# NONLINEAR VIBRATIONS OF IMPERFECT THIN-WALLED CYLINDRICAL SHELLS

D.K. Liu

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## PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR AAN DE TECHNISCHE UNIVERSITEIT DELFT, OP GEZAG VAN DE RECTOR MAGNIFICUS, PROF.DR. J.M. DIRKEN, IN HET OPENBAAR TE VERDEDIGEN TEN OVERSTAAN VAN EEN COMMISSIE AANGEWEZEN DOOR HET COLLEGE VAN DEKANEN OP 31 MAART 1988 TE 14.00 UUR

DOOR

DA-KANG LIU

GEBOREN TE CHINA
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ABSTRACT

In this thesis a theoretical investigation of the nonlinear vibrations of imperfect thin-walled cylindrical shells is presented , which is aimed at two objectives. The first one is to investigate the influence of initial geometric imperfections on the nonlinear vibration behaviour of shells, while the second one is to investigate the effect of different boundary conditions. Donnell shallow shell equations are used with the appropriate damping, inertial and initial geometric imperfection terms included. Galerkin's procedure and the method of averaging are employed in order to reduce the problem to the solution of nonlinear algebraic and nonlinear ordinary differential equations, respectively.

Numerical solutions indicate that the initial geometric imperfections have strong influence on the nonlinear vibrations of shells if certain coupling conditions are satisfied. The imperfections may not only significantly change the natural frequencies and the degree of non-linearity, but also may change the vibration behaviour. Results show that the effect of boundary conditions on the nonlinear vibrations of shells may be significant especially for shorter shells.

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NOMENCLATURE

| | |
|---|---|
| $A_s$, $A_r$ | Cross-sectional area of stringers and rings respectively |
| $A$ | Amplitude function of the driven mode in the dynamic solution |
| $A_o$, $A_1$ | Axisymmetric and asymmetric imperfection respectively |
| $A_t$ | Slowly varying amplitude function of the driven mode |
| $\tilde{A}$ | Vector defined in Eq. (5-3-7) |
| $\bar{A}$ | Average value of $A_t$ |
| $\underset{\sim}{A}$ | Vector defined in (2-2-6) |
| $A_1, A_2, \ldots A_{24}$ | Coefficients defined in Appendix 2-A |
| $a_i, (i=1,2,\ldots 52)$ | Coefficients of Eq. (6-2-12) |
| $a_{1j}, a_{2j} (j=1,2,..7)$ | Coefficients of Eqs. (5-3-1) ~ (5-3-2) |
| $B$ | Amplitude function of the companion mode in the dynamic solution |
| $B_t$ | Slowly varying amplitude function of the companion mode |
| $\bar{B}$ | Average value of $B_t$ |
| $b_j (j=1,2,\ldots 44)$ | Coeffients of Eq. (6-2-13) |
| $C$ | Amplitude function of the axisymmetric mode in the dynamic solution |
| $C_i (i=1,2,3)$ | Components of C |
| $\bar{C}_i (i=1,2,3)$ | Average value of $C_i$ |
| $c$ | $= [3(1-\nu^2)]^{1/2}$ |
| $\bar{c}$ | Damping factor |
| $c_k (k=1,2,\ldots,37)$ | Coefficients of Eq. (6-2-14) |

| | |
|---|---|
| $\bar{c}_i (i=1,2,\ldots,12)$ | Coefficients of Eq. (2-2-7) |
| D | Bending stiffness of shell wall |
| $\bar{D}_{xx}, \bar{D}_{xy}, \bar{D}_{yy}$ | Nondimensional smeared stiffener parameters ($D_{xx} = D.\bar{D}_{xx}$, etc.) |
| $d_i (i=1,2,\ldots,51)$ | Coefficients of Eq. (6-2-15) |
| $\bar{d}_j (j=1,2,\ldots, 8)$ | Coefficients of Eq. (2-2-8) |
| $d_s$, $d_r$ | Spacing of stringers and rings respectively |
| E, $E_s$, $E_r$ | Young's modulus of shell wall, stringers and rings respectively |
| $e_j (j=1,2,\ldots,44)$ | Coefficients of Eq. (6-2-16) |
| $\hat{e}, \hat{e}_1, \hat{e}_2, \hat{e}_3$ | Error functions defined in Eqs. (5-2-1) ~ (5-2-4) |
| $\tilde{e}_i$ (i=0,1,2,3,4) | Coefficients of Eqs. (6-3-8) |
| $F_D$, $F_C$ | Generalized excitations in Eqs. (2-2-7) and (2-2-8) |
| $\bar{F}_D$ | Average value of $F_D$ |
| $F_{SD}$ | Generalized excitation defined in Eq. (4-2-1) |
| $f_{a_i}$, $f_{b_j}$, $f_{c_k}$ | Functions defined in Eqs. (6-2-12) ~ (6-2-16) |
| $\bar{f}_1$, $\bar{f}_2$, $\bar{f}_3$, $\bar{f}_{4j}$ | Functions defined in Eqs. (6-2-29) ~ (6-2-32) |
| $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}_{3j}$ | Functions defined in Eqs. (6-2-33) ~ (6-2-35) |
| $\underset{\sim}{f}$ | Vector defined in Eq. (6-2-41) |
| $\dot{\underset{\sim}{f}}$ | Vector defined in Eq. (6-3-13) |
| $f_e$ | Function defined in Eq. (6-3-11) |

| | |
|---|---|
| $G_s$, $G_r$ | Shear modulus of stringers and rings respectively |
| $G_1$, $G_2$ | Weight functions of Galerkin's method, (see Appendix 2-A.1) |
| $G_{mn}$ | Externally applied radial load in Eq. (2-2-22) |
| $h$ | Shell wall thickness |
| $\bar{H}_{xx}$, $\bar{H}_{xy}$, $\bar{H}_{yy}$ | Nondimensional smeared stiffener parameters ($H_{xx} = \frac{1}{Eh} \bar{H}_{xx}$, etc.) |
| $i$ | Number of axial half-waves of axisymmetric imperfection mode |
| $I$ | Longitudinal inertia of beam defined in Eq. (7-2-1) |
| $I_s$, $I_r$ | Moment of inertia about the centroid of stringers and rings respectively |
| $I_{t_s}$, $I_{t_r}$ | Torsional constants of stringers and rings respectively |
| $\underset{\sim}{I}_i$ | ith unit vector defined in Eq. (6-4-10) |
| $j$ | $= [-1]^{1/2}$ |
| $J$ | Jacobian matrix for Newton's method defined in Eq. (6-4-7) |
| $J'$ | Jacobian matrix defined in Eq. (6-4-11) |
| $k$ | Number of axial half-waves of asymmetric imperfection mode |
| $\ell$ | Number of circumferential full-waves of asymmetric displacement mode |
| $\ell_i, \ell_k, \ell_\ell, \ell_m, \ell_n$ | Normalized wave numbers ($\ell_i = \frac{i\pi}{L}$ ; $\ell_k = \frac{k\pi}{L}$ ; $\ell_\ell = \frac{\ell}{R}$ ; $\ell_m = \frac{m\pi}{L}$ ; $\ell_n = \frac{n}{R}$) |
| $L$ | Length of shell |
| $L_b$ | Length of beam defined in Eq. (7-2-2) |
| $L_D$, $L_Q$, $L_H$ | Linear differential operators |
| $L_{NL}$ | Non-linear differential operator |
| $m$ | Number of axial half-waves of the third term of the dynamic response mode (see Eq. (2-2-2)) |

| | |
|---|---|
| $M$ | Parameter defined in Appendix 2-A.2 $\left(M = \left(\frac{2c}{Rh}\right)^2 \frac{1}{Eh}\right)$ |
| $M_x, M_y, M_{xy}, M_{yx}$ | Total moment resultants |
| $\hat{M}_x, \hat{M}_y, \hat{M}_{xy}, \hat{M}_{yx}$ | Moment resultants of fundamental state |
| $\hat{\hat{M}}_x, \hat{\hat{M}}_y, \hat{\hat{M}}_{xy}, \hat{\hat{M}}_{yx}$ | Moment resultants of dynamic state |
| $[M]$ | Matrix defined in Eq. (4-2-18) |
| $n$ | Number of circumferential full-waves of asymmetric imperfection mode |
| $N$ | Axial tension defined in Eq. (7-2-1) |
| $N_x, N_y, N_{xy}$ | Total force resultants |
| $\hat{N}_x, \hat{N}_y, \hat{N}_{xy}$ | Force resultants of fundamental state |
| $\hat{\hat{N}}_x, \hat{\hat{N}}_y, \hat{\hat{N}}_{xy}$ | Force resultants of dynamic state |
| $N_o$ | Axial compressive load |
| $[N]$ | Matrix defined in Eq. (4-2-18) |
| $P$ | Transverse excitation defined in Eq. (7-2-1) |
| $\tilde{P}$ | Vector defined in Eq. (5-3-7) |
| $P_o$, $P_i$ | Functions defined in Eqs. (6-2-9) and (6-2-10) |
| $Q$ | Generalized force function defined in Eq. (2-2-9) |
| $\tilde{Q}$ | Generalized force function defined in Eq. (6-2-2) |
| $\bar{Q}$ | Average value of $\tilde{Q}$ |
| $\bar{Q}_{xx}, \bar{Q}_{xy}, \bar{Q}_{yy}$ | Nondimensional smeared stiffener parameters $\left(Q_{xx} = \frac{h}{2c} \bar{Q}_{xx}\right.$, etc.) |
| $q$ | Radial dynamic load applied to the surface of the cylinder |
| $r$ | Radius of gyration of cross-section of beam |

| | |
|---|---|
| $R$ | Radius of shell |
| $S$ | Cross sectional area of beam defined in Eq. (7-2-2) |
| $\underset{\sim}{S}$ | Vector defined in Eq. (6-4-4) |
| $\hat{\underset{\sim}{S}}$ | Vector defined in Eq. (6-4-16) |
| $\underset{\sim}{s}$ | Vector defined in Eq. (6-4-1) |
| $\hat{\underset{\sim}{s}}$ | Vector defined in Eq. (6-4-12) |
| $T$ | Period of vibration |
| $t$ | Time |
| $\Delta t$ | Time step used in integration |
| $\underset{\sim}{t}$ | Vector defined in Eq. (6-4-2) |
| $\hat{\underset{\sim}{t}}$ | Vector defined in Eq. (6-4-13) |
| $\underset{\sim}{U}$ | Solution vector defined in Eq. (6-4-3) |
| $\underset{\sim}{u}$ | Vector used in forward integration (see Eq. (6-4-1)) |
| $\hat{\underset{\sim}{u}}$ | Vector used in forward integration (see Eq. (6-4-12)) |
| $u$ | Axial displacement (total), $u = \hat{\hat{u}} + \hat{u}$ |
| $\hat{u}$ | Axial displacement of fundamental state |
| $\hat{\hat{u}}$ | Axial displacement of dynamic state |
| $\tilde{u}$ | Nondimensional axial displacement of dynamic state $\tilde{u} = h\hat{\hat{u}}$ |
| $\underset{\sim}{V}$ | Solution vector defined in Eq. (6-4-3) |
| $\underset{\sim}{v}$ | Vector used in backward integration (see Eq. (6-4-2)) |
| $\hat{\underset{\sim}{v}}$ | Vector used in backward integration (see Eq. (6-4-13)) |
| $v$ | Circumferential displacement (total), $v = \hat{v} + \hat{\hat{v}}$ |

| | |
|---|---|
| $\hat{v}$ | Circumferential displacement of fundamental state |
| $\hat{\hat{v}}$ | Circumferential displacement of dynamic state |
| $W$ | Radial displacement (total), $w = \hat{w} + \hat{\hat{w}}$ |
| $\hat{W}$ | Radial displacement of fundamental state |
| $\hat{\hat{W}}$ | Radial displacement of dynamic state |
| $\bar{W}$ | Initial geometric imperfection |
| $\hat{W}_v, \hat{W}_o, \hat{W}_1$ | Components of $\hat{W}$ |
| $\underset{\sim}{W}_i$ | Vector defined in Eq. (6-4-8) |
| $\underset{\sim}{Y}$ | Unified vector variable defined in Eq. (6-2-40) |
| $\dot{\underset{\sim}{Y}}$ | Unified vector variable defined in Eq. (6-3-12) |
| $x, y, z$ | Coordinates |
| $\bar{x}, \bar{y}$ | Nondimensional coordinates, $\bar{x} = \frac{x}{R}$, $\bar{y} = \frac{y}{R}$ |
| $\alpha_i, \alpha_k, \alpha_\ell, \alpha_m, \alpha_n$ | Nondimensional wave numbers |
| $\alpha_{s1}, \alpha_{s2}, \ldots, \alpha_{s7}$ | Coefficients of Eq. (4-2-1) |
| $\bar{\alpha}_{s1}, \bar{\alpha}_{s2}, \ldots, \bar{\alpha}_{s6}$ | Coefficients of Eq. (4-2-8) |
| $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{13}$ | Coefficients of Eq. (3-2-1) |
| $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_5$ | Wave number parameters defined in Eqs. (6-2-7) and (6-2-8) |
| $\hat{\beta}$ | Stiffener parameter |
| $\beta_1, \beta_2, \ldots, \beta_{10}$ | Coefficients of Eqs. (2-2-15) and (2-2-16) |
| $\beta_{e1}, \beta_{e2}, \ldots, \beta_{e5}$ | Coefficients of Eqs. (2-2-19) and (2-2-20) |

| | |
|---|---|
| $\beta_{n1},\beta_{n2},\ldots,\beta_{n10}$ | Coefficients of Eqs. (2-2-26) and (2-2-27) |
| $\beta_{s1},\beta_{s2},\ldots,\beta_{s6}$ | Coefficients of Eq. (4-2-2) |
| $\bar{\beta}_{n1},\bar{\beta}_{n2},\ldots\bar{\beta}_{n10}$ | Coefficients of Eqs. (2-2-28) and (2-2-29) |
| $\bar{\beta}_{1},\bar{\beta}_{2},\ldots,\bar{\beta}_{10}$ | Coefficients of Eq. (3-2-2) |
| $\bar{\beta}_{s1},\bar{\beta}_{s2},\ldots,\bar{\beta}_{s6}$ | Coefficients of Eq. (4-2-9) |
| $\beta_{i1},\beta_{i2},\ldots,\beta_{i4}$ $(i=1,2)$ | Coefficients of Eq. (5-3-7) |
| $\gamma$ | Nondimensional damping coefficient $\gamma = \bar{c}R\sqrt{\frac{1}{2\bar{\rho}E}}$ |
| $\gamma_s$ | Percentage of critical damping $\gamma_s = \frac{\bar{c}}{2\bar{\rho}\bar{\omega}_{mn}}$ |
| $\gamma_e$ | Coefficient of Eq. (2-2-22) and (2-2-23) |
| $\underset{\sim}{\gamma}$ | Vector defined in Eq. (6-4-16) |
| $\gamma_{D,k,\ell},\gamma_{Q,k,\ell},\gamma_{H,k,\ell}$ | Extended stiffener parameters |
| $\rho,\rho_s,\rho_r$ | Specific mass of shell wall, stringers and rings respectively |
| $\bar{\rho}$ | Specific mass of combined shell wall (including mass of smeared-out stiffness) |
| $\sigma$ | applied axial compressive stress |
| $\sigma_{c\ell}$ | classical buckling stress $\left(= \frac{Et}{cR}\right)$ |
| $\varepsilon$ | Small parameter, $\varepsilon = \left(\frac{n^2h}{R}\right)^2$ |
| $\varepsilon_i\,(i=0,1,\ldots,5)$ | Coefficients of Eq. (6-3-9) |
| $\bar{\varepsilon}_2$ | Average value of component $\varepsilon_2$ of $\varepsilon_i$ |
| $\varepsilon_{c\ell}$ | Parameter used in Chapter 6$\left(\varepsilon_{c\ell} = \frac{1}{c}\frac{h}{R}\right)$ |

| | |
|---|---|
| $\tilde{\varepsilon}_2$ | Parameter defined in Eq. (6-3-11), $\tilde{\varepsilon}_2 = \dfrac{\bar{\varepsilon}_2}{\varepsilon_{c\ell}}$ |
| $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ | Total axial, circumferential and shearing strain respectively, $\varepsilon_x = \hat{\varepsilon}_x + \hat{\hat{\varepsilon}}_x$, $\varepsilon_y = \hat{\varepsilon}_y + \hat{\hat{\varepsilon}}_y$, $\gamma_{xy} = \hat{\gamma}_{xy} + \hat{\hat{\gamma}}_{xy}$ |
| $\hat{\varepsilon}_x, \hat{\varepsilon}_y, \hat{\gamma}_{xy}$ | Strains of fundamental state |
| $\hat{\hat{\varepsilon}}_x, \hat{\hat{\varepsilon}}_y, \hat{\hat{\gamma}}_{xy}$ | Strains of dynamic state |
| $\lambda$ | Nondimensional static load parameter $\left(= \dfrac{\sigma}{\sigma_{c\ell}}\right)$ |
| $\eta(\tau)$ | Small perturbation in the amplitude of companion mode (see Eq. (4-2-5)) |
| $\eta_{01}, \eta_{02}$ | Stiffener parameters |
| $\eta_{t1}, \eta_{t2}$ | Stiffener parameters |
| $\Omega^2$ | $= \omega^2/(E/2\bar{\rho}R^2)$ |
| $\Omega_E$ | Evensen's frequency parameter (see Ref. 30) |
| $\Omega$ | $= \omega/\bar{\omega}_{mn}$ |
| $\Delta\Omega$ | Frequency step used in integration procedure |
| $\omega$ | Circular frequency of vibration |
| $\bar{\omega}_{mn}$ | Linear natural frequency $\bar{\omega}^2_{mn} = \dfrac{1}{2}\dfrac{\tilde{\beta}E}{\bar{\rho}R^2}$ (for $\tilde{\beta}$ see Appendix 2-A4, p. 190) |
| $\mu_1, \mu_2$ | Stiffener parameters |
| $\nu, \nu_s, \nu_r$ | Poisson's ratio of shell wall, stringers and rings respectively |
| $\delta$ | End-shortening (total) |
| $\delta_e$ | Coefficient of Eqs. (2-2-22) and (2-2-23) |

| | |
|---|---|
| $\delta_o$ | End-shortening |
| $\delta_1, \delta_2$ | Amplitudes of axisymmetric and asymmetric imperfection |
| $\hat{\delta}_0, \hat{\delta}_1, \hat{\delta}_2$ | Amplitudes of the fundamental solution |
| $\delta_{ik}$ | Kronecker delta function |
| $\xi$ | Aspect ratio, $\xi = \frac{m\pi/L}{n/R}$ |
| $\xi_1, \xi_2$ | Parameters defined in Eq. (5-2-2) |
| $\chi$ | Eigenvalue defined in Eq. (4-2-19) |
| $\chi_1, \chi_2$ | Stiffener parameters |
| $\zeta_1, \zeta_2$ | Stiffener parameters |
| $\zeta(\tau)$ | Small perturbation in the amplitude of driven mode (See Eq. (4-2-4)) |
| $\Phi$ | Stress function (total), $\Phi = \hat{\Phi} + \hat{\hat{\Phi}}$ |
| $\hat{\Phi}$ | Stress function of the fundamental state |
| $\hat{\hat{\Phi}}$ | Stress function of the dynamic state |
| $\hat{\phi}_i$ (i=0,1,2) | Components of fundamental stress function $\hat{\phi}$ |
| $\hat{\hat{\phi}}_j$ (j=1,2,...,9) | Components of dynamic stress function $\hat{\hat{\Phi}}$ |
| $\hat{\hat{\phi}}_k$ | Components of $\hat{\hat{\phi}}_j$ (k=11,12,13,21,...,93) |
| $\phi$ | Phase angle of driven mode |
| $\underset{\sim}{\phi}$ | Matching function used in Eq. (6-4-4) |
| $\underset{\sim}{\hat{\phi}}$ | Matching function used in Eq. (6-4-16) |
| $\bar{\phi}$ | Average value of $\phi$ |

| | |
|---|---|
| $\{\phi\}$ | Column matrix in Eq. (4-2-18) |
| $\psi$ | Phase angle of companion mode |
| $\bar{\psi}$ | Average value of $\psi$ |
| $\Delta$ | Difference of phase angles, $\Delta = \phi - \psi$ |
| $\bar{\Delta}$ | Average value of $\Delta$ |
| $\tau$ | Nondimensional time, $\tau = \bar{\omega}_{mn} t$ |
| $\varepsilon_{2u}, \varepsilon_{2v}$ | End-shortening in Eqs. (6-4-18) and (6-4-19) |
| $(\ )_{,x}$ | $\frac{\partial(\ )}{\partial x}$ |
| $(\ )_{,xx}$ | $\frac{\partial^2(\ )}{\partial x^2}$ |
| $(\ )'$ | $\frac{d(\ )}{dx}$ |
| $(\dot{\ })$ | $\frac{d(\ )}{dt}$ |

## INTRODUCTION

In modern engineering design, stiffened and unstiffened shells play an important role when it comes to weight critical applications, since these thin walled structures exhibit very favorable strength over weight ratios. Considerable research efforts have been devoted in the past to the strength and stability analysis of such structures. For extensive reviews the reader should consult [79]. The whole dilemma of the stability analysis of axially compressed cylindrical shells is well illustrated in Fig. 1, where some of the available experimental results for isotropic shells have been plotted as a function of the 'thinness' parameter R/h. The cause for the wide experimental scatter and for the poor correlation between the predictions based on a linearized small deflection theory with SS3 ($N_x=v=W=M_x=0$) boundary conditions and the experimental values is attributed to three factors: the influence of nonlinearity of materials, the influence of initial geometric imperfections and the effect of boundary conditions.

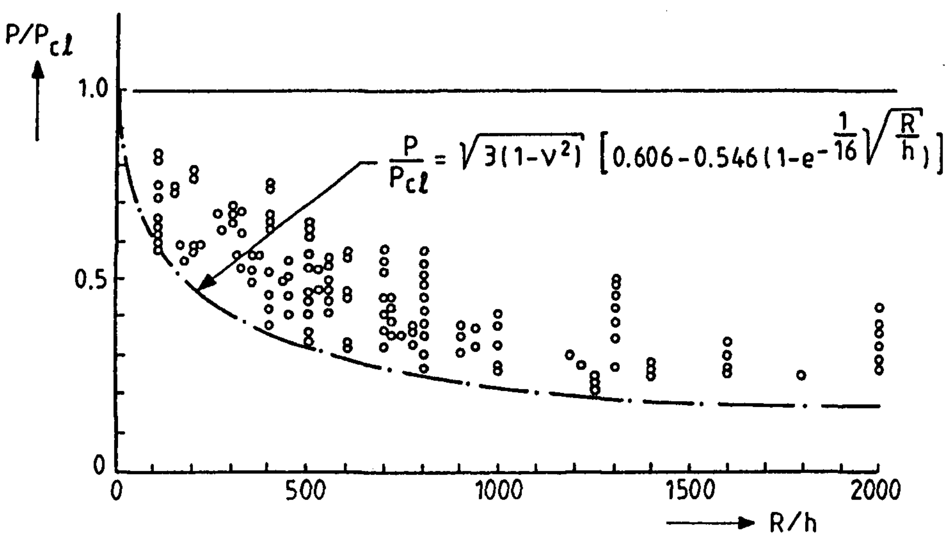



Fig. 1. Test data for isotropic cylindrical shell under axial compression [79].

In recent years the emphasis has been shifting towards the study of the dynamic characteristics of preloaded shell structures. Numerous investigations have been devoted to vibration analysis of shells. An excellent survey prior to 1973 can be found in Ref. [150].

The first paper that dealt with nonlinear vibrations of shells was the pioneering work of Reissner [49]. As stated therein, the earlier investigations of the vibration of thin elastic shells were all based on linearized theories. In Reissner's paper the problems of nonlinear vibrations of a cylinder were analyzed using Donnell's shallow-shell equations. His results indicated that nonlinearity of shell vibration could be either of the hardening or softening type, depending on the geometry of the single half-wave chosen to be analyzed.

Chu [67] employed the same assumed mode shape as that of Reissner's but he proceeded somewhat differently. His results indicated that the nonlinearity was always of the hardening type and could be strong in some cases. Cummings [18] employed a Galerkin procedure and found that the results varied with the region of integration. The results over a single half-wave were the same as those of

Reissner. The results for a complete shell were similar to those of Chu. Thus, it appeared that Reissner's results were characteristic of curved panels whereas Chu's calculations were apparently applicable to complete cylindrical shells. All of these analyses did not investigate the problem of traveling wave and the boundary conditions were only partially satisfied. Also the circumferential periodicity condition was violated by Chu.

Nowinski [91] applied Galerkin's procedure with an additional axisymmetric term in the assumed deflection shape in order to satisfy the circumferential periodicity condition. His results were virtually identical to those of Chu in the isotropic case. However, his assumed deflection shape did not satisfy $W = 0$ at both ends of the shell.

An important contribution to the theory of nonlinear shell vibration was made by Evensen in 1964 [30], who introduced for the first time the companion mode in the vibration analysis of rings to investigate the travelling wave. Subsequently he extended this procedure to the nonlinear vibrations of shells. The mode shape Evensen assumed in his Galerkin procedure satisfied the circumferential periodicity condition rigorously and the simply supported boundary conditions at the shell edges approximately. His results for the shell included the travelling wave response and a stability analysis which indicated the stability region of the standing wave response and travelling wave response in the case without damping. Evensen's results indicated that nonlinearity was either softening or hardening depending upon the aspect ratio $\xi$.

Dowell and Ventres [51] made an analysis similar to that of Evensen with a slightly different axisymmetric mode term in the assumed deflection shape. In this analysis all the simply supported boundary conditions and circumferential periodicity conditions were satisfied 'on the average'. Although no numerical results were given, the modal equations obtained in the limiting case of $L/R \to \infty$ agreed with those of ring equations and $L/R \to 0$ agreed with that of plate equations.
Matsuzaki and Kobayashi [154 ~ 156] carried out an analysis on a cylindrical shell with clamped ends. Their method was also similar to that of Evensen. The results showed the nonlinearity being of the softening type.

It is of interest to review the analysis of the references mentioned above, since these papers used the same Donnell's shallow shell equations and most of them considered simply supported boundary conditions. Also these publications contain the main results of the early investigations of the problem of nonlinear vibrations of shells.

The conclusions emerging from these early studies clearly indicate the following:
1. The mode shapes used had been chosen primarily on the basis of intuition and not by any systematic procedure. They play an important role in the analysis and the results are somewhat dictated by these assumed mode shapes.
2. In-plane inertia effects were generally neglected.
3. Specified boundary conditions are not enforced rigorously.
4. The approximation of shallow shell theory restricted the validity of the analysis to high circumferential wave numbers.

An analysis which corrected many of the above mentioned shortcomings was performed by Bleich and Ginsberg in 1970 [65], who studied nonlinear forced

vibrations of infinitely long cylindrical shells using the so-called modal expansion method.
Their solutions showed that damping has a pronounced influence on the response. Ginsberg subsequently extended his approach to shells of finite length [89].

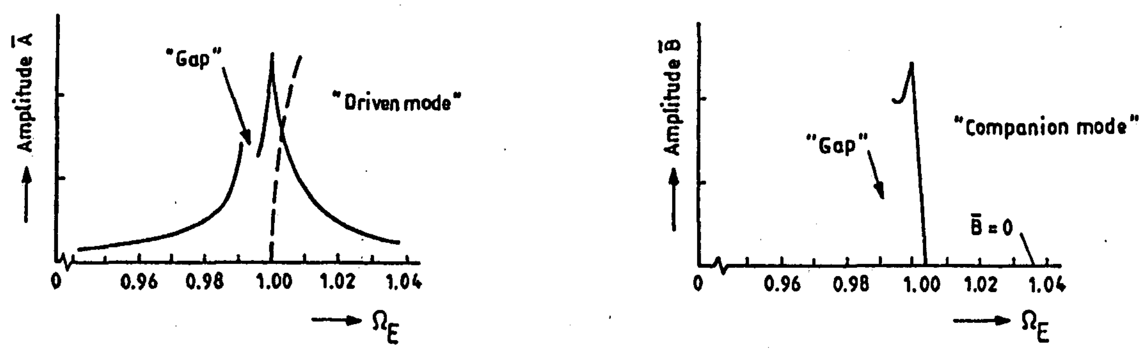
An alternative approach to the problem was taken by Chen [82], who applied a systematic perturbation procedure to the set of governing partial differential equations. By this systematic perturbation approach, for both the differential equations and the boundary conditions, Chen generated an axisymmetric term and the second harmonic terms similar to those encountered by Bleich and Ginsberg. Chen's solution also showed that nonlinear edge effects from both the edge moments and the in-plane boundary condition propagate towards the middle of the shell from the boundaries. Accordingly, when the shells are 'sufficiently long and thin-walled' the boundary effects become negligible.

Some of Evensen, Chen as well as Ginsberg's results are shown in Fig. 2. The differences within them are quite obvious. For example, the 'gap' phenomena in Evensen's solution was not predicted by either Chen or Ginsberg, the peak response obtained by Ginsberg was not discovered by Chen.

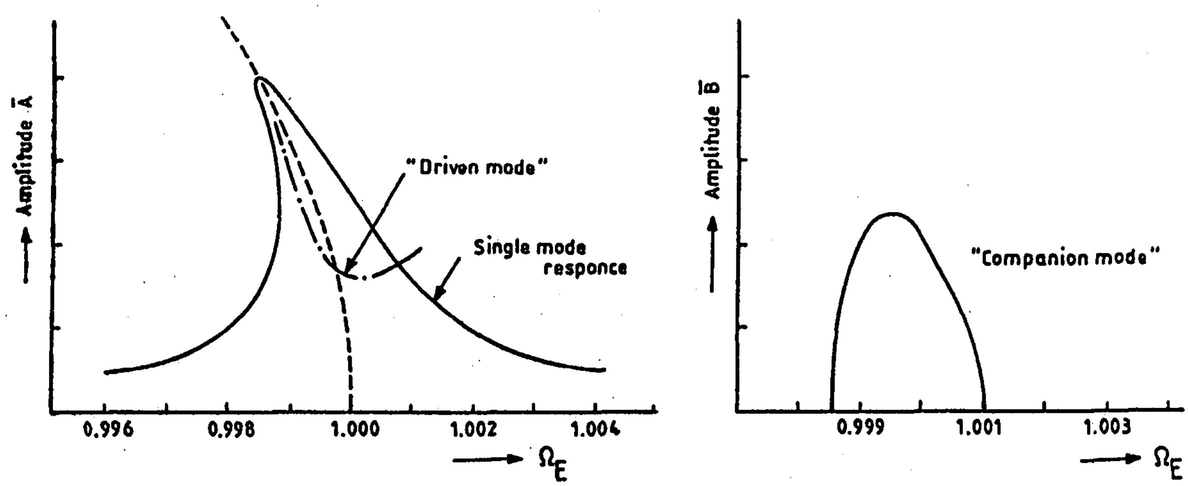
Raju and Rao published a finite element solution to the large amplitude vibrations of thin shells of revolution, obtaining a frequency-amplitude relationship of a hardening nature for a circular cylindrical shell in 1976 [109]. This caused a controversy about the vibration behaviour of shells [36], [61], [62]. Evensen indicated two errors in Raju and Rao's analysis. The main one was that the mode shape selected in Raju and Rao's analysis forced the shell to stretch. This is contradictory to the nature of the problem since thin shells bend more readily than they stretch. Later Ueda [143] studied nonlinear vibration of the conical shell using a finite element method. An axisymmetric term independent of the circumferential coordinate was included in his assumed mode for the radial displacement. His results indicated that nonlinearity was softening.

Still noteworthy are the studies of Atluri [136], Radwan and Genin [68] and Harari [2]. Another paper available is that of Yamaki [129]. He presented a proper formulation of the nonlinear vibrations of shells and outlined two promising methods of solution; however he did not obtain any actual solution. The latest paper available is that of Nayfeh and Raouf [4], in which the modal interactions in the response of shells were studied, which were initiated by McIvor [71,72].
The facts one can observe from all these studies are:

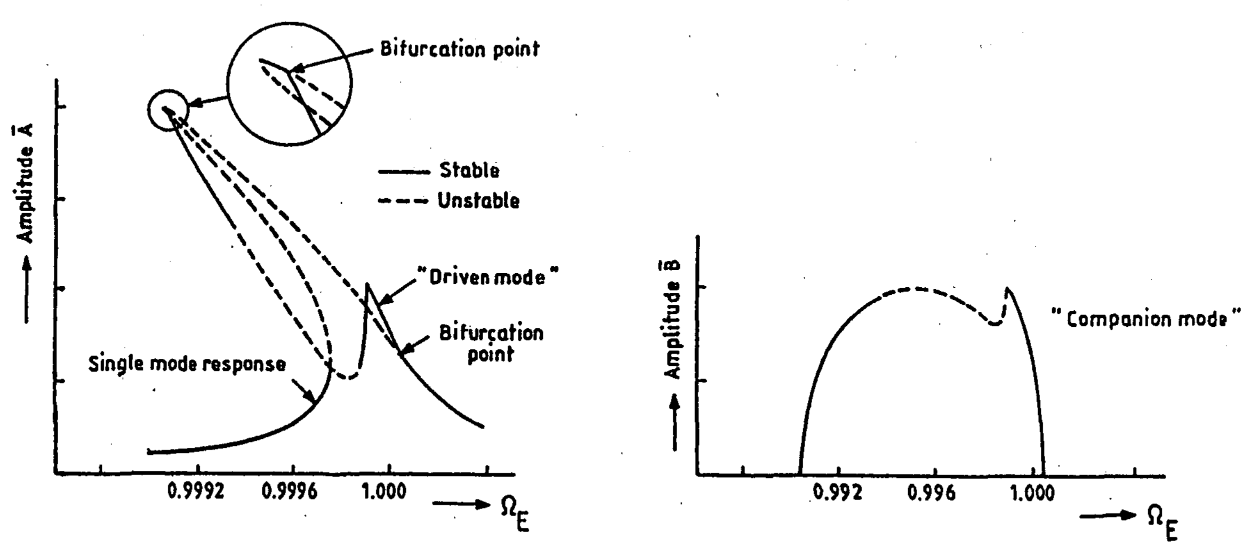
1. Galerkin's method was used in most of the analyses and proved to be by far the simplest method in the investigations of nonlinear vibrations of shells. Galerkin's procedure provides a very powerful approximate method that reduces a system of nonlinear partial differential equations into a system of nonlinear ordinary differential equations which becomes manageable. Also Galerkin's method provides insight into the nonlinear coupling of various vibration modes during the solution procedure. However, its results are highly dependent on the assumed deflection shape. Completely different results can be obtained by differences in the assumed deflection shape as can be seen from the investigations mentioned above.




a. Evensen's results [31]




b. Chen's results [89]




c. Ginsberg's results [82]

Fig. 2. On the non-linear vibrations of shells.

2. Nonlinear effects of large amplitude vibrations of cylindrical shells are demonstrated by two phenomena; namely, the shape of the response-frequency relationship in the vicinity of a resonant frequency (single response) and the occurrence of travelling wave response (coupled-mode response).
3. Agreement between results predicted by different theories and procedures is not satisfactory, expecially for the case of coupled mode response.

There exists certain correlation between buckling and vibration problems of shell since they are both related to the stiffeness of the shell walls. The three factors mentioned before, which could influence buckling behaviour of shell, therefore could also influence vibration behaviour of shell. In fact, the influence of initial geometric imperfection on vibration behaviour of shell has been studied by several investigators in recent years [11,15,102,116].

Rosen and Singer studied the influence of the initial axisymmetric imperfection on the vibration of isotropic shells under axial compression in 1974 [11]. This study was essentially an extension of the Koiter [153] analysis for buckling. The radial inertia term was added to Koiter's formulation directly. It was found that such imperfections have a strong influence on the frequency of the vibration, similar to that on the buckling load of cylindrical shells, not only at high compressive loads but also at zero axial load. The study was extended to asymmetric imperfections [15] and to stiffened shells for both axisymmetric and asymmetric imperfections [102].

Watawala and Nash studied the influence of a single asymmetric imperfection on the nonlinear undamped free and forced vibration problem of simply supported isotropic shells in 1982 [116] by introducing the appropriate terms for the imperfections, the radial inertia and the excitation into the nonlinear Donnell equations of shallow shells. The procedure they used is similar to one by Evensen. The solutions were obtained for the case of single mode response.

A recent study on the influence of both axisymmetric and asymmetric imperfections on the vibration of prestressed orthotropic shells was performed by Hol [94]. In his analysis Hol used the Donnell nonlinear equations written in terms of displacement u, v and W. Utilizing a procedure similar to the one used by Rosen and Singer yields the governing differential equations for the fundamental state and dynamic state respectively. Further, Hol's analysis consisted of two parts. First an approximate solution for the fundamental state governed by the full non-linear equations was obtained. This solution incorporates the effects of the imperfections and the applied axial loading. The in-plane restrictions of the classical simply supported boundary conditions and the periodicity requirement were satisfied 'on the average'. Next a solution for the superposed dynamic state was obtained, based on linearized governing equations in which axial and circumferential inertia were neglected. The in-plane boundary conditions were also satisfied 'on the average'. However, neither in the analysis of the fundamental state nor in the solution of the dynamic state were the out-plane boundary conditions satisfied rigorously.

Fig. 3 shows the relationships of frequencies of vibration vs amplitudes of asymmetric initial geometric imperfections obtained by Rosen and Singer and by Watawala and Nash in the case, where the circumferential wave number $\ell$ of vibration mode is equal to n, the circumferential wave number of initial geometric imperfection mode. The curve showing Hol's linearized results is also included. As shown in Fig. 3 both Rosen and Singer, Watawala and Nash's results indicate that initial geometric imperfection could have a significant influence on the vibration of shells, however they predict completely contradictory behaviour. This considerable discrepecy was attributed to the fact that Rosen and Singer's treatment did not satisfy the circumferential periodicity requirement [116].

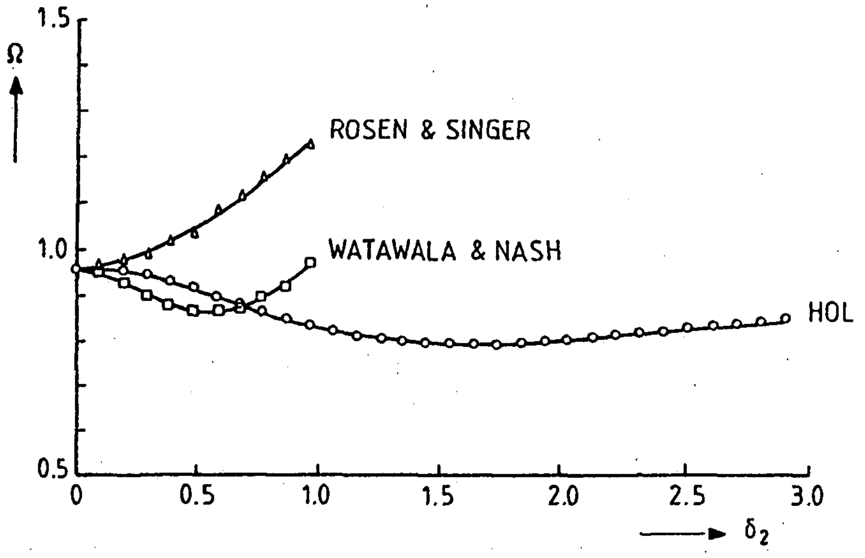



Fig. 3 Natural frequencies vs asymetric imperfections.

Comparing the results of Watawala and Nash with those of Hol's (see Figure 3), reveals that the general trend in the results is the same, but the agreement is not fully satisfactory. This is hard to explain since both analysis used the Donnell theory and satisfied the circumferential periodicity condition. The only difference between them is that Watalwala and Nash's analysis satisfied all the out-plane conditions of classical simply supported boundary except moment free $M_x = 0$ but violated in-plane conditions, while Hol's analysis satisfied the in-plane conditions 'on the average' but violated the out-plane condition $W = 0$ and $M_x = 0$ at the ends of the shell. It is not expected that such differences could result in the disagreement shown in Fig. 3.

Summing up the studies mentioned above one can conclude that:

1. Initial geometric imperfections have a significant influence on the vibrations of thin-walled cylindrical shells.
2. Agreement between the results available is by far unsatisfactory. These results are not yet sufficient to explore fully the behaviour of imperfect shells.
3. Previous investigations were concentrated upon the case of single mode response. No attention has been payed to the coupled mode response.

Boundary conditions have also a considerable influence on the vibration of shell, which have been discussed by many investigators [6]. Yu developed in 1955 [158] the perhaps most simple and general method to obtain natural frequencies and modes for various boundary conditions.

Forsberg carried out extensive studies using linear theory in 1966 [106], in which all sixteen sets of homogeneous boundary conditions were examined at each shell end. The equations of motion developed by Flügge for thin, circular cylindrical shells were used. His results indicated that contrary to the rather common assumption, the condition placed on the axial displacement in many cases is more influential than restrictions on the slope $\partial W/\partial x$ or moment $M_x$.

Nuckolls and Egle investigated the vibrations of a shell with one end on simple supports (SS3) and the other on springs [22]. The effect of varying elastic restraints on the natural frequencies and resonant displacement of a thin circular cylindrical shell excited by a concentrated load with a simple harmonic time history is studied through a Laplace transform solution of the Donnell shell equations. Numerical results for a wide range of three boundary flexibilities (axial, rotational and transverse) show that, for shells with length/radius = 1, the transverse flexibility has the strongest and the axial flexibility the weakest influence on the resonant displacement. Their analysis is also based on the linear theory.

El-Raheb and Babcock [120] studied the vibration of a cylindrical shell with end rings, and found that the end rings noticeably influenced the frequencies and modes of vibrations.

Penzes and Kraus [113] developed a solution for the free vibrations of orthotropic rotating cylindrical shells having arbitrary boundary conditions. The theory includes the combined effects of torsion, normal pressure, axial force. The emphasis of study was placed on the effect of torsion and rotation on natural frequencies.

A study by Greiff [133], which is also based on the linear theory, investigates the vibration characteristics of a cylindrical shell with arbitrary boundary conditions and with several intermediate constraints between the ends. The solution is obtained using a Rayleigh-Ritz procedure in which the axial displacement modes are constructed from simple Fourier series expressions. Geometric boundary conditions that are not identically satisfied are enforced with Lagrange multipliers. Unwanted geometric boundary conditions, forced to be zero due to the nature of the assumed series, are released through the mechanism of Stokes' transformation. Only the effect of intermediate constraint on the natural frequencies was studied in his study.

Harari [2] investigated the non-linear free vibration of prestressed plates and shells in a general form. The analysis includes the effect of in-plane inertia. The analysis is based on the non-linear equations of motion and uses a perturbation procedure. No assumption is made for the form of the time or space mode. The boundary conditions are treated in a general manner including boundary conditions where non-linear stress resultants are specified. In his paper no solution was given except equations.

Scedel developed a new formula, comparable with the one from Yu, for the natural frequencies of circular cylindrical shells in which transverse deflections dominate [151]. It is valid for all boundary conditions for which the roots of the analogous beam problem can be obtained.

Birman and Bert's study presents an exact solution of the problem of free beam-type vibration of a long cylindrical shell subjected to uniform axial tension, uniform internal pressure and elastic axial restraint [144]. The shell is flexurally clamped at the ends. The analysis results in a differential equation with cubic nonlinearities. The effects of flattening, stretching, pressuring and tension on the frequency of the fundamental mode of free vibrations are considered in numerical examples. Their results indicate that when axial restraint is present the frequencies of vibration increase.

Boundary conditions have also considerable influence on the vibrations of stiffened cylindrical shells as demonstrated clearly by Sewall and Naumann [93]. They investigated the effect of different boundary conditions on the vibrations of isotropic and stringer-stiffened cylindrical shells.

The vibrations of axially loaded stiffened shells were also studied theoretically and experimentally by Rosen and Singer [10,12,13,100]. They derived a linear theory for calculation of the influence of elastic edge restraints on the vibrations and buckling of stiffened cylindrical shells. The stiffeners are considered 'smeared' and the edge restraints can be axial, radial, circumferential or rotational. A method of definition of equivalent elastically restrained boundary conditions by use of vibration tests is also discussed. Their results show that boundary conditions have a very significant influence on the vibrations of stringer-stiffened cylindrical shells.

For an authoritative review of the many papers dealing with the vibration characteristics of thin cylindrical shells with different boundary conditions the interested reader should consult Reference [150]. Most of the works considered are based on the linear theory. It appears that the solutions available so far are not yet sufficient to explain fully all aspects of the experimentally observed finite amplitude vibration behaviour of thin walled shells.

As one of most widely used shell geometry the circular cylindrical shell has been thoroughly investigated. The various computer programs currently available (mostly based on the finite element method), allow one to obtain the natural frequencies and vibration modes of any reasonably thin circular shell for any combination of boundary conditions with an accuracy sufficient for most engineering applications. The question might therefore be asked why a further contribution in this area? The answer to this question lies in the fact that in science on should strive always toward a deeper, clearer and more accurate understanding of the physical phenomena involved. One of the conclusions that could be drawn from the solutions of previous studies is that although some basic characteristics on the vibration behaviours of shells have been derived analytically and verified experimentally, there are other areas where still considerable disagreement exists between results obtained by different procedures and between theoretical predictions and experimental evidences. This is especially true in the area of nonlinear vibrations, where the behaviour has not yet been fully explored. Further research therefore is necessary for the complete understanding of all aspects of the problem.

The major purpose of the thesis is to investigate the influence of initial geometric imperfections and the boundary conditions on the nonlinear vibration characteristics of thin cylindrical shells. The thesis consists of two parts. In the first part the nonlinear vibrations of imperfect thin-walled stiffened cylindrical shells is considered with SS3 boundary conditions at both ends, subjected to axial compression $N_o$ and lateral excitation q. Both single and combined initial geometric imperfection modes are considered. One of the objectives of this part is aimed to study the discrepencies existing in the previous investigations, and to obtain a reasonable explanation for them. The emphasis is placed on the influence of geometric imperfections on the coupled mode response a problem for which no solution as yet is available. The Donnell nonlinear differential equations for axially compressed stiffened shell with the simply supported boundary conditions at two ends are used. The 'smeared' theory is applied to treat stiffeners and rings. The Galerkin's method and the method of averaging are employed in sequence to obtain a set of coupled nonlinear algebraic equations, from which the frequency-amplitude relationship can be obtained for various damping ratios, amplitudes of excitations and imperfections. The stability of solutions is studied using the so-called method of slowly varying parameters.

In the second part of the thesis the influence of various boundary conditions on the nonlinear vibrations of imperfect cylindrical shells is investigated, which is the first step of the effort to study the effect of elastic boundary conditions on the nonlinear vibration of shells. The problem of determining the effects of elastic boundary conditions on dynamic response cannot be avoided because in the practical applications 'perfect' boundary conditions, for example the simply supported one, do not usually exist. In reality the boundary conditions are elastic or intermediate between the extreme of fixed and free. Once again Donnell's equations are used. The solution procedure used in this part is an extension of the one used by Arbocz for the buckling problem in Ref. [76]. By employing the same steps as used in part one Donnell's equations are reduced to a set of nonlinear first order ordinary differential equations with two sets of boundary conditions at the shell edges. The problem therefore becomes a 2-point boundary value problem. The numerical integration procedure called 'shooting method' is used in sequence to obtain the frequency-amplitude relationships and vibration modes for various boundary conditions.

# PART I

## Nonlinear Vibrations of Imperfect Thin-walled Cylindrical Shells with Simply Supported Boundary Conditions

# CHAPTER 1 BASIC THEORY AND METHOD

## 1.1 INTRODUCTION

This whole chapter describes the basic theory, assumptions and the methods used in the thesis. In section 1.2 the basic assumptions are explained. The governing equations for thin-walled stiffened cylindrical shells with initial geometric imperfections are developed in Section 1.3, according to Donnell's theory. The equations in terms of radial displacement W and Airy stress function $\Phi$ are then separated into two sets which are, governing the fundamental and dynamical state, respectively. In sections 1.4 and 1.5 the method of averaging and Galerkin's procedure are introduced briefly.

Equations governing both the fundamental and dynamic state in terms of the displacements u, v, W and the relative frequency-amplitude equations are also derived. For the sake of brevity they are not included in the thesis. Interested reader can refer to Ref. [47].

## 1.2 BASIC THEORY AND ASSUMPTIONS

The Donnell shallow shell equations (which involve additional assumptions) are used in the present analysis because of their relative simplicity. Many investigators have discussed their accuracy as compared to the 'exact' solution, for example, of Flügge's equations [106~108, 121,122]. It has been proven that the Donnell assumption is a high frequency approximation. The error introduced by the assumptions asymptotically decreases with increasing circumferential wave numbers $\ell$. The maximum error is small for thin-walled shells with short wave length modes. Consequently, the Donnell assumptions are valid for the dynamics of most finite length thin-walled shells of practical interest in the case of $\ell > 3$.

In order to permit the introduction of an Airy stress function, one neglects the in-plane inertia components in the dynamic equilibrium equations. The practical significance of this assumption was evaluated for isotropic shells in [108,121], where it was shown that it leads to slightly higher natural frequencies and that the magnitude of the error depends mainly on the circumferential wave number $\ell$ (the error decreases asymptotically with increasing values of $\ell$). Ref. [121] indicates that for isotropic shells the error in the natural frequencies introduced by the neglecting of the in-plane inertia components will remain practically unchanged for all boundary conditions for cases where $\ell > 3$. For stiffened shells, where the in-plane displacements are prevented at the ends, the natural frequencies will be significantly influenced by the in-plane inertia components. Since, however, the objective of the first part of the paper is to investigate the influence of initial geometric imperfections on the nonlinear vibration behaviour of stiffened and unstiffened shells, where only the simply supported boundary condition (SS3) is used (therefore the shell ends are free in the in-plane direction) it is expected that neglecting the in-plane inertia components will cause only small error.

Ref. [121] concluded that the error from all Donnell's simplifications and the neglecting of the in-plane inertia is of order $1/\ell^2$ for relatively large $\ell(\ell>3)$. In the present analysis the minimum circumferential wave number is $\ell = 5$, thus the above assumptions do not significantly influence the accuracy of the results,

The stiffeners are treated in the model by 'smeared stiffener' theory which involves the following assumptions [118].

a. The stiffeners are 'distributed over the whole surface of the shell'.
b. The normal strains $\varepsilon_x(z)$ and $\varepsilon_y(z)$ vary linearly in the stiffener as well as in the sheet. The normal strains in the stiffeners and in the sheet are equal at their point of contact.
c. The shear membrane force $N_{xy}$ is carried entirely by the sheet.
d. The torsional rigidity of the stiffener cross-section is added to that of the sheet.

In the present study the amplitudes of vibration are assumed 'finite' which cause geometric non-linearity, but they are still small enough to preclude non-linear material behaviour.

In the thesis the following two-term approximation for the imperfections is used, which contains both an axisymmetric and an asymmetric component:

$$\bar{W} = \delta_1 h \cos(\ell_i x) + \delta_2 h \sin(\ell_k x) \cos(\ell_n y) \qquad (1\text{-}2\text{-}1)$$

where $\delta_1$ and $\delta_2$ are the dimensionless amplitudes of axisymmetric and asymmetric imperfection respectively, h is the thickness of the shell, $\ell_i$, $\ell_k$ and $\ell_n$ are normalized wave numbers, which are defined in Appendix 1-A.2.
It is obvious that the actual shape of imperfections present in practical shell structures is quite arbitrary and cannot possibly be modelled by simple mathematical functions. The shapes vary from one shell to another depending on the fabrication procedure employed. Fig. 1.1 shows the practical distribution of a shell measured by Arbocz [79].

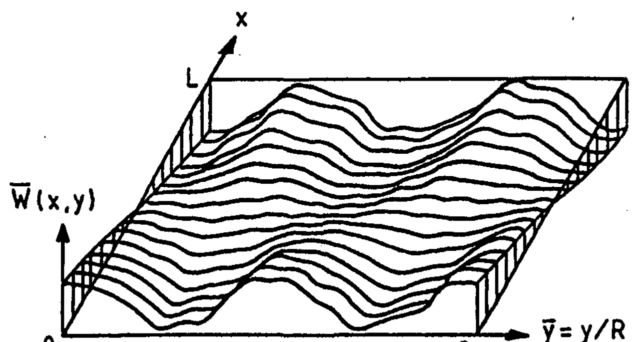



Fig. 1.1 Practical distribution of imperfections

The motivation behind the present investigation is to gain an insight into the problem and help contribute towards an understanding of the overall behaviour of the shell. Hence the simple trigonometric function (1-2-1) is selected in the present analysis.

Finally, the nonlinearity of the shell studied in the paper is assumed weak in order to be able to apply the method of averaging. The assumption comes from the conclusions made by analytical and experimental procedures, which show that the nonlinearity of practical thin-walled shells is indeed weak [82,119].

## 1.3 DEVELOPMENT OF THE BASIC EQUATIONS

The section contains the development of the basic equations that represent the mathematical model of a stiffened thin-walled cylindrical shell with initial geometric imperfections, which is axially compressed by the static load $N_o$ and laterally excited by the dynamic load q. The coordinates x, y, z and the displacements u, v, W and their positive directions are shown in Fig. 1.2a. Notice that the positive direction of W is inward.

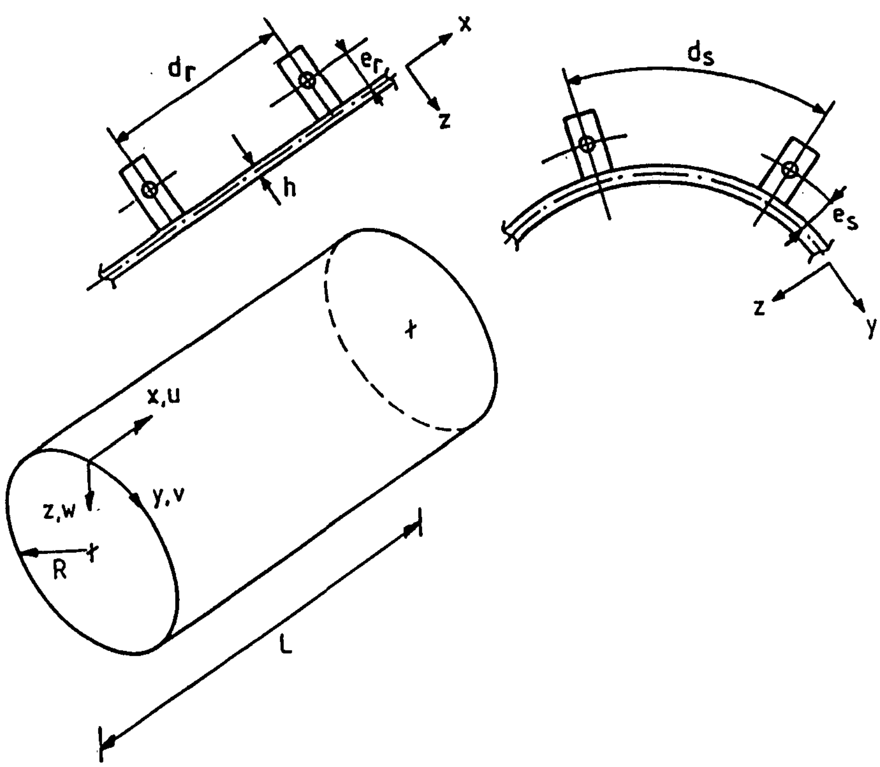



Fig. 1.2.a Shell Geometry

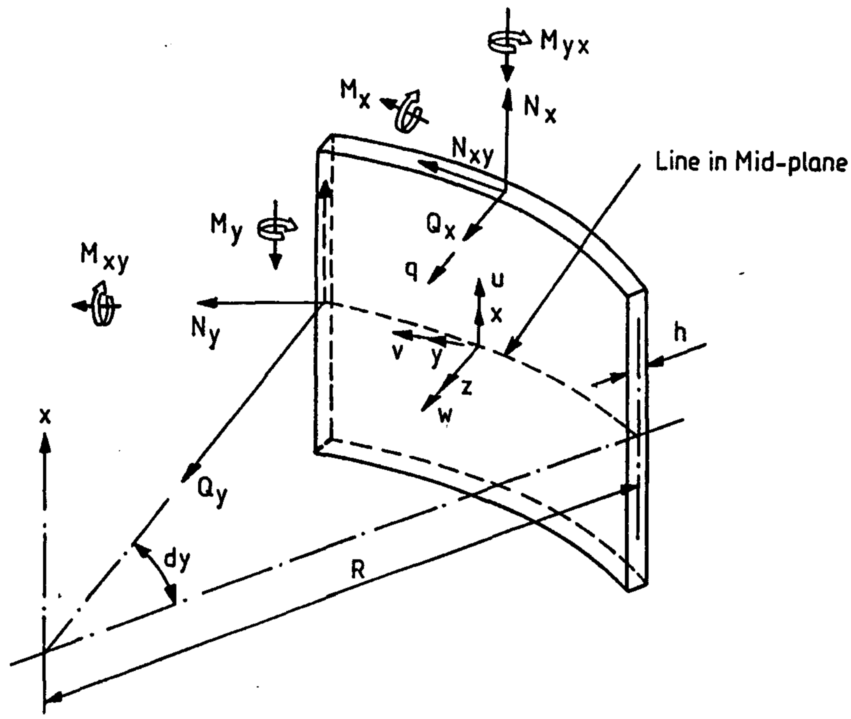



Fig. 1.2b Element of Cylindrical Shell

The development is based on an analytical approach similar to that used by Singer and Prucz [102]. The stiffeners of the shell are treated in the model by the 'smeared stiffener' theory.

Following the Donnell theory, the changes of the curvature and the twisting of the shell considered can be written as

$$\kappa_x = - W,_{xx} \tag{1-3-1}$$

$$\kappa_y = - W,_{yy} \tag{1-3-2}$$

$$\kappa_{xy} = - W,_{xy} \tag{1-3-3}$$

The equilibrium equations of the forces acting on the element shown in Fig. 1.2b in direction x and y are, respectively,

$$N_{x,x} + N_{yx,y} - \bar{\rho}h\ddot{u} - \bar{c}h\dot{u} = 0 \tag{1-3-4}$$

$$N_{xy,x} + N_{y,y} - \bar{\rho}h\ddot{v} - \bar{c}h\dot{v} = 0 \tag{1-3-5}$$

where $\bar{\rho}h$ denotes the mass of the shell per unit area, $(\dot{\ }) = \frac{d(\ )}{dt}$, $\bar{c}$ is the damping factor and t is the time.
The equilibrium equations of the moments acting about the axes x and y are, respectively,

$$M_{y,y} - M_{xy,x} - Q_y = 0 \qquad (1\text{-}3\text{-}6)$$

$$M_{x,x} + M_{yx,y} - Q_x = 0 \qquad (1\text{-}3\text{-}7)$$

The equation of equilibrium of the radial forces can be written,

$$M_{x,xx} + M_{yx,xy} + M_{xy,xy} + M_{y,yy} + \frac{N_y}{R} + N_x(W + \bar{W})_{,xx} + N_y(W + \bar{W})_{,yy} +$$

$$+ 2N_{xy}(W + \bar{W})_{,xy} - \bar{\rho}h\ddot{W} - \bar{c}h\dot{W} + q = 0 \qquad (1\text{-}3\text{-}8)$$

Substitution of $Q_x$ and $Q_y$ from (1-3-6) and (1-3-7) into (1-3-8) yields

$$M_{x,xx} + M_{yx,xy} + M_{xy,xy} + M_{y,yy} + \frac{N_y}{R} + N_x(W + \bar{W})_{,xx} +$$

$$+ N_y(W + \bar{W})_{,yy} + 2N_{xy}(W + \bar{W})_{,xy} - \bar{\rho}h\ddot{W} - \bar{c}h\dot{W} + q = 0 \qquad (1\text{-}3\text{-}9)$$

For isotropic shells, $M_{xy} = M_{yx}$.

The assumption of 'finite' radial displacements used herein requires the consideration of non-linear effects in expressing the relationships between the components of strain and deformation. For an imperfect shell these non-linear relations are

$$\varepsilon_x = u,_x + \frac{1}{2}(W,_x)^2 + W,_x \bar{W},_x \qquad (1\text{-}3\text{-}10)$$

$$\varepsilon_y = v,_y - \frac{W}{R} + \frac{1}{2}(W,_y)^2 + W,_y \bar{W},_y \qquad (1\text{-}3\text{-}11)$$

$$\gamma_{xy} = u,_y + v,_x + W,_x W,_y + W,_x \bar{W},_y + W,_y \bar{W},_x \qquad (1\text{-}3\text{-}12)$$

The compatibility equation for the displacements of an imperfect shell therefore becomes, from Eqs. (1-3-10) ~ (1-3-12)

$$\varepsilon_{x,yy} + \varepsilon_{y,xx} - \gamma_{xy,xy} = (W + \bar{W})^2,_{xy} - (\bar{W},_{xy})^2 -$$

$$- (W + \bar{W}),_{xx}(W + \bar{W}),_{yy} + \bar{W},_{xx} \bar{W},_{yy} - \frac{1}{R} W,_{xx} \qquad (1\text{-}3\text{-}13)$$

Assuming linear elastic behaviour of the shell material, the 'smeared stiffener' model leads to the following relations between the components of stress and strain.

-in the shell:

$$\sigma_x = \frac{E}{1-\nu^2}\left[\varepsilon_x + \nu\varepsilon_y + z(\kappa_x + \nu\kappa_y)\right] \tag{1-3-14}$$

$$\sigma_y = \frac{E}{1-\nu^2}\left[\varepsilon_y + \nu\varepsilon_x + z(\kappa_y + \nu\kappa_x)\right] \tag{1-3-15}$$

$$\tau_{xy} = G(\gamma_{xy} - 2z\kappa_{xy}) \tag{1-3-16}$$

-and in the stiffeners:

$$\sigma_x = E_1(\varepsilon_x + z\kappa_x) \tag{1-3-17}$$

$$\sigma_y = E_2(\varepsilon_y + z\kappa_y) \tag{1-3-18}$$

Substitution of expressions (1-3-10) ~ (1-3-12) into Eqs. (1-3-14) ~ (1-3-18) and integration of the resulting equations from - h/2 to h/2 yields the relationship between the stress resultants and couples and the strain components and curvature changes in the median surface of the shell,

$$N_x = \frac{Eh}{1-\nu^2}\left[(1 + \mu_1)\ \varepsilon_x + \nu\varepsilon_y + \chi_1\kappa_x\right] \tag{1-3-19}$$

$$N_y = \frac{Eh}{1-\nu^2}\left[1 + \mu_2)\ \varepsilon_y + \nu\varepsilon_x + \chi_2\kappa_y\right] \tag{1-3-20}$$

$$N_{xy} = N_{xy} = \frac{Eh}{2(1+\nu)}\ \gamma_{xy} \tag{1-3-21}$$

$$M_x = D\left[(1 + \eta_{01})\ \kappa_x + \nu\kappa_y + \zeta_1\varepsilon_x\right] \tag{1-3-22}$$

$$M_y = D\left[(1 + \eta_{02})\ \kappa_y + \nu\kappa_x + \zeta_2\varepsilon_y\right] \tag{1-3-23}$$

$$M_{xy} = D\left[(1 - \nu) + \eta_{t1}\right]\ \kappa_{xy} \tag{1-3-24}$$

where $D = \dfrac{Eh^3}{12(1-\nu)^2}$ and $\mu_1$, $\mu_2$, $\eta_{02}$, $\zeta_1$, $\zeta_2$, $\eta_{t1}$, $\eta_{t2}$, $\chi_1$ and $\chi_2$ are the 'smeared' stiffener parameters, which are defined in Appendix 1-A.

If in Eqs. (1-3-4) and (1-3-5) one neglects the in-plane inertia and damping terms then the in-plane equilibrium equations can be identically satisfied by introducing an Airy stress function $\Phi$ such that

$$N_x = \Phi,_{yy} \tag{1-3-25}$$

$$N_y = \Phi,_{xx} \tag{1-3-26}$$

$$N_{xy} = -\Phi,_{xy} \tag{1-3-27}$$

The equation of equilibrium of the radial forces (1-3-8) and the compatibility equation (1-3-13) then can be expressed in terms of the two unknowns W and $\Phi$ as

$$L_H(\Phi) - L_Q(W) = -\frac{1}{R} W,_{xx} - \frac{1}{2} L_{NL}(W, W+2\bar{W}) \tag{1-3-28}$$

$$L_Q(\Phi) + L_D(W) = +\frac{1}{R} \Phi,_{xx} + L_{NL}(\Phi, W+\bar{W}) - \bar{\rho}h\ddot{W} - \bar{c}h\dot{W} + q \tag{1-3-29}$$

where

$$\left.\begin{aligned} L_H(\;) &= H_{xx}(\;),_{xxxx} + H_{xy}(\;),_{xxyy} + H_{yy}(\;),_{yyyy} \\ L_Q(\;) &= Q_{xx}(\;),_{xxxx} + Q_{xy}(\;),_{xxyy} + Q_{yy}(\;),_{yyyy} \\ L_D(\;) &= D_{xx}(\;),_{xxxx} + D_{xy}(\;),_{xxyy} + D_{yy}(\;),_{yyyy} \\ L_{NL}(S,T) &= S,_{xx}T,_{yy} - 2S,_{xy}T,_{xy} + S,_{yy}T,_{xx} \end{aligned}\right. \tag{1-3-30}$$

The stiffener parameters $H_{xx}$, $Q_{xx}$, $D_{xx}$, .... etc. are given in Appendix 1-A.

One can also express the basic equations in terms of displacements u, v, W rather than W and $\Phi$. Substituting eqs (1-3-10) ~ (1-3-12) into eqs. (1-3-19) ~ (1-3-24) and then substituting resulted equations into equations (1-3-4), (1-3-5) and (1-3-9) yields the following equations after regrouping

$$\begin{aligned} (1+\mu_1)\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x \partial y} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} &= -(1+\mu_1)\left[\frac{\partial W}{\partial x}\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 \bar{W}}{\partial x^2}\right) + \frac{\partial^2 W}{\partial x^2}\frac{\partial \bar{W}}{\partial x}\right] + \\ &- \nu\left[-\frac{1}{R}\frac{\partial W}{\partial x} + \frac{\partial W}{\partial y}\left(\frac{\partial^2 W}{\partial x \partial y} + \frac{\partial^2 \bar{W}}{\partial x \partial y}\right) + \frac{\partial^2 W}{\partial x \partial y}\frac{\partial \bar{W}}{\partial y}\right] + \chi_1 \frac{\partial^3 W}{\partial x^3} + \\ &- \frac{1-\nu}{2}\left[\frac{\partial^2 W}{\partial x \partial y}\frac{\partial W}{\partial y} + \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial y^2} + \frac{\partial \bar{W}}{\partial y}\frac{\partial^2 W}{\partial x \partial y} + \frac{\partial W}{\partial x}\frac{\partial^2 \bar{W}}{\partial y^2} + \frac{\partial^2 \bar{W}}{\partial x \partial y}\frac{\partial W}{\partial y} + \frac{\partial \bar{W}}{\partial x}\frac{\partial^2 W}{\partial y^2}\right] \end{aligned} \tag{1-3-31}$$

$$\frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x \partial y} + (1+\mu_2)\frac{\partial^2 v}{\partial y^2} = -\nu\left[\frac{\partial W}{\partial x}\left(\frac{\partial^2 \bar{W}}{\partial x \partial y} + \frac{\partial^2 W}{\partial x \partial y}\right) + \frac{\partial^2 W}{\partial x \partial y}\frac{\partial \bar{W}}{\partial x}\right] -$$

$$-(1+\mu_2)\left[-\frac{1}{R}\frac{\partial W}{\partial y} + \frac{\partial W}{\partial y}\left(\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 \bar{W}}{\partial y^2}\right) + \frac{\partial \bar{W}}{\partial y}\frac{\partial^2 W}{\partial y^2}\right] + \chi_2\frac{\partial^3 W}{\partial y^3} +$$

$$-\frac{1-\nu}{2}\left[\frac{\partial^2 W}{\partial x^2}\left(\frac{\partial W}{\partial y} + \frac{\partial \bar{W}}{\partial y}\right) + \frac{\partial W}{\partial x}\left(\frac{\partial^2 W}{\partial x \partial y} + \frac{\partial^2 \bar{W}}{\partial x \partial y}\right) + \frac{\partial^2 \bar{W}}{\partial x^2}\frac{\partial W}{\partial y} + \frac{\partial \bar{W}}{\partial x}\frac{\partial^2 W}{\partial x \partial y}\right] \quad (1\text{-}3\text{-}32)$$

$$L_D(w) + L_Q(\Phi) = \frac{1}{R}\frac{\partial^2 \Phi}{\partial x^2} + L_{NL}(\Phi,\ W+\bar{W}) - \bar{\rho}h\frac{\partial^2 W}{\partial t^2} - \bar{c}h\frac{\partial W}{\partial t} + q \quad (1\text{-}3\text{-}33)$$

According to Koiter's theory [153], the displacements u, v, W and Airy stress function $\Phi$ of the shell while it is vibrating under an axially compressive load and lateral excitation can be expressed as a linear superposition of two independent states of displacement and stress, as shown in the following

$$\Phi = \hat{\Phi} + \hat{\hat{\Phi}} \quad (1\text{-}3\text{-}34)$$

$$u = \hat{u} + \hat{\hat{u}} \quad (1\text{-}3\text{-}35)$$

$$v = \hat{v} + \hat{\hat{v}} \quad (1\text{-}3\text{-}36)$$

$$W = \hat{W} + \hat{\hat{W}} \quad (1\text{-}3\text{-}37)$$

where $\hat{\Phi}$, $\hat{W}$, $\hat{u}$ and $\hat{v}$ are the stress function and displacements of the so-called fundamental, static, geometrical nonlinear state due to the imperfections of the shell and the application of a static axially compressed load $N_o$, and $\hat{\hat{\Phi}}$, $\hat{\hat{W}}$, $\hat{\hat{u}}$ and $\hat{\hat{v}}$ are the stress function and displacements of the so-called dynamic state due to small but not infinitesimal vibration about the fundamental state.

Upon substitution of equations (1-3-34) and (1-3-35) into equations (1-3-28) and (1-3-37), one obtains two sets of differential equations in terms of stress function and radial displacements governing the fundamental and the dynamical state, respectively. For the fundamental state these equations are

$$L_H(\hat{\Phi}) - L_Q(\hat{W}) = -\frac{1}{R}\frac{\partial^2 \hat{W}}{\partial x^2} - \frac{1}{2}L_{NL}(\hat{W},\ \hat{W} + 2\bar{W}) \quad (1\text{-}3\text{-}38)$$

$$L_Q(\hat{\Phi}) + L_D(\hat{W}) = \frac{1}{R}\frac{\partial^2 \hat{\Phi}}{\partial x^2} + L_{NL}(\hat{\Phi}, \hat{W} + \bar{W}) \tag{1-3-39}$$

while for the dynamic state the equations become

$$L_H(\hat{\hat{\Phi}}) - L_Q(\hat{\hat{W}}) = -\frac{1}{R}\frac{\partial^2 \hat{\hat{W}}}{\partial x^2} - \frac{1}{2} L_{NL}(\hat{W}, \hat{\hat{W}}) - \frac{1}{2} L_{NL}(\hat{\hat{W}}, \hat{W} + 2\bar{W}) + $$

$$- \frac{1}{2} L_{NL}(\hat{\hat{W}}, \hat{\hat{W}}) \tag{1-3-40}$$

$$L_Q(\hat{\hat{\Phi}}) + L_D(\hat{\hat{W}}) = \frac{1}{R}\frac{\partial^2 \hat{\hat{\Phi}}}{\partial x^2} + L_{NL}(\hat{\Phi}, \hat{\hat{W}}) + L_{NL}(\hat{\hat{\Phi}}, \hat{W} + \bar{W}) + $$

$$+ L_{NL}(\hat{\hat{\Phi}}, \hat{\hat{W}}) - \bar{\rho} h \frac{\partial^2 \hat{\hat{W}}}{\partial t^2} + q - \bar{c} h \frac{\partial \hat{\hat{W}}}{\partial t} \tag{1-3-41}$$

Similarly, upon substitution of the equations (1-3-34) ~ (1-3-37) into equations (1-3-31) ~ (1-3-33) one obtains two sets of equations in terms of displacements $\hat{u}$, $\hat{v}$, $\hat{W}$ and $\hat{\hat{u}}$, $\hat{\hat{v}}$, $\hat{\hat{W}}$, respectively. In such a case the governing equations of the fundamental state are

$$(1+\mu_1)\frac{\partial^2 \hat{u}}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 \hat{v}}{\partial x \partial y} + \frac{1-\nu}{2}\frac{\partial^2 \hat{u}}{\partial y^2} = -(1+\mu_1)\left[\frac{\partial \hat{W}}{\partial x}\frac{\partial^2 (\hat{W}+\bar{W})}{\partial x^2} + \frac{\partial^2 \hat{W}}{\partial x^2}\frac{\partial \bar{W}}{\partial x}\right] +$$

$$- \nu\left[-\frac{1}{R}\frac{\partial \hat{W}}{\partial x} + \frac{\partial \hat{W}}{\partial y}\frac{\partial^2 (\hat{W}+\bar{W})}{\partial x \partial y} + \frac{\partial^2 \hat{W}}{\partial x \partial y}\frac{\partial \bar{W}}{\partial y} + \chi_1 \frac{\partial^3 \hat{W}}{\partial x^3}\right] +$$

$$- \frac{1-\nu}{2}\left[\frac{\partial^2 \hat{W}}{\partial x \partial y}\frac{\partial \hat{W}}{\partial y} + \frac{\partial \hat{W}}{\partial x}\frac{\partial^2 (\hat{W}+\bar{W})}{\partial y^2} + \frac{\partial^2 \hat{W}}{\partial y \partial x}\frac{\partial \bar{W}}{\partial y} + \frac{\partial \hat{W}}{\partial y}\frac{\partial^2 \bar{W}}{\partial x \partial y} + \frac{\partial \bar{W}}{\partial x}\frac{\partial^2 \hat{W}}{\partial y^2}\right] \tag{1-3-42}$$

$$\frac{1-\nu}{2}\frac{\partial^2 \hat{v}}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 \hat{u}}{\partial x \partial y} + (1+\mu_2)\frac{\partial^2 \hat{v}}{\partial y^2} = -\nu\left[\frac{\partial \hat{W}}{\partial x}\frac{\partial^2(\hat{W}+\bar{W})}{\partial x \partial y} + \frac{\partial^2 \hat{W}}{\partial x \partial y}\frac{\partial \bar{W}}{\partial x}\right] +$$

$$-(1+\mu_2)\left[-\frac{1}{R}\frac{\partial \hat{W}}{\partial y} + \frac{\partial \hat{W}}{\partial y}\frac{\partial^2(\hat{W}+\bar{W})}{\partial y^2} + \frac{\partial^2 \hat{W}}{\partial y^2}\frac{\partial \bar{W}}{\partial y}\right] + \chi_2\frac{\partial^3 \hat{W}}{\partial y^3} +$$

$$-\frac{1-\nu}{2}\left[\frac{\partial^2 \hat{W}}{\partial x^2}\frac{\partial(\hat{W}+\bar{W})}{\partial y} + \frac{\partial \hat{W}}{\partial x}\frac{\partial^2(\hat{W}+\bar{W})}{\partial x \partial y} + \frac{\partial^2 \bar{W}}{\partial x^2}\frac{\partial \hat{W}}{\partial y} + \frac{\partial \bar{W}}{\partial x}\frac{\partial^2 \hat{W}}{\partial x \partial y}\right] \quad (1\text{-}3\text{-}43)$$

$$L_D(\hat{W}) + L_Q(\hat{\Phi}) = \frac{1}{R}\frac{\partial^2 \hat{\Phi}}{\partial x^2} + L_{NL}(\hat{\Phi},\ \hat{W}+\bar{W}) \quad (1\text{-}3\text{-}44)$$

while the equations governing the dynamic state are:

$$(1+\mu_1)\frac{\partial^2 \hat{\hat{u}}}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 \hat{\hat{v}}}{\partial x \partial y} + \frac{1-\nu}{2}\frac{\partial^2 \hat{\hat{u}}}{\partial y^2} = -(1+\mu_1)\left[\frac{\partial \hat{\hat{W}}}{\partial x}\frac{\partial^2(\hat{\hat{W}}+\hat{W}+\bar{W})}{\partial x^2} + \frac{\partial^2 \hat{\hat{W}}}{\partial x^2}\frac{\partial(\hat{W}+\bar{W})}{\partial x}\right] +$$

$$-\nu\left[-\frac{1}{R}\frac{\partial \hat{\hat{W}}}{\partial x} + \frac{\partial \hat{\hat{W}}}{\partial y}\frac{\partial^2(\hat{\hat{W}}+\hat{W}+\bar{W})}{\partial x \partial y} + \frac{\partial^2 \hat{\hat{W}}}{\partial x \partial y}\frac{\partial(\hat{W}+\bar{W})}{\partial y}\right] + \chi_1\frac{\partial^3 \hat{\hat{W}}}{\partial x^3} +$$

$$-\frac{1-\nu}{2}\left[\frac{\partial^2 \hat{\hat{W}}}{\partial x \partial y}\frac{\partial(\hat{\hat{W}}+\hat{W}+\bar{w})}{\partial y} + \frac{\partial \hat{\hat{W}}}{\partial x}\frac{\partial^2(\hat{\hat{W}}+\hat{W}+\bar{W})}{\partial y^2} + \frac{\partial \hat{\hat{W}}}{\partial y}\frac{\partial^2(\hat{W}+\bar{W})}{\partial x \partial y} + \frac{\partial^2 \hat{\hat{W}}}{\partial y^2}\frac{\partial(\hat{W}+\bar{W})}{\partial x}\right] \quad (1\text{-}3\text{-}45)$$

$$\frac{1-\nu}{2}\frac{\partial^2 \hat{\hat{v}}}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 \hat{\hat{u}}}{\partial x \partial y} + (1+\mu_2)\frac{\partial^2 \hat{\hat{v}}}{\partial y^2} = -\nu\left[\frac{\partial \hat{\hat{W}}}{\partial x}\frac{\partial^2(\hat{\hat{W}}+\hat{W}+\bar{W})}{\partial x \partial y} + \frac{\partial^2 \hat{\hat{W}}}{\partial x \partial y}\frac{\partial(\hat{W}+\bar{W})}{\partial x}\right] +$$

$$-(1+\mu_2)\left[-\frac{1}{R}\frac{\partial \hat{\hat{W}}}{\partial y} + \frac{\partial \hat{\hat{W}}}{\partial y}\frac{\partial^2(\hat{\hat{W}}+\hat{W}+\bar{W})}{\partial y^2} + \frac{\partial^2 \hat{\hat{W}}}{\partial y^2}\frac{\partial(\hat{W}+\bar{W})}{\partial y}\right] + \chi_2\frac{\partial^3 \hat{\hat{W}}}{\partial y^3} +$$

$$-\frac{1-\nu}{2}\left[\frac{\partial^2 \hat{\hat{W}}}{\partial x^2}\frac{\partial(\hat{\hat{W}}+\hat{W}+\bar{W})}{\partial y} + \frac{\partial \hat{\hat{W}}}{\partial x}\frac{\partial^2(\hat{\hat{W}}+\hat{W}+\bar{W})}{\partial x \partial y} + \frac{\partial \hat{\hat{W}}}{\partial y}\frac{\partial^2(\hat{W}+\bar{W})}{\partial x^2} + \frac{\partial^2 \hat{\hat{W}}}{\partial x \partial y}\frac{\partial(\hat{W}+\bar{W})}{\partial x}\right] \quad (1\text{-}3\text{-}46)$$

$$L_D(\hat{\hat{W}}) + L_Q(\hat{\hat{\Phi}}) = \frac{1}{R}\frac{\partial^2 \hat{\hat{\Phi}}}{\partial x^2} + L_{NL}(\hat{\Phi},\hat{\hat{W}}) + L_{NL}(\hat{\hat{\Phi}},\hat{W}+\bar{W}) + L_{NL}(\hat{\hat{\Phi}},\hat{\hat{W}}) - \bar{c}h\frac{\partial \hat{\hat{W}}}{\partial t} - \bar{\rho}h\frac{\partial^2 \hat{\hat{W}}}{\partial t^2} + q \quad (1\text{-}3\text{-}47)$$

## 1.4 THE METHOD OF AVERAGING

The method of averaging began to come into use a long time ago in the field of celestial mechanics, where different averaging patterns were applied. The principal idea of these patterns being that some average value is substituted for the perturbing function, so that simpler differential equations are obtained. In mechanics, however, the method of averaging remained unknown until the twenties of the present century, and only after the publication of the well known Van der Pol paper [126] did people take notice of it.

