems in structural design are, what we call, linear. That means that the actual responses on a given input is directly proportional to that input. On the other hand one is not seldom dealing with design criteria where the different linear responses are combined to a non-linear variable. As an introductional example the design criterium for buckling of rectangular plates with in plane stresses will be discussed.

According to Bleich /1/ the buckling of this rectangular plate takes place when the in plane stresses, shown in Fig. (1.1), are combined in the following way:

$$\left(\frac{\sigma_B}{\sigma_{BC}^0}\right)^2 + \left(\frac{\tau}{\tau_C^0}\right)^2 = Z \geq 1 \qquad (1.1)$$

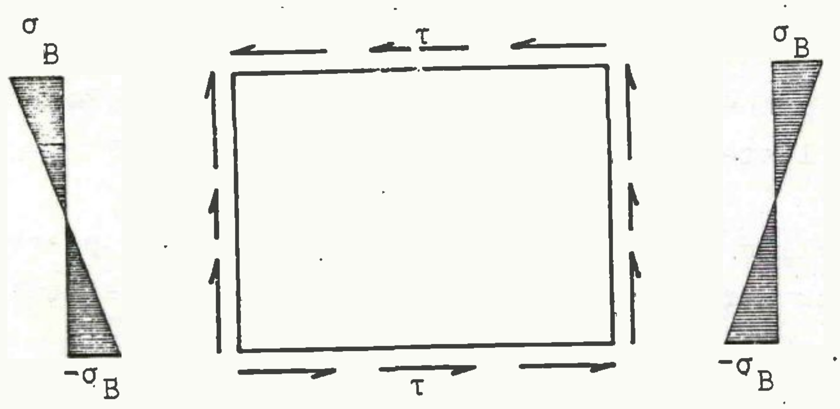


Fig. (1.1) A thin rectangular plate acted upon by in plane stresses.

The computed value of Z has, in the purely deterministic case, to be calculated and compared to 1/n, where n is the safety factor. This procedure works well out for purely static or deterministic variables.

When the variables (σ_B and τ) are given as stochastic processes one has to do the whole calculation in a more complicated way:

In this case Z is a stochastic process too, and one has to examine the individual maxima of Z to find the distribution of these maxima. From this distribution one has to calculate the distribution of the largest maximum of Z (within a predicted time). From this distributional function the probability that Z will exceed 1 (or/and exceed 1/n) is calculated. This probability is in turn compared to some numbers, which is said to be satisfactorial.

The main problem of this design procedure is that the distribution of the individual maxima of non-linear variables of the type shown in Eq. (1.2) is not known, even not when σ_B and τ are Gaussian distributed variables. The first step on the way to get this problem solved is to calculate this distribution, which will be done in this report.

Before leaving these introductorial notes it will be mentioned that variables of the quadratic form will be found in connection with other problems too. Some of them are listed below:

Buckling of thin plates acted upon by in plane shear stresses and constant compressive stresses, σ, (from Bleich /1/):

$$Z = \left(\frac{\tau}{\tau_C^0}\right)^2 + \frac{\sigma}{\sigma^0} \tag{1.3}$$

Combination of stresses according to some plasticity-criterion (von Mises'):

$$Z = \sigma_x{}^2 + 3\tau_{xy}^2 \tag{1.4}$$

Combination of axial forces, N, and bending moments, M, when calculating the condition that plastic hinges will occure in beams:

$$Z = \frac{M}{M_c^0} + \left(\frac{N}{N_c^0}\right)^2 \tag{1.5}$$

or the same combining shear forces, Q, and bending moments, M :

$$Z \simeq \frac{M}{M_c^0} + 0{,}44 \left(\frac{Q}{Q_c^0}\right)^2 \quad (Q/Q_c \leq 0{,}79) \tag{1.6}$$

according to Horne /2/.

One can easily verify that all the variables mentioned will consist of one purely static part, one purely dynamic and one part which is some combination of static and dynamic terms in the following way:

$$Z = \sum_{ij} a_s x_i^s x_j^s + \sum_{ij} b_{sd} x_i^s x_j^d + \sum_{ij} c_d x_i^d x_j^d \tag{1.7}$$

where x_i^s is due to purely static loading and x_i^d is due to dynamic (or stochastic) loading. The static part is in the following assumed to be deterministic, so that Eq. (1.7) can be rewritten:

$$Z = Z_s + \alpha_i X_i + a_{ij} X_i X_j \tag{1.8}$$

where X_i is purely stochastic (with zero mean)

Eq. (1.8) will be discussed in detail in the following and the discussion will separate in four main parts:

1. General theory of stochastic variables.
2. The general theory of the distribution of maxima of stochastic processes.
3. Developments of the distribution of maxima of non-linear variables.
4. Discussion of numerical results.

Most of the mathematical calculations are put into appendices, together with some general theory of some special functions.

2. CHARACTERISTIC FUNCTIONS

The characteristic function , $\Phi(\theta)$, of the variable X (t) is defined as follows:

$$\Phi(\theta) = E\,(e^{i\theta X}) = \int_{-\infty}^{+\infty} e^{i\theta x} f(x)dx \qquad (2.1)$$

where f(x) is the probability density function of X. $\Phi(\theta)$ is here recognized as the Fourier transform of f(x). Because

$$\left\{\int_{-\infty}^{+\infty} |f(x)|\,dx \doteq 1 < \infty\right\} \;\&\; \left\{f(x) \geq 0\right\} \qquad (2.2)$$

the integral in Eq.(2.1) <u>always</u> will exist and hence $\Phi(\theta)$ be defined.
The inversion of Eq.(2.1) leads to:

$$\hat{f}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi(\theta)e^{-i\theta x}\,d\theta \qquad (2.3)$$

which converge to f(x) when f(x) is continous, else to 1/2 (f(x+) + f (x-)) when f(x) is discontinous. Assuming in the following that f(x) is continous for any value of x, one can write:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi(\theta)e^{-i\theta x}\,d\theta \qquad (2.4)$$

Making use of Eq.(2.1) it is easily shown that

$$E(X^m) = \frac{1}{i^m} \frac{d^m}{d\theta^m} E(e^{i\theta x}) \Big|_{\theta=o} = \frac{1}{i^m} \frac{d^m \Phi}{d\theta^m} \Big|_{\theta=o} \qquad (2.5)$$

whenever $E(X^m)$ does exist. Assuming that $E(X^m)$ exist for any m, $\Phi(\theta)$ can be expanded in a Maclaurin series as follows:

$$\Phi(\theta) = 1 + \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} \mu_n \qquad (2.6)$$

where for brevety the following is introduced:

$$\mu_n = E(X^n) \qquad (2.7)$$

From $\Phi(\theta)$ two new functions can be defined:

a) The cumulant generating function:

$$\psi(\theta) = \ln\Phi(\theta) \qquad (2.8)$$

and

b) the moment generating function:

$$M(s) = E(e^{-sX}) = \Phi(is) \qquad (2.9)$$

The cumulant generating function $\psi(\theta)$ can be expanded in a Maclaurin series in the following way:

$$\Psi(\theta) = \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} \kappa_n \qquad (2.1o)$$

where κ_n is called the n-th cumulant of X. Combining Eq. (2.6), Eq. (2.8) and Eq.(2.1o) the following identity is found:

$$\ln\left(1+\sum_{n=1}^{\infty}\frac{(i\theta)^n}{n!}\right) = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{m}\left[\sum_{n=1}^{\infty}\frac{(i\theta)^n}{n!}\mu_n\right]^m$$

$$= \sum_{m=1}^{\infty}\frac{(i\theta)^m}{m!}\kappa_m \qquad (2.11)$$

From Eq.(2.11) κ_m can be found as a function of μ_n , $n \leq m$. The following 4 are easily developed:

$$\kappa_1 = \mu_1$$

$$\kappa_2 = \mu_2 - \mu_1^2$$

$$\kappa_3 = \mu_3 - 3\,\mu_2\mu_1 + 2\mu_1^3$$

$$\kappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\,\mu_2\mu_1^2 - 6\mu_1^4 \qquad (2.12)$$

According to Eq. (2.9), M(s) is defined as the two-sided Laplace-transform of f(x):

$$M(s) = \int_{-\infty}^{+\infty} e^{-sx} f(x)\,dx \qquad (2.13)$$

where μ_n (see Eq.(2.7)) can be found as:

$$\mu_n = (-1)^n \frac{d^n}{ds^n} M(s)\Big|_{s=0} \qquad (2.14)$$

The characteristic function for several jointly distributed random variables is similarly defined as:

$$\Phi(\theta_1,\ldots,\theta_n) = E\left(\exp(i\sum_{m=1}^{n}\theta_m X_m)\right)$$

$$= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} f(x_1,\ldots,x_n)\exp\left(i\sum_{m=1}^{n}\theta_m X_m\right)dx_1\ldots dx_n \quad (2.15)$$

The Maclaurin series expansion of Φ is:

$$\Phi(\theta_1,\ldots,\Theta_n) = \sum_{\substack{m_1=0\\ \vdots \\ m_n=0}}^{\infty} (i\theta_1)^{m_1}\ldots(i\theta_n)^{m_n}\frac{\mu_{m_1,\ldots,m_n}}{m_1!\ \ldots\ m_n!} \quad (2.16)$$

where

$$\mu_{m_1,\ldots,m_n} = E(X_1^{m_1}\cdots\cdot X_n^{m_n}) \quad (2.17)$$

When the variables, X_i, are indepentent, it is easily shown that:

$$\Phi(\theta_1,\ldots,\theta_n) = \Phi_1(\theta_1)\ldots\Phi_n(\theta_n) \quad (2.18)$$

where

$$\Phi_m(\theta_m) = E\left(e^{i\theta_m X_m}\right) \quad (2.19)$$

As for the simple case with one variable, the cumulant generating function can be defined:

$$\psi(\theta_1,\cdots,\theta_n) = \ln \Phi(\theta_1,\cdots,\theta_n)$$

$$= \sum_{\substack{m_1=1\\ \vdots \\ m_n=1}}^{\infty} (i\theta_1)^{m_1}\ldots(i\theta_n)^{m_n}\frac{\kappa_{m_1,\cdots m_n}}{m_1!\ \cdots\ m_n!} \quad (2.20)$$

and if X_i are independent:

$$\Psi(\theta_1,\cdots,\theta_n) = \sum_{m=1}^{n} \psi_m(\theta_m) \tag{2.21}$$

where:

$$\psi_m(\theta_m) = \ln \Phi_m(\theta_m) \tag{2.22}$$

The probability density function of a new set of variables, Y_r, can be found in the following way. Assume that the new variables are defined as follows:

$$Y_r = f_r\,(X_1,\cdots,X_m) \equiv f_r\,(X_i) \tag{2.23}$$

$$r = 1,\cdots,n$$
$$n \leq m$$

The characteristic function of Y_r is given by:

$$\Phi\,(\theta_1,\cdots,\theta_n) = E(\exp(i\sum_{j=1}^{n} \theta_j Y_j))$$

$$= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} f(x_1,\cdots,x_m)\exp(i\sum_{j=1}^{n} \theta_j f_j)dx_1\cdots dx_m \tag{2.24}$$

Inverting Eq.(2.24)the probability density function of Y, $f_y\,(y_1,\cdots,y_n)$ is found:

$$f_y(y_1,\cdots,y_n) = \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \Phi(\theta_1,\cdots,\theta_n)\cdot$$

$$\cdot\exp(-i\sum_{j=1}^{n} \theta_j y_j)\,d\theta_1\cdots d\theta_n \tag{2.25}$$

Take as an example:

$$Y_1 = aX + bX^2 \tag{2.26}$$

where X is Gaussian distributed with zero mean and variance equal to unity , i.e.:

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}x^2) \tag{2.27}$$

(see chapter 3)

According to Eq.(2.24) $\Phi(\theta)$ then becomes:

$$\Phi(\theta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(-\tfrac{1}{2}x^2 + i\theta(ax+bx^2))\, dx \tag{2.28}$$

According to Abramowitz & Stegun /3/

$$\int_{-\infty}^{+\infty} \exp(-\alpha x^2 - 2\beta x)dx = \sqrt{\frac{\pi}{\alpha}} \exp(+\beta^2/\alpha) \tag{2.29}$$

when the real part of α is positive.

Following:

$$\Phi(\theta) = \frac{1}{\sqrt{1 - 2ib\theta}} \cdot \exp\left(-\tfrac{1}{2} \cdot \frac{a^2\theta^2}{1-2ib\theta}\right) \tag{2.30}$$

When $b = o$ $\Phi(\theta)$ takes the value:

$$\Phi_1(\theta) = \exp(\tfrac{1}{2}a^2\theta^2) \tag{2.31}$$

which coinsides with the result for Gaussian distributed variables with variance equal to a^2 (see chapter 3) as should be expected.

When a = o $\Phi(\theta)$ takes the value:

$$\Phi_2(\theta) = \frac{1}{\sqrt{1-2ib\theta}} \tag{2.32}$$

which shows that Y in this case will be Γ-distributed (see Appendix 2). When, in addition, b = 1 Y will be χ^2 -distributed, according to Eq.(2.32) (See Appendix 2)

From the common theory of integral-transforms, it is well known that the behaviour of f(y) as $y \to \infty$ will be given from the behaviour of $\Phi(\theta)$ as $\theta \to o$ (See for instance Doetsch /4/).
The following two expansions of $\Phi(\theta)$ can be performed:

$$\Phi(\theta) \underset{\theta\to o}{\to} \exp(-\tfrac{1}{2}a^2\theta^2) + o(i\theta) \tag{2.33}$$

$$\Phi(\theta) \underset{\theta\to o}{\to} \frac{1}{\sqrt{1-2ib\theta}} + o(\theta^2) \tag{2.34}$$

which shows that the Γ-distribution gives a better fit to $\Phi(\theta)$ than the Gaussion distribution does when $y \to \infty$, which should had been expected from the fact that necessarily $|bx^2| >> |ax|$ when $y \to \infty$.

The exact value of f(y) can be found from Eq.(2.30) by help of Eq. (2.25) and the known integral (see Abramowitz & Stegun /3/):

$$\int_{-\infty}^{+\infty} \exp(-at^2-b/t^2)dt = \sqrt{\frac{\pi}{a}} \exp(-2\sqrt{ab}) \tag{2.35}$$

when the real part of a and b both are positive.

A much simpler way of calculating f(y) in this case is the direct method, by which F(y) is calculated:

$$F(y) = P(Y<y) = P(X>\alpha_1 \cap X<\alpha_2) \tag{2.36}$$

Where α_1 is the smallest and α_2 the largest root of the equation:

$$ax + bx^2 = y \quad , \quad y > -\frac{a^2}{4b} \tag{2.37}$$

Following:

$$F(y) = \frac{1}{\sqrt{2\pi}} \int_{\alpha_1}^{\alpha_2} \exp\ (-\tfrac{1}{2}x^2)dx \tag{2.38}$$

and

$$f(y) = \frac{dF(y)}{dy} = \frac{1}{\sqrt{2\pi}} \left\{ \exp(-\tfrac{1}{2}\ \alpha_2^2)\cdot\frac{d\alpha_2}{dy} \right.$$

$$\left. -\ \exp\ (-\tfrac{1}{2}\ \alpha_1^2)\cdot\frac{d\alpha_1}{dy} \right\} \tag{2.39}$$

according to the common rules for differentiation of integrals (see f.i. Hildebrand /5/)

The solutions of Eq.(2.37) are:

$$\alpha_1 = \frac{a}{2b} \left(-1- \sqrt{1 + \frac{4by}{a^2}} \right)$$

$$\alpha_2 = \frac{a}{2b} \left(-1+ \sqrt{1 + \frac{4by}{a^2}} \right) \qquad (2.40)$$

when (a/b > o) & (b ≠ o)

and according to Eq. (2.39):

$$f(y) = \frac{2}{a\sqrt{2\pi} \cdot \sqrt{1+\frac{4by}{a^2}}} \exp \left(\frac{a^2}{2b^2} + \frac{y}{b} \right) \cdot$$

$$\cdot \cosh\left(\frac{a^2}{2b^2} \sqrt{1 + \frac{4by}{a^2}} \right) \qquad (2.41)$$

Putting $a \to o$ $f(y)$ becomes:

$$f(y) \to \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{by}} \cdot \exp \left(y/b \right) \qquad (2.42)$$

which coinsides with Eq. (2.32)

3. GAUSSIAN DISTRIBUTED VARIABLES

The Gaussian distribution can be defined as the distribution, for which all cumulants, exept for the two **lowest**, vanish identically. In this case the characteristic function can be written:

$$\Phi(\theta) = \exp\left(i\theta\kappa_1 - \frac{\theta^2\kappa_2}{2} \right) \qquad (3.1)$$

From $\Phi(\theta)$ the probability density function, $f(x)$, of the variable X is calculated:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left(i\theta(\kappa_1 - x) - \frac{\theta^2\kappa_2}{2}\right) d\theta \qquad (3.2)$$

According to Abramowitz & Stegun /3/ (p 3o2, Eq.(7.4.6))

$$\int_{-\infty}^{+\infty} \exp(-t^2 + 2ixt)dt = \sqrt{\pi}\,\exp(-x^2) \qquad (3.3)$$

and following:

$$f(x) = \frac{1}{\sqrt{2\pi\kappa_2}} \exp\left(-\frac{1}{2}\frac{(x-\kappa_1)^2}{\kappa_2} \right) \qquad (3.4)$$

Some important properties of Gaussian distributed variables are going to be demonstrated. At first: If X is Gaussian, then any linear transformation of X:

$$Y = a + bX$$

is also Gaussian distributed. This is easily shown, by calculating the characteristic function of Y, by means of Eq. (2.24) and Eq. (3.3)

$$\Phi(\theta) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\kappa_2}} \exp\left(-\tfrac{1}{2}\frac{(x-\kappa_1)^2}{\kappa_2}\right) \exp(i\theta(a+bx))dx$$

$$= \frac{1}{\sqrt{\pi}} \exp(i\theta(a+b\kappa_1)) \int_{-\infty}^{+\infty} \exp(-\xi^2 + i\theta b\sqrt{2\kappa_2}\,\xi)\, d\xi$$

or by use of Eq. (3.3)

$$\Phi(\theta) = \exp(i\,(a+b\kappa_1) - \frac{\theta^2 b^2 \kappa_2}{2}) \qquad (3.6)$$

which shows that Y is Gaussian distributed with mean $(a + b\kappa_1)$ and variance $(b^2\kappa_2)$.