After the appearance of Bogoliuboff's fundamental works [123 ~ 127] dealing with its mathematical foundations, the method of averaging has been applied to a wide variety of problems dealing with nonlinear vibrations. Further, it has been found that using the basic ideas of the method of averaging as a point of departure, one can develop special methods which permit the construction of approximate solutions to any degree of accuracy desired.
In the present work the method of averaging is used in order to obtain a set of reduced equations which can be solved more readily. First of all the method is used to obtain simpler relations for the first and second order derivatives of a function u(t) with slowly varying amplitude a(t) and phase β(t). Thus if

$$u(t) = a(t) \cos [t + \beta(t)] \qquad (1\text{-}4\text{-}1)$$

then

$$\frac{du}{dt} = - a \sin (t+\beta) + \frac{da}{dt} \cos (t+\beta) - a \frac{d\beta}{dt} \sin (t+\beta) \qquad (1\text{-}4\text{-}2)$$

Using the assumption that a and β are slowly varying functions of time yields

$$\frac{da}{dt} \cos (t+\beta) - a \frac{d\beta}{dt} \sin (t+\beta) = 0 \qquad (1\text{-}4\text{-}3)$$

Hence

$$\frac{du}{dt} = - a \sin (t+\beta) \qquad (1\text{-}4\text{-}4)$$

and the second derivative becomes

$$\frac{d^2u}{dt^2} = - a \cos (t+\beta) - \frac{da}{dt} \sin (t+\beta) - a \frac{d\beta}{dt} \cos (t+\beta) \qquad (1\text{-}4\text{-}5)$$

These expressions then are substituted into the governing equation. After some regrouping, in the final state of the analysis, the equation is 'averaged' by integrating over one period of the vibration. In this intergration a(t) and β(t) are approximated by their average value $\bar{a}$ and $\bar{\beta}$. For example

$$\int_0^{2\pi} a(t) \cos^2 t \, dt = \bar{a} \pi \text{ etc.} \qquad (1\text{-}4\text{-}6)$$

See Appendix C for details of the derivations.

## 1.5 GALERKIN'S METHOD

The Galerkin's method, sometimes known as the method of weighting functions, is also used in the present analysis. The method has been proven to be a very powerful and simple approximation tool in reducing a set of nonlinear partial differential equations into a set of nonlinear ordinary differential equations which can be solved more readily. Also Galerkin's method provides insight in the nonlinear coupling of various vibration modes during the solution procedure.

Consider a function $\hat{W}$, which is assumed to be an approximate solution of equations (1-3-40) and (1-3-41)

$$\hat{W} = f(A,B,x,y) \qquad (1\text{-}5\text{-}1)$$

where A and B is the amplitude of assumed vibration modes. Theoretically, the right-hand side L(R) of (1-3-41) should be equal to the left-hand side L(L) after $\hat{W}$ is substituted into it, if $\hat{W}$ is its 'exact' solution.

Generally, however, this is impossible since $\hat{W}$ is only an approximate solution, the 'error' caused therefore is

$$\varepsilon = L(R) - L(L) \neq 0 \qquad (1\text{-}5\text{-}2)$$

The conditions that the weighted error integrated over the domain be zero, according to the Galerkin's method are

$$\int_0^{2\pi R} \int_0^{L} \varepsilon \frac{\partial \hat{W}}{\partial A} \, dxdy = 0 \qquad (1\text{-}5\text{-}3)$$

$$\int_0^{2\pi R} \int_0^{L} \varepsilon \frac{\partial \hat{W}}{\partial B} \, dxdy = 0 \qquad (1\text{-}5\text{-}4)$$

where $\partial\hat{W}/\partial A$ and $\partial\hat{W}/\partial B$ are known as the weighting functions, respectively.

The most important and also the most difficult problem in using the Galerkin's method is to choose accurate solution modes. Completely different results could be obtained by a small difference in the assumed solution modes. For the present analysis, fortunately, the choice is made easier since one can rely on the results obtained by previous investigators.

# CHAPTER 2 UNDAMPED NONLINEAR VIBRATIONS

## 2.1 INTRODUCTION

The undamped nonlinear vibrations of both perfect and imperfect thin cylindrical shells are studied in this Chapter. The initial geometric imperfection is modelled with a combination of one axisymmetric and one asymmetric trigonometric function. The simply supported boundary conditions (SS3) and the circumferential periodicity condition are satisfied. Two vibration modes are assumed in order to satisfy the requirement of the travelling wave, though only one of them is directly excited. Galerkin's procedure is employed to obtain two coupled nonlinear ordinary differential equations for the vibration amplitudes. The approximate natural frequencies and frequency-amplitude relationships for various amplitudes of initial geometric imperfection, and of excitation are calculated from these two equations using the method of averaging. The stability of these solutions then is studied using the method of slowly varying parameters in Chapter 4.

## 2.2 BASIC ASSUMPTIONS

For the present analysis a two-term approximation for initial geometric imperfection, as expressed in (1-2-1) is used.

The displacement mode for the fundamental state is assumed in the form

$$\hat{W} = \hat{\delta}_0 h + \hat{\delta}_1 h \cos(\ell_i x) + \hat{\delta}_2 h \sin(\ell_k x) \cos(\ell_n y) \tag{2-2-1}$$

This choice of the static response mode reflects the fact proven by several authors [94] that the effect of initial geometric imperfections is the strongest when the response mode resembles the initial imperfection mode.
Based on the same consideration mentioned above and considering the requirement of the travelling wave, which has been measured in the experiments before, the vibration mode shape is assumed as

$$\hat{\hat{W}} = Ah \sin(\ell_k x) \cos(\ell_\ell y) + Bh \sin(\ell_k x) \sin(\ell_\ell y) + Ch \sin^2(\ell_m x) \tag{2-2-2}$$

where A, B and C are the time-dependent amplitude functions, $\ell_\ell$ and $\ell_m$ are the normalized wave numbers.

According to the notation of Evensen's paper [31] the first term is called the driven mode and the second term is called the companion mode. It is noted that the above shape satisfies all the boundary conditions of a simply-supported shell, except the moment-free condition at the ends. Therefore the mode shape used herein has boundary conditions that lie somewhere between simply supported and clamped ends.
Details of the solution of the fundamental state have been published. The detailed procedure is not shown here for the sake of brevity. The interested

reader can refer to [94]. Only the analysis for the dynamic state is presented in the present thesis.

### 2.2.1 PERIODICITY REQUIREMENT

The circumferential periodicity requirement

$$\int_{y}^{2\pi R+y} \frac{\partial \hat{\hat{v}}}{\partial y}\, dy = 0 \tag{2-2-3}$$

must be satisfied.

In general, the displacement mode of Eq. (2-2-2) together with the corresponding stress functions do not satisfy the periodicity requirement despite the fact that they are periodic functions. This drawback is eliminated by replacing equation (2-2-2) with

$$\hat{\hat{W}} = Ah \sin(\ell_k x) \cos(\ell_\ell y) + Bh \sin(\ell_k x) \sin(\ell_\ell y) +$$

$$+ \frac{\ell_\ell^2 Rh}{4} \left[A^2 + B^2 + 2\delta_{n,\ell} A(\hat{\delta}_2 + \delta_2)\right] \sin^2(\ell_m x) \tag{2-2-4}$$

where $\delta_{n,\ell}$ is the Kronecker delta function,

$$\delta_{n,\ell} = \begin{matrix} 0 & n \neq \ell \\ 1 & n = \ell \end{matrix} \tag{2-2-5}$$

The detailed derivation of equation (2-2-4) can be found in Appendix 2-B.

### 2.2.2 APPLICATION OF GALERKIN'S METHOD

Before the Galerkin's procedure can be applied, the stress function $\hat{\hat{\Phi}}$ must be determined. Substituting equations (1-2-1), (2-2-1) and (2-2-2) into the compatibility equation (1-3-40) and then solving for $\hat{\hat{\Phi}}$ one obtains the following particular solution:

$$\hat{\hat{\Phi}} = f\,(\underset{\sim}{A}, x, y) \tag{2-2-6}$$

The function f and the vector $\underset{\sim}{A}$ are listed in Appendix 2-A.

At this stage in the analysis, the equations (1-2-1), (2-2-1), (2-2-4) and (2-2-6) are substituted into the equilibrium equation (1-3-41) and then Galerkin's procedure is used. This procedure yields two coupled nonlinear differential equations for A(t) and B(t)

$$\bar{c}_1 \frac{d^2A}{dt^2} + \bar{c}_2 A + \bar{c}_3 \frac{d^2C}{dt^2}\left[A + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\right] - \bar{c}_5 A^2 + \bar{c}_6(A^2+B^2) +$$

$$+ \bar{c}_7(A^2-B^2) + \bar{c}_8 A^3 + \bar{c}_9(A^2+B^2)\,A + \bar{c}_{10}(A^2+B^2)\,A^2 + \bar{c}_{11}(A^2+B^2)^2 +$$

$$+ \bar{c}_{12}(A^2+B^2)^2\,A = F_D(t) \tag{2-2-7}$$

$$\bar{d}_1 \frac{d^2B}{dt^2} + \bar{d}_2 B + \bar{d}_3 B\frac{d^2C}{dt^2} + \bar{d}_4 BA + \bar{d}_5 BA^2 + \bar{d}_6(A^2+B^2)\,B +$$

$$+ \bar{d}_7(A^2+B^2)\,AB + \bar{d}_8(A^2+B^2)^2\,B = F_C(t) \tag{2-2-8}$$

where the $\bar{c}_1$, $\bar{c}_2$, ..., $\bar{c}_{12}$ and $\bar{d}_1$, $\bar{d}_2$, ..., $\bar{d}_8$ are coefficients which are defined in Appendix 2-A. $F_D$ and $F_C$ are the generalized dynamic excitations. They are obtained by evaluating the integrals involving the external excitation q(x,y,t) and Galerkin's weighting functions. In the present study q is assumed to be fixed in space and harmonic in time

$$q(x,y,t) = Q(x,y)\cos\omega t \tag{2-2-9}$$

where Q(x,y) is assumed to be symmetric with respect to y and has zero average value. In this case $F_C(t)$ is identical to zero and $F_D$ is

$$F_D = 2\int_0^L\int_0^{2\pi R} \frac{Q(x,y)\left[\cos(\ell_\ell y)\sin(\ell_k x) + \frac{\ell_\ell^2 Rh}{2}\left[A+\delta_{n,\ell}(\hat{\delta}_2+\delta_2)\right]\sin^2(\ell_m x)\cos\omega t\right.}{\pi RL}.dxdy \tag{2-2-10}$$

2.2.3 APPLICATION OF THE METHOD OF AVERAGING

The coupled nonlinear differential equations (2-2-7) and (2-2-8) cannot be solved exactly. An approximate solution can be obtained by the procedure known as the method of averaging. The unknown functions A(t) and B(t) are taken to be of the form

$$A(t) = A_t(t)\cos(\omega t) \tag{2-2-11}$$

$$B(t) = B_t(t)\sin(\omega t) \tag{2-2-12}$$

Substituting equations (2-2-11) and (2-2-12) into equations (2-2-7) and (2-2-8) and then applying the method of averaging yields the average amplitudes $\bar{A}$ and $\bar{B}$ respectively so that (see the Appendix 2-A for details)

$$A(t) = \bar{A}\cos(\omega t) \tag{2-2-13}$$

$$B(t) = \bar{B}\sin(\omega t) \tag{2-2-14}$$

The average amplitudes $\bar{A}$ and $\bar{B}$, which are time-independent functions can be computed from the following simultaneous, nonlinear, normalized algebraic equations

$$-\Omega^2\{\bar{A} + \beta_1\bar{A}^3 - \beta_1\bar{B}^2\bar{A} + 2\delta_{n,\ell}(\hat{\delta}_2+\delta_2)^2\,\beta_1\bar{A}\} + \beta_2\bar{A} + \beta_3\bar{A}^3 + \beta_4\bar{A}\bar{B}^2 +$$

$$+ \beta_5[5\bar{A}^5 + 2\bar{A}^3\bar{B}^2 + \bar{A}\bar{B}^4] = \bar{F}_D \tag{2-2-15}$$

$$-\Omega^2\{\bar{B} + \beta_6\bar{B}^3 - \beta_6\bar{A}^2\bar{B}\} + \beta_7\bar{B} + \beta_8\bar{A}^2\bar{B} + \beta_9\bar{B}^3 + \beta_{10}[5\bar{B}^5 + 2\bar{B}^3\bar{A}^2 + \bar{A}^4\bar{B}] = 0$$

$$\tag{2-2-16}$$

where $\Omega^2 = \dfrac{2\omega^2\bar{\rho}R^2}{E}$, is the generalized nondimensional frequency,

$$\bar{F}_D = 4R\int_0^L\int_0^{2\pi R}\frac{Q(x,y)\,\{\sin(\ell_k x)\cos(\ell_\ell y) + \frac{\ell_\ell^2 Rh}{2}\,[A+\delta_{n,\ell}(\hat{\delta}_2+\delta_2)]\,\sin^2(\ell_m x)\}}{\pi LE}\,dxdy,$$

is the generalized average excitation, and $\beta_1$, $\beta_2$, ...., $\beta_{10}$ are coefficients which are given in Appendix 2-A.

Note that one possible solution to equations (2-2-15) and (2-2-16) is that $\bar{B}=0$. In this case, these two equations are reduced to

$$-\Omega^2\{1 + \beta_1\bar{A}^2 + 2\delta_{n,\ell}(\hat{\delta}_2+\delta_2)^2 \; \beta_1\} \; \bar{A} + \beta_2\bar{A} + \beta_3\bar{A}^3 + 5\beta_5\bar{A}^5 = \bar{F}_D \tag{2-2-17}$$

The equation governing the amplitude-frequency for various values of amplitude of the imperfection can be obtained from equation (2-2-17)

$$\Omega^2 = \frac{\beta_2 + \beta_3\bar{A}^2 + 5\beta_5\bar{A}^4}{1 + \beta_1\bar{A}^2 + 2\delta_{n,\ell}(\hat{\delta}_2+\delta_2)^2 \; \beta_1} \tag{2-2-18}$$

Notice that $\bar{A}$ can be infinitesimal but cannot be zero.

## 2.3 CHECKING ON THE CORRECTNESS OF THE CURRENT EQUATIONS

The analytical procedure used in the current study is similar to the one used by Evensen [31] and by Watawala and Nash [116].
Evensen investigated a perfect isotropic smooth cylindrical shell. Watawala and Nash also investigated a smooth isotropic shell but it was not perfect, the asymmetric initial geometric imperfection was included. It is obvious that the current general equations and those derived by Watawala and Nash should reduce to Evensen's equations if the appropriate terms are eliminated.

### 2.3.1. REDUCING TO EVENSEN'S EQUATIONS

An unloaded perfect isotropic shell is now considered, which means $\bar{W} = 0$, (that is $\delta_1 = \delta_2 = 0$), $\lambda = 0$ and terms corresponding to stiffeners and rings vanish in equations (2-2-15) and (2-2-16). Introducing the small parameter $\varepsilon = \left(\frac{\ell^2 h}{R}\right)^2$ and the aspect ratio $\xi = \frac{\pi R/L}{\ell/m}$ used by Evensen into these two equations, one obtains

$$-\Omega^2\{\bar{A} + \beta_{e1}\bar{A}^3 - \beta_{e1}\bar{A}\bar{B}^2\} + \beta_{e2}\bar{A} + \beta_{e3}\bar{A}^3 + \beta_{e4}\bar{A}\bar{B}^2 + \beta_{e5}\{5\bar{A}^5 + 2\bar{A}^3\bar{B}^2 + \bar{A}\bar{B}^4\} = \bar{F}_D \tag{2-2-19}$$

$$-\Omega^2\{\bar{B} + \beta_{e1}\bar{B}^3 - \beta_{e1}\bar{A}^2\bar{B}\} + \beta_{e2}\bar{B} + \beta_{e3}\bar{B}^3 + \beta_{e4}\bar{A}^2\bar{B} + \beta_{e5}\{5\bar{B}^5 + 2\bar{B}^3\bar{A}^2 + \bar{A}^4\bar{B}\} = 0 \tag{2-2-20}$$

where $\beta_{e1}$, $\beta_{e2}$, ...., $\beta_{e5}$ are coefficients defined in Appendix 2-C.

Comparing the present frequency parameter $\Omega$ with that used by Evensen one obtains

$$\Omega_E^2 = \frac{\Omega^2}{2\left[\frac{\xi^4}{(\xi^2+1)^2} + \frac{\varepsilon(\xi^2+1)^2}{12(1-\nu)}\right]} \qquad (2\text{-}2\text{-}21)$$

where $\Omega_E$ is Evensen's frequency parameter.

Substituting this parameter into equations (2-2-19) and (2-2-20) yields the equations

$$(1-\Omega_E^2)\,\bar{A} + \frac{3\varepsilon}{16}\,\Omega_E^2\,\bar{A}(\bar{B}^2-\bar{A}^2) - \frac{\varepsilon\gamma_e}{4}\,\bar{A}(3\bar{A}^2+\bar{B}^2) + \frac{\delta_e\varepsilon^2}{8}\,\bar{A}(5\bar{A}^4+2\bar{A}^2\bar{B}^2+\bar{B}^4) = G_{mn} \qquad (2\text{-}2\text{-}22)$$

$$(1-\Omega_E^2)\,\bar{B} + \frac{3\varepsilon}{16}\,\Omega_E^2\,\bar{B}(\bar{A}^2-\bar{B}^2) - \frac{\varepsilon\gamma_e}{4}\,\bar{B}(3\bar{B}^2+\bar{A}^2) + \frac{\delta_e\varepsilon^2}{8}\,\bar{B}(5\bar{B}^4+2\bar{B}^2\bar{A}^2+\bar{A}^4) = 0 \qquad (2\text{-}2\text{-}23)$$

where $\gamma_e$, $\delta_e$ and $G_{mn}$ are the parameters reported by Evensen [31]. Thus equations (2-2-22) and (2-2-23) are indeed Evensen's equations.

## 2.3.2. REDUCING TO WATAWALA AND NASH'S EQUATIONS

Watawala and Nash's frequency-amplitude equations for the case of $n = \ell$ are

$$-\Omega^2\left[\bar{A}+\alpha_1\bar{A}^3+\alpha_1\bar{A}\bar{B}^2\right] + \alpha_2\bar{A} + \alpha_3\bar{A}^3 + \frac{3}{2}\,\alpha_4\bar{A}\bar{B}^2 + \frac{15}{8}\,\alpha_5\bar{A}^5 + \frac{15}{4}\,\alpha_5\bar{A}^3\bar{B}^2 + \frac{15}{8}\,\alpha_5\bar{A}\bar{B}^4 = \bar{C}_{mn} \qquad (2\text{-}2\text{-}24)$$

$$-\Omega^2\left[\bar{B} + \alpha_1\bar{A}^2\bar{B} + \alpha_1\bar{B}^3 + 2\alpha_1\bar{w}_o^2\bar{B}\right] + \alpha_6\bar{B} + \alpha_7\bar{B}^3 + \frac{3}{2}\,\alpha_4\bar{A}^2\bar{B} + \frac{15}{8}\,\alpha_5\bar{B}^5 + \frac{15}{4}\,\alpha_5\bar{A}^2\bar{B}^3 + \frac{15}{8}\,\alpha_5\bar{A}^4\bar{B} = 0 \qquad (2\text{-}2\text{-}25)$$

where $\bar{C}_{mn}$ is the average generalized force, $\bar{W}_o$ is the asymmetric imperfection, and $\alpha_1, \alpha_2, \ldots, \alpha_7$ are coefficients defined in Appendix C of Ref. [116].

The present frequency-amplitude equations can be written for unloaded isotropic shell with the asymmetric imperfection such that $n = \ell$ in the form

$$-\Omega^2[\bar{A} + \beta_{n1}\bar{A}^3 - \beta_{n1}\bar{A}\bar{B}^2 + 2\beta_{n1}\delta_2^2\bar{A}] + \beta_{n2}\bar{A} + \beta_{n3}\bar{A}^3 + \beta_{n4}\bar{A}\bar{B}^2 +$$

$$+ \beta_{n5}[5\bar{A}^5 + 2\bar{A}^3\bar{B}^2 + \bar{A}\bar{B}^4] = \bar{F}_D \qquad (2\text{-}2\text{-}26)$$

$$-\Omega^2[\bar{B} + \beta_{n6}\bar{B}^3 - \beta_{n6}\bar{A}^2\bar{B}] + \beta_{n7}\bar{B} + \beta_{n8}\bar{A}^2\bar{B} + \beta_{n9}\bar{B}^3 +$$

$$+ \beta_{n10}[5\bar{B}^5 + 2\bar{B}^3\bar{A}^2 + \bar{A}^4\bar{B}] = 0 \qquad (2\text{-}2\text{-}27)$$

where $\bar{F}_D$ is the average generalized force as shown in Eq. (2-2-15), $\beta_{n1}$, $\beta_{n2}, \ldots, \beta_{n10}$ are coefficients defined in Appendix 2-C. They correspond in the form to the $\alpha$'s used by Watawala and Nash.

Some differences between the present equations and the ones published by Natawala and Nash have been found. The algebraic sign of the third term in the two sets of equations is opposite. This seems to come from the different assumptions used for the time-dependent functions A and B in the two investigations. In Watawala and Nash's analysis A and B were assumed as

$$A(t) = A_t(t) \cos \omega t$$

and

$$B(t) = B_t(t) \cos \omega t$$

Whereas in the current analysis the forms of A and B are

$$A(t) = A_t(t) \cos \omega t$$

and

$$B(t) = B_t(t) \sin \omega t$$

It appears that the Watawala and Nash's assumption is not appropriate to the present problem since it does not satisfy the requirement of the travelling wave.

For the case of $n \neq \ell$, the present amplitude-frequency relationship can be rewritten in the form of Watawala and Nash's equations

$$-\Omega^2\left(\bar{A}^2 + \bar{\beta}_{n1}\bar{A}^3 - \bar{\beta}_{n1}\bar{A}\bar{B}^2\right) + \bar{\beta}_{n2}\bar{A} + \beta_{n3}\bar{A}^3 + \bar{\beta}_{n4}\bar{A}\bar{B}^2 +$$

$$+ \bar{\beta}_{n5}\left(5\bar{A}^5 + 2\bar{A}^3\bar{B}^2 + \bar{A}\bar{B}^4\right) = \bar{F}_D \tag{2-2-28}$$

$$-\Omega^2\left(\bar{B} + \bar{\beta}_{n6}\bar{B}^3 - \bar{\beta}_{n6}\bar{A}^2\bar{B}\right) + \bar{\beta}_{n7}\bar{B} + \bar{\beta}_{n8}\bar{A}^2\bar{B} + \bar{\beta}_{n9}\bar{B}^3 +$$

$$+ \bar{\beta}_{n10}\left(5\bar{B}^5 + 2\bar{B}^3\bar{A}^2 + \bar{A}^4\bar{B}\right) = 0 \tag{2-2-29}$$

where $\bar{\beta}_{n1}$, $\bar{\beta}_{n2}$, ..., $\bar{\beta}_{n10}$ are coefficients defined in Appendix 2-C.

Upon comparing the current equations with those by Watawala and Nash a similar problem as mentioned in the case of $n=\ell$ was observed. We also found that Watawala and Nash's equations cannot be reduced to those developed by Evensen.

## 2.4 DISCUSSION OF NUMERICAL RESULTS

The equations derived in the thesis are quite general. They can be used to investigate the nonlinear vibration behaviour of orthotropic or isotropic shells. However, in this paper only numerical results for isotropic shells are presented.
For the numerical work those shell geometrics were used, which make comparison with publised results possible.

1. Shell ES1
   An isotropic shell, used by Evensen in the free nonlinear vibration analysis [31]. Its characteristic data are

   $h = 1.0,$

   $R = 225,$

   $L = 150\,\pi,$

   $\nu = 0.3$

2. Shell ES2
   An isotropic shell, also used by Evensen in the forced nonlinear vibration analysis, which has as its characteristic data

$$\varepsilon = \left(\frac{\ell^2 h}{R}\right)^2 = 0.01$$

$$\xi = \frac{\pi R/\ell}{L/m} = 0.1$$

$$\nu = 0.3$$

For purposes of comparison, $\varepsilon = 0.01$ corresponds to $\ell = 5$ and $h/R = 0.004$ and $\xi = 0.1$ corresponds to $m = 1$, $\ell = 5$ and $L/R = 2\pi$.

3. Shell WN
   The isotropic shell WN was used in the present analysis in order to be able to compare with the results of Watawala and Nash [116]. The characteristic data of the shell are

   $h/R = 1/720$,

   $L/R = 2/3$,

   $\nu = 0.272$

4. Shell X-1
   At first the isotropic shell X-1 was studied in the stability analysis by Arbocz and Sechler [76]. Then it was used in the linearized vibration analysis by Hol [94] for the case of combined imperfections and axial compressive load. Its geometric and material parameters are

   $h = 0.004$ in

   $R = 4.0$ in

   $L = 4.0$ in

   $E = 10^7$ lb/in²

   $\nu = 0.272$

   $\rho = 2.60 \cdot 10^{-4}$ lb.sec²/in⁴

The imperfection modes and their amplitudes for shell X-1 were determined experimentally by Arbocz and Babcock. The critical modes plus associated amplitudes for axisymmetric and/or asymmetric imperfections, which they used, are again used here.
A computer program has been developed and an extensive parametric study has been carried out. The computations were aimed at the main aspect, the influence of initial geometric imperfection on the nonlinear vibration of shells. Before reporting the numerical results, it must be mentioned that whenever possible, special combinations of the wave numbers for axisymmetric parts of the vibration mode were chosen. For certain combinations of the wave numbers i, k and m, mode

coupling occurs. For the present analysis, the coupling conditions are i = 2k, i = 2(m+k). The coupling condition i=2k has been proven to be the stronger one [94].

### 2.4.1 PERFECT SHELL

The first series of computations is carried out to determine the natural frequencies of shell ES1 and the response of shell ES2 to an excitation. This is a necessary check on the correctness of the computer program based on the general equations derived in the present study. The results are shown in Figures 2.1 ~ 2.5. A closer look reveals that all the results presented by Evensen are recovered in the current study. In addition, some new and interesting results were found.

Thus, as can be seen from Figs. 2.2 and 2.5, the softening type nonlinear vibration characteristic of the mode m=k=3 is changed into a hardening one for the vibration mode k=3 and m=6. Since experimentally only softening type nonlinear vibration has been observed for moderate size vibration amplitudes (A<2.0, say), therefore when choosing the mathematical models one must not only satisfy the circumferential periodicity condition but als the strong coupling condition m=k between the axisymmetric and the asymmetric vibration modes. Otherwise the predicted nonlinear vibration characteristics become unrealistic.

### 2.4.2 IMPERFECT SHELL

#### 2.4.2.1 SINGLE MODE VIBRATIONS ($\bar{A} \neq 0$, $\bar{B} = 0$)

In this section the influence of asymetric, axisymmetric and combined imperfections as well as of axial compressive load on the nonlinear single mode vibration of a circular cylindrical shell is investigated.

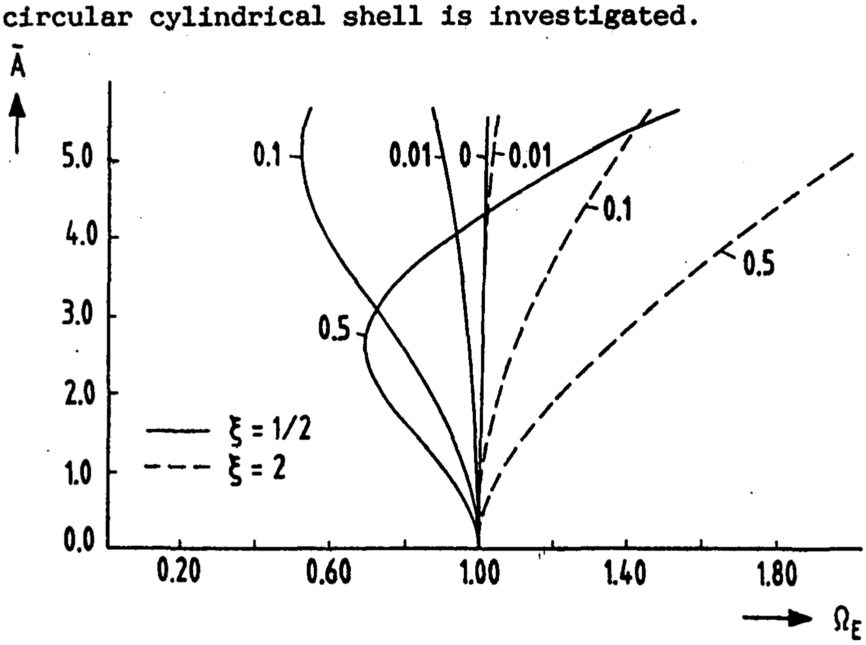



Fig. 2.1 Influence of large amplitude of vibration on natural frequency for various ε. ES1 Shell.

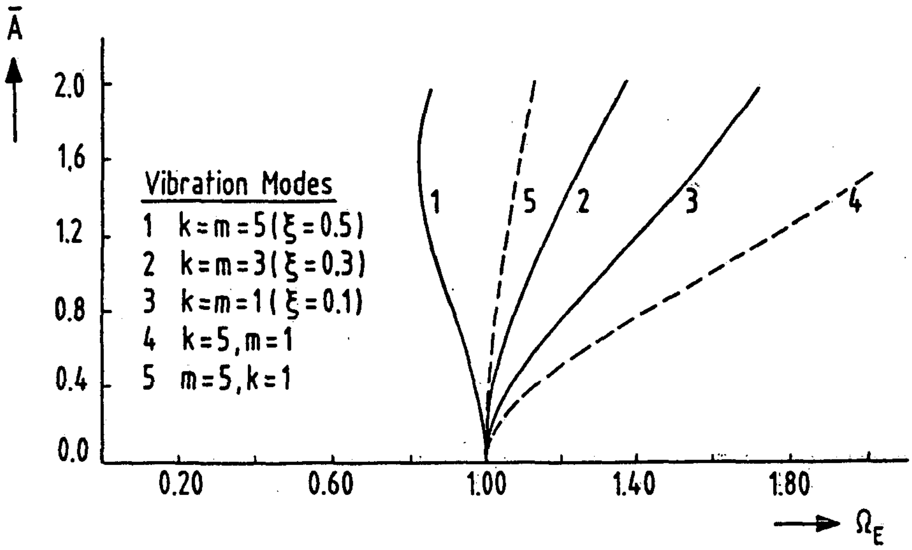



Fig. 2.2 Influence of large amplitude of vibration on natural frequency for different vibration modes. ES1 Shell; $\epsilon = 1.0$.

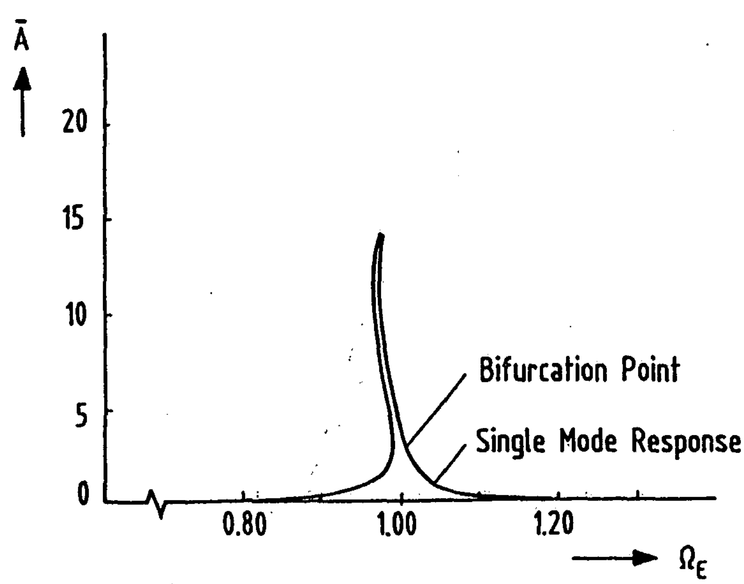



Fig. 2.3 Amplitude-frequency relationship of perfect shell; driven mode response; ES2 Shell.

Excitation $\bar{F}_D = 2 \times 10^{-4}$.

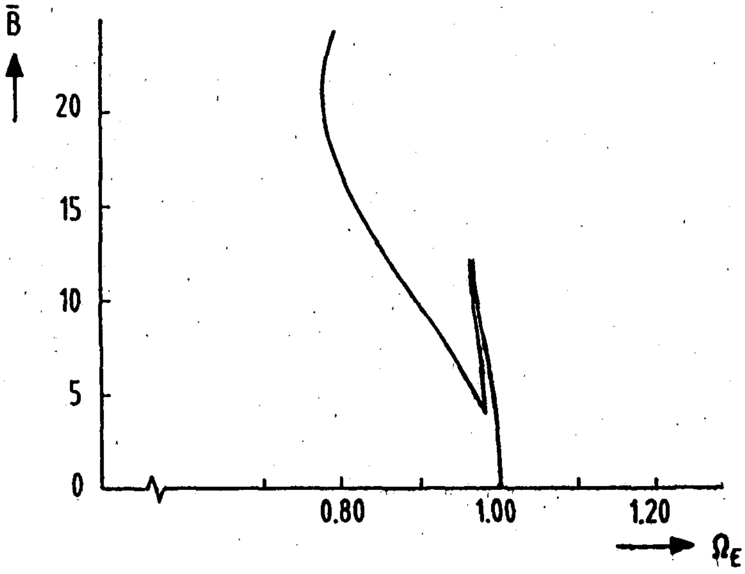



Fig. 2.4 Amplitude-frequency relationship of perfect shell; companion mode response
ES2 Shell; Excitation $\bar{F}_D = 2 \times 10^{-4}$.

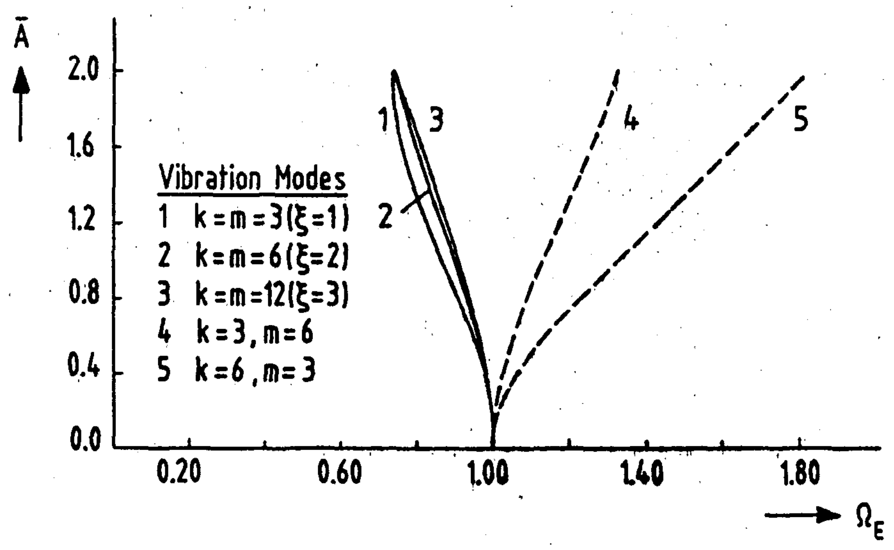



Fig. 2.5 Influence of large amplitude of vibration on natural frequency for different vibration modes.
ES1 Shell; $\varepsilon$ = 1.0.

1. Asymmetric imperfection

At first the free vibrations of shell WN are considered, whereby the circumferential wave patterns of the dynamic response and of the initial imperfection are identical. In order to be able to compare the present numerical results with that of earlier investigations by Rosen and Singer [11,15] and Watawala and Nash [116], we will use the vibration mode with k=m=5 axial half waves and $\ell$=25 circumferential full waves. Notice that this mode is identical to the classical asymmetric buckling mode.
Figure 2.6 shows a comparison between 4 different analyses. The fact that Rosen and Singer predict an increase in the frequency of free vibration $\Omega$ with increasing amplitude of the asymmetric imperfection $\delta_2$ has been traced to the fact that their assumed (infinitesimal) asymmetric vibration mode does not satisfy the circumferential periodicity condition. The large discrepancy between the nonlinear vibration results of Watawala and Nash and the present work are probably due to some error in their frequency-amplitude relationships, which do not reduce to Evensen's perfect shell equations [31] if the amplitude of the initial imperfection is set equal to zero. Notice also that the present nonlinear vibration results show an excellent agreement with Hol's [94] linearized results if the amplitude of vibration $\bar{A}$ and the initial imperfection $\delta_2$ are sufficiently small.
Proceeding with the comparison of the current results with the ones obtained in Reference [116], Figure 2.7 shows the variation of the frequency of free vibration with the amplitude of imperfection for increasing values of the amplitude of vibration $\bar{A}$ obtained by Watawala and Nash.
Notice that increasing the amplitude of vibration results in a softening type behaviour for all values of imperfections considered. This is in part contradicted by the present results shown in Fig. 2.8, where for imperfections greater than a certain critical value (here $\delta_2 \approx 0{,}8$) an increase in the amplitude of vibration results in a hardening type behaviour. Notice also that for imperfections smaller than the critical value a softening type behaviour is predicted if the amplitude of vibration increases.
Figures 2.9 and 2.10 display the relationship between the frequency of free vibration $\Omega$ and the circumferential wave number $\ell$ of the dynamic response mode for different amplitudes of the asymmetric imperfection $\delta_2$. The axial half wave number k and the circumferential full wave number n of the imperfection are kept constant at 5 and 25, respectively. For very small amplitude of vibration ($\bar{A}$=0,001) both the current nonlinear analysis and the one by Watawala and Nash agree very well with the results of the linearized infinitesimal vibration analysis of Rosen and Singer for all values of $\ell$ except for $\ell$=n=25. At this particular value of $\ell$ Ref. [15] predicts a sudden increase in the frequency while Ref. [116] and the present study predict a significant decrease. An explanation of these discrepancies has been given earlier.
Figures 2.11 thru 2.16 are included in order to investigate further the dynamic behaviour of the shell when n=$\ell$. In these plots the axial wave numbers are kept constant (m=k=5) and the circumferential wave numbers n and $\ell$ are varied from 5 to 30. As can be seen at higher aspects ratios ($\xi>\frac{\pi}{2}$, say) the nonlinearity is of the hardening type and the frequencies increase continuously with increasing imperfection amplitude $\delta_2$. However, at lower aspects ratios ($\xi<\frac{\pi}{2}$, say) the

nonlinearity changes to being of the softening type for values of the imperfection $\delta_2$ smaller that a certain critial value. For values of the imperfection $\delta_2$ larger that the critical value the nonlinearity becomes once again of the hardening type. As can be seen from Figures 2.14 - 2.17 the critical value of the imperfection $\delta_2$ becomes smaller as the number of circumferential waves increases (or the aspects ratio $\xi$ decreases).
The influence of asymmetric imperfections on nonlinear vibration for the case of $n \neq \ell$ is shown in Fig. 2.17. The frequencies of free vibration increase with increasing amplitude of imperfection $\delta_2$. The influence of the amplitude of vibration is slight.
The forced response curves of the shell are shown in Figures 2.18 - 2.21. In the figures the dotted lines denote the frequency-amplitude relationships for free vibration (also known as the backbone curve) and the solid lines represent the foced response. As can be seen, in the case of $n \neq \ell$ the type of vibration is governed by the aspect ratio $\xi$ only. Thus for high aspect ratio's ($\xi > \frac{\pi}{2}$, say) one gets a hardening type of behaviour, whereas for low aspect ratio's ($\xi < \frac{\pi}{2}$, say) the nonlinear vibration is of the softening type. If, however, $n=\ell$ then as can be seen from Fig. 2.21 the type of the vibration behaviour depends not only on the aspect ratio $\xi$ but also on the size of the asymmetric imperfection. For instance, an amplitude of imperfection greater than the 'critical value' (0.8, say) alters the vibration behaviour from a softening type to a hardening type.

## 2. Axisymmetric imperfection

In 1974 Rosen and Singer [11] published a paper dealing with the effect of axisymmetric imperfections on the vibrations of cylindrical shells under axial compression. They have shown that geometrical imperfections of the kind which affect buckling have also a large influence on the vibrations of these shells, even at zero axial load. They expressed the hope that this phenomenon will facilitate the evaluation of the effect of the actual imperfections by measuring the deviations in frequencies of the imperfect shell from those of the corresponding perfect one. They assumed the following axisymmetric imperfection

$$\bar{W} = \delta_1 \, h \cos (\ell_i x)$$

and approximated the asymmetric vibration mode as

$$\hat{W} = A \, h \sin (\ell_k x) \cos (\ell_n y)$$

where i=2k. For the isotropic shell X=1 ($\frac{R}{h} = 1000$, $\frac{R}{L} = 1.0$) the classical perfect shell buckling modes are i=18 (the axisymmetric buckling mode) and k=9, n=29 (one of the asymmetric buckling modes). Figure 2.22 displays the frequency-amplitude relationships for shell X-1 at different amplitudes of initial axisymmetric imperfection $\delta_1$. Since the amplitude of vibration is small ($\bar{A}=0.001$) therefore, as expected, the values agree closely with Hol's [44] results.

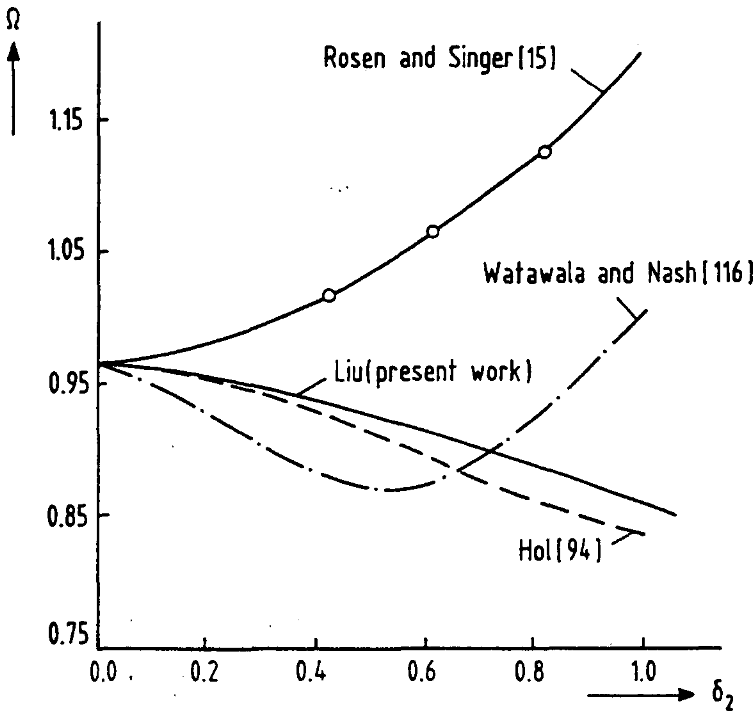



Fig. 2.6 Frequency of free vibration vs amplitude of imperfection.
WN Shell; k=m=5, n=ℓ=5; amplitude of vibration $\bar{A}$=0.001

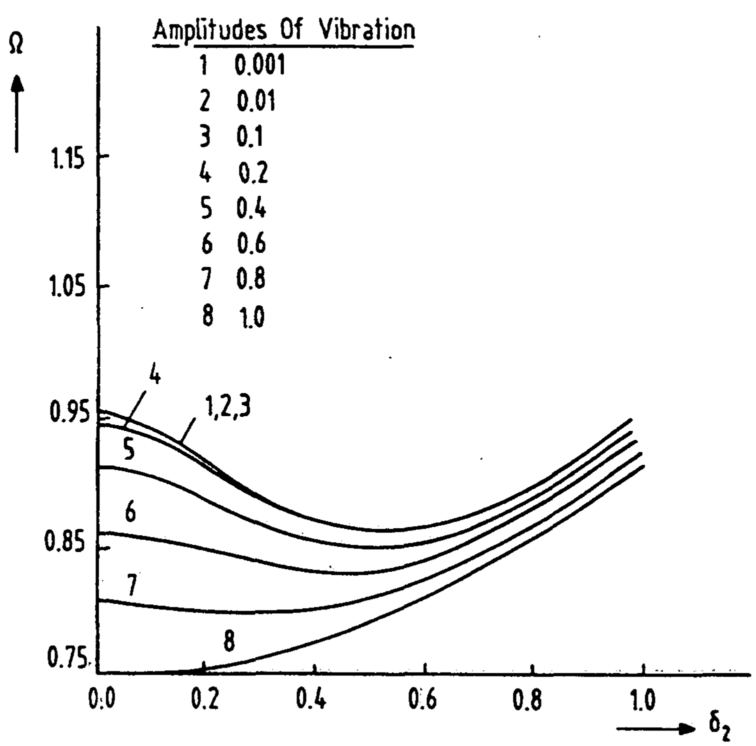



Fig. 2.7 Frequency of free vibration vs amplitude of imperfection for different values of amplitude of vibration from Ref. [116].
WN Shell; k=m=5, n=ℓ=25

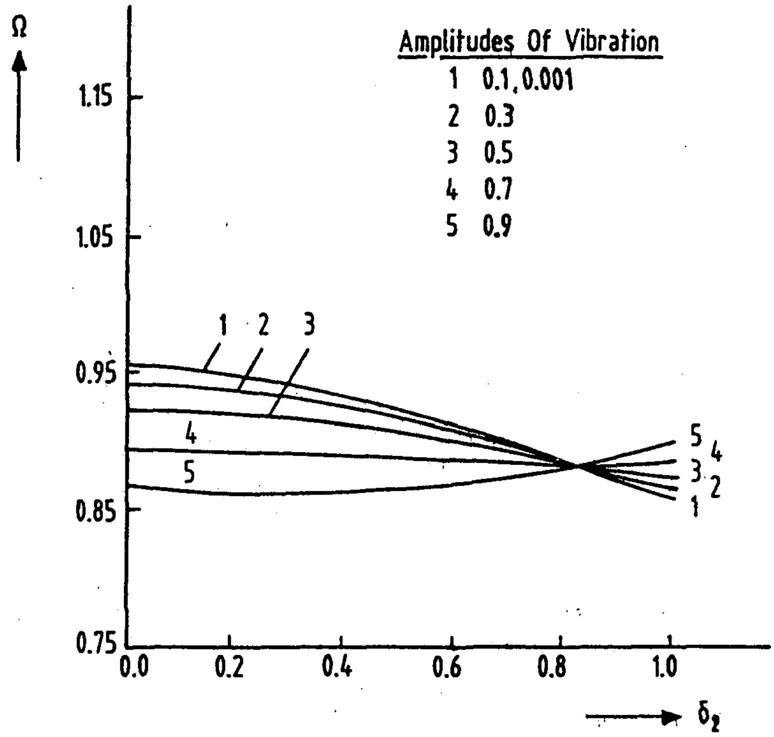



Fig. 2.8 Frequency of free vibration vs amplitude of imperfection for different values of amplitude of vibration (present work).
WN Shell; k=m=5, n=ℓ=25

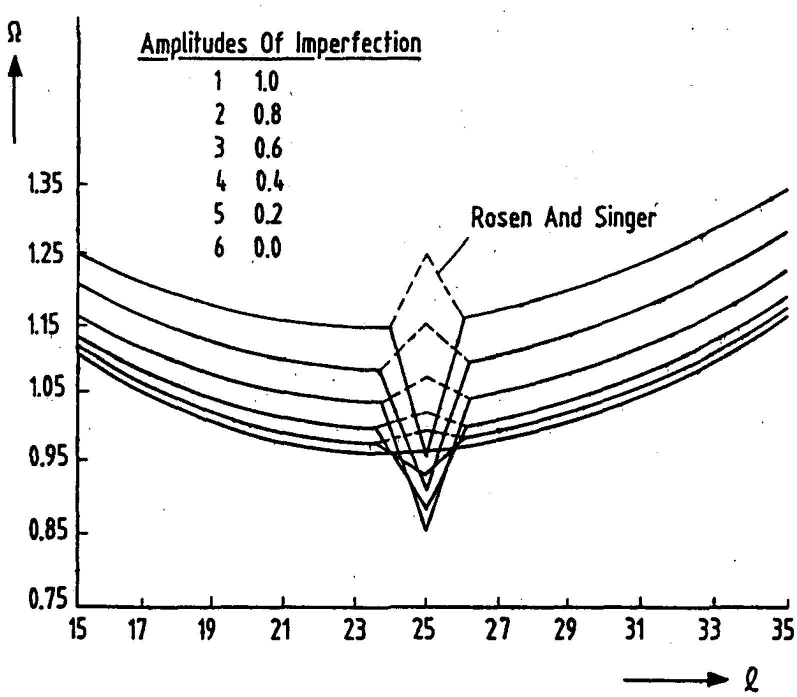



Fig. 2.9 Frequency of free vibration vs circumferential wave number for different imperfections (References [15] and [16]).
WN Shell; k=m=5, n=25; amplitude of vibration $\bar{A}$=0.001

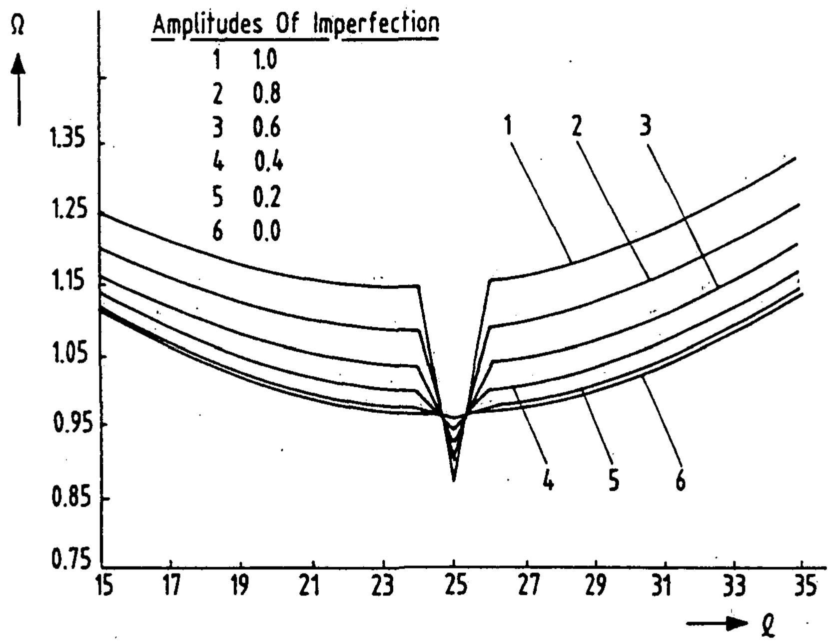



Fig. 2.10 Frequency of free vibration vs circumferential wave number for different imperfections (present work).
WN Shell; k=m=5, n=25; amplitude of vibration A=0.001

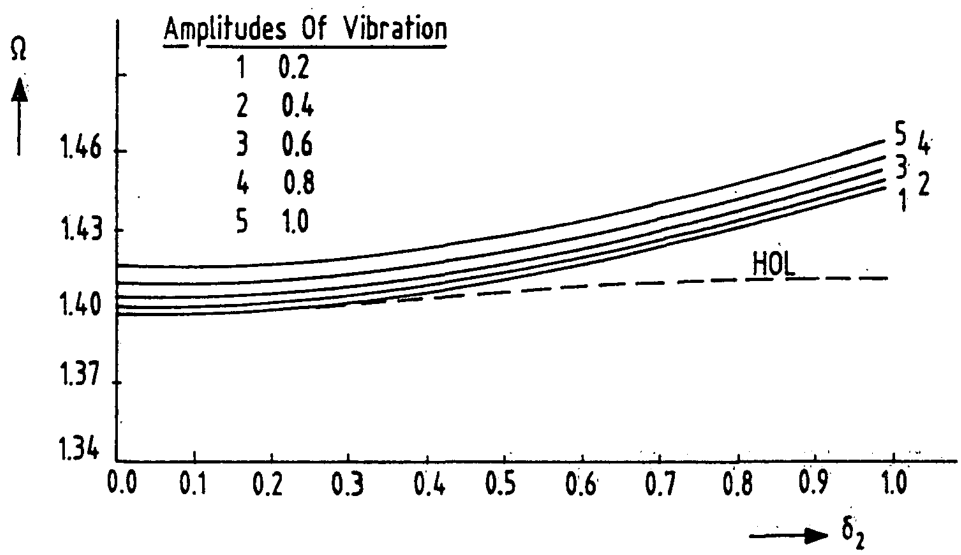



Fig. 2.11 Frequency of free vibration vs amplitude of imperfection for different values of amplitude of vibration.
WN Shell; k=m=5, n=ℓ=5, $\xi=3\pi/2$

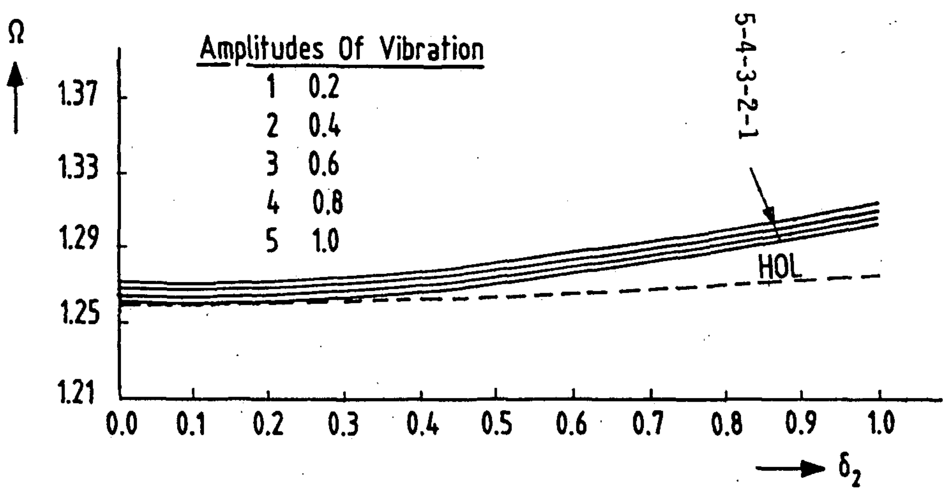



Fig. 2.12 Frequency of free vibration vs amplitude of imperfection for different values of amplitude of vibration.
WN Shell; k=m=5, n=ℓ=10, ξ=3π/4

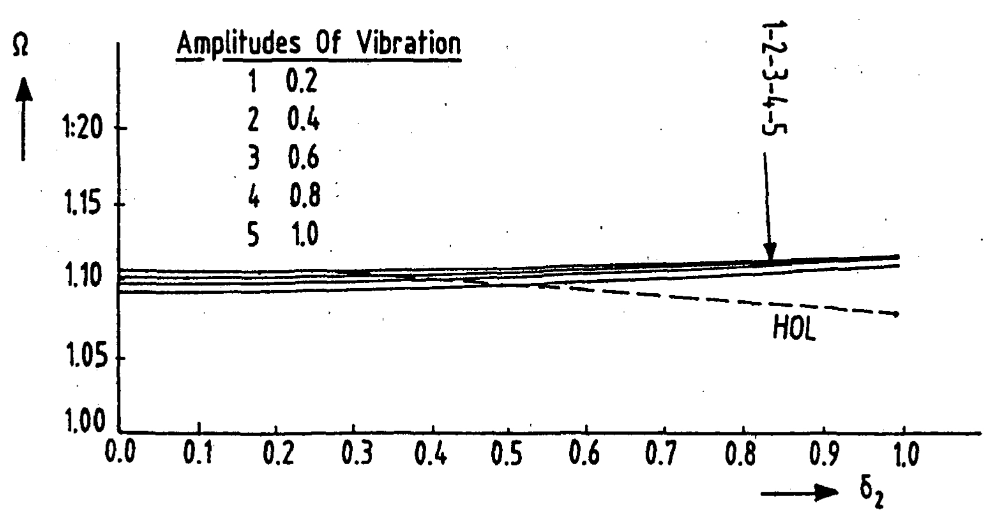



Fig. 2.13 Frequency of free vibration vs amplitude of imperfection for different values of amplitude of vibration.
WN Shell; k=m=5, n=ℓ=15, ξ=π/2

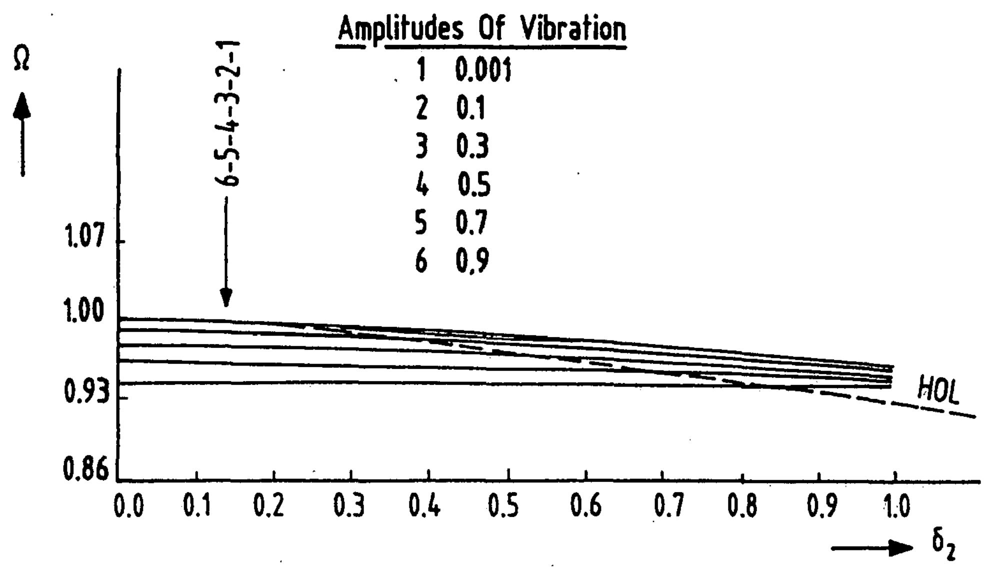



Fig. 2.14 Frequency of free vibration vs amplitude of imperfection for different values of amplitude of vibration.
WN Shell; k=m=5, n=ℓ=20, ξ=3π/8

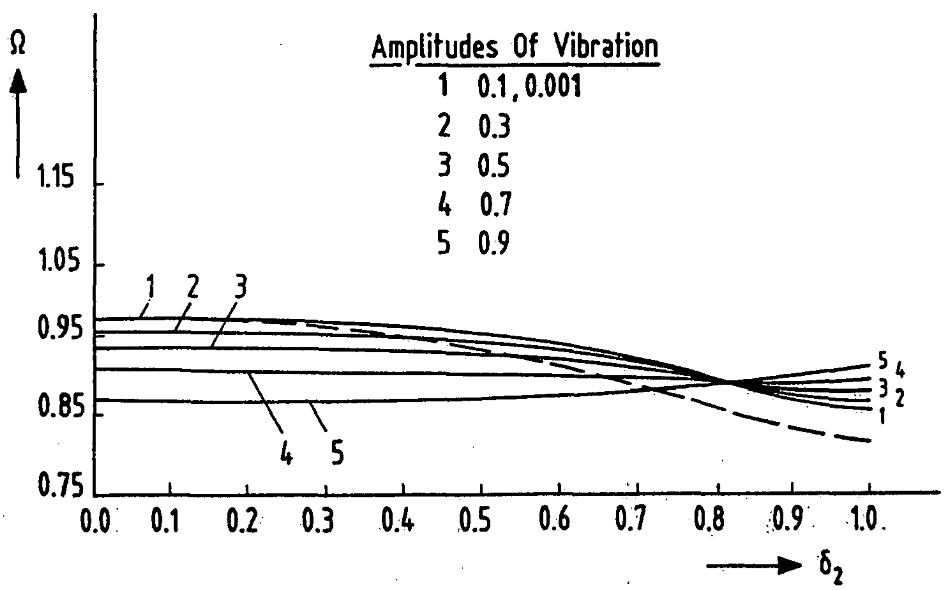



Fig. 2.15 Frequency of free vibration vs amplitude of imperfection for different values of amplitude of vibration
WN Shell; k=m=5, n=ℓ=25, ξ=3π/10

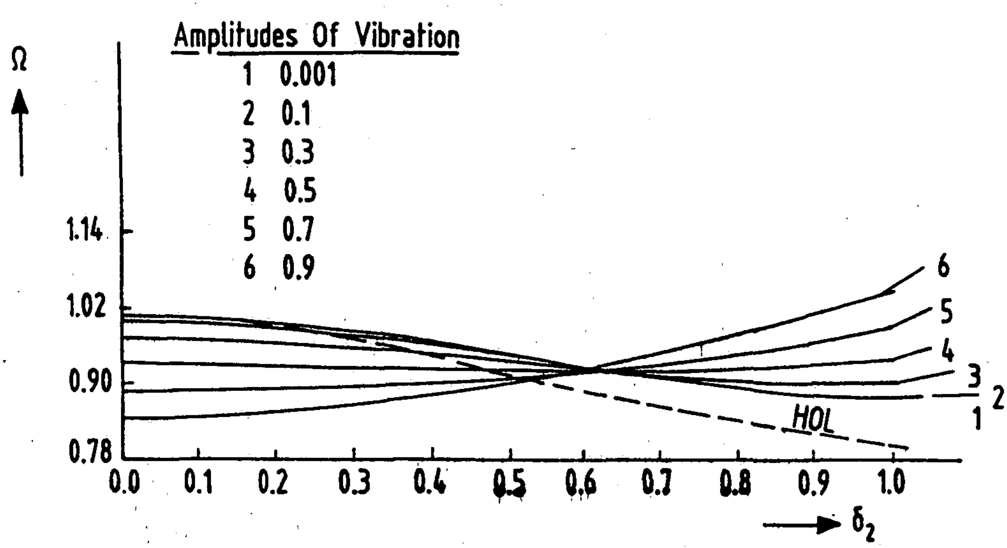



Fig. 2.16 Frequency of free vibration vs amplitude of imperfection for different values of amplitude of vibration.
WN Shell; k=m=5, n=ℓ=30, ξ=π/4

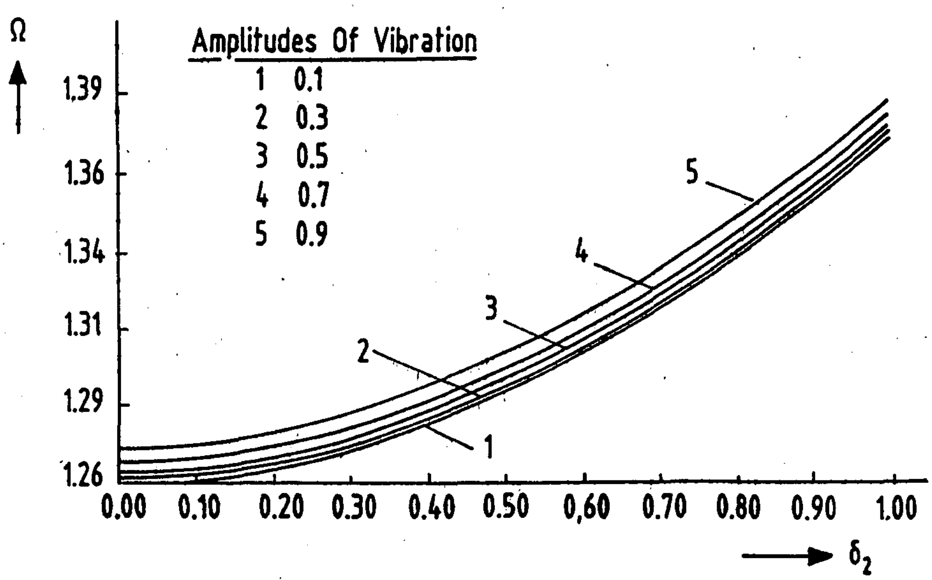



Fig. 2.17 Frequency of free vibration vs amplitude of imperfection for different values of amplitude of vibration.
WN Shell; k=m=5, n=20, ℓ=10

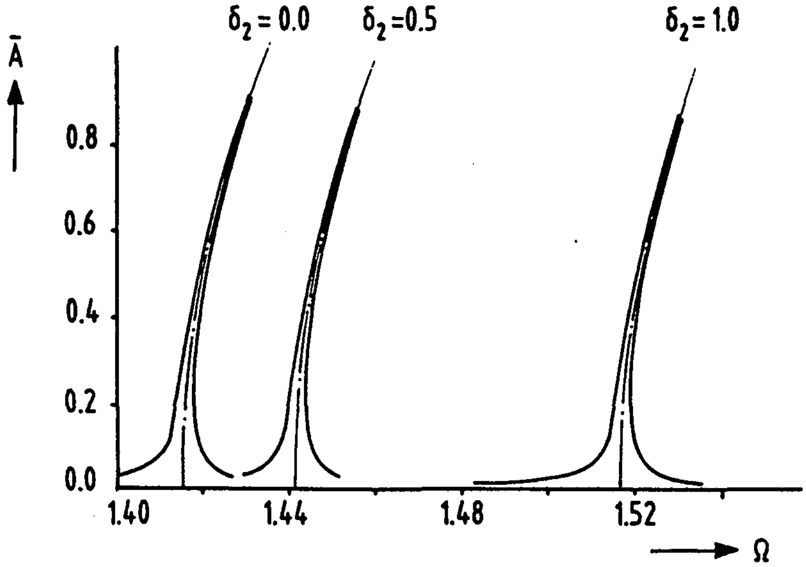



Fig. 2.18 Amplitude of forced vibration vs frequency of excitation for different values of amplitude of imperfection.
WN Shell; k=m=5, n=5, $\ell$=4.
Excitation $\bar{F}_D$=0.001.

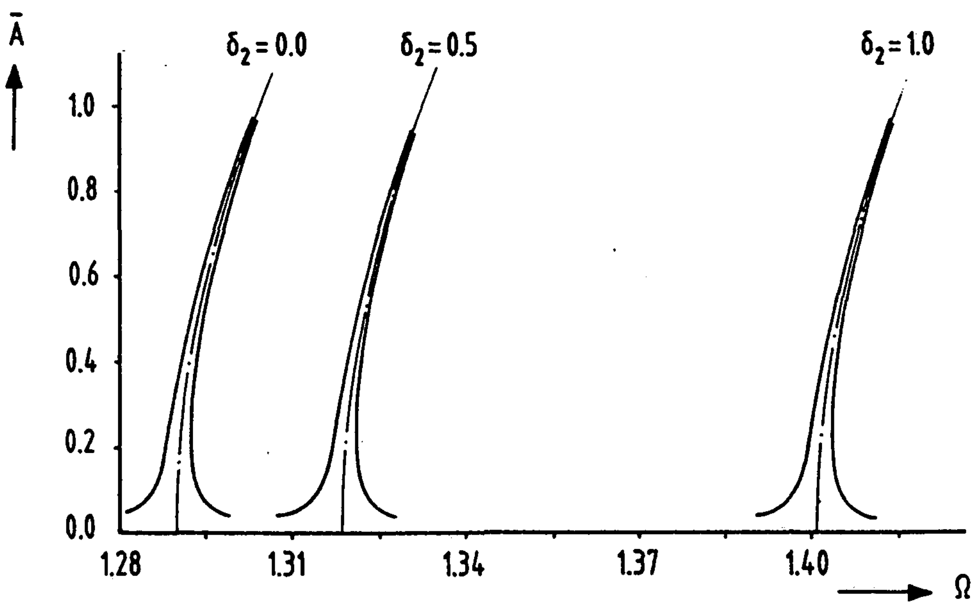



Fig. 2.19 Amplitude of forced vibration vs frequency of excitation for different values of amplitude of imperfection.
WN Shell; i=10, k=m=5, n=10, $\ell$=9.
Excitation $\bar{F}_D$=0.001.

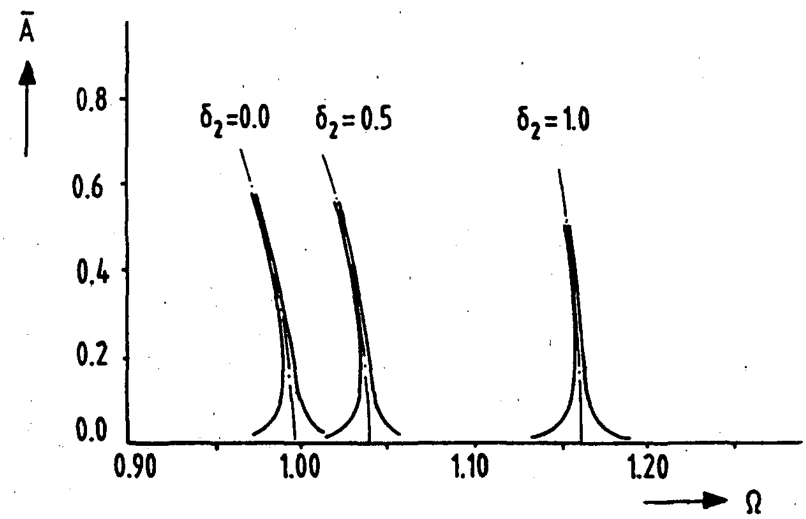



Fig. 2.20 Amplitude of forced vibration vs frequency of excitation for different values of imperfection.
WN Shell; i=10, k=m=5, n=25, ℓ=20.
Excitation $\bar{F}_D$=0.001.

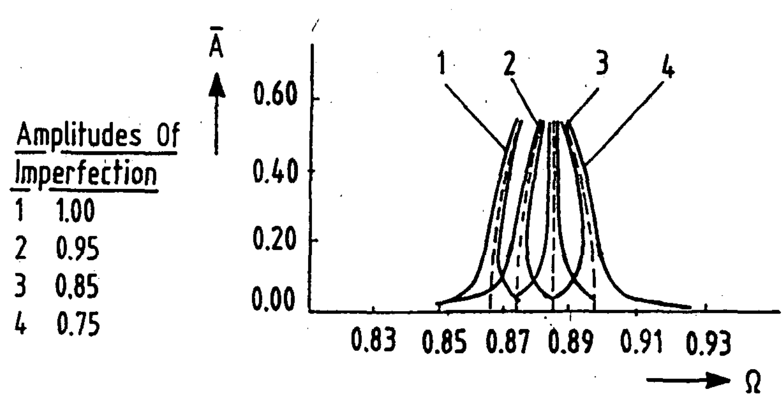



Fig. 2.21 Amplitude of forced vibration vs frequency of excitation for different values of amplitude of imperfection.
WN Shell; k=m=5, n=ℓ=25, ξ=3π/10.
Excitation $\bar{F}_D$=0.001.

Notice that the amplitudes of the axisymmetric imperfections were chosen negative, which means the the axisymmetric imperfection is directed inward over the central portion of the shell, a necessary condition for the buckling load of the imperfect shells to occur at a value less than 1. Considering the solid curves closer it appears that for axisymmetric imperfections greater than $\delta_1 = -0.4$ there is no buckling (the solid curve labeled 4 does not cross the axis $\Omega=0$). This apparent paradox is caused by fixing the circumferential wave numbers of the dynamic mode at $\ell=29$. If one releases $\ell$ (that is one admits other integer values of $\ell$ as possible solutions) then one gets the dashed curve with buckling occurring with $\ell=26$ full waves in the circumferential direction. Actually the Technion Group prefers to plot their results using $\Omega^2$ along the vertical axis. Thus for the ease of comparison Figure 2.22 is so replotted in Figure 2.23.

Figure 2.24 displays the variation of the frequency of free vibration (plotted as $\Omega^2$) with the amplitude of axisymmetric imperfection for increasing values of the amplitude of vibration $\bar{A}$. Notice that for imperfections smaller than a certain critical value (here $\delta_1 \approx 0.4$) an increase in the amplitude of vibration results in a softening type behaviour, whereas for imperfections greater than the critial value a hardening type behaviour is predicted if the amplitude of vibration increases. Thus the effect of an axisymmetric imperfection on the nonlinear vibrations is similar to that of an asymmetric imperfection (see also Fig. 2.15), if the strong coupling condition i=2k is satisfied and the vibration mode considered is affine to a classical asymmetric buckling mode.

### 3. Axial compressive load

Figure 2.25 shows the variation of the frequency of free vibration (plotted as $\Omega^2$) with the axial compressive load for increasing values of the amplitude of vibration $\bar{A}$. At any given axial compressive load level an increase in the amplitude of vibration results in a softening type behaviour.

### 4. Combined imperfection and axial load level

At this stage the present analysis is used to investigate the effect of combined axisymmetric imperfections and of an applied compressive load on the nonlinear vibration. The only known reference data are Hol's [94] linearized results. Shell X-1 and the following combined imperfection model

$$\frac{\bar{W}}{h} = -0.50 \cos \frac{18\pi x}{L} + 0.05 \sin\frac{9\pi x}{L} \cos \frac{29y}{R}$$

are used. Notice that both imperfection modes are classical buckling modes. Figures 2.26 and 2.27 (both plotted as $\Omega^2$) show the influence of the asymmetric and the axisymmetric imperfections acting alone. In both cases for the specified imperfection amplitudes the nonlinear vibration is of the hardening type. If both imperfections act concurrently then the resulting nonlinearities seem to cancel each other partially, because as can be seen from Fig. 2.28 (plotted as $\Omega^2$) for all amplitudes of nonlinear vibration considered the solution curves lie relatively close to the linearized solution of Hol [94].

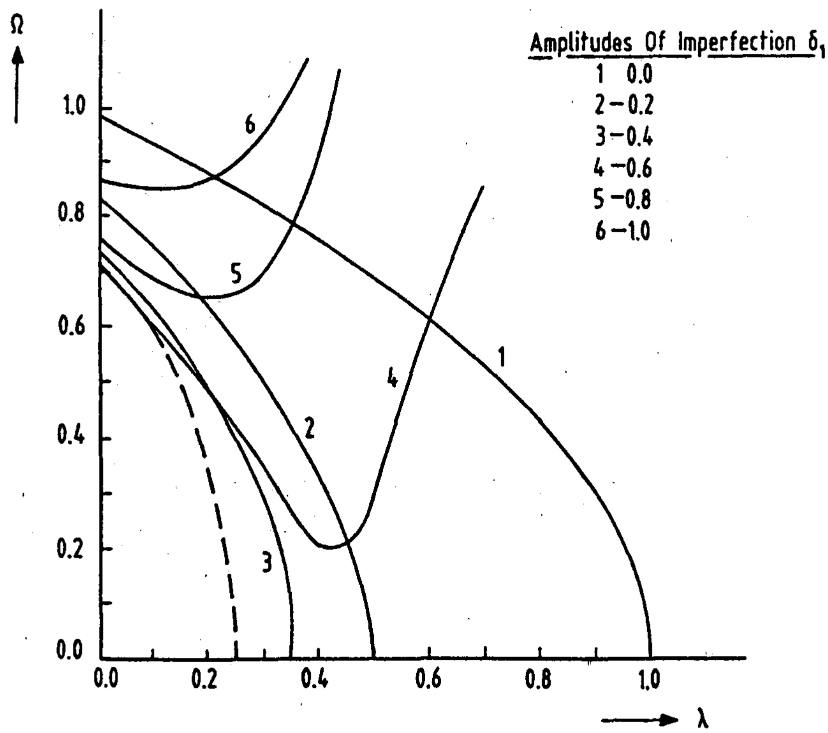



Fig. 2.22 Frequency of free vibration vs axial compressive load for different values of axisymmetric imperfection.
X-1 Shell; i=18, k=m=9, n=ℓ=29; amplitude of vibration $\bar{A}$=0.001.

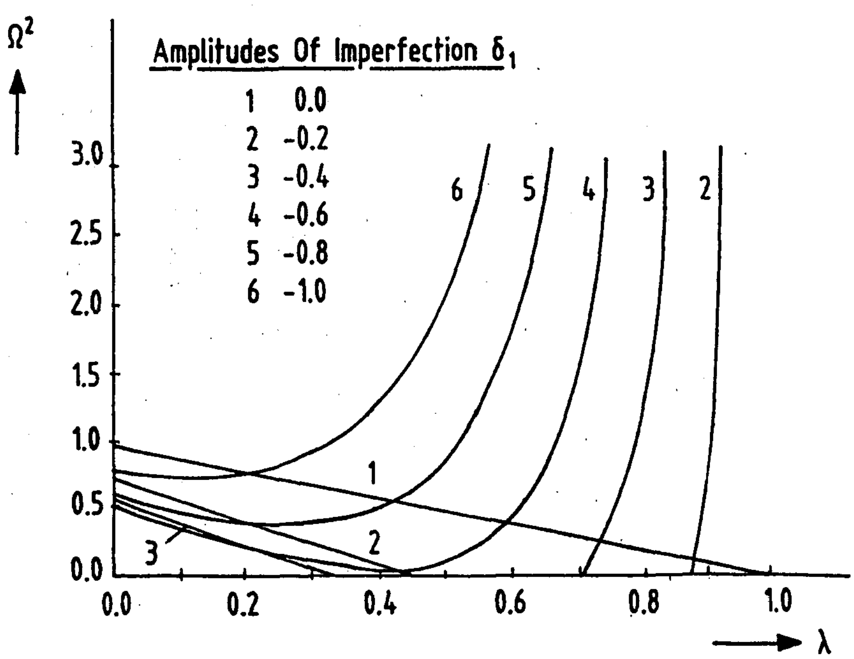



Fig. 2.23 Frequency of free vibration vs axial compressive load for different values of axisymmetric imperfection.
X-1 Shell; i=18, k=m=9, n=ℓ=29; amplitude of vibration $\bar{A}$=0.001

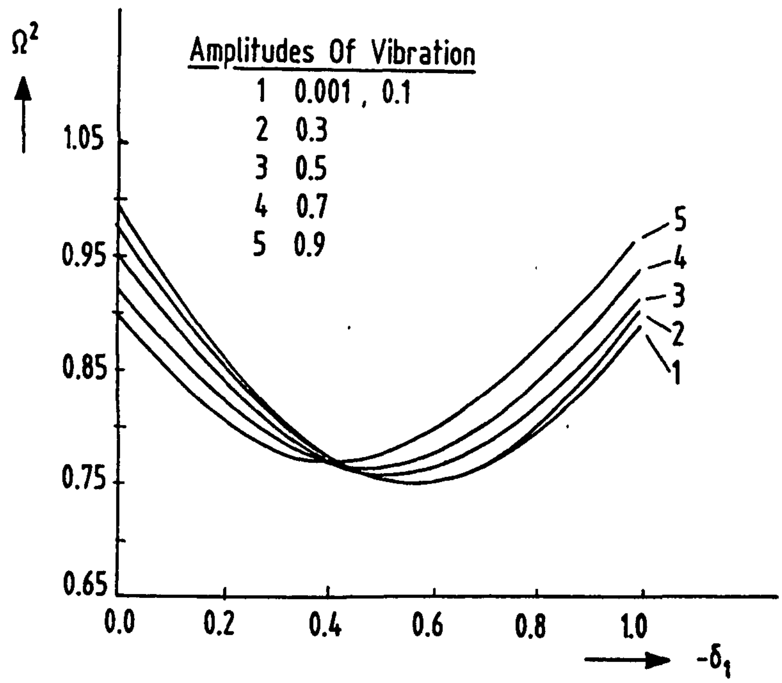



Fig. 2.24 Frequency of free vibration vs amplitude of imperfection for different values of amplitude of vibration.
X-1 Shell; i=18, k=m=9, n=$\ell$=29; $\lambda$=0.

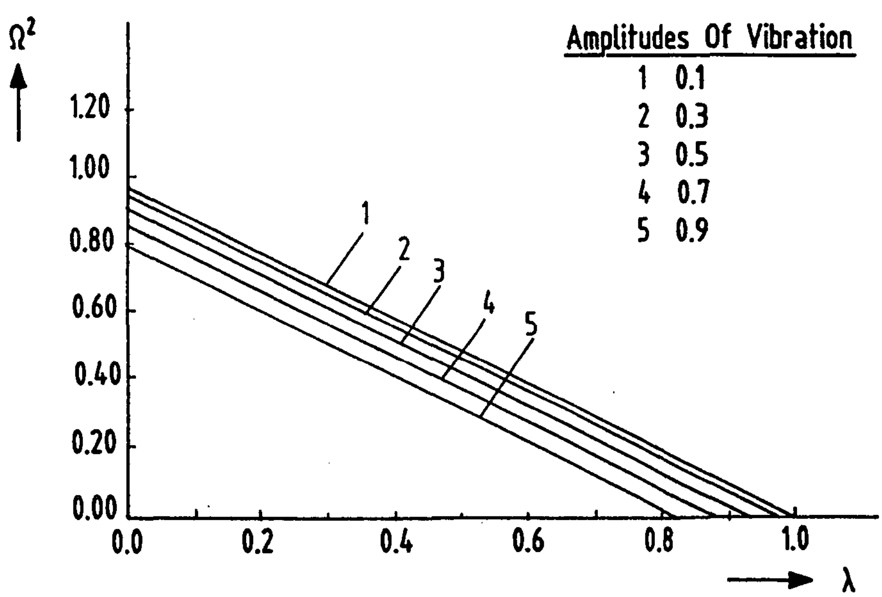



Fig. 2.25 Frequency of free vibration of perfect shell vs axial compressive load.
X-1 Shell; i=18, k=m=9, n=$\ell$=29

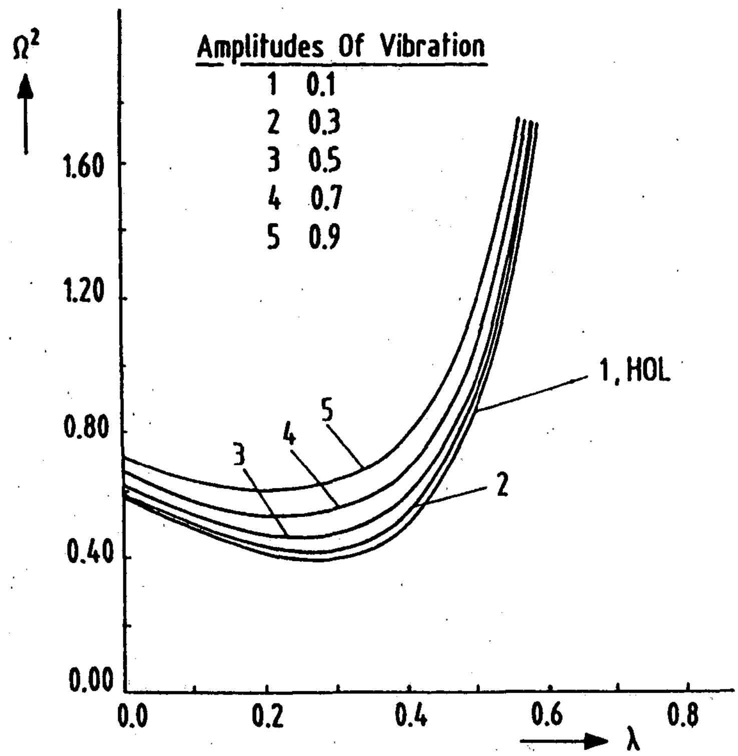



Fig. 2.26 Frequency of free vibration of imperfect shell vs axial compressive load for different values of amplitude of vibration.
X-1 Shell; i=18, k=m=9, n=$\ell$=29; amplitude of imperfection $\delta_2$=0.05.