Secondly the moments, μ_n, of the variable is going to be calculated. By introducing:

$$\frac{d^n}{d\theta^n}\Phi(\theta) = \psi_n(\theta)\Phi(\theta) \qquad (3.7)$$

and making use of Eq. (3.1) and the fact that

$$\psi_1 = -\kappa_2\theta + i\kappa_1$$

$$\frac{d\psi_1}{d\theta} = -\kappa_2$$

$$\frac{d^n\psi_1}{d\theta^n} = 0 \qquad n \geq 2 \qquad (3.8)$$

it can be shown by successive differentiation of Eq. (3.7) that:

$$\psi_n(\theta) = \psi_1(\theta)\cdot \psi_{n-1}(\theta) - \kappa_2\cdot(n-1)\psi_{n-2}(\theta) \qquad (3.9)$$

(This is practically the same equation as Eq. (A1.2))

Bringing to mind Eq. (2.5)

$$\frac{d^n}{d\theta^n} \Phi(\theta)\Big|_{\theta=o} = \psi_n(o) = i^n E(X^n) \qquad (3.10)$$

one finds that:

$$\mu_n = E(X^n) = \kappa_1 E(X^{n-1}) + \kappa_2\cdot(n-1)E(X^{n-2}) \qquad (3.11)$$

By means of Eq.(3.11) all μ_n then can be calculated when introducing:

$$\begin{aligned} E(X^o) &= 1 \\ E(X) &= \kappa_1 \end{aligned} \qquad (3.12)$$

Specially when $\kappa_1 = o$, Eq.(3.11) and Eq. (3.12) give :

$$\begin{aligned} \mu_n &= o && n \text{ odd} \\ &= \kappa_2^{n/2}(n-1)\cdot(n-3)\cdots 3\cdot 1 && n \text{ even} \end{aligned} \qquad (3.13)$$

4. JOINTLY DISTRIBUTED GAUSSIAN VARIABLES.

n variables, X_i, are said to be jointly Gaussian distributed if the joint probability density function is written:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}(\det(\mathbf{S}))^{\frac{1}{2}}} \exp(-\tfrac{1}{2}(\mathbf{x}-\mu)^T\cdot \mathbf{S}^{-1}\cdot(\mathbf{x}-\mu)) \qquad (4.1)$$

where

$$\mathbf{x} = \{x_i\}$$

$$\mu = E(\mathbf{X})$$

and

$$\mathbf{S} = E((\mathbf{X}-\mu)\cdot(\mathbf{X}-\mu)^T) \qquad (4.2)$$

The characteristic function, $\Phi(\psi)$, is in this case given as:

$$\Phi(\psi) = \exp(i\mu^T\psi\cdot - \tfrac{1}{2}\psi^T\cdot\mathbf{S}\cdot\psi) \qquad (4.3)$$

where

$$\psi = \{\theta_i\}$$

In the following $\mu = o$ is assumed. This is no loss of generality, because the variable $\mathbf{Y} = \mathbf{X}-\mu$ could have been discussed in stead. In this case $f(\mathbf{x})$ and $\Phi(\psi)$ can be written:

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}(\det(\mathbf{S}))^{1/2}} \exp(-\tfrac{1}{2}\mathbf{x}^T\mathbf{S}^{-1}\mathbf{x}) \qquad (4.4)$$

$$\Phi(\psi) = \exp(-\tfrac{1}{2}\psi^T\cdot\mathbf{S}\cdot\psi) \qquad (4.5)$$

It is easily stated that variables, Y_i, given as linear combinations of X_i:

$$Y_i = \Lambda_{ij} X_j \tag{4.6}$$

or

$$\mathbf{Y} = \Lambda \cdot \mathbf{X} \tag{4.7}$$

also will be Gaussian distributed, because

$$f(\mathbf{y}) = f(\mathbf{x}(\mathbf{y})) \left\| \frac{\partial(\mathbf{x})}{\partial(\mathbf{y})} \right\| \tag{4.8}$$

where $\left|\frac{\partial(\mathbf{x})}{\partial(\mathbf{y})}\right|$ is the Jacobian determinant , (in this case given as $(\det(\Lambda))^{-1}$) which is independent of $\mathbf{Y}$. Hence:

$$f(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}(\det(\mathbf{S}))^{1/2}\det(\Lambda)} \exp(-\tfrac{1}{2}\mathbf{y}^T\Lambda^T\mathbf{S}^{-1}\Lambda\,\mathbf{y}) \tag{4.9}$$

Before the distribution of $\mathbf{Y}$ is discussed, there will be stated that $\mathbf{S}$ is a second order tensor. $\mathbf{S}$ is according to Eq. (4.2) defined as:

$$\mathbf{S} = E(\mathbf{X} \cdot \mathbf{X}^T) \tag{4.10}$$

In the following assume that $\mathbf{Y}$ is given as:

$$\mathbf{Y} = \Lambda \cdot \mathbf{X} \tag{4.11}$$

where Λ is an orthogonal matrix, with the properties:

$$\det(\Lambda) = 1$$

$$\Lambda^{-1} = \Lambda^T \tag{4.12}$$

then the new variance-matrix is given as:

$$E(\mathbf{Y}\mathbf{Y}^T) = \mathbf{S}' = E(\Lambda \mathbf{X}\mathbf{X}^T \Lambda^T) = \Lambda E(\mathbf{X}\mathbf{X}^T) \Lambda^T$$

or

$$\mathbf{S}' = \Lambda \mathbf{S} \Lambda^T \tag{4.13}$$

which states the tensor properties of $\mathbf{S}$ (See for instance Jaeger /6/). In addition, $\mathbf{S}$ is symmetric, which is easily stated from Eq. (4 .1o):

$$S_{ij} = E(X_i X_j) = E(X_j X_i) = S_{ji} \tag{4.14}$$

In this case it is possible to find at least one Λ which diagonalizes $\mathbf{S}'$:

$$S'_{ij} = V_i \delta_{ij} \qquad \forall_i \tag{4.15}$$

If det($\mathbf{S}$) = o, at least one of V_i is equal to zero. This indicates that X_i are linearly dependent.

$$Y_1 = \Lambda_{1i} X_i = o \tag{4.16}$$

An example is the case when:

$$X_2 = \frac{dX_1}{dt}$$

$$X_3 = \frac{d^2X_1}{dt^2}$$

which both are Gaussian distributed (if $X_1(t)$ is Gaussian). If $X_1(t)$ is given as the response of a harmonic oscillator:

$$\ddot{X}_1 + 2\,\omega_0 \dot{X}_1 + \omega_0^2 X_1 = 0 \qquad (4.17)$$

or

$$\frac{1}{\kappa} X_3 + \frac{2\rho\omega_0}{\kappa} X_2 + \frac{\omega_0^2}{\kappa} X_1 = 0 \qquad (4.18)$$

where

$$\kappa = \sqrt{1 + 4\rho^2\omega_0^2 + \omega_0^4}$$

then Eq.(4.16) is fulfilled.

The occurence of $V_i = 0$ does not make any difference in the following, so no more discussion of this case is necessary.

According to Eq. (4.12) and Eq.(4.13)

$$\mathbf{S}'\Lambda = \Lambda\,\mathbf{S} \qquad (4.19)$$

or on the index-form

$$S'_{ij}\Lambda_{jk} = \Lambda_{ij}S_{jk} \qquad (4.20)$$

Assuming $\mathbf{S}'$ to be diagonal and introducing Eq. (4 .15), Eq.(4.20) is rewritten:

$$V_i \delta_{ij} \Lambda_{jk} = \Lambda_{ij} S_{jk} \qquad \ddagger_i \tag{4.21}$$

or

$$V_i \Lambda_{ik} = \Lambda_{ij} S_{jk} \qquad \ddagger_i \tag{4.22}$$

Introducing:

$$\boldsymbol{\lambda}^i = \{\lambda_k^i\} \tag{4.23}$$

where

$$\lambda_k^i = \Lambda_{ik}$$

Eq.(4.22) can be written in the more compact form:

$$\mathbf{S}\,\boldsymbol{\lambda}^i - V_i\,\boldsymbol{\lambda}^i = o \tag{4.24}$$

which shows that $\boldsymbol{\lambda}^i$ is the eigenvector connected to the eigenvalue V_i of $\mathbf{S}$.

This shows that $\boldsymbol{\Lambda}$ contains the normalized eigenvectors of $\mathbf{S}$ and that the diagonalized $\mathbf{S}'$ contains the eigenvalues of $\mathbf{S}$, such that:

$$S'_{ij} = V_i \delta_{ij} \qquad \ddagger_i \tag{4.25}$$

and

$$\Lambda_{ij} = \lambda_j^i$$

both can be found by means of standard eigenvalue and eigenvector routines.

If one is searching for a set of independent variables, with given values of V_i, the Gram-Schmidt's orthonormalization process can be used. The details on this process can be found in most standard textbooks on numerical analysis, for instance Frødberg /7/.

The process is some sort of a recurrence process. One starts with the first variable:

$$Y_1 = \Lambda_{11} X_1 \tag{4.27}$$

and chooses Λ_{11} so that $E(Y_1^2) = 1$. In the next step Y_2 is assumed in the form:

$$Y_2 = \Lambda_{21} X_1 + \Lambda_{22} X \tag{4.28}$$

and Λ_{21} and Λ_{22} are calculated from the relations:

$$E(Y_1 Y_2) = 0$$

$$E(Y_2^2) = 1 \tag{4.29}$$

In general Y_n is assumed in the form:

$$Y_n = \sum_{i=1}^{n} \Lambda_{ni} X_i \tag{4.30}$$

and Λ_{ni} is calculated according to:

$$E(Y_j Y_n) = 0 \qquad j < n$$

$$E(Y_n^2) = 1 \tag{4.31}$$

In this way Λ is found, and $\mathbf{S}'$ is made equal to the identity matrix. In this case $\Phi(\psi)$ becomes very simple:

$$\Phi(\psi) = \exp(-\tfrac{1}{2}\sum_i \theta_i^2) = \prod_i \exp(-\tfrac{1}{2}\theta_i^2) \qquad (4.32)$$

which also shows that the **uncorrolated** variables are independently distributed (which is the case in general when the variables are Gaussian distributed.).

5. STATIONARY RANDOM PROCESSES.

For the general theory of stochastic processes, the reader is referred to one of the standard textbooks on this field, for instance Cox & Miller /8/ or Sveshnikov /9/.
Here only the main results are going to be given.

A random process $X(t)$ * is said to be strongly stationary if its complete probability structure is independent of a shift in the parameter t, i.e.:

$$f(x_1 ; t_1) = f(x_1 ; t_1+a) \tag{5.1}$$

$$f(x_1, x_2 ; t_1, t_2) = f(x_1, x_2 ; t_1+a, t_2+a) \tag{5.2}$$

$$f(x_1, x_2, \cdots, x_n ; t_1, t_2, \cdots, t_n)$$

$$= f(x_1, x_2, \cdots, x_n ; t_1+a, t_2+a, \cdots, t_n+a) \tag{5.3}$$

where

$f(x_1, \cdots, x_n ; t_1, \cdots, t_n)$ is the joint probability density function of the variables $X_i(t_i)$.

When Eq.(5.1) and Eq.(5.2) are satisfied only, $X(t)$ is said to be weakly stationary.

*) In the following $X(t)$ is assumed continuous both as a variable and with respect to t.

The above discussion can be extended to several jointly distributed random processes. For example take X(t) and Y(u). In this case X(t) and Y(u) are said to be weakly stationary if:

$$f(x_1, y_1 \;;\; t_1, u_1) = f(x_1, y_1 \;;\; t_1 + a, u_1 + a) \qquad (5.4)$$

and strongly stationary if the generalization of Eq.(5.4) is satisfied.

In the case of Gaussian random processes with zero mean the sufficient and necessary conditions for weakly stationarity of X(t) and Y(t) are that:

$$E(X(t_1) \cdot X(t_2)) = R_{xy}(t_1 - t_2) \qquad (5.5)$$

$$E(X(t_1) \cdot X(t_2)) = R_{xx}(t_1 - t_2) \qquad (5.6)$$

$$E(Y(t_1) \cdot Y(t_2)) = R_{yy}(t_1 - t_2) \qquad (5.7)$$

It has generally been proved that the Fourier transform of R_{xx} and R_{xy} exist:

$$S_{xx} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\omega\tau)\, R_{xx}(\tau)\, d\tau \qquad (5.8)$$

$$S_{xy} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(-i\omega\tau) R_{xy}(\tau)\, d\tau \qquad (5.9)$$

S_{xx} and S_{xy} are called the spectral density function (or shortly the power spectrum or the spectrum) of X and the cross-spectrum of X and Y, respectively.

According to the theory of Fourier transforms, R_{xx} and R_{xy} can be found by means of S_{xx} and S_{xy} in the following way:

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} \exp(+i\omega\tau)\, S_{xx}(\omega)\, d\omega \tag{5.10}$$

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} \exp(+i\omega\tau)\, S_{xy}(\omega)\, d\omega \tag{5.11}$$

$R_{xx}(\tau)$ and $R_{xy}(\tau)$ possess the following properties:

$$|R_{xx}(\tau)| \leq R_{xx}(o) \tag{5.12}$$

$$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(o)\, R_{yy}(o)} \tag{5.13}$$

It is proved that the sufficient and necessary condition that the time derivative of $X(t)$,$\dot{X}(t)$, exists is that:

$$\left.\frac{d^2}{dt^2} R_{xx}(\tau)\right|_{\tau=o} \quad \text{exists and is unique} \tag{5.14}$$

It also follows that:

$$R_{x\dot{x}}(\tau) = \frac{d}{d\tau} R_{xx}(\tau) \tag{5.15}$$

From condition (5.14) and Eq.(5.15) it follows that

$$E(x(t)\dot{x}(t)) = R_{x\dot{x}}(o) = o \tag{5.16}$$

which includes statistical independence in the case of jointly distributed Gaussian variables with zero mean.

For real random variables which are given from X(t) by means of linear differential or integral operators, L( ), the spectral densities can be found in the following way:

$$Y(t) = L(X(t)) \tag{5.17}$$

$$\overline{Y}(\omega) = \phi(\omega)\cdot \overline{X}(\omega) \tag{5.18}$$

where $\overline{X}$ and $\overline{Y}$ stand for the Fourier transforms of X(t) and Y(t) and $\phi(\omega)$ is called the transfer function.

If Y(t) is assumed to be (weakly) stationary, its spectral density is given as:

$$S_{yy}(\omega) = \phi(\omega)\cdot\phi^*(\omega)\cdot S_{xx}(\omega) \tag{5.19}$$

where the asterisk stands for complex conjugation.

According to Eq.(5.20). $R_{yy}(\tau)$ is found as:

$$R_{yy}(\tau) = \int_{-\infty}^{+\infty} \exp(+i\omega\tau)\phi(\omega)\cdot\phi^*(\omega)S_{xx}(\omega)d\omega \tag{5.20}$$

and specially:

$$E(Y^2(t)) = R_{yy}(o) = \int_{-\infty}^{+\infty} \phi(\omega)\cdot\phi^*(\omega)\cdot S_{xx}(\omega)d\omega \tag{5.21}$$

A generalization of Eq.(5.19) is given below: Assume:

$$Y_i(t) = L_i(X(t)) \tag{5.22}$$

Which implies:

$$\bar{Y}_i(\omega) = \phi_i(\omega)\bar{X}(\omega) \tag{5.23}$$

and following:

$$S_{y_i y_j} = \phi_i(\omega)\cdot\phi_j^*(\omega)\, S_{xx}(\omega) \tag{5.24}$$

According to Eq. (5.11):

$$R_{y_i y_j}(\tau) = \int_{-\infty}^{+\infty} \exp(i\omega\tau)\phi_i\phi_j^* S_{xx}(\omega)d\omega \tag{5.25}$$

and

$$E(Y_i(t)Y_j(t)) = R_{y_i y_j}(o) = \int_{-\infty}^{+\infty} \phi_i(\omega)\phi_j^*(\omega)S_{xx}(\omega)d\omega \tag{5.26}$$

6. THRESHOLD CROSSINGS.

Let in the following $X(t)$ be a continous random process (also with respect to time) and $I(\xi, t_1, t_2)$ be an associated counting process, which counts the numbers of times $X(t)$ crosses the threshold, ξ, from below, within the time interval $\langle t_1, t_2]$. In addition to $X(t)$ the following random process will be constructed:

$$\begin{aligned} Y(t) &= 1 \qquad \text{when } X(t) \geq \xi \\ &= o \qquad \text{when } X(t) < o \end{aligned} \tag{6.1}$$

This can be written in a more compact form:

$$Y(t) = H(X(t) - \xi) \tag{6.2}$$

where $H(s)$ is the socalled Heaviside's step function, defined according to Eq.(6.1). The derivative of $H(s)$ is given as:

$$\frac{dH(s)}{ds} = \delta(s) \tag{6.3}$$

where $\delta(s)$ is Dirac's delta function, defined as:

$$\delta(s) = o \qquad s \neq o$$

$$\int_{-\varepsilon_1}^{+\varepsilon_1} \phi(s) \cdot \delta(s) ds = \phi(o) \quad (\varepsilon_1 \neq o \ \& \ \varepsilon_2 \neq o \ \& \ \phi(o) \text{ finite}) \tag{6.4}$$

According to this:

$$\dot{Y}(t) = \frac{d}{dX}(H(X(t)-\xi)) \cdot \frac{dX}{dt} = \dot{X} \cdot \delta(X(t)-\xi) \quad (6.5)$$

Fig. (6.1) shows a sample function of the process X(t), together with the associated y(t) and $\dot{y}(t)$.

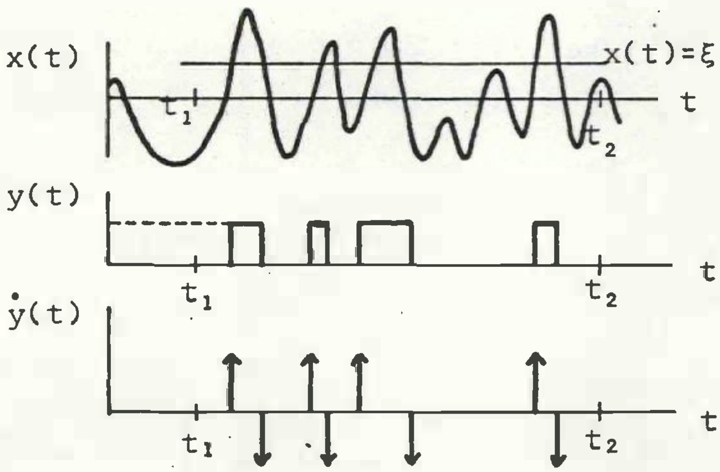


Fig.(6.1)

By introducing the process Z(t)

$$Z(t) = \dot{Y}(t) \qquad \dot{Y}(t) > 0$$

$$= 0 \qquad \dot{Y}(t) \leq 0 \qquad (6.6)$$

$I(\xi;t_1,t_2)$ can be expressed as:

$$I(\xi,t_1,t_2) = \int_{t_1^+}^{t_0^+} Z(t)dt \qquad (6.7)$$

Taking the mathematical expectation of Eq.(6.7) the following is obtained:

$$E(I(\xi,t_1,t_2)) = \int_{t_1^+}^{t_2^+} E(Z(t))dt$$

$$= \int_{t_1^+}^{t_2^+} \int_{x=-\infty}^{+\infty} \int_{\dot{x}=0}^{\infty} \dot{x}\delta(x-\xi)\, f_{X,\dot{X}}(x,\dot{x},t)d\dot{x}dxdt$$

$$= \int_{t_1^+}^{t_2^+} \int_{0}^{\infty} \dot{x}\, f_{X,\dot{X}}(\xi,\dot{x},t)d\dot{x}dt \qquad (6.8)$$

where $f_{X,\dot{X}}(x,\dot{x},t)$ is the probability density function of $(X(t), \dot{X}(t))$. To get the finale result Eq.(6.4) is used.