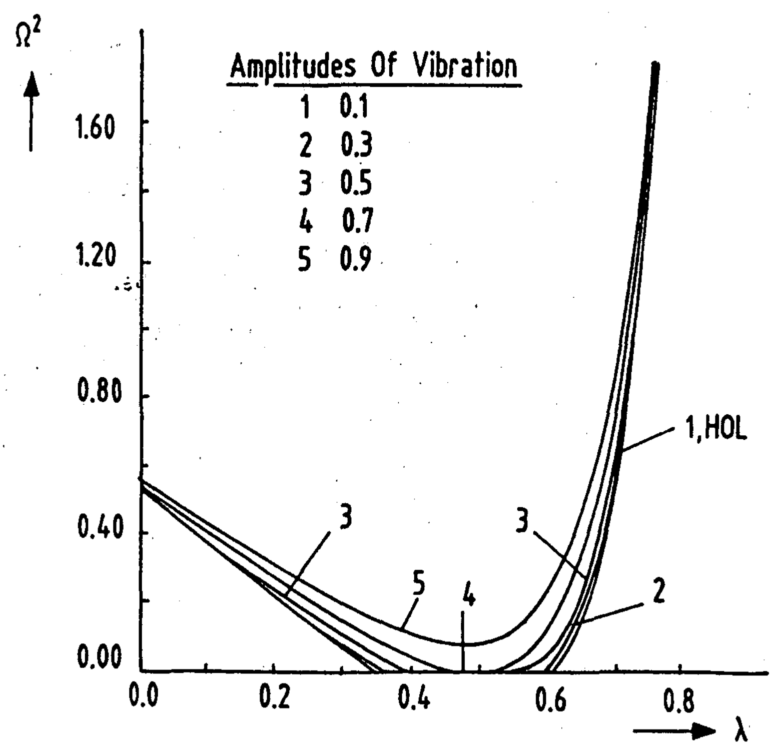



Fig. 2.27 Frequency of free vibration of imperfect shell vs axial compressive load for different values of amplitude of vibration.
X-1 Shell; i=18, k=m=9, n=$\ell$=29; amplitude of imperfection $\delta_1$= -0.50.

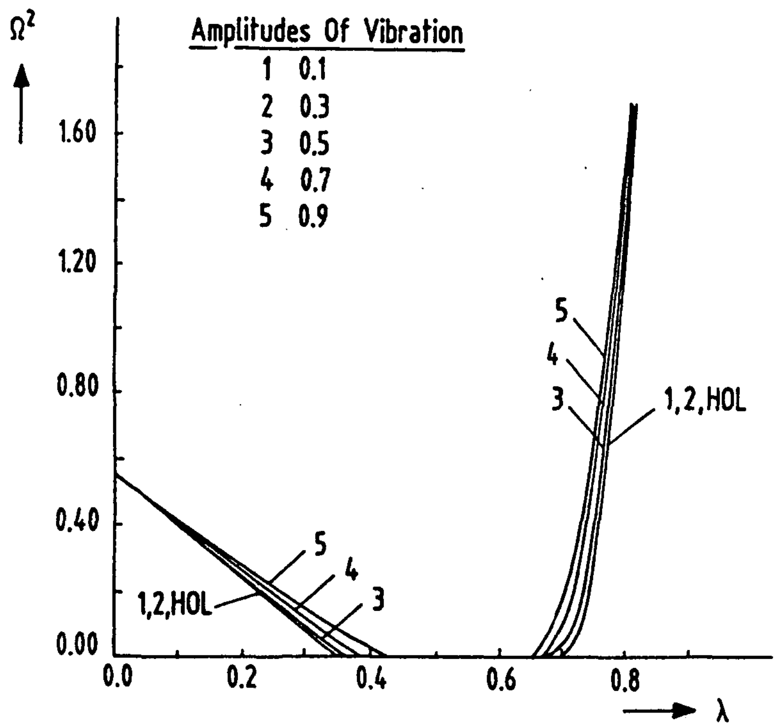



Fig. 2.28 Frequency of free vibration of imperfect shell vs axial compressive load for different values of amplitude of vibration.
X-1 Shell; i=18, k=m=9, n=ℓ=29; amplitude of imperfections $\delta_1$=-0.50, $\delta_2$=0.05.

## 2.4.2.2 VIBRATION OF COUPLED MODE ($\bar{A} \neq 0$, $\bar{B} \neq 0$)

There are two possible cases of the nonlinear vibration of a cylindrical shell, the single mode vibration ($\bar{A} \neq 0$, $\bar{B} = 0$) and coupled mode vibration ($\bar{A} \neq 0$, $\bar{B} \neq 0$). It has been demonstrated above that the initial geometric imperfections may significantly influence the amplitude-frequency relationships of the single mode vibration. Thus, it can be conjectured that initial geometric imperfections may also exhibit equally significant effect on the coupled mode vibration.No reference data for this case are available up to date.

Figure 2.29 displays the single mode and the coupled mode undamped free and forced vibrations of the perfect shell ES2 from Evensen [31]. The stability of the approximate solutions given by Eqs. 2-2-13 and 2-2-14 was examined by the usual techniques of perturbation analysis (see Chapter 4 for details). The stable branches are indicated by solid curves and the unstable parts by dashed curves.
Considering the single mode response curve the first instability region 'a' coincides with the locus of vertical tangents to the response curves, which indicates the well-known jump phenomena. The second instability region 'b' indicates the area where the single mode response is unstable because of the nonlinear coupling with the companion mode. If adequate solutions are to be obtained in region 'b', it is necessary to consider motions where both modes vibrate. Notice that the 'backbone curve' represents the amplitude frequency relation for free vibrations of a single mode.

Since the peak responses of the driven mode and the companion mode occur at about the same value of $\Omega$ one says that the driven and the companion mode are 'symmetric'.

The effect of asymmetric imperfection on the single mode and coupled mode undamped free and forced vibrations of shell ES2 are shown in Fig. 2.30. A direct comparison with the perfect shell results displayed in Fig. 2.29 reveals a few minor changes in the shape of the curves. Also the bifurcation point on the upper branch of the single mode response curve occurs for the imperfect shell at $\Omega=1.00145$, a value slightly higher than that of the perfect shell of $\Omega=1.00030$.

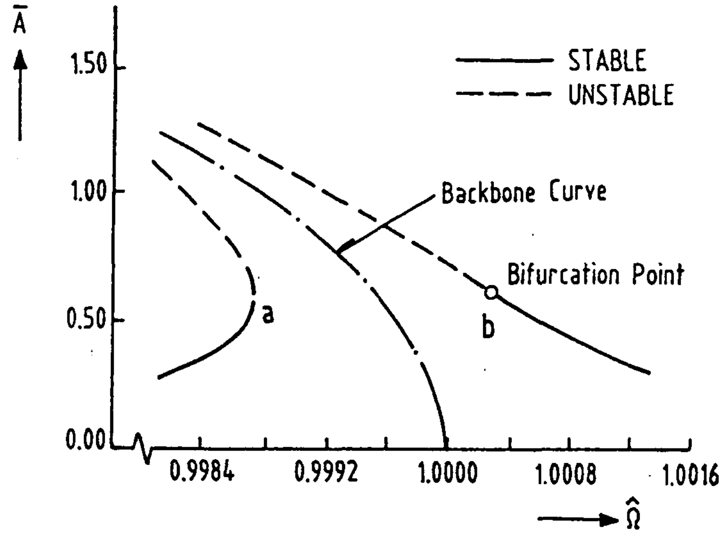
On the other hand the effect of a small axisymmetric imperfection is, as can be seen from Fig. 2.31, much more pronounced if the coupling condition $i=2k$ $(=2m)$ is satisfied. Thus besides the bifurcation point on the upper branch of the single mode response curve (which now occurs at $\Omega=0.99733$) there are now also 2 bifurcation points on the lower branch of the single mode response curve, one at $\Omega=0.99563$ and the other at $\Omega=0.99703$.

Finally, Fig. 2.32 displays the effect of both an axisymmetric and an asymmetric imperfection. Notice that the combined imperfection model

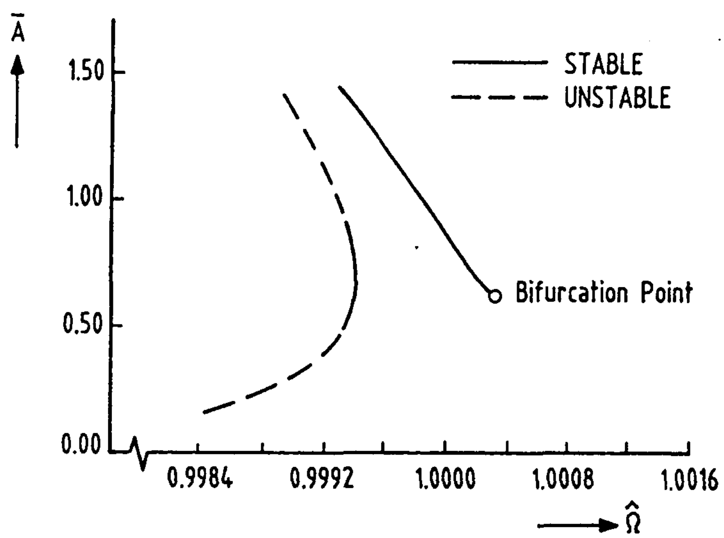
$$\frac{\bar{W}}{h} = -\,0.04 \cos \frac{2\pi x}{L} + 0.05 \sin \frac{\pi x}{L} \cos \frac{5y}{R}$$

satisfies the strong coupling condition $i=2k$; also the axisymmetric imperfection is directed inward over the central portion of the shell. However, neither the axisymmetric nor the asymmetric mode are affine to classical buckling modes. (For shell ES2 such modes would be $i=58$ and $k=29$, $\ell=14$.)

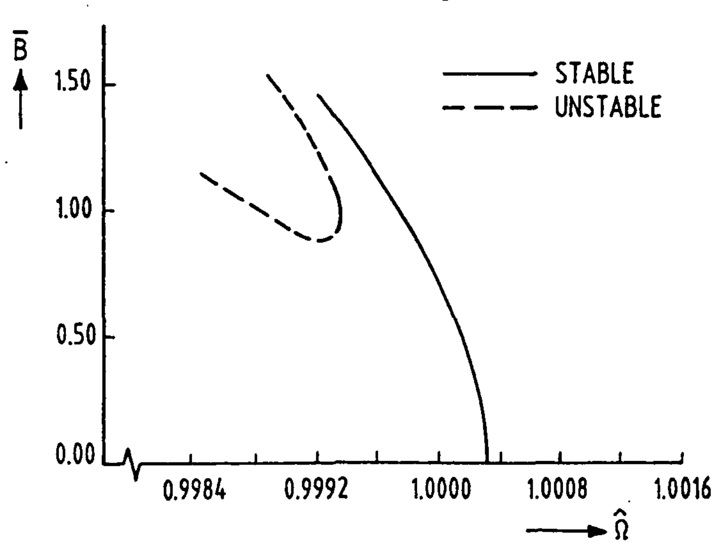
This was done on purpose, because experimental measurements [80] on thin-walled seamless isotropic cylinders indicate that the amplitudes of the measured imperfection harmonics decay exponentially with increasing wave numbers. The imperfection model used is thought to be representative of the quality attainable with carefully made laboratory scale shell specimens.




a. Single Mode Response

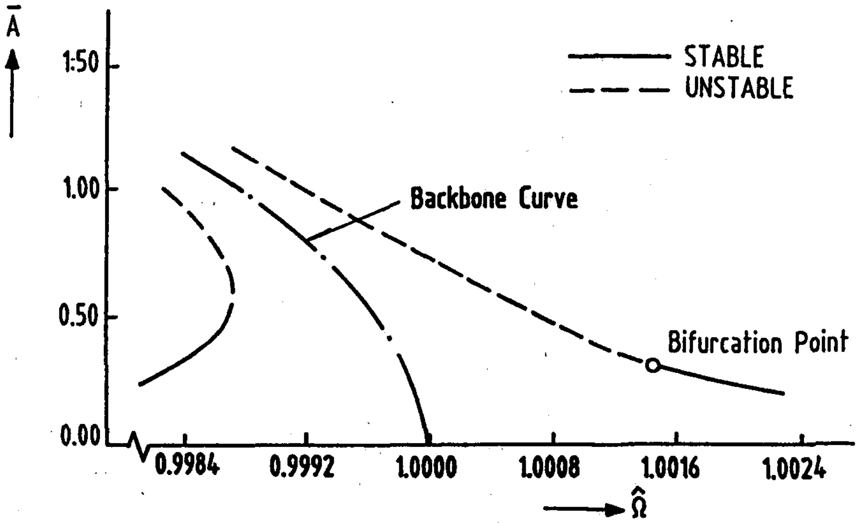



b. Driven Mode Response

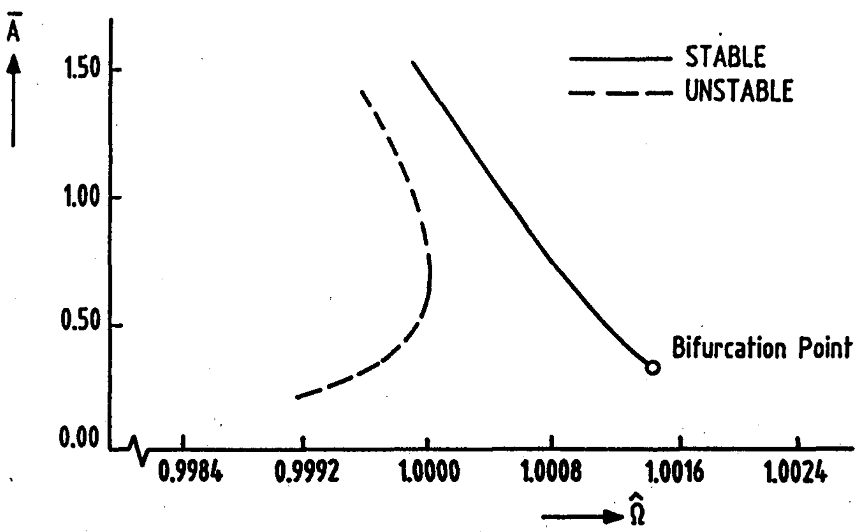



c. Companion Mode Response

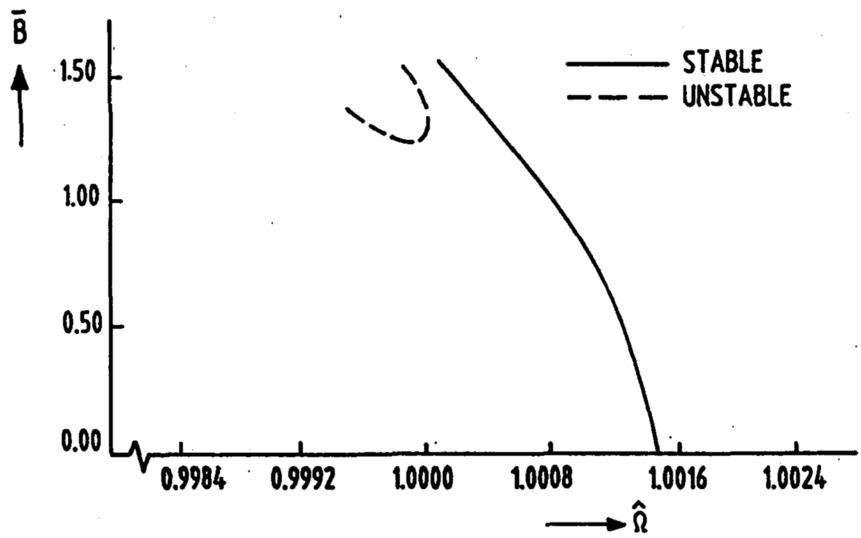
Fig. 2.29 Amplitude-frequency relationship of undamped vibration of perfect shell.
ES2 Shell; k=m=1, n=ℓ=5; excitation $\bar{F}_D=2\times10^{-6}$




a. Single Mode Response



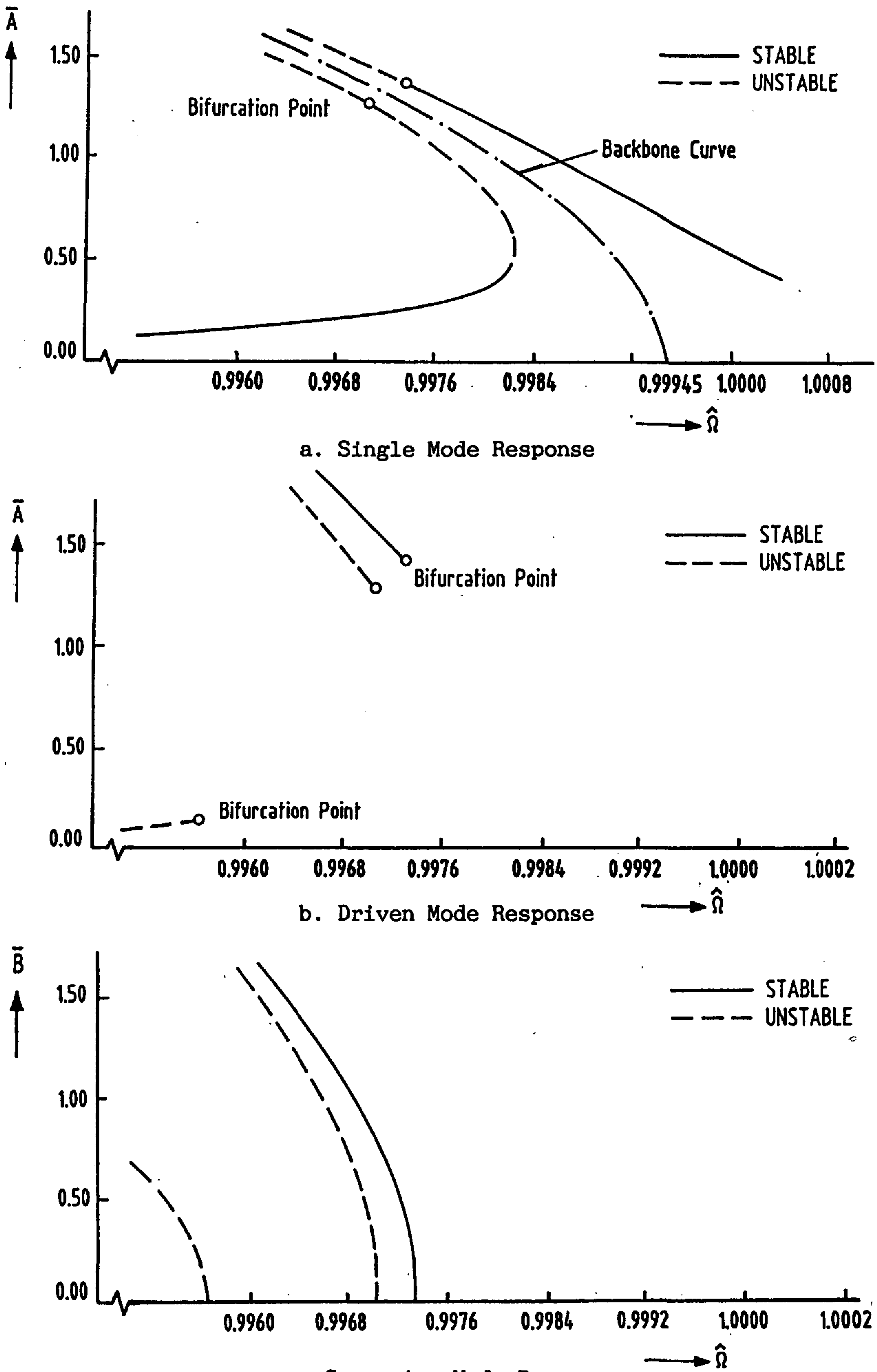

b. Driven Mode Response




c. Companion Mode Response

Fig. 2.30 Amplitude-frequency relationship of undamped vibration of imperfect shell with $\delta_1$ = 0.0 and $\delta_2$ = 0.05.

ES2 Shell; k=m=1, n=$\ell$=5; excitation $\bar{F}_D$=2x10$^{-6}$



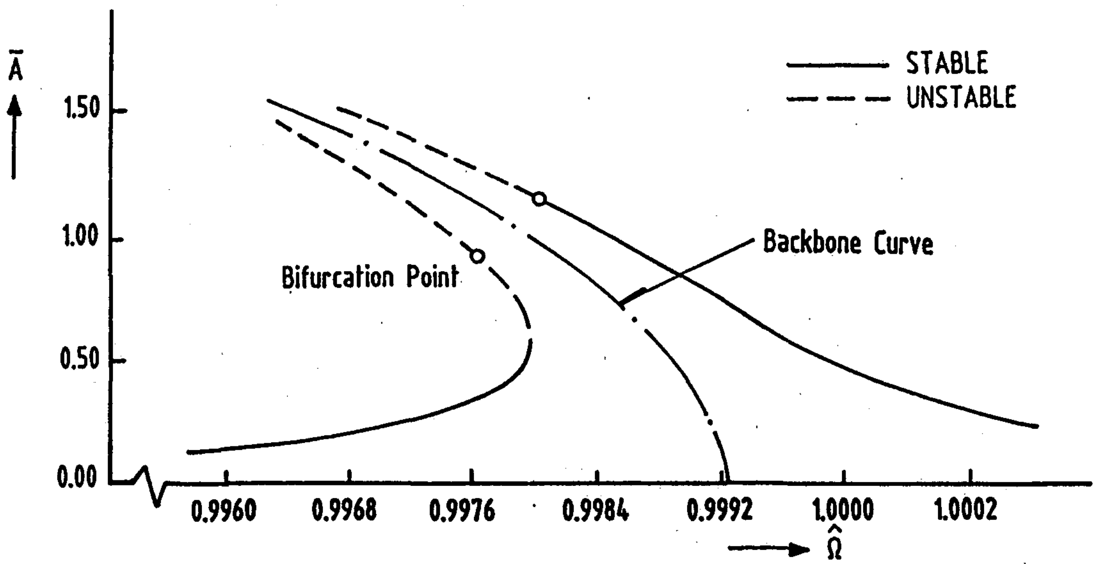

a. Single Mode Response

b. Driven Mode Response

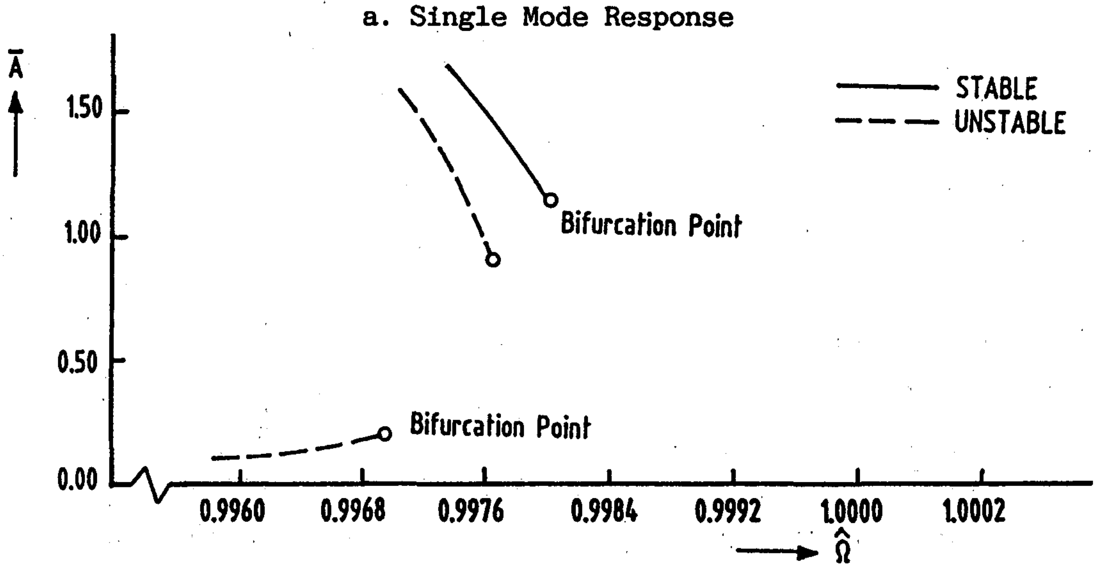
c. Companion Mode Response

Fig. 2.31 Amplitude-frequency relationship of undamped vibration of imperfect shell with $\delta_1 = -0.04$ and $\delta_2 = 0.00$.

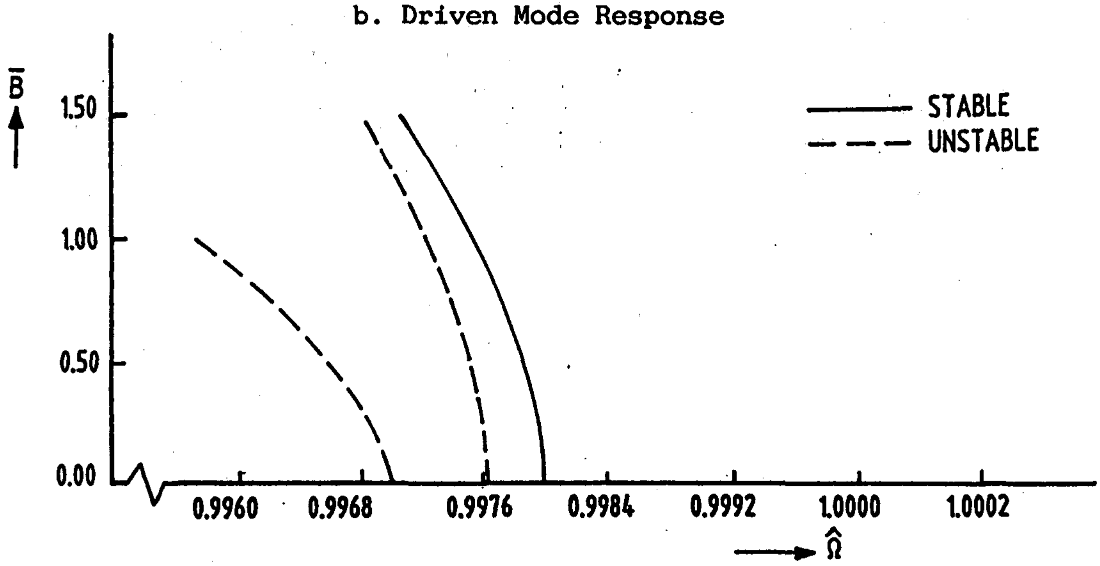
ES2 Shell; i=2, k=m=1, n=ℓ=5; excitation $\bar{F}_D=2x10^{-6}$




a. Single Mode Response




b. Driven Mode Response




c. Companion Mode Response

Fig. 2.32 Amplitude-frequency relationship of undamped vibration of imperfect shell with $\delta_1 = -0.04$ and $\delta_2 = 0.05$.

ES2 Shell; i=2, k=m=1, n=$\ell$=5; excitation $\bar{F}_D = 2 \times 10^{-6}$

## 2.4.3 CONCLUSIONS

The problem of the undamped nonlinear vibrations of the thin-walled smooth stiffened cylindrical shell with initial geometric imperfections was treated in this chapter. Donnell's nonlinear equations were used to formulate the problem mathematically. The stiffeners were incorporated by the 'smeared' approach. An idealized model consisting of one axisymmetric and one asymmetric component is assumed for the initial geometric imperfections. Hol's solutions of the fundamental state were directly adopted. Galerkin's method and the method of averaging were employed in sequence to obtain a set of coupled nonlinear algebraic equations, from which the amplitude-frequency relationships were derived. These equations are rather general and they can be used to study the nonlinear vibration behaviour of orthotropic or isotropic, of perfect or imperfect, of axially loaded or unloaded circular cylindrical shells. In the present study, only axially compressed isotropic shells with initial geometric imperfections were studied numerically.
The principal conclusions of the present study are

1. Initial geometric imperfections may have a significant influence on the nonlinear vibrations of cylindrical shells. This influence depends mainly on the enforcing of certain coupling conditions between the initial imperfection and the vibration mode shapes.

2. The general observation concerning the single mode vibration is that if the circumferential wave number n of the asymmetric imperfection is not identical to $\ell$, the circumferential wave number of the vibration mode, then the frequencies will increase about equally for the different amplitudes of vibration with increasing amplitude of imperfection. The basic vibration behaviour is not changed.

3. In the case of $n = \ell$, which means that the shell is vibrating in the same pattern as that of the asymmetric imperfection, the influence of the asymmetric imperfection on the single mode vibration is quite strong and it depends on the aspect ratio $\xi$ of the vibration and on the amplitude of the initial imperfection.
   For high values of aspect ratios ($\xi > \pi/2$, say) the nonlinearity is of the hardening type and the frequencies increase continuously with increasing imperfection amplitude. On the other hand for low values of aspect ratios ($\xi < \pi/2$, say) the nonlinearity changes to being of the softening type for values of imperfection smaller than a certain critical value. For asymmetric imperfections larger than the critical value the nonlinearity switches once again to being of the hardening type.

4. The influence of the axisymmetric imperfection on the vibration is found to be significant only if the coupling condition $i = 2k$ is satisfied. The 'critical value' and the 'shifting' phenomena found in the case of asymmetric imperfection, were also found for the axisymmetric imperfection. This means that the presence of the axisymmetric imperfection can also change the degree of nonlinearity and the vibration behaviour.

5. The influence of the combined imperfections on the vibration can become complicated due to the nonlinear coupling between the asymmetric and the axisymmetric imperfections. It seems that the total effect of combined imperfections, in general, cannot be obtained by superposition of the separate effects of the asymmetric and the axisymmetric imperfections acting alone.

6. It was found that the presence of the axial compressive load results in a softening type behaviour.

# CHAPTER 3 DAMPED NONLINEAR VIBRATIONS

## 3.1 INTRODUCTION

The damped nonlinear vibrations of imperfect thin walled cylindrical shells with SS3 boundary conditions are studied in this chapter. The analytical procedures and assumptions discussed in chapter 2 are used in this chapter without any further explanation. Only the new ones are specifically mentioned.

In engineering the study of damped vibrations is of great importance since any realistic structure has some inherent material damping. The results available so far show that the damping has a pronounced influence on the nonlinear vibration of shells [30,82,89].

As mentioned in chapter 2, one of the conclusions that can be drawn from the results of previous studies is that although some basic characteristics of the damped vibration behaviour of shells have been derived analytically and also verified experimentally, there are certain situations where considerable disagreement still exists between results obtained by different analytical procedures and also between theoretical predictions and experimental evidence.

The objective of this chapter is to investigate the effect of the initial geometric imperfections on the damped nonlinear vibration of shells. This objective is the natural extension of chapter 2. The emphasis of the current work is placed on the influence of initial geometric imperfections on the coupled mode response for which no solution is as yet available. In addition, the author also intends to study the discrepancies between the results of earlier studies and to get a reasonable explanation, if it is possible.

## 3.2 ANALYSIS

Through the appropriate operations and the application of Galerkin's procedure to the equations (1-3-40) and (1-3-41) one obtains two coupled nonlinear differential equations for A(t), B(t) and C(t):

$$\bar{\alpha}_1 \frac{d^2A}{dt^2} + \bar{\alpha}_2 \frac{dA}{dt} + \bar{\alpha}_3 A + \bar{\alpha}_4 \frac{d^2C}{dt^2}[A + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \bar{\alpha}_5 \frac{dC}{dt}[A + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] +$$
$$+ \bar{\alpha}_6 A^2 + \bar{\alpha}_7(A^2+B^2) + \bar{\alpha}_8(A^2-B^2) + \bar{\alpha}_9 A^3 + \bar{\alpha}_{10}(A^2+B^2)A +$$
$$+ \bar{\alpha}_{11}(A^2+B^2)A^2 + \bar{\alpha}_{12}(A^2+B^2)^2 + \bar{\alpha}_{13}(A^2+B^2)^2 A = F_D \qquad (3\text{-}2\text{-}1)$$

$$\bar{\beta}_1 \frac{d^2B}{dt^2} + \bar{\beta}_2 \frac{dB}{dt} + \bar{\beta}_3 B + \bar{\beta}_4 \frac{d^2C}{dt^2} + \bar{\beta}_5 B \frac{dC}{dt} + \bar{\beta}_6 AB + \bar{\beta}_7 AB + \bar{\beta}_8(A^2+B^2)B +$$
$$+ \bar{\beta}_9(A^2+B^2)AB + \bar{\beta}_{10}(A^2+B^2)^2 B = F_C \qquad (3\text{-}2\text{-}2)$$

where the $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_{13}$ and $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_{10}$ are coefficients which are defined in Appendix 3-A and $F_D$ and $F_C$ are generalized dynamic forces.

An expression for C comes from the periodicity condition yielding

$$C = \frac{Rh}{4} \ell^2_\ell \left[A^2 + B^2 + 2\delta_{n,\ell} (\hat{\delta}_2 + \delta_2)\right] \tag{3-3-3}$$

A(t) and B(t) are taken to be of the form

$$A(t) = A_t(t) \cos(\omega t + \phi) \tag{3-3-4}$$

and

$$B(t) = B_t(t) \sin(\omega t + \psi) \tag{3-3-5}$$

where $\phi$ and $\psi$ are the phase angles between the driven mode and the companion mode and the excitation respectively. These angles are functions of time. Substituting equations (3-3-4), (3-3-5) and (3-3-3) into equations (3-2-1) and (3-2-2) and then applying the method of averaging yield the approximate solutions for A(t) and B(t)

$$A(t) = \bar{A}(t) \cos(\omega t + \bar{\phi}) \tag{3-3-6}$$

$$B(t) = \bar{B}(t) \sin(\omega t + \bar{\psi}) \tag{3-3-7}$$

where $\bar{A}$, $\bar{B}$, $\bar{\phi}$ and $\bar{\psi}$ are the average value of the A(t), B(t), $\phi(t)$ and $\psi(t)$ over one period respectively. They can be obtained by solving the following simultaneous nonlinear algebraic equations for a given average excitation $\bar{F}_D$, damping $\gamma$ and forcing frequency $\Omega$

$$\begin{aligned}
-\Omega^2\bar{A} \{1 &+ \beta_1[\bar{A}^2 - \bar{B}^2 \cos 2\bar{\Lambda} + 2\,\delta_{n,\ell}\,(\delta_2 + \hat{\delta}_2)^2]\} + \beta_2\bar{A} - \gamma\Omega\bar{A}\bar{B}^2\beta_1 \sin 2\bar{\Lambda} + \\
&+ \beta_3\bar{A}^3 + 2\beta_4\bar{A}\bar{B}^2\,(1 - \tfrac{1}{2}\cos 2\bar{\Lambda}) + \beta_5\bar{A}\,[5\bar{A}^4 + 4\bar{A}^2\bar{B}^2\,(\tfrac{3}{2} - \cos 2\bar{\Lambda}) + \\
&+ 2\bar{B}^4\,(\tfrac{3}{2} - \cos 2\bar{\Lambda})] = \bar{F}_D \cos\bar{\phi}
\end{aligned} \tag{3-3-8}$$

$$\begin{aligned}
\{\bar{A}\bar{B}^2[\beta_1\Omega^2 &- \beta_4 - 2\,\beta_5\bar{A}(\bar{A}^2 + \bar{B}^2)]\} \sin 2\bar{\Lambda} - \Omega\gamma\,\{2\bar{A} + \beta_1[\bar{A}^3 - \bar{A}\bar{B}^2 \cos 2\bar{\Lambda} + \\
&+ 4\bar{A}\delta_{n,\ell}\,(\hat{\delta}_2 + \delta_2)^2]\} = \bar{F}_D \sin\bar{\phi}
\end{aligned} \tag{3-3-9}$$

$$\begin{aligned}
-\,\Omega^2\bar{B}\{1 &+ \beta_6(\bar{B}^2 - \bar{A}^2\cos 2\bar{\Lambda})\} + \beta_7\bar{B} + \Omega\gamma\bar{A}^2\bar{B}\beta_6 \sin 2\bar{\Lambda} + \beta_9\bar{B}^3 + \\
&+ 2\beta_8\bar{A}^2\bar{B}(1 - \tfrac{1}{2}\cos 2\bar{\Lambda}) + \beta_{10}\bar{B}\{5\bar{B}^4 + [4\bar{A}^2\bar{B}^2 + 2\bar{A}^4]\,(\tfrac{3}{2} - \cos 2\bar{\Lambda})\} = 0
\end{aligned} \tag{3-3-10}$$

$$\{\bar{A}^2\bar{B}\,[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)]\}\sin 2\bar{\Delta} + \Omega\gamma\{2\bar{B} + \beta_6[\bar{B}^3 - \bar{A}^2\bar{B}\cos 2\Delta]\} = 0 \tag{3-3-11}$$

where

$\gamma = cR\sqrt{\frac{1}{2\bar{\rho}E}}$, is the generalized nondimensional damping, $\bar{\Delta} = \bar{\Phi} - \bar{\psi}$, is the "average difference of phase angles", and $\beta_1, \beta_2, \ldots, \beta_{10}$ are coefficients which are given in the Appendix 2-A.

The details of derivations of equations (3-3-8) ~ (3-3-11) are given in Appendix 3-C.

The analysis is carried out for two separate cases.

### 3.2.1 SINGLE MODE RESPONSE ($\bar{A} \neq 0$, $\bar{B} = 0$)

As can be seen $\bar{B} = 0$ is a possible solution of equations (3-3-8) ~ (3-3-11). In this case they become

$$-\Omega^2\bar{A}\,\{1 + \beta_1\bar{A}^2 + 2\beta_1\delta_{n,\ell}\,(\delta_2 + \hat{\delta}_2)^2\} + \beta_2\bar{A} + \beta_3\bar{A}^3 + 5\beta_5\bar{A}^5 = \bar{F}_D\cos\bar{\Phi} \tag{3-3-12}$$

$$-\Omega\gamma\,\{2\bar{A} + \beta_1\bar{A}^3 + 4\beta_1\bar{A}\delta_{n,\ell}\,(\hat{\delta}_2 + \delta_2)^2\} = \bar{F}_D\sin\bar{\Phi} \tag{3-3-13}$$

A single equation governing the amplitude-frequency relationship of the single mode response is obtained by first squaring both equations, then adding them and finally using the identity

$$\sin^2\bar{\Phi} + \cos^2\bar{\Phi} = 1$$

This yields

$$\alpha_1\Omega^4 + \alpha_2\Omega^2 + \alpha_3 = 0 \tag{3-3-14}$$

where

$$\alpha_1 = \bar{A}^2\{1 + \beta_1\,[\bar{A}^2 + 2\delta_{n,\ell}\,(\hat{\delta}_2 + \delta_2)^2]\}^2$$

$$\alpha_2 = -\,2\bar{A}^2\,\{1 + \beta_1\,[\bar{A}^2 + 2\delta_{n,\ell}\,(\hat{\delta}_2 + \delta_2)^2]\}\,\{\beta_2 + \beta_3\bar{A}^2 + 5\beta_5\bar{A}^4\} + \\ + \gamma^2\bar{A}^2\,\{2 + \beta_1\bar{A}^2 + 4\beta_1\delta_{n,\ell}(\hat{\delta}_2 + \delta_2)^2\}^2$$

$$\alpha_3 = \bar{A}^2\,\{\beta_2 + \beta_3\bar{A}^2 + 5\beta_5\bar{A}^4\}^2 - \bar{F}_D^2$$

It is obvious that one can obtain not only the damped response for various damping and excitation levels but also the free vibration (or undamped response) if one lets the damping and excitation terms vanish in equation (3-3-14).

### 3.2.2 COUPLED MODE RESPONSE ($\bar{A} \neq 0$, $\bar{B} \neq 0$)

Another possible solution of equations (3-3-8) ~ (3-3-11) is $\bar{A} \neq 0$ and $\bar{B} \neq 0$, namely the damped coupled mode response.
For the damped couple mode response a direct simultaneous solution of equations (3-3-8) ~ (3-3-11)) is too cumbersome. A further simplification can be carried out as follows. Initially one solves equations (3-3-10) and (3-3-11) for $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$ in terms of $\bar{A}$ and $\bar{B}$, respectively. Then one uses the identity $\sin^2 2\bar{\Delta} + \cos^2 2\bar{\Delta} = 1$, which results in a single equation with the unknowns $\bar{A}$ and $\bar{B}$. Next one back-subtitutes for $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$ in equations (3-3-8) and (3-3-9) and then uses the identity $\sin^2\bar{\Phi} + \cos^2\bar{\Phi} = 1$, which yields a second equation with the unknowns $\bar{A}$ and $\bar{B}$.

The amplitude-frequency relationship of damped, coupled response then can be obtained by solving these two nonlinear algebraic equations simultaneously for given values of damping, imperfection and excitation. The equations can be expressed in the form

$$\begin{aligned} &\langle -\Omega^2\bar{A}\,\{1 + \beta_1[\bar{A}^2 - \bar{B}^2 \cos 2\bar{\Delta} + 2\delta_{n,\ell}\,(\hat{\delta}_2 + \delta_2)^2]\} + \beta_2\bar{A} - \gamma\Omega\bar{A}\bar{B}^2\beta_1 \sin 2\bar{\Delta} + \\ &+ \beta_3\bar{A}^3 + 2\beta_4\bar{A}\bar{B}^2\,(1 - \tfrac{1}{2}\cos 2\bar{\Delta}) + \beta_5\bar{A}\,[5\bar{A}^4 + 4\bar{A}^2\bar{B}^2\,(\tfrac{3}{2} - \cos 2\bar{\Delta}) + \\ &+ 2\bar{B}^4(\tfrac{3}{2} - \cos 2\bar{\Delta})]\rangle^2 + \langle\{\bar{A}\bar{B}^2[\beta_1\Omega^2 - \beta_4 - 2\,\beta_5\,(\bar{A}^2 + \bar{B}^2)]\}\sin 2\bar{\Delta} + \\ &- \Omega\gamma\,\{2\bar{A} + \beta_1\,[\bar{A}^3 - \bar{A}\bar{B}^2\cos 2\bar{\Delta} + 4\bar{A}\delta_{n,\ell}\,(\hat{\delta}_2 + \delta_2)^2]\}\rangle^2 - \bar{F}_D^2 = 0 \qquad (3\text{-}3\text{-}15) \end{aligned}$$

and

$$\hat{\alpha}_1\bar{B}^{12} + \hat{\alpha}_2\bar{B}^{10} + \hat{\alpha}_3\bar{B}^8 + \hat{\alpha}_4\bar{B}^6 + \hat{\alpha}_5\bar{B}^4 + \hat{\alpha}_6\bar{B}^2 + \hat{\alpha}_7 = 0 \qquad (3\text{-}3\text{-}16)$$

or

$$\hat{\beta}_1\bar{A}^{12} + \hat{\beta}_2\bar{A}^{10} + \hat{\beta}_3\bar{A}^8 + \hat{\beta}_4\bar{A}^6 + \hat{\beta}_5\bar{A}^4 + \hat{\beta}_6\bar{A}^2 + \hat{\beta}_7 = 0 \qquad (3\text{-}3\text{-}17)$$

where

$$\begin{aligned} \sin 2\bar{\Delta} = &- \gamma\Omega\{2[\beta_6\Omega^2 - \beta_8 - \beta_{10}(4\bar{B}^2 + 2\bar{A}^2)] + \beta_6[\beta_7 - \Omega^2 + \bar{B}^2(\beta_9 - \beta_8) + \\ &+ 2\beta_8\bar{A}^2 + \beta_{10}(\bar{B}^4 + 4\bar{A}^2\bar{B}^2 + 3\bar{A}^4)]\}/\,S_d \qquad (3\text{-}3\text{-}18) \end{aligned}$$

$$\cos 2\bar{\lambda} = \{\Omega^2\gamma^2\beta_6(2 + \beta_6\bar{B}^2) + [\Omega^2(1+\beta_6\bar{B}^2) - \beta_7 - \beta_9\bar{B}^2 - 2\beta_8\bar{A}^2 +$$

$$- \beta_{10}(5\bar{B}^4+6\bar{A}^2\bar{B}^2 + 3\bar{A}^4)][\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)]\}/ S_d \quad (3\text{-}3\text{-}19)$$

and

$$S_d = (\bar{A}\beta_6\gamma\Omega)^2 + \bar{A}^2[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)][\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + 2\bar{B}^2)] \quad (3\text{-}3\text{-}20)$$

Here, $\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_7$ are functions of $\bar{A}$ only and $\hat{\beta}_1, \hat{\beta}_2, \ldots, \hat{\beta}_7$ are functions of $\bar{B}$ only and which are given in Appendix 3-B.

One can get solutions for the driven mode $\bar{A}$ by numerically solving equations (3-3-15) and (3-3-16) for given amplitudes of the companion mode $\bar{B}$ or get solutions for the companion mode $\bar{B}$ from equations (3-3-15) and (3-3-17) for given amplitudes of the driven mode $\bar{A}$.

## 3.3 DISCUSSION OF RESULTS

Equations (3-3-15) and (3-3-16) and equations (3-3-15) and (3-3-17) are two sets of nonlinear algebraic equations for the two unknowns $\bar{A}$ and $\bar{B}$. A direct solution for $\bar{A}$ and $\bar{B}$ as functions of $\bar{F}_D$, $\Omega$ and $\gamma$ is quite difficult. Therefore the normal procedure is to calculate $\bar{B}$ from equation (3-3-16) for given values of $\bar{A}$, $\Omega$ and $\gamma$ (or to calculate $\bar{A}$ from equation (3-3-17) for given $\bar{B}$, $\Omega$, and $\gamma$), then calculate the generalized excitation $\bar{F}_D$ from equation (3-3-15) upon substituting $\bar{B}$ (or $\bar{A}$) for given values of $\bar{A}$ (or $\bar{B}$), $\Omega$ and $\gamma$. By cross-plotting the results, it is possible to obtain curves for $\bar{A}$ vs. $\Omega$ and $\bar{B}$ vs. $\Omega$ for constant $\bar{F}_D$.

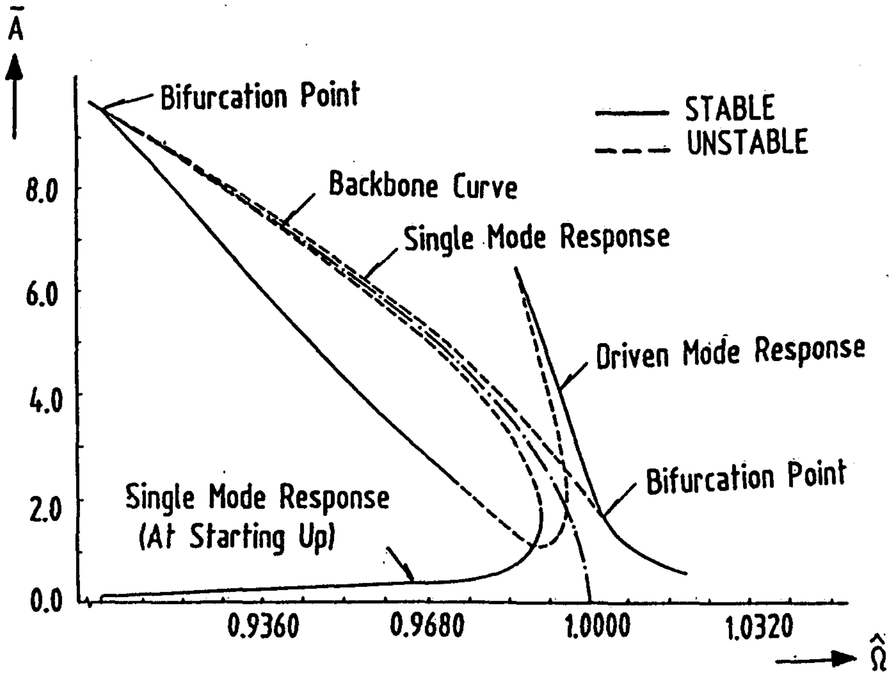
To obtain the necessary accuracy in the solutions, the Newton-Raphson procedure is used in the present analysis, which takes the results obtained from cross-plotting as the starting values.

The isotropic shell used in chapter 2, called ES2 is used herein.The wave numbers of vibration modes are chosen such that

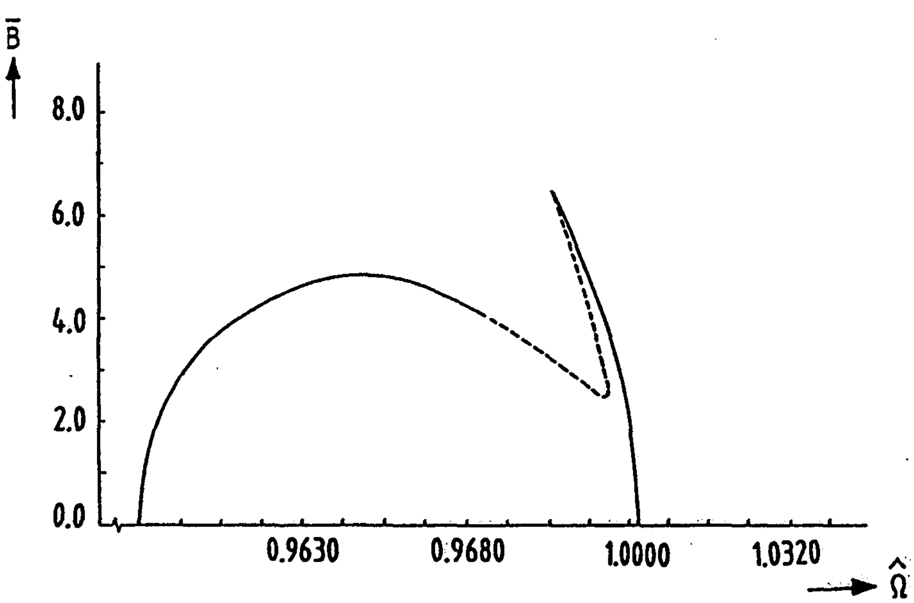
a. They satisfy the accuracy requirements of Donnell's equations, namely the circumferential wave number must be greater than 4;
b. They would constitute lower order modes which can be excited easily into the nonlinear region to make experimental verification possible.

In the present analysis the mode k=1, $\ell$=5 was selected. Various values of i and n were selected depending on the different coupling conditions.

A series of computations were performed for damped single and coupled mode responses. In order to facilitate understanding, the discussions of these numerical results are divided into five categories.

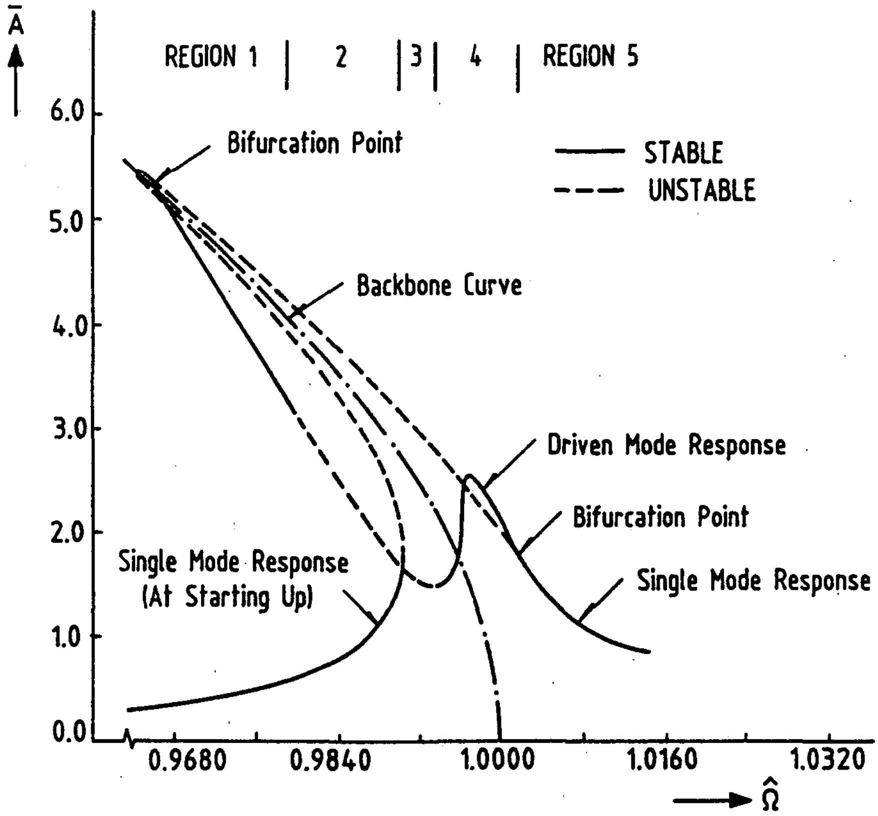



a. Single mode and driven mode response

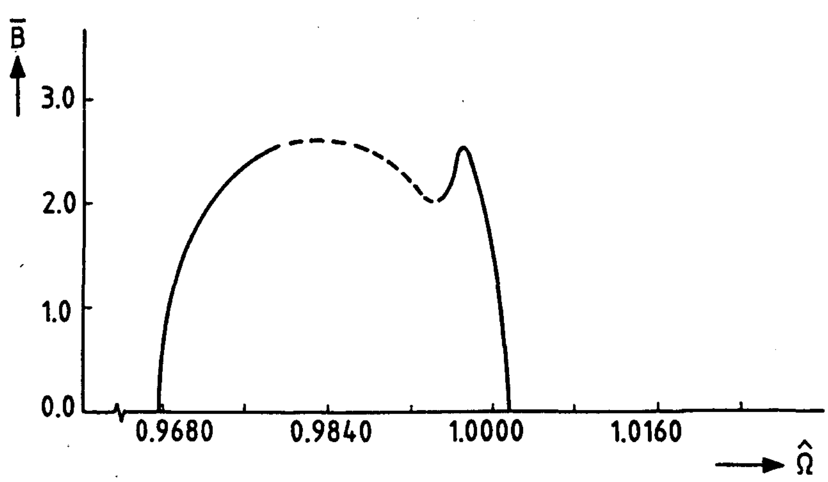



b. Companion mode response

Fig. 3.1 Amplitude-frequency relationship of damped vibration of perfect shell. Damping $\gamma=5\times10^{-5}$, excitation $\bar{F}_D=4.25\times10^{-5}$.
ES2 Shell; k=m=1, n=ℓ=5

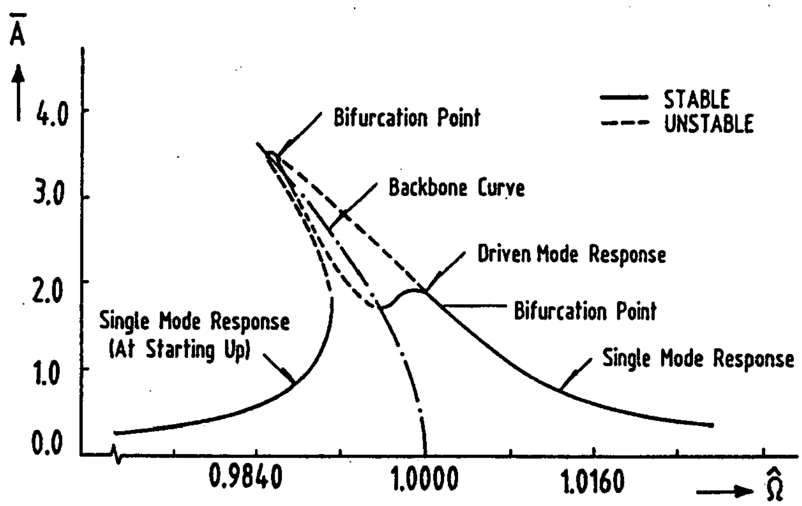



a. Single mode and driven mode response

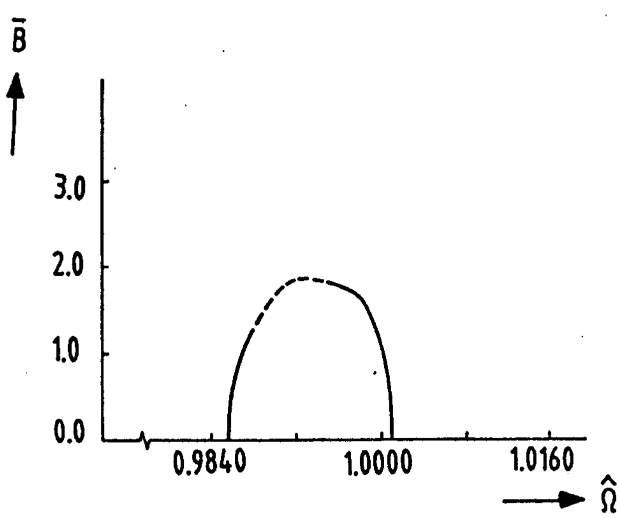



b. Companion mode response

Fig. 3.2 Amplitude-frequency relationship of damped vibration of perfect shell. Damping $\gamma=9\text{x}10^{-5}$, excitation $\bar{F}_D=4.25\text{x}10^{-5}$.
ES2 Shell; k=m=1, n=$\ell$=5

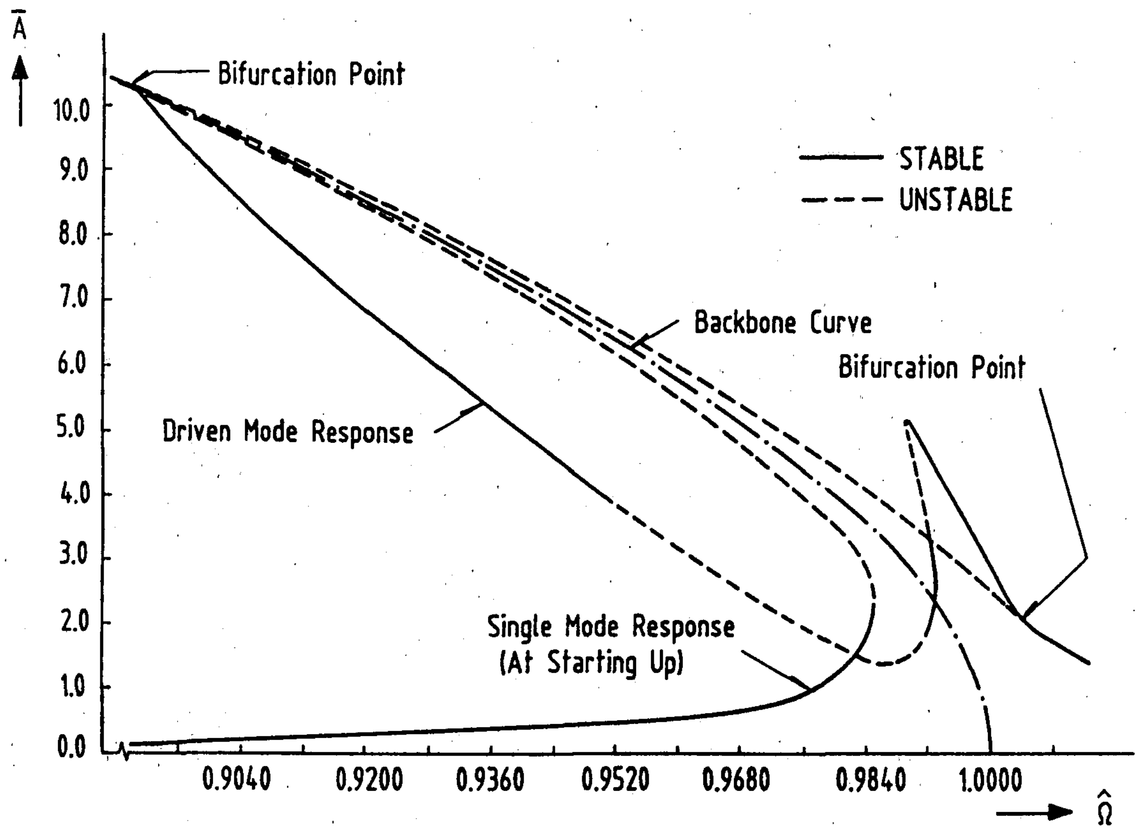



a. Single mode and driven mode response

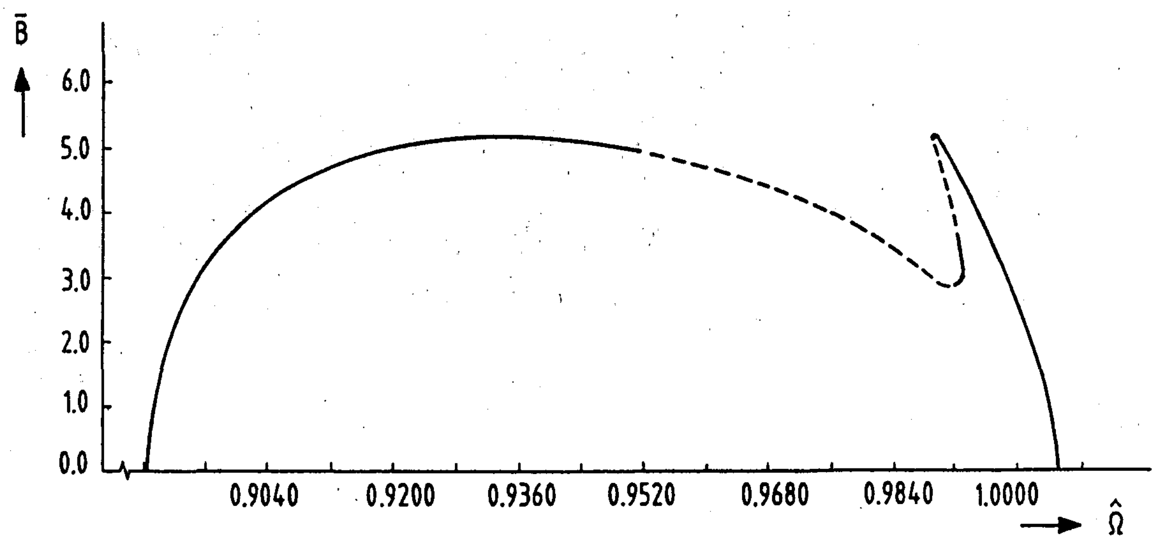



b. Companion mode response

Fig. 3.3 Amplitude-frequency relationship of damped vibration of perfect shell. Damping $\gamma$=1.35x10$^{-4}$, excitation $\bar{F}_D$=4.25x10$^{-5}$. ES2 Shell; k=m=1, n=$\ell$=5




a. Single mode and driven mode response




b. Companion mode response

Fig. 3.4 Amplitude-frequency relationship of damped vibration of perfect shell. Damping $\gamma=9\times10^{-5}$, excitation $\bar{F}_D=8.50\times10^{-5}$.
ES2 Shell; $k=m=1$, $n=\ell=5$

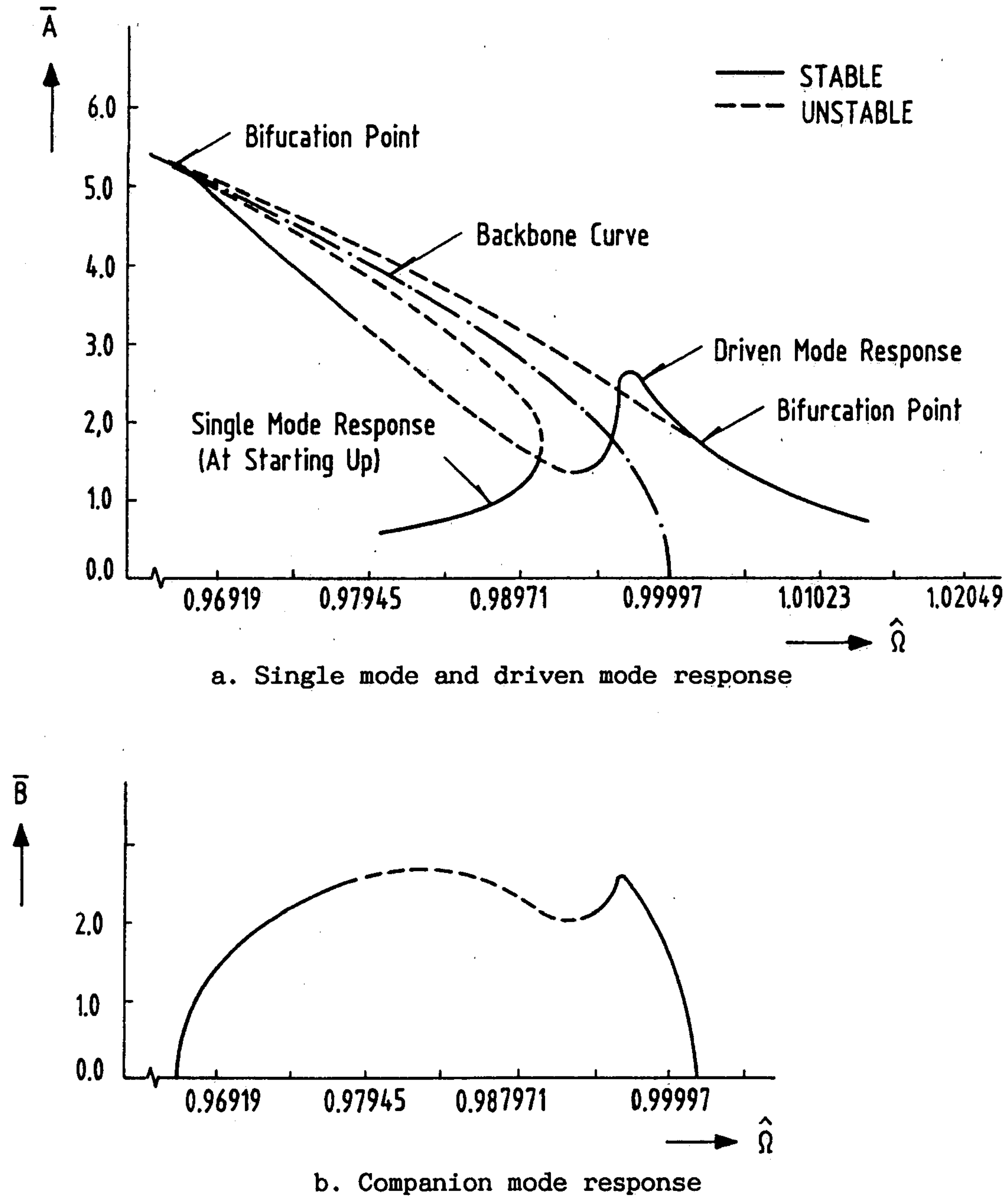
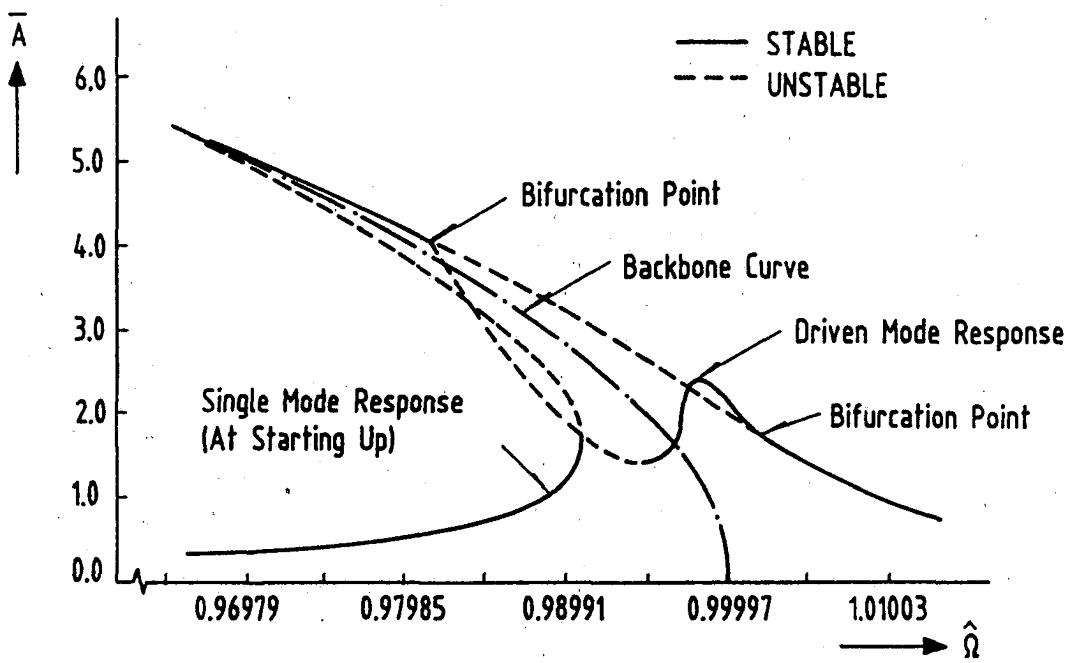


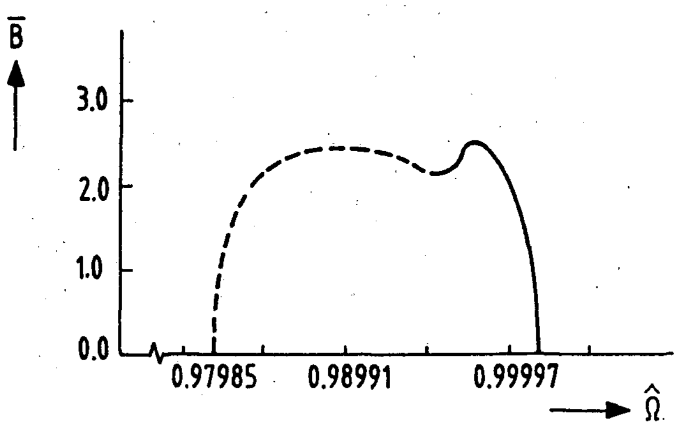

Fig. 3.5 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=0.00 and $\delta_2$=0.01.

Damping $\gamma=9\mathrm{x}10^{-5}$, excitation $\bar{F}_D=4.25\mathrm{x}10^{-5}$.

ES2 Shell; k=m=1, n=ℓ=5




a. Single mode and driven mode response

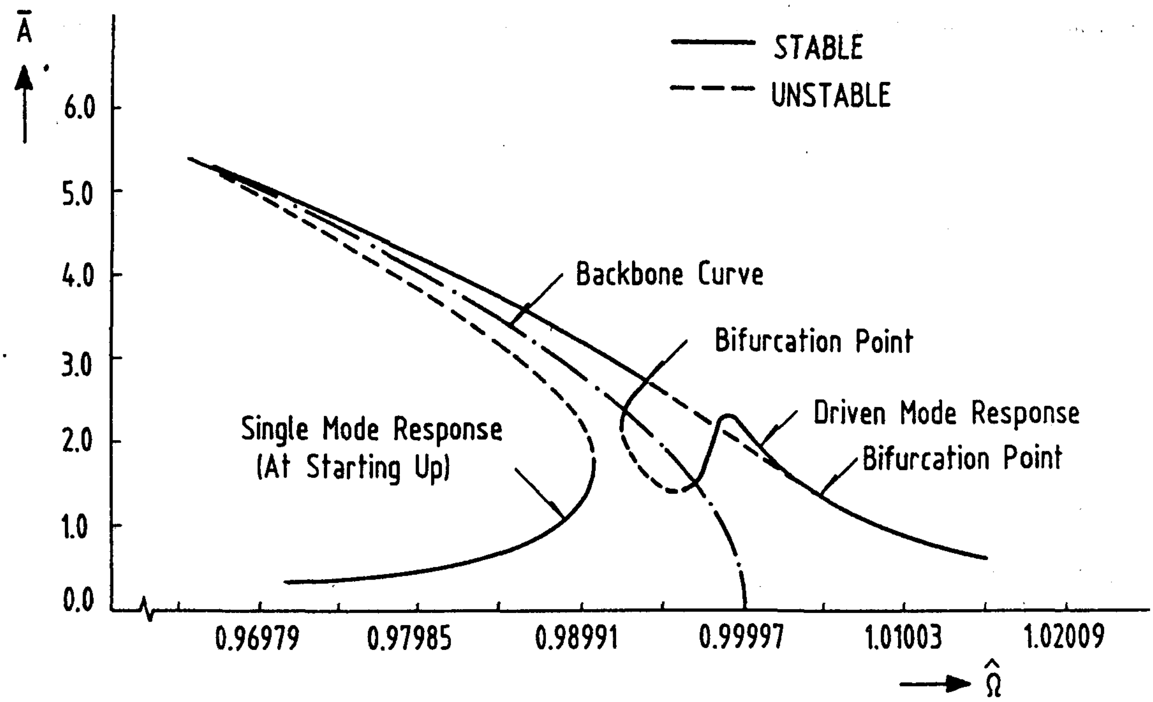



b. Companion mode response

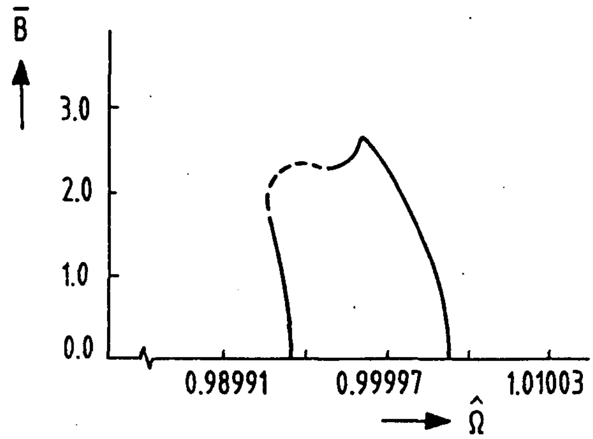
Fig. 3.6 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=0.00 and $\delta_2$=0.05.

Damping $\gamma=9\text{x}10^{-5}$, excitation $\bar{F}_D=4.25\text{x}10^{-5}$.

ES2 Shell; k=m=1, n=ℓ=5




a. Single mode and driven mode response

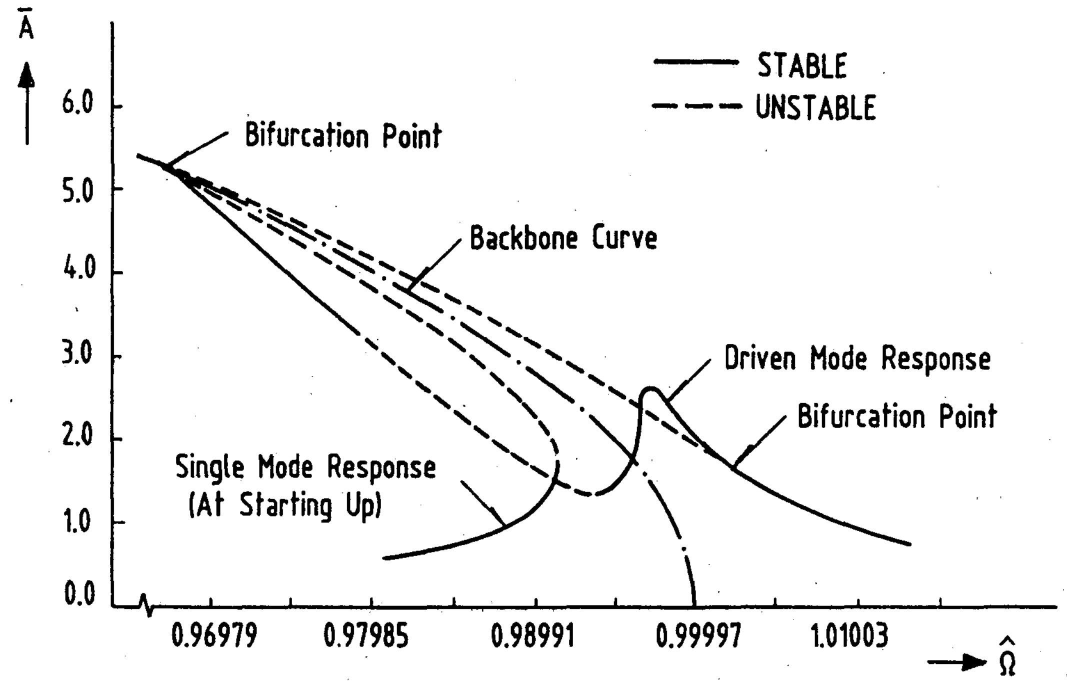



b. Companion mode response

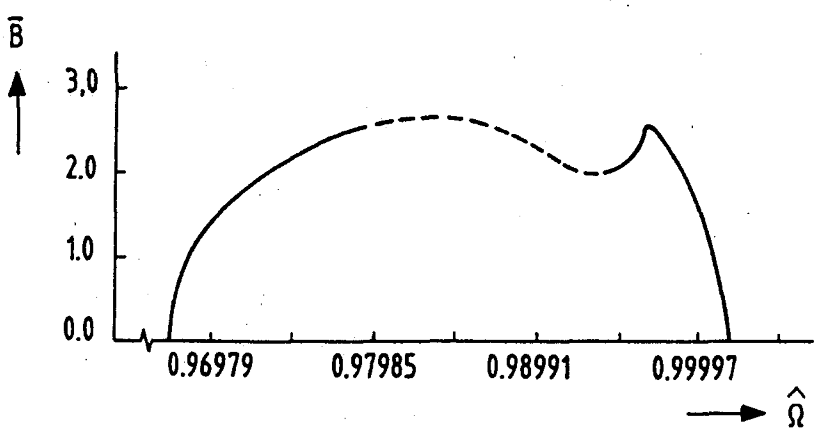
Fig. 3.7 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=0.00 and $\delta_2$=0.07.

Damping $\gamma=9\times10^{-5}$, excitation $\bar{F}_D=4.25\times10^{-5}$.

ES2 Shell; k=m=1, n=$\ell$=5




a. Single mode and driven mode response




b. Companion mode response

Fig. 3.8 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=0.00 and $\delta_2$=0.05.

Damping $\gamma=9\text{x}10^{-5}$, excitation $\bar{F}_D=4.25\text{x}10^{-5}$.

ES2 Shell; k=m=1, n=10, $\ell$=5

## 3.3.1 INFLUENCE OF DAMPING AND EXCITATION

Figures* 3.1 to 3.4 show the amplitude-frequency relationships of a perfect shell for different values of damping and excitation. One can draw the conclusion that the damping has a strong influence on the behaviour of the coupled mode response after comparing the present results with those of chapter 2 (see Fig. 2.29). The presence of even very small damping changes the shape of the undamped coupled mode response completely. As can be seen from Figures 3.2 and 3.3, increasing the damping can quickly round-off the coupled-mode response peak. Similar results have been found by Ginsberg [87].

An interesting fact can be deduced from the results shown in figures 3.1 (or 3.2) and 3.4; namely, that increasing the amplitude of excitation (or decreasing the value of damping) can disrupt the stability of the coupled mode response peak. This fact has been found by Chen [82] in his careful experiments of fifteen years ago, but has not been predicted by theoretical analysis before.

## 3.3.2 INFLUENCE OF ASYMMETRIC IMPERFECTIONS

The amplitude-frequency relationships of a shell with the asymmetric imperfections are shown in Figures 3.5 to 3.8, where the damping and excitation are held constant. It can be observed that asymmetric imperfections have a significant influence on the coupled mode response if the coupling condition $n=\ell$ is satisfied. Increasing the amplitude of the asymmetric imperfection can quickly decrease the region where the coupled mode response occurs. The asymmetric imperfection also changes the stability characteristics of the coupled-mode response. But, as it can be seen from Fig. 3.8, the influence of the asymmetric imperfections on the coupled-mode response is minimal if the coupling condition $n=\ell$ is not satisfied. It should be noted that the influence of the asymmetric imperfections on the shape of the single mode amplitude-frequency curve is also practically nil.