If assuming $X(t)$ to be (weakly) stationary, then $f_{X,\dot{X}}(x,\dot{x},t)$ is independent of t and $E(I(\xi,t_1,t_2))$ becomes:

$$E(I(\xi,t_1,t_2)) = (t_2-t_1) \int_0^{\infty} \dot{x}f_{X,\dot{X}}(\xi,\dot{x})d\dot{x} \qquad (6.9)$$

In this case it is more convenient to deal with the expected rate of crossings per unit time:

$$N(\xi) = \frac{1}{t_2-t_1} \cdot E(I(\xi,t_1,t_2)) = \int_0^{\infty} \dot{x}f_{X,\dot{X}}(\xi,\dot{x})d\dot{x} \qquad (6.10)$$

Eq.(6.1o) was first found by Rice/10/ in his celebrated work from 1944 and 1945.

As an example, let X(t) be (weakly) stationary and Gaussian with zero mean. In this case X and $\dot{X}$ are independent (The proof is found in any textbook on stochastic processes, for instance Sveshnikov / 9/) and the joint probability function is found as:

$$f_{x,\dot{x}} = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}} \exp(-\tfrac{1}{2}(\frac{x}{\sigma_x})^2 - \tfrac{1}{2}(\frac{\dot{x}}{\sigma_{\dot{x}}})^2) \qquad (6.11)$$

Making use of the following integral expression:

$$\int_0^\infty z e^{-\frac{1}{2}z^2} dz = 1 \qquad (6.12)$$

$N(\xi)$ is found as:

$$N(\xi) = \frac{1}{2\pi} \cdot \frac{\sigma_{\dot{x}}}{\sigma_x} \cdot \exp(-\tfrac{1}{2}(\frac{\xi}{\sigma_x})^2) \qquad (6.13)$$

Of special interest is the expected rate of zero crossings, N(o)

$$N(o) = \frac{1}{2\pi} \cdot \frac{\sigma_{\dot{x}}}{\sigma_x} \qquad (6.14)$$

7. PEAK DISTRIBUTION.

A peak, or a local maximum, of a continuous random process $x(t)$ occurs when $\dot{x}(t) = 0$ and simultaneously $\ddot{x}(t) < 0$. this suggests that the information about the distribution of the peaks of $x(t)$ can be obtained from the joint probability distribution of $X(t)$, $\dot{X}(t)$ and $\ddot{X}(t)$. As for threshold crossing a counting process, $J(\xi,t_1,t_2)$ is defined, which counts the local maxima of $X(t)$ above a level ξ within the time interval $<t_1,t_2]$.

By defining

$$Y(t) = H(\dot{X}(t)) \tag{7.1}$$

and

$$Z(t) = -\dot{Y}(t) \qquad \dot{Y} < 0$$
$$= 0 \qquad \dot{Y} \geq 0 \tag{7.2}$$

$J(\xi,t_1,t_2)$ is found as:

$$J(\xi,t_1,t_2) = \int_{t_1+}^{t_2+} Z(t) \cdot H(X(t)-\xi)dt \tag{7.3}$$

Taking the mathematical expectation of Eq.(7.3) $E(J(\xi,t_1,t_2))$ becomes:

$$E(J(\xi,t_1,t_2)) = \int_{t_1+}^{t_2+} \int_{x=\infty}^{+\infty} \int_{\dot{x}=-\infty}^{+\infty} \int_{-\infty}^{0} -\ddot{x}(t)\delta(\dot{x}) \cdot$$

$$H(x(t)-\xi)f_{x,\dot{x},\ddot{x}}(x,\dot{x},\ddot{x},t)d\ddot{x}d\dot{x}dxdt$$

$$= \int_{t_1^+}^{t_2^+} \int_{\xi}^{\infty} \int_{-\infty}^{0} -\ddot{x}(t) f_{x,\dot{x},\ddot{x}}(x,o,\ddot{x},t)\,d\ddot{x}\,dx\,dt \qquad (7.4)$$

In the case, when X(t) is (weakly) stationary, Eq. (7.4) is simplified to:

$$E(J(\xi,t_1,t_2)) = (t_2-t_1)\cdot M(\xi) \qquad (7.5)$$

where $M(\xi)$ is the expected number of local maxima above the level ξ per unit time:

$$M(\xi) = \int_{\xi}^{\infty} \int_{-\infty}^{0} -\ddot{x} f_{x,\dot{x},\ddot{x}}(x,o,\ddot{x})\,d\ddot{x}\,dx \qquad (7.6)$$

The expected total number of local maxima per unit time then becomes:

$$M_T = M(-\infty) = \int_{-\infty}^{+\infty} \int_{-\infty}^{0} -\ddot{x} f_{x,\dot{x},\ddot{x}}(x,o,\ddot{x})\,d\ddot{x}\,dx \qquad (7.7)$$

Following: The probability that any single local maximum will fall above the level ξ becomes:

$$1-F(\xi) = \frac{M(\xi)}{M_T} \qquad (7.8)$$

where $F(\xi)$ is the probability distribution function of the local maxima.
The probability density function of the local maxima then can be found by differentiating $M(\xi)$ with respect to ξ , which gives:

$$f(\xi) = -\frac{d}{d\xi}(1-F(\xi)) = -\frac{1}{M_T}\frac{d}{d\xi}M(\xi) =$$

$$\frac{1}{M_T} \cdot \int_{-\infty}^{o} -\ddot{x} f_{X,\dot{X},\ddot{X}}(\xi, o, \ddot{x}) d\ddot{x} \qquad (7.9)$$

In the case when X(t) is (weakly) stationary Gaussian stochastic process with zero mean Rice /10/ has found the solution for $f(\xi)$. His results have been discussed in details by Cartwright & Longuet-Higgins /11/, from which the following is taken:

$$f(\eta) = \frac{1}{\sqrt{2\pi}} \left\{ \varepsilon \cdot \exp(-\tfrac{1}{2}(\frac{\eta}{\varepsilon})^2) + \sqrt{1-\varepsilon^2} \cdot \eta \cdot \exp(-\tfrac{1}{2}\eta^2) \right.$$

$$\left. \cdot \operatorname{erf}(\frac{\eta}{\varepsilon} \cdot \sqrt{1-\varepsilon^2}) \right\} \qquad (7.10)$$

where

$$\varepsilon^2 = 1 - \frac{(Var(\dot{X}))^2}{Var(X) \cdot Var(\ddot{X})} \qquad o \leq \varepsilon \leq 1$$

$$\eta = \xi / \sqrt{Var(X)}$$

and

$$\operatorname{erf}(x) \doteq \sqrt{\frac{2}{\pi}} \cdot \int_{o}^{x} \exp(-\tfrac{1}{2}y^2) dy$$

It can easily be shown that

a) $$f(\eta) \to \eta \exp(-\tfrac{1}{2}\eta^2) \qquad \eta \geq o$$
$$\to o \qquad \eta < o \qquad (7.11)$$

$\varepsilon \to o$

b) $$f(\eta) \to \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}\eta^2)$$

$$\varepsilon \to 1 \qquad (7.12)$$

c) $$f(\eta) \to \sqrt{1-\varepsilon^2}\; \eta \exp(-\tfrac{1}{2}\eta^2) \qquad (7.13)$$

$$\eta \to \infty$$

$$\varepsilon \nrightarrow 1$$

d) $\varepsilon \to o \Leftrightarrow$ The effect spectrum of X, $S(\omega)$ takes the form: $S(\omega) \to \frac{1}{2}m_o(\delta(\omega-\omega_o) + \delta(\omega+\omega_o))$ where $m_o = Var(X)$

For most observed ocean surface wave spectra, ε is found to take a values between o.3 and 0.6, which indicates a value between 0.8 and 0.95 for $\sqrt{1-\varepsilon^3}$. For responses to ocean waves the transfer function (RAO) acts as a band-pass filter, and hence gives a lower value on ε for the response spectrum than for the ocean wave spectrum. Point c) above then indicates that the Rayleigh distribution:

$$\begin{aligned} f(\eta) &= \eta\exp(-\tfrac{1}{2}\eta^2) && \eta \geq o \\ &= o && \eta < o \qquad (7.14) \end{aligned}$$

will be a good approximation to the probability density function of local maxima of the (weakly) stationary Gaussian stochastic process with zero mean, $X(t)$, when $X(t)$ is a response to surface waves and η is somewhat larger than one. Fig.(7.1) which is taken from Cartwright & Longuet-Higgins/11/ and shows $f(\eta)$ for different values of ε , indicates the same.

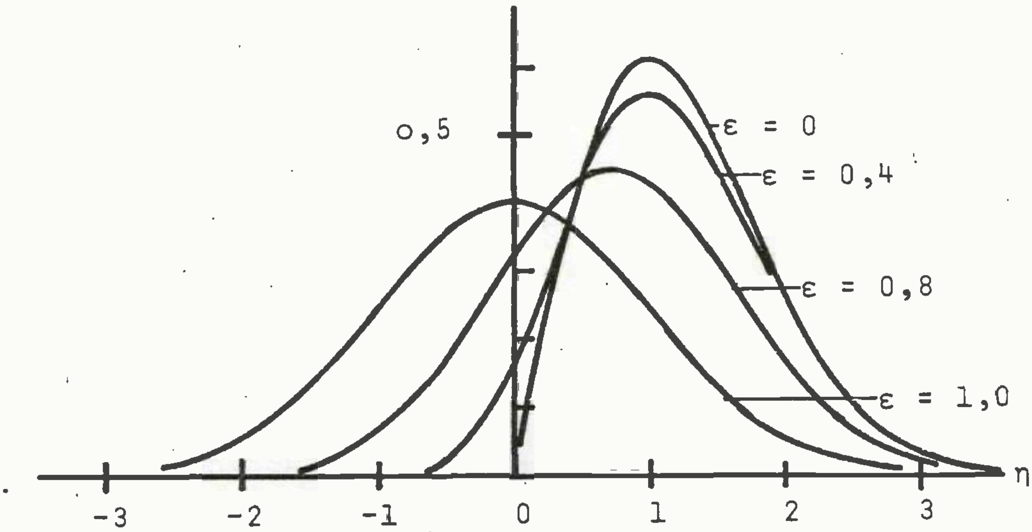


Fig. (7.1)

When assuming from the beginning that the effect spectrum of X(t) to be narrow (i.e. $\varepsilon^2 << 1$) the distribution of the local maxima can be approximated in a some simpler way. This approximation neglects the possibility of positive local minima or, when restricting the results to be valied for $\eta > \eta_0$, neglects the possibility of local minima for $\eta > \eta_0$.

In this case

$$M(\xi) = N(\xi) \tag{7.15}$$

where $N(\xi)$ is given from Eq. (6.10). If, in addition M_T is put equal to N(0), one gets:

$$1-F(\xi) = \frac{\int_0^\infty \dot{x} f_{x,\dot{x}}(\xi,\dot{x})d\dot{x}}{\int_0^\infty \dot{x} f_{x,\dot{x}}(o,\dot{x})d\dot{x}} \tag{7.16}$$

As an example let X(t) be stationary Gaussian, like before. $N(\xi)$ is then found from Eq.(6.13), and following:

$$1-F(\xi) = \exp(-\tfrac{1}{2}(\frac{\xi}{\sigma_x})^2) \tag{7.17}$$

or

$$f(\xi) = -\frac{d}{d\xi} \exp(-\tfrac{1}{2}(\frac{\xi}{\sigma_x})^2) = \frac{\xi}{\sigma_x^2} \exp(-\tfrac{1}{2}(\frac{\xi}{\sigma_x})^2) \qquad (7.18)$$

Which coincides with Eq.(7.14)

When the spectrum is not strictly narrow banded, M_T is overestimated when putting it equal to:

$$M_{T_0} = \int_0^\infty \dot{x} f_{x,\dot{x}}(o,\dot{x}) d\dot{x} \qquad (7.19)$$

This, in turn, shows that Eq.(7.16) gives a conservative result when applied to the distribution of peaks at a high level of ξ.

The same result is found when comparing Eq.(7.13) and Eq. (7.18)

8. THE DISTRIBUTION OF ONE SLIGHTLY NON-LINEAR VARIABLE

Assume in the following that the variables Y_i are independently Gaussian distributed with zero mean and with variance V_i, respectively. The non-linear variable Σ is in the following assumed to be given as:

$$\Sigma = \alpha_i Y_i + a_{ij} Y_i Y_j \qquad (8.1)$$

where

$$i = 1,\cdots,N$$
$$j = 1,\cdots,N$$

The "slightlyness" of the non-linearity of the variable Σ cannot be precisely defined according to Eq.(8.1). When introducing the new variables:

$$X_i = Y_i / \sqrt{V_i} \qquad (8.2)$$

$$Z = \Sigma / \sqrt{\nu} \qquad (8.3)$$

where

$$\nu = \sum_i \alpha_i^2 V_i \qquad (8.4)$$

Eq. (8.1) is transformed into

$$Z = \lambda_i X_i + \varepsilon_{ij} X_i X_j \qquad (8.5)$$

where

$$\lambda_i = \alpha_i / \sqrt{\nu}$$

such that $\lambda_i \lambda_i = 1$ $\qquad (8.6)$

and

$$\varepsilon_{ij} = \frac{a_{ij}\sqrt{V_i V_j}}{\sqrt{\nu}} \qquad \ddagger_{ij} \tag{8.7}$$

The only parameters left to be used in defining the "slightlyness" of the non-linearity are ε_{ij}. In the limit $\varepsilon_{ij} \to o$, Z becomes Gaussian, and hence: Z is said to be slightly non-linear when for any pair (i,j):

$$\varepsilon_{ij} << 1 \tag{8.8}$$

Introducing Eq(8.4) into Eq. (8.7), ε_{ij} is expressed as:

$$\varepsilon_{ij} = \frac{a_{ij}\sqrt{V_i V_j}}{\sqrt{\sum_k \alpha_k^2 V_k}} \qquad \ddagger_{ij} \tag{8.9}$$

In the following the parameter ε, which is assumed small, is defined as:

$$\varepsilon = (\sum_{ij} \varepsilon_{ij}\varepsilon_{ji})^{\frac{1}{2}} = (\sum_{ij} \frac{a_{ij}a_{ji}V_i V_j}{\sum_k \alpha_k^2 V_k})^{\frac{1}{2}} \tag{8.10}$$

Hence Z can be expressed as:

$$Z = \lambda_i X_i + \varepsilon \Lambda_{ij} X_i X_j \tag{8.11}$$

where

$$\Lambda_{ij} = \frac{1}{\varepsilon}\,\varepsilon_{ij} \tag{8.12}$$

which is normed as follows:

$$\Lambda_{ij}\Lambda_{ji} = 1 \qquad (8.13)$$

The distribution of Z now can be found in one of the following three ways:

a) Introducing Eq.(8.11) into Eq.(2.24), then expanding $\exp(i\theta Z(X_j))$ into a power-series in ε and then integrating Eq.(2.24) term by term, $\Phi(\theta)$ is found in the form

$$\Phi(\theta) = \sum_m \varepsilon^m P_m(i\theta)\exp(-\tfrac{1}{2}\theta^2) \qquad (8.14)$$

where $P_m(i\theta)$ is a polynomial in $(i\theta)$ of order m. Separating $P_m(i\theta)$ in powers of $(i\theta)$ and using Eq.(A1.9), f(z) is found as:

$$f(z) = \frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}z^2)(1+\sum_{m=1}^{\infty} B_m \, He_n(z)) \qquad (8.15)$$

b) Calculating the values of $\mu_m = E(Z^m)$. Then $\Phi(\theta)$ is given as:

$$\Phi(\theta) = \sum_{m=o}^{\infty} \frac{\mu_m}{m!}(i\theta)^m \qquad (8.16)$$

Separating $\exp(-\frac{1}{2}\theta^2)$ from Φ in the following way:

$$\Phi(\theta) = \exp(-\tfrac{1}{2}\theta^2)(\sum_{n=o}^{\infty} \frac{1}{n!}(\tfrac{1}{2}\theta^2)^n)(\sum_{m=o}^{\infty} \frac{\mu_m}{m!}(i\theta)^m) \qquad (8.17)$$

where it is taken into account that:

$$\exp(\tfrac{1}{2}\theta^2) = \sum_{n=o}^{\infty} (\frac{1}{n!}(\tfrac{1}{2}\theta^2)^n) \qquad (8.18)$$

Making one series expansion out of the product of the two series in Eq.(8.17)gives:

$$\Phi(\theta) = \exp(-\tfrac{1}{2}\theta^2)(\sum_m A_m He_m(i\theta)\exp(-\tfrac{1}{2}\theta^2)) \qquad (8.19)$$

The rest of the calculation follows method a), when in stead of separating $P_m(i\theta)$,separating $He_m(i\theta)$

c) Calculating the cumulants of Z, κ_m, and writing $\Phi(\theta)$ in the following way:

$$\Phi(\theta) = \exp(\sum_{m=1}^{\infty} \frac{\kappa_m}{m!}(i\theta)^m) \qquad (8.20)$$

and then expanding $\exp(\sum_{m=1}^{\infty} \frac{\kappa_m}{m!}(i\theta)^m)$ in a power series in ε , which gives:

$$\Phi(\theta) = \sum_{m=o}^{\infty} \varepsilon^m P_m(i\theta)\exp(-\tfrac{1}{2}\theta^2) \qquad (8.21)$$

The rest of the calculation follows method a).

Of these three methods, method b) is the least tractable: There is a lot of work to calculate all μ_m and there is difficult to predict which terms that will cancel in $P_m(i\theta)$, this is first found when $P_m(i\theta)$ is calculated.

Method a) looks to be the most direct method, and for a problem as simple as the one indicated here, it is the most tractable.Trying to apply this method to two jointly distributed slightly non-linear variables it is found that the method becomes untractable.

Usually the cumulants are calculated according to the moments (μ_m). If this was the only way of doing is, method c) would have been less tractable than method a). In Appendix 3 the method of finding κ_m is shown more directly. This simple calculation makes method c) very tractable. In the following the calculation of f(z) from κ_m is shown in detail.

$\Phi(\theta)$ is assumed to be given as:

$$\Phi(\theta) \;=\; \exp\left(\sum_{m=o}^{\infty} \frac{\kappa_m}{m!} (i\theta)^m\right) \qquad (8.20)$$

where κ_m is (according to Appendix 3) given as:

$$\kappa_m \;=\; k_m \varepsilon^{m-2} + h_m \varepsilon^m \qquad m \geq 2 \qquad (8.22)$$

$$\kappa_0 \;=\; k_1 \;=\; o$$

$$k_2 \;=\; 1$$

so that $\Phi(\theta)$ can be written in the following way:

$$\Phi(\theta) \;=\; \exp(-\tfrac{1}{2}\theta^2)\cdot\exp\left(\sum_{n=1}^{\infty} \varepsilon^n (k_{n+2}(i\theta)^{n+2} + h_n (i\theta)^n\right) \qquad (8.23)$$

Calculating the series-expansion of $\exp(\sum_{n=1}^{\infty} \varepsilon^n (k_{n-2}(i\theta)^{n-2} + h_n(i\theta)^n))$, $\Phi(\theta)$ is found as:

$$\Phi(\theta) \;=\; \exp(-\tfrac{1}{2}\theta^2)\Big[1+\sum_{n=1}^{\infty} \varepsilon^n (k_{n+2}(i\theta)^{n+2} + h_n (i\theta)^n)$$

$$+\tfrac{1}{2}\Big(\sum_{n=1}^{\infty} \varepsilon^n (k_{n+2}(i\theta)^{n+2} + h_n (i\theta)^n\Big)^2\Big) + \cdots\Big]$$

$$= \exp(-\tfrac{1}{2}\theta^2)\Big[\sum_{m=o}^{\infty} \varepsilon^m P_m(i\theta)\Big] \qquad (8.24)$$

$P_m(i\theta)$ up to m=4, are calculated in Appendix 4.

According to Eq.((2.3) f(z) now can be found as:

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi(\theta)\, \exp(-i\theta z)d\theta = \sum_{m=o}^{\infty} \varepsilon^m \frac{1}{2\pi}\cdot$$

$$\cdot\int_{-\infty}^{+\infty} \exp(-\tfrac{1}{2}\theta^2 - i\theta z)\cdot P_m(i\theta)d\theta \qquad (8.25)$$

In Appendix 3 it is shown that $P_m(i\theta)$ is given in the form:

$$P_m(i\theta) = \sum_{n=m,m+2}^{3m} H_n^m\cdot(i\theta)^n \qquad (8.26)$$

and hence:

$$f(z) = \sum_{m=o}^{\infty} \varepsilon^m \sum_{n=m,m+2}^{3m} H_n^m\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-\tfrac{1}{2}\theta^2 - i\theta z)(i\theta)^n d\theta \qquad (8.27)$$

According to Eq.(A1.10) f(z) now can be found as:

$$f(z) = \frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}z^2)(1+\sum_{m=1}^{\infty} \varepsilon^m \sum_{n=m,m+2}^{3m} H_n^m\cdot(-1)^n He_n(z)) \qquad (8.28)$$

By means of Eq.(A1.20) the probability distribution function, F(z), now can be found:

$$F(z) = \int_{-\infty}^{z} f(\zeta)d\zeta = G(z) + \frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}z^2)$$

$$\cdot(\sum_{m=1}^{\infty} \varepsilon^m\cdot\sum_{n=m,m+2}^{3m} H_n^m\cdot(-1)^{n-1}He_{n-1}(z)) \qquad (8.29)$$

where

$$G(z) = \int_{\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}\zeta^2)d\zeta \tag{8.30}$$

It is easily shown that when $\varepsilon \to o$

$$f(z) \to \frac{1}{\sqrt{2\pi}} \exp(-\tfrac{1}{2}z^2)$$

and

$$F(z) \to G(z)$$

as expected.

9. THE DISTRIBUTION OF TWO JOINTLY DISTRIBUTED, SLIGHTLY NON-LINEAR VARIABLES.

In the following assume that the two non-linear variables Z_1, Z_2 are given in the following way:

$$Z_1 = \lambda_i X_i + \varepsilon \Lambda_{ij} X_i X_j \tag{9.1}$$

$$Z_2 = \gamma_i X_i + \varepsilon \Gamma_{ij} X_i X_j \tag{9.2}$$

where X_i are independently Gaussian distributed with zero mean and variance equal to 1. ε is assumed to be a small parameter.

The calculation of the probability density function of Z_1 and Z_2 is similar to the one for one single non-linear variable given in chapter 8.

In the following one simplification is made:

> The part of the cumulant κ_{11}, which is of order ε^0 is assumed equal to zero.

This will be shown to be the case for the problems to be solved by means of the present method.

According to Appendix 3 and Appendix 4 it is shown that the characteristic function of Z_1 and Z_2, $\Phi(\theta_1,\theta_2)$, can be written:

$$\Phi(\theta_1,\theta_2) = \exp(-\tfrac{1}{2}\nu_{20}\theta_1^2)\exp(-\tfrac{1}{2}\nu_{02}\theta_2^2)$$

$$\cdot\left(1+\sum_{m=1}^{\infty}\varepsilon^m \sum_{\substack{i=m\\ j=m}}^{3m} H_{ij}^m\cdot(i\theta_1)^i(i\theta_2)^j\right) \tag{9.3}$$

where ν_{20} is the part of κ_{20} which is of order ε^0 and ν_{02} is the part of κ_{02} which is of order ε^0.

According to Eq.(2.25), the probability density function of Z_1 and Z_2 is:

$$f(z_1,z_2) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{+\infty}\Phi(\theta_1,\theta_2)\exp(-i\theta_1 z_1 - i\theta_2 z_2)d\theta_1 d\theta_2 \tag{9.4}$$

and following Eq.(9.3) and Eq.(A1.10), $f(z_1,z_2)$ is found to be:

$$f(z_1,z_2) = \frac{1}{2\pi}\frac{1}{\sqrt{\nu_{20}\nu_{02}}}\exp\left(-\tfrac{1}{2}\frac{z_1^2}{\nu_{20}} - \tfrac{1}{2}\frac{z_2^2}{\nu_{02}}\right)$$

$$\cdot\left(1+\sum_{m=1}^{\infty}\varepsilon^m\sum_{\substack{i=m\\ j=m}}^{3m} H_{ij}^m \nu_{20}^{-i/2}\nu_{02}^{-j/2}(-1)^{i+j}He_i(z_1/\sqrt{\nu_{20}})\right.$$

$$\left.\cdot He_j(z_2/\sqrt{\nu_{02}})\right) \tag{9.5}$$

where H_{ij}^m is given in Appendix 4 for m less than 5 .

The calculation of H_{ij}^m in Appendix 4 is based on a straight forward series expansion. An automated procedure, well fit for electronic computers, is given in Appendix 5.

10. THE DISTRIBUTION OF LOCAL MAXIMA OF ONE WEAKLY STATIONARY, SLIGHTLY NON-LINEAR RANDOM PROCESS.

Assume in the following that the variable $Y_1(t)$ is given as:

$$Y_1(t) = \delta_i X_i(t) + \Delta_{ij} X_i(t) X_j(t) \tag{10. 1}$$

where $X_i(t)$ is assumed to be a weakly stationary Gaussian random process with zero mean. In addition the variance of $(\delta_i X_i)$ is equal to unity.

The derivative of $Y_1(t)$ with respect to time, $\dot{Y}_1(t) = Y_2(t)$, is given as:

$$Y_2(t) = \delta_i \dot{X}_i(t) + 2\varepsilon\Delta_{ij} X_i(t) \cdot \dot{X}_j(t) \tag{10. 2}$$

when assuming $\Delta_{ij} = \Delta_{ji}$.

$\dot{X}_i(t)$ also becomes weakly stationary and Gaussian with zero mean, coupled to the variables $X_j(t)$ and $\dot{X}_j(t)$ (even though uncoupled to $X_i(t)$).

By transforming the variables $\{X_i, \dot{X}_i\}$ crthogonally to a set of independent variables and deviding those by the square ro t of **their** variance, a set of new variables, $Z_i(t)$, are found, which are independent Gaussian with zero mean and with variance equal to unity. Then the variable $\chi_i(t)$ can be written:

$$\chi_1(t) = \lambda_i Z_i(t) + \varepsilon\Lambda_{ij} Z_i(t) Z_j(t) \tag{10. 3}$$

$$\chi_2(t) = \gamma_i Z_i(t) + \varepsilon \Gamma_{ij} Z_i(t) Z_j(t) \tag{10. 4}$$

where

$$\sum_i \lambda_i^2 = 1 \tag{10. 5}$$

and following:

$$\chi_1(t) = \frac{Y_1(t)}{\sqrt{E(\delta_i \delta_j X_i X_j)}}$$

and

$$\chi_2(t) = \frac{Y_2(t)}{\sqrt{E(\delta_i \delta_j X_i X_j)}} \cdot$$

and ε is given such that

$$\sum_{ij} \Lambda_{ij} \Lambda_{ji} = 1 \tag{10. 6}$$

In addition ε is assumed to be a small parameter.

From Eq.(2.16) and Eq. (2.20) it is easily found that:

$$\sum_{m,n \neq o,o}^{\infty} \frac{\kappa_{mn}}{m!n!} (i\theta_1)^m (i\theta_2)^n = \sum_{j=1}^{\infty} \left(\sum_{m,n \neq o,o}^{\infty} \frac{\mu_{mn}}{m!n!} (i\theta_1)^m (i\theta_2)^n \right)^j \tag{10. 7}$$

when separating terms of $(i\theta_1)$ and $(i\theta_2)$, κ_{m_1} will be given as a series in μ_{mn}, where every term is proportional to μ_{ν_1}, where ν takes some value less than, or equal to m.
So: If it is proved that μ_{ν_1}=o for all ν then follows that: κ_{m_1}=o

The proof for μ_{ν_1}=o goes as follows:

$$\mu_{\nu_1} = E(\chi^{\nu}\dot{\chi}) = \frac{1}{n+1} E(\frac{d}{dt} \chi^{n+1}) = \frac{1}{n+1} \frac{d}{dt}(E\chi^{n+1}) = o \quad (10.8)$$

because Y is stationary and the operations E( ) and

$$\frac{df}{dt} = \lim_{n \to o} \left(\frac{f(t+h)-f(t)}{h}\right)$$ commute.

According to the assumptions: Eq.(10.5) and $E(Z_i^2) = 1$, ν_{20} is found to become equal to unity and

$$\nu_{02} = \sum_i \dot{\gamma}_i^2 = \sigma^2 \quad (10.9)$$

According to chapter 9 the probability density function, $f(\xi_1,\xi_2)$, then is given as:

$$f(\xi_1,\xi_2) = \frac{1}{2\pi\sigma} \exp(-\tfrac{1}{2}\xi_1^2 - \tfrac{1}{2}(\tfrac{\xi_2}{\sigma})^2) \cdot (1 + \sum_{m=1}^{\infty} \varepsilon^m \cdot$$

$$\cdot \sum_{i+j=m}^{3m} H_{ij}^m \sigma^{-j} He_i(\xi_1) He_j(\xi_2/\sigma)) \quad (10.10)$$

According to chapter 7, the distribution of local maxima, η, of χ_1 is found (approximately) as:

$$F(\eta) = 1 - \frac{1}{M_T} \int_0^{\infty} \xi_2 f(\eta,\xi_2) d\xi_2 \quad (10.11)$$

where $F(\eta)$ is the probability distribution function of η and M_T is given as

$$M_T = \int_0^{\infty} \xi_2 f(o,\xi_2) d\xi_2 \quad (10.12)$$

The integral

$$I_1(\eta) = \int_0^\infty \xi_2 f(\eta,\xi_2)d\xi_2 \tag{10.13}$$

can be calculated according to Eq.(A1.12), Eq.(A1.14) and Eq.(A1.16) in the following way:

$$I_1 = \frac{1}{2\pi\sigma}\exp(-\tfrac{1}{2}\eta^2)(\sum_{m=o}^{\infty}\varepsilon^m\sum_{i+j=m}^{3m} H^m_{ij}\sigma^{-j}He_i(\eta)$$

$$\cdot\int_0^\infty \xi_2 He_j(\xi_2/\sigma)\exp(-\tfrac{1}{2}\xi_2/\sigma)d\xi_2) \tag{10.14}$$

where

$$H^o_{oo} = 1 \tag{10.15}$$

Hence:

$$I_1(\eta) = \frac{\sigma}{2\pi}\exp(-\tfrac{1}{2}\eta^2)(\sum_{m=o}^{\infty}\varepsilon^m\sum_{i+j=m}^{3m} H^m_{ij}\sigma^{-j}He_i(\eta)\ J_j) \tag{10.16}$$

where

$$J_k = \int_0^\infty \xi He_k(\xi)\exp(-\tfrac{1}{2}\xi^2)d\xi \tag{10.17}$$

or following Appendix 1:

$$\begin{aligned} J_o &= 1 \\ J_1 &= \sqrt{\pi/2} \\ J_{2n} &= He_{2n-2}(o) = (-1)^{n-1}\cdot\frac{(2n-2)!}{2^{n-1}(n-1)!} \qquad n\geq 1 \\ J_{2n-1} &= He_{2n-3}(o) = 0 \qquad n\geq 2 \end{aligned} \tag{10.18}$$

From Eq.(10.12)it is given that:

$$M_T = I_1(o) = \frac{\sigma}{2\pi} \sum_{m=o}^{\infty} \varepsilon^m \sum_{i+j=m}^{3m} H_{ij}^m \sigma^{-j} He_i(o) \cdot J_j \tag{10.19}$$

and hence according to Eq.(10.11)

$$F(\eta) = 1 - \exp(-\tfrac{1}{2}\eta^2) \frac{\sum_{m=0}^{\infty} \varepsilon^m \sum_{i+j=m}^{3m} H_{ij}^m \sigma^{-j} He_i(\eta) \cdot J_j}{\sum_{m=o}^{\infty} \varepsilon^m \sum_{i+j=m}^{3m} H_{ij}^m \sigma^{-j} He_i(o) \cdot J_j} \tag{10.20}$$

The probability density function of η, $f(\eta)$, now can be found as:

$$f(\eta) = \frac{dF(\eta)}{d\eta} = - \frac{\sum_{m=o}^{\infty} \varepsilon^m \sum_{i+j=m}^{3m} H_{ij}^m \sigma^{-j} J_j \frac{d}{d\eta}(\exp(-\tfrac{1}{2}\eta^2) He_i(\eta))}{\sum_{m=o}^{\infty} \varepsilon^m \sum_{i+j=m}^{3m} H_{ij}^m \sigma^{-j} He_i(o) \cdot J_j} \tag{10.21}$$

or according to Eq.(A1.18):

$$f(\eta) = \exp(-\tfrac{1}{2}\eta^2) \frac{\sum_{m=o}^{\infty} \varepsilon^m \sum_{i+j=m}^{3m} H_{ij}^m \sigma^{-j} He_{i+1}(\eta)\, J_j}{\sum_{m=o}^{\infty} \varepsilon^m \sum_{i+j=m}^{3m} H_{ij}^m \sigma^{-j} He_i(o)\, J_j} \tag{10.22}$$

By taking $\varepsilon \to o$ in Eq.(10.22) it is found that:

$$f(\eta) \to He_1(\eta) \exp(-\tfrac{1}{2}\eta^2) = \eta \exp(-\tfrac{1}{2}\eta^2) \tag{10.23}$$

which coincides with the Rayleigh-distribution, as expected.

When calculating the next term in Eq.(10.22) one gets:

$$f(\eta) \simeq \exp(-\tfrac{1}{2}\eta^2)(\eta+\varepsilon((H^1_{10}+\frac{1}{\sigma^2}H^1_{12})(\eta^2-1)$$

$$+H^1_{30}(\eta^4-6\eta^2+3) \qquad (10.24)$$

where:

$$H^1_{10} = \Lambda_{ii}$$

$$H^1_{12} = \gamma_i\Lambda_{ij}\gamma_i+2\lambda_i\Gamma_{ij}\gamma_j \qquad (10.25)$$

$$H^1_{30} = \lambda_i\Lambda_{ij}\lambda_j$$

11. NUMERICAL RESULTS AND DISCUSSION.

No computer program for calculation of the probability density function is jet available, but it is expected to be the case within few months (from may 1974). On the other hand the autor has calculated by hand the probability density function of one special case, namely when:

$$Y_1 = X + \varepsilon X^2$$
$$Y_2 = \dot{X} + \varepsilon \cdot 2X \cdot \dot{X} \qquad (11.\ 1)$$

Fortunately Lin /15/ has calculated the probability density function of maxima of Y_1 in this case and the results of the present calculations are compared to Lin's results.

Without loss of generality EX^2 is in this case put equal to 1 and $E\dot{X}^2$ equal to σ^2.

For calculation of $\theta(\eta)$ according to the present approximate method the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ and the matrices Λ and Γ have to be calculated.

According to Eq. (11.1)

$$\lambda = \{1,0\} \qquad (11.\ 2)$$

and

$$\Lambda = \begin{Bmatrix} 1 & 0 \\ 0 & 0 \end{Bmatrix} \qquad (11.\ 3)$$

Differentiating Eq. (11.1) gives:

$$Y_2 = \dot{X} + 2\varepsilon X \cdot \dot{X} \tag{11.4}$$

and following:

$$\gamma = \sigma\{o,1\} \tag{11.5}$$

and

$$\Gamma = \sigma\left\{\begin{matrix} o & 1 \\ 1 & o \end{matrix}\right\} \tag{11.6}$$

where $\sigma^2 = \mathrm{Var}(\dot{X})$

According to chapter 10 and Appendix 4 $\theta_4(\eta)$, which is the approximation to $\theta(\eta)$ up to order ε^4 is given as:

$$\begin{aligned}\bar{\theta}_4(\eta) &= \exp(-\tfrac{1}{2}\eta^2)[1 + \varepsilon(3He_1(\eta) + He_3(\eta)) \\ &+\varepsilon^2(\tfrac{15}{2}He_2(\eta) + 5He_4(\eta) + \tfrac{1}{2}He_6(\eta)) \\ &+\varepsilon^3(\tfrac{35}{2}He_3(\eta) + \tfrac{35}{2}He_5(\eta) + \tfrac{7}{2}He_7(\eta) + \tfrac{1}{6}He_9(\eta)) \\ &+\varepsilon^4(\tfrac{315}{8}He_4(\eta) + \tfrac{105}{2}He_6(\eta) + \tfrac{63}{4}He_8(\eta) + \tfrac{3}{2}He_{10}(\eta) + \tfrac{1}{24}He_{12}(\eta))]\end{aligned} \tag{11.7}$$

On Fig.(11.1) $\bar{\theta}_n(\eta)$,for $\eta \leq 4$, have been plotted together with the exact solution, $\theta(\eta)$ for ε = o.1. The plot of $\theta(\eta)$ (and $\bar{\theta}_n(\eta)$)is (as easily seen) in logaritmic scale. Even if $\bar{\theta}$ for larger values of η ,the values of $\tilde{\eta}_n$ given at different probability levels of $\bar{\theta}_n(\eta)$ are remarkably close to the exact value. For instance:

$$\frac{\tilde{\eta} - \tilde{\eta}_4}{\tilde{\eta}} = 3.3 \cdot 10^{-2} \qquad (11.8)$$

at the level 10^{-4}.

On the other hand:

$$\frac{\tilde{\eta} - \tilde{\eta}_4}{\tilde{\eta}} = \frac{1}{10} \qquad (11.9)$$

at the level 10^{-5}, which indicates that the calculation of long term distribution of slightly non-linear variables must be done by means of $\bar{\theta}_n(\eta)$, where n is larger or equal to 5.