## 3.3.3 INFLUENCE OF AXISYMMETRIC IMPERFECTIONS

Figures 3.10 to 3.12 display the amplitude-frequency relationships of the ES2 shell with axisymmetric imperfections. As can be seen from figures 3.10 to 3.11, if the coupling condition $i=2k$ is satisfied then the axisymmetric imperfection has a strong influence on the nonlinear vibration behaviour. Otherwise as shown in Fig. 3.12 the influence is quite slight. Notice that in the case of $i=2k$, the left bifurcation point is now on the lower branch of the associated single mode response curve rather than on the upper branch as in the case of an asymmetric imperfection. In addition increasing axisymmetric imperfections have a stabilizing effect on the stability characteristics of the coupled-mode response curves if the strong coupling condition $i=2k$ is satisfied.

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* Notice that the frequencies in the present figures were normalized by dividing the forcing frequency by the frequency of free vibration (linear theory) of the perfect unloaded shell.

### 3.3.4 INFLUENCE OF COMBINED IMPERFECTIONS

Figures 3.13-3.16 show the amplitude-frequency relationship of shell ES2 with the combined imperfections ($\delta_1 \neq 0$, $\delta_2 \neq 0$). Comparing the results of Fig. 3.13 with that of Fig. 3.9 (or the results of Fig. 3.14 with that of Fig. 3.10), both of which have the same axisymmetric imperfections, then one must conclude that the addition of an asymmetric imperfection of about identical amplitude has resulted in minor changes only. Since in these cases the characteristics of the nonlinear responses are basically those of a shell with axisymmetric imperfection only, one can conclude that an axisymmetric imperfection has a stronger influence on the nonlinear vibration than an asymmetric imperfection of about the same amplitude. Further, upon comparing the results shown in Fig. 3.6 (imperfect shell with $\delta_1$=0.00, $\delta_2$=0.05), in Fig. 3.10 (imperfect shell with $\delta_1$=-0.06, $\delta_2$=0.00) and in Fig. 3.14 (imperfect shell with $\delta_1$=-0.06, $\delta_2$=0.05) one must conclude that the influence of the combined imperfections on the nonlinear vibration is not simply the superposition of the influence of the asymmetric and the axisymmetric imperfections acting alone. For instance, the region of the coupled mode response in the case of combined imperfections is larger than the corresponding region in the case of the asymmetric imperfection acting alone (see Figures 3.6 and 3.14). On the other hand, the region of coupled mode response in the case of combined imperfections is smaller than the corresponding region in the case of the axisymmetric imperfection acting alone (see Figures 3.1 and 3.14).
The same combined imperfection case as shown in Fig. 3.14 has been rerun twice more with slightly modified wave numbers. Thus Fig. 3.15 shows the results of the case where the asymmetric imperfection mode (with n=5) and the asymmetric response mode (with $\ell$=10) are not affine. Notice the rather large shift in frequencies and the change in the coupled mode response. Figure 3.16 displays the results of the case where the axisymmetric and the asymmetric imperfections do not satisfy the strong coupling condition i=2k. A comparison with Fig. 3.6 reveals that in this case besides a noticeable shift in frequencies, the shape of the amplitude-frequency relationship resembles that of a single asymmetric imperfection.

### 3.3.5 INFLUENCE OF AXIAL COMPRESSIVE LOAD

Investigation of the effect of an axial compressive load on the nonlinear vibrations of a shell is of importance in engineering since many shells used in practice carry such load. Figures 3.17 to 3.19 indicate the dynamic behaviour of perfect shells for the cases of $\lambda$ = 0.1, $\lambda$ = 0.3 and $\lambda$ = 0.5 respectively. By studying these figures one can draw the conclusion that increasing the axial compressive load has the following effects on the nonlinear vibration:

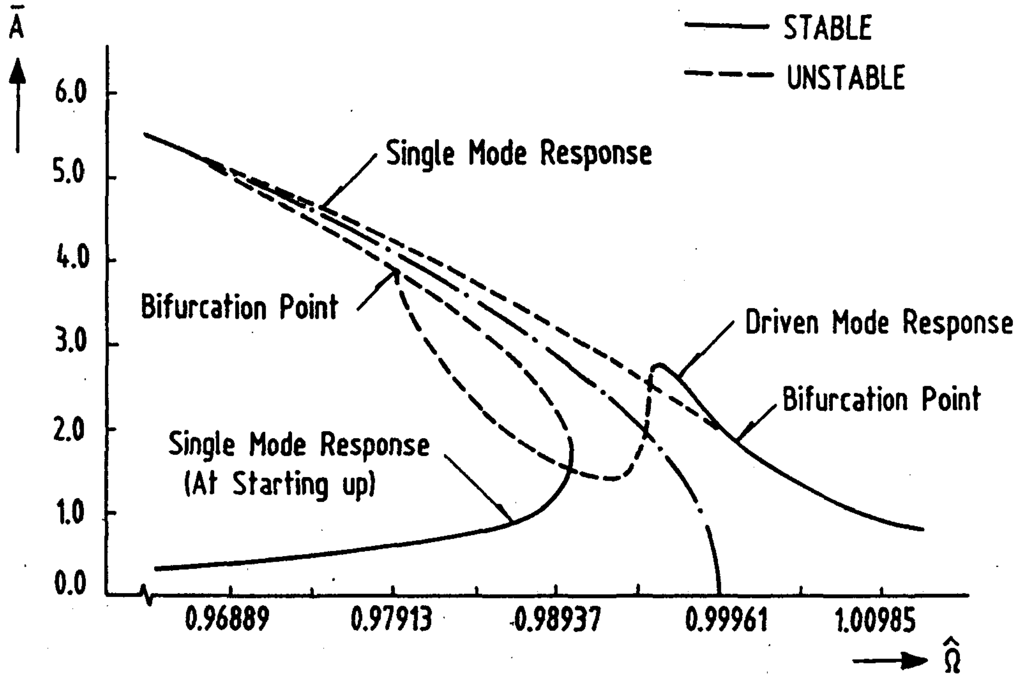
. It increases the amplitude of response (which is equivalent to decreasing the damping),
. It increases the region of the coupled mode response.

It is clear that the presence of an axial compressive load does not change the vibration behaviour.

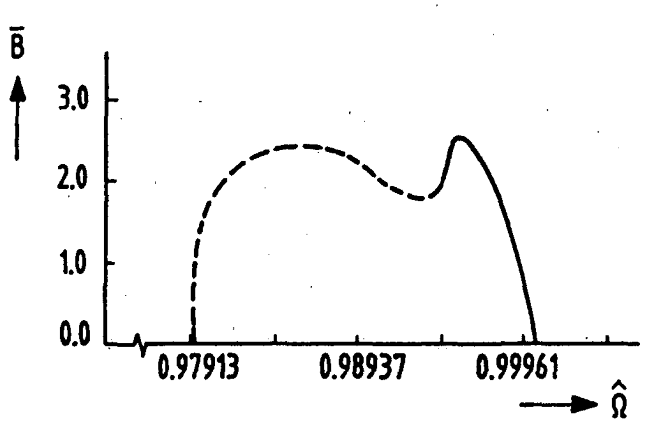
## 3.4 CONCLUSIONS

The nonlinear flexural vibration of perfect and imperfect thin-walled cylindrical shells with viscous damping were analyzed by using Donnell's nonlinear shell equations. Numerical solutions were obtained by applying Galerkin's method together with the method of averaging. The study yields the following conclusions:

a. A good agreement between the present perfect shell results and Ginsberg's [89] analysis is obtained. The "gap" found in Evensen's analysis [30], which is the major difference between Evensen and Ginsberg, is not found in the present analysis. It is the present author's opinion that the "gap" resulted because Evensen neglected the negative values of cos $2\bar{A}$ in his ring analysis. Therefore, one can now say that no qualitative difference exists between the results of the different solution procedures which are (a) Galerkin's method (Evensen [30,35] and the present analysis), (b) the small parameter perturbation method (Chen [82]), and (c) the special perturbation technique (Ginsberg [89]).
b. The general characteristics of the damped response of perfect shells found by Ginsberg are confirmed by the present analysis, namely
   1. the damping has a pronounced influence on the coupled mode response. Increasing damping can completely eliminate coupled mode response peaks;
   2. the damped response of a perfect shell can be divided into five regions, as shown in Fig. 3.2. In region 3 both the single mode and the coupled mode responses are unstable. The coupled mode response peak however is stable;
   3. the single mode response between the two bifurcation points is unstable. One of the extra results obtained by the present analysis is that the stability of the coupled mode response in region 4 is not always stable. It depends on the magnitude of damping (or excitation), as shown in Figs. 3.1 and 3.2.
c. Initial geometric imperfections have a significant influence on the damped vibrations of either the single or the coupled mode responses under certain coupling conditions. The general influence of imperfections is quite similar to that of damping. That is, increasing the amplitude of imperfections can quickly eliminate the coupled mode response. In addition, the presence of initial geometric imperfections changes the stability characteristics of the solutions. It is noted that the influence of combined imperfection modes cannot be obtained simply by superposition of the individually determined effects of the axisymmetric and asymmetric imperfection modes.
d. Axial compressive loads have an influence on the nonlinear vibration of perfect shells. Such loads increase both the amplitudes of response as well as the region of coupled mode response. However the addition of the axial compression does not change the vibration behaviour.




a. Single mode and driven mode response

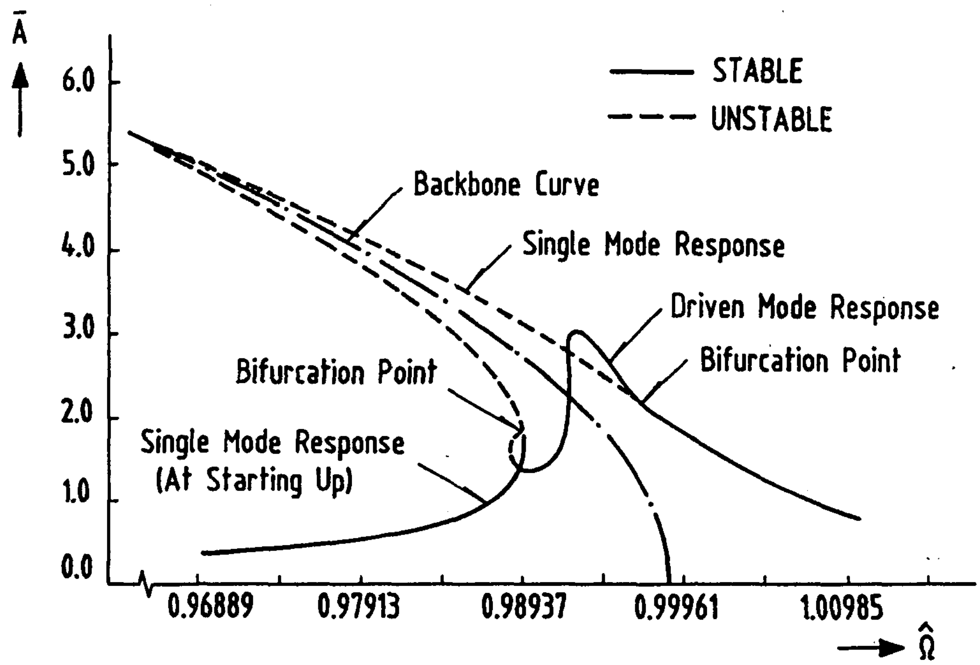



b. Companion mode response

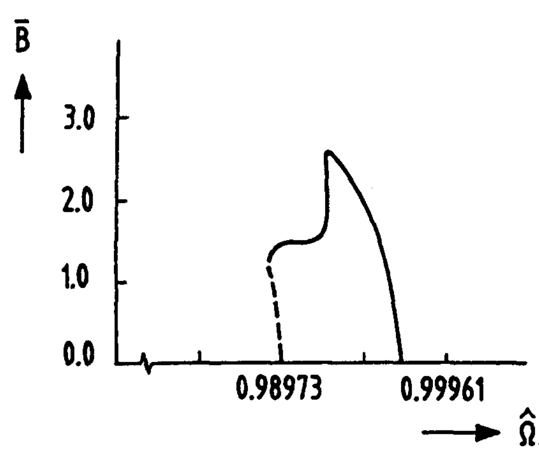
Fig. 3.9 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=-0.02 and $\delta_2$=0.00.

Damping $\gamma=9\text{x}10^{-5}$, excitation $\bar{F}_D=4.25\text{x}10^{-5}$.

ES2 Shell; i=2, k=m=1, n=ℓ=5




a. Single mode and driven mode response

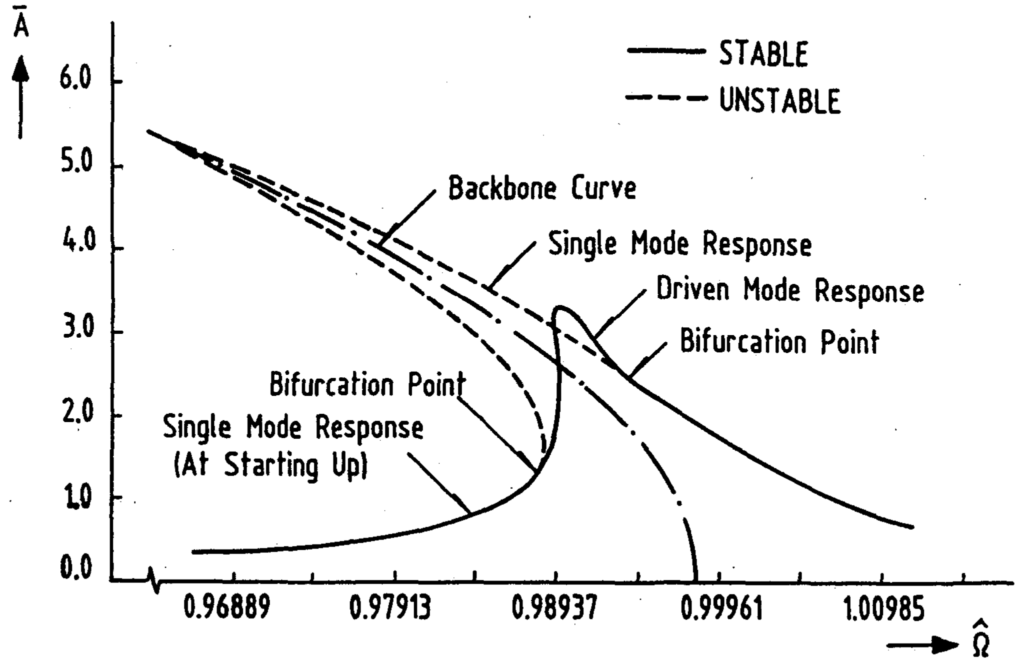



b. Companion mode response

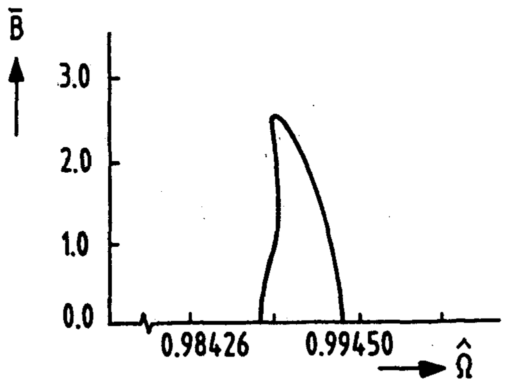
Fig. 3.10 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=-0.06 and $\delta_2$=0.00.

Damping $\gamma=9\text{x}10^{-5}$, excitation $\bar{F}_D=4.25\text{x}10^{-5}$.

ES2 Shell; i=2, k=m=1, n=ℓ=5




a. Single mode and driven mode response

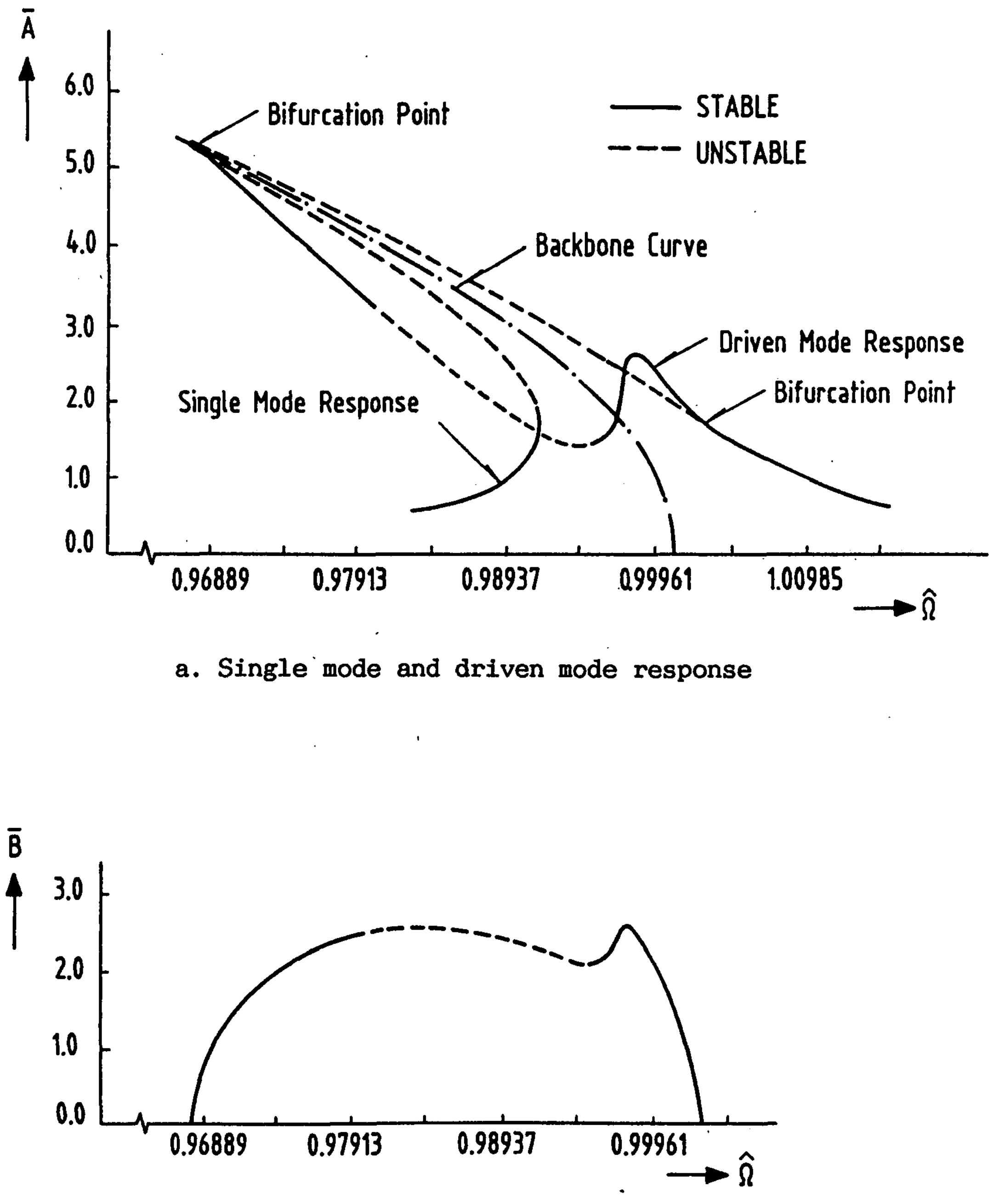



b. Companion mode response

Fig. 3.11 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=-0.10 and $\delta_2$=0.00.

Damping $\gamma$=9x10$^{-5}$, excitation $\bar{F}_D$=4.25x10$^{-5}$.

ES2 Shell; i=2, k=m=1, n=$\ell$=5




a. Single mode and driven mode response

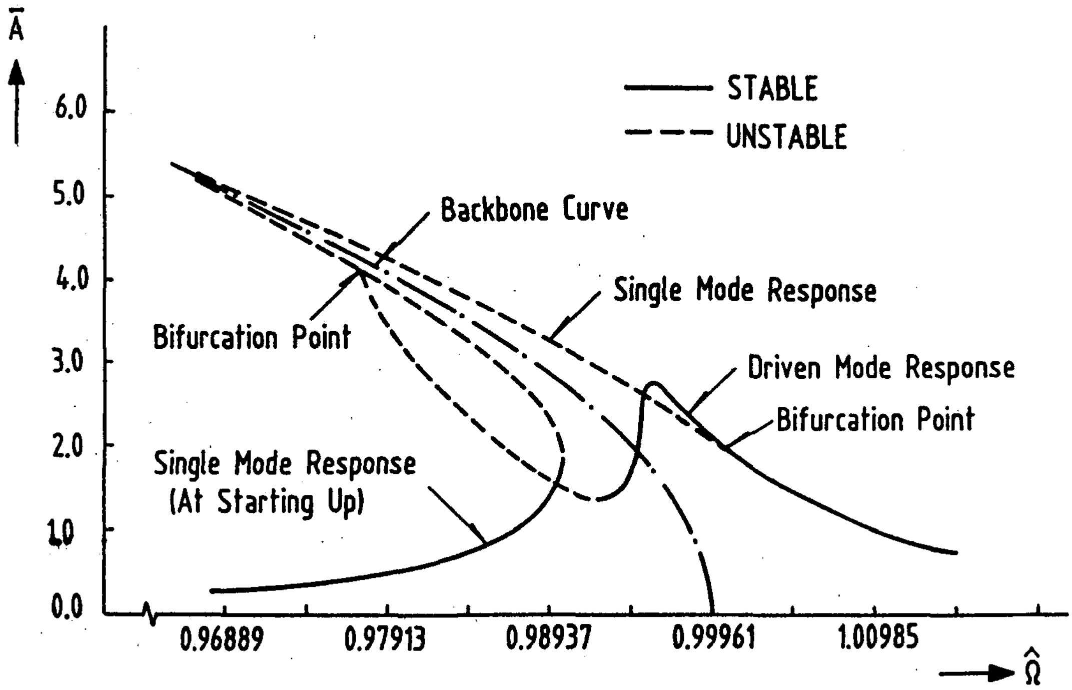


b. Companion mode response

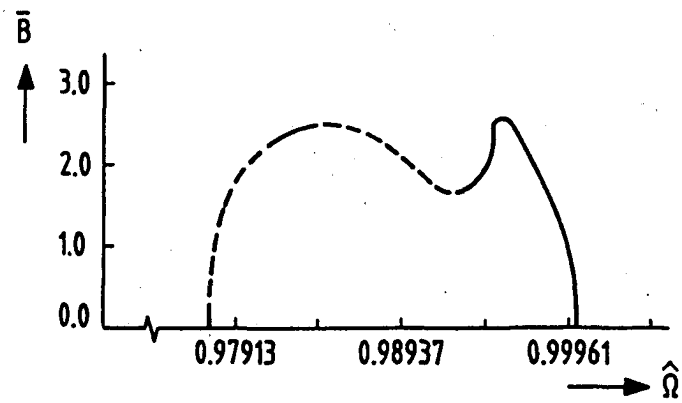
Fig. 3.12 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=-0.06 and $\delta_2$=0.00.

Damping $\gamma=9\times10^{-5}$, excitation $\bar{F}_D=4.25\times10^{-5}$.

ES2 Shell; i=7, k=m=1, n=$\ell$=5




a. Single mode and driven mode response

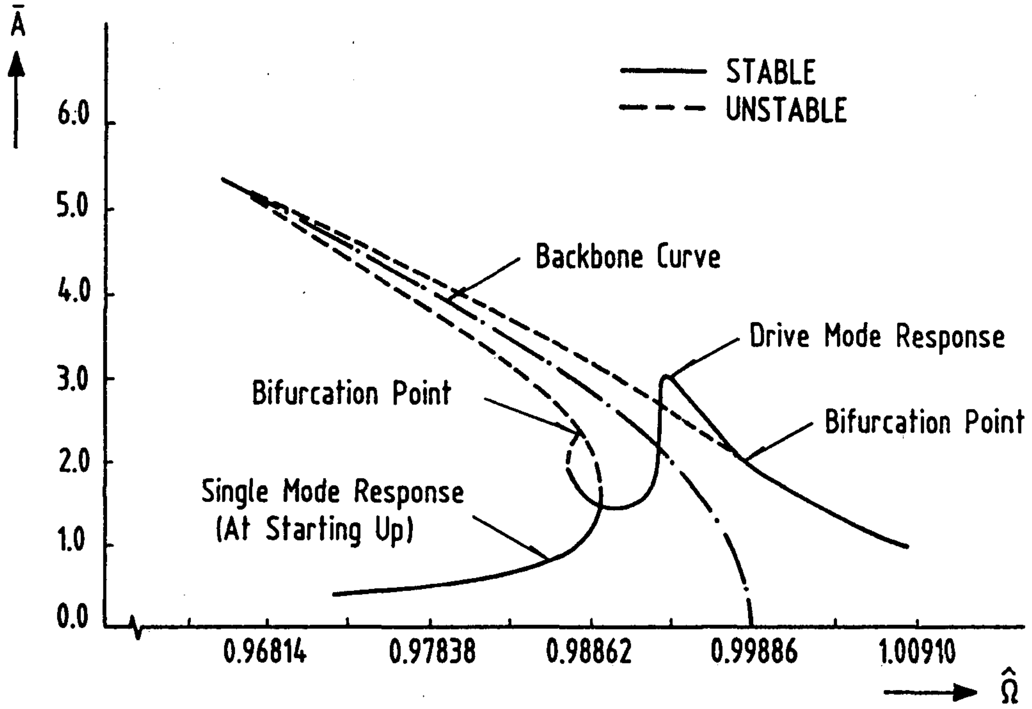



b. Companion mode response

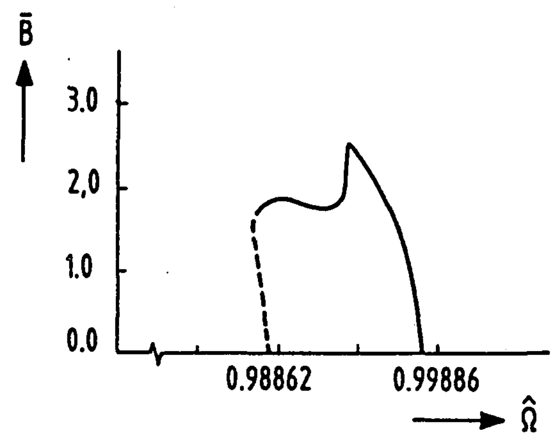
Fig. 3.13 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=-0.02 and $\delta_2$=0.02.

Damping $\gamma$=9x10$^{-5}$, excitation $\bar{F}_D$=4.25x10$^{-5}$.

ES2 Shell; i=2, k=m=1, n=$\ell$=5




a. Single mode and driven mode response

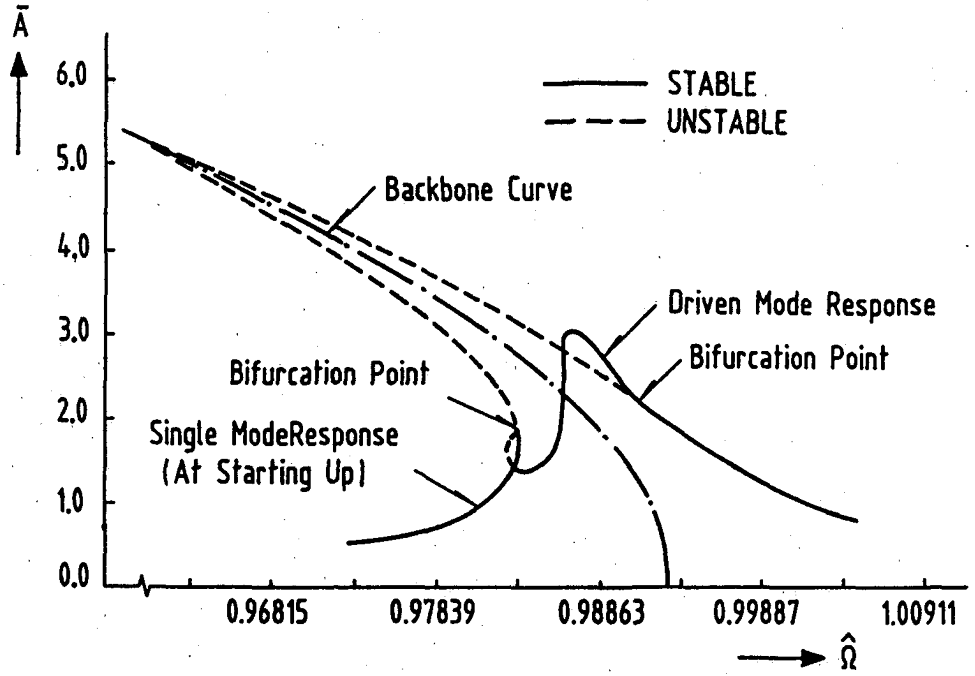



b. Companion mode response

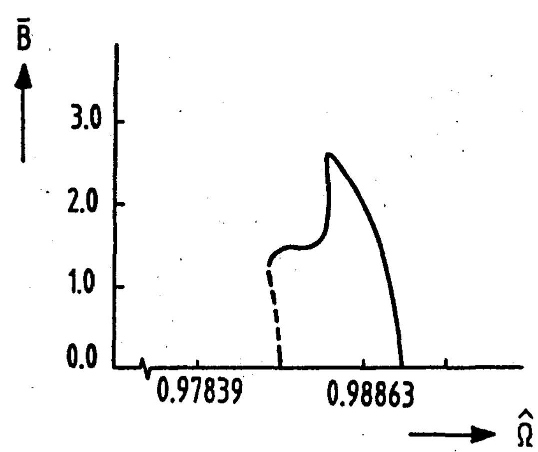
Fig. 3.14 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=-0.06 and $\delta_2$=0.05.

Damping $\Upsilon=9\times10^{-5}$, excitation $\bar{F}_D=4.25\times10^{-5}$.

ES2 Shell; i=2, k=m=1, n=$\ell$=5




a. Single mode and driven mode response

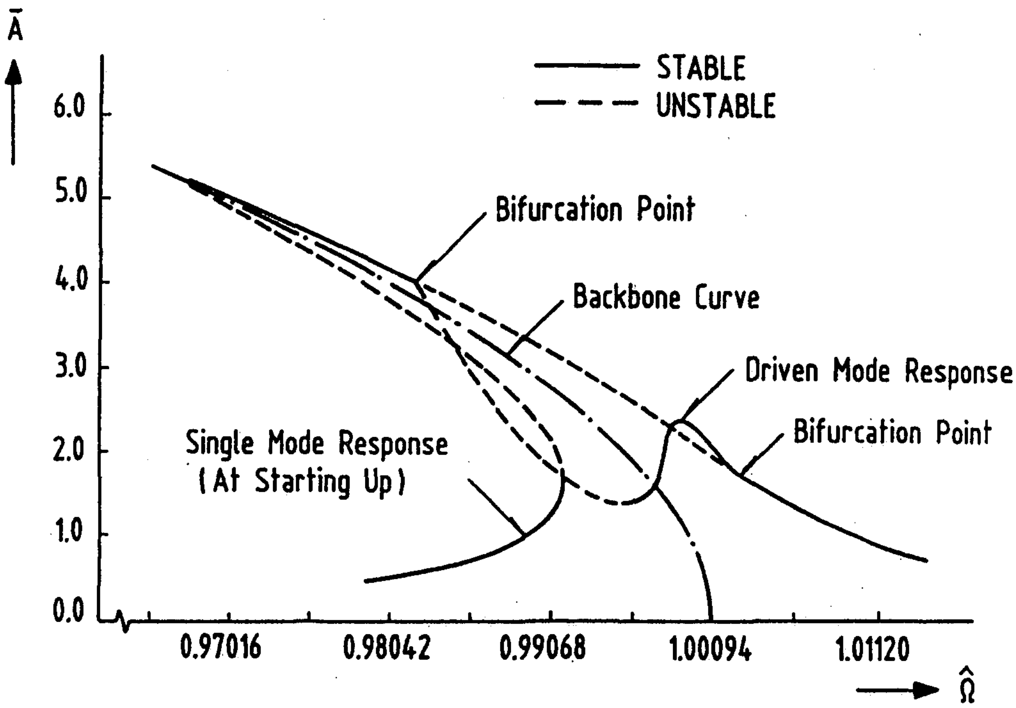



b. Companion mode response

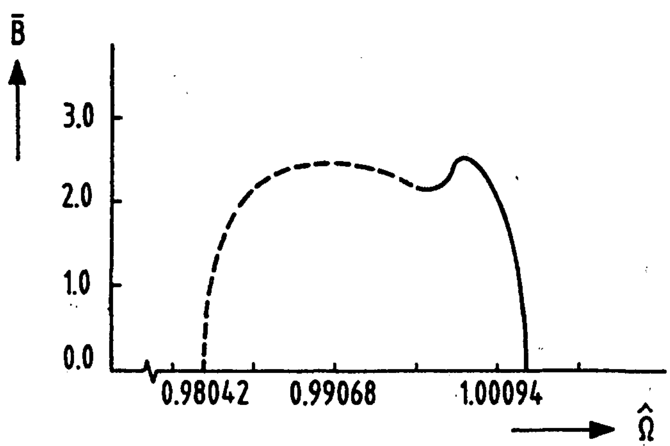
Fig. 3.15 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=-0.06 and $\delta_2$=0.05.

Damping $\gamma$=9x10$^{-5}$, excitation $\bar{F}_D$=4.25x10$^{-5}$.

ES2 Shell; i=2, k=m=1, n=10, $\ell$=5




a. Single mode and driven mode response

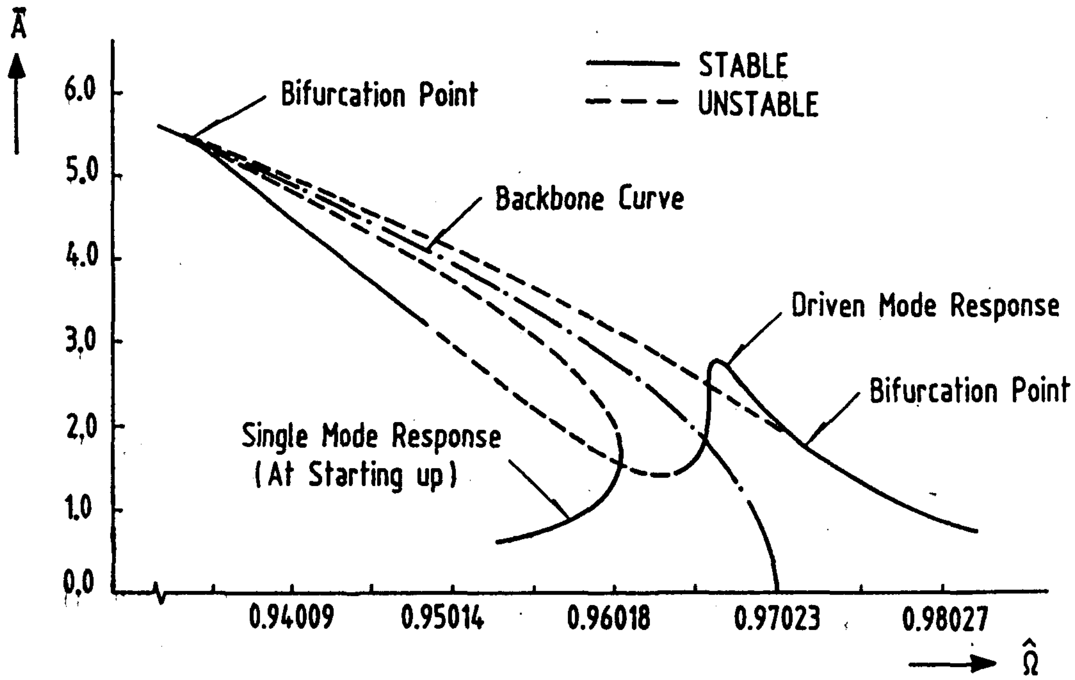



b. Companion mode response

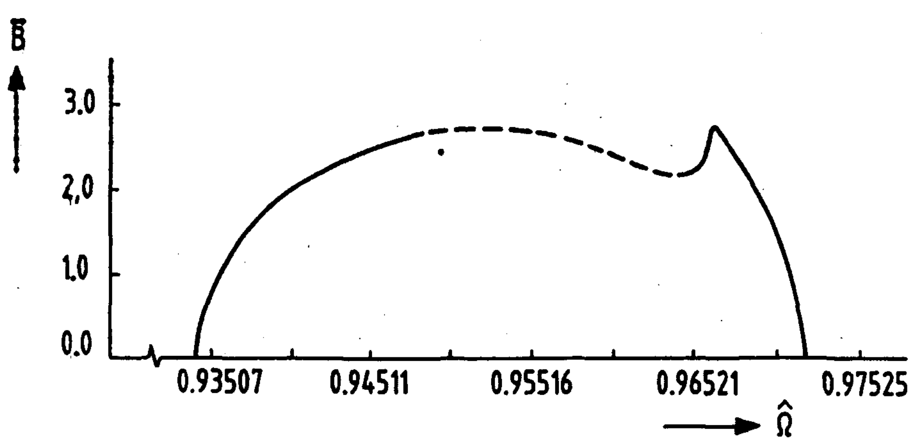
Fig. 3.16 Amplitude-frequency relationship of damped vibration of imperfect shell with $\delta_1$=-0.06 and $\delta_2$=0.05.

Damping $\gamma=9\times10^{-5}$, excitation $\bar{F}_D=4.25\times10^{-5}$.

ES2 Shell; i=7, k=m=1, n=$\ell$=5




a. Single mode and driven mode response

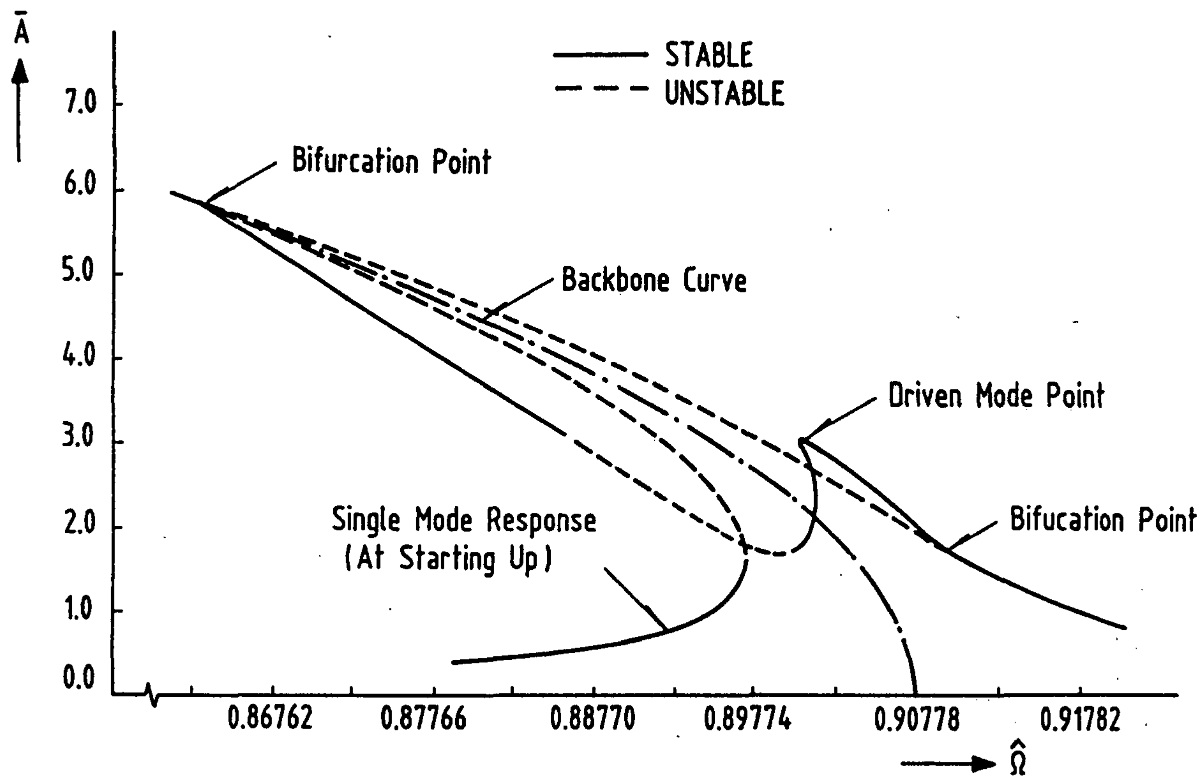



b. Companion mode response

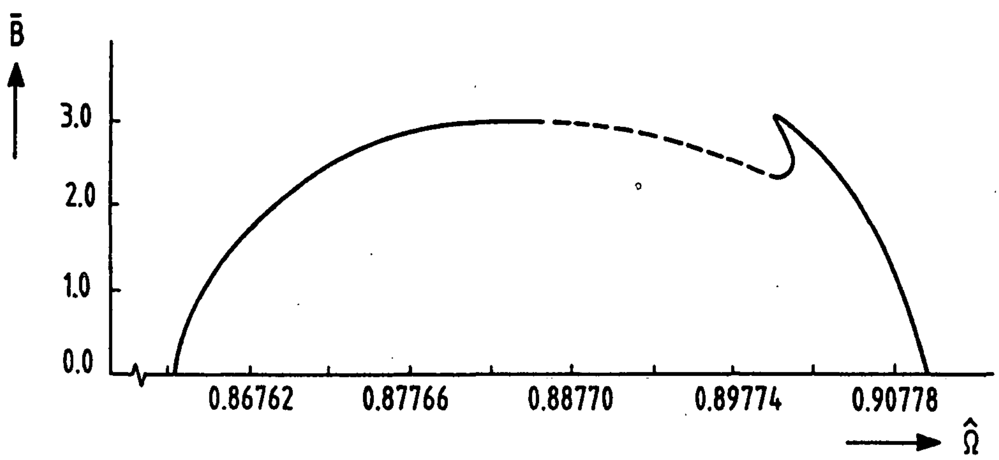
Fig. 3.17 Amplitude-frequency relationship of damped vibration of perfect shell, axial compressive load $\lambda=0.1$.

Damping $\gamma=9\times10^{-5}$, excitation $\bar{F}_D=4.25\times10^{-5}$.

ES2 Shell; $k=m=1$, $n=\ell=5$

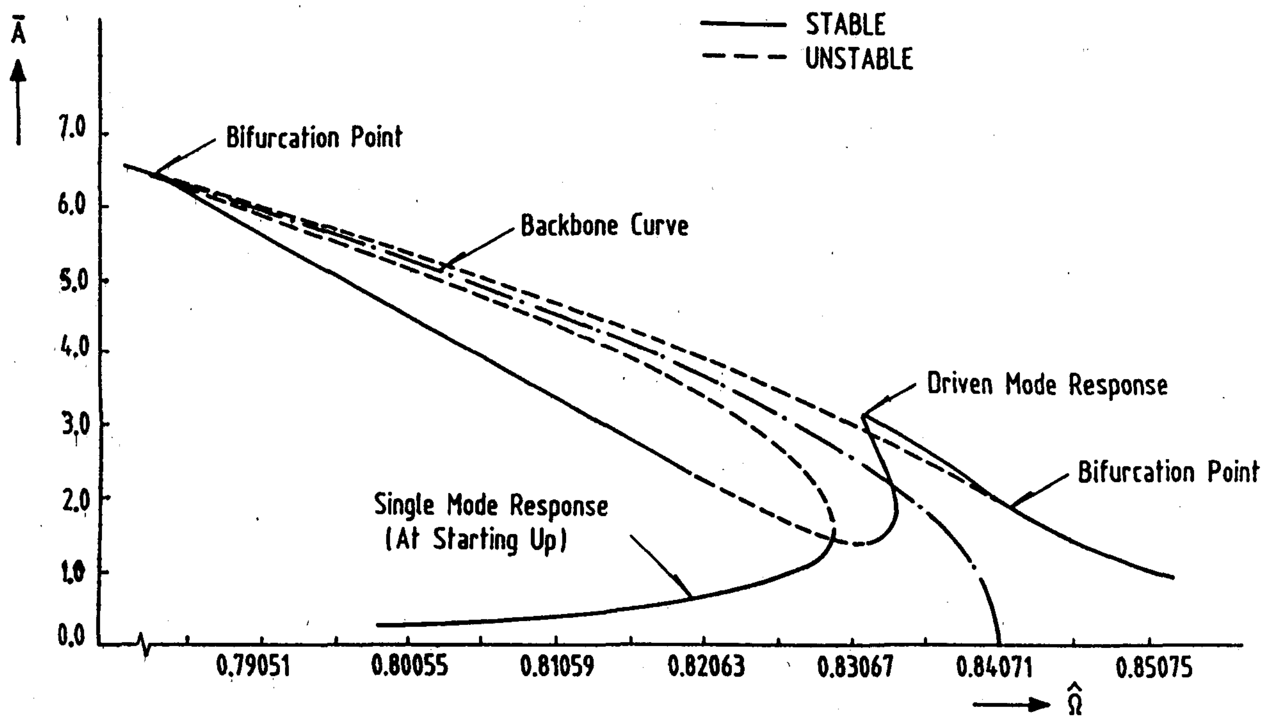



a. Single mode and driven mode response

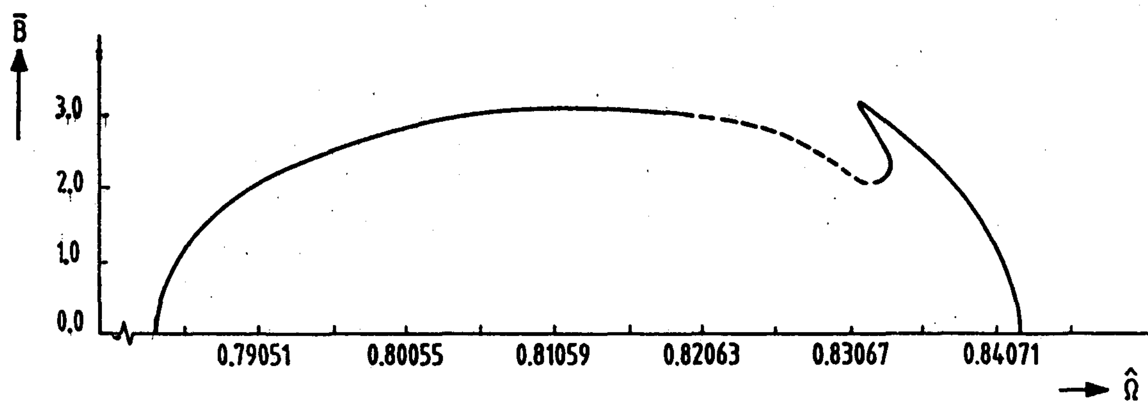



b. Companion mode response

Fig. 3.18 Amplitude-frequency relationship of damped vibration of perfect shell, axial compressive load $\lambda$=0.3.
Damping $\gamma=9\times10^{-5}$, excitation $\bar{F}_D=4.25\times10^{-5}$.
ES2 Shell; k=m=1, n=$\ell$=5




a. Single mode and driven mode response




b. Companion mode response

Fig. 3.19 Amplitude-frequency relationship of damped vibration of perfect shell, axial compressive load $\lambda=0.5$.
Damping $\gamma=9\times10^{-5}$, excitation $\bar{F}_D=4.25\times10^{-5}$.
ES2 Shell; $k=m=1$, $n=\ell=5$

# CHAPTER 4 STABILITY ANALYSIS

## 4.1 INTRODUCTION

The previous results indicate that the frequency-amplitude relationships admit more than one solution for some values of frequency. One solution that is always possible is $\bar{B} = 0$, namely, single mode response, in which case the nodal lines of radial displacement form a stationary spatial pattern and the motion is symmetric about the planes $y = 0$. We may also sometimes find real, nonzero values of $\bar{B}$ which satisfy the frequency-amplitude relationships. Such solutions represent the travelling wave moving around the circumference resulting in a moving nodal pattern. In order to ascertain whether the single or coupled mode response will actually occur, we must consider the stability of the responses obtained by the theoretical analysis.

The first work dealing with the stability of response was done by Evensen for rings in 1964 [30]. He used the method of slowly varying parameters [130] in his investigation. This method was also used later by Ginsberg in the stability analysis of the nonlinear vibration of shells [87].

The purpose of this phase of the present study is to investigate the stability characteristics of the frequency-amplitude relationships derived for various cases, which are presented in chapter 2 and chapter 3. The method of slowly varying parameters is used. The stability of both the single mode as well as the coupled mode responses are investigated. In order to ensure the necessary accuracy the Newton-Rapson method was used. The results of the calculations are presented and discussed in chapter 2 and chapter 3. Only the process of the investigation and the associated equations are presented here.

## 4.2 ANALYSIS

To study stability of the solutions presented in chapter 2 and 3, equations (3-2-1) and (3-2-2) are rewritten

$$\frac{d^2A}{d\tau^2} + 2\gamma_s \frac{dA}{d\tau} + A + \frac{3}{8}\varepsilon[(\frac{dA}{d\tau})^2 + A\frac{d^2A}{d\tau^2} + (\frac{dB}{d\tau})^2 + B\frac{d^2B}{d\tau^2} +$$

$$+ \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\frac{d^2A}{d\tau^2}][A+\delta_{n,\ell}(\delta_2+\hat{\delta}_2)] +$$

$$+ \frac{3}{4}\gamma_s\varepsilon[A\frac{dA}{d\tau} + B\frac{dB}{d\tau} + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\frac{dA}{d\tau}][A+\delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \alpha_{s1}A^2 + \alpha_{s2}B^2 +$$

$$+ \alpha_{s3}A^3 + \alpha_{s4}AB^2 + \alpha_{s5}(A^2+B^2)A^2 + \alpha_{s6}(A^2+B^2)^2 + \alpha_{s7}(A^2+B^2)^2A$$

$$= F_{sD}\cos\Omega_s\tau \qquad (4\text{-}2\text{-}1)$$

$$\frac{d^2B}{d\tau^2} + 2\gamma_s \frac{dB}{d\tau} + \beta_{s1}B + \frac{3}{8}\varepsilon B[(\frac{dA}{d\tau})^2 + A\frac{d^2A}{d\tau^2} + (\frac{dB}{d\tau})^2 + B\frac{d^2B}{d\tau^2}$$

$$+ \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\frac{d^2A}{d\tau^2}] + \frac{3}{4}\gamma_s\varepsilon B[A\frac{dA}{d\tau} + B\frac{dB}{d\tau} + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\frac{dA}{d\tau}]$$

$$+ \beta_{s2}AB + \beta_{s3}A^2B + \beta_{s4}B^3 + \beta_{s5}(A^2+B^2)AB + \beta_{s6}(A^2+B^2)^2B = 0 \qquad (4\text{-}2\text{-}2)$$

where $\Omega_s$, $\gamma_s$, $\tau$ and $F_{SD}$ are nondimensional frequency, damping, time and excitation, defined as follows:

$$\Omega_s = \omega/\bar{\omega}_{mn}$$

$$\gamma_s = \frac{\bar{c}}{2\bar{\rho}\bar{\omega}_{mn}}$$

$$\tau = \bar{\omega}_{mn}t$$

$$F_{sD} = F_D/\tilde{\beta} \qquad (4\text{-}2\text{-}3)$$

$\bar{\omega}^2_{mn} = \frac{1}{2}\frac{\tilde{\beta}E}{\bar{\rho}R^2}$ is the linear natural frequency of free vibration of the imperfect shell, and

$$\varepsilon = (\frac{\ell^2h}{R})^2$$

The coefficients $\alpha_{s1}$, $\alpha_{s2}$, ...., $\alpha_{s7}$ and $\beta_{s1}$, $\beta_{s2}$, ...., $\beta_{s6}$ are defined in Appendix 4-A. $\tilde{\beta}$ is defined in Appendix 2-A.4.

As a test of the stability of the response, the following small perturbations $\zeta(\tau)$ and $\eta(\tau)$ are introduced

$$A(\tau) = \bar{A}\cos\varphi_1 + \zeta(\tau) \qquad (4\text{-}2\text{-}4)$$

$$B(\tau) = \bar{B}\cos\varphi_2 + \eta(\tau) \qquad (4\text{-}2\text{-}5)$$

where

$$\varphi_1 = \Omega_s\tau + \bar{\phi} \qquad (4\text{-}2\text{-}6)$$

$$\varphi_2 = \Omega_s\tau + \bar{\psi} \qquad (4\text{-}2\text{-}7)$$

The $\bar{\Phi}$ and $\bar{\psi}$ are the average value (over one period) of the phase angle defined in chapter 3.

These expressions are then substituted into equations (4-2-1) and (4-2-2). The first order terms in the perturbations are retained. This procedure results in two coupled differential equations for $\zeta(\tau)$ and $\eta(\tau)$, namely,

$$\bar{\alpha}_{s1}\frac{d^2\zeta}{d\tau^2} + \bar{\alpha}_{s2}\frac{d\zeta}{d\tau} + \bar{\alpha}_{s3}\zeta + \bar{\alpha}_{s4}\frac{d^2\eta}{d\tau^2} + \bar{\alpha}_{s5}\frac{d\eta}{d\tau} + \bar{\alpha}_{s6}\eta = 0 \tag{4-2-8}$$

$$\bar{\beta}_{s1}\frac{d^2\eta}{d\tau^2} + \bar{\beta}_{s2}\frac{d^2\eta}{d\tau} + \bar{\beta}_{s3}\eta + \bar{\beta}_{s4}\frac{d^2\zeta}{d\tau^2} + \bar{\beta}_{s5}\frac{d\zeta}{d\tau} + \bar{\beta}_{s6}\zeta = 0 \tag{4-2-9}$$

where the coefficients $\bar{\alpha}_{s1}$, $\bar{\alpha}_{s2}$, ..., $\bar{\alpha}_{s6}$ and $\bar{\beta}_{s1}$, $\bar{\beta}_{s2}$, ..., $\bar{\beta}_{s6}$ are defined in Appendix 4-B. The details of derivation of the equations (4-2-8) can be found in Appendix 4-C.

It is obvious that no close solutions of these equations are known. However, approximate solutions can be obtained by using numerical integration procedures directly or indirect numerical procedures. In the present analysis, the method of slowly varying perturbations [130] is employed, in which, the perturbations $\zeta(\tau)$ and $\eta(\tau)$ are assumed in the form

$$\zeta(\tau) = \zeta_1(\tau)\cos\varphi_1 + \zeta_2(\tau)\sin\varphi_1 \tag{4-2-10}$$

$$\eta(\tau) = \eta_1(\tau)\sin\varphi_2 + \eta_2(\tau)\cos\varphi_2 \tag{4-2-11}$$

where $\zeta_1$, $\zeta_2$, $\eta_1$ and $\eta_2$ are assumed to be slowly varying functions of $\tau$. Then the derivatives $\frac{d\zeta}{d\tau}$, $\frac{d^2\zeta}{d\tau^2}$, $\frac{d\eta}{d\tau}$ and $\frac{d^2\eta}{d\tau^2}$ are replaced by

$$\frac{d\zeta}{d\tau} = -\zeta_1\Omega_s\sin\varphi_1 + \zeta_2\Omega_s\cos\varphi_1 \tag{4-2-12}$$

$$\frac{d\eta}{d\tau} = \eta_1\Omega_s\cos\varphi_2 - \eta_2\Omega_s\sin\varphi_2 \tag{4-2-13}$$

$$\frac{d^2\zeta}{d\tau^2} = -\zeta_1\Omega_s^2\cos\varphi_1 - \zeta_2\Omega_s^2\sin\varphi_1 - \frac{d\zeta_1}{d\tau}\Omega_s\sin\varphi_1 + \frac{d\zeta_2}{d\tau}\Omega_s\cos\varphi_1 \tag{4-2-14}$$

$$\frac{d^2\eta}{d\tau^2} = \frac{d\eta_1}{d\tau}\Omega_s\cos\varphi_2 - \frac{d\eta_2}{d\tau}\Omega_s\sin\varphi_2 - \eta_1\Omega_s^2\sin\varphi_2 - \eta_2\Omega_s^2\cos\varphi_2 \tag{4-2-15}$$

together with the auxiliary conditions

$$\frac{d\zeta_1}{d\tau}\cos\varphi_1 + \frac{d\zeta_2}{d\tau}\sin\varphi_1 = 0 \tag{4-2-16}$$

$$\frac{d\eta_1}{d\tau}\sin\varphi_2 + \frac{d\eta_2}{d\tau}\cos\varphi_2 = 0 \tag{4-2-17}$$

These expressions for the derivatives are then substituted into equations (4-2-8) and (4-2-9), and the procedure described in Appendix 4-D is used. This procedure yields four linear differential equations for $\bar{\zeta}_1$, $\bar{\zeta}_2$, $\bar{\eta}_1$ and $\bar{\eta}_2$, which can be put in matrix form as follows:

$$[M]\{\phi\} = [N]\{\frac{d\phi}{d\tau}\} \tag{4-2-18}$$

where

$$\{\phi\} = \begin{bmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \\ \bar{\eta}_1 \\ \bar{\eta}_2 \end{bmatrix}$$

and [M] and [N] are 4x4 matrices respectively. The elements that are contained in these matrices are functions of $\bar{A}$, $\bar{B}$, $\Omega$ and $\Upsilon$, which are defined in Appendix 4-E. Equations (4-2-18) are derived for the case of the coupled mode response. The equations for the case of single mode response can be obtained easily from this matrix equation by letting $\bar{B} = 0$ in the matrix elements and replacing $\sin 2\bar{\Lambda}$ and $\cos 2\bar{\Lambda}$ by $\sin\bar{\Phi}$ and $\cos\bar{\Phi}$ respectively as shown in Appendix 4-E.

It is indicated that $\{\phi\} = \{\phi_0\}\, e^{\chi\tau}$ is a possible solution of the matrix equation (4-2-18), where $\{\phi_0\}$ is a constant column matrix whose matrix elements will be determined.

Substituting $\{\phi\} = \{\phi_0\}\, e^{\chi\tau}$ into equation (4-2-18) leads to a standard eigenvalue problem for determining the $\chi$'s,

$$|N - \chi M| = 0 \tag{4-2-19}$$

It is demonstrated in Appendix 4-E that the matrices [M] and [N] are nonsymmetric. That means the solutions of equation (4-2-19) are complex. If any of the $\lambda$'s have a positive real part, then the corresponding perturbations will increase exponentially with time. In this case, the associated response is said to be unstable; conversely, if none of the $\lambda$'s have a positive real part, the response is said to be stable.

Stability of both the single mode response as well as the coupled mode response for various values of damping, excitation and shell geometry were investigated by using equation (4-2-19). In order to ensure the required accuracy in the present analysis, the Newton-Raphson method was used to refine the values of $\bar{A}$ and $\bar{B}$ obtained by cross-plotting. These refined values are then substituted into equation (4-19) along with the $\Omega$, $\gamma$ and associated parameters $\sin 2\bar{\lambda}$ and $\cos 2\bar{\lambda}$. For each case the eigenvalues were examined to determine whether or not they had a positive real part. In this manner, the stability of the responses plotted in chapter 2 and chapter 3 was determined.

# CHAPTER 5 ERROR CHARACTERIZATION

## 5.1 INTRODUCTION

As mentioned before, the solutions of the present analysis are obtained based on a number of simplifications, assumptions and approximations, by which certain errors could be introduced if they are used incorrectly. First of all is the use of the relatively simple Donnell shallow shell equations, in which the axial and circumferential inertia contributions are neglected. Next one is the violation of the simply supported boundary conditions.

The most interesting one is the application of the method of averaging (or the method of slowly varying parameters) based on the assumption that the variation of amplitudes of vibration during one period is quite small. In this chapter, the effect of all these factors on the accuracy of solutions will be discussed in detail, except the violation of boundary conditions which will be discussed in Part II of this thesis.

## 5.2 ERROR CHARACTERISTICS

In Donnell's shallow-shell equations the neglecting of the in-plane displacements in the curvature relations and of the transverse shear force in the in-plane equilibrium equations comes from the shallow shell approximation in which the circumferential wave length $2\pi R/\ell$ ($\ell$ is the circumferential wave number) of the deformation mode is always small compared to the radius R. Essentially, it is saying that the curvature and in-plane equilibrium of the cylindrical shell are the same as for a flat plate. The in-plane inertias $\partial^2 u/\partial t^2$ and $\partial^2 v/\partial t^2$ are neglected because of the assumption that the flexural motion is predominant in the present study. Rotary inertia is neglected because the wave lengths $2\pi R/\ell$ and $2L/k$ are large compared to the shell thickness h.

El Raheb made a detailed comparison for the case of linear vibrations between the results of Donnell's equations and the 'exact' equations of motion derived by Koiter based on the Love-Kirchoff hypothesis for shell theory in which all the quantities neglected in the Donnell's equations have been retained [121]. He found that the errors in frequency obtained by the approximation were

$$\hat{e} = \frac{\omega_{approx.}}{\omega_{exact}} - 1 \tag{5-2-1}$$

and concluded that:

1. The maximum error due to neglecting in-plane displacements in curvature and transverse shear in in-plane equilibrium is

$$(\hat{e}_1)_{max} \simeq \frac{3^{\frac{3}{4}}\xi_1^{\frac{1}{2}}}{4(1-\nu^2)\ \xi_2} \tag{5-2-2}$$

where

$$\xi_1 = \frac{h}{\sqrt{12}\,R} \quad \text{and} \quad \xi_2 = \frac{k\pi R}{L}$$

2. The maximum error due to neglecting tangential inertia is

$$\hat{e}_2 \simeq \frac{1}{2\ell^2} \quad \text{for} \quad \ell >> 3 \tag{5-2-3}$$

3. The error due to neglecting rotary inertia is

$$\hat{e}_3 = \frac{\xi_1^2(\xi_1^2+\ell^2)}{2} \tag{5-2-4}$$

With these estimations in errors due to the approximations in the Donnell shallow-shell equations, the present study, in general, will be limited to shells with the following configurations same as those used by Chen [82]

$$\frac{h}{\sqrt{12}\,R} < \frac{1}{100} \tag{5-2-5}$$

and

$$\frac{L}{R} \leq 10 \tag{5-2-6}$$

Also, the vibration mode will be limited to

$$4 \leq \ell \leq 30 \tag{5-2-7}$$

and

$$1 \leq k \leq 6 \tag{5-2-8}$$

so as to keep the error within the acceptable range.

Although El Raheb's error estimation for Donnell's shallow-shell equations has been made for the linear vibration case, it is assumed that it will also be approximately applicable to nonlinear vibrations since the nonlinearity of vibration in the present analysis is quite small.

It should be noted that the vibration modes with $k = 9$ or $\ell = 35$ were used in the present study. This may lead to larger errors, but this loss of accuracy is not expected to alter the qualitative behaviour of the solutions.

In the present study the analysis of the fundamental state was performed in such a way that the in-plane conditions are rigorously satisfied, whereas the out-of-plane boundary conditions are satisfied only approximately. Therefore the errors introduced are not expected to be large.

It should be noted that the present theory imposes no restrains on the axial in-plane displacement at the ends of the cylinder in the analysis of the dynamic state. Ginsberg's results, which satisfied the in-plane boundary conditions as well as out-of-plane boundary conditions, show that these violations did not alter the vibration behaviour [89].

## 5.3 ERROR CHARACTERISTICS OF THE METHOD OF AVERAGING

The method of averaging has been widely used in the analysis of nonlinear vibrations of thin cylindrical shells. Its basic concept is based on the assumption that the variation of vibration amplitude during one period is so slow that it can be replaced by its 'average value'. Besides the method of averaging many other methods are also used in the field. For example, Chen [82] applied a perturbation procedure to the problem, in which no such assumption (variation of amplitude is slow during one period) as well as mode shape was made a priori. One would expect that their solutions would be similar since they investigated the same problem. But, unfortunately, as shown in Fig. 2 the agreement between Chen's solutions and those of Evensen is not satisfactory. The 'gap' phenomena and the peak response of companion mode in the Evensen's solution are not predicted by Chen. This raises the question naturally about the method of averaging. Is it appropriate to apply such a method to the present problem?

To be able to answer the question a direct numerical integration procedure, the Runge-Kutta-Gill method [57], is used in this section to the integrate system of Eqs. (2-2-7) and (2-2-8). The solutions are compared with those of chapter 4. It should be noticed that Figs. 5.2, 5.4, 5.5 and 5.6 are simulations of those plotted by computer rather than the original which are too difficult to redraw by hand.

### 5.3.1 FORMULATION OF THE PROBLEM

To be able to apply the numerical integration procedure equations (2-2-7) and (2-2-8) are rewritten as

$$a_{11}\frac{d^2A}{d\tau^2} + a_{12}\frac{d^2B}{d\tau^2} + a_{13}\left(\frac{dA}{d\tau}\right)^2 + a_{14}\left(\frac{dB}{d\tau}\right)^2 + a_{15}\frac{dA}{d\tau} + a_{16}\frac{dB}{d\tau} + a_{17} = 0 \qquad (5\text{-}3\text{-}1)$$

$$a_{21}\frac{d^2A}{d\tau^2} + a_{22}\frac{d^2B}{d\tau^2} + a_{23}\left(\frac{dA}{d\tau}\right)^2 + a_{24}\left(\frac{dB}{d\tau}\right)^2 + a_{25}\frac{dA}{d\tau} + a_{26}\frac{dB}{d\tau} + a_{27} = 0 \qquad (5\text{-}3\text{-}2)$$

where $a_{11}$, $a_{12}$, ..., $a_{16}$ and $a_{21}$, $a_{22}$, ..., $a_{26}$ are the functions of the amplitudes A and B, damping $\gamma_s$ and initial geometric imperfections $\tilde{W}$, they are defined in Appendix 5-A, and

$$a_{17} = \alpha_{s1}A^2 + \alpha_{s2}B^2 + \alpha_{s4}AB^2 + \alpha_{s5}(A^2+B^2)\ A^2 + \alpha_{s6}(A^2+B^2)^2$$

$$+ \alpha_{s7}(A^2 + B^2)^2\ A + A - F_{sD}\cos\Omega\tau \tag{5-3-3}$$

$$a_{27} = \beta_{s2}AB + \beta_{s3}A^2B + \beta_{s4}B^3 + \beta_{s5}(A^2+B^2)\ AB + \beta_{s6}(A^2+B^2)\ B$$

$$+ \beta_{s1}B \tag{5-3-4}$$

where $\alpha_{s1}$, $\alpha_{s2}$, ..., $\alpha_{s7}$ and $\beta_{s1}$, $\beta_{s2}$, ..., $\beta_{s6}$ can be found in Appendix 4-A

Solving $\frac{d^2A}{d\tau^2}$ and $\frac{d^2B}{d\tau^2}$ from equations (5-3-1) and (5-3-2), one obtains

$$\frac{d^2A}{d\tau^2} = \beta_{11}\left(\frac{dA}{d\tau}\right)^2 + \beta_{12}\left(\frac{dB}{d\tau}\right)^2 + \beta_{13}\frac{dA}{d\tau} + \beta_{14} \tag{5-3-5}$$

$$\frac{d^2B}{d\tau^2} = \beta_{21}\left(\frac{dA}{d\tau}\right)^2 + \beta_{22}\left(\frac{dB}{d\tau}\right)^2 + \beta_{23}\frac{dB}{d\tau} + \beta_{24} \tag{5-3-6}$$

The coefficients $\beta_{11}$, $\beta_{12}$, ..., $\beta_{14}$ and $\beta_{21}$, $\beta_{22}$, ... $\beta_{24}$ are defined in Appendix 5-A.

Introducing the new parameters

$$\frac{dA}{d\tau} = p_1$$

$$\frac{dB}{d\tau} = p_2$$

and substituting them into equations (5-3-5) and (5-3-6) yields

$$\frac{d\tilde{A}}{d\tau} = \tilde{p} \tag{5-3-7}$$

where

$$\tilde{A} = \{A,\ P_1,\ B,\ P_2\}$$

$$\tilde{P} = \{P_1,\ \tilde{\beta}_1,\ P_2,\ \tilde{\beta}_2\}$$

and

$$\tilde{\beta}_1 = \beta_{11}P_1^2 + \beta_{13}P_1 + \beta_{14}$$

$$\tilde{\beta}_2 = \beta_{21}P_1^2 + \beta_{22}P_2^2 + \beta_{23}P_2 + \beta_{24}$$

## 5.3.2. DISCUSSIONS

The numerical integration procedure developed in this chapter and the related computer program MDEF had to be verified by comparison with the available exact linear solutions before they can be employed with confidence. One of the examples checked is

$$\ddot{x} + 2 \times 0.19808 \times 10^{-2}\ \dot{x} + x = 0.20587 \times 10^{-1} \times \cos\ (0.92000\ t)$$

The exact solution of this equation is $x = 0.13399$. The result obtained using MDEF, letting $\Delta t = \frac{T}{20}$, where $T = \frac{2\pi}{0.92}$, achieves also 0.13399 after $t = 600T$ (T is one period). This very good agreement leads to confidence in the program MDEF.

### 1. Single mode response ($A \neq 0$, $B = 0$)

The frequency-amplitude relationship of single mode response obtained using the method of averaging for the given damping ($\gamma = 9\text{x}10^{-5}$) and excitation ($\bar{F}_D = 4.25\text{x}10^{-5}$) is shown in Figure 5.1.

It is obvious that the curve consists of two branches, upper branch EF and lower one EG. Additionly, the curve can be divided into two regions, the region 1, where at each frequency there are 2 or 3 possible vibration amplitudes, and the region 2, where each frequency is associated with a single vibration amplitude.

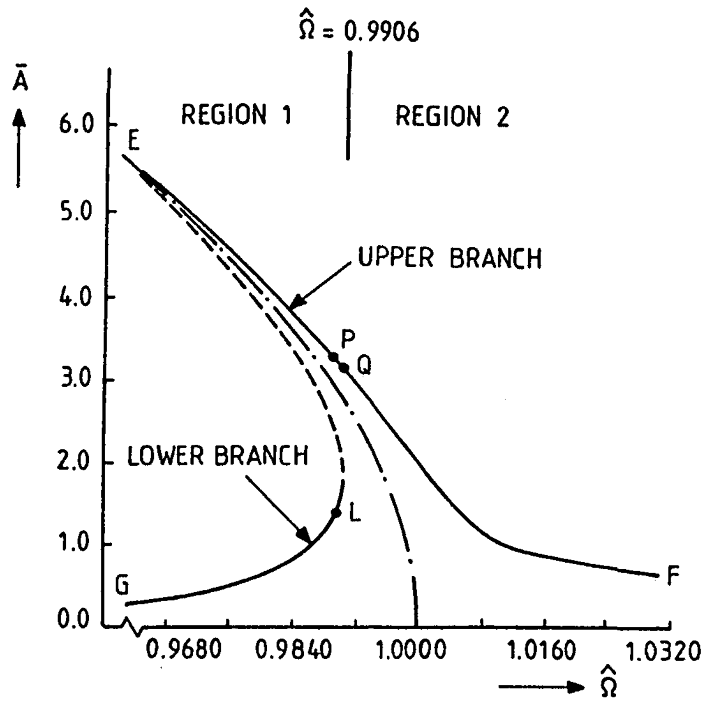



Figure 5.1 Amplitude-frequency relationship of damped vibration of perfect ES2 Shell. Damping $\gamma = 9 \times 10^{-5}$, excitation $\bar{F}_D = 4.25 \times 10^{-5}$.

In Table 5.1 and 5.2, the frequencies obtained in present study after t = 800T are compared with those obtained using the method of averaging for upper and lower branches, respectively.

Table 5.1. Comparison of frequencies obtained using the numerical integration procedure with $\Delta t = \frac{T}{20}$ at t = 800T with those obtained using the method of averaging (upper branch).

| AMPLITUDES | $\hat{\Omega}$ NUMERICAL INTEGRATION | $\hat{\Omega}$ AVERAGE METHOD |
|---|---|---|
| 1.05820 | 1.0080 | 1.0081 |
| 1.95694 | 1.0000 | 1.0000 |
| 2.95078 | 0.9920 | 0.9920 |
| 3.04144 | 0.9912 | 0.9912 |
| 3.13133 | 0.9904 | 0.9904 |
| 3.30496 | 0.9888 | 0.9888 |
| 3.39551 | 0.9879 | 0.9880 |
| 3.43462 | 0.9875 | 0.9876 |
| 3.48256 | 0.9870 | 0.9872 |

Table 5.2 Comparison of frequencies obtained using the numerical integration procedure with $\Delta t = \frac{T}{20}$ at $t = 800T$ with those obtained using the method of averaging (lower branch).

| AMPLITUDES | $\hat{\Omega}$ NUMERICAL INTEGRATION | $\hat{\Omega}$ AVERAGE METHOD |
|---|---|---|
| 1.38305 | 0.9904 | 0.9904 |
| 1.52748 | 0.9906 | 0.9906 |
| 3.06315 | 0.9910 | 0.9910 |

It is evident that the agreement is excellent. The maximum errors are less than 0.2% for the present time step $\Delta t$, and they would be expected to become less if $\Delta t$ is taken smaller ($\frac{1}{2}\Delta t$, say).

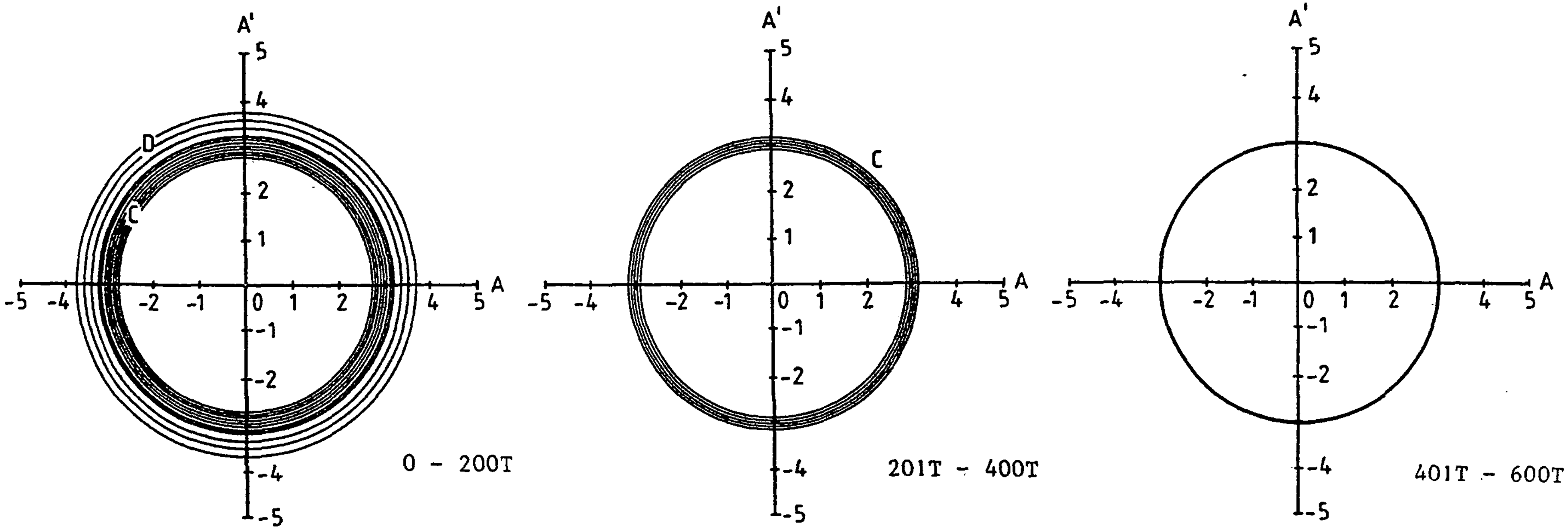
Figure 5.2 shows the phase plane of the response. It indicates that the shape of the limit cycle is nearly a perfect circle, which means that the variation of the amplitude in one period is very small indeed (less than 0.08%).

It should be noted that the converging process of the present numerical integration procedure is sinusoidally asymptotical. This makes the calculations of the amplitude in Region 1 quite difficult, even impossible in the neighbourhood of the peak of the curve. The initial conditions must be chosen carefully. Slight differences in the initial conditions will result in different solutions. For example, as shown in Table 5.3, the different initial conditions result in different solutions corresponding to the points L and Q at $\hat{\Omega} = 0.9904$ in Figure 5.1, respectively.

Table 5.3 Comparison of amplitude obtained using different initial conditions (at $\hat{\Omega} = 0.9904$, $t = 800T$)

| INITIAL CONDITION | SOLUTION OF NUMERICAL INTEGRATION |
|---|---|
| $A = 8.95814 \times 10^{-1}$<br>$A' = 1.25314 \times 10^{2}$ | $A = 3.13071$ |
| $A = 8.95814 \times 10^{-1}$<br>$A' = 1.25314$ | $A = 1.38307$ |

Also, when tracing the curves in Fig. 5.1 the frequency step $\Delta\hat{\Omega}$ must be kept small enough, especially in the Region 1. For instance, if one uses the amplitude and velocity at $\hat{\Omega}=0.9888$ of the point P along the upper branch as the initial conditions and one employs a frequency step of $\Delta\hat{\Omega}=0.0009$ then the solution converges at point Q along the upper branch. If, however, the frequency step is chosen slightly higher, say $\Delta\hat{\Omega}=0.0011$, one obtains a solution along the lower branch, as is shown in Table 5.4.




D : The transient response region;

C : The region where the stable response exists.

Fig. 5.2 Phase plane of single mode response ($A \neq 0$, $B = 0$). Frequency $\hat{\Omega} = 0.9912$.

Table 5.4 Effect of frequency step $\Delta\hat{\Omega}$.
(at $\hat{\Omega}$ = 0.9888)

| FREQUENCY STEP $\Delta\hat{\Omega}$ | SOLUTION OF NUMERICAL INTEGRATION |
|---|---|
| $\Delta\hat{\Omega}$ = -0.0011 | A = $8.9581314 \times 10^{-1}$ |
| $\Delta\hat{\Omega}$ = -0.0009 | A = 3.39551 |

2. Coupled mode response (A ≠ 0, B ≠ 0)

The amplitude-frequency relationship of the coupled mode response obtained using the method of averaging is shown in Figure 5.3. A comparison of the two sets of solutions is given in Table 5.5.

Table 5.5 Comparison of frequency obtained using the numerical integration procedure with $\Delta t = \frac{T}{20}$ with those obtained using the method of averaging.

| CASE | $\hat{\Omega}$ | NUMERICAL INTEGRATION | | METHOD OF AVERAGING | |
|---|---|---|---|---|---|
| | | A | B | A | B |
| a* | 1.0010 | 1.87383 (S)** | 1.03442 (S) | 1.88965 (S) | 1.04786 (S) |
| b | 1.0000 | 2.10926 (S) | 1.60653 (S) | 2.10774 (S) | 1.59821 (S) |
| c | 0.9960 | 1.81733 (S) | 2.19571 (S) | 1.82036 (S) | 2.20031 (S) |
| d | 0.9920 | N.S.** | N.S. | N.S. | N.S. |

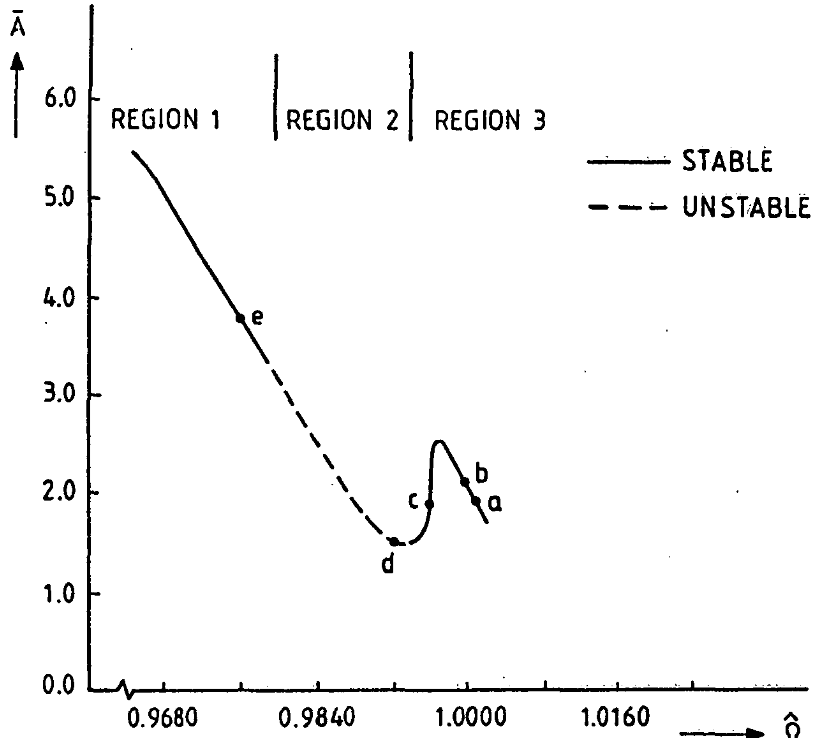
* a, b, c, d: see Figure 5.3.
** S means solution is stable, N.S. instable

As can be seen the agreement is once again very good, even for the large time step $\frac{T}{20}$. It is evident that by using a smaller time step the agreement could be further improved.