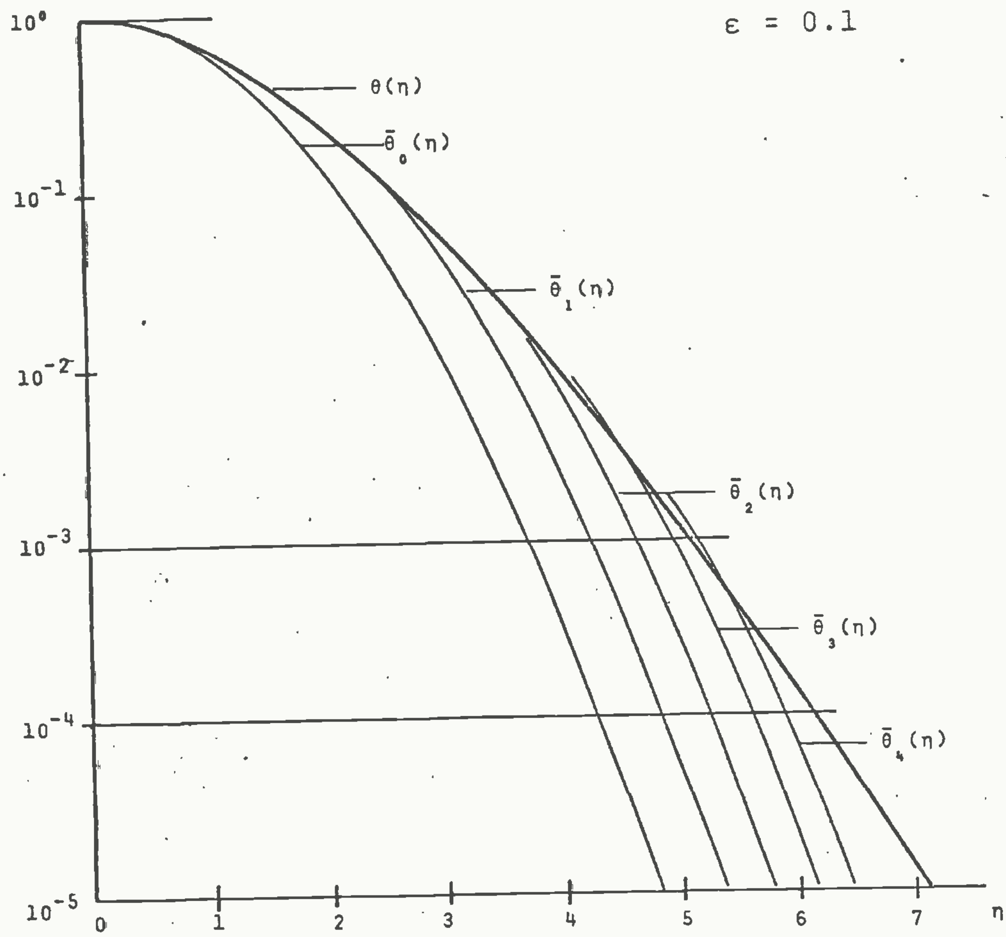


Fig.(11.1) Plot of $\theta(\eta)$ and $\bar{\theta}_n(\eta)$ for $n \leq 4$ and $\varepsilon = 0.1$

A plot of $\bar{\theta}_n(\eta)/\theta(\eta)$ for $\varepsilon = 0.1$ and $n \leq 4$ is given in Fig.(11.2) As shown, the $\bar{\theta}_n$ falls rapidly from a slightly correct value down to a low value within a narrow band of η. This indicates that one has to be careful with the choise of n, not to make it too small. On the other hand the computation work is raising very rapidly with n, approximately as n^6, which indicates that one has to be careful not to choose n too large.

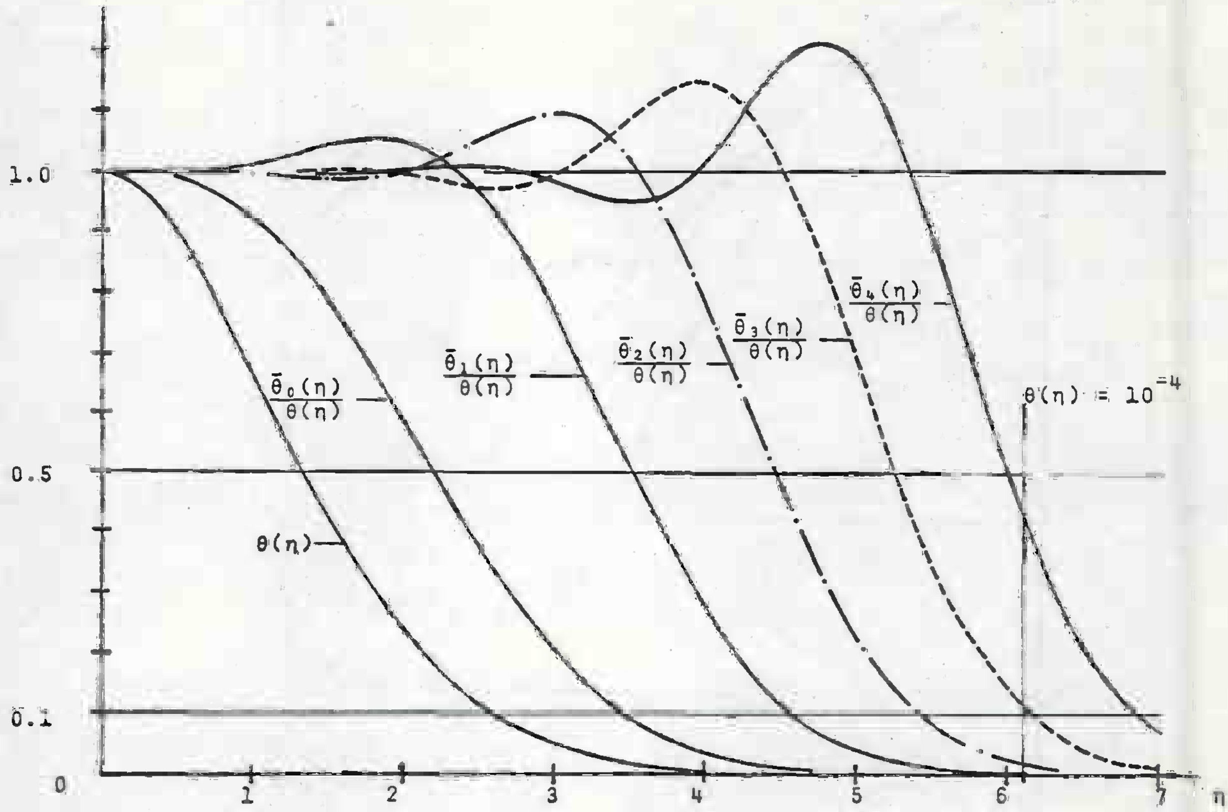


Fig.(I1.2) Plot of $\bar{\bar\theta}_n(\eta)/\theta(\eta)$ for $n \leq 4$ and $\varepsilon = o.1$ together with $\theta(\eta)$.

To give an indication of how sensitive the results are for change in ε, $\tilde\eta_n$ have been calculated for $n \leq 4$ and $Q = 10^{-4}$ ($\tilde\eta_n$ given by: $\bar\theta_n(\tilde\eta_n) = Q$). The result is given in Fig.(11.4), where it is shown that for $\varepsilon > o,125$ the results show significant errors, and that this error is rapidly growing with decreasing n. $\tilde\eta$ in the figure is given by $\theta(\tilde\eta) = Q$

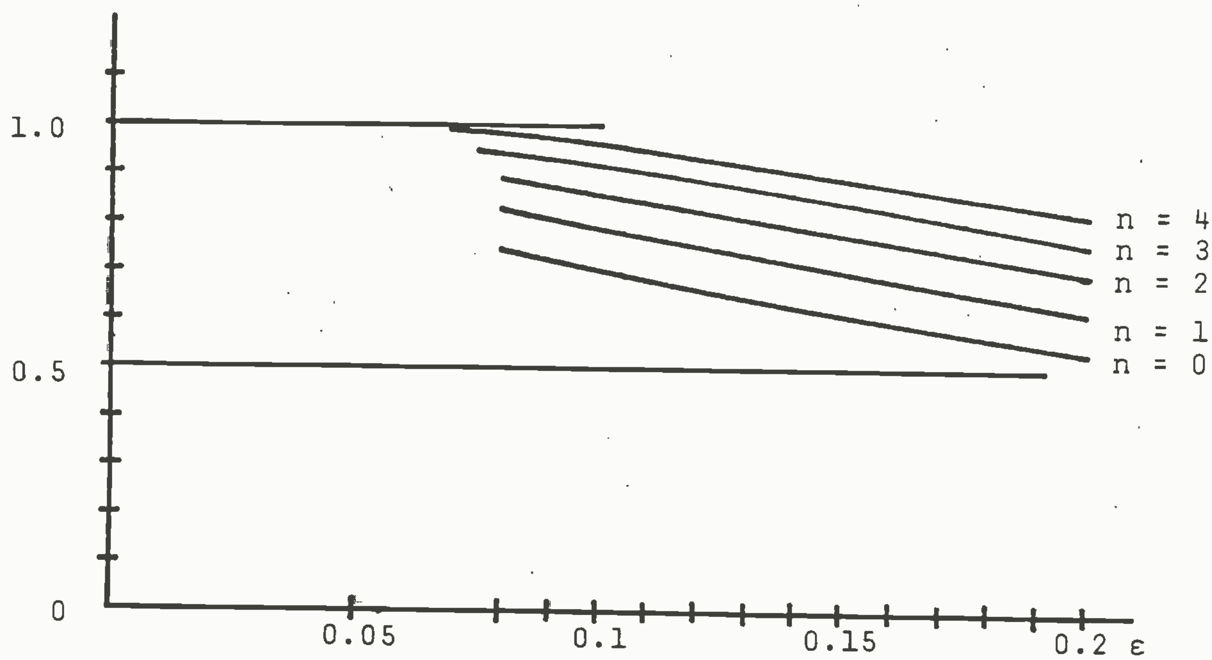


Fig.(11.3) $\tilde{\eta}_n/\tilde{\eta}$ as a function of ε for n ≤ 4

In Fig.(11.4) it is shown how $\tilde{\eta}_4/\tilde{\eta}$ is varying with ε for $Q = 10^{-3}$, 10^{-4} and 10^{-5}. The results for $Q = 10^{-3}$ are remarkably near to the correct value ($\tilde{\eta}_4 = \tilde{\eta}$), while the result for $Q = 10^{-5}$ show significant errors for larger values of ε.

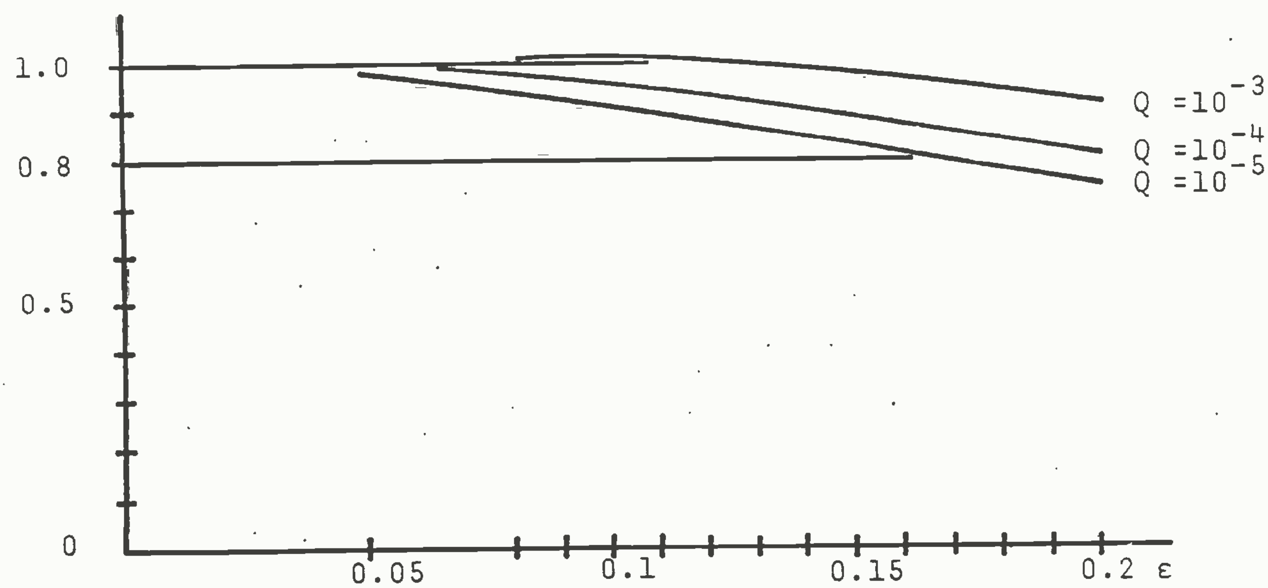


Fig.(11.4) $\tilde{\eta}_4/\tilde{\eta}$ as a function of ε for $Q=10^{-3}, 10^{-4}$ and 10^{-5}.

These results mainly coinside with those given in the preceding part of this chapter:

For larger values of ε ($\varepsilon > o.125$) n has to be raised above 4 to get saticfactorial results for larger values of η ($\eta > 6,5$)

APPENDIX 1 - HERMITE POLYNOMIALS

The following two functions are characterized as Hermite polynomials: $H_n(x)$ and $He_n(x)$. They are defined as follows:

$$e^{-x^2} \quad H_n(x) = (-1)^n \frac{d^n}{dx^n} (e^{-x^2})$$

$$e^{-\frac{1}{2}x^2} \quad He_n(x) = (-1)^n \frac{d^n}{dx^n} (e^{-\frac{1}{2}x^2}) \tag{A1.1}$$

From Eq.(A1.1) the following recurrence relations are found:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

$$He_{n+1}(x) = xHe_n(x) - nHe_{n-1}(x) \tag{A1.2}$$

The following values are easily calculated from Eq.(A1.1):

$$H_o(x) = 1$$

$$He_o(x) = 1 \tag{A1.3}$$

$$H_1(x) = 2x$$

$$He_1(x) = x \tag{A1.4}$$

From Eq.(A1.2) and Eq.(A1.3) it can be shown that

$$H_n(x) = 2^{n/2} He_n(\sqrt{2}\, x) \tag{A1.5}$$

and

$$He_n(x) = 2^{-n/2} H_n(x/\sqrt{2}\,) \tag{A1.6}$$

The special values for x = o are given as:

$$H_n(o) = o \qquad n \text{ odd}$$

$$= (-1)^{\frac{n}{2}} \frac{n!}{(\frac{n}{2})!} \qquad n \text{ even} \tag{A1.7}$$

$$He_n(o) = o \qquad n \text{ odd}$$

$$= (-1)^{n/2} \cdot \frac{n!}{2^{n/2}(\frac{n}{2})!} \qquad n \text{ even} \tag{A1.8}$$

The following integrals are useful for the present calculations:

I)
$$\int_{-\infty}^{+\infty} \exp(-t^2+2ixt)\,(it)^n dt = \frac{1}{2^n}\frac{d^n}{dx^n}\int_{-\infty}^{+\infty} \exp(-t^2+ixt)dt$$

The last integral is known from the litterature (see Abramowitz & Stegun /3/ p 302, Eq.(7.4.6))

$$\int_{-\infty}^{+\infty} \exp(-t^2+2ixt)dt = \sqrt{\pi}\,\exp(-x^2)$$

and following:

$$\int_{-\infty}^{+\infty} \exp(-t^2+2ixt)(it)^n dx = \sqrt{\pi}\,(-1)^n \frac{1}{2^n} H_n(x)e^{-x^2} \quad (A1.9)$$

and

$$\int_{-\infty}^{+\infty} \exp(-\tfrac{1}{2}t^2+ixt)(it)^n dt = \sqrt{2\pi}\,(-1)^n He_n(x)e^{-\frac{1}{2}x^2} \quad (A1.10)$$

II)

$$\int_0^\infty tH_n(t)e^{-t^2}dt = (-1)^n \int_0^\infty t\frac{d^n}{dt^n}(e^{-t^2})dt$$

$$= (-1)^{n-1} \int_0^\infty \frac{d^{n-1}}{dt^{n-1}}(e^{-t^2})dt = (-1)^{n-1} \Big| _0^\infty \frac{d^{n-2}}{dt^{n-2}}(e^{-t^2})$$

$$= H_{n-2}(o) \quad (A1.11)$$

$$\int_0^\infty tHe_n(t)e^{-t^2/2}dt = He_{n-2}(o) \quad (A1.12)$$

Both valied for $n \geq 2$

For $n = 1$ and $n = o$ the integrals have to be calculated separatly:

$$\int_0^\infty tH_1(t)e^{-t^2}dt = \int_0^\infty 2t^2e^{-t^2}dt = \int_0^\infty \sqrt{x}\,e^{-x}dx = \frac{\sqrt{\pi}}{2} \quad (A1.13)$$

and

$$\int_0^\infty tHe_1(t)e^{-t^2/2}dt = \int_0^\infty t^2e^{-t^2/2}dt = \sqrt{2}\int \sqrt{x}\,e^{-x}dx = \sqrt{\frac{\pi}{2}} \quad (A1.14)$$

$$\int_0^\infty tH_0(t)e^{-t^2}dt = \int_0^\infty te^{-t^2}dt = \tfrac{1}{2} \tag{A1.15}$$

and

$$\int_0^t tHe_0(t)e^{-\frac{t^2}{2}}\,dt = \int_0^\infty te^{-\frac{t^2}{2}}\,dt = 1 \tag{A1.16}$$

From Eq.(A1.1) the following rules of differensiation are found:

$$\frac{d}{dx}(H_n(x)e^{-x^2}) = (-1)^n \frac{d^{n+1}}{dx^{n+1}}(e^{-x^2}) = H_{n+1}(x)e^{-x^2} \tag{A1.17}$$

$$\frac{d}{dx}(He_n(x)e^{-x^2/2}) = -He_{n+1}(x)\cdot e^{-x^2/2} \tag{A1.18}$$

According to those two equations, the following are deduced:

$$\int H_n(x)e^{-x^2}dx = -H_{n-1}(x)e^{-x^2} \tag{A1.19}$$

$$\int He_n(x)e^{-x^2/2}dx = -He_{n-1}(x)e^{-x^2/2} \tag{A1.20}$$

APPENDIX 2 - THE GAMMA-DISTRIBUTION

A variable, X, is said to be Γ-distributed if it's probability density function is written:

$$F(x) = \frac{1}{\Gamma(\alpha+1)\beta^{\alpha+1}} x^{\alpha} e^{-x/\beta} \qquad x \geq o$$
$$= o \qquad x < o \tag{A2.1}$$

Where $\alpha > -1$ and $\beta > o$.
and $\Gamma(\xi)$ is the Γ-function, defined as:

$$\Gamma(\xi) = \int_{o}^{\infty} x^{\xi-1} e^{-x} dx \tag{A2.2}$$

The characteristic function is written:

$$\Phi(\theta) = (1-i\beta\theta)^{-(\alpha+1)} \tag{A2.3}$$

When $\beta = 2$ and $\alpha = \frac{n}{2} - 1$, where n is an integer, x is said to be χ^2 distributed with n degrees of freedom, with characteristic function:

$$\Phi(\theta) = (1 - i2\theta)^{-n/2} \tag{A2.4}$$

It is easily shown that if X_1 and X_2 are independently Γ-distributed with the same parameter β and with parameters α_1 and α_2, $Z = X_1+X_2$ is Γ-distributed with parameters $\beta_Z = \beta$ and $\alpha_Z = \alpha_1+\alpha_2+1$. In the case when X_1 and X_2 are independently χ^2-distributed with n_1 and n_2 degrees of freedom, $Z = X_1+X_2$ is χ^2distributed with (n_1+n_2) degrees of freedom.

From Eq.(2.24) it is found that when X is Gaussian-distributed with zero mean and variance σ^2, $Z = X^2$ is Γ-distributed with $\alpha = -\frac{1}{2}$ and $\beta = 2\sigma^2$. For $\sigma = 1$, Z then becomes χ^2distributed with 1 degree of freedom. The proof follows:

$$\Phi(\theta) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}} e^{i\theta x^2} dx = (1-i2\sigma^2\theta)^{-\frac{1}{2}} \tag{A2.5}$$

according to Eq.(2.30).

APPENDIX 3 - CALCULATION OF THE CUMULANTS.

The following formulation is taken from Longuet-Higgins /12/,/13/, while the method of calculation is developed by the autor.

Assume that the two variables Z_1 and Z_2 can be formally written in the following way:

$$Z_1 = A_1 + A_2 \tag{A3.1}$$

$$Z_2 = B_1 + B_2 \tag{A3.2}$$

where A_1 and B_1 contain only linear terms in the variables X_i and A_2 and B_2 the quadratic terms.

$$A_1 = \lambda_i X_i \tag{A3.3}$$

$$B_1 = \gamma_i X_i \tag{A3.4}$$

$$A_2 = \varepsilon_{ij} X_i X_j \tag{A3.5}$$

$$B_2 = \delta_{ij} X_i X_j \tag{A3.6}$$

where

$$\varepsilon_{ij} = \varepsilon \Lambda_{ij} \tag{A3.7}$$

$$\delta_{ij} = \varepsilon \Gamma_{ij} \tag{A3.8}$$

For the following calculation the terms: Reducable forms and irreducable forms are introduced. These terms are used in connection with the expected values of products of Z_1 and Z_2, in the way that (i.e) $E(A_1^2 B_2)$ contains both reducable and irreducable terms.

A straight-forward calculation shows that

$$E(A_1^2 B_2) = \sum_{ijkl} \lambda_i \lambda_j \delta_{kl} \, E(X_i X_j X_k X_l) \qquad \text{(A3.9)}$$

which in turn only takes values when:

$i = j = k = l$

$i = j \; , \; k = l \neq i$

$i = k \; , \; j = l \neq i$

$i = l \; , \; j = k \neq i$

or

$$E(A_1^2 B_2) = \sum_i \lambda_i \lambda_i \delta_{ii} \, E(X_i^4) + \sum_{\substack{i \\ k}} \lambda_i \lambda_i \delta_{kk} \, E(X_i^2 X_k^2)$$

$$+ 2 \sum_{\substack{i \\ j \neq i}} \lambda_i \lambda_j \delta_{ij} E(X_i^2 X_j^2) - \sum_i \lambda_i \lambda_i \delta_{ii} E(X_i^2) \cdot E(X_i^2) \qquad \text{(A3.10)}$$

In this case $\sum_{ik} \lambda_i \lambda_i \delta_{kk} E(X_i^2 X_k^2)$ is said to be reducable, because it can be reduced into two separat groups:

$$\sum_{ik} \lambda_i \lambda_i \delta_{kk} E(X_i^2 X_k^2) = (\sum_i \lambda_i \lambda_i E(X_i^2)) \cdot (\sum_k \delta_{kk} E(X_k^2)) \qquad \text{(A3.11)}$$

On the other hand, the rest of $E(A_1^2 B_2)$ can be shown to form an irreducable group, because:

$$E(X_i^4) - E(X_i^2)\,E(X_i^2) = 2E(X_i^2)\,E(X_i^2) \qquad \text{(A3.12)}$$

due to Eq.(3.13). Hence:

$$2\sum_{\substack{i \\ j\neq i}} \lambda_i \lambda_j \delta_{ij}\, E(X_i^2 X_j^2) + \sum_i \lambda_i \lambda_i \delta_{ii} E(X_i^4)$$

$$-\sum_i \lambda_i \lambda_i \delta_{ii}\, E(X_i^2) E(X_i^2)$$

$$= 2\sum_{ij} \lambda_i \lambda_j \delta_{ij}\, E(X_i^2) E(X_j^2) \qquad \text{(A3.13)}$$

which cannot be separated as with Eq.(A3.11), and is therefor said to be irreducable.

According to Longuet - Higgins /12/ it can be shown that the cumulants, κ_{mn}, of Z_1 and Z_2 can be written:

$$\kappa_{mn} = \sum_{\substack{p_1 \cdots p_m \\ q_1 \cdots q_n}} (Ap_1 Ap_2 \cdots Ap_m Bq_1 Bq_2 \cdots Bq_n) \qquad \text{(A3.14)}$$

where p_i and q_i are integers and

$(Ap_1 Ap_2 \cdots Ap_m Bq_1 Bq_2 \cdots Bq_n)$ stands for the irreducable part of $E(Ap_1 \cdot Ap_2 \cdots \cdot Ap_m \cdot Bq_1 \cdot Bq_2 \cdots Bq_n)$

It is clear that some of the terms in the series expansion of κ_{mn} will coincide and it is not neccesary to calculate all the braquets separatly. For instance

$$(A_1 A_2 B_1) = (A_2 A_1 B_1) \qquad \text{(A3.15)}$$

and so on.

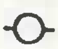
It is more complicated to calculate the value of the remaining braquets. The autor has developed a method for calculation, which simplifies the calculations and makes it less time-consuming.

In the following A_1, B_1, A_2 and B_2 are given by the symbols:

A_1 = 

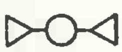
B_1 = 

A_2 = 

B_2 = 

Remark that A_1 and B_1 only have one "arm" and that A_2 and B_2 both have two "arms". Those "arms" will be used as "junctions" to other "bodies". For instance

$(A_1 A_1 B_2)$ is formally given as:

$(A_1 A_1 B_2)$ =  $= 2\sum_{ij} \lambda_i \lambda_j \delta_{ij}$ (A3.16)

when

$$E(X_i^2) = 1 \qquad \text{(A3.17)}$$

is assumed.

Remark the number "2", which indicates the number of ordered regrouping of A_1 and B_2 ; the special "j-arm" of B_2 can be "joint" to two different "arms" of the two A_1 bodies. When this is selected, the other "junction" is automatically given.

From the definition of irreducable groups the following is clear:

> An irreducable group can only consist of zero or two "one-armed bodies". The reason for this is that the "bodies" either have to form an open chain with a "one-armed body" at each end or a closed chain without any "one-armed bodies".

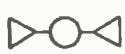
From this it is simple to show that κ_{mn} will be given in the following way:

$$\kappa_{mn} = k_{mn}\varepsilon^{m+n-2} + h_{mn}\varepsilon^{m+n} \qquad \text{(A3.18)}$$

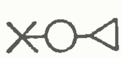
according to Eq.(A3.7) & Eq.(A3.8)

Before calculating the irreducable groups, there will be stated that a lot of them can be calculated from others by changing "bodies".

As an example take:

$(A_1 A_1 B_2)$ =  (A3.19)

this is calculated from

$(A_1 B_1 B_2)$ =  (A3.20)

by changing B_1 by A_1 (or  by ).

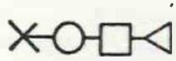
Hence the following irreducable groups are calculated to be:

$(A_1 B_1) =$  $=\lambda_i \gamma_i$

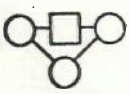
$(A_2 B_2) =$  $= 2\varepsilon^2 \Lambda_{ij} \Gamma_{ji}$

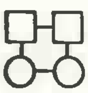
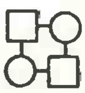
$(A_1 B_1 A_2) =$  $=2\varepsilon \lambda_i \Lambda_{ij} \gamma_j$

$(A_2 A_2 B_2) =$  $=8\varepsilon^3 \Lambda_{ij} \Lambda_{jk} \Gamma_{ki}$

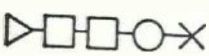
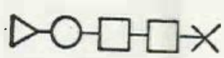
$(A_1 B_1 A_2 B_2) =$  $+$  $=$

$4\varepsilon^3 \gamma_i \Lambda_{ij} \Gamma_{jk} \lambda_k + 4\varepsilon^2 \gamma_i \Gamma_{ij} \Lambda_{jk} \lambda_k$

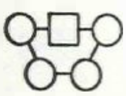
$(A_2 B_2^3) =$  $=48\varepsilon^4 \Lambda_{ij} \Gamma_{jk} \Gamma_{kl} \Gamma_{li}$

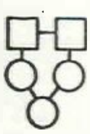
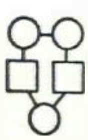
$(A_2^2 B_2^2) =$  $+$  $=$

$16\varepsilon^4 \Lambda_{ij} \Gamma_{jk} \Lambda_{kl} \Gamma_{li} + 32\varepsilon^4 \Lambda_{ij} \Lambda_{jk} \Gamma_{kl} \Gamma_{li}$

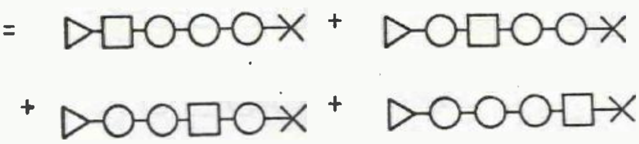
$(A_1 B_1 A_2^2 B_2) =$  $+$  $+$ 

$= 16\varepsilon^3 \lambda_i \Lambda_{ij} \Lambda_{jk} \Gamma_{kl} \gamma_l + 16\varepsilon^3 \lambda_i \Lambda_{ij} \Gamma_{jk} \Lambda_{kl} \gamma_l + 16\varepsilon^3 \lambda_i \Gamma_{ij} \Lambda_{jk} \Lambda_{kl} \gamma_l$

$(A_2 B_2^4) =$  $= 384\varepsilon^5 \Lambda_{ij} \Gamma_{jk} \Gamma_{kl} \Gamma_{lm} \Gamma_{mi}$

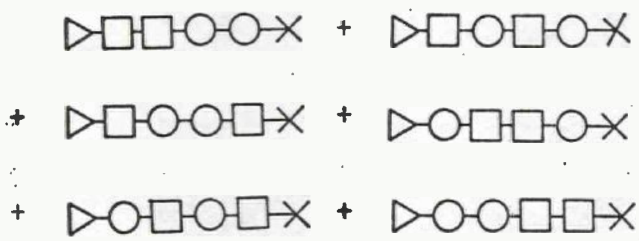
$(A_2^2 B_2^3) =$  $+$ 

$=192\varepsilon^5 \Lambda_{ij} \Lambda_{jk} \Gamma_{kl} \Gamma_{lm} \Gamma_{mi} + 192\varepsilon^5 \Lambda_{ij} \Gamma_{jk} \Lambda_{kl} \Gamma_{lm} \Gamma_{mi}$

$(A_1 B_1 A_2 B_2^3) =$ 

$= 96\varepsilon^4 \lambda_i \Lambda_{ij} \Gamma_{jk} \Gamma_{kl} \Gamma_{lm} \gamma_m + 96\varepsilon^4 \lambda_i \Gamma_{ij} \Lambda_{jk} \Gamma_{kl} \Gamma_{lm} \gamma_m$

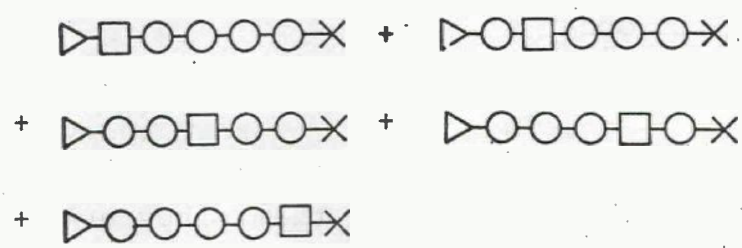
$+ 96\varepsilon^4 \lambda_i \Gamma_{ij} \Gamma_{jk} \Lambda_{kl} \Gamma_{lm} \gamma_m + 96\varepsilon^4 \lambda_i \Gamma_{ij} \Gamma_{jk} \Gamma_{kl} \Lambda_{lm} \gamma_m$

$(A_1 B_1 A_2^2 B_2^2) =$ 

$= 64\varepsilon^4 \lambda_i \Lambda_{ij} \Lambda_{jk} \Gamma_{kl} \Gamma_{lm} \gamma_m + 64\varepsilon^4 \lambda_i \Lambda_{ij} \Gamma_{jk} \Lambda_{kl} \Gamma_{lm} \gamma_m$

$+ 64\varepsilon^4 \lambda_i \Lambda_{ij} \Gamma_{jk} \Gamma_{kl} \Lambda_{lm} \gamma_m + 64\varepsilon^4 \lambda_i \Gamma_{ij} \Lambda_{jk} \Lambda_{kl} \Gamma_{lm} \gamma_m$

$+ 64\varepsilon^4 \lambda_i \Gamma_{ij} \Lambda_{jk} \Gamma_{kl} \Lambda_{lm} \gamma_m + 64\varepsilon^4 \lambda_i \Gamma_{ij} \Gamma_{jk} \Lambda_{kl} \Lambda_{lm} \gamma_m$

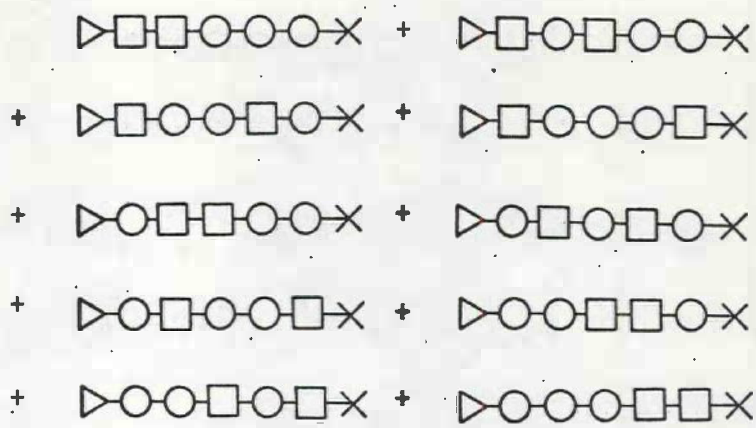
$(A_1 B_1 A_2^2 B_2^4) =$ 

$= 768\varepsilon^5 \lambda_i \Lambda_{ij} \Gamma_{jk} \Gamma_{kl} \Gamma_{lm} \Gamma_{mn} \gamma_n + 768\varepsilon^5 \lambda_i \Gamma_{ij} \Lambda_{jk} \Gamma_{kl} \Gamma_{lm} \Gamma_{mn} \gamma_n$

$= 768\varepsilon^5 \lambda_i \Gamma_{ij} \Gamma_{jk} \Lambda_{kl} \Gamma_{lm} \Gamma_{mn} \gamma_n + 768\varepsilon^5 \lambda_i \Gamma_{ij} \Gamma_{jk} \Gamma_{kl} \Lambda_{lm} \Gamma_{mn} \gamma_n$

$= 768\varepsilon^5 \lambda_i \Gamma_{ij} \Gamma_{jk} \Gamma_{kl} \Gamma_{lm} \Lambda_{mn} \gamma_n$

$(A_1 B_1 A_2^2 B_2^3) =$



$$= 384\varepsilon^5\lambda_i\Lambda_{ij}\Lambda_{jk}\Gamma_{kl}\Gamma_{lm}\Gamma_{mn}\gamma_n + 384\varepsilon^5\lambda_i\Lambda_{ij}\Gamma_{jk}\Lambda_{kl}\Gamma_{lm}\Gamma_{mn}\gamma_n$$

$$+ 384\varepsilon^5\lambda_i\Lambda_{ij}\Gamma_{jk}\Gamma_{kl}\Lambda_{lm}\Gamma_{mn}\gamma_n + 384\varepsilon^5\lambda_i\Lambda_{ij}\Gamma_{jk}\Gamma_{kl}\Gamma_{lm}\Lambda_{mn}\gamma_n$$

$$+ 384\varepsilon^5\lambda_i\Gamma_{ij}\Lambda_{jk}\Lambda_{kl}\Gamma_{lm}\Gamma_{mn}\gamma_n + 384\varepsilon^5\lambda_i\Gamma_{ij}\Lambda_{jk}\Gamma_{kl}\Lambda_{lm}\Gamma_{mn}\gamma_n$$

$$+ 384\varepsilon^5\lambda_i\Gamma_{ij}\Lambda_{jk}\Gamma_{kl}\Gamma_{lm}\Lambda_{mn}\gamma_n + 384\varepsilon^5\lambda_i\Gamma_{ij}\Gamma_{jk}\Lambda_{kl}\Lambda_{lm}\Gamma_{mn}\gamma_n$$

$$+ 384\varepsilon^5\lambda_i\Gamma_{ij}\Gamma_{jk}\Lambda_{kl}\Gamma_{lm}\Lambda_{mn}\gamma_n + 384\varepsilon^5\lambda_i\Gamma_{ij}\Gamma_{jk}\Gamma_{kl}\Lambda_{lm}\Lambda_{mn}\gamma_n$$

which contain information about all irreducable groups up to order ε^5.