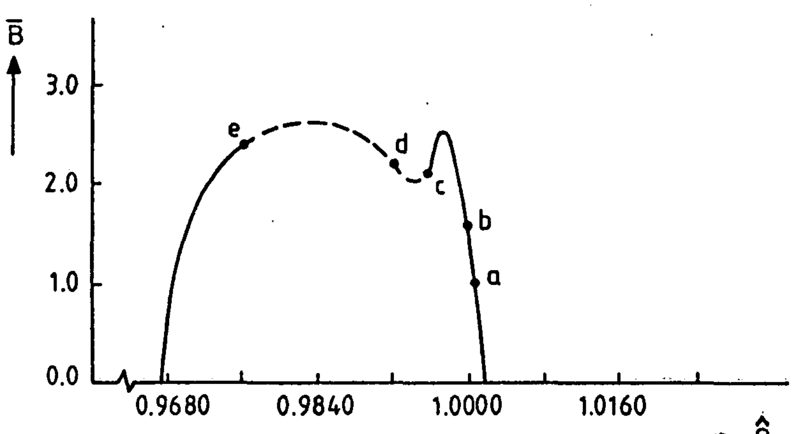
Figure 5.4 shows the phase plane of the response at $\hat{\Omega}$ = 1.00. It is quite clear that solution of such a case is unique and stable.

It can be seen from Figure 5.3 that in Region 2 of the response curve the vibration is unstable. The corresponding phase plane obtained using numerical integration is shown in Figure 5.5. It indicates that for the given initial conditions there is no stable limit circle. That means neither the coupled mode response nor the single mode response is stable at this frequency (at $\hat{\Omega}$ = 0.9940).

It is quite difficult to get a solution for the frequencies in Region 1 using the present numerical integration, since in this region the amplitudes of the single mode response and the coupled mode response are so close that the numerical integration procedure cannot identify them. Figure 5.6 shows the phase plane at $\hat{\Omega}$ = 0.9760 corresponding to the point e in Figure 5.3. As can be seen although the solution finally converged to the lower branch, one still can observe that there are solutions between A = 4 ~ 6.



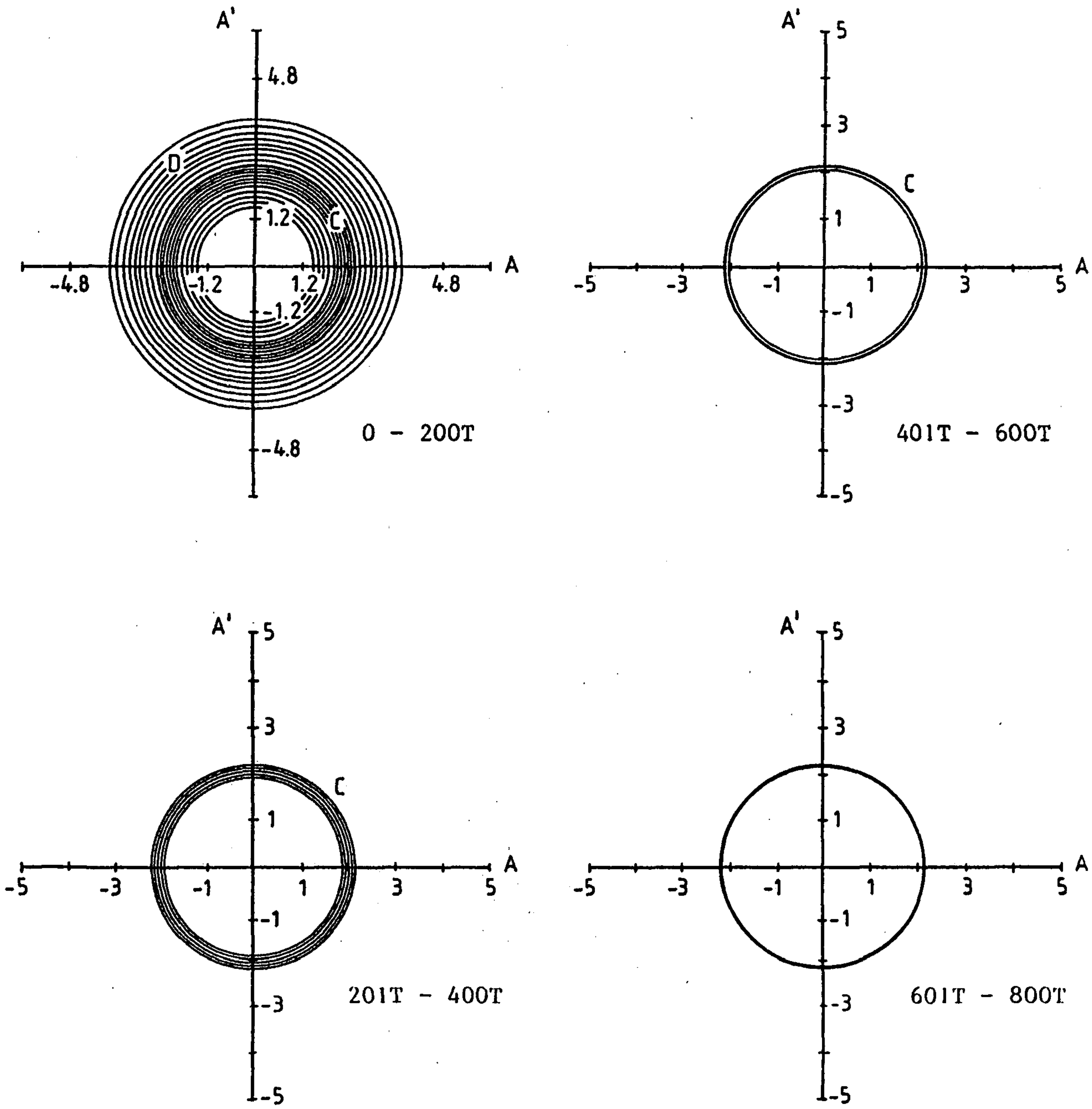

a. Single mode and driven mode response




b. Companion mode response

Fig. 5.3 Amplitude-frequency relationship of damped vibration of perfect ES2 Shell.

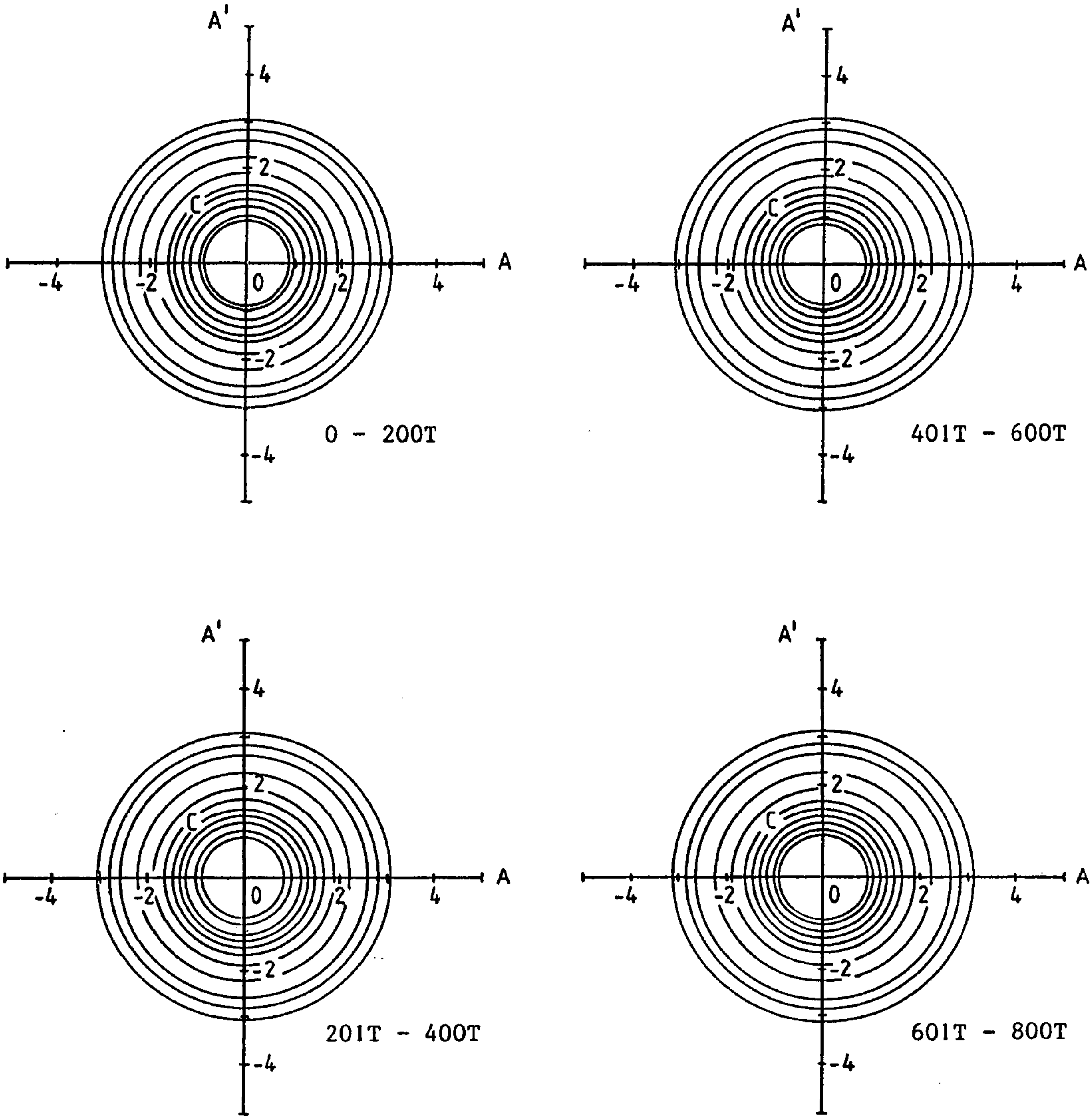
Damping $\gamma = 9 \times 10^{-5}$, excitation $F_D = 4.25 \times 10^{-5}$.




C : The region where the stable response exists.

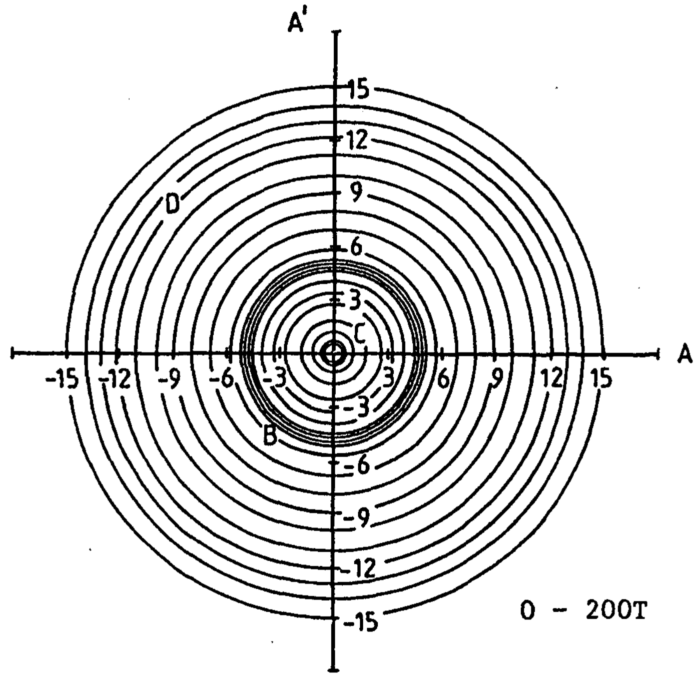
D : The transient response region.

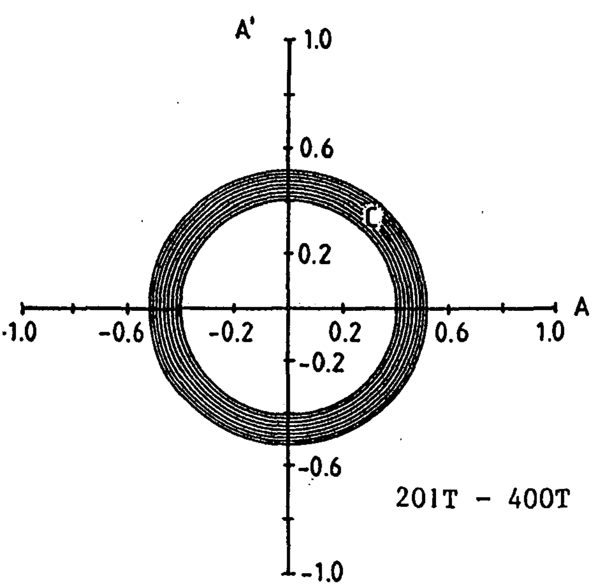
Fig. 5.4 The phase plane of coupled mode response (driven mode)
Frequency $\hat{\Omega}$ = 1.0000.

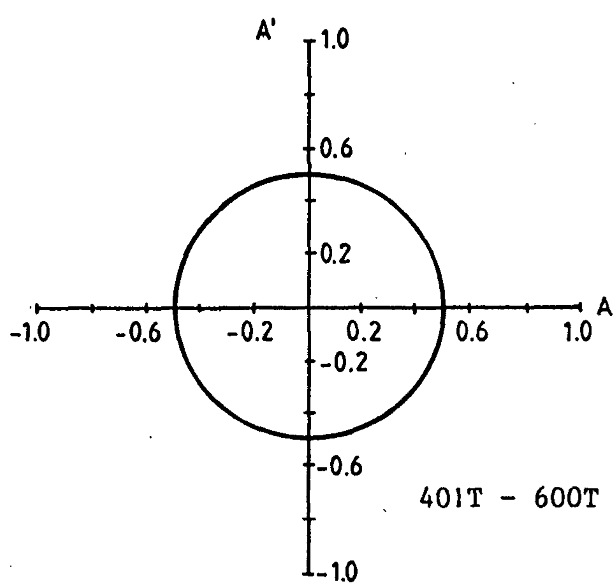



C : The region where no response is stable.

Fig. 5.5 The phase plane of coupled mode response (driven mode)
Frequency $\hat{\Omega} = 0.9940$




D : The transient response region

C : The region where the stable response exists

B : The region where no response is stable

Fig. 5.6 The phase plane of coupled mode response (driven mode)
Frequency $\hat{\Omega}$ = 0.9760.

# PART II

# Nonlinear Vibrations of Imperfect Thin-walled Cylindrical Shells with Different Boundary Conditions

# CHAPTER 6 DEVELOPMENT OF GOVERNING EQUATIONS

## 6.1 INTRODUCTION

It is the aim of Part II to study the effect of boundary conditions on the nonlinear vibration behaviour of imperfect thin-walled cylindrical shells. The governing equations of the problem are derived briefly in this chapter. The details of derivation are given in the relevant appendix. To be able to obtain a complete nonlinear amplitude-frequency relationship the concept known as the 'end-shortening' is introduced in Section 6.3, which is a natural extension of Ref. [76] to the dynamic state. Of course, other methods, such as the incremental method described in Ref. [53~55] could also be used to overcome the problem. It was decided to use the parallel shooting method, which is presented in Section 6.4, since it has been proven to be very succesful for solving nonlinear problems [63]. The basic idea in the parallel shooting method is to partition the interval into subintervals and to compute the solution over each of them (more or less) independently. The initial guesses involved are then improved iteratively while one satisfies the given boundary conditions and the relevant continuity conditions, which are imposed at each subinterval interface.

It should be mentioned that unlike in Part I, only the dynamic state is considered in this part. The solution procedure for the fundamental state is a direct extension of the present procedure.

## 6.2. DEVELOPMENT OF BASIC THEORY

The equations (1-3-47) and (1-3-48) can be written as

$$L_H(\hat{\hat{\Phi}}) - L_Q(\hat{\hat{W}}) = -R\hat{\hat{W}}_{,\bar{x}\bar{x}} - \frac{1}{2}L_{NL}(\hat{W},\hat{\hat{W}}) - \frac{1}{2}L_{NL}(\hat{\hat{W}},\hat{W}+2\bar{W}) - \frac{1}{2}L_{NL}(\hat{\hat{W}},\hat{\hat{W}}) \tag{6-2-1}$$

$$L_Q(\hat{\hat{\Phi}}) + L_D(\hat{\hat{W}}) = R\hat{\hat{\Phi}}_{,\bar{x}\bar{x}} + L_{NL}(\hat{\Phi},\hat{\hat{W}}) + L_{NL}(\hat{\hat{\Phi}},\hat{W}+\bar{W}) + L_{NL}(\hat{\hat{\Phi}},\hat{\hat{W}})$$

$$- \bar{\rho}hR^4\hat{\hat{w}}_{,tt} + qR^4 \tag{6-2-2}$$

where $\bar{x} = \frac{x}{R}$ , $\bar{y} = \frac{y}{R}$ and $q = \tilde{Q}(\bar{x},\bar{y})\cos\omega t$.

If we assume that the initial imperfection surface can be represented as,

$$\bar{W}(\bar{x},\bar{y}) = hA_o(\bar{x}) + hA_1(\bar{x})\ \cos n\bar{y} \tag{6-2-3}$$

where $A_o(\bar{x})$ and $A_1(\bar{x})$ are known functions of $\bar{x}$, then the solutions for the fundamental state are [76],

$$\hat{W}(\bar{x},\bar{y}) = h\hat{W}_v + h\hat{W}_o(\bar{x}) + h\hat{W}_1(\bar{x}) \cos n\bar{y} \tag{6-2-4}$$

$$\hat{\Phi}(\bar{x},\bar{y}) = \frac{ERh^2}{c} \{-\frac{\lambda}{2}\bar{y}^2 + \hat{\phi}_o(\bar{x}) + \hat{\phi}_1(\bar{x})\cos n\bar{y} + \hat{\phi}_2(\bar{x})\cos 2n\bar{y}\} \tag{6-2-5}$$

Assuming that the dynamic solution has the form

$$\hat{\hat{W}}(\bar{x},\bar{y},t) = hA(\bar{x},t)\cos\ell\bar{y} + hB(\bar{x},t)\sin\ell\bar{y} + hC(\bar{x},t) \tag{6-2-6}$$

and introducing equations (6-2-6) and (6-2-3)~(6-2-5) into (6-2-1) yields

$$L_H(\hat{\hat{\Phi}}) = \bar{a}_o + \sum_{i=1}^{5} \bar{a}_i \cos(\tilde{\alpha}_i\bar{y}) + \sum_{j=1}^{4} \bar{b}_j \sin(\tilde{\alpha}_j\bar{y}) \tag{6-2-7}$$

To allow separation of variables of equation (6-2-7) $\hat{\hat{\Phi}}$ should have the form

$$\hat{\hat{\Phi}} = \frac{ERh^2}{c} \{\hat{\hat{\phi}}_o + \sum_{i=1}^{5} \hat{\hat{\phi}}_i \cos(\tilde{\alpha}_i\bar{y}) + \sum_{j=1}^{4} \hat{\hat{\phi}}_j \sin(\tilde{\alpha}_j\bar{y})\} \tag{6-2-8}$$

where $a_o$ and $a_i$ are the functions of the unknowns A, B, C and their derivatives, $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ are the wave parameters, as shown in Appendix 6-A.1.

Substituting the expressions for $\hat{\hat{W}}$, $\hat{W}$ and $\hat{\hat{\Phi}}$ into the compatibility equation (6-2-1), using some trigonometric identities and finally equating coefficients of like terms results in the following system of 10 nonlinear partial differential equations,

$$\hat{\hat{\phi}}_{o,\overline{xxxx}} = P_o \tag{6-2-9}$$

$$\hat{\hat{\phi}}_{i,\overline{xxxx}} = P_i \quad (i = 1,2,\ldots,9) \tag{6-2-10}$$

where both $P_o$ and $P_i$ are the functions of A, B, C and their derivatives, which are defined in the Appendix 6-A.2. The equation (6-2-9) can be integrated twice to yield

$$\hat{\hat{\Phi}}_{o,\overline{\overline{xx}}} = \frac{1}{2}\frac{h}{R}\frac{\bar{Q}_{xx}}{\bar{H}_{xx}} C_{,\overline{\overline{xx}}} - \frac{c}{\bar{H}_{xx}} C + \frac{1}{4}\frac{h}{R}\frac{c}{\bar{H}_{xx}} \ell^2 (A^2+B^2)$$
$$+ \delta_{n,\ell}\left[\frac{1}{2}\frac{h}{R}\frac{c}{\bar{H}_{xx}}\ell^2 (\hat{W}_1+A_1)A - \hat{\hat{\Phi}}_{6,\overline{\overline{xx}}}\right] \qquad (6\text{-}2\text{-}11)$$

where the constants of integration are already set to zero in order to satisfy periodicity condition, as indicated in Appendix 6-B.

Substituting in turn the expressions for $\hat{\hat{W}}$, $\hat{W}$, $\bar{W}$, $\hat{\Phi}$ and $\hat{\hat{\Phi}}$ into the equilibrium equation (6-2-2) and applying Galerkin's method yields the following system of nonlinear partial differential equations,

$$A_{,\overline{\overline{\overline{\overline{xxxx}}}}} = \sum_{i=1}^{52} a_i f_{a_i} \qquad (6\text{-}2\text{-}12)$$

$$B_{,\overline{\overline{\overline{\overline{xxxx}}}}} = \sum_{j=1}^{44} b_j f_{b_j} \qquad (6\text{-}2\text{-}13)$$

$$C_{,\overline{\overline{\overline{\overline{xxxx}}}}} = \sum_{k=1}^{37} c_k f_{c_k} \qquad (6\text{-}2\text{-}14)$$

The $a_i$, $b_j$, $c_k$ are the functions of geometric parameters, while the $f_{a_i}$, $f_{b_j}$ and $f_{c_k}$ are the functions of the unknowns and their derivatives. All these coefficients and parameters are shown in Appendix 6-A.3.

Eliminating the terms of $A_{,\overline{\overline{\overline{\overline{xxxx}}}}}$ and $B_{,\overline{\overline{\overline{\overline{xxxx}}}}}$ from Eqs. (6-2-10) with the help of equations (6-2-12) and (6-2-13) yields the final equations of $\hat{\hat{\Phi}}_{1,\overline{\overline{\overline{\overline{xxxx}}}}}$ and $\hat{\hat{\Phi}}_{2,xxxx}$,

$$\hat{\hat{\Phi}}_{1,\overline{\overline{\overline{\overline{xxxx}}}}} = \sum_{i=1}^{51} d_i f_{a_i} \qquad (6\text{-}2\text{-}15)$$

$$\hat{\phi}_{2,\overline{xxxx}} = \sum_{j=1}^{44} e_j f_{b_j} \qquad (6\text{-}2\text{-}16)$$

where the $d_i$ and $e_j$ are listed in Appendix 6-A.4.
The problem thus involves the system of 12 nonlinear partial differential equations (6-2-10) and (6-2-12) ~ (6-2-16).

If we assume that the dynamic response A and B can be represented as

$$A(\bar{x},t) = A_o(\bar{x},t)\cos\omega t \qquad (6\text{-}2\text{-}17)$$

$$B(\bar{x},t) = B_o(\bar{x},t)\sin\omega t \qquad (6\text{-}2\text{-}18)$$

then with the help of equations (6-2-10) and (6-2-11) we can assume further the form for the unknowns $\hat{\phi}_i$ and C,

$$C(\bar{x},t) = C_1(\bar{x},t) + C_2(\bar{x},t)\cos 2\omega t + \delta_{n,\ell} C_3(\bar{x},t)\cos\omega t \qquad (6\text{-}2\text{-}19)$$

$$\hat{\phi}_1(\bar{x},t) = \hat{\phi}_{11}(\bar{x},t)\cos\omega t + \hat{\phi}_{12}(\bar{x},t)\cos 3\omega t + \delta_{n,\ell}\hat{\phi}_{13}(\bar{x},t)\cos^2\omega t \qquad (6\text{-}2\text{-}20)$$

$$\hat{\phi}_2(\bar{x},t) = \hat{\phi}_{21}(\bar{x},t)\sin\omega t + \hat{\phi}_{22}(\bar{x},t)\sin 3\omega t + \delta_{n,\ell}\hat{\phi}_{23}(\bar{x},t)\sin 2\omega t \qquad (6\text{-}2\text{-}21)$$

$$\hat{\phi}_3(\bar{x},t) = \hat{\phi}_{31}(\bar{x},t) + \hat{\phi}_{32}(\bar{x},t)\cos 2\omega t \qquad (6\text{-}2\text{-}22)$$

$$\hat{\phi}_4(\bar{x},t) = \hat{\phi}_{41}(\bar{x},t)\sin 2\omega t \qquad (6\text{-}2\text{-}23)$$

$$\hat{\phi}_5(\bar{x},t) = \hat{\phi}_{51}(\bar{x},t)\cos\omega t \qquad (6\text{-}2\text{-}24)$$

$$\hat{\phi}_6(\bar{x},t) = \hat{\phi}_{61}(\bar{x},t)\cos\omega t \qquad (6\text{-}2\text{-}25)$$

$$\hat{\phi}_7(\bar{x},t) = \hat{\phi}_{71}(\bar{x},t)\sin\omega t \qquad (6\text{-}2\text{-}26)$$

$$\hat{\hat{\phi}}_8(\bar{x},t) = \hat{\hat{\phi}}_{81}(\bar{x},t)\sin\omega t \tag{6-2-27}$$

$$\hat{\hat{\phi}}_9(\bar{x},t) = \hat{\hat{\phi}}_{91}(\bar{x},t) + \hat{\hat{\phi}}_{92}(\bar{x},t)\cos 2\omega t + \delta_{n,\ell}\hat{\hat{\phi}}_{93}(\bar{x},t)\cos\omega t \tag{6-2-28}$$

substituting all of these expressions into equations (6-2-10) and (6-2-12) ~ (6-2-16) and applying the method of averaging to the resulting equations in sequence yields the following system of 21 nonlinear ordinary differential equations,

$$\bar{\hat{\hat{\phi}}}_i^{IV} = \bar{f}_i \qquad (i = 11,12,13,21,22,23,31,32,41,51,61,71,81,91,92,93) \tag{6-2-29}$$

$$\bar{A}^{IV} = \bar{f}_2 \tag{6-2-30}$$

$$\bar{B}^{IV} = \bar{f}_3 \tag{6-2-31}$$

$$\bar{C}_j^{IV} = \bar{f}_{4j} \qquad (j = 1,2,3) \tag{6-2-32}$$

where $\bar{f}_i$, $\bar{f}_2$, $\bar{f}_3$ and $\bar{f}_{4j}$ are functions of the geometric parameters, the excitations and the unknowns $\bar{A}$, $\bar{B}$, $\bar{C}_j$, $\bar{\hat{\hat{\phi}}}_i$ and their derivatives. They are listed in Appendix 6-A.5. $\bar{\hat{\hat{\phi}}}_i$, $\bar{A}$, $\bar{B}$ and $\bar{C}_j$ are the average values of $\hat{\hat{\phi}}_i$, A, B and $C_j$ over one period of vibration respectively.

For the case of single mode response, where $\bar{B} = 0$, these equations can be reduced into a system of 11 nonlinear ordinary differential equations,

$$\bar{\hat{\hat{\phi}}}_i^{IV} = \tilde{f}_i \qquad (i = 11,12,13,31,51,61,91,93) \tag{6-2-33}$$

$$\bar{A}^{IV} = \tilde{f}_2 \tag{6-2-34}$$

$$\bar{C}_j^{IV} = \tilde{f}_{3j} \qquad (j = 1,2) \tag{6-2-35}$$

The $\tilde{f}_i$, $\tilde{f}_2$ and $\tilde{f}_{3j}$ are also listed in Appendix 6-A.6. Notice in such a case we have new assumptions,

$$\bar{C} = 2\bar{C}_1\cos^2\omega t + \delta_{n,\ell}\bar{C}_3\cos\omega t \tag{6-2-34}$$

$$\bar{\hat{\hat{\Phi}}}_{31} = \bar{\hat{\hat{\Phi}}}_{32} \tag{6-2-37}$$

$$\bar{\hat{\hat{\Phi}}}_9 = 2\bar{\hat{\hat{\Phi}}}_{91}\cos^2\omega t + \delta_{n,\ell}\ \bar{\hat{\hat{\Phi}}}_{93}\cos\omega t \tag{6-2-38}$$

Once the boundary conditions are specified, the problem described by Eqs. (6-2-29)~(6-2-32) is well posed. The boundary conditions used in the thesis are summarized in Table 6.1. The derivation of these boundary conditions are given in Appendix 6-C.

Table 6.1 List of boundary conditions

| Case | Symbol | Boundary conditions | | | |
|---|---|---|---|---|---|
| 1 | SS1 | $\hat{\hat{W}} = 0$ | $\hat{\hat{M}}_x = 0$ | $\hat{\hat{N}}_{xy} = 0$ | $\hat{\hat{N}}_x = 0$ |
| 2 | SS2 | $\hat{\hat{W}} = 0$ | $\hat{\hat{M}}_x = 0$ | $\hat{\hat{N}}_{xy} = 0$ | $\hat{\hat{u}} = 0$ |
| 3 | SS3 | $\hat{\hat{W}} = 0$ | $\hat{\hat{M}}_x = 0$ | $\hat{\hat{v}} = 0$ | $\hat{\hat{N}}_x = 0$ |
| 4 | SS4 | $\hat{\hat{W}} = 0$ | $\hat{\hat{M}}_x = 0$ | $\hat{\hat{v}} = 0$ | $\hat{\hat{u}} = 0$ |
| 5 | C1 | $\hat{\hat{W}} = 0$ | $\hat{\hat{W}}_{,x} = 0$ | $\hat{\hat{N}}_{xy} = 0$ | $\hat{\hat{N}}_x = 0$ |
| 6 | C2 | $\hat{\hat{W}} = 0$ | $\hat{\hat{W}}_{,x} = 0$ | $\hat{\hat{N}}_{xy} = 0$ | $\hat{\hat{u}} = 0$ |
| 7 | C3 | $\hat{\hat{W}} = 0$ | $\hat{\hat{W}}_{,x} = 0$ | $\hat{\hat{v}} = 0$ | $\hat{\hat{N}}_x = 0$ |
| 8 | C4 | $\hat{\hat{W}} = 0$ | $\hat{\hat{W}}_{,x} = 0$ | $\hat{\hat{v}} = 0$ | $\hat{\hat{u}} = 0$ |

In the following study the boundary condition SS3 is used to illustrate the solution of the problem. Notice that the derivation of the reduced boundary conditions requires both the application of Galerkin's method to eliminate the y dependence and the use of the method of averaging to eliminate the time dependence. For the SS-3 boundary condition one obtaines for the case of single mode response ($\bar{B} = 0$)

$$
\begin{aligned}
&\bar{A} = \bar{C}_1 = \bar{C}_3 = \bar{\hat{\bar{\phi}}}_i = 0 \\
&\qquad\qquad (i = 11,12,13,31,51,91,93) \quad \text{at } \bar{x} = 0, \tfrac{L}{R} \\
&\bar{A}'' = \bar{C}_1'' = \bar{C}_3'' = \bar{\hat{\bar{\phi}}}_i'' = 0
\end{aligned}
\tag{6-2-39}
$$

Introducing now as a unified variable the 44-dimensional vector $\underset{\sim}{Y}$ defined as follows,

$$
Y_1 = \bar{\hat{\bar{\phi}}}_{11},\ Y_2 = \bar{\hat{\bar{\phi}}}_{12},\ \ldots,\ Y_{44} = \bar{C}_3''' \tag{6-2-40}
$$

then the system of equations (6-2-33) ~ (6-2-35) and (6-2-39) can be reduced to the following nonlinear 2-point boundary value problem

$$
\begin{aligned}
&\frac{d}{d\bar{x}} \underset{\sim}{Y} = \underset{\sim}{f}(\bar{x}; \underset{\sim}{Y}, \Omega) && \text{for} \quad 0 \leq \bar{x} \leq \frac{L}{R} \\
&Y_i = 0 && (i = 1,2,\ldots,11) \quad \text{at } \bar{x} = 0 \\
&Y_j = 0 && (j = 23,24,\ldots,33) \quad \text{at } \bar{x} = \frac{L}{R}
\end{aligned}
\tag{6-2-41}
$$

The solution of this nonlinear 2-point boundary value problem will then yield the form the amplitude-frequency relationships and the vibration modes for the case of single mode reponse. For the coupled mode response case the system of equations (6-2-29) ~ (6-2-32) can be reduced to the solution of another nonlinear 2-point boundary value problem by proceeding as described above.

## 6.3 CONCEPT OF END-SHORTENING

A typical frequency-amplitude relationship of nonlinear vibration of a shell is shown in Fig. 6.1.

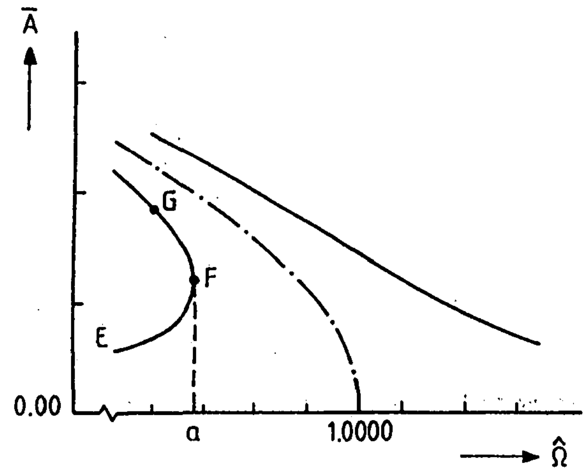



Fig. 6.1 Typical frequency-amplitude relationship of nonlinear vibration of a shell

It is quite clear that using the frequency increments in the present analysis one cannot obtain the solutions in the segment GF since the problem is multi-valued there. A closer look at the curve shown in Fig. 6.1 reveals, however, that one would be able to extend the curve beyond the point F using increments in deformation instead of increments in frequency. Such technique is developed and discussed in detail in this section. For the sake of brevity, only the equations for the case of single mode response are shown here.
The method developed following Ref. [76] uses the concept of 'end-shortening', which is defined as

$$\delta = - \frac{1}{2\pi RL} \int_0^{2\pi R} \int_0^{L} \hat{\hat{u}}_{,x} \, dxdy \qquad (6\text{-}3\text{-}1)$$

where

$$\hat{\hat{u}}_{,x} = \hat{\hat{\varepsilon}}_x - \frac{1}{2} (\hat{\hat{W}}_{,x})^2 - (\hat{\hat{W}}_{,x})^2 - (\hat{\hat{W}}_{,x} \bar{W}_{,x}) \qquad (6\text{-}3\text{-}2)$$

and

$$\hat{\hat{\varepsilon}}_x = \hat{\beta} \frac{1-\nu^2}{Eh} [(1+\mu_2)\hat{\hat{N}}_x - \nu\hat{\hat{N}}_y] - (1+\mu_2)\chi_1\hat{\beta}\hat{\hat{\kappa}}_x + \hat{\beta}\nu\chi_2\hat{\hat{\kappa}}_y \qquad (6\text{-}3\text{-}3)$$

Substituting for $\hat{\hat{\Phi}}$ and $\hat{\hat{W}}$ into Eq. (6-3-3) and (6-3-2), eliminating the y-dependence by integrating over y and then making it nondimensional yields,

$$\tilde{u}_{,\bar{x}} = \frac{1}{Rh}\{-\hat{\beta}(1-\nu^2)\frac{\nu}{c}Rh\,\hat{\hat{\phi}}_{0,\bar{x}\bar{x}} + (1+\mu_2)\chi_1\hat{\beta}h\,C_{,\bar{x}\bar{x}} - \frac{1}{4}h^2A^2_{,\bar{x}} - \frac{1}{2}C^2_{,\bar{x}}$$

$$- h^2A_{0,\bar{x}}C_{,\bar{x}}\} + \delta_{n,\ell}\frac{1}{Rh}\{\hat{\beta}(1-\nu^2)(1+\mu_2)Rh\frac{1}{c}\hat{\hat{\phi}}_{6,\bar{x}\bar{x}} - \frac{1}{2}h^2\,A_{1,\bar{x}}A_{,\bar{x}}\} \tag{6-3-4}$$

This equation can be reduced further by introducing $\hat{\hat{\Phi}}_{0,\bar{x}\bar{x}}$ from the equation (6-2-11). This yields

$$\tilde{u}_{,\bar{x}} = \frac{1}{2}\frac{1}{c}\frac{h}{R}\bar{Q}_{xx}\left[-\frac{\nu}{1+\mu_1} + \frac{1+\mu_2}{\nu}\right]C_{,\bar{x}\bar{x}} - \frac{1}{2}\frac{h}{R}C^2_{,\bar{x}} - \frac{h}{R}A_{0,\bar{x}}C_{,\bar{x}} + \frac{\nu}{1+\mu_1}C$$

$$- \frac{1}{4}\frac{h}{R}\frac{\nu}{1+\mu_1}\ell^2A^2 - \frac{1}{4}\frac{h}{R}A^2_{,\bar{x}} - \delta_{n,\ell}\left\{\frac{1}{2}\frac{h}{R}\frac{\nu}{1+\mu_1}\ell^2A_1A + \frac{1}{2}\frac{h}{R}A_{1,\bar{x}}A_{,\bar{x}}\right\} \tag{6-3-5}$$

where

$$\tilde{u} = h\hat{\hat{u}}$$

Substituting Eq. (6-3-5) into (6-3-1) and then expressing the resulting equation in the form of a differential equation one obtains

$$\frac{d\delta}{d\bar{x}} = -\frac{h}{L}\left\{\frac{1}{2}\frac{1}{c}\frac{h}{R}\bar{Q}_{xx}\left[-\frac{\nu}{1+\mu_1} + \frac{1+\mu_2}{\nu}\right]C_{,\bar{x}\bar{x}} - \frac{1}{2}\frac{h}{R}C^2_{,\bar{x}} - \frac{h}{R}A_{0,\bar{x}}C_{,\bar{x}}\right.$$

$$+ \frac{\nu}{1+\mu_1}C - \frac{1}{4}\frac{h}{R}\frac{\nu}{1+\mu_1}\ell^2A^2 - \frac{1}{4}\frac{h}{R}A^2_{,\bar{x}}$$

$$\left. - \delta_{n,\ell}\left[\frac{1}{2}\frac{h}{R}\frac{\nu}{1+\mu_1}\ell^2A_1A + \frac{1}{2}\frac{h}{R}A_{1,\bar{x}}A_{,\bar{x}}\right]\right\} \tag{6-3-6}$$

Recalling that

$$C(\bar{x},t) = 2C_1(\bar{x},t)\cos^2\omega t + \delta_{n,\ell}C_3(\bar{x},t)\cos\omega t$$

$$A(\bar{x},t) = A(\bar{x},t)\cos\omega t \tag{6-3-7}$$

and substituting them into Eq. (6-3-6) yields

$$\frac{d}{d\bar{x}}\,\delta = \tilde{e}_o(\bar{x},t) + \sum_{i=1}^{4} \tilde{e}_i\,(\bar{x},\ t)\ \cos i\omega t$$

and

$$\delta(o,t) = 0$$

$$\delta(\tfrac{L}{R},\ t) = \delta_L \qquad (6\text{-}3\text{-}8)$$

where the $\tilde{e}_o$ and $\tilde{e}_i$ are listed in Appendix 6-D.

Basing on the concept of harmonic balance the form of $\varepsilon$ could be guessed as

$$\delta(\bar{x},t) = \varepsilon_o(\bar{x},\ t) + \sum_{i=1}^{4} \varepsilon_i\,(\bar{x},\ t)\ \cos i\omega t \qquad (6\text{-}3\text{-}9)$$

Theoretically, either $\varepsilon_o$ or any one of the $\varepsilon_i$'s could be chosen as the expression for 'end-shortening', for example $\varepsilon_2(\bar{x},t)$. Substituting equations (6-3-9) and (6-3-7) into (6-3-6), and then applying the method of averaging to the resulting equation using cos2ωt as a weighting function one obtains

$$\frac{d}{d\bar{x}}\,\bar{\varepsilon}_2 = \frac{1}{2}\,\frac{1}{c}\,\bar{Q}_{xx}\ \left[\frac{\nu}{1+\mu_1} - \frac{1+\mu_2}{\nu}\right] \bar{C}_1'' + \frac{h^2}{RL}\,A_{0,\bar{x}}\,\bar{C}_1' - \frac{h}{L}\,\frac{\nu}{1+\mu_1}\,\bar{C}_1 + \frac{h^2}{LR}\,(\bar{C}'_1)^2$$

$$+ \delta_{n,\ell}\,\frac{1}{4}\,\frac{h^2}{LR}\,(\bar{C}_3')^2 + \frac{1}{8}\,\frac{h^2}{LR}\,\frac{\nu}{1+\mu_1}\,\ell^2\bar{A}^2 + \frac{1}{8}\,\frac{h^2}{LR}\,(\bar{A}')^2 \qquad (6\text{-}3\text{-}10)$$

and

$$\bar{\varepsilon}_2(o) = 0$$

$$\bar{\varepsilon}_2(\tfrac{L}{R}) = \bar{\varepsilon}_{2L}$$

Introducing the parameter $\varepsilon_{c\ell} = \frac{1}{c}\,\frac{h}{R}$ and letting $\tilde{\varepsilon}_2 = \dfrac{\bar{\varepsilon}_2}{\varepsilon_{c\ell}}$ one obtains

$$\frac{d}{d\bar{x}} \tilde{\varepsilon}_2 = f_e \tag{6-3-11}$$

with the boundary conditions

$$\tilde{\varepsilon}_2(o) = 0$$

$$\tilde{\varepsilon}_2(\frac{L}{R}) = \delta_o$$

where $\delta_0 = \frac{\bar{\varepsilon}_{2L}}{\varepsilon_{cL}}$, and the function $f_e$ is of the form

$$f_e = \frac{1}{2}\,\frac{h}{L}\,[\frac{\nu}{1+\mu_1} - \frac{1+\mu_2}{\nu}]\bar{Q}_{xx}\bar{C}_1'' + \frac{h}{L}\,cA_{0,\bar{x}}\,\bar{C}_1' - \frac{R}{L}\,c\,\frac{\nu}{1+\mu_1}\,\bar{C}_1 + \frac{h}{L}\,c\,(\bar{C}_1')^2$$

$$+ \delta_{n,\ell}[\frac{1}{4}\,\frac{h}{L}\,c\,(\bar{C}_3')^2] + \frac{1}{8}\,\frac{h}{L}\,c\,[\frac{\nu}{1+\mu_1}\,\ell^2\,\bar{A}^2 + (\bar{A}')^2]$$

Hence one must find the solution for the system of Eqs. (6-2-33) ~ (6-2-35) (for the case of single mode response) under the restriction that the solution must also satisfy the constraint condition given by Eq. (6-3-11). With Eqs. (6-2-33) ~ (6-2-35) and (6-3-11) together one obtains a new system of nonlinear ordinary differential equations, in which the frequencies as well as amplitudes are both unknowns.

Introducing now a new vector $\hat{\underset{\sim}{Y}}$

$$\hat{\underset{\sim}{Y}} = \{\underset{\sim}{Y},\ Y_{45}\} \tag{6-3-12}$$

where

$$Y_{45} = \tilde{\varepsilon}_2$$

the system then reduces to the following nonlinear 2-point boundary value problem,

$$\frac{d}{d\bar{x}}\hat{\underset{\sim}{Y}} = \hat{\underset{\sim}{f}}\,(\bar{x};\ \hat{\underset{\sim}{Y}},\ \Omega) \qquad \text{for } 0 \leq \bar{x} \leq \frac{L}{R}$$

$$\hat{Y}_i = 0 \qquad (i = 1,2,\ldots,11) \qquad \text{at } \bar{x} = 0$$

$$\hat{Y}_j = 0 \qquad (j = 23,24,\ldots,32,33) \qquad \text{at } \bar{x} = \frac{L}{R} \tag{6-3-13}$$

## 6.4. GENERAL NUMERICAL PROCEDURE

Due to the nonlinear nature of the problem, anything but a numerical solution is out of the question. There are many numerical methods which are available to solve the problem, for example, the finite difference method, the perturbation method and the shooting method, etc. Trying to make use of the readily available coded subroutines for solving nonlinear initial value problems it was decided to employ the 'shooting method' [57] to solve the present problem. Though, due to numerical instability, it becomes necessary to employ parallel shooting over 10 intervals to carry out the integration over the shell lengths used. For the purpose of describing the method, let us consider just 'double shooting' or 'parallel shooting over 2 intervals'.

### 6.4.1 GENERAL PROCEDURE

Let us associate the following 2 initial value problems with the 2-point nonlinear boundary value problem described by equation (6-2-41),

$$\frac{d\underset{\sim}{u}}{d\bar{x}} = \underset{\sim}{f}(\bar{x},\ \Omega;\underset{\sim}{u}) \qquad \text{for } 0 \leq \bar{x} \leq \bar{x}_o \qquad \underline{\text{Forward Integration}}$$

$$\underset{\sim}{u}(o) = \underset{\sim}{s} = \{0,0,0,\ldots,0;\ s_1,s_2,\ldots,s_{11};\ 0,0, \quad ,0;\ s_{12},s_{13},\ldots s_{22}\} \tag{6-4-1}$$

and

$$\frac{d\underset{\sim}{v}}{d\bar{x}} = \underset{\sim}{f}(\bar{x},\Omega;\underset{\sim}{v}) \qquad \text{for } \bar{x}_o \leq \bar{x} \leq \frac{L}{R} \qquad \underline{\text{Backward Integration}}$$

$$\underset{\sim}{v}(\tfrac{L}{R}) = \underset{\sim}{t} = \{0,0,\ldots,0;\ t_1,t_2,\ldots,t_{11};\ \ 0,\ 0,\ldots,\ 0;\ t_{12},t_{13},\ldots,t_{22}\} \tag{6-4-2}$$

Under appropriate smoothness conditions on the nonlinear vector function $\underset{\sim}{f}(x,\Omega;\underset{\sim}{Y})$ we are assured of the existence of unique solutions of these initial value problems, here denoted by

$$\underset{\sim}{U}(\bar{x},\Omega;\underset{\sim}{s}) \text{ and } \underset{\sim}{V}(\bar{x},\Omega;\underset{\sim}{t}) \tag{6-4-3}$$

These solutions must satisfy matching conditions at $\bar{x} = \bar{x}_o$. Introducing the new vector function $\underset{\sim}{\phi}$ these conditions can be written as

$$\underset{\sim}{\phi}(\underset{\sim}{S}) = \underset{\sim}{U}(\bar{x} = \bar{x}_0,\Omega;\underset{\sim}{s}) - \underset{\sim}{V}(\bar{x} = \bar{x}_0,\Omega;\underset{\sim}{t}) = 0 \tag{6-4-4}$$

where

$$\underset{\sim}{S} = \binom{\underset{\sim}{s}}{\underset{\sim}{t}}$$

Thus the solution of the nonlinear 2-point boundary-value problem (6-2-41) has been transformed to the solution of the two associated initial value problems (6-4-1) ~ (6-4-2) and to the finding of the roots $\underset{\sim}{S}$ of the system of simultaneous equations defined by equation (6-4-4).

Using Newton's method for finding the roots of $\underset{\sim}{\phi}(\underset{\sim}{S}) = 0$ we have the following iteration scheme,

$$\underset{\sim}{S}^{\mu+1} = \underset{\sim}{S}^{\mu} + \Delta\underset{\sim}{S}^{\mu} \tag{6-4-5}$$

where $\Delta\underset{\sim}{S}^{\mu}$ is the solution of the 44th-order linear algebraic system

$$\frac{\partial}{\partial S}\underset{\sim}{\phi}(\underset{\sim}{S}^{\mu}).\Delta\underset{\sim}{S}^{\mu} = -\underset{\sim}{\phi}(\underset{\sim}{S}^{\mu}) \tag{6-4-6}$$

To apply Newton's method we must be able to find the Jacobian matrix J

$$J(\underset{\sim}{S}^{\mu}) = \frac{\partial}{\partial \underset{\sim}{S}}\underset{\sim}{\phi}(\underset{\sim}{S}^{\mu}) = \begin{bmatrix} \frac{\partial\phi_1}{\partial S_1} & \frac{\partial\phi_1}{\partial S_2} \cdots \cdots & \frac{\partial\phi_1}{\partial S_{44}} \\ \frac{\partial\phi_2}{\partial S_1} \cdots \cdots & & \\ \cdots \cdots \cdots \cdots & & \\ \frac{\partial\phi_{44}}{\partial S_1} & \cdots \cdots & \frac{\partial\phi_{44}}{\partial S_{44}} \end{bmatrix} \tag{6-4-7}$$

In order to solve for the components of the Jacobian matrix J let us introduce the following new vectors

$$\underset{\sim}{W}_i = \frac{\partial\underset{\sim}{u}}{\partial S_i} = \frac{\partial\underset{\sim}{u}}{\partial s_i} \text{ for } i = 1,2,\ldots,22$$

$$\underset{\sim}{W}_i = \frac{\partial\underset{\sim}{v}}{\partial S_i} = \frac{\partial\underset{\sim}{v}}{\partial t_j} \text{ for } i = 23,24,\ldots,44 \text{ and } j = i-22 \tag{6-4-8}$$

These vectors then are found as the solutions of the corresponding variational equations obtained by implicit differentiation of the associated initial value problems. Thus, for i = 1,2,...,22 we must solve

$$\frac{d\underset{\sim}{W}_i}{d\bar{x}} = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}} (\bar{x},\Omega;\underset{\sim}{u}) \cdot \underset{\sim}{W}_i \qquad \text{for } 0 \leq \bar{x} \leq \bar{x}_o \qquad \underline{\text{Forward Integration}}$$

$$\underset{\sim}{W}_i(0) = \underset{\sim}{I}_i \tag{6-4-9}$$

and for i = 23,24,...,44

$$\frac{d\underset{\sim}{W}_i}{d\bar{x}} = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{v}} (\bar{x},\Omega;\underset{\sim}{v}) \cdot \underset{\sim}{W}_i \qquad \text{for } \bar{x}_o \leq \bar{x} \leq \frac{L}{R} \qquad \underline{\text{Backward Integration}}$$

$$\underset{\sim}{W}_i\left(\frac{L}{R}\right) = \underset{\sim}{I}_i \tag{6-4-10}$$

were $\underset{\sim}{I}_i$ = {0, ..., 0, 1, 0, ..., 0} is the ith unit vector in the n-space. The components of the Jacobian matrix J' can be calculated analytically

$$J' = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{u}} (\bar{x},\Omega;\underset{\sim}{u}) = \frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{v}} (\bar{x},\Omega;\underset{\sim}{v}) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \cdots \cdots & \frac{\partial f_1}{\partial u_{44}} \\ \frac{\partial f_2}{\partial u_1} \cdots \cdots & & \\ \cdots \cdots \cdots \cdots & & \\ \frac{\partial f_{44}}{\partial u_1} \cdots \cdots & & \frac{\partial f_{44}}{\partial u_{44}} \end{bmatrix} \tag{6-4-11}$$

Since the Jacobian matrix J' is a function of $\underset{\sim}{u}$ (or $\underset{\sim}{v}$), the variational equations (6-4-9) depend step-by-step on the results of the associated initial value problem (6-4-1), and the variational equations (6-4-10) depend step-by-step on the results of the associated initial value problem (6-4-2). Thus the variational equations depend on the initial guess $\underset{\sim}{S}$. Also, it is advantageous to integrate the 22 variational equations simultaneously with the corresponding associated initial value problem. This results, for double shooting, in a 1012 dimensional, 1st-order, nonlinear differential equation.

## 6.4.2 PROCEDURE INCLUDING END-SHORTENING

For the problem described by equation (6-3-13), if the end-shortening is considered, the 2 initial value problems associated with it become

$$\frac{d}{d\bar{x}} \hat{\underset{\sim}{u}} = \hat{\underset{\sim}{f}} \ (\bar{x},\Omega;\hat{\underset{\sim}{u}}) \qquad \text{for } 0 \leq \bar{x} \leq \bar{x}_0 \qquad \underline{\text{Forward Integration}}$$

$$\hat{\underset{\sim}{u}}(0) = \hat{\underset{\sim}{s}} \ \{0,0,\ldots,0;\ s_1,s_2,\ldots s_{11};\ 0,0\ldots,0;\ s_{12},s_{13},\ldots,s_{22};\ 0\}$$

(6-4-12)

and

$$\frac{d\hat{\underset{\sim}{v}}}{d\bar{x}} = \hat{\underset{\sim}{f}} \ (\bar{x},\Omega;\hat{\underset{\sim}{v}}) \qquad \text{for } \bar{x}_0 \leq \bar{x} \leq \frac{L}{R} \qquad \underline{\text{Backward Integration}}$$

$$\hat{\underset{\sim}{v}}\left(\frac{L}{R}\right) = \hat{\underset{\sim}{t}} = \{0,0\ldots,0;\ t_1,t_2,\ldots,t_{11};\ 0,0,\ldots,\ 0;\ t_9,t_{10},\ldots,t_{22};\ 0\}$$

(6-4-13)

where

$$\hat{\underset{\sim}{u}} = \begin{bmatrix} \underset{\sim}{u} \\ \delta_u \end{bmatrix} \quad ; \quad \hat{\underset{\sim}{v}} = \begin{bmatrix} \underset{\sim}{v} \\ \delta_v \end{bmatrix} \quad ; \quad \hat{\underset{\sim}{f}} = \begin{bmatrix} \underset{\sim}{f} \\ f_e \end{bmatrix}$$

Under appropriate smoothness conditions on the nonlinear vector function $\hat{\underset{\sim}{f}} \ (\bar{x};\hat{\underset{\sim}{Y}},\Omega)$ we are assured of the existence of unique solutions of these initial value problems, here denoted by

$$\hat{\underset{\sim}{U}} \ (\bar{x},\Omega;\hat{\underset{\sim}{s}}) \text{ and } \hat{\underset{\sim}{V}} \ (\bar{x},\Omega;\hat{\underset{\sim}{t}}) \qquad (6\text{-}4\text{-}14)$$

In this case these solutions satisfy the following matching conditions at $\bar{x} = \bar{x}_o$:

$$U_i \ (\bar{x} = \bar{x}_0,\Omega;\hat{\underset{\sim}{s}}) = V_i(\bar{x} = \bar{x}_0,\Omega;\hat{\underset{\sim}{t}}) \qquad \text{for } i = 1,2,\ldots,44$$

(6-4-15)

$$U_{45} \ (\bar{x} = \bar{x}_0,\Omega;\hat{\underset{\sim}{s}}) - V_{45} \ (\bar{x} = \bar{x}_0,\Omega;\hat{\underset{\sim}{t}}) = \delta_0$$

Introducing a new vector function $\hat{\underset{\sim}{\phi}}$ these matching conditions can be written as:

$$\hat{\underset{\sim}{\phi}}\ (\hat{\underset{\sim}{S}}) = \hat{\underset{\sim}{U}}\ (\bar{x}_0,\Omega;\hat{\underset{\sim}{s}}) - \hat{\underset{\sim}{V}}\ (\bar{x} = \bar{x}_0,\Omega;\hat{\underset{\sim}{t}}) - \underset{\sim}{\Upsilon} = 0 \qquad (6\text{-}4\text{-}16)$$

where

$$\hat{\underset{\sim}{S}} = \begin{bmatrix} \hat{\underset{\sim}{s}} \\ \hat{\underset{\sim}{t}} \\ \Omega \end{bmatrix} \quad \text{and} \quad \underset{\sim}{\Upsilon} = \begin{bmatrix} \underset{\sim}{Q} \\ \delta_0 \end{bmatrix}$$

Using Newton's method for finding the roots of $\hat{\underset{\sim}{\phi}}\ (\hat{\underset{\sim}{S}}) = 0$ we have the following iteration scheme

$$\hat{\underset{\sim}{S}}^{\ \mu+1} = \hat{\underset{\sim}{S}}^{\mu} + \Delta\hat{\underset{\sim}{S}}^{\mu}$$

where $\Delta\hat{\underset{\sim}{S}}^{\mu}$ is the solution of the 45th-order linear algebraic system.

$$\frac{\partial\hat{\underset{\sim}{\phi}}}{\partial\hat{\underset{\sim}{S}}}\ (\hat{\underset{\sim}{S}}^{\mu}) \cdot \Delta\hat{\underset{\sim}{S}}^{\mu} = -\ \hat{\underset{\sim}{\phi}}(\hat{\underset{\sim}{S}}^{\mu}) \qquad (6\text{-}4\text{-}17)$$

The components of the Jacobian matrix $\hat{J} = \dfrac{\partial\hat{\underset{\sim}{\phi}}\ (\hat{\underset{\sim}{S}}^{\mu})}{\partial\hat{\underset{\sim}{S}}}$ are once again calculated from the appropriate variational equations. However, here the dimension of the system of variational equations derived in equation (6-4-9) must be increased by one. Thus for $i = 1,2,\ldots,22$ one must include as the 45th equation

$$\frac{d}{d\bar{x}}\ \frac{\partial\tilde{\varepsilon}_{2u}}{\partial S_i} = \frac{\partial f_e}{\partial\underset{\sim}{u}}\ \frac{\partial\underset{\sim}{u}}{\partial S_i} = \frac{\partial f_e}{\partial\underset{\sim}{u}}\ \underset{\sim}{W}_i \qquad \text{for } 0 \leq \bar{x} \leq \bar{x}_0 \ \ \underline{\text{Forward Integration}}$$

$$\frac{\partial}{\partial S_i}\ \tilde{\varepsilon}_{2u}(0) = 0 \qquad (6\text{-}4\text{-}18)$$

whereas for $i = 23,24,\ldots,44$ the 45th equation is given by

$$\frac{d}{d\bar{x}}\ \frac{\partial\tilde{\varepsilon}_{2v}}{\partial S_i} = \frac{\partial f_e}{\partial\underset{\sim}{v}}\ \frac{\partial\underset{\sim}{v}}{\partial S_i} = \frac{\partial f_e}{\partial\underset{\sim}{v}}\ \underset{\sim}{W}_i \qquad \text{for } \bar{x}_0 \leq \bar{x} \leq \frac{L}{R} \ \ \underline{\text{Backward Integration}}$$

$$\frac{\partial}{\partial S_i}\ \tilde{\varepsilon}_{2v}\ (\tfrac{L}{R}) = 0 \qquad (6\text{-}4\text{-}19)$$

In addition we must also solve the following inhomogeneous variational equations

$$\frac{d}{d\bar{x}}\left(\frac{\partial \hat{\underset{\sim}{u}}}{\partial \Omega}\right) = \frac{\partial \hat{\underset{\sim}{f}}}{\partial \hat{\underset{\sim}{u}}}\left(\frac{\partial \hat{\underset{\sim}{u}}}{\partial \Omega}\right) + \frac{\partial \hat{\underset{\sim}{f}}}{\partial \Omega} \quad \text{for } 0 \leq \bar{x} \leq \bar{x}_0 \quad \underline{\text{Forward Integration}}$$

$$\frac{\partial}{\partial \Omega}\hat{\underset{\sim}{u}}(0) = \underset{\sim}{I}_\nu \tag{6-4-20}$$

and

$$\frac{d}{d\bar{x}}\left(\frac{\partial \hat{\underset{\sim}{v}}}{\partial \Omega}\right) = \frac{\partial \hat{\underset{\sim}{f}}}{\partial \hat{\underset{\sim}{v}}}\left(\frac{\partial \hat{\underset{\sim}{v}}}{\partial \Omega}\right) + \frac{\partial \hat{\underset{\sim}{f}}}{\partial \Omega} \quad \text{for } \bar{x}_0 \leq \bar{x} \leq \frac{L}{R} \quad \underline{\text{Backward Integration}}$$

$$\frac{\partial}{\partial \Omega}\hat{\underset{\sim}{v}}\left(\frac{L}{R}\right) = \underset{\sim}{I}_\nu \tag{6-4-21}$$

Here $\underset{\sim}{I}_\nu = \{0,0\ldots,0\}$.

In these equations the Jacobian matrix $\hat{J}' = \frac{\partial \hat{\underset{\sim}{f}}}{\partial \hat{\underset{\sim}{u}}} = \frac{\partial \hat{\underset{\sim}{f}}}{\partial \hat{\underset{\sim}{v}}}$ and the vectors $\frac{\partial f_e}{\partial \underset{\sim}{u}} = \frac{\partial f_e}{\partial \Omega}$ and $\frac{\partial \underset{\sim}{f}}{\partial \Omega}$ can be calculated analytically.

The solutions of these additional variational equations are then used to add one row and column to the 44-dimensional Jacobian $\frac{\partial \underset{\sim}{\phi}}{\partial \underset{\sim}{S}}$ derived in (6-4-11).

Schematically the resulting 45th dimensional linear algebraic system (6-4-17) can then be represented as:

$$\begin{bmatrix} \frac{\partial \underset{\sim}{\phi}}{\partial \underset{\sim}{S}} & \frac{\partial \underset{\sim}{\phi}}{\partial \Omega} \\ \frac{\partial \delta}{\partial \underset{\sim}{S}} & \frac{\partial \delta}{\partial \Omega} \end{bmatrix} \begin{bmatrix} \Delta \hat{\underset{\sim}{s}}^{\mu} \\ \Delta \hat{\underset{\sim}{t}}^{\mu} \\ \Delta \Omega^{\nu} \end{bmatrix} = - \hat{\underset{\sim}{\phi}}\,(\hat{\underset{\sim}{S}}^{\mu}) \tag{6-4-22}$$

Since the Jacobian $\hat{J}$ is a function of $\hat{\underset{\sim}{u}}$ (or $\hat{\underset{\sim}{v}}$) therefore, as pointed out above, the variational equations depend step-by-step on the results of the associated initial value problems. It implies that the variational equations depend on the initial guess $\hat{\underset{\sim}{S}}$. Thus once again we must iterate at a prescribed value of $\delta_0$, the value of the end-shortening, until all components of the ratio $\left|\frac{\Delta\hat{\underset{\sim}{S}}^{\mu}}{\hat{\underset{\sim}{S}}^{\mu}}\right|$ are smaller than some preselected positive quantity.

It has been shown in [76] that the shooting method is slower than the standard finite difference or finite element schemes. However, if the length of the intervals in integration is properly chosen so that numerical instabilities are avoided, this method gives more accurate results.

### 6.4.3 PROBLEM OF STARTING VALUES

One of the greatest difficulties in 'shooting' consists of obtaining a starting estimate of the initial data which is sufficiently close to the exact initial data so that the iteration scheme used to find the solution of the nonlinear problem will converge. In the case of single mode response, fortunately, the nonlinear solution approaches the linearized solution asymptotically for values of the frequency $\Omega$ sufficiently far from resonant point ($\Omega \to 0$ or $\Omega \to \infty$). Thus, for sufficiently low (or high) values of the frequencies one can use the linearized solution as the starting values for the nonlinear iteration scheme. The governing equations for the linearized problem can be reduced by a procedure similar to the one described for the nonlinear problem above, and they can also be solved by 'parallel shooting'. It is well known that for the linear problems the 'parallel shooting' will yield the correct solution directly, no iteration is required [76].

For the case of coupled mode response, as shown in Fig. 2.29c the problem of obtaining a starting estimate for the initial data becomes complicated since the left branch of companion mode response seems to approach infinity while the frequency $\Omega$ decreases towards zero. A careful look at the picture reveals, however, that the curve tends toward the backbone curve of the relevant single mode response if the amplitude of imperfection is small enough. This fact makes the problem solvable since one can use the values of the backbone curve as a starting estimate for companion mode. This works as long as one is far enough removed from the resonant frequency. As to the right branch of the companion mode, Fig. 2.29c shows that it approaches the values of both the driven mode and of the single mode response. So one can use values of the single mode response as a starting estimate for it. The problem of obtaining a starting value for the initial data for the driven mode presents no difficulty, since the nonlinear solution approaches the linear solution asymptotically as long as the frequency is far from the resonant point ($\Omega \to 0$, $\Omega \to \infty$), which is quite similar to the case of single mode response mentioned above.

# CHAPTER 7 CHECKING THE CORRECTNESS OF THE PROPOSED SOLUTION PROCEDURE

## 7.1 INTRODUCTION

The combined analytical/numerical procedure, which has been successfully employed for the solution of buckling problems by Arbocz is now used to obtain the dynamic response of thin cylindrical shells with different boundary conditions. As usual, the solution procedure developed in the thesis has to be checked out by using some examples the solutions of which have been proven to be correct. In the present thesis two examples are used. First, one considers the problem of nonlinear vibrations of beams with different boundary conditions, with which the correctness and applicability of the solution procedure can be demonstrated. Second, one looks at the problem of linearized vibrations of thin cylindrical shells with different boundary conditions. These known solutions are employed to check the program developed by the author for nonlinear problems and to supply the initial estimates for the starting data needed in the case of nonlinear calculations.

## 7.2 NONLINEAR VIBRATIONS OF A BEAM WITH DIFFERENT BOUNDARY CONDITIONS

When the longitudinal inertia term is neglected the equation of motion of a uniform beam with immovable end supports is [32],

$$EI \frac{\partial^4 W}{\partial x^4} - N \frac{\partial^2 W}{\partial x^2} + \rho \frac{\partial^2 W}{\partial t^2} = P(x) \cos\omega t \tag{7-2-1}$$

where N is the axial tension, given by

$$N(t) = \frac{EA}{2L_b} \int_0^{L_b} \left(\frac{\partial W}{\partial x}\right)^2 dx \tag{7-2-2}$$

W is the lateral deflection, A is the cross-section area, $L_b$ is the length and P(x) is the external forcing function which is uniformly distributed along the x-axis. The other symbols have their usual meaning.

We are seeking a solution to Eqs. (7-2-1) and (7-2-2) which satisfies the preselected boundary conditions, for instance, hinged-hinged one,

$$W = \frac{\partial^2 W}{\partial x^2} = 0 \qquad (x = 0)$$

$$W = \frac{\partial^2 W}{\partial x^2} = 0 \qquad (x = L_b) \tag{7-2-3}$$

Defining the 'end-shortening' as

$$U_b(x) = \int_0^{L_b} \frac{\partial U_b}{\partial x}\, dx \tag{7-2-4}$$

then using the relationship

$$\frac{\partial U_b}{\partial x} = \frac{N}{EA} \tag{7-2-5}$$

and applying in sequence the procedure described in Section 6.4 to these equations yields the solution of the problem.

The accuracy of the procedure is verified by comparing the natural frequency obtained using the current theory with those of exact solutions. For example, the exact fundamental frequency of a beam with hinged-hinged boundary condition is $\Omega_b = \pi^2 = 9.8696$ [115], while the present approximate solution is $\Omega_b = 9.8701$ for the case of small vibration ($\tilde{W} = \frac{W}{r} = 0.00315$, $\Delta\bar{x} = 0.05$, where r is the radius of gyration of cross-section, $\Delta\bar{x}$ is the integration step). Here the relative error is only 0.005% and it could be decreased further if one specifies smaller integration steps $\Delta\bar{x}$.

Table 7.1 Natural frequencies of beam with different boundary conditions

| | Hinged-Hinged | Hinged-Clamped | Clamped-Clamped |
|---|---|---|---|
| Amplitude $(\frac{\tilde{W}}{r})$ | 0.00315 | 0.00126 | 0.00195 |
| Frequency ($\Omega_b$) | 9.8701 | 15.6006 | 22.3712 |

Table 7.1 lists the fundamental frequencies for the beam with the different boundary conditions, where the excitation is fixed of a constant value of $\bar{F}_B$=1.1 and where $\bar{F}_B$= $(P/EI)(L^4/r)$. The amplitudes are those of the middle point of the beam. As can be seen the fundamental frequency of the hinged-hinged beam is lower than that of the clamped-clamped beam, which indicates that the stronger boundary conditions make a beam 'stiffer'.

The curves shown in Fig. 7.1 are the backbone curves of the beam with different boundary conditions. As can be seen the clamped-clamped beam exhibits the least nonlinearity. These results are quite similar to those reported by Evensen [32], who applied the perturbation method to the same problem. Notice that in Figures 7.1 and 7.2 the frequencies are normalized using their respective nondimensional fundamental frequencies from Table 7.1 as the normalizing factor.

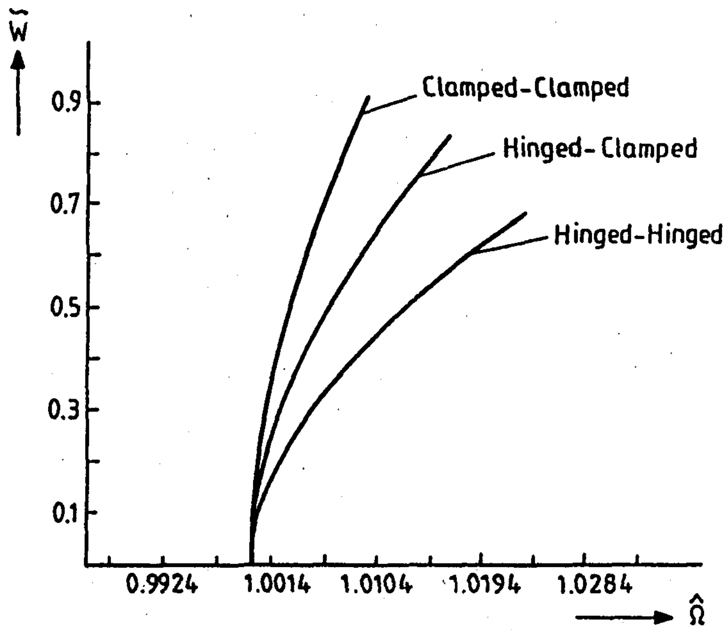



Fig. 7.1 The amplitude-frequency relationships of free vibration for various boundary conditions.

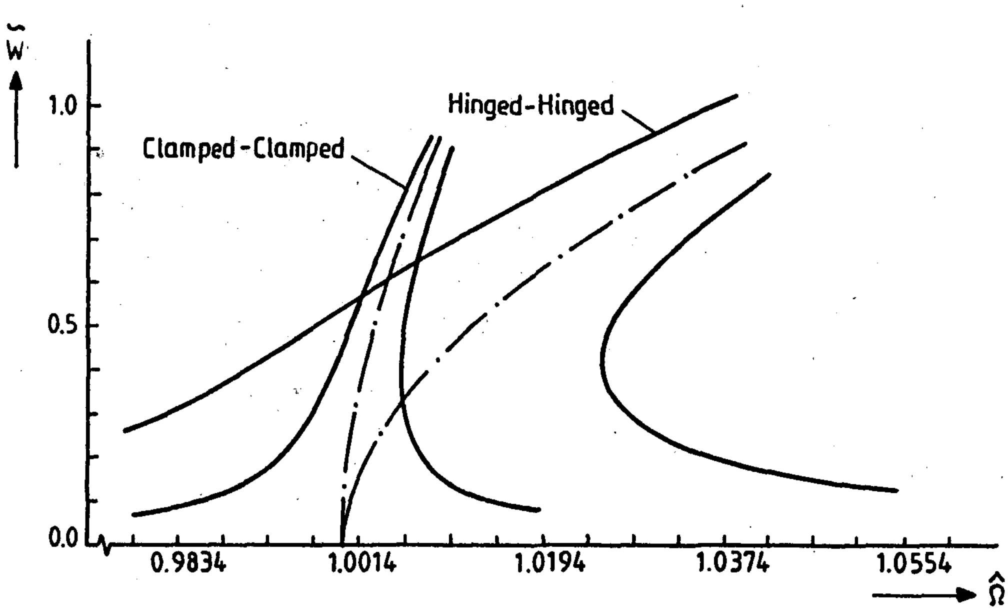



Fig. 7.2 The amplitude-frequency relationships of forced vibration for various boundary conditions.

Excitation $\bar{F}_B = 1.1$

The response curves of the clamped-clamped and the hinged-hinged beams are shown in Fig. 7.2. As can be seen from this figure the response of the hinged-hinged beam to a given excitation is much stronger than that of the clamped-clamped beam. The 'resonant region' of the hinged-hinged beam is wider than that of the clamped-clamped beam.

The modal shapes in the $\bar{x}$ direction for different cases considered are shown in Fig. 7.3. As expected, the maximum amplitude occurs at $\bar{x} = 0.5$ for the symmetric boundary conditions, whereas for the unsymmetric boundary condition the position of the maximum amplitude is shifted away from the middle point of the beam.

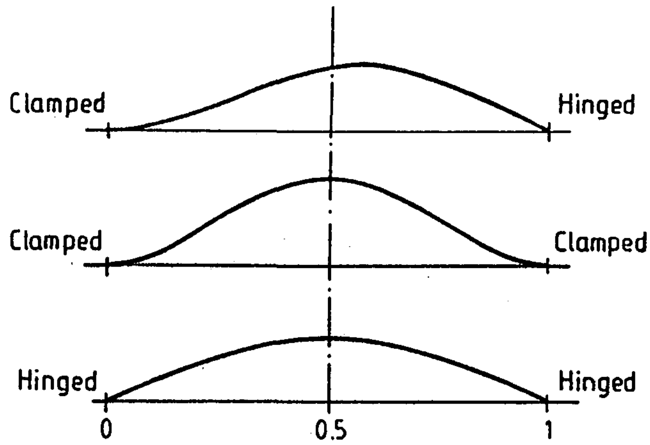



Fig. 7.3 Modal shapes of beams with different boundary conditions

As mentioned before the objective of the present analysis in to prove the `applicability' of the solution procedure developed in the thesis. Therefore the comparison is limited to the case of the lowest order vibration. There is no difficulty in solving problems of higher order vibration of beam using the present procedure.
The results presented for the problem of nonlinear vibrations of beams with different boundary conditions show that the solution procedure developed in the thesis is reliable. Next it will be applied directly to the problem of nonlinear vibrations of imperfect thin cylindrical shells with different boundary conditions.

## 7.3 LINEARIZED VIBRATIONS OF THIN CYLINDRICAL SHELLS

The nonlinear vibrations of imperfect thin walled cylindrical shells with different boundary conditions as treated in this thesis involve the solution of a response problem. However, if one lets the external excitation $\bar{F}_D$ approach zero the forced vibration curves approach ever more the backbone curve representing the case of free vibration. Thus, as can be seen fron Fig. 7.4, it is possible to simulate the linearized vibration problems of perfect cylindrical shells under different boundary conditions.
As has been pointed out earlier (see p. 84) the nondimensional frequency parameter $\Omega$ is obtained by dividing the forcing frequency $\omega$ by the frequency of free vibration (linear theory) of the perfect unloaded shell $\sqrt{E/2\bar{\rho}R^2}$. Assuming that Evensen's solution of Ref. [30] represents the exact solution for SS3 boundary conditions, that is

$$\Omega_E^2 = \frac{\omega^2}{\omega_{mn}^2} = 1.0$$

where

$$\omega_{mn}^2 = \frac{E}{\bar{\rho}R^2}\left\{\frac{\xi^4}{(\xi^2+1)^2} + \epsilon\,\frac{(\xi^2+1)^2}{12(1-\nu^2)}\right\}$$

then

$$\frac{\Omega^2}{\Omega_E^2} = 2\left\{\frac{\xi^4}{(\xi^2+1)^2} + \epsilon\,\frac{(\xi^2+1)^2}{12(1-\nu^2)}\right\}$$

Hence for $\xi=0.1$, $\epsilon=0.01$ and $\Omega_E^2=1.0$ one obtains

$$\Omega^2 = 2.06437 \times 10^{-2} \quad (\rightarrow \Omega = 4.54354 \times 10^{-2})$$

In Fig. 7.4 the response curves of the mode with one half-wave along the $\bar{x}$-axis and $\ell=5$ full waves in the circumferential direction are shown for different values of the excitation $\bar{F}_D$.

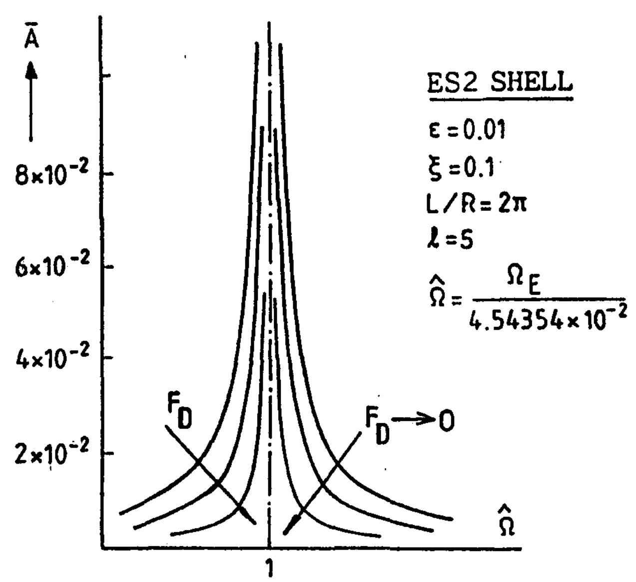



Fig. 7.4 Simulation of the natural frequencies

Notice that the frequency $\hat{\Omega}$ along the horizontal axis has been normalized by the natural frequency of the SS3 boundary condition ($\Omega = 4.54354 \times 10^{-2}$).

Table 7.2 lists the upper- and lower bounds of the lowest natural frequencies ($m=1$, $\ell=5$) for the usual 8 boundary conditions listed in detail on p. 122.

Table 7.2 Natural frequencies of Shell ES2

| B.C. | Resonant Frequency | |
|---|---|---|
| | Approximate solution ($\times 10^{-2}$) | Exact solution ($\times 10^{-2}$) |
| SS1-SS1 | 4.54140 ~ 4.54180 | - |
| SS2-SS2 | 5.26075 ~ 5.26120 | - |
| SS3-SS3 | 4.54353 ~ 4.54360 | 4.54354 |
| SS4-SS4 | 5.28375 ~ 5.28381 | - |
| C1-C1 | 4.55975 ~ 4.55987 | - |
| C2-C2 | 5.28300 ~ 5.28400 | - |
| C3-C3 | 4.56320 ~ 4.56345 | - |
| C4-C4 | 5.28770 ~ 5.28777 | - |
| SS3-C1 | 4.54867 ~ 4.55733 | - |

It is not surprising that the C4-C4 (the strongest boundary condition) has the highest natural frequency. As a matter of fact all boundary conditions with a strong axial constraint (with $\hat{\hat{u}}=0$) have just about equally high natural frequencies.

# CHAPTER 8 NUMERICAL SOLUTIONS

## 8.1 INTRODUCTION

Two computer programs were developed based on the present approach for the two different cases of $n = \ell$ and $n \neq \ell$. They provide a powerful tool for examining the influence of both various boundary conditions and initial geometric imperfections on the nonlinear vibration behaviour of thin-walled isoptropic or orthotropic shells. In this chapter two problems are considered. One of them is the influence of different boundary conditions on the nonlinear vibrations of a particular shell. Here the shell ES2 with a slight change in the charateristic data is used. The second one is to study the effect of initial geometric imperfection on the nonlinear vibration of the shell WN with different boundary conditions.

In studying the nonlinear vibration of a system, generally three factors, the natural frequencies, the backbone curve and the modal shape are considered since these factors outline its principal dynamic behaviour. Following this approach, in this chapter we shall consider the influence of the different boundary conditions and the inital geometric imperfections on the natural frequencies, the backbone curves and the modal shapes.

There is no unique approach for comparing and discussing the solutions for the different boundary conditions. Normally, one way is to compare the results based on the fundamental frequencies which probably have different circumferential wave numbers for different boundary conditions [106]. On the other hand, one can also compare the solutions for different boundary conditions based on a constant circumferential wave number, keeping in mind that the frequencies under such cases may be not the fundamental ones. In the present study the later way is followed.

## 8.2 NUMERICAL SOLUTIONS AND DISCUSSIONS

The characteristic data of the ES2 shell are slightly changed to make the calculations simpler. The new data are as follows,

$$\varepsilon = \left(\frac{\ell^2 h}{R}\right)^2 = 0.00999998 \simeq 0.010$$

$$\xi = \frac{k\pi/L}{\ell/R} = 0.10471976 \simeq 0.105$$

$$\nu = 0.3$$

The associated geometric and vibration mode data are

$$R = L = 1$$

$$h = 1.111 \times 10^{-4}$$

$$\ell = 30$$

The way the nonlinear modal shapes were calculated for the different boundary conditions is illustrated in Fig. 8.1. As starting values one uses the results of the linearized response calculations for the desired boundary condition (say SS3).