Now the cumulants have to be calculated. From Eq.(A3.14) it is easy to compute them. They are as given below:

$$\kappa_{00} = 0$$

$$\kappa_{10} = \sum_{p_1} (A_{p_1}) = (A_2) = \varepsilon\Lambda_{ii}$$

$$\kappa_{01} = \sum_{q_1} (Bq_1) = (B_2) = \varepsilon\Gamma_{ii}$$

$$\kappa_{20} = \sum_{p_1, p_2} (A_{p_1} A_{p_2}) = (A_1^2) + (A_2^2)$$

$$\kappa_{11} = \sum_{p_1, q_1} (A_{p_1} B_{q_1}) = (A_1 B_1) + (A_2 B_2)$$

$$\kappa_{02} = (B_1^2) + (B_2^2)$$

$$\kappa_{30} = \sum_{p_1, p_2 p_3} (A_{p_1} A_{p_2} A_{p_3}) = 3(A_1^2 A_2) + (A_2^3)$$

$$\kappa_{21} = \sum_{\substack{p_1 p_2 \\ q_1}} (A_{p_1} A_{p_2} B_{q_1}) = (A_1^2 B_2) + 2(A_1 B_1 A_2) + (A_2^2 B_2)$$

$$\kappa_{12} = (B_1^2 A_2) + 2(A_1 B_1 B_2) + (A_2 B_2^2)$$

$$\kappa_{03} = 3(B_1^2 B_2) + (B_2^3)$$

$$\kappa_{40} = \sum_{\substack{p_1 p_2 \\ p_3 p_4}} (A_{p_1} A_{p_2} A_{p_3} A_{p_4}) = 6(A_1^2 A_2^2) + (A_2^4)$$

$$\kappa_{31} = \sum_{\substack{p_1 p_2 p_3 \\ q_1}} (A_{p_1} A_{p_2} A_{p_3} B_{q_1}) = 3(A_1^2 A_2 B_2) + 3(A_1 B_1 A_2^2)$$

$$+ (A_2^3 B_2)$$

$$\kappa_{22} = \sum_{\substack{p_1 p_2 \\ q_1 q_2}} (A_{p_1} A_{p_2} B_{q_1} B_{q_2}) = (A_1^2 B_2^2) + 4(A_1 B_1 A_2 B_2)$$

$$+ (B_1^2 A_2^2) + (A_2^2 B_2^2)$$

$$\kappa_{13} = 3(B_1^2 A_2 B_2) + 3(A_1 B_1 B_2^2) + (A_2 B_2^3)$$

$$\kappa_{04} = 6(B_1^2 B_2^2) + (B_2^4)$$

$$\kappa_{50} = \sum_{p_1 p_2 p_3 p_4 p_5} (A_{p_1} A_{p_2} A_{p_3} A_{p_4} A_{p_5}) = 10(A_1^2 A_2^3) + (A_2^5)$$

$$\kappa_{41} = 4(A_1 B_1 A_2^3) + 6(A_1^2 A_2^2 B_2) + (A_2^4 B_2)$$

$$\kappa_{32} = 3(A_1^2 A_2 B_2^2) + 6(A_1 B_1 A_2^2 B_2) + (B_1^2 A_2^3) + (A_2^3 B_2^2)$$

$$\kappa_{23} = 3(B_1^2 B_2 A_2^2) + 6(A_1 B_1 B_2^2 A_2) + (A_1^2 B_2^3) + (B_2^3 A_2^2)$$

$$\kappa_{14} = 4(A_1 B_1 B_2^3) + 6(B_1^2 A_2 B_2^2) + (A_2 B_2^4) .$$

$$\kappa_{05} = 10(B_1^2 B_2^3) + (B_2^5)$$

$$\kappa_{60} = 15(A_1^2 A_2^4) + o(\varepsilon^6)$$

$$\kappa_{51} = 10(A_1^2 A_2^3 B_2) + 5(A_1 B_1 A_2^4) + o(\varepsilon^6)$$

$$\kappa_{42} = 6(A_1^2 A_2^2 B_2^2) + 8(A_1 B_1 A_2^3 B_2) + (B_1^2 A_2^4) + o(\varepsilon^6)$$

$$\kappa_{33} = 3(A_1^2 A_2 B_2^3) + 9(A_1 B_1 A_2^2 B_2^2) + 3(B_1^2 A_2^3 B_2) + o(\varepsilon^6)$$

$$\kappa_{24} = 6(B_1^2 A_2^2 B_2^2) + 8(A_1 B_1 A_2 B_2^3) + (A_1^2 B_2^4) + o(\varepsilon^6)$$

$$\kappa_{15} = 10(B_1^2 A_2 B_2^3)+5(A_1 B_1 B_2^4) + o(\varepsilon^6)$$

$$\kappa_{06} = 15(B_1^2 B_2^4) + o(\varepsilon^6)$$

$$\kappa_{70} = 21(A_1^2 A_2^5) + o(\varepsilon^7)$$

$$\kappa_{61} = 15(A_1^2 A_2^4 B_2) + 6(A_1 B_1 A_2^5) + o(\varepsilon^7)$$

$$\kappa_{52} = 10(A_1^2 A_2^3 B_2^2) + 10(A_1 B_1 A_2^4 B_2)+(B_1^2 A_2^5) + o(\varepsilon^7)$$

$$\kappa_{43} = 6(A_1^2 A_2^2 B_2^3)+12(A_1 B_1 A_2^3 B_2^2)+ 3(B_1^2 A_2^4 B_2) + o(\varepsilon^7)$$

$$\kappa_{34} = 6(B_1^2 A_2^2 B_2^3)+12(A_1 B_1 A_2^2 B_2^3)+3(A_1^2 A_2 B_2^4) + o(\varepsilon^7)$$

$$\kappa_{25} = 10(B_1^2 A_2^2 B_2^3)+10(A_1 B_1 A_2 B_2^4)+(A_1^2 B_2^5) + o(\varepsilon^7)$$

$$\kappa_{16} = 15(B_1^2 A_2 B_2^4)+ 6(A_1 B_1 B_2^5) + o(\varepsilon^7)$$

$$\kappa_{07} = 21(B_1^2 B_2^2) + o(\varepsilon^7)$$

Remark that the cumulants for the case of one variable can be found as:

$$\kappa_m = \kappa_{m_0} \tag{A3.16}$$

APPENDIX 4 - CALCULATION OF THE P_m - FUNCTIONS

The functions $P_m(i\theta_1, i\theta_2)$ are defined as follows:

$$\Phi(\theta_1,\theta_2) = \exp(-\tfrac{1}{2}\nu_{20}\theta_1^2)\exp(-\tfrac{1}{2}\nu_{02}\theta_2^2)(1+\sum_{m=1}^{\infty}\varepsilon^m P_m(i\theta_1,i\theta_2)) \tag{A4. 1}$$

where Φ is given as:

$$\Phi(\theta_1,\theta_2)=\exp(\sum_{\substack{k=o\\l=o}}^{\infty}\frac{\kappa_{kl}}{k!l!}(i\theta_1)^k(i\theta_1)^l) \tag{A4. 2}$$

According to chapter 9, $\kappa_{00} = o$ and the parts of κ_{10} and κ_{01} which are of order ε^0 are equal to zero. ν_{20} is the part of κ_{20} which is of order ε^0 and ν_{02} the part of κ_{02} which is of order ε^0 . In addition, the part of κ_{11}, which is of order ε^0, is assumed equal to zero

Comparing Eq.(A4. 1) and Eq.(A4. 2), one finds that

$$\sum_{m=1}^{\infty}\varepsilon^m P_m(s,t) = \sum_{n=1}^{\infty}\frac{1}{n!}(\sum_{\substack{k=o\\l=o}}^{\infty}\frac{\kappa_{kl}}{k!l!}s^k t^l - \tfrac{1}{2}\nu_{20}s^2-\tfrac{1}{2}\nu_{02}t^2)^n \tag{A4. 3}$$

Writing for κ_{kl}:

$$\kappa_{kl} = k_{kl}\varepsilon^{k+l-2} + h_{kl}\varepsilon^{k+l} \tag{A4. 4}$$

the paranthes is at the right hand side of Eq.(A4.3) can be rewritten:

$$\sum_{\substack{k=o\\l=o}}^{\infty} \frac{\kappa_{kl}}{k!\,l!} s^k t^l - \tfrac{1}{2}s^2 - \tfrac{1}{2}t^2 = \sum_{m=1}^{\infty} \varepsilon^m K_m(s,t) \qquad \text{(A4. 5)}$$

where K_m is given as follows:

$$K_1 = h_{10} s + h_{01} t + \frac{1}{6}k_{30} s^3 + \tfrac{1}{2}k_{21} s^2 t + \tfrac{1}{2}k_{12} st^2 + \frac{1}{6}k_{03} t^3$$

$$K_2 = \tfrac{1}{2}h_{20} s^2 + h_{11} st + \tfrac{1}{2}h_{02} t^2 + \frac{1}{24}k_{40} s^4 + \frac{1}{6}k_{31} s^3 t + \frac{1}{4}k_{22} s^2 t^2 + \frac{1}{6}k_{13} st^3 + \frac{1}{24} k_{04} t^4$$

$$K_3 = \frac{1}{6}h_{30} s^3 + \tfrac{1}{2}h_{21} s^2 t + \tfrac{1}{2}h_{12} st^2 + \frac{1}{6}h_{03} t^3 + \frac{1}{120}k_{50} s^5 + \frac{1}{24}k_{41} s^4 t + \frac{1}{12}k_{32} s^3 t^2 + \frac{1}{12}k_{23} s^2 t^3 + \frac{1}{24}k_{14} st^4 + \frac{1}{120}k_{05} t^5$$

$$K_4 = \frac{1}{24} h_{40} s^4 + \frac{1}{6}h_{31} s^3 t + \frac{1}{4}h_{22} s^2 t^2 + \frac{1}{6}h_{13} st^3 + \frac{1}{24}h_{04} t^4 + \frac{1}{720}k_{60} s^6 + \frac{1}{120}k_{51} s^5 t + \frac{1}{48}k_{42} s^4 t^2 + \frac{1}{36}k_{33} s^3 t^3 + \frac{1}{48}k_{24} s^2 t^4 + \frac{1}{120}k_{15} st^5 + \frac{1}{720}k_{06} t^6$$

$$K_5 = \frac{1}{120}h_{50} s^5 + \frac{1}{24}h_{41} s^4 t + \frac{1}{12}h_{32} s^3 t^2 + \frac{1}{12}h_{23} s^2 t^2 + \frac{1}{24}h_{14} st^4 + \frac{1}{120}h_{05} t^5 + \frac{1}{5040}k_{70} s^7 + \frac{1}{720}k_{61} s^6 t + \frac{1}{240}k_{52} s^5 t^2 + \frac{1}{144}k_{43} s^4 t^3 + \frac{1}{144}k_{34} s^3 t^4 + \frac{1}{240} k_{25} s^2 t^5 + \frac{1}{720}k_{16} st^6 + \frac{1}{5040}k_{07} t^7$$

The values of $P_m(s,t)$ can now be calculated in terms of K_m according to Eq.(A4. 3):

$$P_1 = K_1 \tag{A4. 6}$$

$$P_2 = K_2 + \tfrac{1}{2}K_1^2 \tag{A4. 7}$$

$$P_3 = K_3 + K_2K_1 + \frac{1}{6}K_1^3 \tag{A4. 8}$$

$$P_4 = K_4 + K_3\cdot K_1 + \tfrac{1}{2}K_2^2 + \tfrac{1}{2}K_2K_1^2 + \frac{1}{24}K_1^4 \tag{A4. 9}$$

$$P_5 = K_5 + K_4\cdot K_1 + K_3K_2 + \tfrac{1}{2}K_3K_1^2 + \tfrac{1}{2}K_2^2K_1 + \frac{1}{6}K_2K_1^4 + \frac{1}{120}K_1^5 \tag{A4.10}$$

Using the expressions of K_m, P_m can be found in the following way:

$$P_m(s.t) = \sum_{i+j=m}^{3m} H_{ij}^m s^i t^j \tag{A4.11}$$

where H_{ij}^m, which are not equal to zero, are given as:

$$H_{10}^1 = h_{10}$$

$$H_{01}^1 = h_{01}$$

$$H_{30}^1 = \frac{1}{6}k_{30}$$

$$H_{21}^1 = \tfrac{1}{2}k_{21}$$

$$H_{12}^1 = \tfrac{1}{2}k_{12}$$

$$H_{03}^1 = \frac{1}{6}k_{03}$$

$$H^2_{20} = \tfrac{1}{2}h_{20} + \tfrac{1}{2}h^2_{10}$$

$$H^2_{11} = \tfrac{1}{2}h_{11} + h_{10}h_{01}$$

$$H^2_{02} = \tfrac{1}{2}h_{02} + \tfrac{1}{2}h^2_{01}$$

$$H^2_{40} = \frac{1}{24}k_{40} + \frac{1}{6}h_{10}k_{30}$$

$$H^2_{31} = \frac{1}{6}k_{31} + \tfrac{1}{2}h_{10}k_{21} + \frac{1}{6}h_{01}k_{30}$$

$$H^2_{22} = \frac{1}{4}k_{22} + \tfrac{1}{2}h_{10}k_{12} + \tfrac{1}{2}h_{01}k_{21}$$

$$H^2_{13} = \frac{1}{6}k_{13} + \tfrac{1}{2}h_{01}k_{12} + \frac{1}{6}h_{10}k_{03}$$

$$H^2_{04} = \frac{1}{24}k_{04} + \frac{1}{6}\,h_{01}k_{03}$$

$$H^2_{60} = \frac{1}{72}k^2_{30}$$

$$H^2_{51} = \frac{1}{12}k_{30}k_{21}$$

$$H^2_{42} = \frac{1}{12}k_{30}k_{12} + \frac{1}{8}\,k^2_{21}$$

$$H^2_{33} = \frac{1}{36}k_{30}k_{03} + \frac{1}{4}k_{21}k_{12}$$

$$H^2_{24} = \frac{1}{12}k_{03}k_{21} + \frac{1}{8}k^2_{12}$$

$$H^2_{15} = \frac{1}{12}k_{03}k_{12}$$

$$H^2_{06} = \frac{1}{72}k^2_{03}$$

$$H^3_{30} = \frac{1}{6}h_{30} + \tfrac{1}{2}h_{20}h_{10} + \frac{1}{6}h^3_{10}$$

$$H^3_{21} = \tfrac{1}{2}h_{21} + \tfrac{1}{2}h_{20}h_{01} + h_{11}h_{10} + \tfrac{1}{2}h^2_{10}h_{01}$$

$$H^3_{12} = \tfrac{1}{2}h_{12} + \tfrac{1}{2}h_{01}h_{10} + h_{11}h_{10} + \tfrac{1}{2}h^2_{01}h_{10}$$

$$H^3_{03} = \frac{1}{6}h_{03} + \tfrac{1}{2}h_{02}h_{01} + \frac{1}{6}h^3_{01}$$

$$H^3_{50} = \frac{1}{120}k_{50} + \frac{1}{12}k_{30}h_{20} + \frac{1}{24}k_{40}h_{10} + \frac{1}{12}k_{30}h^2_{10}$$

$$H^3_{41} = \frac{1}{24}k_{41} + \frac{1}{4}k_{21}h_{20} + \frac{1}{6}k_{30}h_{11} + \frac{1}{24}k_{40}h_{01} + \frac{1}{6}k_{31}h_{10} + \frac{1}{4}h^2_{10}k_{21} + \frac{1}{6}h_{10}h_{01}k_{30}$$

$$H^3_{32} = \frac{1}{12}k_{32} + \frac{1}{4}h_{10}k_{22} + \frac{1}{12}k_{30}h_{02} + \frac{1}{6}h_{01}k_{31} + \tfrac{1}{2}h_{11}k_{21} + \frac{1}{4}h_{20}k_{12} + \frac{1}{12}k_{30}h^2_{01} + \tfrac{1}{2}h_{10}h_{01}k_{21} + \frac{1}{4}h^2_{10}k_{12}$$

$$H^3_{23} = \frac{1}{12}k_{23} + \frac{1}{4}h_{01}k_{22} + \frac{1}{12}k_{03}h_{20} + \frac{1}{6}h_{10}k_{13} + \tfrac{1}{2}h_{11}k_{12} + \frac{1}{4}h_{02}k_{21} + \frac{1}{12}k_{03}h^2_{10} + \tfrac{1}{2}h_{10}h_{01}k_{12} + \frac{1}{4}h^2_{01}k_{21}$$

$$H^3_{14} = \frac{1}{24}k_{14} + \frac{1}{4}k_{12}h_{02} + \frac{1}{6}k_{03}h_{11} + \frac{1}{24}k_{04}h_{10} + \frac{1}{6}h_{01}k_{13} + \frac{1}{4}h^2_{01}k_{12} + \frac{1}{6}h_{10}h_{01}k_{03}$$

$$H^3_{05} = \frac{1}{120}k_{05} + \frac{1}{12}k_{03}h_{02} + \frac{1}{24}k_{04}h_{01} + \frac{1}{12}k_{03}h^2_{01}$$

$$H^3_{70} = \frac{1}{144}k_{40}k_{30} + \frac{1}{72}k^2_{30}h_{10}$$

$$H^3_{61} = \frac{1}{36}k_{31}k_{30} + \frac{1}{48}k_{21}k_{40} + \frac{1}{72}k^2_{30}h_{01} + \frac{1}{12}k_{21}k_{30}h_{10}$$

$$H^3_{52} = \frac{1}{24}k_{22}k_{30} + \frac{1}{12}k_{31}k_{21} + \frac{1}{48}k_{40}k_{12} + \frac{1}{8}h_{10}k^2_{21}$$

$$+ \frac{1}{12}h_{10}k_{12}k_{30} + \frac{1}{12}h_{01}k_{21}k_{30}$$

$$H^3_{43} = \frac{1}{144}k_{03}k_{40} + \frac{1}{12}k_{31}k_{12} + \frac{1}{8}k_{21}k_{22} + \frac{1}{36}k_{30}k_{13}$$

$$+ \frac{1}{36}h_{10}k_{03}k_{30} + \frac{1}{4}h_{10}k_{21}k_{12} + \frac{1}{12}h_{01}k_{30}k_{12} + \frac{1}{8}h_{01}k^2_{21}$$

$$H^3_{34} = \frac{1}{144}k_{30}k_{04} + \frac{1}{12}k_{13}k_{21} + \frac{1}{8}k_{12}k_{22} + \frac{1}{36}k_{03}k_{31}$$

$$+ \frac{1}{36}h_{01}k_{03}k_{30} + \frac{1}{4}h_{01}k_{12}k_{21} + \frac{1}{12}h_{10}k_{03}k_{21} + \frac{1}{8}h_{10}k^2_{12}$$

$$H^3_{25} = \frac{1}{24}k_{22}k_{03} + \frac{1}{12}k_{13}k_{12} + \frac{1}{48}k_{04}k_{21} + \frac{1}{8}h_{01}k^2_{12}$$

$$+ \frac{1}{12}h_{01}k_{21}k_{03} + \frac{1}{12}h_{10}k_{12}k_{03}$$

$$H^3_{16} = \frac{1}{36}k_{13}k_{03} + \frac{1}{48}k_{12}k_{04} + \frac{1}{72}k^2_{03}h_{10} + \frac{1}{12}k_{12}k_{03}h_{01}$$

$$H^3_{07} = \frac{1}{144}k_{04}k_{03} + \frac{1}{72}k^2_{03}h_{01}$$

$$H^3_{90} = \frac{1}{1296}k^3_{30}$$

$$H^3_{81} = \frac{1}{144}k^2_{30}k_{21}$$

$$H^3_{72} = \frac{1}{144}k^2_{30}k_{12} + \frac{1}{48}k_{30}k^2_{21}$$

$$H^3_{63} = \frac{1}{432}k^2_{30}k_{03} + \frac{1}{24}k_{30}k_{21}k_{12} + \frac{1}{48}k^3_{21}$$

$$H^3_{54} = \frac{1}{48}k_{30}k^2_{12} + \frac{1}{72}k_{03}k_{21}k_{30} + \frac{1}{16}k^2_{21}k_{12}$$

$$H^3_{45} = \frac{1}{48}k_{03}k^2_{21} + \frac{1}{72}k_{30}k_{12}k_{03} + \frac{1}{16}k^2_{12}k_{21}$$

$$H^3_{36} = \frac{1}{432}k^2_{03}k_{30} + \frac{1}{24}k_{03}k_{12}k_{21} + \frac{1}{48}k^3_{12}$$

$$H^3_{27} = \frac{1}{144}k^2_{03}k_{21} + \frac{1}{48}k_{03}k^2_{21}$$

$$H^3_{18} = \frac{1}{144}k^3_{03}k_{12}$$

$$H^3_{09} = \frac{1}{1296}k^3_{03}$$

$$H^4_{40} = \frac{1}{24}h_{40} + \frac{1}{6}h_{30}h_{10} + \frac{1}{8}h^2_{20} + \frac{1}{4}h_{20}h^2_{10} + \frac{1}{24}h^4_{10}$$

$$H^4_{31} = \frac{1}{6}h_{31} + \frac{1}{6}h_{30}h_{01} + \tfrac{1}{2}h_{21}h_{10} + \tfrac{1}{2}h_{20}h_{11} + \tfrac{1}{2}h_{20}h_{10}h_{11}$$