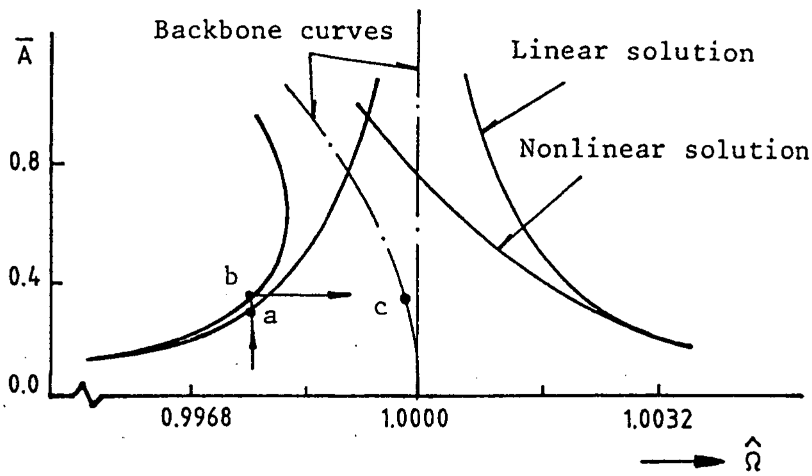



Fig. 8.1 Steps involved in nonlinear response calculations

Thus when attempting to find the linearized natural frequencies of the modified shell ES2, by inputing the values of $\Omega_a$ and $\bar{F}_D$ one obtains the linearized modal shapes at point 'a'. Using the same values of $\Omega_a$ and $\bar{F}_D$ and the linearized modal shapes as an initial guess one can solve the nonlinear problem iteratively and obtain the nonlinear modal shapes and the value of the control variable $\bar{\varepsilon}_2$, also called 'end-shortening' at point 'b'. Next one sets $\bar{F}_D$=0 and uses $\Omega_a$ and the nonlinear shapes at point 'b' as initial guesses to calculate the value of $\Omega$ and the nonlinear modal shapes at point 'c' while keeping the value of the control variable $\bar{\varepsilon}_2$ fixed at the value found at point 'b'. Finally by increasing or decreasing the value of the control variable $\bar{\varepsilon}_2$ one can trace out the whole backbone curve. For very small values of $\bar{A}_{max}$ (the maximum amplitude of the modal shape $\bar{A}$) the value of $\Omega$ corresponding to the backbone curve is the (linearized) natural frequency for the specified boundary condition.

Table 8.1 lists the lowest natural frequencies of the modified ES2 shell for 8 different boundary conditions, which were all calculated by the nonlinear procedure described above. The exact linearized natural frequency given by Evensen's formula [30] for the SS3 boundary condition is $4.59044 \times 10^{-2}$. As can be seen from Table 8.1 this value is matched quite accurately. For a detailed list of the different boundary conditions refer to Tabel 6.1 on p.122.

Table 8.1 The natural frequencies of modified shell ES2 with different boundary conditions

| CASE | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| B.C. | SS1 | SS2 | SS3 | SS4 | C1 | C2 | C3 | C4 |
| $\Omega x10^{-2}$ | 4.58753 | 5.41926 | 4.59041 | 5.44970 | 4.60936 | 5.44784 | 4.61304 | 5.45454 |

Considering the relative changes in the lowest natural frequencies introduced by varying the different constraints one finds that varying the axial constraint ($\hat{u}=0$ vs. $\hat{N}_{xy}=0$) results in a change of about 18%, varying the circumferential constraint ($\hat{v}=0$ vs. $\hat{N}_{xy}=0$) produces a change of only about 0.1%, whereas varying the rotational constraint ($\hat{w},_{\bar{x}}=0$ vs. $\hat{M}_x=0$) gives rise to a change of about 0.5%. Thus it is clear that the axial constraint $\hat{u}=0$ has a very strong influence on the value of the lowest natural frequency. In comparison, the effect of the rotational constraint $\hat{w},_{\bar{x}}=0$ is small and the effect of the circumferential constraint $\hat{v}=0$ is nearly negligible.

The backbone curves for the different boundary conditions are plotted in Fig. 8.2. Notice that the frequencies $\hat{\Omega}$ for all curves have been normalized by their respective natural frequencies listed in Tabel 8.1.

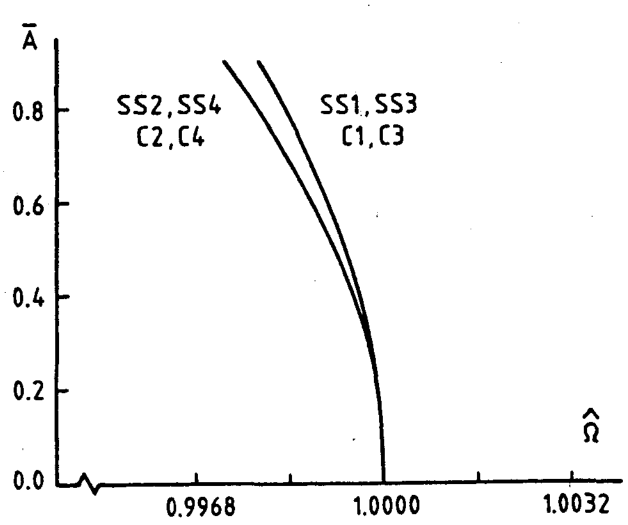



Fig. 8.2 Backbone curves of the modified ES2 shell for different boundary conditions

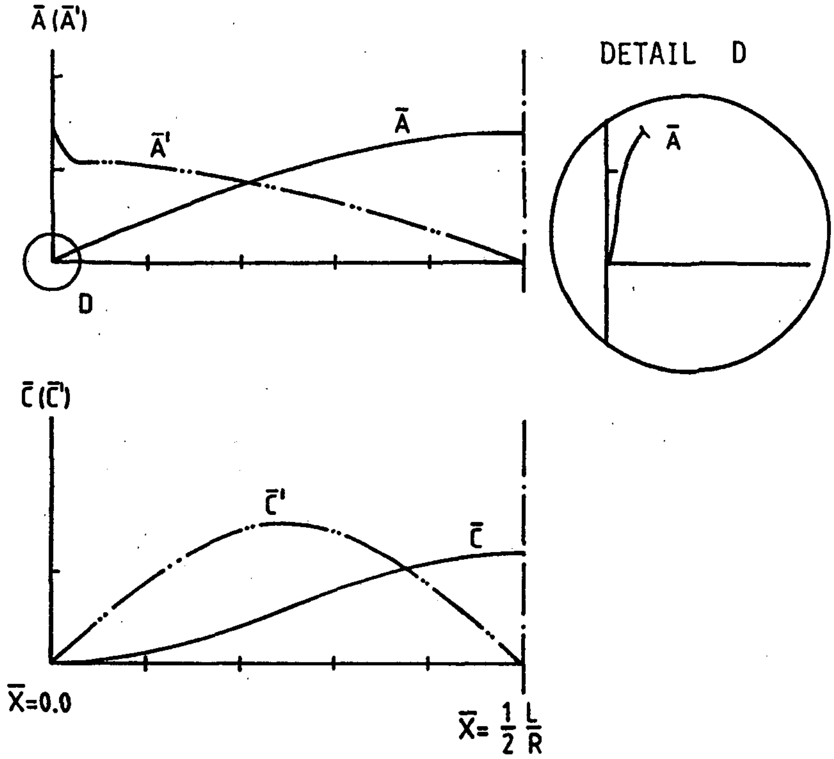
Clearly all curves show a softening type behaviour, which agrees with the previously obtained solutions where the SS3 boundary condition was satisfied approximately. This means that the different boundary conditions do not change the basic nonlinear vibration characteristics of the modified ES2 shell. A closer look at these curves reveals the fact that the backbone curves for the boundary conditions SS2, SS4, C2, C4 and for the boundary conditions SS1, SS3, C1, C3 are practically identical. This clearly means that the circumferential constraint $\hat{v}=0$ and the rotational constraint $\hat{w}_{,\bar{x}}=0$ have less influence on the nonlinearity of vibration of the modified ES2 shell then the axial constraint $\hat{u}=0$, but this influence becomes significant only for the cases of large amplitude vibrations ($\bar{A}_{max} > 0.3$, say).

A study of the modal shapes for single mode response of the perfect shall ES2 reveal the changes in the modal shapes produced by the different boundary conditions. Since in these comparisons one is especially interested in the differences of the modal shapes, therefore Figures 8.3 ~ 8.6 are drawn about the same sizes and there are no scales indicated along the vertical axes. To help in understanding the differences besides the modal shapes $\bar{A}$ and $\bar{C}$ also their first derivatives are plotted.

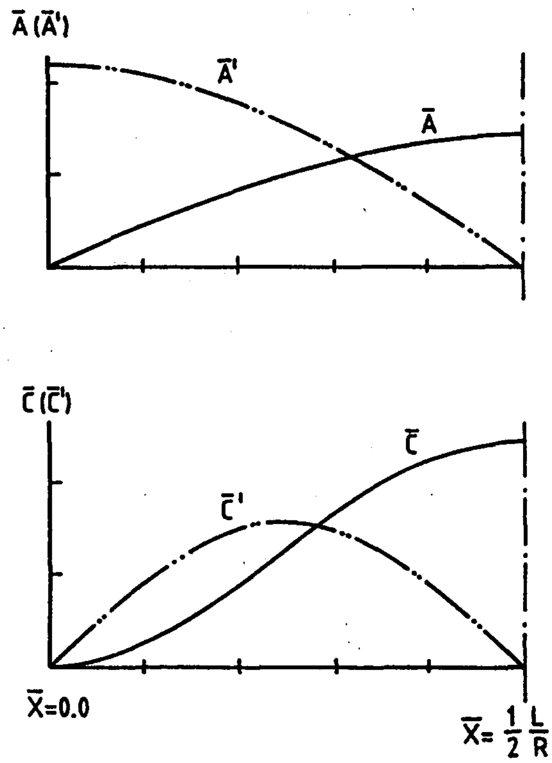
As can be seen from Fig. 8.3b, for the SS3 boundary condition the modal shape of the driven mode $\bar{A}$ is a perfect half wave sine, whereas the modal shape of the axisymmetric term $\bar{C}$, needed to satisfy the circumferential periodicity condition, is a perfect cosine term plus a constant value. These results agree with the assumptions made in Part I of this thesis (see Eq. 2-2-4, p. 39). Changing the circumferential constraint condition from $\hat{v}=0$ to $\hat{N}_{xy}=0$ (from SS3 to SS1 boundary condition) one gets the results shown in Fig. 8.3a. As expected, the modal shapes are practically identical except within a very narrow region next to the edge of the shell.

Looking now at Fig. 8.4 one sees immediately the strong influence of the axial constraint condition $\hat{u}=0$. The modal shape of the driven mode $\bar{A}$ no longer resembles a half wave sine but is more like a full wave cosine. This statement is confirmed by the shape of the first derivative of $\bar{A}$, which looks like a sine wave except in a narrow region close to the edge of the shell. Notice also that the modal shapes of the axisymmetric terms $\bar{C}$ are no longer pure trigonometric shapes.

Considering now the clamped boundary conditions, as can be seen from Fig. 8.5 the modal shape of the driven mode $\bar{A}$ resembles once again closely a half wave sine shape, except in a very narrow region close to the edge of the shell. If, however, one changes from $\hat{N}_x=0$ to the axial constraint condition $\hat{u}=0$, then as can be seen from Fig. 8.6 both modal shapes $\bar{A}$ and $\bar{C}$ deviate considerably from a simple trigonometric function, especially in a narrow region close to the edge of the shell.

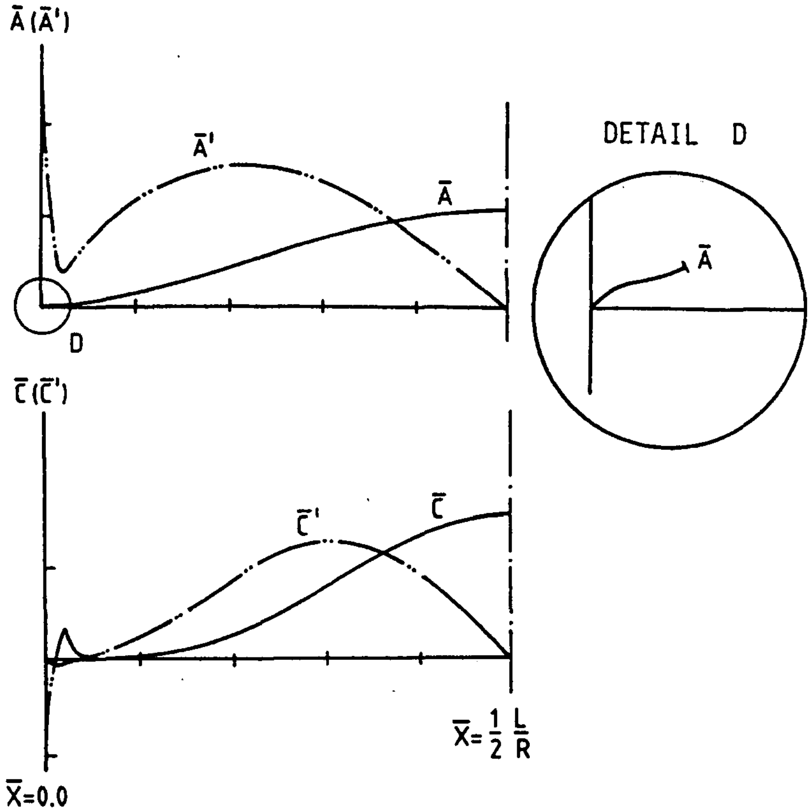



a. SS1 boundary condition ($\hat{\hat{N}}_x=0$, $\hat{\hat{N}}_{xy}=0$)

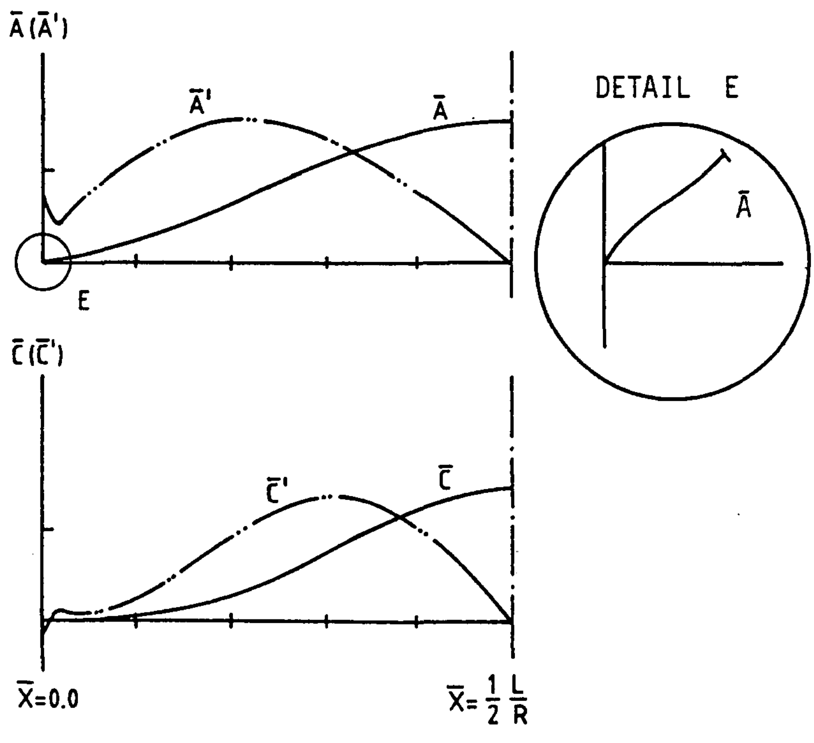



b. SS3 boundary condition ($\hat{\hat{N}}_x=0$, $\hat{\hat{v}}=0$)

Fig. 8.3 Modal shapes for single mode response using simply supported boundary conditions. Modified ES2 Shell; $\ell=30$.

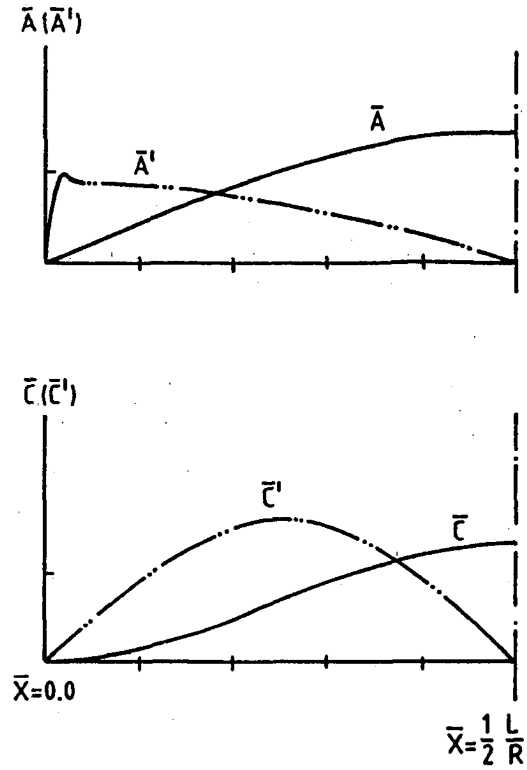



a. SS2 boundary condition ($\hat{\hat{u}}=0$, $\hat{\hat{N}}_{xy}=0$)

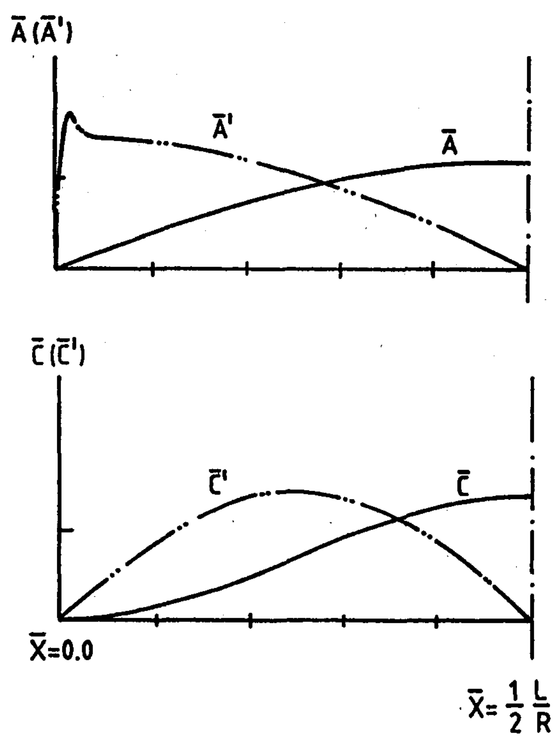



b. SS4 boundary condition ($\hat{\hat{u}}=0$, $\hat{\hat{v}}=0$)

Fig. 8.4 Modal shapes for single mode response using simply supported boundary conditions. Modified ES2 Shell; $\ell=30$.

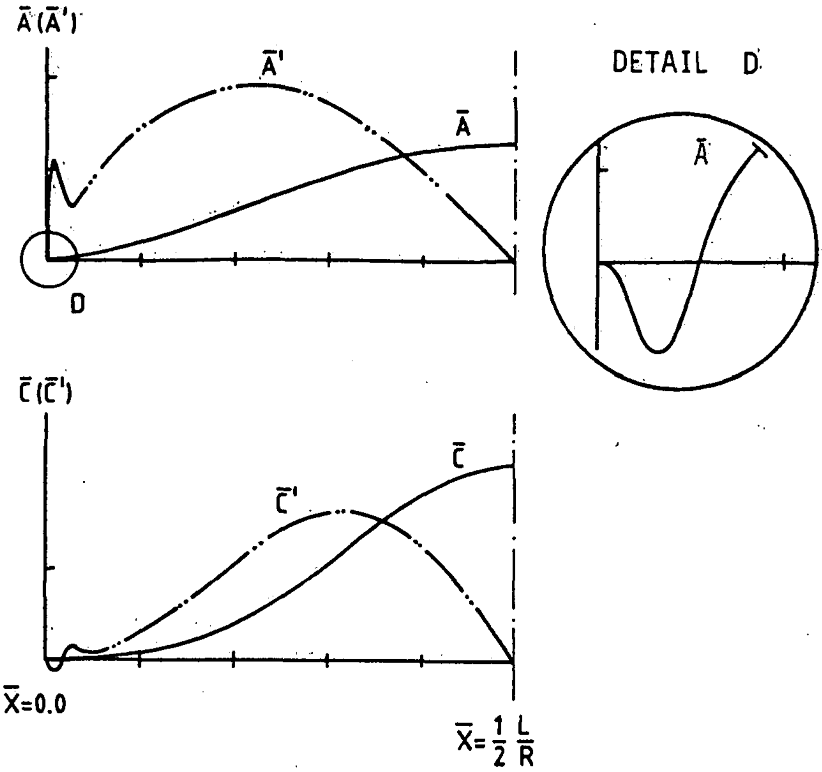



a. C1 boundary condition ($\hat{\hat{N}}_x=0$, $\hat{\hat{N}}_{xy}=0$)

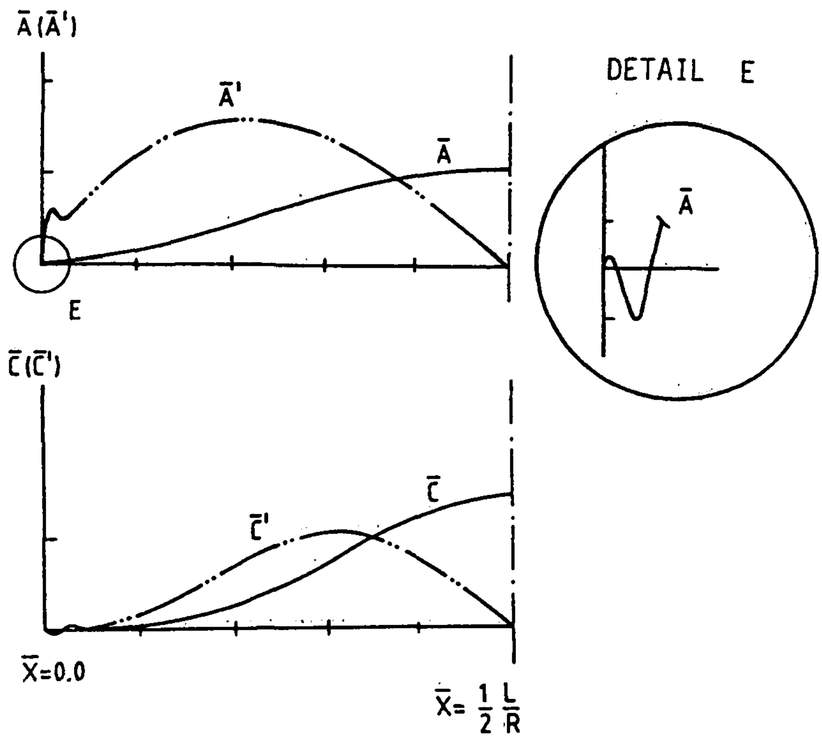



b. C3 boundary condition ($\hat{\hat{N}}_x=0$, $\hat{\hat{v}}=0$)

Fig. 8.5 Modal shapes for single mode response using clamped boundary conditions. Modified ES2 Shell; $\ell=30$.




a. C2 boundary condition ($\hat{\hat{u}}=0$, $\hat{\hat{N}}_{xy}=0$)




b. C4 boundary condition ($\hat{\hat{u}}=0$, $\hat{\hat{v}}=0$)

Fig. 8.6 Modal shapes for single mode response using clamped boundary conditions. Modified ES2 Shell; $\ell=30$.

The second shell investigated in this chapter is the WN shell. Its geometric data, as introduced before, are

$$\frac{R}{h} = 720$$

$$\frac{L}{R} = \frac{2}{3}$$

and

$$\nu = 0.272$$

Since the asymmetric imperfection used in this study is

$$\frac{\bar{W}}{h} = 0.4 \sin \frac{5\pi x}{L} \cos 25 \frac{y}{R}$$

therefore one expects the dominant single mode vibration to have similar characteristics. As a matter of fact, for SS3 boundary condition the corresponding perfect shell vibration mode is expected to closely approximate the following form

$$\frac{\hat{W}}{h} = A \sin \frac{5\pi x}{L} \cos 25 \frac{y}{R} + C \sin^2 \frac{5\pi x}{L}$$

With this vibration characteristics

$$\varepsilon = (\ell^2 \frac{h}{R})^2 = 0.09425$$

$$\xi = \frac{k\pi/L}{\ell/R} = 0.07535$$

and the exact linearized natural frequency for SS3 boundary condition is given by Evensen's formula [30] yielding

$$\Omega_{SS3} = 0.96236$$

Table 8.2 lists the natural frequencies of the perfect WN shell which correspond to the vibration mode k=m=5, $\ell$=25 for the 8 different boundary conditions listed. As can be seen the SS3 value matches closely the exact result. For a detailed list of the different boundary conditions refer to Table 6.1 on p. 122.

Table 8.2 The natural frequencies of shell WN with different boundary conditions.

| CASE | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| B.C. | SS1 | SS2 | SS3 | SS4 | C1 | C2 | C3 | C4 |
| $\Omega$ | 0.9288 | 0.9288 | 0.9624 | 0.9689 | 0.9918 | 0.9932 | 1.0017 | 1.0082 |

To understand the relative changes in these higher order natural frequencies introduced by varying the different constraints it is helpful to present the data of Table 8.2 regrouped so as to better visualize the results. Thus Table 8.3 shows the relative changes in higher order natural frequencies produced by varying the rotational constraints ($\hat{\hat{w}}_{,x}=0$ vs $\hat{\hat{M}}_x=0$), whereas Table 8.4 displays the relative changes caused by varying the axial constraint ($\hat{\hat{u}}=0$ vs $\hat{\hat{N}}_x=0$), and finally Table 8.5 lists relative changes produced by varying the circumferential constraint ($\hat{\hat{v}}=0$ vs $\hat{\hat{N}}_{xy}=0$).

Table 8.3 Relative change in the higher order natural frequencies (k=m=1, $\ell$=25)
($\hat{\hat{w}}_{,x}=0$ vs $\hat{\hat{M}}_x=0$)

$$\frac{C1-SS1}{SS1} = 6.78\% \qquad - \hat{\hat{N}}_{xy} = 0$$

$$\frac{C2-SS2}{SS2} = 6.43\% \qquad - \hat{\hat{N}}_{xy} = 0$$

$$\frac{C3-SS3}{SS3} = 4.08\% \qquad - \hat{\hat{v}} = 0$$

$$\frac{C4-SS4}{SS4} = 4.06\% \qquad - \hat{\hat{v}} = 0$$

Table 8.4 Relative changes in the higher order natural frequencies (k=m=1, $\ell$=25)
($\hat{\hat{u}}=0$ vs $\hat{\hat{N}}_x=0$)

$$\frac{SS2-SS1}{SS1} = 0 \qquad - \hat{\hat{N}}_{xy} = 0$$

$$\frac{C2-C1}{C1} = 0.14 \qquad - \hat{\hat{N}}_{xy} = 0$$

$$\frac{SS4-SS3}{SS3} = 0.68 \qquad - \hat{\hat{v}} = 0$$

$$\frac{C4-C3}{C3} = 0.65 \qquad - \hat{\hat{v}} = 0$$

Table 8.5 Relative changes in the higher order natural frequencies ($k=m=1$, $\ell=25$) ($\hat{v}=0$ vs $\hat{N}_{xy}=0$)

$$\frac{SS3-SS1}{SS1} = 3.62\% \qquad - \hat{N}_x = 0$$

$$\frac{SS4-SS2}{SS2} = 4.32\% \qquad - \hat{u} = 0$$

$$\frac{C3-C1}{C1} = 1.00\% \qquad - \hat{N}_x = 0$$

$$\frac{C4-C2}{C2} = 1.51\% \qquad - \hat{u} = 0$$

From these results it is obvious that sofar as the influence on the higher order natural frequencies is concerned, variations in the rotational constraint have the largest effect (about 4 ~ 7%). The effect appears to be the strongest if the shell is unrestrained in the circumferential direction. On the other hand the effect of the axial constraint on the higher order natural frequencies appears to be negligible (only about 0.1%) if the shell is unrestrained in the circumferential direction. Even if $\hat{v}=0$ the effect is less than 1%.

This result is somewhat surprising in view of the earlier results obtained for the lowest natural frequencies of the very thin modified ES2 shell ($\frac{R}{h}$ ~ 9000).

Finally, the circumferential constraint has a noticeable effect on the higher order natural frequencies, which is stronger for the simply supported boundary conditions (about 4%) than for the clamped ones (about 1.5%).

The backbone curves for the different boundary conditions are plotted in Fig. 8.7. Notice that the frequencies $\hat{\Omega}$ for all curves have been normalized by their respective natural frequencies listed in Table 8.2.

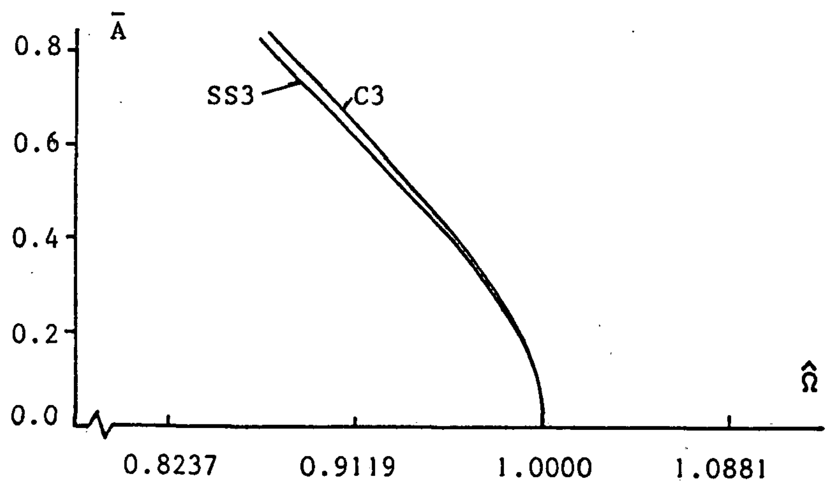



Fig. 8.7 Backbone curves of the WN shell for different boundary conditions

Comparing the results of the backbone curve for the higher order modes (Fig. 8.7) with those of the lower order modes (Fig. 8.2) it becomes evident that the higher order modes exhibit a much stronger softening type nonlinearity than the lower order modes. (Notice the difference between the scales used in Figures 8.7 and 8.2.)

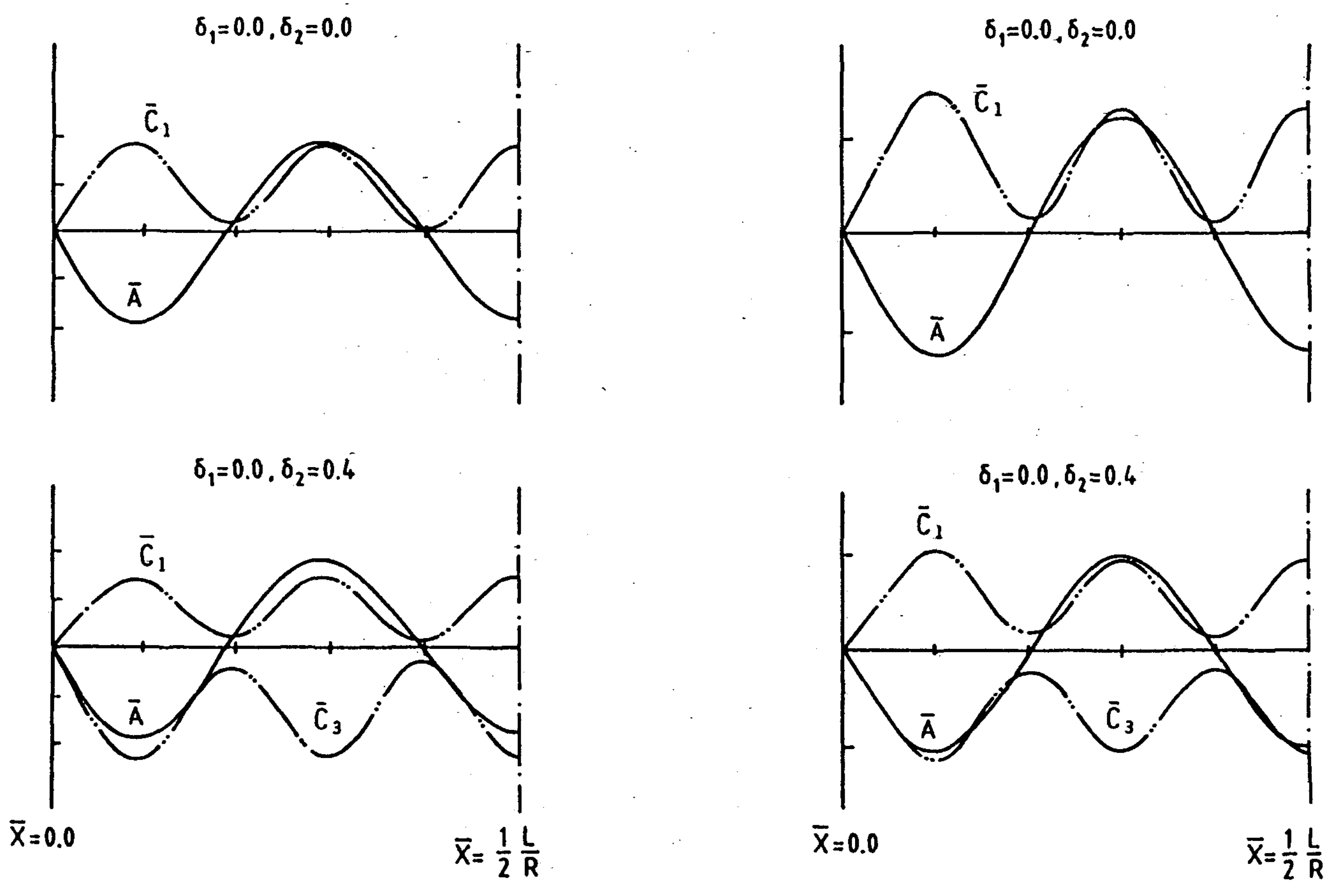
A study of the higher order modal shapes for single mode response of the perfect and imperfect WN shell reveal the changes in the higher order modal shapes produced by the different boundary conditions and by the inclusion of moderate size initial imperfections.

As can be seen from Figures 8.8 and 8.9, for the 4 different simply supported boundary conditions the changes in the higher order modal shapes are slight. Notice also that the inclusion of a moderate size asymmetric imperfection ($\delta_2$=0.4) results only in small changes. There is now, however, also the response term $\bar{C}_3$ present, which accounts for the circumferential periodicity correction necessary because of the inclusion of the asymmetric imperfection.

Considering the modal shapes for single mode response using clamped boundary conditions, as can be seen from Figures 8.10 and 8.11 the effect of the rotational restraint is restricted to a narrow region next to the shell edges. Otherwise the higher order clamped modal shapes ressemble closely the simply supported ones. Also the inclusion of a moderate size asymmetric imperfection produces once again only small changes. However, the presence of a initial asymmetric imperfection will result in noticeable distortion of the higher order modal shaped as can be seen in Fig. 8.12.

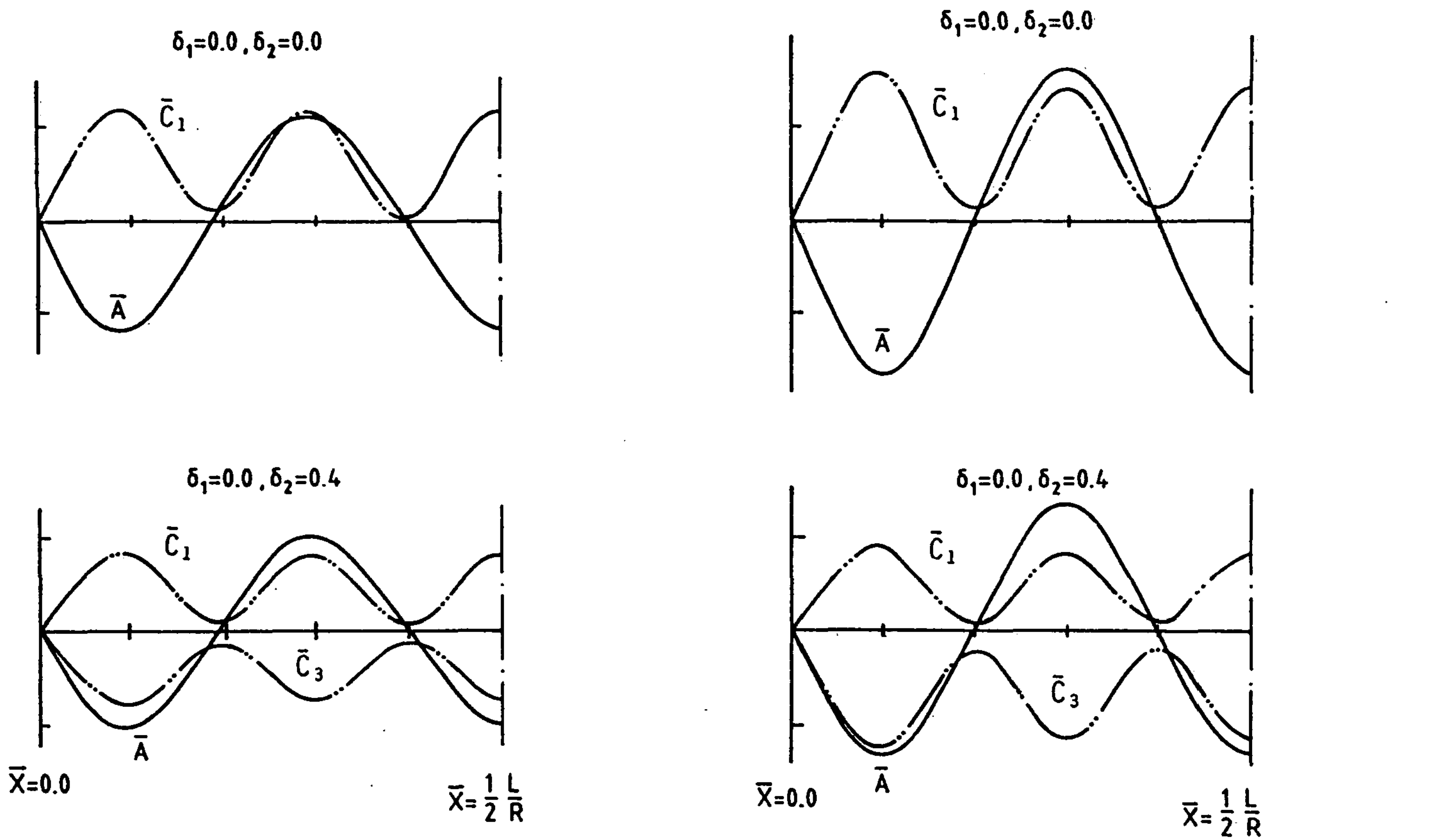
On the other hand, as can be seen from Fig. 8.13, the size of the maximum amplitude of the nonlinear vibration response $\bar{A}_{max}$ has only a negligible influence on the shape of the higher order modal shapes. Notice that the amplitudes of the different modal components have been normalized so that identical amplitudes at x=L/2 resulted.

Finally Fig. 8.14 displays the frequency-asymmetric imperfection relationships for various boundary conditions. Notice that in this figure the frequencies of the different curves are normalized using respective nondimensional natural frequencies listed in Table 8.2 as normalizing factors. Proceeding this way the 4 curves for simply supported boundary conditions and the 4 curves for clamped boundary conditions practically collapse into a single curve each for moderate size asymmetric imperfections (for $\delta_2 < 0.5$, say). However, the effects of the different boundary conditions become more and more pronounced for larger imperfections. Initially the general tendency is that the natural frequency decreases with increasing values of the asymmetric imperfection. It appears that for very large imperfections ($\delta_2 > 2.0$, say) this trend is reversed, the frequency begins to increase with increasing asymmetric imperfections.




a. SS1 Boundary condition ($\hat{\hat{N}}_x$=0)  b. SS3 boundary condition ($\hat{\hat{N}}_x$=0)

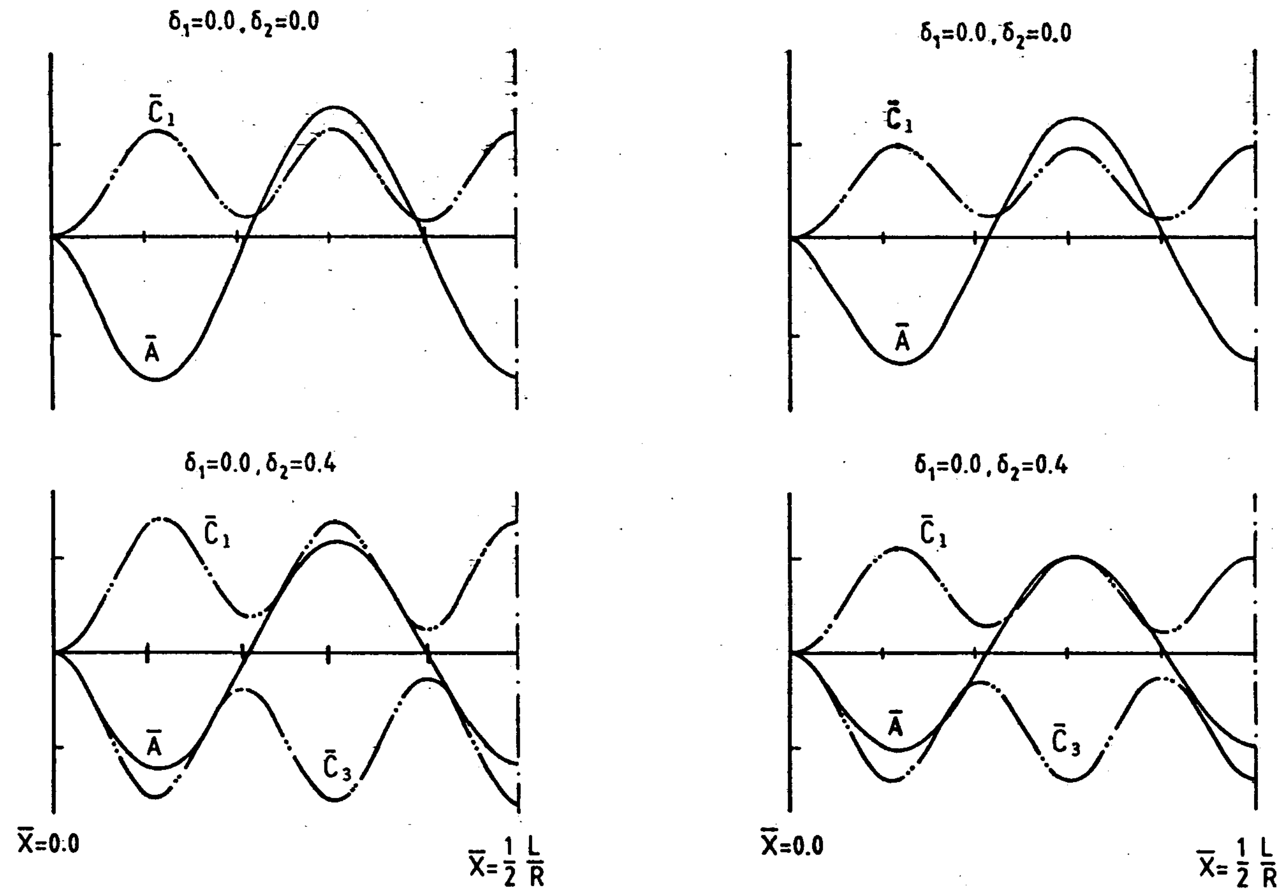
Fig. 8.8 Modal shapes for single mode response using simply supported boundary conditions
WN Shell; k=5, n=ℓ=25




a. SS2 Boundary condition ($\hat{u}$=0)

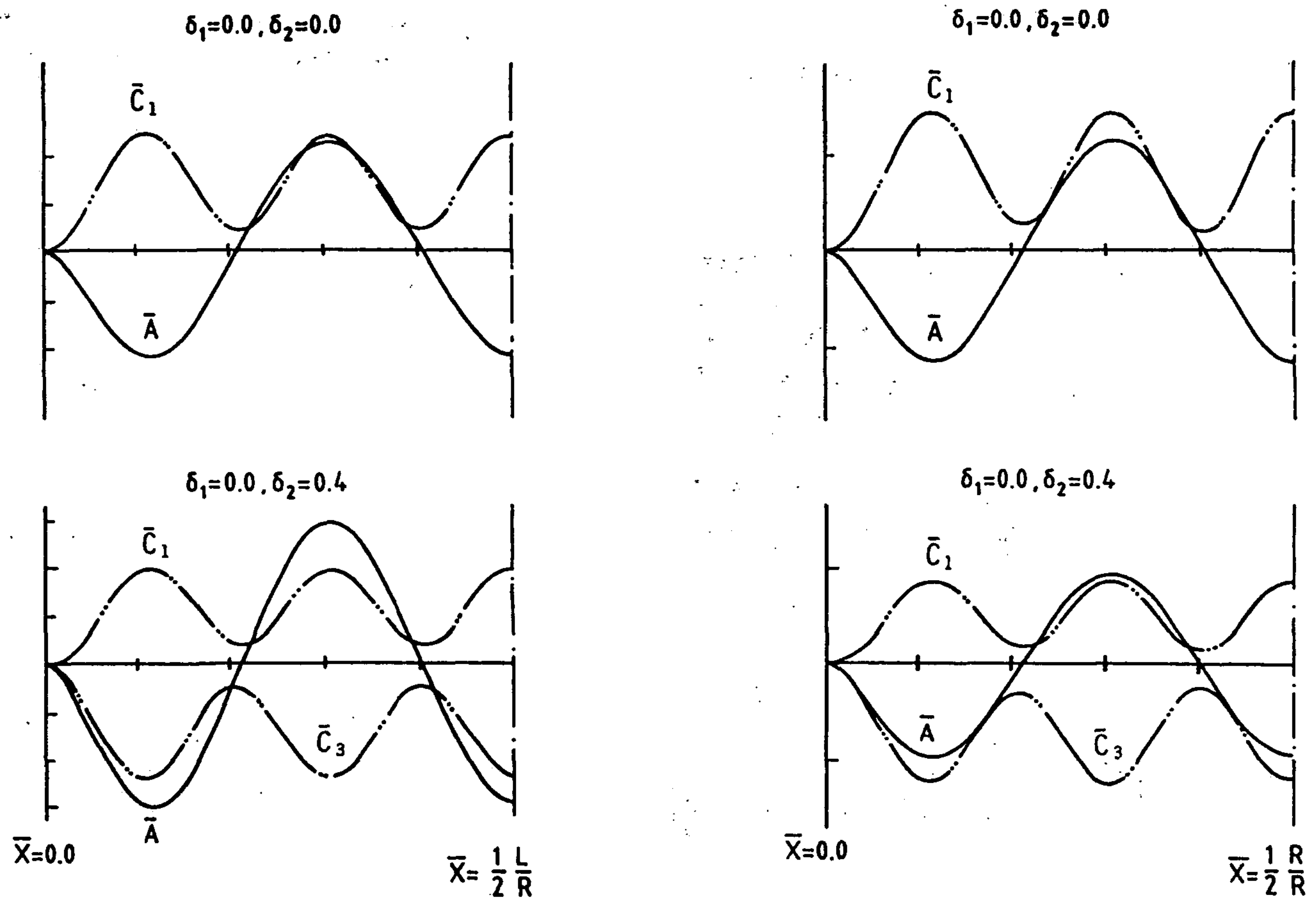
b. SS4 boundary condition ($\hat{u}$=0)

Fig. 8.9 Modal shapes for single mode response using simply supported boundary conditions WN Shell; k=5, n=$\ell$=25




a. C1 Boundary condition ($\hat{\hat{N}}_x$=0) b. C3 boundary condition ($\hat{\hat{N}}_x$=0)

Fig. 8.10 Modal shapes for single mode response using damped boundary conditions
WN Shell; k=5, n=ℓ=25




a. C2 Boundary condition ($\hat{\hat{u}}$=0) b. C4 boundary condition ($\hat{\hat{u}}$=0)

Fig. 8.11 Modal shapes for single mode response using damped boundary conditions
WN Shell; k=5, n=ℓ=25

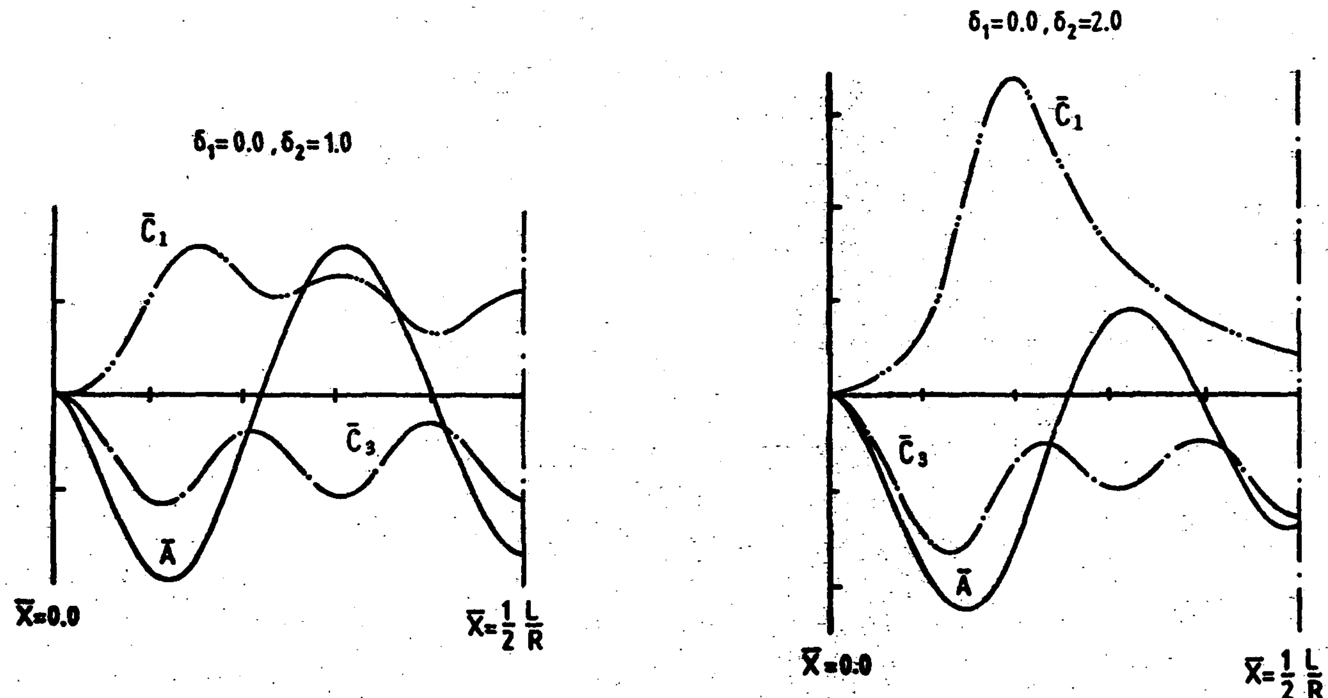



Fig. 8.12 Modal shapes for single mode response using C2 boundary conditions
Imperfect WN Shell; $k=5$, $n=\ell=25$

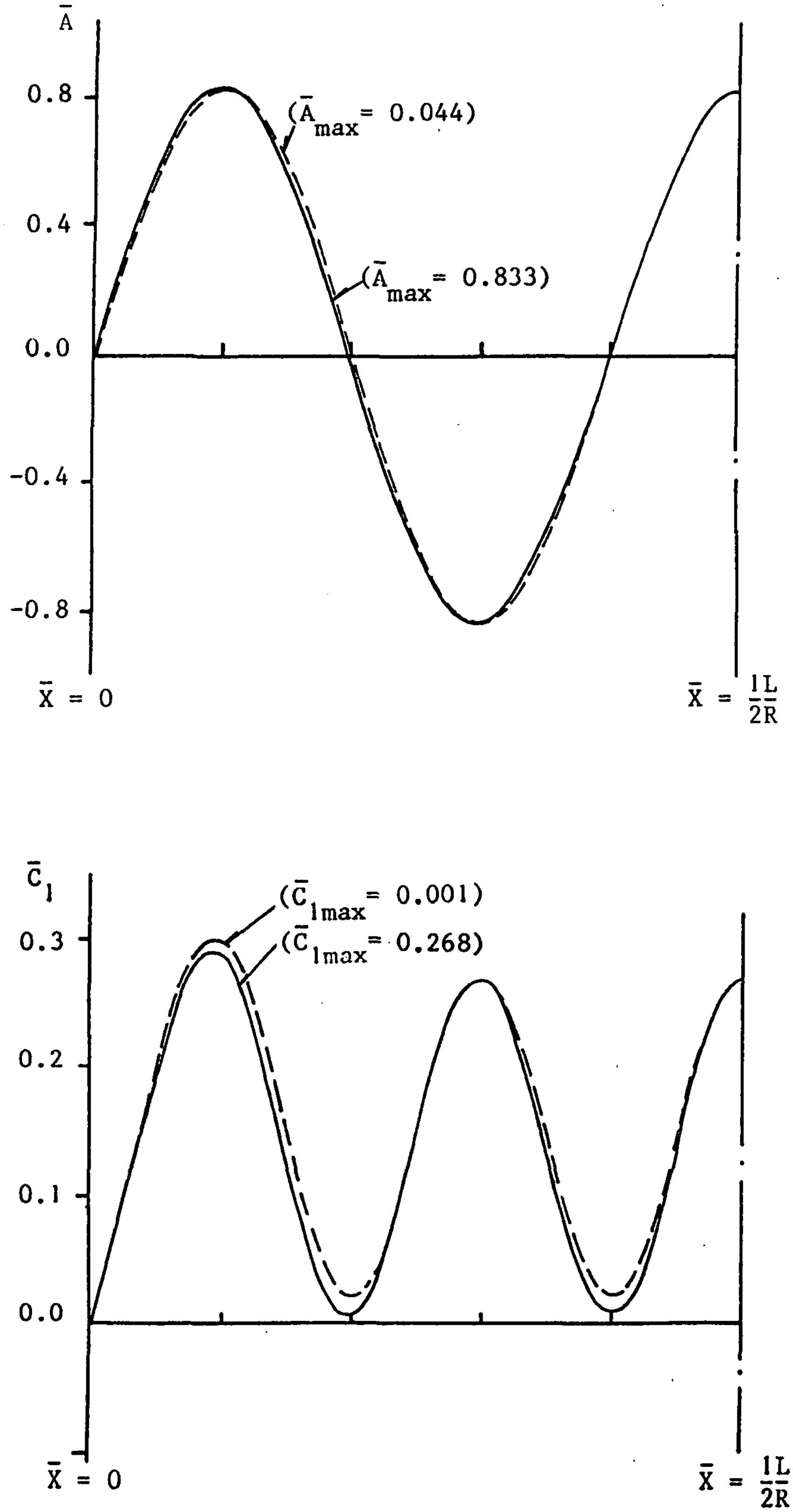
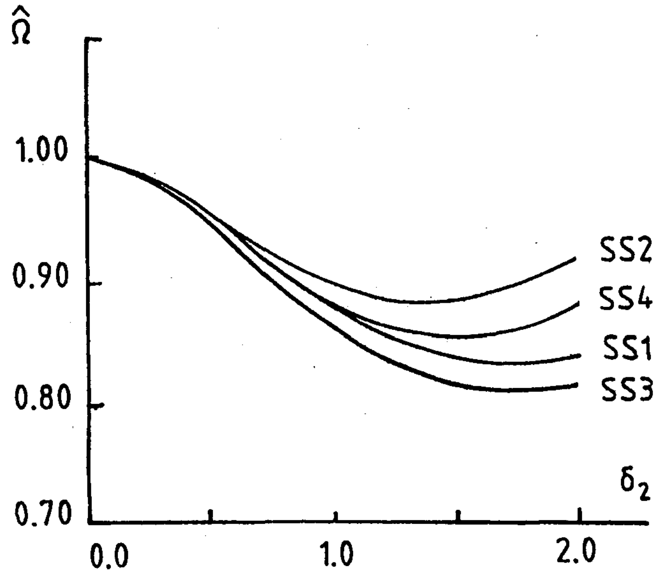
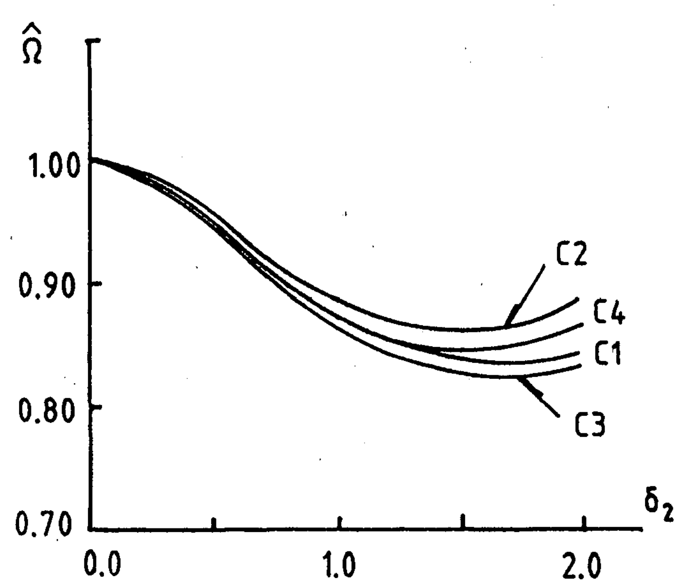



Fig. 8.13 Modal shapes for single mode response using SS3 boundary condition.
WN Shell; k=5, n=ℓ=25




a. Simply supported boundary conditions




b. Clamped boundary conditions

**Fig. 8.14** The frequency-asymmetric imperfection relationships for various boundary conditions.

WN Shell; $k=5$, $n=\ell=25$; amplitude of vibration $\bar{A}=0.001$

## 8.3 CONCLUSIONS

The present approach demonstrates a technique for the examination of the influence of different boundary conditions on the nonlinear vibration behaviour of thin-walled perfect and imperfect cylindrical shells. In principle, there is no difficulty in extending this method to the investigation of the nonlinear vibrations of cylindrical shells with elastic edge restraints.

Numerical results for two isotropic shells are presented. From these initial results it appears that the effect of boundary conditions on the nonlinear vibration behaviour of cylindrical shells varies wether one is dealing with the lower order or the higher order modes. Further it appears that the presence of not all too large asymmetric imperfections ($\delta_2 < 1.5$, say) always results in a lowering of the natural frequencies if $n=\ell$.

REFERENCES


1. H.S. Tsien — 'The buckling of thin cylindrical shells under axial compression'. J. Aeron. Sci. 8 (1941) pp. 303-312.

2. A. Harari — 'Generalized non-linear free vibrations of pre-stressed plates and shells'. Int. J. Non-linear Mech., 11 (1976), pp. 169-181.

3. A.H. Nayfeh — 'Introduction to Perturbation techniques'. Wiley-Interscience, New York, 1981.

4. A.H. Nayfeh and R.A. Raouf — 'Nonlinear oscillation of circular cylindrical shells'. AIAA/ASME/ASCE/AHS 27th Structures, Structural Dynamics and Materials Conference, Paper 86-0964, San Antonio, Texas, 1986.

5. A.K. Aziz (Editor) — 'Numerical solutions of boundary value problems for ordinary differential equations'. Proceedings of Symposium held at the University of Maryland, Academic Press, Inc., London, New York, June, 1974.

6. A. Maewal — 'Nonlinear flexural vibration of an elastic ring'. ASME J. Appl. Mech., 45 (1978) pp. 128-129.

7. A. Maewal — 'Finite element analysis of steady nonlinear harmonic oscillations of axisymmetric shells'. Comp. Meths. Appl. Mech. Eng., 58 (1986), pp. 37-50.

8. A. Maewal — 'Chaos in a harmonically excited elastic beam'. ASME J. Appl. Mech., 53 (1986), pp. 628-632.

9. A. Maewal and W. Nachbar — 'Lyapunov-Schmidt method for analysis of post-buckling behaviour and steady nonlinear harmonic oscillations of elastic structures', in: Proceedings, Eighth U.S. National Congress of Applied Mechanics, University of California at Los Angeles, CA. 1978.

10. A. Rosen and J. Singer — 'Vibrations of axially loaded stiffened cylindrical shells: Part II - Experimental Analysis'. TAE Rept. No. 163, Technion-Israel Institute of Technology, Dept. of Aerospace Engineering, Haifa, Israel, Aug. 1973.

11. A. Rosen and J. Singer — 'The effect of axisymmetric initial imperfections on the vibration of cylindrical shells under axial compression'. AIAA J., 12 (1974), pp. 995-997.

12. A. Rosen and J. Singer 'Vibrations of axially loaded stiffened cylindrical shells: Part I - Theoretical Analysis'. TAE Rept. No. 162, Technion-Israel Institute of Technology, Dept. of Aeronautical Engineering, Haifa, Israel, Feb. 1974.

13. A. Rosen and J. Singer 'Vibrations of axially loaded stiffened cylindrical shells with elastic restraints'. TAE Rept. No. 208, Technion-Israel Institute of Technology, Dept. of Aerospace Engineering, Haifa, Israel, Jan. 1975.

14. A. Rosen and J. Singer 'Experimental studies of vibrations and buckling of heavily stiffened cylindrical shells with elastic restraints'. TAE Rept. No. 210, Technion-Israel Institute of Technology, Dept. of Aeronautical Engineering, Haifa, 1975.

15. A. Rosen and J. Singer 'Influence of asymmetric imperfections on the vibration of axially compressed cylindrical shells'. Israel Journal of Technology, 14 (1976), pp. 23-26.

16. B. Childs, M. Scott and J.M. Daniel (Editors) 'Codes for boundary-value problems in ordinary differential equations'. Springer-Verlag, Berlin, Heidelberg, New York, 1979.

17. B. Carnaham, H.A. Luther and J.O. Wilkers 'Applied Numerical Methods'. John Wiley and Sons, Inc., New York, London, 1969.

18. B.E. Cummings 'Some nonlinear vibration and response problems of cylindrical shells'. Report SM 62-32 (AFOSR 3123), Graduate Aero. Labs., Caltech.) Pasadena, CA., June 1962.

19. B.O. Almroth 'Influence of edge conditions on the stability of axially compressed cylindrical shells'. NASA-CR-161, 1965.

20. C.A. Felippa and K.C. Park 'Direct time integration method in nonlinear structural dynamics'. Comp. Meths. Appl. Mech. Eng., 17/18 (1979), pp. 277-313.

21. C.B. Sharma 'Vibration characteristics of thin circular cylinders'. J. Sound and Vibration, 63(4) (1979), pp. 581-592.

22. C.E. Nuckolls and D.M. Egle — 'The influence of elastic boundary conditions on the resonant response of thin cylindrical shells' AIAA/ASME 12 th Structures, Structural Dynamics and Materials Conference, Paper 71-335, Anaheim, California, 1971.

23. C. Lakshmikantham and Tien-Yu Tsu — 'Dynamic stability of axially stiffened imperfect cylindrical shells under step loading'. AIAA J., 12(2) (1974), pp. 163-169.

24. C.L. Dym — 'Stability theory and its applications to structural mechanics'. Noordhoff International Publishing - Leyden, 1973.

25. C.L. Sun and S.Y. Lu — 'Nonlinear dynamic behaviour of heated conical and cylindrical shells'. Nuclear Engineering and Design, 7 (1968), pp. 113-122.

26. C. Massalas and N. Kafousias — 'Non-linear vibrations of a shallow cylindrical panel on a non-linear elastic foundation'. J. Sound and Vibration, 66(4) (1979), pp. 507-512.

27. C.P. Vendhan — 'An investigation into non-linear vibrations of thin plates'. Int. J. Non-linear Mech., 12 (1977) pp. 209-221.

28. C.W. Bert and D.M.Egle — 'Dynamics of composite, sandwich and stiffened shell-type structures'. J. Spacecraft and Rockets, 6(12) (1969), pp. 1345-1361.

29. D.A. Evensen — 'Some observations on the nonlinear vibration of thin cylindrical shells'. AIAA J., 1(12) (1963), pp. 2857-2858.

30. D.A. Evensen — 'A theoretical and experimental study of the nonlinear flexural vibrations of thin circular rings'. NASA TR R-227, 1965.

31. D.A. Evensen — 'Nonlinear flexural vibrations of thin-walled circular cylinders'. NASA TN D-4090, 1967.

32. D.A. Evensen — 'Nonlinear vibrations of beams with various boundary conditions'. AIAA J., 6(2) (1968), pp. 370-372.

33. D.A. Evensen — 'Nonlinear vibration of an infinitely long cylindrical shell'. AIAA J., 7(7) (1968), pp. 1401-1403.

34. D.A. Evensen and R. Aprabamian — 'Holographic measurement of wave propagation in axi-symmetric shells'. NASA CR-2063, 1972.

35. D.A. Evensen — 'Nonlinear vibration of circular cylindrical shells'. In: 'Thin-Shell Structures - Theory, Experiment and Design'. Editors: Y.C. Fung and E.E. Sechler, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.

36. D.A. Evensen — 'Comment on large amplitude asymmetric vibration of some thin shells of revolution'. Int. J. Sound Vibration, 52(3) (1977), pp. 453-455.

37. D.E. Lipouskii and V.M. Tokarenko — 'Effect of initial imperfections on free oscillation frequencies of cylindrical shells'. 6th All-Union Conference on the Theory of Shells and Plates, Baku, Azerbaidzhan SSR, Sept. 15-20, 1966. Transactions. Moscow, Izdatel'stvo Nauka, 1966, pp. 542-547.

38. D.F. Lockhart — 'Dynamic buckling of a damped externally pressurized imperfect cylindrical shells'. ASME J. Appl. Mech., 46(2) (1979), pp. 372-376.

39. D.F. Lockhart — 'Post-buckling dynamic behaviour of periodically supported imperfect shells'. Non-linear Mech. J., 17(3) (1983), pp. 165-174.

40. D. Hui, R.C. Tennyson and J.S. Hansen — 'Mode interaction of axially stiffened cylindrical shells: effects of stringer axial stiffness, torsional rigidity, and eccentricity'. ASME J. Appl. Mech., 48 (1981), pp. 915-922.

41. D. Hui and A.W. Leissa — 'Effects of uni-directional geometric imperfections on vibrations of pressurized shallow spherical shells'. Int. J. Non-linear Mechanics, 18(4) (1983), pp. 279-285.

42. D. Hui — 'Effects of geometric imperfections on large amplitude vibrations of rectangular plates with hysteresis damping'. ASME J. Appl. Mech., 51 (1984) pp. 216-220.

43. D. Hui — 'Influence of geometric imperfections and in-plane constraints on nonlinear vibrations of simply supported cylindrical panels'. ASME J. Appl. Mech., 51 (1984), pp. 383-390.

44. D. Hui — 'Large-amplitude vibrations of geometrically imperfect shallow sperical shells with structural damping'.
AIAA J., 21(12) (1983), pp. 1736-1741.

45. D.J. Gorman — 'On use of the Dirac Delta function for representing concentrated forces and moments in structural mechanics'.
Int. J. Mechanical Engineering Education, 10(3) (1982), pp. 177-183.

46. D.K. Liu — 'Survey of and suggestions for experiments of nonlinear vibrations of thin-walled cylindrical shells'.
Report LR-458, Dept. of Aerospace Engineering, TU-Delft, The Netherlands, 1985.

47. D.K. Liu and J. Arbocz — 'The nonlinear frequency-amplitude equations of imperfect stiffened cylindrical shells'.
Report LR-499, Dept. of Aerospace Engineering, TU-Delft, The Netherlands, 1986.

48. D.W. Jordan and P. Smith — 'Nonlinear ordinary differential equations'.
Clarendon Press, Oxford, 1977.

49. E. Reissner — 'Non-linear effect in vibration of cylindrical shells'.
Rept. AM 5-6, The RamoWooldridge Corp., Sept. 1955.

50. E.H. Dowell — 'On the nonlinear flexural vibrations of rings'.
AIAA J., 5(8) (1967), pp. 1508-1509.

51. E.H. Dowell and C.S. Ventres — 'Modal equations for the nonlinear flexural vibrations of a cylindrical shell'.
Int. J. Solids Structures, 4(10), (1968), pp. 975-991.

52. E.H. Dowell — 'Free vibrations of an arbitrary structure in terms of component modes'.
ASME J. Appl. Mech., 39(3) (1972), pp. 727-732.

53. E. Riks — 'The application of Newton's method to the problem of elastic stability.'.
ASME J. Appl. Mech., 39(4) (1972), pp. 1060-1065.

54. E. Riks — 'A unified method for the computation of critical equilibrium states of nonlinear elastic system'.
NLR MP 77041 U, National Aerospace Lab., The Netherlands.

55. E. Riks — 'An incremental approach to the solution of snapping and buckling problems'.
Int. J. Solids Structures, 15 (1979), pp. 529-551.

56. F. Badrakhan — 'Approximate backbone curve for non-linear systems with several degrees of freedom'. Int. J. Non-linear Mech., 18(5) (1983) pp. 385-394.

57. G. Dahlquist and A. Björk — 'Numerical Methods'. Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1974.

58. G.D. Smith — 'Numerical solution of partial differential equations'. Clarendon Press, Oxford, 1978.

59. G.F. Gerald and P.O. Wheatley — 'Applied Numerical Analysis'. Addison-Wesley Publishing Company, New York, 1984.

60. G. Hall and J.M. Watt — 'Modern numerical methods for ordinary differential equations'. Clarendon Press, Oxford, 1976.

61. G. Prathap — 'Comment on the large amplitude asymmetric vibrations of some thin shells of revolution'. Int. J. Sound Vibration, 56(2) (1978), pp. 303-308.

62. G. Prathap — 'On the large amplitude vibration of circular cylindrical shells'. Int. J. Sound Vibration, 59(2) (1978), pp. 295-298.

63. H.B. Keller — 'Numerical solution of boundary value problems for ordinary differential equations: Survey and some recent results on different methods'. In: 'Numerical solutions of boundary value problems for ordinary differential equations'. A.K. Aziz (Editor), Academic Press, New York, 1975.

64. H.F. Bauer — 'Nonlinear response of elastic plates to pulse excitations'. ASME J. of Appl. Mech., 35(3) (1968), pp. 47-52.

65. H.H. Bleich and J.H. Ginsberg — 'Nonlinear forced vibrations of infinitely long cylindrical shells'. TR-46 Dept. of Civil Engineering and Engineering Mechanics, Colombia University, New York, 1970.

66. H. Kraus — 'Thin elastic shells'. John Wiley and Sons, Inc., New York, London 1967, pp. 66-68.

67. H.N. Chu — 'Influence of large amplitudes on flexural vibrations of a thin circular cylindrical shell'. J. Aerospace Sci. 28 (1961), pp. 602-609.

68. H.R. Radwan and J. Genin — 'Nonlinear vibration of thin cylinders'. ASME J. Appl. Mech., 43 (1976), pp. 370-372.

69. I.A. Mitropolsky — 'Averaging method in non-linear mechanics'. Int. J. Non-linear Mechanics, 2 (1967), pp. 69-96.

70. I. Gladwell and D.K. Sayers (Editors) — 'Computational techniques for ordinary differential equations'. Academic Press, London, New York, 1978.

71. I.K. McIvor — 'Dynamic stability and nonlinear oscillations of cylindrical shells (plane strain) subjected to impulsive pressure'. Ph.D. Dissertation, Stanford University, Stanford, CA., 1962.

72. I.K. McIvor — 'The elastic cylindrical shell under radial impulse'. ASME Journal of Applied Mech., 33 (1966), pp. 831-837.

73. I.S. Habib — 'Engineering analysis methods'. Lexington Books, London, 1975.

74. I.V. Ojalvo and M. Newman — 'Natural vibrations of a stiffened pressurized cylinder with an attached mass'. 4th Aerospace Sciences Meeting, Paper 66-449, Los Angeles, 1966.

75. J. Arbocz and C.D. Babcock — 'The effect of general imperfections on the buckling of cylindrical shells'. ASME J. Appl. Mech. 36 (1969), pp. 28-38.

76. J. Arbocz and E.E. Sechler — 'On the buckling of axially compressed ring and stringer stiffened imperfect cylindrical shells'. Report SM 73-10, Graduate Aero. Labs., Caltech, Pasadena, CA., December 1973.

77. J. Arbocz and E.E. Sechler — 'On the buckling of axially compressed imperfect cylindrical shells'. ASME J. Appl. Mech., 41(3) (1974), pp. 737-743.

78. J. Arbocz — 'The effect of initial imperfections on shell stability'. In: 'Thin-Shell Structures-Theory, Experiment and Design'. Editors Y.C. Fung and E.E. Sechler, Prentice Hall, Englewood Cliffs, New Jersey, 1974, pp. 205-245.

79. J. Arbocz — 'Past, present and future of shell stability analysis'. Z. Flugwiss, Weltraumforsch. 5(6) (1981).

80. J. Arbocz and C.D. Babcock — 'Prediction of buckling loads based on experimentally measured initial imperfections'. In: 'Buckling of Structures'. B. Budiansky (Editor) Springer-Verlag, Berlin, 1976, pp. 291-331.

81. J.A. Bennett and J.G. Eisley — 'A multiple degree-of-freedom approach to nonlinear beam vibrations'. AIAA J., 8(4) (1970), pp. 734-739.

82. J.C. Chen and C.D. Babcock — 'Nonlinear vibration of cylindrical shells'. AIAA J., 13(7) (1975), pp. 868-876.

83. J.G. Simmonds — 'Accurate nonlinear equations and a perturbation solution of the free vibrations of a circular elastic ring'. ASME J. Appl. Mech. 46 (1979), pp. 156-160.

84. J.H. Balbi — 'Estimation of the relative error for the averaging method'. Int. J. Non-linear Mech., 15 (1980) pp. 13-20.

85. J.H. Babli — 'Averaging and reduction'. Int. J. Non-linear Mech., 17(5/6) (1982), pp. 343-353.

86. J.H. Ginsberg — 'An equivalent principle for systems with secondary generalized coordinates'. J. of Sound and Vibration, 25 (1972), pp. 83-93.

87. J.H. Ginsberg — 'The effects of damping on a non-linear system with two degrees of freedom'. Int. J. Non-linear Mech., 7 (1972), pp. 323-336.

88. J.H. Ginsberg — 'Nonlinear resonant vibrations of infinitely long cylindrical shells'. AIAA J., 10(8) (1972), pp. 979-980.

89. J.H. Ginsberg — 'Large amplitude forced vibrations of simply supported thin cylindrical shells'. ASME J. Appl. Mech. 40 (1973), pp. 461-477.

90. J.H. Ginsberg — 'Dynamic stability of transverse axisymmetric waves in circular cylindrical shells'. ASME J. Appl. Mech., 41(1) (1974), pp. 77-82.

91. J.L. Nowinski — 'Nonlinear transverse vibrations of orthotropic cylindrical shells'. AIAA J., 1(3) (1963), pp. 617-620.

92. J.L. Sanders — 'Nonlinear theory for thin shells'. Quart, Appl. Math., 21 (1963), pp. 21-36.

93. J.L. Sewall and E.C. Naumann — 'An experimental and analytical vibration study of thin cylindrical shells with and without longitudinal stiffeners'. NASA TN D-4705, 1968.

94. J.M.A.M. Hol — 'The effect of general imperfections on the natural frequencies of axially compressed stiffened cylinders'. ir. Thesis, Department of Aerospace Engineering, TH-Delft, The Netherlands, 1983.

95. J.M. Ortega and W.C. Rheinboldt — 'Iterative solution of nonlinear equations in several variables'. Academic Press, London, New York 1970.

96. J.M.T. Thompson and G.W. Hunt (Editors) — 'Collapse: the buckling of structures in theory and practice'. IUTAM Symposium, Cambridge University Press, 1983.

97. J. Padovan — 'Travelling waves vibrations and buckling of rotating anisotropic shells of revolution by finite elements'. Int. J. Solids Structures, 11 (1975), pp. 1367-1380.

98. J. Padovan — 'Nonlinear vibrations of general structures'. J. Sound and Vibration, 72(4) (1980), pp. 427-441.

99. J. Singer, J. Arbocz and C.D. Babcock — 'Buckling of imperfect stiffened cylindrical shells under axial compression'. AIAA J., 9 (1971), pp. 68-75.

100. J. Singer — 'Recent studies on the correlation between vibration and buckling of stiffened cylindrical shells'. Z. Flugwiss. und Weltraumforsch. 3(6) (1979), pp. 333-344.

101. J. Singer, O. Rand and A. Rosen — 'Vibrations of axially loaded stiffened cylindrical panels with elastic restraints'. TAE Rept. No. 439, Technion-Israel Institute of Technology, Dept of Aeronautical Engineering, Haifa, Israel 1981.

102. J. Singer and J. Prucz — 'Influence of initial geometrical imperfections on vibrations of axially compressed stiffened cylindrical shells'. J. Sound and Vibration, 80(1) (1982), pp. 117-143.

103. J. Singer — 'Vibrations and buckling of imperfect stiffened shells: recent developments'. In: 'Collapse: the buckling of structures in theory and practice'. Editors: J.M.T. Thompson and G.W. Hunt, Cambridge University Press, Cambridge, 1983.

104. K.A.V. Pandulai and M. Sathyamoorthy — 'On the modal equations of large amplitude flexural vibration of beams, plates, rings and shells'. Int. J. Non-linear Mechanics, 8 (1973), pp. 213-218.

105. K.C. Park — 'Practical aspects of numerical time integration'. Computer & Structures, 7 (1977), pp. 343-353.

106. K. Forsberg — 'Influence of boundary conditions on the modal response of thin cylindrical shells'. AIAA J., 2(12) (1964), pp. 2150-2157.

107. K. Forsberg — 'Axially symmetric and beam type vibrations of thin cylindrical shells'. Proceedings 4th Aerospace Sciences Meeting, June 1966.

108. K. Forsberg — 'A review of analytical methods used to determine the modal characteristic of cylindrical shells'. NASA CP-163, 1966, 67 pp.

109. K.K. Raju and G.V. Rao — 'Large amplitude asymmetric vibrations of some thin shells of revolution'. Int. J. Sound Vibration, 44(3) (1976), pp. 327-333.

110. K. Nagai and N. Yamaki — 'Dynamic stability of circular cylindrical shells under periodic compressive forces'. J. Sound and Vibration, 58(3) (1978), pp. 425-441.

111. K. Suzuki, S. Tamura, T. Kosawada and S. Takahashi — 'Vibrations of nonlinear cylindrical shells'. Bulletin of the JSME. 26(215) (1983), pp. 215-214.

112. L.E. Penzes — 'Effect of boundary conditions on flexural vibrations of thin orthogonally stiffened cylindrical shells'. The Journal of the Acoustical Society of America, 42(4) (1967), pp. 901-903.

113. L.E. Penzes and H. Kraus — 'Free vibration of prestressed cylindrical shells having arbitrary homogeneous boundary conditions'. AIAA J., 10(10) (1972), pp. 1309-1313.

114. L.H. Donnell — 'A new theory for the buckling of thin cylinders under axial compression and bending'. Trans. ASME, 56 (1934), pp. 795-806.

115. L. Meirovitch — 'Elements of vibration analysis'. Mc Graw-Hill Book Co, New York, 1975.

116. L. Watawala and W.A. Nash — 'Influence of initial geometric imperfections on vibrations of thin circular cylindrical shells'. Technical Report, Grant CEE 76-14833, Dept. of Civil Engineering, University of Massachussetts, Amherst, May 1982.

117. L.W. Rehfield — 'Nonlinear free vibrations of elastic structures'. Int. J. Solids Structures, 9 (1973) pp. 581-590.

118. M. Baruch and J. Singer 'Effect of eccentricity of stiffeners on the general instability of stiffened cylindrical shells under hydrostatic pressure'.
J. of Mech. Engineering Science, 5(1) (1963), pp. 23-27.

119. M.D. Olson 'Some experimental observations on the nonlinear vibration of cylindrical shells'.
AIAA J., 3(9) (1965), pp. 1775-1777.

120. M. El Raheb and C.D. Babcock 'The effect of end rings on the natural frequencies of cylindrical shells'.
Report SM 71-1, Graduate Aero. Labs, Caltech, Pasadena, CA., January 1971.

121. M. El Raheb and C.D. Babcock 'Some approximations in the linear dynamic equations of thin cylinders,
J. of Sound and Vibration, 76(4) (1981), pp. 543-559.

122. N.J. Hoff and N.Y. Brooklyn 'The accuracy of Donnell's equations'.
ASME J. Appl. Mech., 22(3) (1955), pp. 329-334.

123. N. Kryloff and N.N. Bogoliuboff 'Introduction to non-linear mechanics'.
Ann. Math. Studies, No. 11, Princeton U. Press, Princeton (1947).

124. N.N. Bogoliuboff 'On some statistical methods in mathematical physics'.
USSR Academy of Sciences (1945).

125. N.N. Bogoliuboff 'Theory of perturbations in nonlinear mechanics'.
Institute for Structural Mechanics. UkrSSR Academy of Sciences, Collection of papers, No. 14, pp. 9-34 (1950).

126. N.N. Bogoliuboff and D.N. Zubarev 'The method of asymptotic approximation for systems with rotating phase and its application to the motion of charged particles in magnetic field'.
UMJVII, 5 (1955).

127. N.N. Bogoliuboff and I.A. Mitropolski 'Asymptotic methods in the theory of non-linear oscillations'.
Gordon and Breach Sci. Publ. Inc., New York 1961.

128. N. Yamaki, K. Otomo and M. Chiba 'Non-linear vibrations of a clamped circular plate with initial deflection and initial edge displacement, Part 1: Theory.
J. Sound and Vibration, 79(1) (1981), pp. 23-42.

129. N. Yamaki — 'Dynamic behaviour of thin-walled circular cylindrical shell'. In: 'Developments in Thin-walled Structures-I' Editors: J. Rhodes and A.C. Walker, Applied Science Publishers, London, 1983.