$$+ \tfrac{1}{2}h^2_{10}h_{11} + \frac{1}{6}h^3_{10}h_{01}$$

$$H^4_{22} = \frac{1}{4}h_{22} + \tfrac{1}{2}h_{21}h_{01} + \tfrac{1}{2}h_{12}h_{10} + \tfrac{1}{2}h^2_{11} + \frac{1}{4}h_{20}h_{02}$$

$$+ \frac{1}{4}h_{20}h^2_{01} + h_{11}h_{01}h_{10} + \frac{1}{4}h_{02}h^2_{10} + \frac{1}{4}h^2_{10}h^2_{01}$$

$$H^4_{13} = \frac{1}{6}h_{13} + \frac{1}{6}h_{03}h_{10} + \tfrac{1}{2}h_{12}h_{01} + \tfrac{1}{2}h_{02}h_{11} + \tfrac{1}{2}h_{02}h_{01}h_{10}$$

$$+ \tfrac{1}{2}h^2_{01}h_{11} + \frac{1}{6}h^3_{01}h_{10}$$

$$H^4_{04} = \frac{1}{24}h_{04} + \frac{1}{6}h_{03}h_{01} + \frac{1}{8}h^2_{02} + \frac{1}{4}h_{02}h^2_{01} + \frac{1}{24}h^4_{01}$$

$$H^4_{60} = \frac{1}{720}k_{60} + \frac{1}{36}k_{30}h_{30} + \frac{1}{120}k_{50}h_{10} + \frac{1}{48}h_{20}k_{40}$$

$$+ \frac{1}{12}h_{10}k_{30}h_{20} + \frac{1}{48}k_{40}h^2_{10} + \frac{1}{36}h^3_{10}k_{30}$$

$$H^4_{51} = \frac{1}{120}k_{51} + \frac{1}{24}k_{41}h_{10} + \frac{1}{120}k_{50}h_{01} + \frac{1}{24}k_{40}h_{11}$$

$$+ \frac{1}{12}k_{31}h_{20} + \frac{1}{12}h_{20}h_{01}k_{30} + \frac{1}{4}h_{20}h_{10}k_{21} + \frac{1}{6}h_{11}h_{10}k_{30}$$

$$+ \frac{1}{24}k_{40}h_{10}h_{01} + \frac{1}{12}k_{31}h^2_{10} + \frac{1}{12}k_{30}h^2_{10}h_{01} + \frac{1}{16}k_{21}h^3_{10}$$

$$H^4_{42} = \frac{1}{48}k_{42} + \frac{1}{24}h_{01}k_{41} + \frac{1}{12}h_{10}k_{32} + \frac{1}{12}h_{30}k_{12}$$

$$+ \frac{1}{4}h_{21}k_{21} + \frac{1}{12}h_{12}k_{30} + \frac{1}{48}h_{02}k_{40} + \frac{1}{6}h_{11}k_{31}$$

$$+ \frac{1}{8}h_{20}k_{22} + \frac{1}{4}h_{20}k_{21}h_{01} + \frac{1}{4}h_{20}k_{12}h_{10}$$

$$+ \frac{1}{6}h_{11}k_{30}h_{01} + \tfrac{1}{2}h_{11}k_{21}h_{10} + \frac{1}{12}h_{01}k_{30}h_{10}$$

$$+ \frac{1}{48}k_{40}h^2_{01} + \frac{1}{6}k_{31}h_{10}h_{01} + \frac{1}{8}k_{22}h^2_{10} + \frac{1}{12}k_{30}h_{10}h^2_{01}$$

$$+ \frac{1}{4}k_{21}h^2_{10}h_{01} + \frac{1}{12}k_{12}h^3_{10}$$

$$H^4_{33} = \frac{1}{36}k_{33} + \frac{1}{36}h_{30}k_{03} + \frac{1}{4}h_{21}k_{12} + \frac{1}{4}h_{12}k_{21} + \frac{1}{36}h_{03}k_{30}$$
$$+ \frac{1}{12}k_{32}h_{01} + \frac{1}{12}k_{23}h_{10} + \frac{1}{12}h_{20}k_{13} + \frac{1}{4}h_{11}k_{22}$$
$$+ \frac{1}{12}h_{02}k_{31} + \frac{1}{12}k_{30}h_{01}h_{02} + \frac{1}{4}k_{21}h_{10}h_{02}$$
$$+ \frac{1}{2}k_{21}h_{11}h_{01} + \frac{1}{2}k_{12}h_{11}h_{10} + \frac{1}{4}k_{12}h_{01}h_{20}$$
$$+ \frac{1}{36}k_{31}h^2_{01} + \frac{1}{4}k_{21}h_{10}h^2_{01} + \frac{1}{4}k_{12}h_{01}h^2_{10}$$
$$+ \frac{1}{36}k_{03}h^3_{10}$$

H^4_{24} ,H^4_{15} and H^4_{06} can be calculated from H^4_{42} , H^4_{51} and H^4_{60} by changing the indices of k_{ij} and h_{ij} (k_{ij} is replaced by k_{ji} and so on).

$$H^4_{80} = \frac{1}{720}k_{30}k_{50} + \frac{1}{1152}k^2_{40} + \frac{1}{144}h_{10}k_{30}k_{40}$$
$$+ \frac{1}{144}h_{20}k^2_{30} + \frac{1}{144}k^2_{30}h^2_{10}$$

$$H^4_{71} = \frac{1}{144}k_{41}k_{30} + \frac{1}{240}k_{50}k_{21} + \frac{1}{144}k_{40}k_{31} + \frac{1}{144}k_{40}k_{30}h_{01}$$
$$+ \frac{1}{48}k_{40}k_{21}h_{10} + \frac{1}{36}k_{31}k_{30}h_{10} + \frac{1}{24}k_{30}k_{21}h^2_{10} + \frac{1}{72}k^2_{30}h_{10}h_{01}$$

$$H^4_{62} = \frac{1}{240}k_{50}k_{12} + \frac{1}{48}k_{41}k_{21} + \frac{1}{72}k_{32}k_{30} + \frac{1}{96}k_{40}k_{22}$$
$$+ \frac{1}{72}k^2_{31} + \frac{1}{48}k_{40}k_{21}h_{01} + \frac{1}{48}k_{40}k_{12}h_{10} + \frac{1}{12}k_{31}k_{21}h_{10}$$
$$+ \frac{1}{36}k_{31}k_{30}h_{01} + \frac{1}{24}k_{22}k_{30}h_{10} + \frac{1}{24}h_{20}k_{30}k_{12}$$
$$+ \frac{1}{16}h_{20}k^2_{21} + \frac{1}{12}h_{11}k_{30}k_{21} + \frac{1}{144}h_{02}k^2_{30}$$
$$+ \frac{1}{144}k^2_{30}h^2_{01} + \frac{1}{12}k_{21}k_{30}h_{01}h_{10} + \frac{1}{24}k_{30}k_{12}h^2_{10}$$
$$+ \frac{1}{16}k^2_{21}h^2_{10}$$

$$H^4_{53} = \frac{1}{720}k_{50}k_{03} + \frac{1}{48}k_{41}k_{12} + \frac{1}{24}k_{32}k_{21} + \frac{1}{72}k_{23}k_{30}$$
$$+ \frac{1}{144}k_{40}k_{13} + \frac{1}{24}k_{31}k_{22} + \frac{1}{144}k_{40}k_{03}h_{10} + \frac{1}{48}k_{40}k_{12}h_{01}$$
$$+ \frac{1}{12}k_{31}k_{12}h_{10} + \frac{1}{12}k_{31}k_{21}h_{01} + \frac{1}{8}k_{22}k_{21}h_{10} + \frac{1}{24}k_{11}k_{30}h_{01}$$
$$+ \frac{1}{72}h_{20}k_{30}k_{03} + \frac{1}{8}h_{20}k_{21}k_{12} + \frac{1}{12}h_{11}k_{30}k_{12} + \frac{1}{8}h_{11}k^2_{21}$$
$$+ \frac{1}{48}h_{02}k_{30}k_{21} + \frac{1}{8}h^2_{10}k_{21}k_{12} + \frac{1}{72}h^2_{10}k_{30}k_{03}$$
$$+ \frac{1}{8}h_{10}h_{01}k^2_{21} + \frac{1}{12}h_{10}h_{01}k_{30}k_{12} + \frac{1}{24}h^2_{01}k_{30}k_{21}$$

$$H^4_{44} = \frac{1}{144}k_{41}k_{03} + \frac{1}{24}k_{32}k_{12} + \frac{1}{24}k_{23}k_{21} + \frac{1}{144}k_{14}k_{30}$$
$$+ \frac{1}{576}k_{40}k_{04} + \frac{1}{36}k_{31}k_{13} + \frac{1}{32}k^2_{22} + \frac{1}{144}k_{40}h_{01}k_{03}$$
$$+ \frac{1}{36}k_{31}h_{10}k_{03} + \frac{1}{12}k_{31}h_{01}h_{12} + \frac{1}{8}k_{22}h_{10}k_{12} + \frac{1}{8}k_{22}h_{01}k_{21}$$
$$+ \frac{1}{12}k_{13}h_{10}k_{21} + \frac{1}{36}k_{13}h_{01}k_{30} + \frac{1}{144}k_{04}h_{10}k_{30}$$
$$+ \frac{1}{24}h_{20}k_{03}k_{21} + \frac{1}{24}h_{02}k_{30}k_{12} + \frac{1}{36}h_{11}k_{03}k_{30}$$
$$+ \frac{1}{4}h_{11}k_{12}k_{21} + \frac{1}{16}h_{02}k^2_{21} + \frac{1}{16}h_{20}k^2_{12}$$
$$+ \frac{1}{24}h^2_{10}k_{03}k_{21} + \frac{1}{16}h^2_{10}k^2_{12} + \frac{1}{4}h_{10}h_{01}k_{12}k_{21}$$
$$+ \frac{1}{36}h_{10}h_{01}k_{30}k_{03} + \frac{1}{24}h^2_{01}k_{30}k_{12} + \frac{1}{16}h^2_{01}k^2_{21}$$

H^4_{35} , H^4_{26} , H^4_{17} and H^4_{08} can be calculated from H^4_{53}, H^4_{62} , H^4_{71} and H^4_{80} by changing the indices of k_{ij} and h_{ij}

$$H^4_{10,0} = \frac{1}{1728}k_{40}k^2_{30} + \frac{1}{1296}h_{10}k^3_{30}$$

$$H^4_{91} = \frac{1}{288}k_{40}k_{30}k_{21} + \frac{1}{432}k_{31}k^2_{30} + \frac{1}{1296}h_{01}k^3_{30}$$
$$+ \frac{1}{144}h_{10}k_{21}k^2_{30}$$

$$H^4_{82} = \frac{1}{192}k_{40}k^2_{21} + \frac{1}{288}k_{40}k_{30}k_{12} + \frac{1}{72}k_{31}k_{30}k_{21}$$
$$+ \frac{1}{288}k_{22}k^2_{30} + \frac{1}{144}k^2_{30}h_{10}k_{12} + \frac{1}{144}k^2_{30}h_{01}k_{21}$$
$$+ \frac{1}{48}k_{30}k^2_{21}h_{10}$$

$$H^4_{73} = \frac{1}{864}k_{40}k_{30}k_{03} + \frac{1}{96}k_{40}k_{21}k_{12} + \frac{1}{48}k_{31}k^2_{21}$$
$$+ \frac{1}{72}k_{31}k_{30}k_{12} + \frac{1}{48}k_{22}k_{30}k_{21} + \frac{1}{48}h_{10}k^3_{21}$$
$$+ \frac{1}{432}h_{10}k^2_{30}k_{03} + \frac{1}{24}h_{10}k_{30}k_{21}k_{12} + \frac{1}{144}h_{01}k^2_{30}k_{12}$$
$$+ \frac{1}{48}h_{01}k_{30}k^2_{21}$$

$$H^4_{64} = \frac{1}{192}k_{40}k^2_{12} + \frac{1}{288}k_{40}k_{21}k_{03} + \frac{1}{216}k_{31}k_{30}k_{03}$$
$$+ \frac{1}{24}k_{31}k_{12}k_{21} + \frac{1}{32}k_{22}k^2_{21} + \frac{1}{48}k_{22}k_{30}k_{12}$$
$$+ \frac{1}{72}k_{13}k_{30}k_{21} + \frac{1}{48}h_{10}k_{30}k^2_{12} + \frac{1}{1728}k_{04}k^2_{30}$$
$$+ \frac{1}{72}h_{10}k_{30}k_{21}k_{03} + \frac{1}{16}h_{10}k^2_{21}k_{12} + \frac{1}{48}h_{01}(k_{21})^3$$
$$+ \frac{1}{24}h_{01}k_{30}k_{12}k_{21} + \frac{1}{432}h_{01}k^2_{30}k_{03}$$

$$H^4_{55} = \frac{1}{288}k_{40}k_{12}k_{03} + \frac{1}{48}k_{31}k^2_{12} + \frac{1}{72}k_{31}k_{21}k_{03}$$
$$+ \frac{1}{144}k_{22}k_{30}k_{03} + \frac{1}{16}k_{22}k_{12}k_{21} + \frac{1}{72}k_{13}k_{12}k_{30}$$
$$+ \frac{1}{48}k_{13}k^2_{21} + \frac{1}{288}k_{04}k_{21}k_{30} + \frac{1}{72}h_{10}k_{30}k_{12}k_{03}$$
$$+ \frac{1}{48}h_{10}k^2_{21}k_{03} + \frac{1}{16}h_{10}k_{21}k^2_{12} + \frac{1}{16}h_{01}k_{12}k^2_{21}$$
$$+ \frac{1}{48}h_{01}k^2_{12}k_{30} + \frac{1}{72}h_{01}k_{03}k_{21}k_{30}$$

$H^4_{46}, H^4_{37}, H^4_{28}, H^4_{19}$ and $H^4_{0,10}$ can be calculated from $H^4_{64}, H^4_{73}, H^4_{82}, H^4_{91}$ and $H^4_{10,0}$ by changing the indices of k_{ij} and h_{ij}.

$$H^4_{12,0} = \frac{1}{31104} k^4_{30}$$

$$H^4_{11,1} = \frac{1}{2592} k^3_{30} k_{21}$$

$$H^4_{10,2} = \frac{1}{2592} k^3_{30} k_{12} + \frac{1}{576} k^2_{30} k^2_{21}$$

$$H^4_{93} = \frac{1}{7776} k^3_{30} k_{03} + \frac{1}{288} k^2_{30} k_{12} k_{21} + \frac{1}{288} k_{30} k^3_{21}$$

$$H^4_{84} = \frac{1}{864} k^2_{30} k_{21} k_{03} + \frac{1}{576} k^2_{30} k^2_{12} + \frac{1}{96} k_{30} k^2_{21} k_{12} + \frac{1}{384} k^4_{21}$$

$$H^4_{75} = \frac{1}{864} k^2_{30} k_{12} k_{03} + \frac{1}{288} k_{30} k^2_{21} k_{03} + \frac{1}{96} k_{30} k_{21} k^2_{12} + \frac{1}{96} k^3_{21} k_{12}$$

$$H^4_{66} = \frac{1}{5184} k^2_{30} k^2_{03} + \frac{1}{144} k_{30} k_{21} k_{12} k_{03} + \frac{1}{288} k^3_{21} k_{03} + \frac{1}{288} k_{30} k^3_{12} + \frac{1}{64} k^2_{21} k^2_{12}$$

$H^4_{57}, H^4_{48}, H^4_{39}, H^4_{2,10}, H^4_{1,11}$ and $H^4_{0,12}$ can be calculated from $H^4_{7,5}, H^4_{8,4}, H^4_{9,3}, H^4_{10,2}, H^4_{11,1}$ and $H^4_{12,0}$ by changing the indices of k_{ij} and h_{ij}.

Here only the terms ut to order ε^4 have been calculated.

It is clear that the P_m-functions for one variable easily can be found as:

$$P_m(s) = \sum_{i=m_m}^{3m} H_{io}^m \, s^i \tag{A4. 7}$$

where H_{io}^m can be found above. Remark that during this calculation only k_{mo} and h_{mo} shall be taken into account.

APPENDIX 5 - AN AUTOMATED PROCEDURE FOR CALCULATION OF H^m_{ij}.

In the following we will assume that some function, $f^m(t,s)$, is known to be a polynomial of order m in t and s:

$$f^m(t,s) = \sum_{i=o}^{m} \sum_{j=o}^{m} H^m_{ij} t^i s^j \qquad \text{(A5 1)}$$

When f^m is known to be a polynomial, it can formally be written:

$$f^m(t,s) = \sum_{\substack{i=o\\ j=o}}^{m} \alpha_{ij} P_i(t) \cdot P_j(s) \qquad \text{(A5. 2)}$$

where $P_i(t)$ is the Legendre polynomial of order i (See f.i./3/). Due to the orthogonality relations of $P_i(t)$:

$$\int_{-1}^{+1} P_i(t) \cdot P_n(t)dt = o \qquad i \neq n$$
$$= \frac{2}{2i+1} \qquad i = n \qquad \text{(A5.3)}$$

α_{ij} is given as follows:

$$\alpha_{ij} = (i + \tfrac{1}{2})(j + \tfrac{1}{2})\ I^m_{ij} \qquad \text{(A5. 4)}$$

where

$$I^m_{ij} = \iint_{-1}^{+1} f^m(t,s)P_i(t)\cdot P_j(s)\cdot dtds \qquad \text{(A5 5)}$$

$f^m(t,s)$ is assumed to be given in one or another way (f.i. by means of Eq.(A4.6) - Eq.(A4.10)); but when knowing of its polynomial nature, I^m_{ij} can be calculated exactly by means of Gauss-quadrature.(see f.i. /3/ or /7/). The result of such an integration is given as:

$$I_{ij} = \sum_{p\ q} w_p w_q P_i(x_p)\cdot P_j(x_q) f^m(x_p,x_q) \qquad \text{(A5. 6)}$$

where p and q are summed over

$$(p,q) \leq (\frac{m}{2} + 1, \frac{m}{2} + 1) \qquad \text{(A5. 7)}$$

According to Abramowitz & Stegun /3/ the Legendre polynomials are given as:

$$P_i(t) = \sum_{n=o}^{i} a^i_n \cdot t^n \qquad \text{(A5. 8)}$$

where a^i_n can be found by a simple recurrence relation.

Hence $f^m(t,s)$ can be found according to Eq.(A5 2) and Eq.(A5.8) as:

$$f^m(t,s) = \sum_{\substack{i=o\\j=o}}^{m} \alpha_{ij}(\sum_{n=o}^{i} a^i_n t^n)\cdot(\sum_{k=o}^{i} a^j_k s^k) \qquad \text{(A5. 9)}$$

Rewriting Eq.(A5.9) now gives:

$$f^m(t,s) = \sum_{n=o}^{m} \sum_{k=o}^{m} t^n s^k (\sum_{\substack{i=n \\ j=k}}^{m} \alpha_{ij} a_n^i a_k^j) \tag{A5.10}$$

and hence according to Eq.(A5. 1)

$$H_{ij}^m = \sum_{\substack{n=1 \\ k=j}}^{m} \alpha_{nk} a_i^n a_j^k \tag{A5.11}$$

APPENDIX 6 - REFERENCES

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