130. N.W. McLachlan — 'Ordinary nonlinear differential equations in engineering and physical sciences'. Clarendon Press, Oxford, 1956.

131. P.M. Anselone and R.H. Moore — 'An extension of the Newton-Kantorovic method for solving nonlinear equations with an application to elasticity'. J. Mathematical Analysis and Applications. 13 (1966), pp. 467-501..

132. R. Delpak and W.M. Hague — 'An experimental and theoretical investigation of the frequencies and mode shapes of axisymmetric shell models'. J. Sound and Vibration, 72(2) (1980), pp. 235-249.

133. R. Greif and H. Chung — 'Vibrations of constrained cylindrical shells'. AIAA J., 13(9) (1975), pp. 1190-1198.

134. R. Miserentino and L.F. Vosteen — 'Vibration test of pressurized thin-walled cylindrical shells'. NASA TN D-3066, 1965.

135. R.W. Hornbeck — 'Numerical methods'. Quantum Publishers Inc., New York, 1975.

136. S. Atluri — 'A perturbation analysis of the non-linear free flexural vibrations of a circular cylindrical shell'. Int. J. Solids Structures, 8 (1972), pp. 549-569.

137. S. Kobayashi — 'The influence of the boundary conditions on the buckling load of cylindrical shells under axial compression'. NASA CR-558, 1966.

138. S.L. Lau and Y.K. Cheung — 'Amplitude incremental variational principle for nonlinear vibration of elastic systems'. ASME J. Appl. Mech. 48 (1981), pp. 959-964.

139. S.M. Schapiro and P.R. Sethna — 'An estimate for the small parameter in the asymptotic analysis of non-linear systems by the method of averaging'. Int. J. Non-linear Mech., 12 (1977), pp. 127-140.

140. S.N. Liu — 'On the generalized variational principle and some problems in the theory of elastic shallow shell'. in Chinese, ACTA Mechanica Sinica, 6(1) (1961).

141. S. Saigal, T.Y. Yang and R.K. Kapania — 'Dynamic buckling of imperfection-sensitive shell structures'. AIAA/ASME/ASCE/AHS 27th Structures, Structural Dynamics and Materials Conference, Paper 86-0966, San Antonio, 1986.

142. T. Irie, G. Yamada and Y. Muramoto — 'Free vibration of a circular cylindrical shell elastically restrained by axially spaced springs'. ASME J. Appl. Mech., 50 (1983), pp. 544-548.

143. T. Ueda — 'Non-linear free vibrations of conical shells'. Int. J. Sound Vibration, 64(1) (1979), pp. 85-95.

144. V. Birman and C.W. Bert — 'Non-linear beam-type vibrations of long cylindrical shells'. AIAA/ASME/ASCE/AHS 27th Structures, Structural Dynamics and Materials Conference, Paper 86-0965, San Antonio, 1986.

145. V.I. Weingarten, E.J. Morgan and P. Seide — 'Elastic stability of thin-walled cylindrical and conical shells under axial compression'. AIAA J., 3 (1965), pp. 500-505.

146. V.V. Novozhilov — 'Thin shell theory'. Translated from the second Russian edition by P.G. Lowe. Wolters-Noordhoff Publishing, Groningen 1970.

147. W.A. Nash and J.R. Modeer — 'Certain approximate analyses of nonlinear behavior of plates and shallow shells'. In: 'Theory of thin elastic shells', 1960, North-Holland Publishing Company, Amsterdam, pp. 331-354.

148. W.C. Chei — 'Variation and finite element method'. in Chinese, Scientific Press, 1980.

149. W.C.L. Hu — 'Vibration analysis of stiffened cylinder including inter-ring motion'. J. Acoustical Soc. of America, 43(5) (1968), pp. 1005-1016.

150. W. Leissa — 'Vibration of Shells'. NASA SP-288, 1973.

151. W. Scedel — 'A new frequency formula for closed circular cylindrical shells for a large variety of boundary conditions'. Journal of Sound and Vibration 70(3) (1980), pp. 309-317.

152. W. Scedel — 'Vibrations of shells and plates'. Marcel Dekker Inc., New York and Basel, 1981.

153. W.T. Koiter 'The effect of axisymmetric imperfections on the buckling of cylindrical shells under axial compression'.
Proc. Kon. Ned. Ak. Wet. Ser. B, 66, 1963, pp. 265-279.

154. Y. Matsuzaki and S. Kobayashi 'An analytical study of the non-linear flexural vibration of thin circular cylindrical shells'.
J. Japan Soc. Aeronaut. Space Sci., 17(187) (1969), pp. 308-315.

155. Y. Matsuzaki and S. Kobayashi 'Nonlinear analysis of supersonic panel flutter of circular cylindrical shells'.
J. Japan Soc. Aeronaut. Space Sci., 18(184) (1970), pp. 103-111.

156. Y. Matsuzaki and S. Kobayashi 'A theoretical and experimental study of the nonlinear flexural vibration of thin circular shells with clamped ends'.
J. Japan Soc. Aeronaut. Space Sci., 12(21) (1969), pp. 55-62.

157. Y.S. Tamura 'Dynamic stability of cylindrical shells under step loading'.
Report SM 73-5 (AFOSR RT-73), Graduate Aero. Labs., Caltech, Pasadena, CA., 1973.

158. Y.Y. Yu 'Free vibrations of thin cylindrical shells having finite lengths with freetly supported and clamped edges'.
ASME Journal of Applied Mech., 22 (1955), pp. 547-552.

159. Y.Y. Yu 'Dynamic equations of Donnell type for cylindrical shells with application to vibration problems'.
Ninth Congres. Intern. Mecan. Appl., Univ. Bruxelles, 7 (1957), pp. 261-278.

APPENDIX 1-A

## DEFINITION OF CONSTANTS AND PARAMETERS

### 1-A.1 Smeared stiffener definitions

The smeared stiffener approach is often used to account for the effect of eccentric stringers and rings [102]. If the basic stiffener data, such as material properties and geometric dimensions and constants are known, a number of parameters can be defined.

(a) Parameters to represent the increase of the effective cross-sectional area of the shell due to stringers and rings respectively

$$\mu_1 = (1 - \nu^2)\,\frac{E_s A_s}{Ehd_s}$$

$$\mu_2 = (1 - \nu^2)\,\frac{E_r A_r}{Ehd_r}$$

(b) Parameters to represent the change of the extensional stiffness of the shell due to eccentricity of stringers and rings respectively

$$\chi_1 = (1 - \nu^2)\,\frac{E_s A_s}{Ehd_s}\,e_s = \mu_1 e_s$$

$$\chi_2 = (1 - \nu^2)\,\frac{E_r A_r}{Ehd_r}\,e_r = \mu_2 e_r$$

(c) Parameters to represent the increase in flexural stiffness of the shell due to stringers and rings respectively

$$\eta_{01} = \frac{E_s}{d_s D}\,(I_s + e_s^2 A_s)$$

$$\eta_{02} = \frac{E_r}{d_r D}\,(I_r + e_r^2 A_r)$$

(d) Parameters to represent the increase of torsional stiffness of the shell due to stringers and rings respectively

$$\eta_{t1} = \frac{G_s I_{t_s}}{d_s D}$$

$$\eta_{t2} = \frac{G_r I_{t_r}}{d_r D}$$

where

$$G_s = \frac{E_s}{2(1+\nu_s)}$$

and

$$G_r = \frac{E_r}{2(1+\nu_r)}$$

are the shear moduli of stringers and rings respectively.

(e) parameters to represent the change of flexural stiffness of the shell due to the eccentricity of stringers and rings respectively

$$\zeta_1 = \frac{E_s A_s}{d_s D} e_s$$

$$\zeta_2 = \frac{E_r A_r}{d_r D} e_r$$

In these parameters the bending stiffness is used. This is defined as

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

(f) The specific mass of the combination of shell wall, stringers and rings is represented as follows

$$\bar{\rho} = \rho + \rho_s \frac{A_s}{d_s h} + \rho_r \frac{A_r}{d_r h}$$

## 1-A.2 Normalized wave numbers

The normalized wave numbers are defined as

$$\alpha_i^2 = i^2 \frac{Rh}{2c} \left(\frac{\pi}{L}\right)^2 = \ell_i^2 \frac{Rh}{2c} \qquad (i = i,\ k,\ m)$$

$$\alpha_\ell^2 = \ell^2 \frac{Rh}{2c} \left(\frac{\ell}{R}\right)^2 = \ell_\ell^2 \frac{Rh}{2c} \qquad (\ell = \ell,\ n)$$

Using the normalized wave numbers the following extended stiffener parameters, which also account for the wave numbers and hence the deformation patterns, can be defined:

$$\gamma_{D,k,\ell} = \bar{D}_{xx}\alpha_k^4 + \bar{D}_{xy}\alpha_k^2\alpha_\ell^2 + \bar{D}_{yy}\alpha_\ell^4$$

$$\gamma_{Q,k,\ell} = \bar{Q}_{xx}\alpha_k^4 + \bar{Q}_{xy}\alpha_k^2\alpha_\ell^2 + \bar{D}_{yy}\alpha_\ell^4$$

$$\gamma_{H,k,\ell} = \bar{H}_{xx}\alpha_k^4 + \bar{H}_{xy}\alpha_k^2\alpha_\ell^2 + \bar{H}_{yy}\alpha_\ell^4$$

## 1-A.3 Load parameter

The load parameter (axial compression only) is defined as

$$\lambda = \frac{Rc}{Eh^2} N_o$$

APPENDIX 2-A

## COEFFICIENTS OF CHAPTER 2

### 2-A.1 The weighting functions for Galerkin's method

The weighting functions which were used in Galerkin's method are as follows:

$$G_1 = \frac{\partial \hat{\hat{w}}}{\partial A} = h\{\cos \ell_\ell y \sin \ell_k x + \frac{h\, \ell_\ell^2\, R}{2} [A + \delta_{n,\ell}(\delta_2 + \hat{\delta}_2) \sin^2 \ell_m x]\}$$

$$G_2 = \frac{\partial \hat{\hat{w}}}{\partial B} = h\{\sin \ell_\ell y \sin \ell_k x + \frac{h\, \ell_\ell^2\, R}{2} B \sin^2 \ell_m x)\}$$

### 2-A.2 Coefficients of equation (2-2-6)

$f(\underset{\sim}{A},x,y) =$

$$A_1\cos(\ell_{2\ell}y) + A_2\sin(\ell_{2\ell}y) + A_3\cos(\ell_{n+\ell}y) + A_4\sin(\ell_{n+\ell}y) + A_5\cos(\ell_{2k}x) +$$

$$+ A_6\cos(\ell_{2k}x)\cos(\ell_{n-\ell}y) + A_7\cos(\ell_{2k}x)\sin(\ell_{n-\ell}y) + A_8\cos(\ell_{n-\ell}y) +$$

$$+ A_9\sin(\ell_{2m+k}x)\cos(\ell_\ell y) + A_{10}\sin(\ell_{2m+k}x)\cos(\ell_n y) + A_{11}\sin(\ell_{2m-k}x)\cos(\ell_\ell y) +$$

$$+ A_{12}\sin(\ell_{2m-k}x)\cos(\ell_n y) + A_{13}\sin(\ell_{2m+k}x)\sin(\ell_\ell y) + A_{14}\sin(\ell_{2m-k}x)\sin(\ell_\ell y) +$$

$$+ A_{15}\cos(\ell_{2k}x)\cos(\ell_{n+\ell}y) + A_{16}\cos(\ell_{2k}x)\sin(\ell_{n+\ell}y) + A_{17}\sin(\ell_{n-\ell}y) +$$

$$+ A_{18}\sin(\ell_k x)\cos(\ell_\ell y) + A_{19}\sin(\ell_k x)\sin(\ell_\ell y) + A_{20}\cos(\ell_{2m}x) +$$

$$+ A_{21}\sin(\ell_{k+1}x)\cos(\ell_\ell y) + A_{22}\sin(\ell_{k-i}x)\cos(\ell_\ell y) + A_{23}\sin(\ell_{k+i}x)\sin(\ell_\ell y) +$$

$$+ A_{24}\sin(\ell_{k-1}x)\sin(\ell_\ell y)$$

where the $A_1$, $A_2$, ..., $A_{24}$ are functions of the time-dependent amplitudes A and B, the imperfection terms $\delta_1$ and $\delta_2$, and the fundamental response $\hat{\delta}_1$ and $\hat{\delta}_2$. These functions are given as follows.

$$A_1 = -\frac{1}{32}\frac{(A^2-B^2)\,h^2\,\ell_k^2}{H_{yy}\,\ell_\ell^2}$$

$$A_2 = -\frac{1}{32}\frac{AB\,h^2\,\ell_k^2}{H_{yy}\,\ell_\ell^2}$$

$$A_3 = -\frac{1}{4}\frac{(\ell_n^2+\ell_\ell^2)^2}{H_{yy}\,\ell_{n+\ell}^4}\,h^2\,\ell_k^2(\delta_2+\hat{\delta}_2)\;A$$

$$A_4 = -\frac{1}{4}\frac{\ell_k^2}{H_{yy}\,\ell_{n+\ell}^2}\,h^2(\delta_2+\hat{\delta}_2)\;B$$

$$A_5 = \frac{1}{32}\frac{h^2\,\ell_k^2}{H_{xx}\,\ell_k^2}\,(A^2+B^2)$$

$$A_6 = \frac{1}{4}\frac{h^2\,\ell_k^2\,\ell_{n+\ell}^2}{M(\gamma_{H,2k,n-\ell})}\,(\delta_2+\hat{\delta}_2)\;A$$

$$A_7 = -\frac{1}{4}\frac{h^2\,\ell_k^2\,\ell_{n+\ell}^2}{M(\gamma_{H,2k,n-\ell})}\,(\delta_2+\hat{\delta}_2)\;B$$

$$A_8 = \begin{cases} 0 & n = \ell \\ -\dfrac{1}{4}\dfrac{h^2\,\ell_k^2}{H_{yy}\,\ell_{n-\ell}^2}\,(\delta_2+\hat{\delta}_2)\;A & n \neq \ell \end{cases}$$

$$A_9 = \frac{h^2\,\ell_\ell^2\,\ell_m^2}{M(\gamma_{H,2m+k,\ell})}\,AC$$

$$A_{10} = \frac{h^2 \ell_n^2 \ell_m^2}{M(\gamma_{H,2m+k,n})} (\delta_2 + \hat{\delta}_2) \; C$$

$$A_{11} = - \frac{h^2 \ell_\ell^2 \ell_m^2}{M(\gamma_{H,2m-k,\ell})} \; AC$$

$$A_{12} = - \frac{h^2 \ell_n^2 \ell_m^2}{M(\gamma_{H,2m-k,n})} (\delta_2 + \hat{\delta}_2) \; C$$

$$A_{13} = \frac{h^2 \ell_\ell^2 \ell_m^2}{M(\gamma_{H,2m+k,\ell})} \; BC$$

$$A_{14} = - \frac{h^2 \ell_\ell^2 \ell_m^2}{M(\gamma_{H,2m-k,\ell})} \; BC$$

$$A_{15} = \frac{1}{4} \frac{h^2 \ell_k^2 \ell_{n-\ell}^2}{M(\gamma_{H,2k,n+\ell})} (\delta_2 + \hat{\delta}_2) \; A$$

$$A_{16} = \frac{1}{4} \frac{h^2 \ell_k^2 \ell_{n-\ell}^2}{M(\gamma_{H,2k,n+\ell})} (\delta_2 + \hat{\delta}_2) \; B$$

$$A_{17} = \begin{cases} \dfrac{1}{4} \dfrac{h^2 \ell_k^2}{H_{yy} \ell_{n-\ell}^2} (\delta_2 + \hat{\delta}_2) \; B & n \neq \ell \\ 0 & n = \ell \end{cases}$$

$$A_{18} = \frac{1}{R} \frac{\left(\ell_k^2 h + \frac{2c}{R} \gamma_{Q,k,\ell}\right)}{M(\gamma_{H,k,\ell})} \; A$$

$$A_{19} = \frac{1}{R} \frac{\left(\ell_k^2 h + \frac{2c}{R} \gamma_{Q,k,\ell}\right)}{M(\gamma_{H,k,\ell})} \; B$$

$$A_{20} = - \frac{1}{16} \frac{h(\frac{2}{R} + 8\, Q_{xx} \ell_m^2)}{H_{xx} \ell_m^2} C$$

$$A_{21} = - \frac{1}{2} \frac{h^2 \ell_i^2 \ell_\ell^2}{M(\gamma_{H,k+i,\ell})} (\delta_1 + \hat{\delta}_1) A$$

$$A_{22} = - \frac{1}{2} \frac{h^2 \ell_i^2 \ell_\ell^2}{M(\gamma_{H,k-i,\ell})} (\delta_1 + \hat{\delta}_1) A$$

$$A_{23} = - \frac{1}{2} \frac{h^2 \ell_i^2 \ell_\ell^2}{M(\gamma_{H,k+i,\ell})} (\delta_1 + \hat{\delta}_1) B$$

$$A_{24} = - \frac{1}{2} \frac{h^2 \ell_i^2 \ell_\ell^2}{M(\gamma_{H,k-i,\ell})} (\delta_1 + \hat{\delta}_1) B$$

where, the parameter $\gamma$ is defined in appendix 1-A.2 and the parameter M is defined as

$$M = \left(\frac{2c}{Rh}\right)^2 \frac{1}{Eh}$$

<u>2-A.3 Coefficients of equations (2-2-7) and (2-2-8)</u>

$$\bar{c}_1 = \bar{\rho} h^2$$

$$\bar{c}_2 = \frac{Eh^2}{R^2} \left( \frac{(\alpha_k^2 + \gamma_{Q,k,\ell})^2}{\gamma_{H,k,\ell}} + \gamma_{D,k,\ell} - \lambda \left[2\alpha_k^2 + 2c^2 \alpha_m^2 \alpha_\ell^4 (\delta_2 + \hat{\delta}_2)^2 \;\; \delta_{n,\ell}\right] + \right.$$

$$\left. + \frac{1}{4} c^2 \frac{\alpha_\ell^4}{\bar{H}_{xx}} (\hat{\delta}_2^2 + 2\delta_2 \hat{\delta}_2) + c^2 \left[\frac{1-\delta_{i,k}}{\gamma_{H,k-i,\ell}} + \frac{1}{\gamma_{H,k+i,\ell}}\right] \alpha_i^4 \alpha_\ell^4 (\delta_1 + \hat{\delta}_1)^2 \right.$$

$$+ \left( \begin{array}{l} c^2 \left[\dfrac{1}{\bar{H}_{yy}}\right] \alpha_k^4 (\delta_2+\hat{\delta}_2)^2 \Big|_{n\neq\ell} \\ \dfrac{c^2}{2} \dfrac{\alpha_k^4}{\bar{H}_{yy}} (\delta_2+\hat{\delta}_2)^2 \Big|_{n=\ell} \end{array} \right) + \frac{c^2}{4} \left[\frac{\alpha_{n+\ell}^4 (1+\delta_{n,\ell})}{\gamma_{H,2k,n-\ell}} + \right.$$

$$\left. + \frac{\alpha_{n-\ell}^4}{\gamma_{H,2k,n+\ell}}\right] \alpha_k^4 (\delta_2+\hat{\delta}_2)^2 + \delta_{i,2k} \left\langle 2c \left[\frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}}\right] \alpha_i^2 \alpha_\ell^2 (\delta_1+\hat{\delta}_1)^2 + \right.$$

$$\left. + c\left[\frac{1+\bar{Q}_{xx}\alpha_i^2}{\bar{H}_{xx}}\right] \alpha_\ell^2 \hat{\delta}_1 \right\rangle - \delta_{n,\ell}\delta_{m,k} \left\langle 4c^2 \left[\frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}}\right] \alpha_k^2 \alpha_\ell^4 (\delta_2+\hat{\delta}_2)(\delta_2+2\hat{\delta}_2) + \right.$$

$$\left. + c^2 \left[\frac{\alpha_{2k}^2+\gamma_{Q,2k,n-\ell}}{\gamma_{H,2k,n-\ell}}\right] \alpha_k^2 \alpha_\ell^4 (\delta_2+\hat{\delta}_2)^2 + \frac{1}{2} c^2 \frac{\alpha_\ell^4}{\bar{H}_{xx}} [1+\bar{Q}_{xx}\alpha_{2\ell}^2](\delta_2+\hat{\delta}_2)^2 \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(k-m)} \left\langle 4c^3 \frac{\alpha_i^2\alpha_\ell^6\alpha_m^2}{\gamma_{H,k-i,\ell}} (2\hat{\delta}_1\hat{\delta}_2+2\hat{\delta}_1\delta_2+2\hat{\delta}_2\delta_1+\delta_1\delta_2)(\delta_2+\hat{\delta}_2) \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(m-k)} \left\langle 4c^3 \frac{\alpha_i^2\alpha_\ell^6\alpha_m^2}{\gamma_{H,k+i,\ell}} (2\hat{\delta}_1\hat{\delta}_2+2\hat{\delta}_1\delta_2+2\hat{\delta}_2\delta_1+\delta_1\delta_2)(\delta_2+\hat{\delta}_2) \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(m+k)} \left\langle 4c^3 \frac{\alpha_i^2\alpha_\ell^6\alpha_m^2}{\gamma_{H^2,m+k,\ell}} (2\hat{\delta}_1\hat{\delta}_2+2\hat{\delta}_1\delta_2+2\hat{\delta}_2\delta_1+\delta_1\delta_2)(\delta_2+\hat{\delta}_2) \right\rangle +$$

$$- \delta_{n,\ell}\delta_{k,2m} \left\langle 4c^4 \frac{1}{\bar{H}_{yy}} \alpha_\ell^4 \alpha_m^4 (\delta_2+\hat{\delta}_2)^4 \right\rangle +$$

$$- \delta_{n,\ell}\delta_{i,2m} \left\langle c^3 \left[\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1}{\gamma_{H,k-i,\ell}}\right] \alpha_i^4 \alpha_\ell^6 (2\hat{\delta}_1\hat{\delta}_2+2\hat{\delta}_1\delta_2+2\hat{\delta}_2\delta_1+\delta_1\delta_2) \right.$$

$$\left. (\delta_2+\hat{\delta}_2) + 2c^2 \frac{\alpha_i^2\alpha_k^2}{\bar{H}_{yy}} (\delta_1+\hat{\delta}_1)(\delta_2+\hat{\delta}_2)^2 \right\rangle +$$

$$+ \delta_{n,\ell} \left\langle 4c^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}\right] \alpha_n^4 \alpha_m^4 \alpha_\ell^4 (\delta_2 + \hat{\delta}_2)^4 + \right.$$

$$\left. + \frac{1}{2} c^2 \left[\frac{(1+\bar{Q}_{xx}\alpha_{2m}^2)^2}{\bar{H}_{xx}} + \bar{D}_{xx}\alpha_{2m}^2\right] \alpha_\ell^4 (\delta_2 + \hat{\delta}_2)^2 + \frac{1}{4} c^2 \frac{\alpha_k^4}{\bar{H}_{yy}} (\hat{\delta}_2^2 + 2\delta_2\hat{\delta}_2) \right\rangle \Bigg]$$

$$\bar{c}_3 = \frac{3}{4} \bar{\rho} h^3 \ell_\ell^2 R$$

$$\bar{c}_5 = \frac{Eh^2}{R^2} \Bigg[ - \delta_{n,\ell}\delta_{m,k} \left\langle c^2 \frac{(1 + \bar{Q}_{xx}\alpha_{2k}^2)}{\bar{H}_{xx}} \alpha_\ell^4 (\delta_2 + \hat{\delta}_2) + \right.$$

$$+ 6c^2 \frac{\alpha_k^2 + \gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}} \alpha_\ell^4 \alpha_m^2 (\delta_2 + \hat{\delta}_2) +$$

$$\left. + 2c^2 \frac{\alpha_k^2 + \gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}} \alpha_\ell^4 \alpha_m^2 \hat{\delta}_2 \right\rangle +$$

$$- \delta_{n,\ell}\delta_{i,2m} \left\langle \frac{1}{2} c^3 \alpha_i^4 \alpha_\ell^6 \left[\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1}{\gamma_{H,k-i,\ell}}\right] [2\hat{\delta}_1\hat{\delta}_2 + 2\hat{\delta}_1\delta_2 + 2\hat{\delta}_2\delta_1 + \delta_1\delta_2 + \delta_1 + \hat{\delta}_1] + \right.$$

$$\left. + c^3 \frac{\alpha_i^2 \alpha_k^2 \alpha_\ell^2}{\bar{H}_{yy}} (\delta_1 + \hat{\delta}_1)(\delta_2 + \hat{\delta}_2) \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(k-m)} \left\langle 2c^3 \alpha_i^2 \alpha_\ell^6 \alpha_m^2 \frac{1}{\gamma_{H,2m-k,\ell}} (\delta_1 + \hat{\delta}_1)(4\hat{\delta}_2 + 3\delta_2) \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(m-k)} \left\langle 2c^3 \alpha_i^2 \alpha_\ell^6 \alpha_m^2 \frac{1}{\gamma_{H,2m-k,\ell}} (\delta_1 + \hat{\delta}_1)(4\hat{\delta}_2 + 3\delta_2) \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(m+k)} \left\langle 2c^3 \alpha_i^2 \alpha_\ell^6 \alpha_m^2 \frac{1}{\gamma_{H,2m+k,\ell}} (\delta_1+\hat{\delta}_1)(4\hat{\delta}_2+3\delta_2) \right\rangle +$$

$$- \delta_{n,\ell}\delta_{k,2m} \left\langle 16c^4 \alpha_m^4 \alpha_\ell^4 \frac{1}{\bar{H}_{yy}} (\delta_2+\hat{\delta}_2)^3 \right\rangle +$$

$$+ \delta_{n,\ell} \left\langle 16c^4 \alpha_\ell^8 \alpha_m^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}\right] (\delta_2+\hat{\delta}_2)^3 + 2c^4 \frac{\alpha_k^2\alpha_\ell^4\alpha_m^2}{\bar{H}_{yy}} (\delta_2+\hat{\delta}_2)^3 + \right.$$

$$\left. + \frac{1}{2} c^2 \alpha_\ell^4 \left[\frac{(1+\bar{Q}_{xx}\alpha_{2m}^2)}{\bar{H}_{xx}} + \bar{D}_{xx}\alpha_{2m}^4\right] (\delta_2+\hat{\delta}_2) - 4\lambda\alpha_m^2\alpha_\ell^4 c^2 (\delta_2+\hat{\delta}_2) \right\rangle \Bigg]$$

$$\bar{c}_6 = \frac{Eh^2}{R^2} \Bigg[ \delta_{n,\ell} \left\langle \frac{1}{2} \frac{c^2\alpha_\ell^4}{\bar{H}_{xx}} (\delta_2+\hat{\delta}_2) + \frac{1}{2} \frac{c^2\alpha_k^4}{\bar{H}_{yy}} (\delta_2+\hat{\delta}_2) - 2\lambda\alpha_m^2\alpha_\ell^4 c^2 (\delta_2+\hat{\delta}_2) + \right.$$

$$+ 2c^4 \alpha_n^4 \alpha_\ell^4 \alpha_m^4 \left[\frac{1}{\gamma_{H,2m+k,n}} + \frac{1}{\gamma_{H,2m-k,n}}\right] (\delta_2+\hat{\delta}_2)^3 +$$

$$\left. + \frac{1}{4} c^2 \alpha_\ell^4 \left[\frac{(1+\bar{Q}_{xx}\alpha_{2m}^2)^2}{\bar{H}_{xx}} + \bar{D}_{xx}\alpha_{2m}^4\right] (\delta_2+\hat{\delta}_2) \right\rangle +$$

$$- \delta_{n,\ell}\delta_{m,k} \left\langle \frac{1}{2} c^2 \alpha_\ell^4 \frac{(1+\bar{Q}_{xx}\alpha_{2m}^2)}{\bar{H}_{xx}} (\delta_2+\hat{\delta}_2) + c^2 \alpha_\ell^4 \alpha_m^2 \frac{(\alpha_k^2+\gamma_{Q,k,\ell})}{\gamma_{H,k,\ell}} (4\hat{\delta}_2+3\delta_2) \right\rangle +$$

$$- \delta_{n,\ell}\delta_{i,2m} \left\langle \frac{1}{4} c^3 \alpha_i^4 \alpha_\ell^6 \left[\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1}{\gamma_{H,k-i,\ell}}\right] (4\hat{\delta}_1\delta_2+4\delta_1\hat{\delta}_2+4\hat{\delta}_1\hat{\delta}_2+3\delta_1\delta_2) \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(m+k)} \left\langle c^3 \alpha_i^2 \alpha_m^2 \alpha_\ell^6 \frac{1}{\gamma_{H,k-i,\ell}} (4\hat{\delta}_1\delta_2+4\delta_1\hat{\delta}_2+4\hat{\delta}_1\hat{\delta}_2+3\delta_1\delta_2) \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(k-m)} \left\langle c^3 \alpha_i^2 \alpha_m^2 \alpha_\ell^6 \frac{1}{\gamma_{H,k-i,\ell}} (4\hat{\delta}_1\delta_2+4\delta_1\hat{\delta}_2+4\hat{\delta}_1\hat{\delta}_2+3\delta_1\delta_2) \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(m-k)} \left\langle c^3 \alpha_i^2 \alpha_m^2 \alpha_\ell^6 \frac{1}{\gamma_{H,k+i,\ell}} (4\hat{\delta}_1\delta_2+4\delta_1\hat{\delta}_2+4\hat{\delta}_1\hat{\delta}_2+3\delta_1\delta_2) \right\rangle +$$

$$- \delta_{n,\ell}\delta_{k,2m} \left\langle 2c^4 \alpha_\ell^8 \alpha_m^4 \frac{1}{\gamma_{H,2m-k,\ell}} (\delta_2+\hat{\delta}_2)^3 \right\rangle \Bigg]$$

$$\bar{c}_7 = \frac{Eh^2}{R^2} \Bigg( \delta_{n,\ell} \left\langle \frac{1}{4} c^2 \left[\frac{\alpha_k^4}{\bar{H}_{yy}} + \frac{\alpha_\ell^4}{\bar{H}_{xx}}\right] (\delta_2+\hat{\delta}_2) \right\rangle + \delta_{n,3\ell} \left\langle \frac{3}{4} \frac{c^2 \alpha_k^4}{\bar{H}_{yy}} (\delta_2+\hat{\delta}_2) \right\rangle \Bigg]$$

$$\bar{c}_8 = \frac{Eh^2}{R^2} \Bigg( \delta_{n,\ell} \left\langle 8c^4 \alpha_\ell^8 \alpha_m^4 \left[\frac{3}{\gamma_{H,2m+k,\ell}} + \frac{3-\delta_{k,2m}}{\gamma_{H,2m-k,\ell}}\right] (\delta_2+\hat{\delta}_2)^2 \right\rangle +$$

$$- \delta_{n,\ell}\delta_{k,2m} \left\langle \frac{c^4}{2} \frac{\alpha_k^4 \alpha_\ell^4}{\bar{H}_{yy}} (\delta_2+\hat{\delta}_2)^2 \right\rangle \Bigg]$$

$$\bar{c}_9 = \frac{Eh^2}{R^2} \Bigg( \frac{c^2}{4} \left[\frac{\alpha_\ell^4}{\bar{H}_{xx}} + \frac{\alpha_k^4}{\bar{H}_{yy}}\right] - 2\lambda \alpha_m^2 \alpha_\ell^4 + \frac{c^2 \alpha_\ell^4}{4} \left[\frac{(1+\bar{Q}_{xx}\alpha_{2m}^2)^2}{\bar{H}_{xx}} + \bar{D}_{xx}\alpha_{2m}^4\right] +$$

$$+ 2c^4 \alpha_n^4 \alpha_m^4 \alpha_\ell^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}\right] (\delta_2+\hat{\delta}_2)^2 +$$

$$- \delta_{i,2m} \left\langle c^3 \alpha_\ell^6 \alpha_i^4 \left[\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1}{\gamma_{H,k-i,\ell}}\right] (\delta_1+\hat{\delta}_1) \right\rangle +$$

$$+ \delta_{i,2(m+k)} \left\langle 4c^3 \alpha_i^2 \alpha_\ell^6 \alpha_m^2 \left[\frac{1}{\gamma_{H,2m+k,\ell}}\right] (\delta_1+\hat{\delta}_1) \right\rangle +$$

$$+ \delta_{i,2(k-m)} \left\langle 4c^3 \alpha_i^2 \alpha_\ell^6 \alpha_m^2 \left[\frac{1}{\gamma_{H,2m-k,\ell}}\right] (\delta_1+\hat{\delta}_1) \right\rangle +$$

$$+ \delta_{i,2(m-k)} \left\langle 4c^3 \alpha_i^2 \alpha_\ell^6 \alpha_m^2 \left[\frac{1}{\gamma_{H,2m-k,\ell}}\right] (\delta_1+\hat{\delta}_1) \right\rangle +$$

$$- \delta_{m,k} \left\langle \frac{c^2}{2} \alpha_\ell^4 \frac{(1+\bar{Q}_{xx}\alpha_{2m}^2)}{\bar{H}_{xx}} + 4c^2 \alpha_k^2 \alpha_\ell^4 \left[\frac{\alpha_k^2 + \gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}}\right] \right\rangle +$$

$$- \delta_{k,2m} \left\langle 2c^4 \alpha_n^4 \alpha_\ell^4 \alpha_m^4 \frac{1}{\gamma_{H,2m-k,n}} (\delta_2 + \hat{\delta}_2)^2 \right\rangle +$$

$$- \delta_{n,\ell} \delta_{k,2m} \left\langle \frac{3}{4} c^4 \alpha_k^4 \alpha_\ell^4 \frac{1}{\bar{H}_{yy}} (\delta_2 + \hat{\delta}_2)^2 \right\rangle +$$

$$+ \delta_{n,\ell} \left\langle 12c^4 \alpha_\ell^8 \alpha_m^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}\right] (\delta_2 + \hat{\delta}_2)^2 \right\rangle \Bigg]$$

$$\bar{c}_{10} = \frac{Eh^2}{R^2} \Bigg[ \delta_{n,\ell} \left\langle 8c^4 \alpha_\ell^8 \alpha_m^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}\right] (\delta_2 + \hat{\delta}_2) + \right.$$

$$\left. + 4c^4 \alpha_\ell^8 \alpha_m^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1-\delta_{k,2m}}{\gamma_{H,2m-k,\ell}}\right] (\delta_2 + \hat{\delta}_2) \right\rangle +$$

$$- \delta_{n,\ell} \delta_{k,2m} \left\langle \frac{1}{4} \frac{c^4 \alpha_k^4 \alpha_\ell^4}{\bar{H}_{yy}} (\delta_2 + \hat{\delta}_2) + c^4 \frac{\alpha_m^2 \alpha_k^2 \alpha_\ell^4}{\bar{H}_{yy}} (\delta_2 + \hat{\delta}_2) \right\rangle \Bigg]$$

$$\bar{c}_{11} = \frac{Eh^2}{R^2} \Bigg[ \delta_{n,\ell} \left\langle 3c^4 \alpha_\ell^8 \alpha_m^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}\right] (\delta_2 + \hat{\delta}_2) \right\rangle +$$

$$- \delta_{n,\ell} \delta_{k,2m} \left\langle 6c^4 \alpha_\ell^8 \alpha_m^4 \frac{1}{\bar{H}_{yy}} (\delta_2 + \hat{\delta}_2) \right\rangle \Bigg]$$

$$\bar{c}_{12} = \frac{Eh^2}{R^2} \left[ c^4 \alpha_\ell^8 \alpha_m^4 \left[\frac{3}{\gamma_{H,2m+k,\ell}} + \frac{3-\delta_{k,2m}}{\gamma_{H,2m-k,\ell}}\right] - \delta_{k,2m} \left\langle 2c^4 \alpha_m^4 \alpha_\ell^4 \frac{1}{\bar{H}_{yy}} \right\rangle \right]$$

$$\bar{d}_1 = \bar{c}_1$$

$$\bar{d}_2 = \frac{Eh^2}{R^2}\left(\left[\frac{(\alpha_k^2+\gamma_{Q,k,\ell})^2}{\gamma_{H,k,\ell}} + \gamma_{D,k,\ell}\right] - 2\lambda\alpha^2{}_k + \frac{c^2}{4}\,\frac{\alpha_\ell^4}{\bar{H}_{xx}}\,(\hat{\delta}_2+2\hat{\delta}_2\delta_2) +\right.$$

$$+ c^2\alpha_i^2\alpha_\ell^4\left[\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1-\delta_{i,k}}{\gamma_{H,k-i,\ell}}\right](\delta_1 + \hat{\delta}_2)^2 +$$

$$+ \left(\begin{array}{l} c^2\,\dfrac{1-\frac{1}{2}\delta_{n,\ell}}{\bar{H}_{yy}}\,\alpha_k^2(\delta_2+\hat{\delta}_2)^2\Big|_{n\neq\ell} \\[2ex] c^2\,\dfrac{1}{\bar{H}_{yy}}\,\alpha_k^2(\delta_2+\hat{\delta}_2)^2\Big|_{n=\ell}\end{array}\right) +$$

$$+ \frac{c^2}{4}\,\alpha_k^4\left[\frac{\alpha_{n+\ell}^4(1-\delta_{n,\ell})}{\gamma_{H,k,n-\ell}} + \frac{\alpha_{n-\ell}^4}{\gamma_{H,2k,n+\ell}}\right](\delta_2+\hat{\delta}_2)^2 +$$

$$+ \delta_{i,2k}\left\langle 2c\alpha_i^2\alpha_\ell^2\,\frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}}\,(\delta_1+\hat{\delta}_1) + c\alpha_\ell^2\,\frac{1+\bar{Q}_{xx}\alpha_i^2}{\bar{H}_{yy}}\,\hat{\delta}_1\right\rangle +$$

$$\left. - \delta_{n,\ell}\left\langle\frac{c^2}{4}\,\frac{\alpha_k^2}{\bar{H}_{yy}}\,(\hat{\delta}_2^2+2\delta_2\hat{\delta}_2)\right\rangle\right)$$

$$\bar{d}_3 = \frac{3}{4}\,\bar{\rho}h^3\,\ell_\ell^2 R$$

$$\bar{d}_5 = \frac{Eh^2}{R^2}\left(\delta_{n,\ell}\left\langle\frac{c^2}{2}\left[\frac{a_k^4}{\bar{H}_{yy}} + \frac{\alpha_\ell^4}{\bar{H}_{xx}}\right](\delta_2+\hat{\delta}_2) +\right.\right.$$

$$+ 4c^4\alpha_n^4\alpha_m^4\alpha_\ell^4\left[\frac{1}{\gamma_{H,2m+k,n}} + \frac{1}{\gamma_{H,2m-k,n}}\right](\delta_2+\hat{\delta}_2)^3 +$$

$$\left. + \frac{1}{2}\,c^2\alpha_\ell^4\left[\frac{(1+\bar{Q}_{xx}\alpha_{2m}^2)^2}{\bar{H}_{xx}} + \bar{D}_{xx}\alpha_{2m}^4\right] - 2\lambda\,\alpha_m^2\alpha_\ell^4(\delta_2+\hat{\delta}_2)\right\rangle +$$

$$- \delta_{n,\ell}\delta_{i,2m} \left\langle \frac{c^3}{2} \alpha_i^4 \alpha_\ell^6 \left[\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1}{\gamma_{H,k-i,\ell}}\right][4\delta_1\hat{\delta}_2+4\hat{\delta}_1\delta_2+4\hat{\delta}_1\hat{\delta}_2+3\delta_1\delta_2] + \right.$$

$$\left. + \frac{c^3 \alpha_i^2 \alpha_k^2 \alpha_\ell^2}{\bar{H}_{yy}} (\delta_1\delta_2+\hat{\delta}_1\hat{\delta}_2+\delta_1\hat{\delta}_2+\hat{\delta}_1\delta_2) \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(m+k)} \left\langle 2c^3 \alpha_i^2 \alpha_\ell^6 \alpha_m^2 \left[\frac{1}{\gamma_{H,k-i,\ell}}\right] [4\hat{\delta}_1\hat{\delta}_2+4\hat{\delta}_1\delta_2+4\delta_1\hat{\delta}_2+3\delta_1\delta_2] \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(k-m)} \left\langle 2c^3 \alpha_i^2 \alpha_\ell^6 \alpha_m^2 \left[\frac{1}{\gamma_{H,2m-k,\ell}}\right] [4\hat{\delta}_1\hat{\delta}_2+4\hat{\delta}_1\delta_2+4\delta_1\hat{\delta}_2+3\delta_1\delta_2] \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(m-k)} \left\langle 2c^3 \alpha_i^2 \alpha_\ell^6 \alpha_m^2 \left[\frac{1}{\gamma_{H,2m+k,\ell}}\right] [4\hat{\delta}_1\hat{\delta}_2+4\hat{\delta}_1\delta_2+4\delta_1\hat{\delta}_2+3\delta_1\delta_2] \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{m,k} \left\langle \frac{1}{2} c^2 \frac{\alpha_{2k}^2+\gamma_{Q,k,n-\ell}}{\gamma_{H,2k,n-\ell}} \alpha_k^2 \alpha_{n+\ell}^2 \alpha_\ell^2 (\delta_2+\hat{\delta}_2) + \right.$$

$$+ \frac{c^2}{2} \frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}} \alpha_k^2 \alpha_{n+\ell}^2 \alpha_\ell^2 (\delta_2+\hat{\delta}_2) +$$

$$+ 2c^2 \frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}} \alpha_\ell^4 \alpha_k^2 \hat{\delta}_2 + 4c^2 \frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}} \alpha_m^2 \alpha_\ell^4 (\delta_2+\hat{\delta}_2) +$$

$$\left. + \frac{c^2}{2} \frac{1+\bar{Q}_{xx}\alpha_{2m}^2}{\bar{H}_{xx}} \alpha_\ell^4 (\delta_2+\hat{\delta}_2) \right\rangle +$$

$$- \delta_{n,\ell}\delta_{k,2m} \left\langle c^4 \alpha_n^4 \alpha_{2m-2k}^2 \alpha_\ell^4 \alpha_m^2 \left[\frac{1}{\gamma_{H,2m-k,n}}\right] (\delta_2+\hat{\delta}_2)^3 \right\rangle +$$

$$\left. - \delta_{n,3\ell} \left\langle \frac{3c^2}{2} \frac{\alpha_k^4}{\bar{H}_{yy}} (\delta_2+\hat{\delta}_2) \right\rangle \right]$$

$$\bar{d}_6 = \frac{Eh^2}{R^2}\left[\;\delta_{n,\ell}\left\langle 4c^4\alpha_\ell^8\alpha_m^4\;[\frac{3}{\gamma_{H,2m+k,\ell}} + \frac{3-\delta_{k,2m}}{\gamma_{H,2m-k,\ell}}]\;(\delta_2+\hat{\delta}_2)^2\right\rangle + \right.$$

$$\left. - \delta_{n,\ell}\delta_{k,2m}\left\langle \frac{c^4}{2}\;\frac{\alpha_k^4\alpha_\ell^4}{\bar{H}_{yy}}\;(\delta_2+\hat{\delta}_2)^2\right\rangle\right]$$

$$\bar{d}_7 = \frac{Eh^2}{R^2}\left[\;\frac{c^2}{4}\;[\frac{\alpha_\ell^4}{\bar{H}_{xx}} + \frac{\alpha_k^4}{\bar{H}_{yy}}] + 2c^4\alpha_n^4\alpha_m^4\;[\frac{1}{\gamma_{H,2m+k,n}} + \frac{1}{\gamma_{H,2m-k,n}}]\;(\delta_2+\hat{\delta}_2)^2 + \right.$$

$$+ \frac{c^2}{4}\;[\frac{(1+\bar{Q}_{xx}\alpha_{2m}^2)^2}{\bar{H}_{xx}} + \bar{D}_{xx}\alpha_{2m}^4]\;\alpha_\ell^4 - 2\lambda\;\alpha_m^2\alpha_\ell^4 +$$

$$- \delta_{i,2m}\left\langle c^3\;[\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1}{\gamma_{H,k-i,\ell}}]\;\alpha_i^4\alpha_\ell^6(\delta_1+\hat{\delta}_1)\right\rangle +$$

$$+ \delta_{i,2(m+k)}\left\langle 4c^3\alpha_i^2\alpha_\ell^6\alpha_m^2\;[\frac{1}{\gamma_{H,2m+k,\ell}}]\;(\delta_1+\hat{\delta}_1)\right\rangle +$$

$$+ \delta_{i,2(k-m)}\left\langle 4c^3\alpha_i^2\alpha_\ell^6\alpha_m^2\;[\frac{1}{\gamma_{H,k-i,\ell}}]\;(\delta_1+\hat{\delta}_1)\right\rangle +$$

$$+ \delta_{i,2(m-k)}\left\langle 4c^3\alpha_i^2\alpha_\ell^6\alpha_m^2\;[\frac{1}{\gamma_{H,2m-k,\ell}}]\;(\delta_1+\hat{\delta}_1)\right\rangle +$$

$$- \delta_{m,k}\left\langle \frac{c^2}{2}\;\alpha_\ell^4\;\frac{(1+\bar{Q}_{xx}\alpha_{2k}^2)}{\bar{H}_{xx}} + 4c^2\alpha_k^2\alpha_\ell^4\;[\frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}}]\right\rangle +$$

$$\left. - \delta_{k,2m}\left\langle \frac{c^4}{2}\;\alpha_n^4\alpha_m^2\alpha_{2m-2k}^2\alpha_\ell^4\;[\frac{1}{\gamma_{H,2m-k,\ell}}]\right\rangle\right]$$

$$\bar{d}_8 = \frac{Eh^2}{R^2}\left[\;\delta_{n,\ell}\left\langle 4c^3\alpha_\ell^8\alpha_m^4\;[\frac{3}{\gamma_{H,2m+k,\ell}} + \frac{3-\delta_{k,2m}}{\gamma_{H,2m-k,\ell}}]\;(\delta_2+\hat{\delta}_2)\right\rangle + \right.$$

$$- \delta_{n,\ell}\delta_{k,2m} \left\langle \frac{1}{2} c^4 \frac{\alpha_k^4 \alpha_\ell^4}{\bar{H}_{yy}} (\delta_2 + \hat{\delta}_2) \right\rangle \Big]$$

$$\bar{d}_9 = \frac{Eh^2}{R^2} \left( c^4 \alpha_\ell^4 \alpha_m^4 \left[\frac{3}{\gamma_{H,2m+k,\ell}} + \frac{3-\delta_{k,2m}}{\gamma_{H,2m-k,\ell}}\right] - 2\delta_{k,2m}\, c^4 \alpha_\ell^4 \alpha_m^4 \left[\frac{1}{\gamma_{H,2m-k,\ell}}\right] \right)$$

2-A.4 Coefficients of equations (2-2-15) and (2-2-16)

$$\tilde{\beta} = 2 \left\{ \gamma_{D,k,\ell} + \frac{(\alpha_k^2 + \gamma_{Q,k,\ell})^2}{\gamma_{H,k,\ell}} \right\}$$

$$\beta_1 = \frac{3}{16} \left(\frac{h}{R}\right)^2 \ell^4$$

$$\beta_2 = 2 \left[\frac{(\alpha_k^2 + \gamma_{Q,k,\ell})^2}{\gamma_{H,k,\ell}} + \gamma_{D,k,\ell}\right] + \left( \begin{array}{l} 2c^2 \dfrac{1 + \frac{1}{2}\delta_{n,\ell}}{\bar{H}_{yy}} \alpha_k^4 (\delta_2 + \hat{\delta}_2)^2 \Big|_{n \neq \ell} \\ \\ c^2 \dfrac{\alpha_k^4}{\bar{H}_{yy}} (\delta_2 + \hat{\delta}_2)^2 \Big|_{n=\ell} \end{array} \right) +$$

$$+ \frac{c^2}{2} \left[\frac{\alpha_{n+\ell}^4 (1+\delta_{n,\ell})}{\gamma_{H,2k,n-\ell}} + \frac{\alpha_{n-\ell}^4}{\gamma_{H,2k,n+\ell}}\right] \alpha_k^4 (\delta_2 + \hat{\delta}_2)^2 +$$

$$+ \delta_{n,\ell} \left\langle 8c^4 \left[\frac{1}{\gamma_{H,2m+k,n}} + \frac{1}{\gamma_{H,2m-k,n}}\right] \alpha_n^4 \alpha_m^4 \alpha_\ell^4 (\delta_2 + \hat{\delta}_2)^4 + \frac{c^2}{2} \frac{\alpha_k^4}{\bar{H}_{yy}} (\hat{\delta}_2^2 + 2\delta_2 \hat{\delta}_2) + \right.$$

$$\left. + c^2 \left[\frac{(1 + \bar{Q}_{xx} \alpha_{2m}^2)^2}{\bar{H}_{xx}} + \bar{D}_{xx} \alpha_{2m}^4\right] \alpha_\ell^4 (\delta_2 + \hat{\delta}_2)^2 \right\rangle +$$

$$- \delta_{n,\ell}\delta_{m,k} \left\langle 8c^2 \left[\frac{\alpha_k^2 + \gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}}\right] \alpha_k^2 \alpha_\ell^4 (\delta_2 + \hat{\delta}_2)(\delta_2 + 2\hat{\delta}_2) + \right.$$

$$+ c^2 \frac{\alpha_\ell^4}{\bar{H}_{xx}} (1+\bar{Q}_{xx}\alpha_{2m}^2)(\delta_2+\hat{\delta}_2)^2 +$$

$$+ 4c^2 \left[\frac{\alpha_{2k}^2+\gamma_{Q,2k,n-\ell}}{\gamma_{H,2k,n-\ell}} \alpha_k^2\alpha_\ell^2(\delta_2+\hat{\delta}_2)^2\right\rangle +$$

$$- \lambda[4\alpha_k^2 + 4\delta_{n,\ell}c^2\alpha_m^2\alpha_\ell^4(\delta_2+\hat{\delta}_2)^2] + \frac{c^2}{2} \frac{\alpha_\ell^4}{\bar{H}_{xx}} (\hat{\delta}_2+2\delta_2\hat{\delta}_2) +$$

$$+ 2c^2 \left[\frac{1-\delta_{i,k}}{\gamma_{H,k-i,\ell}} + \frac{1}{\gamma_{H,k+i,\ell}}\right] \alpha_i^4\alpha_\ell^4(\delta_1+\hat{\delta}_1)^2 +$$

$$+ \delta_{i,2k} \left[4c \frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}} \alpha_i^2\alpha_\ell^2(\delta_1+\hat{\delta}_1) + 2c \frac{1+\bar{Q}_{xx}\alpha_i^2}{\bar{H}_{xx}} \alpha_\ell^2\hat{\delta}_1\right] +$$

$$+ \delta_{n,\ell}\delta_{i,2(k-m)} \left[8c^3 \frac{\alpha_i^2\alpha_\ell^6\alpha_m^2}{\gamma_{H,k-i,\ell}} (2\hat{\delta}_1\hat{\delta}_2+2\delta_1\hat{\delta}_2+2\hat{\delta}_2\delta_1+\delta_1\delta_2)(\delta_2+\hat{\delta}_2)\right] +$$

$$- \delta_{n,\ell}\delta_{i,2m} \left\langle 2c^3 \left[\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1}{\gamma_{H,k-i,\ell}}\right] \alpha_i^4\alpha_\ell^6(2\hat{\delta}_1\hat{\delta}_2+2\delta_1\hat{\delta}_2+2\hat{\delta}_2\delta_1+\delta_1\delta_2)(\delta_2+\hat{\delta}_2) +\right.$$

$$\left. + 2c^2 \frac{\alpha_i^2\alpha_k^2}{\bar{H}_{yy}} (\delta_1+\hat{\delta}_1)\ (\delta_2+\hat{\delta}_2)^2 \right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(m-k)} \left\langle 8c^3 \frac{1}{\gamma_{H,i+k,\ell}} \alpha_i^2\alpha_\ell^6\alpha_m^2(2\hat{\delta}_1\hat{\delta}_2+2\hat{\delta}_2\delta_1+2\hat{\delta}_1\delta_2+\delta_1\delta_2)(\delta_2+\hat{\delta}_2)\right\rangle +$$

$$+ \delta_{n,\ell}\delta_{i,2(m+k)} \left\langle 8c^3 \frac{1}{\gamma_{H,2m+k,\ell}} \alpha_i^2\alpha_\ell^6\alpha_m^2(2\hat{\delta}_1\hat{\delta}_2+2\hat{\delta}_2\delta_1+2\hat{\delta}_1\delta_2+\delta_1\delta_2)(\delta_2+\hat{\delta}_2)\right\rangle +$$

$$- \delta_{n,\ell}\delta_{k,2m} \left\langle 8c^4 \frac{\alpha_\ell^4\alpha_m^4}{\bar{H}_{yy}} (\delta_2+\hat{\delta}_2)^4\right\rangle$$

$$\beta_3 = \frac{3}{4}\left( \frac{c^2}{2}\left[\frac{\alpha_\ell^4}{\bar{H}_{xx}} + \frac{\alpha_k^4}{\bar{H}_{yy}}\right] + 4c^4\left[\frac{1}{\gamma_{H,2m+k,n}} + \frac{1}{\gamma_{H,2m-k,n}}\right] \alpha_\ell^4\alpha_n^4\alpha_m^4(\delta_2+\hat{\delta}_2)^2 \right. +$$

$$+ \frac{c^2}{2}\left[\frac{(1+\bar{Q}_{xx}\alpha_{2m}^2)^2}{\bar{H}_{xx}} + \bar{D}_{xx}\alpha_{2m}^4\right]\alpha_\ell^4 - 4c^2\lambda\alpha_m^2\alpha_\ell^4 +$$

$$+ \delta_{n,\ell}\left\langle 40c^4\left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}\right]\alpha_\ell^8\alpha_m^4(\delta_2+\hat{\delta}_2)^2 \right. +$$

$$\left. + 8c^4\left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1-\delta_{k,2m}}{\gamma_{H,2m-k,\ell}}\right]\alpha_\ell^8\alpha_m^4(\delta_2+\hat{\delta}_2)^2\right\rangle +$$

$$- \delta_{m,k}\left\langle c^2\,\frac{(1+\bar{Q}_{xx}\alpha_{2k}^2)}{\bar{H}_{xx}}\,\alpha_\ell^4 + 8c^2\left[\frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}}\right]\alpha_k^2\alpha_\ell^4\right\rangle +$$

$$- \delta_{n,\ell}\delta_{k,2m}\left[\frac{5}{2}c^4\,\frac{\alpha_k^4\alpha_\ell^4}{\bar{H}_{yy}}(\delta_2+\hat{\delta}_2)^2\right] + \delta_{i,2(k+m)}\left[\frac{8c^3\alpha_i^2\alpha_\ell^6\alpha_m^2}{\gamma_{H,2m+k,\ell}}(\delta_1+\hat{\delta}_1)\right] +$$

$$- \delta_{i,2m}\left\langle 2c^3\left[\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1}{\gamma_{H,k-i,\ell}}\right]\alpha_i^4\alpha_\ell^6(\delta_1+\hat{\delta}_1)\right\rangle +$$

$$+ \delta_{i,2(k-m)}\left[8c^3\,\frac{1}{\gamma_{H,2m-k,\ell}}\,\alpha_i^2\alpha_\ell^6\alpha_m^2(\delta_1+\hat{\delta}_1)\right] +$$

$$+ \delta_{i,2(m-k)}\left[8c^3\,\frac{1}{\gamma_{H,2m-k,\ell}}\,\alpha_i^2\alpha_\ell^6\alpha_m^2(\delta_1+\hat{\delta}_1)\right] -$$

$$- \delta_{k,2m}\left[4c^4\,\frac{\alpha_m^4\alpha_\ell^4}{\bar{H}_{yy}}(\delta_2+\hat{\delta}_2)\right] +$$

$$\left. + \delta_{i,2(m+k)}\left[8c^3\,\frac{1}{\gamma_{H,2m+k,\ell}}\,\alpha_i^2\alpha_\ell^6\alpha_m^2(\delta_1+\hat{\delta}_1)\right]\right)$$

$$\beta_4 = \frac{1}{3}\Bigg[ \beta_3 - \frac{3}{4}\,\delta_{n,\ell} \Bigg\langle 16c^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}\right] \alpha_\ell^8 \alpha_m^4 (\delta_2+\hat{\delta}_2)^2 +$$

$$+ 8c^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1-\delta_{k,2m}}{\gamma_{H,2m-k,\ell}}\right] \alpha_\ell^8 \alpha_m^4 (\delta_2+\hat{\delta}_2)^2 + 4c^4\, \frac{\alpha_k^2\alpha_m^2}{\bar{H}_{yy}}\, \alpha_\ell^4 (\delta_2+\hat{\delta}_2)^2 \Bigg\rangle +$$

$$+ \frac{3}{4}\,\delta_{n,\ell}\delta_{k,2m} \left[c^4\, \frac{\alpha_k^4\alpha_m^4}{\bar{H}_{yy}}\, (\delta_2+\hat{\delta}_2)^2\right]\Bigg]$$

$$\beta_5 = \frac{1}{8}\Bigg[ 2c^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1-\delta_{k,2m}}{\gamma_{H,2m-k,\ell}}\right] \alpha_\ell^8 \alpha_m^4 +$$

$$+ 4c^4 \left[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}\right] \alpha_\ell^8 \alpha_m^4 - \delta_{k,2m} \left[4c^4\, \frac{\alpha_\ell^4\alpha_m^4}{\bar{H}_{yy}}\right]\Bigg]$$

$$\beta_6 = \beta_1$$

$$\beta_7 = 2\left[\frac{(\alpha_k^2+\gamma_{Q,k,\ell})^2}{\gamma_{H,k,\ell}} + \gamma_{D,k,\ell}\right] - 4\lambda\alpha_k^2 + \left(\begin{array}{l} 2c^2\, \dfrac{1-\frac{1}{2}\delta_{n,\ell}}{\bar{H}_{yy}}\, \alpha_k^4 (\delta_2+\hat{\delta}_2)^2 \Big|_{n\neq\ell} \\ \\ c^2\, \dfrac{\alpha_k^4}{\bar{H}_{yy}}\, (\delta_2+\hat{\delta}_2)^2 \Big|_{n=\ell} \end{array}\right) +$$

$$+ \frac{c^2}{2}\, \frac{\alpha_\ell^4}{\bar{H}_{xx}}\, (\hat{\delta}_2+2\delta_2\hat{\delta}_2) + 2c^2 \left[\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1-\delta_{i,k}}{\gamma_{H,k-i,\ell}}\right] \alpha_i^4\alpha_\ell^4 (\delta_1+\hat{\delta}_1)^2 +$$

$$- \delta_{n,\ell} \left[\frac{c^2}{2}\, \frac{\alpha_k^4}{\bar{H}_{yy}}\, (\hat{\delta}_2^2+2\delta_2\hat{\delta}_2)\right] + \delta_{i,2k}\Big[4c\, \frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}}\, \alpha_i^2\alpha_\ell^2 (\delta_1+\hat{\delta}_1) +$$

$$+ 2c \frac{1+\bar{Q}_{xx}\alpha_i^2}{\bar{H}_{xx}} \alpha_\ell^2 \hat{\delta}_1]$$

$$\beta_8 = \frac{1}{4} \left\{ \frac{c^2}{2} [\frac{\alpha_\ell^4}{\bar{H}_{xx}} + \frac{\alpha_k^4}{\bar{H}_{yy}}] + 4c^4 [\frac{1}{\gamma_{H,2m+k,n}} + \frac{1}{\gamma_{H,2m-k,n}}] \alpha_n^4 \alpha_\ell^4 \alpha_m^4 (\delta_2+\hat{\delta}_2)^2 + \right.$$

$$+ \frac{c^2}{2} [\frac{(1+\bar{Q}_{xx}\alpha_{2m}^2)^2}{\bar{H}_{xx}} + \bar{D}_{xx}\alpha_{2m}^4] \alpha_\ell^4 - 4c^2 \lambda \alpha_m^2 \alpha_\ell^4 +$$

$$+ \delta_{n,\ell} \left\langle 8c^4 [\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}] \alpha_\ell^8 \alpha_m^4 (\delta_2+\hat{\delta}_2)^2 + \right.$$

$$\left. + 16c^4 [\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}] \alpha_\ell^8 \alpha_m^4 (\delta_2+\hat{\delta}_2)^2 \right\rangle +$$

$$- \delta_{n,\ell}\delta_{k,2m} [c^4 \frac{\alpha_k^4\alpha_\ell^4}{\bar{H}_{yy}} (\delta_2+\hat{\delta}_2)^2] - \delta_{m,k}[c^2 \frac{(1+\bar{Q}_{xx}\alpha_{2k}^2)}{\bar{H}_{xx}} \alpha_\ell^4 +$$

$$+ 8c^2 \frac{\alpha_k^2+\gamma_{Q,k,\ell}}{\gamma_{H,k,\ell}} \alpha_k^2\alpha_\ell^4] - \delta_{k,2m}[4c^4 \frac{\alpha_m^4\alpha_\ell^4}{\bar{H}_{yy}} (\delta_2+\hat{\delta}_2)^2] +$$

$$- \delta_{i,2m} \left\langle 2c^3 [\frac{1}{\gamma_{H,k+i,\ell}} + \frac{1}{\gamma_{H,k-i,\ell}}] \alpha_i^4 \alpha_\ell^6 (\delta_1+\hat{\delta}_1) \right\rangle +$$

$$+ \delta_{i,2(m+k)} \left\langle 8c^3 \frac{1}{\gamma_{H,2m+k,\ell}} \alpha_i^2\alpha_\ell^6\alpha_m^2(\delta_1+\hat{\delta}_1) \right\rangle +$$

$$+ \delta_{i,2(m-k)} \left\langle 8c^3 \frac{1}{\gamma_{H,2m-k,\ell}} \alpha_i^2\alpha_\ell^6\alpha_m^2(\delta_1+\hat{\delta}_1) \right\rangle +$$

$$\left. + \delta_{i,2(k-m)} \left\langle 8c^3 \frac{1}{\gamma_{H,2m-k,\ell}} \alpha_i^2\alpha_\ell^6\alpha_m^2(\delta_1+\hat{\delta}_1) \right\rangle \right]$$

$$\beta_9 = 3 \Bigg[ \beta_8 - \frac{1}{4}\,\delta_{n,\ell} \Big\langle 8c^4 \Big[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1-\delta_{k,2m}}{\gamma_{H,2m-k,\ell}}\Big]\, \alpha_\ell^8 \alpha_m^2 (\delta_2 + \hat{\delta}_2)^2 +$$

$$+ 16c^4 \Big[\frac{1}{\gamma_{H,2m+k,\ell}} + \frac{1}{\gamma_{H,2m-k,\ell}}\Big]\, \alpha_\ell^8 \alpha_m^4 (\delta_2 + \hat{\delta}_2)^2 \Big\rangle +$$

$$+ \frac{1}{4}\,\delta_{n,\ell}\delta_{k,2m}\Big[c^4\, \frac{\alpha_k^2 \alpha_\ell^4}{\bar{H}_{yy}}\, (\delta_2 + \hat{\delta}_2)^2\Big] \Bigg]$$

$$\beta_{10} = \beta_5$$

APPENDIX 2-B

## PERIODICITY REQUIREMENT FOR THE CIRCUMFERENTIAL DISPLACEMENT

Since the cylindrical shell is assumed to be complete and circular, the displacements, slope, moments, shears, and stress must satisfy continuity requirements. In particular, the continuity requirement on the circumferential displacement v results in conditions which the functions w and $\hat{\hat{\Phi}}$ must satisfy. The condition for continuity of v in the circumferential direction is:

$$\int_{0}^{2\pi R} \frac{\partial \hat{\hat{v}}}{\partial y}\, dy = 0 \tag{2-B-1}$$

Substituting $w = \hat{\hat{w}}$ into equation (A-3) yields:

$$\hat{\hat{\varepsilon}}_y = \hat{\hat{v}}_{,y} - \frac{\hat{\hat{W}}}{R} + \frac{1}{2}\hat{\hat{W}}^2_{,y} + \hat{\hat{W}}_{,y}\bar{W}_{,y} \tag{2-B-2}$$

Hence:

$$\hat{\hat{v}}_{,y} = \hat{\beta}\,\frac{1-\nu^2}{Eh}\left[-\nu\,\hat{\hat{\Phi}}_{,yy} + (1+\mu_1)\,\hat{\hat{\Phi}}_{,xx}\right] - \nu\hat{\beta}\,X_1\,\hat{\hat{W}}_{,xx} + (1+\mu_1)\,\hat{\beta}\,X_2\,\hat{\hat{W}}_{,yy} +$$

$$+\frac{1}{R}\hat{\hat{W}} - \frac{1}{2}\hat{\hat{W}}^2_{,y} - \hat{\hat{W}}_{,y}(\hat{W}+\bar{W}) \tag{2-B-3}$$

Substituting for $\hat{\hat{\Phi}}$, $\hat{\hat{W}}$, $\hat{W}$ and $\bar{W}$ yields:

(1) Case 1 ($n=\ell$)

$$\hat{\hat{v}}_{,y} = \hat{\beta}\,\frac{1-\nu^2}{Eh}\,(1+\mu_1)\left[-A_5\ell^2_{2k}\cos\ell_{2k}X - A_6\ell^2_{2k}\cos\ell_{2k}X - A_{20}\ell^2_{2m}\cos\ell_{2m}X\right] +$$

$$-\,2Ch\ell^2_m\nu\hat{\beta}X_1\cos\ell_{2m}X + \frac{1}{R}\,Ch\sin^2\ell_m X - \frac{1}{8}\,(A^2+B^2)\,h^2\ell^2_\ell +$$

$$+ \frac{1}{8} (A^2+B^2)\ h^2 \ell_\ell^2 \cos \ell_{2k} X - \frac{1}{4} (\delta_2+\hat{\delta}_2)\ Ah^2 \ell_\ell^2 +$$

$$+ \frac{1}{4} (\delta_2+\hat{\delta}_2)\ h^2 \ell_\ell^2 A \cos \ell_{2k} X \tag{2-B-4}$$

(2) Case 2 ($n \neq \ell$)

$$\hat{\hat{v}}_{,y} = \hat{\beta}\, \frac{1-\nu^2}{Eh} \quad (1+\mu_1)\ [-A_5 \ell_{2k}^2 \cos \ell_{2k} X - A_{20} \ell_{2m}^2 \cos \ell_{2m} X]\ +$$

$$- 2Ch\ell_m^2 \nu\hat{\beta}X_1 \cos \ell_{2m} X + \frac{1}{R}\ Ch \sin^2 \ell_m X - \frac{1}{8} (A^2+B^2)\ h^2 \ell_\ell^2 +$$

$$+ \frac{1}{8} (A^2+B^2)\ h^2 \ell_\ell^2 \cos \ell_{2k} X \tag{2-B-5}$$

Substituting for $\frac{\partial \hat{\hat{v}}}{\partial y}$ and carrying out the integration results in:

(1) Case 1 ($n = \ell$)

$$C = \frac{\ell_\ell^2 Rh}{4} [(A^2+B^2) + 2A(\delta_2+\hat{\delta}_2)] \tag{2-B-6}$$

(2) Case 2 ($n \neq \ell$)

$$C = \frac{\ell_\ell^2 Rh}{4} (A^2+B^2) \tag{2-B-7}$$

The Eq. (2-B-6) and (2-B-7) then can be rewritten

$$C = \frac{\ell_\ell^2 Rh}{4} [(A^2+B^2) + 2\delta_{n,\ell} A(\delta_2+\hat{\delta}_2)] \tag{2-B-8}$$

where

$$\delta_{n,\ell} = \begin{cases} 0 & n \neq \ell \\ 1 & n = \ell \end{cases}$$ is the Kronecker Delta function.

APPENDIX 2-C

## COEFFICIENTS OF AMPLITUDE-FREQUENCY EQUATIONS

### 2-C.1 Coefficients of equations (2-2-26) and (2-2-27)

$$\beta_{n1} = \frac{3}{16}\left(\frac{h}{R}\right)^2 \ell^4$$

$$\beta_{n2} = \left[\frac{m^2\pi^2}{(L/R)^2} + \ell^2\right]^2 \frac{1}{6(1-\nu^2)}\left(\frac{h}{R}\right)^2 + \frac{2}{\left[1 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2} + \frac{1}{4}\frac{m^4\pi^4}{(L/R)^4}\left(\frac{h}{R}\right)^2 \delta_2^2 +$$

$$+ \frac{1}{2}\ell^8 \left(\frac{h}{R}\right)^4 \left\langle \frac{1}{\left[9 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2} + \frac{1}{\left[1 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2} \right\rangle \delta_2^4 +$$

$$+ \frac{\ell^4 m^4 \pi^4}{3(1-\nu^2)}\left(\frac{h}{L}\right)^4 \delta_2^2 - \frac{2m^4\ell^4\pi^4}{\left[1 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2}\left(\frac{h}{L}\right)^2 \delta_2^2 + \frac{1}{4}\ell^4\left(\frac{h}{R}\right)^2 \delta_2^2$$

$$\beta_{n3} = \frac{3}{4}\left[\frac{1}{8}\frac{m^4\pi^4}{(L/R)^4}\left(\frac{h}{R}\right)^2 + \frac{13}{4}\ell^8\left(\frac{h}{R}\right)^4 \left[\frac{1}{\left(9 + \frac{\ell^2 L^2}{m^2\pi^2R^2}\right)^2} + \frac{1}{\left(1 + \frac{\ell^2 L^2}{m^2\pi^2R^2}\right)^2}\right]\delta_2^2 + \right.$$

$$\left. + \frac{\ell^4 m^4\pi^4}{6(1-\nu^2)}\left(\frac{h}{L}\right)^4 - \frac{2\ell^4}{\left[1 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2}\left(\frac{h}{R}\right)^2\right]$$

$$\beta_{n4} = \frac{1}{3}\left[\bar{\beta}_3 - \frac{9}{8}\left[\frac{1}{\left(9 + \frac{\ell^2L^2}{m^2\pi^2R^2}\right)^2} + \frac{1}{\left(1 + \frac{\ell^2L^2}{m^2\pi^2R^2}\right)^2}\right]\ell^8\left(\frac{h}{R}\right)^4\right]$$

$$\beta_{n5} = \frac{3}{64}\ell^8\left(\frac{h}{R}\right)^4\left[\frac{1}{\left[9 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2} + \frac{1}{\left[1 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2}\right]$$

$$\beta_{n6} = \beta_{n1}$$

$$\beta_{n7} = \left[\frac{m^2\pi^2}{(L/R)^2} + \ell^2\right]^2 \frac{1}{6(1-\nu^2)} \left(\frac{h}{R}\right)^2 + \frac{2}{\left[1 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2} + \frac{1}{4}\frac{m^4\pi^4}{(L/R)^4}\left(\frac{h}{R}\right)^2 \delta_2^2$$

$$\beta_{n8} = \frac{1}{4}\left\{\frac{1}{8}\frac{m^4\pi^4}{(L/R)^4}\left(\frac{h}{R}\right)^2 + \frac{7}{4}\ell^8\left(\frac{h}{R}\right)^4\left[\frac{1}{\left(9 + \frac{\ell^2L^2}{m^2\pi^2R^2}\right)^2} + \frac{1}{\left(1 + \frac{\ell^2L^2}{m^2\pi^2R^2}\right)^2}\right]\delta_2^2 + \right.$$

$$\left. + \frac{\ell^4 m^4\pi^4}{6(1-\nu^2)}\left(\frac{h}{L}\right)^4 - \frac{2\ell^4}{\left[1 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2}\left(\frac{h}{R}\right)^2\right\}$$

$$\beta_{n9} = 3\left\{\bar{\beta}_8 - \frac{3}{8}\left[\frac{1}{\left(9 + \frac{\ell^2L^2}{m^2\pi^2R^2}\right)^2} + \frac{1}{\left(1 + \frac{\ell^2L^2}{m^2\pi^2R^2}\right)^2}\right]\ell^8\left(\frac{h}{R}\right)^4\right\}$$

$$\beta_{n10} = \beta_{n5}$$

<u>2-C.2 Coefficients of equations (2-2-28) and (2-2-29)</u>

$$\bar{\beta}_{n1} = \frac{3}{16}\left(\frac{h}{R}\right)^2 \ell^4$$

$$\bar{\beta}_{n2} = \left[\frac{m^2\pi^2}{(L/R)^2} + \ell^2\right]^2 \left(\frac{h}{R}\right)^2 \frac{1}{6(1-\nu^2)} + \frac{1}{2}\frac{m^4\pi^4}{(L/R)^4}\left(\frac{h}{R}\right)^2\delta_2^2 + \frac{2}{\left[1 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2} +$$

$$+ \frac{1}{8}\frac{\pi^4 m^4}{(L/R)^4}\left(\frac{h}{R}\right)^2\delta_2^2\left\langle\frac{(\ell+n)^4}{\left[(\ell-n)^2 + \frac{4\pi^2m^2}{(L/R)^2}\right]^2} + \frac{(\ell-n)^4}{\left[(\ell+n)^2 + \frac{4\pi^2m^2}{(L/R)^2}\right]^2}\right\rangle$$

$$\bar{\beta}_{n3} = \frac{3}{4}\left\{\frac{1}{8}\frac{\pi^4m^4}{(L/R)^4}\left(\frac{h}{R}\right)^2 + \frac{\ell^4}{4}n^4\left(\frac{h}{R}\right)^4\left\langle\frac{1}{9 + \left(\frac{nL}{m\pi R}\right)^{2\,2}} + \right.\right.$$

$$\left.\left. + \frac{1}{1 + \left(\frac{nL}{m\pi R}\right)^{2\,2}}\right\rangle\delta_2^2 + \frac{\ell^4m^4\pi^4}{6(1-\nu^2)}\left(\frac{h}{L}\right)^4 - \frac{2\ell^4}{\left[1 + \left(\frac{\ell L}{m\pi R}\right)^2\right]^2}\left(\frac{h}{R}\right)^2\right\}$$

$$\bar{\beta}_{n4} = \frac{1}{3}\,\bar{\beta}_{n3}$$

$$\bar{\beta}_{n5} = \frac{3}{64}\,\ell^{8}\,\left(\frac{h}{R}\right)^{4} \left\langle \frac{1}{[9 + \frac{\ell^2}{m^2\pi^2}\,(L/R)^2]^2} + \frac{1}{[1 + \frac{\ell^2}{m^2\pi^2}\,(L/R)^2]^2} \right\rangle$$

$$\bar{\beta}_{n6} = \bar{\beta}_{n1}$$

$$\bar{\beta}_{n7} = \bar{\beta}_{n2}$$

$$\bar{\beta}_{n8} = \frac{1}{3}\,\bar{\beta}_{n3}$$

$$\bar{\beta}_{n9} = \bar{\beta}_{n3}$$

$$\bar{\beta}_{n10} = \bar{\beta}_{n5}$$

<u>2-C.3 Coefficients of equation (2-2-19) and (2-2-20)</u>

$$\beta_{e1} = \frac{3}{16}\,\varepsilon$$

$$\beta_{e2} = 2\left[\frac{\varepsilon}{12(1-\nu^2)}\,(\xi^2+1)^2 + \frac{\xi^4}{(\xi^2+1)^2}\right]$$

$$\beta_{e3} = \frac{3}{32}\,\xi^4\varepsilon - \frac{3}{2}\,\varepsilon\,\frac{\xi^4}{(\xi^2+1)^2} + \frac{3}{2}\,\frac{\xi^4\varepsilon^2}{12(1-\nu^2)}$$

$$\beta_{e4} = \frac{1}{3}\,\beta_{e3}$$

$$\beta_{e5} = \frac{3}{64}\,\varepsilon^2\left[\frac{\xi^4}{(9\xi^2+1)^2} + \frac{\xi^4}{(\xi^2+1)^2}\right]$$

APPENDIX 3-A

## COEFFICIENTS OF CHAPTER 3

| | |
|---|---|
| $\bar{\alpha}_1 = \bar{c}_1$ | $\bar{\beta}_1 = \bar{d}_1$ |
| $\bar{\alpha}_2 = ch^2$ | $\bar{\beta}_2 = ch^2$ |
| $\bar{\alpha}_3 = \bar{c}_2$ | $\bar{\beta}_3 = \bar{d}_2$ |
| $\bar{\alpha}_4 = \bar{c}_3$ | $\bar{\beta}_4 = \bar{d}_3$ |
| $\bar{\alpha}_5 = \frac{3}{4} ch^3 \ell_\ell^2 R$ | $\bar{\beta}_5 = \frac{3}{4} ch^2 \ell_\ell^3 R$ |
| $\bar{\alpha}_6 = \bar{c}_4$ | $\bar{\beta}_6 = \bar{d}_4$ |
| $\bar{\alpha}_7 = \bar{c}_5$ | $\bar{\beta}_7 = \bar{d}_5$ |
| $\bar{\alpha}_8 = \bar{c}_6$ | $\bar{\beta}_8 = \bar{d}_6$ |
| $\bar{\alpha}_9 = \bar{c}_7$ | $\bar{\beta}_9 = \bar{d}_7$ |
| $\bar{\alpha}_{10} = \bar{c}_8$ | $\bar{\beta}_{10} = \bar{d}_8$ |
| $\bar{\alpha}_{11} = \bar{c}_9$ | |
| $\bar{\alpha}_{12} = \bar{c}_{10}$ | |
| $\bar{\alpha}_{13} = \bar{c}_{11}$ | |

where $\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_{11}$ and $\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_8$ are the coefficients defined in 2-A.3

APPENDIX 3-B

## COEFFICIENTS OF CHAPTER 3

$$\hat{\alpha}_1 = 100\ \beta_{10}^4$$

$$\hat{\alpha}_2 = 440\beta_{10}^4\bar{A}^2 - 40\beta_{10}^3(\beta_6\Omega^2-\beta_9) - 100\beta_{10}^3(\beta_6\Omega^2-\beta_8)$$

$$\hat{\alpha}_3 = 4\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 25\ \beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 + 40\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) +$$

$$- 40\beta_{10}^3(\Omega^2-\beta_7) + 21\ \gamma^2\Omega^2\beta_6^2\beta_{10}^2 + \bar{A}^2[80\beta_8\beta_{10}^3-128\beta_{10}^3(\beta_6\Omega^2-\beta_9) +$$

$$- 340\beta_{10}^3(\beta_6\Omega^2-\beta_8)] + 780\beta_{10}^4\bar{A}^4$$

$$\hat{\alpha}_4 = 40\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 8\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) - 4\beta_{10}(\beta_6\Omega^2-\beta_9)^2(\beta_6\Omega^2-\beta_8)+$$

$$- 10\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9)+ 2\gamma^2\Omega^2\beta_6\beta_{10}[12\beta_{10}-5\beta_6(\beta_6\Omega^2-\beta_8) - 2\beta_6(\beta_6\Omega^2-\beta_9)+$$

$$+ \beta_6(\beta_9-\beta_8)] + \bar{A}^2[8\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 60\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 128\beta_{10}^3(\Omega^2-\beta_7) +$$

$$+ 88\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 16\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9) - 80\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8) +$$

$$+ 52\gamma^2\Omega^2\beta_6^2\beta_{10}^2] + \bar{A}^4[256\beta_8\beta_{10}^3 - 160\beta_{10}^3(\beta_6\Omega^2-\beta_9) - 408\beta_{10}^3(\beta_6\Omega^2-\beta_8)] +$$

$$+ 720\beta_{10}^4\bar{A}^6$$

$$\hat{\alpha}_5 = [64\beta_{10}^2+\beta_6^2(\beta_9-\beta_8)^2 + 6\beta_6^2\beta_{10}(\beta_7-\Omega^2) - 16\beta_6\beta_{10}(\beta_6\Omega^2-\beta_8) - 16\beta_6\beta_{10}(\beta_9-\beta_8)+$$

$$- 8\beta_6\beta_{10}(\beta_6\Omega^2-\beta_9) + 2\beta_6^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9)]\gamma^2\Omega^2 + (\Omega\gamma\beta_6)^4 + 4\beta_{10}^2(\Omega^2-\beta_7)^2 +$$

$$+ (\beta_6\Omega^2-\beta_9)^2(\beta_6\Omega^2-\beta_8)^2 - 8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) +$$

$$- 10\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)^2 +$$

$$+ \bar{A}^2\{4\beta_6\beta_{10}\gamma^2\Omega^2[3\beta_6\beta_8 + 2\beta_6-(\beta_9-\beta_8) + 4\beta_{10}-\beta_6(\beta_6\Omega^2-\beta_9) - 3\beta_6(\beta_6\Omega^2-\beta_8)] +$$

$$+ 88\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 16\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) - 16\,\beta_8\beta_{10}^2(\Omega^2-\beta_7) +$$

$$- 4\beta_{10}(\beta_6\Omega^2-\beta_9)^2(\beta_6\Omega^2-\beta_8) + 16\,\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) +$$

$$- 12\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9) + 20\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2\} +$$

$$+ \bar{A}^4\{42\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 4\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 16\beta_8^2\beta_{10}^2 + 14\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 +$$

$$- 160\beta_{10}^3(\Omega^2-\beta_7) + 72\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 32\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9) +$$

$$- 176\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8)\} +$$

$$+ \bar{A}^6\{320\beta_8\beta_{10}^3 - 96\beta_{10}^3(\beta_6\Omega^2-\beta_9) - 200\beta_{10}^3(\beta_6\Omega^2-\beta_8)\} + 380\beta_{10}^4\,\bar{A}^8$$

$$\hat{\alpha}_6 = \{2\gamma^2\Omega^2[\beta_6^2(\beta_9-\beta_8)(\beta_7-\Omega^2) - 16\beta_{10}(\beta_6\Omega^2-\beta_8) + 2\beta_6(\beta_9-\beta_8)(\beta_6\Omega^2-\beta_8) +$$

$$- 4\beta_{10}\beta_6(\beta_7-\Omega^2) + 2\beta_6(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) + \beta_6^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)] + 4\beta_6^3\Omega^4\gamma^4 +$$

$$- 4\beta_{10}(\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8) + 2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9)\} +$$

$$+ \bar{A}^2\{4\gamma^2\Omega^2[16\beta_{10}^2 + 3\beta_{10}\beta_6^2(\beta_7-\Omega^2) + \beta_6^2\beta_8(\beta_9-\beta_8) - 2\beta_6\beta_{10}(\beta_6\Omega^2-\beta_8) +$$

$$- 2\beta_6\beta_{10}(\beta_9-\beta_8) - 4\beta_6\beta_8\beta_{10} - 2\beta_6\beta_{10}(\beta_6\Omega^2-\beta_9) - \beta_6^2\beta_8(\beta_6\Omega^2-\beta_8)] +$$

$$+ 8\beta_{10}^2(\Omega^2-\beta_7)^2 - 8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 16\beta_8\beta_{10}(\beta_7-\Omega^2)(\beta_6\Omega^2-\beta_8)+$$

$$- 12\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)^2 - 4\beta_8(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9))\} +$$

$$+ \bar{A}^4\{2\beta_6\beta_{10}\gamma^2\Omega^2[12\beta_6\beta_8 + 3\beta_6(\beta_9-\beta_8) - 4\beta_{10} + 3\beta_6(\beta_6\Omega^2-\beta_8)] +$$

$$+ 72\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 8\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) - 32\beta_8\beta_{10}^2(\Omega^2-\beta_7) +$$

$$+ 16\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 6\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9) - 16\beta_8^2\beta_{10}(\beta_6\Omega^2-\beta_8)+$$

$$+ 24\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2 + 12\beta_{10}(\beta_6\Omega^2-\beta_8)^3\} +$$

$$+ \bar{A}^6\{12\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 32\beta_8^2\beta_{10}^2 - 36\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 96\beta_{10}^3(\Omega^2-\beta_7) +$$

$$+ 24\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 144\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8) - 16\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9))\} +$$

$$\bar{A}^8\{192\beta_8\beta_{10}^3 - 24\beta_{10}^3(\beta_6\Omega^2-\beta_9) - 36\beta_{10}^3(\beta_6\Omega^2-\beta_8))\} + 120\beta_{10}^4\bar{A}^{10}$$

$$\hat{\alpha}_7 = \{\gamma^2\Omega^2[4(\beta_6\Omega^2-\beta_8)^2 + \beta_6^2(\beta_7-\Omega^2)^2] + 4\beta_6^2\gamma^4\Omega^4 + (\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8)^2\} +$$

$$\bar{A}^2\{4\gamma^2\Omega^2[\beta_6^2\beta_8(\beta_7-\Omega^2) - 4\beta_{10}(\beta_6\Omega^2-\beta_8)] - 4\beta_{10}(\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8) +$$

$$- 4\beta_8(\beta_6\Omega^2-\beta_8)^2(\Omega^2-\beta_7))\} +$$

$$+ \bar{A}^4\{2\gamma^2\Omega^2[8\beta_{10}^2 + 2\beta_6^2\beta_8^2 - \beta_6^2(\beta_6\Omega^2-\beta_8)^2 + 3\beta_6^2\beta_{10}(\beta_7-\Omega^2)] +$$

$$+ 4\beta_{10}^2(\Omega^2-\beta_7)^2 + 4\beta_8^2(\beta_6\Omega^2-\beta_8)^2 + 16\beta_8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) +$$

$$- 6\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)^2 - (\beta_6\gamma\Omega)^4 - (\beta_6\Omega^2-\beta_8)^4] +$$

$$+ \bar{A}^6\{4\beta_6\beta_{10}\gamma^2\Omega^2[3\beta_6\beta_8 + 2\beta_6(\beta_6\Omega^2-\beta_8)] + 24\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) +$$

$$- 16\beta_8\beta_{10}^2(\Omega^2-\beta_7) - 16\beta_8^2\beta_{10}(\beta_6\Omega^2-\beta_8) + 12\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2 + 8\beta_{10}(\beta_6\Omega^2-\beta_8)^3\} +$$

$$\bar{A}^8\{\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 16\beta_8^2\beta_{10}^2 - 15\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 48\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8) +$$

$$- 24\beta_{10}^3(\Omega^2-\beta_7)\} +$$

$$+ \bar{A}^{10}\{48\beta_{10}^3\beta_8 - 4\beta_{10}^3(\beta_6\Omega^2-\beta_8)\} + 20\beta_{10}^4\ \bar{A}^{12}$$

$$\hat{\beta}_1 = 20\beta_{10}^4$$

$$\hat{\beta}_2 = \{48\beta_8\beta_{10}^3 - 4\beta_{10}^3(\beta_6\Omega^2-\beta_8) + 120\beta_{10}^4\bar{B}^2$$

$$\hat{\beta}_3 = \{\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 16\beta_8^2\beta_{10}^2 - 15\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 48\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8) +$$

$$-24\beta_{10}^3(\Omega^2-\beta_7)\} + \bar{B}^2\{192\beta_8\beta_{10}^3 - 36\beta_{10}^3(\beta_6\Omega^2-\beta_8) - 24\beta_{10}^3(\beta_6\Omega^2-\beta_9)\} +$$

$$+ 380\beta_{10}^4\bar{B}^4$$

$$\hat{\beta}_4 = \{4\beta_6^2\beta_{10}\gamma^2\Omega^2[3\beta_8+2(\beta_6\Omega^2-\beta_8)] + 24\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) - 16\beta_8\beta_{10}^2(\Omega^2-\beta_7) +$$

$$- 16\beta_8^2\beta_{10}(\beta_6\Omega^2-\beta_8) + 12\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2 + 8\beta_{10}(\beta_6\Omega^2-\beta_8)^3\} +$$

$$+ \bar{B}^2\{12(\beta_6\beta_{10}\gamma\Omega)^2 + 32\beta_8^2\beta_{10}^2 - 36\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 96\beta_{10}^3(\Omega^2-\beta_7) +$$

$$+ 24\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 144\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8) - 16\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9)\} +$$

$$+ \bar{B}^4\{320\beta_8\beta_{10}^3 - 200\beta_{10}^3(\beta_6\Omega^2-\beta_8) - 96\beta_{10}^3(\beta_6\Omega^2-\beta_9)\} + 720\beta_{10}^4\bar{B}^6$$

$$\hat{\beta}_5 = \{\gamma^2\Omega^2[16\beta_{10}^2 + 4\beta_6^2\beta_8^2 + 6\beta_6^2\beta_{10}(\beta_7-\Omega^2) - 2\beta_6^2(\beta_6\Omega^2-\beta_8)^2] - (\beta_6\gamma\Omega)^4 +$$

$$+ 4\beta_{10}^2(\Omega^2-\beta_7)^2 + 4\beta_8^2(\beta_6\Omega^2-\beta_8)^2 + 16\beta_{10}\beta_8(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) +$$

$$- 6\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) - (\beta_6\Omega^2-\beta_8)^4\} +$$

$$+ \bar{B}^2\{\gamma^2\Omega^2[24\beta_6^2\beta_8\beta_{10} + 6\beta_6^2\beta_{10}(\beta_9-\beta_8) - 8\beta_6\beta_{10}^2 + 6\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_8)] +$$

$$+ 72\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 8\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) - 32\beta_8\beta_{10}^2(\Omega^2-\beta_7) +$$

$$+ 16\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 6\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9) - 16\beta_8^2\beta_{10}(\beta_6\Omega^2-\beta_8) +$$

$$+ 24\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2 + 12\beta_{10}(\beta_6\Omega^2-\beta_8)^3\} +$$

$$+ \bar{B}^4\{42\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 4\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 16\beta_8^2\beta_{10}^2 + 14\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 +$$

$$- 160\beta_{10}^3(\Omega^2-\beta_7) + 72\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 32\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9) +$$

$$- 176\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8)\} +$$

$$+ \bar{B}^6\{196\beta_8\beta_{10}^3 - 408\beta_{10}^3(\beta_6\Omega^2-\beta_8) - 160\beta_{10}^3(\beta_6\Omega^2-\beta_9)\} + 780\beta_{10}^4\bar{B}^8$$

$$\hat{\beta}_6 = \{4\gamma^2\Omega^2[\beta_6^2\beta_8(\beta_7-\Omega^2) - 4\beta_{10}(\beta_6\Omega^2-\beta_8)] - 4\beta_{10}(\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8) +$$

$$- 4\beta_8(\beta_6\Omega^2-\beta_8)^2(\Omega^2-\beta_7)\} +$$

$$+ \bar{B}^2\{4\gamma^2\Omega^2[16\beta_{10}^2 + 3\beta_6^2\beta_{10}(\beta_7-\Omega^2) + \beta_6^2\beta_8(\beta_9-\beta_8) - 2\beta_6\beta_{10}(\beta_6\Omega^2-\beta_8) +$$

$$- 2\beta_6\beta_{10}(\beta_6\Omega^2-\beta_9) - 2\beta_6\beta_{10}(\beta_9-\beta_8) - 4\beta_6\beta_8\beta_{10}-\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_8)] +$$

$$+ 8\beta_{10}^2(\Omega^2-\beta_7)^2 - 8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) + 16\beta_8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)+$$

$$- 12\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) - 4\beta_8(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9)\} +$$

$$+ \bar{B}^4\{4\gamma^2\Omega^2[3\beta_6^2\beta_8\beta_{10}+ 2\beta_6^2\beta_{10}(\beta_9-\beta_8) + 4\beta_6\beta_{10}^2-\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_9) +$$

$$- 3\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_8)] + 88\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 16\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) +$$

$$- 16\beta_8\beta_{10}^2(\Omega^2-\beta_7) - 4\beta_{10}(\beta_6\Omega^2-\beta_9)^2(\beta_6\Omega^2-\beta_8) + 16\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) +$$

$$- 12\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9) + 20\beta_8\beta_{10}(\beta_6\Omega^2-\beta_8)^2\} +$$

$$+ \bar{B}^6\{52\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 8\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 60\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 128\beta_{10}^3(\Omega^2-\beta_7) +$$

$$+ 88\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) - 16\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_9) - 80\beta_8\beta_{10}^2(\beta_6\Omega^2-\beta_8)\} +$$

$$+ \bar{B}^8\{80\beta_8\beta_{10}^3 - 340\beta_{10}^3(\beta_6\Omega^2-\beta_8) - 128\beta_{10}^3(\beta_6\Omega^2-\beta_9)\} + 440\beta_{10}^4\bar{B}^{10}$$

$$\hat{\beta}_7 = \{\gamma^2\Omega^2[4(\beta_6\Omega^2-\beta_8)^2 + \beta_6^2(\beta_7-\Omega^2)^2] + 4\beta_6^2(\gamma\Omega)^4+(\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8)^2\} +$$

$$+ \bar{B}^2\{2\gamma^2\Omega^2[\beta_6^2(\beta_7-\Omega^2)(\beta_9-\beta_8) + 2\beta_6(\beta_6\Omega^2-\beta_8)(\beta_9-\beta_8) - 4\beta_6\beta_{10}(\beta_7-\Omega^2) +$$

$$- 16\beta_{10}(\beta_6\Omega^2-\beta_8) + 2\beta_6(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9) + \beta_6^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)] + 4\beta_6^3(\gamma\Omega)^4 +$$

$$- 4\beta_{10}(\Omega^2-\beta_7)^2(\beta_6\Omega^2-\beta_8) + 2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9)\} +$$

$$+ \bar{B}^4\{\gamma^2\Omega^2[64\beta_{10}^2 + \beta_6^2(\beta_9-\beta_8)^2 + 6\beta_6^2\beta_{10}(\beta_7-\Omega^2) - 16\beta_6\beta_{10}(\beta_6\Omega^2-\beta_8) +$$

$$- 16\beta_6\beta_{10}(\beta_9-\beta_8) - 8\beta_6\beta_{10}(\beta_6\Omega^2-\beta_9) + 2\beta_6^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9)] + \beta_6^4(\gamma\Omega)^4 +$$

$$+ 4\beta_{10}^2(\Omega^2-\beta_7)^2 + (\beta_6\Omega^2-\beta_9)^2(\beta_6\Omega^2-\beta_8)^2 - 10\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) +$$

$$- 8\beta_{10}(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9)\} +$$

$$+ \bar{B}^6\{\gamma^2\Omega^2[2\beta_6^2\beta_{10}(\beta_9-\beta_8) - 24\beta_6\beta_{10}^2 - 10\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_8) - 4\beta_6^2\beta_{10}(\beta_6\Omega^2-\beta_9)] +$$

$$+ 40\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_8) + 8\beta_{10}^2(\Omega^2-\beta_7)(\beta_6\Omega^2-\beta_9) - 4\beta_{10}(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9)^2 +$$

$$- 10\beta_{10}(\beta_6\Omega^2-\beta_8)^2(\beta_6\Omega^2-\beta_9)\} +$$

$$+ \bar{B}^8\{21\beta_6^2\beta_{10}^2\gamma^2\Omega^2 + 4\beta_{10}^2(\beta_6\Omega^2-\beta_9)^2 + 25\beta_{10}^2(\beta_6\Omega^2-\beta_8)^2 - 40\beta_{10}^3(\Omega^2-\beta_7) +$$

$$+ 40\beta_{10}^2(\beta_6\Omega^2-\beta_8)(\beta_6\Omega^2-\beta_9)\} +$$

$$- \bar{B}^{10}\{40\beta_{10}^3(\beta_6\Omega^2-\beta_9) + 100\beta_{10}^3(\beta_6\Omega^2-\beta_8)\} + 100\beta_{10}^4\,\bar{B}^{12}$$

## APPENDIX 3-C DERIVATION OF EQUATIONS (3-3-8) TO (3-3-11)

The equations are derived from equations (3-2-1) and (3-3-2) using the method of averaging.
To apply the method of averaging to equations (3-2-1) and (3-2-2), let

$$A = A_0(t)\cos[\omega t + \phi_0(t)] = A_0 \cos\chi_1 \qquad \text{(C1-1)}$$

$$B = B_0(t)\sin[\omega t + \psi_0(t)] = B_0 \sin\chi_2 \qquad \text{(C1-2)}$$

where $A_0$, $B_0$, $\phi_0$ and $\psi_0$ are assumed to be slowly varying functions of time t, and

$$\chi_1 = \omega t + \phi_0(t)$$

$$\chi_2 = \omega t + \psi_0(t)$$

Taking the derivative of A and B respectively gives

$$\frac{dA}{dt} = \frac{dA_0}{dt}\cos\chi_1 - A_0\omega\sin\chi_1 - A_0\frac{d\phi_0}{dt}\sin\chi_1 \qquad \text{(C2-1)}$$

$$\frac{dB}{dt} = \frac{dB_0}{dt}\sin\chi_2 + B_0\omega\cos\chi_2 + B_0\frac{d\psi_0}{dt}\cos\chi_2 \qquad \text{(C2-2)}$$

Using the assumptions that $A_0$, $B_0$, $\phi_0$ and $\psi_0$ are slowly varying functions of time gives

$$\frac{dA_0}{dt}\cos\chi_1 - A_0\frac{d\phi_0}{dt}\sin\chi_1 = 0 \qquad \text{(C3-1)}$$

$$\frac{dB_0}{dt}\sin\chi_2 + B_0\frac{d\psi_0}{dt}\cos\chi_2 = 0 \qquad \text{(C3-2)}$$

and

$$\frac{dA}{dt} = -A_0\omega\sin\chi_1 \qquad \text{(C4-1)}$$

$$\frac{dB}{dt} = B_0\omega\cos\chi_2 \qquad \text{(C4-2)}$$

The second derivatives $\frac{d^2A}{dt^2}$ and $\frac{d^2B}{dt^2}$ are then computed from equations (C4)

$$\frac{d^2A}{dt^2} = - \frac{dA_0}{dt} \omega\sin \chi_1 - A_0\omega^2\cos \chi_1 - A_0 \frac{d\phi_0}{dt} \omega\cos \chi_1 \tag{C5-1}$$

$$\frac{d^2B}{dt^2} = + \frac{dB_0}{dt} \omega\cos \chi_2 - B_0\omega^2\sin \chi_2 - B_0 \frac{d\phi_0}{dt} \omega\sin \chi_2 \tag{C5-2}$$

Substituting the equations (C5), (C4) and (C1) into equations (3-2-1) and (3-2-2) yields

$$\bar{\alpha}_1\{- \frac{dA_0}{dt} \omega\sin\chi_1 - A_0\omega^2\cos\chi_1 - A_0 \frac{d\phi_0}{dt} \omega\cos\chi_1\} + \bar{\alpha}_2\{-A_0\omega\sin\chi_1\} + \bar{\alpha}_3\{A_0\cos\chi_1\} +$$

$$+ \bar{\alpha}_4 \frac{\ell_\ell^2 Rh}{2}\{(A_0\omega\sin\chi_1)^2 - A_0 \frac{dA_0}{dt}\omega\sin\chi_1\cos\chi_1 - (A_0\omega\cos\chi_1)^2 - A_0^2 \frac{d\phi_0}{dt} \omega\cos^2\chi_1 +$$

$$+ (B_0\omega\cos\chi_2)^2 + B_0 \frac{dB_0}{dt}\omega\sin\chi_2\cos\chi_2 - (B_0\omega\sin\chi_2)^2 - B_0^2 \frac{d\phi_0}{dt}\omega\sin^2\chi_2 +$$

$$+ \delta_{n,\ell}(- \frac{dA_0}{dt} \omega\sin\chi_1 - A_0\omega^2\cos\chi_1 - A_0 \frac{d\phi_0}{dt}\omega\cos\chi_1)(\delta_2+\hat{\delta}_2)\} \{A_0\cos\chi_1 +$$

$$+ \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\} +$$

$$+ \bar{\alpha}_5 \frac{\ell_\ell^2 Rh}{2} \{- A_0^2\omega\sin\chi_1\cos\chi_1 + B_0^2\omega\cos\chi_2\sin\chi_2 - A_0\omega\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\chi_1\}$$

$$\{A_0\cos\chi_1 + \delta_{n,\ell} (\delta_2+\hat{\delta}_2)\} +$$

$$+ \bar{\alpha}_6 A_0^2\cos^2\chi_1 + \bar{\alpha}_7 \{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\} + \bar{\alpha}_8 \{(A_0\cos\chi_1)^2 +$$

$$- (B_0\sin\chi_2)^2\} + \bar{\alpha}_9 (A_0\cos\chi_1)^3 + \bar{\alpha}_{10} \{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\} A_0\cos\chi_1 +$$

$$+ \bar{\alpha}_{11} \{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\} (A_0\cos\chi_1)^2 + \bar{\alpha}_{12} \{(A_0\cos\chi_1)^2 +$$

$$+ (B_0\sin\chi_2)^2\}^2 + \bar{\alpha}_{13}\{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\}^2 A_0\cos\chi_1 = F_D\cos(\chi_1-\phi_0)$$

(C6-1)

and

$$\bar{\beta}_1\{+ \frac{dB_0}{dt}\omega\cos\chi_2 - B_0\omega^2\sin\chi_2 - B_0\frac{d\psi_0}{dt}\omega\sin\chi_2\} + \bar{\beta}_2\{B_0\omega\cos\chi_2\} + \bar{\beta}_3\{B_0\sin\chi_2\} +$$

$$+ \bar{\beta}_4\frac{\ell_\ell^2 Rh}{2}\{(A_0\omega\sin\chi_1)^2 - A_0\frac{dA_0}{dt}\omega\sin\chi_1\cos\chi_1 - (A_0\omega\cos\chi_1)^2 +$$

$$- A_0^2\frac{d\phi_0}{dt}\omega\cos^2\chi_1 + B_0^2\omega^2\cos^2\chi_2 + B_0\frac{dB_0}{dt}\omega\sin\chi_2\cos\chi_2 - (B_0\omega\sin\chi_2)^2 +$$

$$- B_0^2\frac{d\psi_0}{dt}\omega\sin^2\chi_2 + \delta_{n,\ell}(-\frac{dA_0}{dt}\omega\sin\chi_1 - (A_0\omega^2\cos\chi_1 - A_0\frac{d\phi_0}{dt}\omega\cos\chi_1)(\delta_2+\hat{\delta}_2))\}$$

$$B_0\sin\chi_2 +$$

$$+ \bar{\beta}_5\frac{\ell_\ell^2 Rh}{2}\{-A_0^2\omega\sin\chi_1\cos\chi_1 + B_0^2\omega\sin\chi_2\cos\chi_2 - A_0\omega\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\chi_1\}B_0\sin\chi_2 +$$

$$+ \bar{\beta}_6 A_0B_0\sin\chi_2\cos\chi_1 + \bar{\beta}_7 A_0^2B_0\sin\chi_2\cos\chi_1^2 + \bar{\beta}_8\{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\}$$

$$B_0\sin\chi_2$$

$$+ \bar{\beta}_9\{(A_0\cos\chi_1)^2 + (B_0\sin\chi_2)^2\}A_0B_0\sin\chi_2\cos\chi_1 + \bar{\beta}_{10}\{(A_0\cos\chi_1)^2 +$$

$$+ (B_0\sin\chi_2)^2\}B_0\sin\chi_2 = 0 \qquad \text{(C6-2)}$$

Both sides of equation (C6-1) are multiplied by $\cos\chi_1$, and the results are added to equation (C3-1) after the latter has been multiplied by $\omega\sin\chi_1$. This procedure yields

$$\bar{\alpha}_1\{- A_0\omega^2\cos^2\chi_1 - A_0\frac{d\phi_0}{dt}\omega\} + \bar{\alpha}_2\{- A_0\omega\sin\chi_1\cos\chi_1\} + \bar{\alpha}_3\{A_0\cos^2\chi_1\} +$$

$$+ \bar{\alpha}_4 \frac{\ell_\ell^2 Rh}{2} \{A_0^3\omega^2\sin^2\chi_1\cos^2\chi_1 - A_0^2 \frac{dA_0}{dt} \omega\sin\chi_1\cos^3\chi_1 - A_0^3\omega^2\cos^4\chi_1 - A_0^3 \frac{d\phi_0}{dt}$$

$$\omega\cos^4\chi_1 +$$

$$+ B_0^2A_0\omega^2\cos^2\chi_1\cos^2\chi_2 + A_0B_0 \frac{dB_0}{dt} \omega\sin\chi_2\cos\chi_2\cos^2\chi_1 - A_0B_0^2\omega^2\sin^2\chi_2\cos^2\chi_1 +$$

$$+ B_0^2A_0 \frac{d\psi_0}{dt} \omega\sin^2\chi_2\cos^2\chi_1 + \delta_{n,\ell}\{-A_0 \frac{dA_0}{dt} \omega\sin\chi_1\cos^2\chi_1 - A_0^2\omega^2\cos^3\chi_1 +$$

$$- A_0^2 \frac{d\phi_0}{dt} \omega\cos^3\chi_1\}(\delta_2+\hat{\delta}_2) + \delta_{n,\ell}\{A_0^2\omega^2\sin^2\chi_1\cos\chi_1 - A_0 \frac{dA_0}{dt} \omega\sin\chi_1\cos^2\chi_1 +$$

$$- A_0^2\omega^2\cos^3\chi_1 - A_0^2 \frac{d\phi_0}{dt} \omega\cos^3\chi_1 + B_0^2\omega^2\cos^2\chi_2\cos\chi_1 + B_0 \frac{dB_0}{dt}\omega\sin\chi_2\cos\chi_2\cos\chi_1 +$$

$$- B_0^2\omega^2\sin^2\chi_2\cos\chi_1 - B_0^2 \frac{d\psi_0}{dt}\omega\sin^2\chi_2\cos\chi_1 - \frac{dA_0}{dt} \omega\sin\chi_1\cos\chi_1 - A_0\omega^2\cos^2\chi_1 +$$

$$- A_0 \frac{d\phi_0}{dt} \omega\cos^2\chi_1\} \ (\delta_2+\hat{\delta}_2)^2\} +$$

$$+ \bar{\alpha}_5 \frac{\ell_\ell^2 Rh}{4} \{- A_0^3\omega\sin\chi_1\cos^3\chi_1 + A_0B_0^2\omega\sin\chi_2\cos\chi_2\cos^2\chi_1 - \delta_{n,\ell} A_0^2\omega\sin\chi_1\cos^2\chi_2$$

$$(\delta_2+\hat{\delta}_2) +$$

$$+ \delta_{n,\ell}\{-A_0^2\omega\sin\chi_1\cos^2\chi_1 + B_0^2\omega\sin\chi_2\cos\chi_2\cos\chi_1 - A_0\omega(\delta_2+\hat{\delta}_2)\sin\chi_1\cos\chi_1\}$$

$$(\delta_2+\hat{\delta}_2)\} +$$

$$+ \bar{\alpha}_6 A_0^2\cos^3\chi_1 + \bar{\alpha}_7 \{A_0^2\cos^3\chi_1 + B_0^2\sin^2\chi_2\cos\chi_1\} + \bar{\alpha}_8 \{A_0^2\cos^3\chi_1 +$$

$$- B_0^2\sin^2\chi_2\cos\chi_1\} + \bar{\alpha}_9\ A_0^3\cos^4\chi_1 + \bar{\alpha}_{10}\{A_0^3\cos^4\chi_1 + A_0B_0^2\sin^2\chi_2\cos^2\chi_1\} +$$

$$+ \bar{\alpha}_{11}\{A_0^4\cos^5\chi_1 + A_0^2B_0^2\sin^2\chi_2\cos^3\chi_1\} + \bar{\alpha}_{12}\{A_0^2\cos^2\chi_1 + B_0^2\sin^2\chi_2\}^2A_0\cos\chi_1 +$$

$$+ \bar{\alpha}_{13}\{A_0^2\cos^2\chi_1 + B_0^2\sin^2\chi_2\}^2A_0^2\cos^2\chi_1 = F_D\cos(\chi_1-\phi_0)\cos\chi_1 \qquad \text{(C7)}$$

At this state of the analysis, this equation is "averaged" by integrating over one period on $\chi_1$ or $\chi_2$. In the integration, $A_0$, $B_0$, $\phi_0$ and $\psi_0$ are approximated by their average values $\bar{A}$, $\bar{B}$, $\bar{\Phi}$ and $\bar{\psi}$. For example:

$$\int_0^{2\pi} A_0(t)\cos^2\chi_1 d\chi_1 \approx \int_0^{2\pi} \bar{A}\cos^2\chi_1 d\chi_1 = \bar{A}\pi$$

$$\int_0^{2\pi} A_0^3(t)\ \frac{d\phi_0}{dt}\ \cos^3\chi_1 d\chi_1 \approx \int_0^{2\pi} \bar{A}^3\ \frac{d\bar{\Phi}}{dt}\ \cos^3\chi_1 d\chi_1 = 0$$

$$\int_0^{2\pi} F_D\cos\phi_0 d\chi_1 = 2\pi\bar{F}_d\cos\bar{\Phi}$$

$$\int_0^{2\pi} A_0B_0^2\cos2(\phi_0-\psi_0)d\chi_1 = 2\pi\bar{A}\bar{B}^2\cos2\bar{\Lambda}$$

where $d\bar{\Phi}/dt$ and $\bar{\Lambda}$ are the average values of $d\phi_0/dt$ and $\phi_0-\psi_0$, respectively. When equation (C7) is averaged in this fashion, it becomes

$$\bar{\alpha}_1\{-2\pi\bar{A}\ \frac{d\bar{\Phi}}{dt}\ \omega - \pi\bar{A}\omega^2\} + \bar{\alpha}_2\ \{0\} + \bar{\alpha}_3\ \{\pi\bar{A}\} + \bar{\alpha}_4\ \frac{\ell_\ell^2 Rh}{2}\ \{-\ \frac{1}{2}\ \pi\bar{A}^3\omega^2 - \frac{3}{4}\ \pi\bar{A}^3\ \frac{d\bar{\Phi}}{dt}\ \omega +$$

$$+ \frac{1}{2}\ \pi\bar{A}\bar{B}^2\omega^2(1+ \frac{1}{2}\ \cos2\bar{\Lambda})\ -\frac{1}{4}\pi\bar{A}\bar{B}\ \frac{d\bar{B}}{dt}\ \omega\sin2\bar{\Lambda} - \frac{1}{2}\ \pi\bar{A}\bar{B}^2\omega^2\ (1-\frac{1}{2}\cos2\bar{\Lambda}) +$$

$$+ \frac{1}{2}\ \pi\bar{A}\bar{B}^2\ \frac{d\bar{\psi}}{dt}\ \omega\ (1- \frac{1}{2}\ \cos2\bar{\Lambda}) + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2\ \{-\pi\bar{A}\omega^2 - \pi\bar{A}\ \frac{d\bar{\Phi}}{dt}\ \omega\} +$$

$$+ \bar{\alpha}_5 \frac{\ell_\ell^2 Rh}{2} \{- \frac{\pi}{4} \bar{A}\bar{B}^2 \omega \sin 2\bar{\Delta}\} + \bar{\alpha}_6 \{0\} + \bar{\alpha}_7 \{0\} + \bar{\alpha}_8 \{0\} + \bar{\alpha}_9 \{\frac{3}{4} \pi \bar{A}^3\} +$$

$$+ \bar{\alpha}_{10} \{\frac{3}{4} \pi \bar{A}^3 + \frac{\pi}{2} \bar{A}\bar{B}^2 (1 - \frac{1}{2} \cos 2\bar{\Delta})\} + \bar{\alpha}_{11} \{0\} + \bar{\alpha}_{12} \{0\} +$$

$$+ \bar{\alpha}_{13} \{\frac{5}{8} \pi \bar{A}^5 + \frac{1}{2} \pi \bar{A}^3 \bar{B}^2 (\frac{3}{2} - \cos 2\bar{\Delta}) + \frac{1}{4} \pi \bar{A}\bar{B}^4 (\frac{3}{2} - \cos 2\bar{\Delta})\} = \pi \bar{F}_d \cos\bar{\Phi} \tag{C8}$$

It should be noted that the steady-state vibrations are studied in the present analysis, which means the average values $\bar{A}$ and $\bar{\Phi}$ remain steady (i.e., constant) with time. In this case, the average derivatives $d\bar{A}/dt$, $d\bar{B}/dt$, $d\bar{\Phi}/dt$ and $d\bar{\psi}/dt$ are identically zero, and equation (C8) can be reduced to

$$- \bar{\alpha}_1 \bar{A}\omega^2 + \bar{\alpha}_3 \bar{A} + \bar{\alpha}_4 \frac{\ell_\ell^2 Rh}{2} \{- \frac{1}{2} \bar{A}^3 \omega^2 + \frac{1}{2} \bar{A}\bar{B}^2 \omega^2 (1 + \frac{1}{2} \cos 2\bar{\Delta}) - \frac{1}{2} \bar{A}\bar{B}^2 \omega^2$$

$$(1 - \frac{1}{2} \cos 2\bar{\Delta}) - \delta_{n,\ell} \bar{A}\omega^2 (\delta_2 + \hat{\delta}_2)^2\} + \bar{\alpha}_5 \frac{\ell_\ell^2 Rh}{2} \{- \frac{1}{4} \bar{A}\bar{B}^2 \omega \sin 2\bar{\Delta}\} + \bar{\alpha}_9 \frac{3}{4} \bar{A}^3 +$$

$$+ \bar{\alpha}_{10} \{\frac{3}{4} \bar{A}^3 + \frac{1}{2} \bar{A}\bar{B}^2 (1 - \frac{1}{2} \cos 2\bar{\Delta})\} + \bar{\alpha}_{13} \{\frac{5}{8} \bar{A}^5 + \frac{1}{2} \bar{A}^3 \bar{B}^2 (\frac{3}{2} - \cos 2\bar{\Delta}) + \frac{1}{4} \bar{A}\bar{B}^4$$

$$(\frac{3}{2} - \cos 2\bar{\Delta})\} = \bar{F}_d \cos\bar{\Phi} \tag{C9}$$

In nondimensional form. equation (C9) is

$$- \Omega^2 \bar{A} [1 + \beta_1 (\bar{A}^2 - \bar{B}^2 \cos 2\bar{\Delta} + 2\delta_{n,\ell} (\delta_2 + \hat{\delta}_2)^2)] + \beta_2 \bar{A} - \gamma \Omega \bar{A}\bar{B}^2 \beta_1 \sin 2\bar{\Delta} +$$

$$+ \beta_3 \bar{A}^3 + 2\beta_4 \bar{A}\bar{B}^2 (1 - \frac{1}{2} \cos 2\bar{\Delta}) + \beta_5 \{5\bar{A}^5 + 4\bar{A}^3 \bar{B}^2 (\frac{3}{2} - \cos 2\bar{\Delta}) + 2\bar{A}\bar{B}^4 (\frac{3}{2} - \cos 2\bar{\Delta})\}$$

$$= \bar{F}_D \cos\bar{\Phi} \tag{C10}$$

where

$$\Omega^2 = \frac{2\bar{\rho}R^2\omega^2}{E}$$

$$\bar{F}_D = \frac{2R^2\bar{F}_d}{Eh^2}$$

and

$$\gamma = c/\sqrt{\frac{2E\bar{\rho}}{R^2}}$$

In a similar fashion, equation (3-3-9) is obtained by
(1) Multiplying both sides of equation (C6-1) by sin $\chi_1$
(2) Adding this result to equation (C3-1) after multiplying the latter by $-\omega\cos\chi_1$
(3) Averaging the final equation by the method of averaging.

These manipulations give equation (3-3-9).

Similarly, the equations (3-3-10) and (3-3-11) can be obtained by multiplying both sides of equation (C6-2) by $\sin\chi_2$ and $\cos\chi_2$respectively $\cos\chi_2$, and then using the procedure mentioned above.

APPENDIX 4-A COEFFICIENTS OF EQUATIONS (4-2-1) AND (4-2-2)

$$\alpha_{s1} = (\beta_{15} + \beta_{16} + \beta_{17})/\beta_2$$

$$\alpha_{s2} = (\beta_{16} - \beta_{17})/\beta_2$$

$$\alpha_{s3} = \frac{4}{3}\,\beta_3/\beta_2$$

$$\alpha_{s4} = 4\beta_4/\beta_2$$

$$\alpha_{s5} = \beta_{110}/\beta_2$$

$$\alpha_{s6} = \beta_{111}/\beta_2$$

$$\alpha_{s7} = 8\beta_5/\beta_2$$

$$\beta_{s1} = \beta_7/\beta_2$$

$$\beta_{s2} = \beta_{25}/\beta_2$$

$$\beta_{s3} = 4\beta_8/\beta_2$$

$$\beta_{s4} = \frac{4}{3}\,\beta_9/\beta_2$$

$$\beta_{s5} = \beta_{28}/\beta_2$$

$$\beta_{s6} = 8\beta_{10}/\beta_2$$

where $\beta_2$, $\beta_3$, ..., $\beta_{10}$ are defined in Appendix 2-A.4, whereas $\beta_{15}$, $\beta_{16}$, $\beta_{17}$, $\beta_{25}$, $\beta_{28}$, $\beta_{110}$ and $\beta_{111}$ are defined as follows:

$$\beta_{15} = \frac{2R^2}{Eh^2}\,\bar{c}_5$$

$$\beta_{16} = \frac{2R^2}{Eh^2}\,\bar{c}_6$$

$$\beta_{17} = \frac{2R^2}{Eh^2}\,\bar{c}_7$$

$$\beta_{110} = \frac{2R^2}{Eh^2}\,\bar{c}_{10}$$

$$\beta_{111} = \frac{2R^2}{Eh^2}\,\bar{c}_{11}$$

$$\beta_{25} = \frac{2R^2}{Eh^2}\,\bar{d}_4$$

$$\beta_{28} = \frac{2R^2}{Eh^2}\,\bar{d}_7$$

where $\bar{c}_5$, $\bar{c}_6$, $\bar{c}_7$, $\bar{c}_{10}$, $\bar{c}_{11}$, $\bar{d}_4$ and $\bar{d}_7$ are defined in Appendix 2-A.3.

APPENDIX 4-B COEFFICIENTS OF EQUATIONS (4-2-8) AND (4-2-9)

$$\bar{\alpha}_{s1} = 1 + \frac{3}{8}\,\varepsilon[\bar{A}^2\cos^2\varphi_1 + 2\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\,\bar{A}\,\cos\varphi_1 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2]$$

$$\bar{\alpha}_{s2} = 2\gamma_s - \frac{3}{4}\,\varepsilon\Omega_s\bar{A}^2\sin\varphi_1[\cos\varphi_1 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \frac{3}{4}\,\gamma_s\varepsilon\bar{A}[\bar{A}\cos^2\varphi_1 +$$

$$+ 2\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\cos\varphi_1 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2]$$

$$\bar{\alpha}_{s3} = 1 + \frac{3}{8}\,\varepsilon[\Omega_s^2\bar{A}^2(\sin^2\varphi_1-2\cos^2\varphi_1) + \Omega_s^2\bar{B}^2(\cos^2\varphi_2-\sin^2\varphi_2) +$$

$$- 2\Omega_s^2\bar{A}\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\cos\varphi_1] - \frac{3}{4}\,\gamma_s\varepsilon\Omega_s[\bar{A}^2\sin(2\varphi_1) - \frac{1}{2}\,\bar{B}^2\sin(2\varphi_2) +$$

$$- 2\bar{A}\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\varphi_1] + 2\alpha_{s1}\bar{A}\cos\varphi_1 + 3\alpha_{s3}\bar{A}^2\cos^2\varphi_1 + \alpha_{s4}\bar{B}^2\sin^2\varphi_2 +$$

$$+ \alpha_{s5}[4\bar{A}^3\cos^3\varphi_1 + \bar{B}^2\sin^2\varphi_2 + 2\bar{A}\bar{B}^2\sin^2\varphi_2\cos\varphi_1] +$$

$$+ 2\alpha_{s6}[2\bar{A}^3\cos^3\varphi_1 + \bar{B}^2\sin^2\varphi_2 + 2\bar{A}\bar{B}^2\sin^2\varphi_2\cos\varphi_1] +$$

$$+ \alpha_{s7}[5\bar{A}^4\cos^4\varphi_1 + 6\bar{A}^2\bar{B}^2\sin^2\varphi_2\cos^2\varphi_1 + \bar{B}^4\sin^4\varphi_2]$$

$$\bar{\alpha}_{s4} = \frac{3}{8}\,\varepsilon\bar{B}[\bar{A}\cos\varphi_1\sin\varphi_2 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\varphi_2]$$

$$\bar{\alpha}_{s5} = \frac{3}{4}\,\gamma_s\varepsilon\bar{B}[\bar{A}\cos\varphi_1\sin\varphi_2 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\varphi_2] + \frac{3}{4}\,\varepsilon\Omega_s\bar{B}[\bar{A}\cos\varphi_1\cos\varphi_2 +$$

$$+ \delta_{n,\ell}(\delta_2+\hat{\delta}_2)\cos\varphi_2]$$

$$\bar{\alpha}_{s6} = -\frac{3}{8}\,\varepsilon\Omega_s^2\,\bar{B}\sin\varphi_2[\bar{A}\cos\varphi_1 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \frac{3}{4}\,\varepsilon\gamma_s\Omega_s\bar{B}\cos\varphi_2[\bar{A}\cos\varphi_1 +$$

$$+ \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + 2\alpha_{s2}\bar{B}\sin\varphi_2 + 2\alpha_{s4}\bar{A}\bar{B}\cos\varphi_1\sin\varphi_2 + 2\alpha_{s5}\bar{A}^2\bar{B}\cos^2\varphi_1\sin\varphi_2 +$$

$$+ 4\alpha_{s6}[\bar{B}^3\sin^3\varphi_2 + \bar{A}^2\bar{B}\cos^2\varphi_1\sin\varphi_2] + 4\alpha_{s7}[\bar{A}^3\bar{B}\cos^3\varphi_1\sin\varphi_2 +$$

$$+ \bar{A}\bar{B}^3\sin^3\varphi_2\cos\varphi_1]$$

$$\bar{\beta}_{s1} = 1 + \tfrac{3}{8}\varepsilon\bar{B}^2\sin^2\chi_2$$

$$\bar{\beta}_{s2} = 2\gamma_s + \tfrac{3}{8}\varepsilon\Omega_s\bar{B}^2\sin(2\varphi_2) + \tfrac{3}{4}\gamma_s\varepsilon\bar{B}^2\sin\varphi_2$$

$$\bar{\beta}_{s3} = \beta_7/\beta_2 + \tfrac{3}{8}\varepsilon[\Omega_s^2\bar{A}^2(\sin^2\varphi_1 - \cos^2\varphi_1) + \Omega_s^2\bar{B}^2(\cos^2\varphi_2 - \sin^2\varphi_2) +$$

$$- \Omega_s^2\bar{A}\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\cos\varphi_1] +$$

$$- \frac{3}{4}\,\gamma_s\varepsilon\Omega_s[\frac{1}{2}\,\bar{A}^2\sin(2\varphi_1) - \bar{B}^2\sin(2\varphi_2) + \bar{A}\delta_{n,\ell}(\delta_2+\hat{\delta}_2)\sin\varphi_1] + \beta_{s2}\bar{A}\cos\varphi_1 +$$

$$+ \beta_{s3}\,\bar{A}^2\cos^2\varphi_1 + 3\beta_{s4}\,\bar{B}^2\sin^2\varphi_2 + \beta_{s5}[\bar{A}^3\cos^3\varphi_1 + 3\bar{A}\bar{B}^2\cos\varphi_1\sin^2\varphi_2] +$$

$$+ \beta_{s6}[\bar{A}^4\cos^4\varphi_1 + 5\bar{B}^4\sin^4\varphi_2 + 6\bar{A}^2\bar{B}^2\cos^2\varphi_1\sin^2\varphi_2]$$

APPENDIX 4-C DERIVATION OF EQUATIONS (4-2-8) and (4-2-9)

The details of the derivation are demonstrated here by deriving equation (4-2-8) from equation (4-2-1).
Substituting equations (4-2-4) and (4-2-5) into equation(4-2-1) and keeping only the first order terms of the perturbations in the resulting equation one obtains

$$f_1(\Omega_s,\gamma_s,\bar{A},\bar{B},F_{sD}) + f_2(\Omega_s,\gamma_s,\zeta_1,\zeta_2,\eta_1,\eta_2,\bar{A},\bar{B}) = 0 \qquad \text{(B-1)}$$

where

$$\begin{aligned} f_1 = &- \Omega_s^2\bar{A}\cos\varphi_1 + 2\gamma_s(-\Omega_s\bar{A}\sin\varphi_1) + \bar{A}\cos\varphi_1 + \frac{3}{8}\varepsilon[\Omega_s^2\bar{A}^2\sin^2\varphi_1 - \Omega_s^2\bar{A}^2\cos^2\varphi_1 + \\ &+ \Omega_s^2\bar{B}^2\cos^2\varphi_2 - \Omega_s^2\bar{B}^2\sin^2\varphi_2 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)(-\Omega_s^2\bar{A}\cos\varphi_1)][\bar{A}\cos\varphi_1 + \\ &+ \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \\ &+ \frac{3}{4}\gamma_s\varepsilon[-\Omega_s\bar{A}^2\sin\varphi_1\cos\varphi_1 + \Omega_s\bar{B}^2\sin\varphi_2\cos\varphi_2 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)(-\Omega_s\bar{A}\sin\varphi_1)] \\ &[\bar{A}\cos\varphi_1 + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)] + \alpha_{s1}\bar{A}^2\cos^2\varphi_1 + \alpha_{s2}\bar{B}^2\sin^2\varphi_2 + \alpha_{s3}\bar{A}^3\cos^3\varphi_1 + \\ &+ \alpha_{s4}\bar{A}\bar{B}^2\sin^2\varphi_2\cos\varphi_1 + \alpha_{s5}[\bar{A}^4\cos^4\varphi_1 + \bar{A}^2\bar{B}^2\cos^2\varphi_1\sin^2\varphi_2] + \alpha_{s6}[\bar{A}^4\cos^4\varphi_1 + \\ &+ \bar{B}^4\sin^4\varphi_2 + 2\bar{A}^2\bar{B}^2\cos^2\varphi_1\sin^2\varphi_2] + \alpha_{s7}[\bar{A}^4\cos^4\varphi_1 + \bar{B}^4\sin^4\varphi_2 + \\ &+ 2\bar{A}^2\bar{B}^2\cos^2\varphi_1\sin^2\varphi_2]\bar{B}\sin\varphi_2 - F_{sD}\cos\Omega\tau \end{aligned} \qquad \text{(B-2)}$$

and

$$f_2 = \bar{\alpha}_{s1}\frac{d^2\zeta}{d\tau^2} + \bar{\alpha}_{s2}\frac{d\zeta}{d\tau} + \bar{\alpha}_{s3}\,\zeta + \bar{\alpha}_{s4}\frac{d^2\eta}{d\tau^2} + \bar{\alpha}_{s5}\frac{d\eta}{d\tau} + \bar{\alpha}_{s6}\,\eta \qquad \text{(B-3)}$$

It is evident that the function $f_1$ is identically equal to zero since equations (4-2-4) and (4-2-5) are its solution. Only $f_2$ remains. Equation (4-2-9) can be obtained using a similar procedure.

## APPENDIX 4-D DERIVATION OF EQUATION (4-2-18)

The purpose of this appendix is to give the details of the derivation of equation (4-2-18).
Substituting expressions for the derivatives (4-2-12) to (4-2-15) into equations (4-2-8) and (4-2-9) yields two coupled equations:

$$\bar{\alpha}_{s1}\{-\zeta_1\Omega_s^2\cos\varphi_1 - \zeta_2\Omega_s^2\sin\varphi_1 - \frac{d\zeta_1}{d\tau}\Omega_s\sin\varphi_1 + \frac{d\zeta_2}{d\tau}\Omega_s\cos\varphi_1\} +$$

$$+ \bar{\alpha}_{s2}\{-\zeta_1\Omega_s\sin\varphi_1 + \zeta_2\Omega_s\cos\varphi_1\} + \bar{\alpha}_{s3}\{\zeta_1\cos\varphi_1 + \zeta_2\sin\varphi_1\} +$$

$$+ \bar{\alpha}_{s4}\{\frac{d\eta_1}{d\tau}\Omega_s\cos\varphi_2 - \frac{d\eta_2}{d\tau}\Omega_s\sin\varphi_2 - \eta_1\Omega_s^2\sin\varphi_2 - \eta_2\Omega_s^2\cos\varphi_2\} +$$

$$+ \bar{\alpha}_{s5}\{\eta_1\Omega_s\cos\varphi_2 - \eta_2\Omega_s\sin\varphi_2\} + \bar{\alpha}_{s6}\{\eta_1\sin\varphi_2 + \eta_2\cos\varphi_2\} = 0 \qquad \text{(D-1)}$$

$$\bar{\beta}_{s1}\{\frac{d\eta_1}{d\tau}\Omega_s\cos\varphi_2 - \frac{d\eta_2}{d\tau}\Omega_s\sin\varphi_2 - \eta_1\Omega_s^2\sin\varphi_2 - \eta_2\Omega_s^2\cos\varphi_2\} +$$

$$+ \bar{\beta}_{s2}\{\eta_1\Omega_s\cos\varphi_2 - \eta_2\Omega_s\sin\varphi_2\} + \bar{\beta}_{s3}\{\eta_1\sin\varphi_2 + \eta_2\cos\varphi_2\} +$$

$$+ \bar{\beta}_{s4}\{-\zeta_1\Omega^2{}_s\cos\varphi_1 - \zeta_2\Omega_s^2\sin\varphi_1 - \frac{d\zeta_1}{d\tau}\Omega_s\sin\varphi_1 + \frac{d\zeta_2}{d\tau}\Omega_s\cos\varphi_1\} +$$

$$+ \bar{\beta}_{s5}\{-\zeta_1\Omega_s\sin\varphi_1 + \zeta_2\Omega_s\cos\varphi_1\} + \bar{\beta}_{s6}\{\zeta_1\cos\varphi_1 + \zeta_2\sin\varphi_1\} = 0 \qquad \text{(D-2)}$$

Equation (D-1) is multiplied by $\sin\varphi_1$ and is added to the auxiliary condition (4-2-16) after the latter has been multiplied by $(-\bar{\alpha}_{s1}\Omega_s\cos\varphi_1)$. This procedure yields the following equation:

$$-\bar{\alpha}_{s1}\frac{d\zeta_1}{d\tau}\Omega_s + \bar{\alpha}_{s4}\frac{d\eta_1}{d\tau}\Omega_s\sin\varphi_1\cos\varphi_2 - \alpha_{s4}\frac{d\eta_2}{d\tau}\Omega_s\sin\varphi_1\sin\varphi_2$$

$$= \{\frac{1}{2}\bar{\alpha}_{s1}\Omega_s^2\sin2\varphi_1 + \bar{\alpha}_{s2}\Omega_s\sin^2\varphi_1 - \frac{1}{2}\bar{\alpha}_{s3}\sin2\varphi_1\}\zeta_1 +$$

$$+ \left(\bar{\alpha}_{s1}\Omega_s^2\sin^2\varphi_1 - \frac{1}{2}\bar{\alpha}_{s2}\Omega_s\sin 2\varphi_1 + \bar{\alpha}_{s3}\sin^2\varphi_1\right)\zeta_2 +$$

$$+ \left(\bar{\alpha}_{s4}\Omega_s^2\sin\varphi_2\sin\varphi_1 - \bar{\alpha}_{s5}\Omega_s\sin\varphi_1\cos\varphi_2 - \bar{\alpha}_{s6}\sin\varphi_2\sin\varphi_1\right)\eta_1 +$$

$$+ \left(\bar{\alpha}_{s4}\Omega_s^2\sin\varphi_1\cos\varphi_2 + \bar{\alpha}_{s5}\Omega_s\sin\varphi_1\sin\varphi_2 - \bar{\alpha}_{s6}\sin X_1\cos\varphi_2\right)\eta_2 \qquad \text{(D-3)}$$

At this stage of the analysis, the coefficients $\bar{\alpha}_{s1}$, $\bar{\alpha}_{s2}$, ..., $\bar{\alpha}_{s6}$ are substituted into equation (D-3) and then the resulting equation is "averaged" by integrating $\tau$ from 0 to $2\pi$. This procedure yields

$$m_{11}\frac{\bar{d}\zeta_1}{d\tau} + m_{12}\frac{\bar{d}\zeta_2}{d\tau} + m_{13}\frac{\bar{d}\eta_1}{d\tau} + m_{14}\frac{\bar{d}\eta_2}{d\tau} = n_{11}\bar{\zeta}_1 + n_{12}\bar{\zeta}_2 + n_{13}\bar{\eta}_1 + n_{14}\bar{\eta}_2 \qquad \text{(D-4)}$$

Similarly, if both sides of equation (D-1) are multiplied by $\cos\varphi_1$ and the resulting equation is added to the auxiliary condition (4-2-16) after the latter has been multiplied by $(-\bar{\alpha}_{s1}\sin\varphi_1)$, and then the average procedure mentioned above is used again, one can obtain a second equation as

$$m_{21}\frac{\bar{d}\zeta_1}{d\tau} + m_{22}\frac{\bar{d}\zeta_2}{d\tau} + m_{23}\frac{\bar{d}\eta_1}{d\tau} + m_{24}\frac{\bar{d}\eta_2}{d\tau} = n_{21}\bar{\zeta}_1 + n_{22}\bar{\zeta}_2 + n_{23}\bar{\eta}_1 + n_{24}\bar{\eta}_2 \qquad \text{(D-5)}$$

Applying the same procedure to equation (D-2) yields another two equations.

$$m_{31}\frac{\bar{d}\zeta_1}{d\tau} + m_{32}\frac{\bar{d}\zeta_2}{d\tau} + m_{33}\frac{\bar{d}\eta_1}{d\tau} + m_{34}\frac{\bar{d}\eta_2}{d\tau} = n_{31}\bar{\zeta}_1 + n_{32}\bar{\zeta}_2 + n_{33}\bar{\eta}_1 + n_{34}\bar{\eta}_2 \qquad \text{(D-6)}$$

and

$$m_{41}\frac{\bar{d}\zeta_1}{d\tau} + m_{42}\frac{\bar{d}\zeta_2}{d\tau} + m_{43}\frac{\bar{d}\eta_1}{d\tau} + m_{44}\frac{\bar{d}\eta_2}{d\tau} = n_{41}\bar{\zeta}_1 + n_{42}\bar{\zeta}_2 + n_{43}\bar{\eta}_1 + n_{44}\bar{\eta}_2 \qquad \text{(D-7)}$$

where $\bar{\zeta}_1$, $\bar{\zeta}_2$, $\bar{\eta}_1$ and $\bar{\eta}_2$ are average values of $\zeta_1$, $\zeta_2$, $\eta_1$ and $\eta_2$ respectively and $m_{ij}$ are coefficients defined in Appendix 4-E.

These four linear differential equations can be put in matrix form as follows:

$$[M]\{\phi\} = [N]\left\{\frac{d\phi}{d\tau}\right\} \qquad \text{(D-8)}$$

APPENDIX 4-E COMPONENTS OF MATRIX [M] AND [N]

$$m_{11} = -\Omega_s\{1 + \frac{3}{16}\varepsilon\bar{A}^2 + \frac{3}{8}\varepsilon\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2\}$$

$$m_{12} = m_{21} = 0$$

$$m_{13} = \frac{3}{64}\varepsilon\Omega_s\bar{A}\bar{B}\cos 2\bar{\Delta}$$

$$m_{14} = m_{13}$$

$$m_{22} = -m_{11}$$

$$m_{23} = -\frac{3}{64}\varepsilon\Omega_s\bar{A}\bar{B}\sin 2\bar{\Delta}$$

$$m_{24} = -\frac{3}{32}\varepsilon\Omega_s\bar{A}\bar{B}(1-\frac{1}{2}\cos 2\bar{\Delta})$$

$$m_{31} = -m_{13}$$

$$m_{32} = m_{23}$$

$$m_{33} = \Omega_s\{1 + \frac{3}{16}\varepsilon\bar{B}^2\}$$

$$m_{34} = m_{43} = 0$$

$$m_{41} = -m_{23}$$

$$m_{42} = -m_{24}$$

$$m_{44} = -m_{33}$$

$$n_{11} = \Omega_s\gamma_s\{1 + \frac{3}{8}\varepsilon[-\frac{3}{4}\bar{A}^2 + \frac{1}{4}\bar{B}^2\cos 2\bar{\Delta} + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2]\} - \frac{1}{8}\bar{B}^2\{\frac{3}{4}\varepsilon\Omega_s^2 + \alpha_{s4} +$$

$$+ \alpha_{s7}[3\bar{A}^2 - \bar{B}^2]\}\sin 2\bar{\Delta}$$

$$n_{12} = \frac{3}{8}\bar{A}^2\{\alpha_{s3} - \frac{1}{8}\varepsilon\Omega_s^2\} - \frac{3}{32}\varepsilon\Omega_s^2\bar{B}^2\cos 2\bar{\Delta} + \frac{3}{32}\varepsilon\gamma_s\Omega_s\bar{B}^2\sin 2\bar{\Delta} +$$

$$+ \frac{1}{4}\alpha_{s4}\bar{B}^2(1+\frac{1}{2}\cos 2\bar{\Delta}) + \frac{1}{16}\alpha_{s7}\{5\bar{A}^4 + 6\bar{A}^2\bar{B}^2 + \bar{B}^4(3 + 2\cos 2\bar{\Delta}) +$$

$$- \Omega_s^2\{\frac{1}{2} + \frac{3}{32}\,\varepsilon\bar{A}^2(1-\frac{1}{2}\cos^2\bar{\Delta}) + \frac{3}{16}\,\varepsilon\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2\} + \frac{1}{2}$$

$$n_{13} = -\frac{1}{4}\,\bar{A}\bar{B}\{\alpha_{s4} + \alpha_{s7}[\bar{A}^2 + 2\bar{B}^2] - \frac{3}{4}\,\varepsilon\Omega_s^2\}\sin 2\bar{\Delta} + \frac{3}{16}\,\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\cos 2\Delta$$

$$n_{14} = \frac{1}{4}\,\bar{A}\bar{B}\{\alpha_{s4} + \alpha_{s7}[\bar{A}^2 + \bar{B}^2] - \frac{3}{4}\,\varepsilon\Omega_s^2\}\cos 2\bar{\Delta} + \frac{3}{16}\,\varepsilon\,\gamma_s\Omega_s\bar{A}\bar{B}\sin 2\Delta$$

$$n_{21} = \frac{9}{16}\,\bar{A}^2\{\alpha_{s3} + \frac{1}{4}\,\varepsilon\Omega_s^2\} - \frac{3}{32}\,\varepsilon\Omega_s^2\bar{B}^2\cos 2\bar{\Delta} - \frac{3}{32}\,\varepsilon\Omega_s\gamma_s\bar{B}^2\sin 2\bar{\Delta} +$$

$$+ \frac{1}{4}\,\alpha_{s4}\bar{B}^2(1-\frac{1}{2}\cos 2\bar{\Delta}) + \frac{1}{16}\,\alpha_{s7}\{25\bar{A}^4 + 12\bar{A}^2\bar{B}^2(\frac{3}{2}-\cos 2\bar{\Delta})\} + 2\bar{B}^4(\frac{3}{2}-\cos 2\bar{\Delta}) +$$

$$+ \Omega_s^2\{\frac{1}{2} + \frac{9}{64}\,\varepsilon\bar{A}^2 + \frac{3}{16}\,\varepsilon\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2\} + \frac{1}{2}$$

$$n_{22} = -\Omega_s\gamma_s\{1 + \frac{3}{8}\,\varepsilon[\frac{5}{4}\,\bar{A}^2 - \frac{1}{4}\,\bar{B}^2\cos 2\bar{\Delta} + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2\} +$$

$$+ \frac{1}{8}\,\bar{B}^2\{\frac{3}{4}\,\varepsilon\Omega_s^2 - \alpha_{s4} - \alpha_{s7}[3\bar{A}^2 + \bar{B}^2]\}\sin 2\Delta$$

$$n_{23} = -\frac{1}{2}\,\bar{A}\bar{B}\{\alpha_{s4}(1-\frac{1}{2}\cos 2\bar{\Delta}) + \alpha_{s7}[\bar{A}^2 + \bar{B}^2](\frac{3}{2}-\cos 2\bar{\Delta}) + \frac{3}{8}\,\varepsilon\Omega_s^2\cos 2\bar{\Delta}\} +$$

$$+ \frac{3}{16}\,\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\sin 2\Delta$$

$$n_{24} = \frac{1}{4}\,\bar{A}\bar{B}\{\alpha_{s4} + \alpha_{s7}[2\bar{A}^2 + \bar{B}^2] - \frac{3}{4}\,\varepsilon\Omega_s^2\}\sin 2\bar{\Delta} - \frac{3}{16}\,\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\cos 2\Delta$$

$$n_{31} = \frac{1}{4}\,\bar{A}\bar{B}\{\beta_{s3} + \beta_{s6}[2\bar{A}^2 + \bar{B}^2] - \frac{3}{4}\,\varepsilon\Omega_s^2\}\sin 2\bar{\Delta} + \frac{3}{16}\,\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\cos 2\Delta$$

$$n_{32} = -\bar{A}\bar{B}\{\beta_{s5} + \frac{1}{4}\,\beta_{s6}[\bar{A}^2 + \bar{B}^2] - \frac{3}{16}\,\varepsilon\Omega_s^2\}\cos 2\bar{\Delta} + \frac{3}{16}\,\varepsilon\gamma_s\Omega_s\bar{A}\bar{B}\sin 2\Delta$$

$$n_{33} = \frac{1}{8}\,\bar{A}^2\{\beta_{s3} + \beta_{s6}[\bar{A}^2+3\bar{B}^2] - \frac{3}{4}\,\varepsilon\Omega_s^2\}\sin 2\bar{\Delta} + \frac{3}{32}\,\varepsilon\gamma_s\Omega_s\bar{A}^2\cos 2\Delta +$$

$$- \frac{9}{32} \varepsilon \gamma_s \Omega_s \bar{B}^2 - \gamma_s \Omega_s$$

$$n_{34} = \frac{1}{2} \Omega_s^2 \{1 + \frac{3}{16} \varepsilon[\bar{B}^2 + \bar{A}^2 \cos 2\bar{\Delta}]\} + \frac{3}{32} \varepsilon \gamma_s \Omega_s \bar{A}^2 \sin 2\Delta - \frac{1}{4} \beta_{s3} \bar{A}^2 (1 + \frac{1}{2} \cos 2\bar{\Delta}) +$$

$$- \frac{1}{8} \beta_{s6} [\bar{A}^4 (\frac{3}{2} + \cos 2\bar{\Delta}) + \frac{5}{2} \bar{B}^4 + 3\bar{A}^2 \bar{B}^2] - \frac{1}{2} \beta_{s1} - \frac{3}{8} \beta_{s4} \bar{B}^2$$

$$n_{41} = - \frac{3}{16} \varepsilon \bar{A}\bar{B} \{\Omega_s^2 \cos 2\bar{\Delta} + \gamma_s \Omega_s \sin 2\bar{\Delta}\} - \frac{1}{2} \beta_{s3} \bar{A}\bar{B} (1 - \frac{1}{2} \cos 2\bar{\Delta}) +$$

$$- \frac{1}{2} \beta_{s6} \bar{A}\bar{B} [\bar{A}^2 + \bar{B}^2] (\frac{3}{2} - \cos 2\bar{\Delta})$$

$$n_{42} = - \frac{3}{16} \varepsilon \bar{A}\bar{B} \{\Omega_s^2 \sin 2\bar{\Delta} - \gamma_s \Omega_s \cos 2\bar{\Delta}\} + \frac{1}{4} \beta_{s3} \bar{A}\bar{B} \sin 2\bar{\Delta} + \frac{1}{4} \beta_{s6} \bar{A}\bar{B} [\bar{A}^2 + 2\bar{B}^2] \sin 2\bar{\Delta}$$

$$n_{43} = \frac{1}{2} \Omega_s^2 \{1 + \frac{3}{16} \varepsilon [3\bar{B}^2 - \bar{A}^2 \cos 2\bar{\Delta}]\} - \frac{3}{32} \varepsilon \gamma_s \Omega_s \bar{A}^2 \sin 2\bar{\Delta} - \frac{1}{4} \beta_{s3} \bar{A}^2 (1 - \frac{1}{2} \cos 2\bar{\Delta}) +$$

$$- \frac{1}{8} \beta_{s6} [\bar{A}^4 (\frac{3}{2} - \cos 2\bar{\Delta}) + \frac{25}{2} \bar{B}^4 + 6\bar{A}^2 \bar{B}^2 (\frac{3}{2} - \cos 2\bar{\Delta})] - \frac{1}{2} \beta_{s1} - \frac{9}{8} \beta_{s4} \bar{B}^2$$

$$n_{44} = \frac{1}{8} A^2 (\beta_{s3} + \beta_{s6} [\bar{A}^2 + 3\bar{B}^2] - \frac{3}{4} \varepsilon \Omega_s^2) \sin 2\bar{\Delta} + \frac{3}{32} \varepsilon \gamma_s \Omega_s \bar{A}^2 \cos 2\bar{\Delta} - \frac{3}{32} \varepsilon \gamma_s \Omega_s \bar{B}^2 +$$

$$+ \gamma_s \Omega_s$$

Notice that in the case of $\bar{A} \neq 0$ and $\bar{B} \neq 0$, namely, the case of coupled mode response $\sin 2\bar{\Delta}$ and $\cos 2\bar{\Delta}$ assume the values defined in Chapter 2, which are repeated here for convenience:

$$\sin 2\bar{\Delta} = - \gamma \Omega \{2[\beta_6 \Omega^2 - \beta_8 - 2\beta_{10} (2\bar{B}^2 + \bar{A}^2)] + \beta_6 [\beta_7 - \Omega^2 + \bar{B}^2 (\beta_9 - \beta_8) +$$

$$+ 2\beta_8 \bar{A}^2 + \beta_{10} (\bar{B}^4 + 4\bar{A}^2 \bar{B}^2 + 3\bar{A}^4)]\} / S_d$$

$$\cos 2\bar{\Delta} = \{\Omega^2 \gamma^2 \beta_6 (2 + \beta_6 \bar{B}^2) + [\Omega^2 (1 + \beta_6 \bar{B}^2) - \beta_7 - \beta_9 \bar{B}^2 - 2\beta_8 \bar{A}^2$$

$$- \beta_{10}(5\bar{B}^4 + 6\bar{A}^2\bar{B}^2 + 3\bar{A}^4)][\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)]\}/S_d$$

$$S_d = \bar{A}^2\{(\beta_6\gamma\Omega)^2 + [\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)]^2 +$$

$$- 2\beta_{10}\bar{A}^2\bar{B}^2[\beta_6\Omega^2 - \beta_8 - 2\beta_{10}(\bar{A}^2 + \bar{B}^2)]\}$$

But for the case of $\bar{A} \neq 0$ and $\bar{B} = 0$, namely the case of single mode response the quantities $\sin 2\bar{A}$ and $\cos 2\bar{A}$ must be replaced by $\sin\bar{\Phi}$ and $\cos\bar{\Phi}$, respectively; where

$$\sin\bar{\Phi} = \{-\Omega^2\bar{A}[1 + \beta_1\bar{A}^2 + 2\beta_1\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2] + \beta_2\bar{A} + \beta_3\bar{A}^3 + 5\beta_5\bar{A}^5\}/\bar{F}_D$$

$$\cos\bar{\Phi} = -\Omega\gamma[2\bar{A} + \beta_1\bar{A}^3 + 4\beta_1\bar{A}\delta_{n,\ell}(\delta_2+\hat{\delta}_2)^2]/\bar{F}_D$$

# APPENDIX 5-A COEFFICIENTS OF CHAPTER 5

## 5-A.1

$$a_{11} = 1 + \frac{3}{8} \varepsilon \left[A + \delta_{n,\ell}(\delta_2 + \hat{\delta}_2)\right]^2$$

$$a_{12} = \frac{3}{8} B \left[A + \delta_{n,\ell}(\delta_2 + \hat{\delta}_2)\right]$$

$$a_{13} = \frac{3}{8} \varepsilon \left[A + \delta_{n,\ell}(\delta_2 + \hat{\delta}_2)\right]$$

$$a_{14} = a_{13}$$

$$a_{15} = 2\gamma_3 + \frac{3}{4} \gamma_s \varepsilon \left[A + \delta_{n,\ell}(\delta_2 + \hat{\delta}_2)\right]^2$$

$$a_{16} = \frac{3}{4} \gamma_s \varepsilon B \left[A + \delta_{n,\ell}(\delta_2 + \hat{\delta}_2)\right]$$

$$a_{21} = a_{12}$$

$$a_{22} = 1 + \frac{3}{8} \varepsilon B^2$$

$$a_{23} = \frac{3}{8} \varepsilon B$$

$$a_{24} = a_{23}$$

$$a_{25} = a_{26}$$

$$a_{26} = 2\gamma_s + \frac{3}{4} \gamma_s \varepsilon B^2$$

where $\gamma_s$ is defined by Eq.2-3) on p. 99.

5-A.2

$$\beta_{11} = \frac{3}{8}\,\varepsilon\,[A + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)]/\eta$$

$$\beta_{12} = \beta_{11}$$

$$\beta_{13} = 2\gamma_s\,\{1 + \frac{3}{8}\,\varepsilon\,[A + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)]^2 + \frac{3}{8}\,\varepsilon\,B^2\}/\eta$$

$$\beta_{14} = (a_{17}a_{22} - a_{27}a_{12})/\eta$$

$$\beta_{21} = \frac{3}{8}\,\varepsilon\,B/\eta$$

$$\beta_{22} = \beta_{21}$$

$$\beta_{23} = \beta_{13}$$

$$\beta_{24} = (a_{27}a_{11} - a_{17}a_{21})/\eta$$

where

$$\eta = -\{1 + \frac{3}{8}\,\varepsilon\,B^2 + \frac{3}{8}\,\varepsilon\,[A + \delta_{n,\ell}(\delta_2+\hat{\delta}_2)]^2\}$$

# APPENDIX 6-A COEFFICIENTS AND FUNCTIONS OF CHAPTER 6

## 6-A.1 COEFFICIENTS OF EQUATION (6-2-7)

$$\bar{a}_o = \frac{1}{4} h^2 \ell^2 (A^2+B^2)_{,xx} + \frac{h^2}{2c} \bar{Q}_{xx} C_{,xxxx} - Rh\, C_{,xx}$$

$$\bar{a}_1 = h^2 \ell^2 [A(\hat{W}_o+A_o)_{,xx} + AC_{,xx}] + \frac{h^2}{2c} [\bar{Q}_{xx} A_{,xxxx} - \bar{Q}_{xy} \ell^2 A_{,xx} + \bar{Q}_{yy} \ell^4 A] - RhA_{,xx}$$

$$\bar{b}_1 = h^2 \ell^2 [B(\hat{W}_o+A_o)_{,xx} + BC_{,xx}] + \frac{h^2}{2c} [\bar{Q}_{xx} B_{,xxxx} - \bar{Q}_{xy} \ell^2 B_{,xx} + \bar{Q}_{yy} \ell^4 B] - RhB_{,xx}$$

$$\bar{a}_2 = \frac{1}{2} h^2 \ell^2 (AA_{,xx} - BB_{,xx} - A^2_{,x} + B^2_{,x})$$

$$\bar{b}_2 = \frac{1}{2} h^2 \ell^2 (AB_{,xx} + BA_{,xx} - 2A_{,x} B_{,x})$$

$$\bar{a}_3 = \frac{1}{2} h^2 \ell^2 (\hat{W}_1+A_1)_{,xx} A + \frac{1}{2} h^2 n^2 (\hat{W}_1+A_1) A_{,xx} - h^2 n \ell (\hat{W}_1+A_1)_{,x} A_{,x}$$

$$\bar{a}_4 = \frac{1}{2} h^2 \ell^2 (\hat{W}_1+A_1)_{,xx} A + \frac{1}{2} h^2 n^2 (\hat{W}_1+A_1) A_{,xx} + h^2 n \ell (\hat{W}_1+A_1)_{,x} A_{,x}$$

$$\bar{b}_3 = \frac{1}{2} h^2 \ell^2 (\hat{W}_1+A_1)_{,xx} B + \frac{1}{2} h^2 n^2 (\hat{W}_1+A_1) B_{,xx} - h^2 n \ell (\hat{W}_1+A_1)_{,x} B_{,x}$$

$$\bar{b}_4 = \frac{1}{2} h^2 \ell^2 (\hat{W}_1+A_1)_{,xx} B + \frac{1}{2} h^2 n^2 (\hat{W}_1+A_1) B_{,xx} + h^2 \ell n (\hat{W}_1+A_1)_{,x} B_{,x}$$

$$\bar{a}_5 = h^2 n^2 (\hat{W}_1+A_1) C_{,xx}$$

$$\tilde{\alpha}_1 = \ell$$

$\tilde{\alpha}_2 = 2\ell$

$\tilde{\alpha}_3 = n+\ell$

$\tilde{\alpha}_4 = \ell-n$

$\tilde{\alpha}_5 = n$

## 6-A.2 FUNCTIONS OF EQUATIONS (6-2-9) AND (6-2-10)

$$p_o = \frac{c}{\bar{H}_{xx}} \frac{1}{Rh} \bar{a}_o$$

$$p_i = \frac{\bar{H}_{xy}}{\bar{H}_{xx}} (\tilde{\alpha}_i)^2 \, \hat{\hat{\Phi}}_{i,\overline{xx}} - \frac{\bar{H}_{yy}}{\bar{H}_{xx}} (\tilde{\alpha}_i)^4 \, \hat{\hat{\Phi}}_i + \frac{c}{\bar{H}_{xx}} \frac{1}{Rh} \bar{a}_i \qquad (i=1,2,\ldots,5)$$

$$p_{j+5} = \frac{\bar{H}_{xy}}{\bar{H}_{xx}} (\tilde{\alpha}_j)^2 \, \hat{\hat{\Phi}}_{j+5,\overline{xx}} - \frac{\bar{H}_{xy}}{\bar{H}_{xx}} (\tilde{\alpha}_j)^4 \, \hat{\hat{\Phi}}_{j+5} + \frac{c}{\bar{H}_{xx}} \frac{1}{Rh} \bar{b}_j \qquad (j=1,2,\ldots,4)$$

## 6-A.3 COEFFICIENTS AND FUNCTIONS OF EQUATIONS (6-2-12) AND (6-2-14)

$$A_{,\overline{xxxx}} = A_{,\overline{xxxx}} \text{ (Linear part)} + A_{,\overline{xxxx}} \text{ (Nonlinear part)}$$

$$= \{a_1 \hat{\hat{\Phi}}_{1,\overline{xx}} + a_2 \hat{\hat{\Phi}}_{,\bar{x}} + a_3 \hat{\hat{\Phi}}_1 + a_4 A_{,\overline{xx}} + a_5 A_{,\bar{x}} + a_6 A + a_7 C + a_{13} C_{,\overline{xx}}$$

$$+ a_{16} \hat{\hat{\Phi}}_{3,\overline{xx}} + a_{17} \hat{\hat{\Phi}}_{3,\bar{x}} + a_{18} \hat{\hat{\Phi}}_3 + a_{25} \hat{\hat{\Phi}}_{5,\overline{xx}} + a_{26} \hat{\hat{\Phi}}_{5,\bar{x}} + a_{27} \hat{\hat{\Phi}}_5$$

$$+ a_{28} \hat{\hat{\Phi}}_{6,\overline{xx}} + a_{29} \hat{\hat{\Phi}}_{6,\bar{x}} + a_{30} \hat{\hat{\Phi}}_6 + a_{44} \hat{\hat{\Phi}}_{9,\overline{xx}} + a_{45} \hat{\hat{\Phi}}_{9,\bar{x}} + a_{36} \hat{\hat{\Phi}}_9$$

$$+ a_{51}A_{,tt} + a_{52}q\} +$$

$$+ \{a_8A^2 + a_9B^2 + a_{10}AC + a_{11}A^3 + a_{12}AB^2 + a_{14}AC_{,\overline{xx}} + a_{15}\hat{\hat{\phi}}_1C_{,\overline{xx}}$$

$$+ a_{19}A\hat{\hat{\phi}}_{3,\overline{xx}} + a_{20}A_{,\bar{x}}\hat{\hat{\phi}}_{3,\bar{x}} + a_{21}A_{,\overline{xx}}\hat{\hat{\phi}}_3 + a_{22}B\hat{\hat{\phi}}_{4,\overline{xx}} + a_{23}\hat{\hat{\phi}}_{4,\bar{x}}B_{,\bar{x}}$$

$$+ a_{24}\hat{\hat{\phi}}_4B_{,\overline{xx}} + a_{31}A\hat{\hat{\phi}}_{5,\overline{xx}} + a_{32}A_{,\bar{x}}\hat{\hat{\phi}}_{5,\bar{x}} + a_{33}A_{,\overline{xx}}\hat{\hat{\phi}}_5 + a_{34}A\hat{\hat{\phi}}_{6,\overline{xx}}$$

$$+ a_{35}A_{,\bar{x}}\hat{\hat{\phi}}_{6,\bar{x}} + a_{36}A_{,\overline{xx}}\hat{\hat{\phi}}_6 + a_{37}\hat{\hat{\phi}}_6C_{,\overline{xx}} + a_{38}B\hat{\hat{\phi}}_{7,\overline{xx}} + a_{39}B_{,\bar{x}}\hat{\hat{\phi}}_{7,\bar{x}}$$

$$+ a_{40}B_{,\overline{xx}}\hat{\hat{\phi}}_7 + a_{41}B\hat{\hat{\phi}}_{8,\overline{xx}} + a_{42}B_{,\bar{x}}\hat{\hat{\phi}}_{8,\bar{x}} + a_{43}B_{,\overline{xx}}\hat{\hat{\phi}}_8 + a_{47}A\hat{\hat{\phi}}_{9,\overline{xx}}$$

$$+ a_{48}A_{,\bar{x}}\hat{\hat{\phi}}_{9,\bar{x}} + a_{49}A_{,\overline{xx}}\hat{\hat{\phi}}_9\}$$

$$B_{,\overline{xxxx}} = B_{,\overline{xxxx}}\text{ (Linear part)} + B_{,\overline{xxxx}}\text{ (Nonlinear part)}$$

$$= \{b_1\hat{\hat{\phi}}_{2,\overline{xx}} + b_2\hat{\hat{\phi}}_{2,\bar{x}} + b_3\hat{\hat{\phi}}_2 + b_4B_{,\overline{xx}} + b_5B_{,\bar{x}} + b_{13}\hat{\hat{\phi}}_{4,\overline{xx}}$$

$$+ b_{14}\hat{\hat{\phi}}_{4,\bar{x}} + b_{15}\hat{\hat{\phi}}_4 + b_{22}\hat{\hat{\phi}}_{7,\overline{xx}} + b_{23}\hat{\hat{\phi}}_{7,\bar{x}} + b_{24}\hat{\hat{\phi}}_7 + b_{25}\hat{\hat{\phi}}_{8,\overline{xx}}$$

$$+ b_{26}\hat{\hat{\phi}}_{8,\bar{x}} + b_{27}\hat{\hat{\phi}}_8 + b_{44}B_{,tt}\} +$$

$$+ \{b_7BC + b_8BA + b_9BA^2 + b_{10}B^3 + b_{11}BC_{,\overline{xx}} + b_{16}B\hat{\hat{\phi}}_{3,\overline{xx}} + b_{17}B_{,\bar{x}}\hat{\hat{\phi}}_{3,\bar{x}} +$$

$$+ b_{18}B,_{\bar{x}\bar{x}}\hat{\hat{\phi}}_3 + b_{19}A\hat{\hat{\phi}}_{4,\bar{x}\bar{x}} + b_{20}A,_{\bar{x}}\hat{\hat{\phi}}_{4,\bar{x}} + b_{21}\hat{\hat{\phi}}_4A,_{\bar{x}\bar{x}} + b_{28}B\hat{\hat{\phi}}_{5,\bar{x}\bar{x}} +$$

$$+ b_{29}B,_{\bar{x}}\hat{\hat{\phi}}_{5,\bar{x}} + b_{30}B,_{\bar{x}\bar{x}}\hat{\hat{\phi}}_5 + b_{31}\hat{\hat{\phi}}_{6,\bar{x}\bar{x}}B + b_{32}B,_{\bar{x}}\hat{\hat{\phi}}_{6,\bar{x}} + b_{33}\hat{\hat{\phi}}_6B,_{\bar{x}\bar{x}}$$

$$+ b_{34}A\hat{\hat{\phi}}_{7,\bar{x}\bar{x}} + b_{35}A,_{\bar{x}}\hat{\hat{\phi}}_{7,\bar{x}} + b_{36}A,_{\bar{x}\bar{x}}\hat{\hat{\phi}}_7 + b_{37}A\hat{\hat{\phi}}_{8,\bar{x}\bar{x}} + b_{38}A,_{\bar{x}}\hat{\hat{\phi}}_{8,\bar{x}}$$

$$+ b_{39}\hat{\hat{\phi}}_8A,_{\bar{x}\bar{x}} + b_{40}\hat{\hat{\phi}}_8C,_{\bar{x}\bar{x}} + b_{41}B\hat{\hat{\phi}}_{9,\bar{x}\bar{x}} + b_{42}\hat{\hat{\phi}}_{9,\bar{x}}B,_{\bar{x}} + b_{43}\hat{\hat{\phi}}_9B,_{\bar{x}\bar{x}}\}$$

$$C,_{\bar{x}\bar{x}\bar{x}\bar{x}} = C,_{\bar{x}\bar{x}\bar{x}\bar{x}}\text{ (Linear part) } + C,_{\bar{x}\bar{x}\bar{x}\bar{x}}\text{ (Nonlinear part)}$$

$$= \{c_1C,_{\bar{x}\bar{x}} + c_2C + c_3A + c_5A,_{\bar{x}} + c_6A,_{\bar{x}\bar{x}} + c_{10}\hat{\hat{\phi}}_{1,\bar{x}\bar{x}} + c_{11}\hat{\hat{\phi}}_{1,\bar{x}} + c_{12}\hat{\hat{\phi}}_1$$

$$+ c_{19}\hat{\hat{\phi}}_{3,\bar{x}\bar{x}} + c_{20}\hat{\hat{\phi}}_{3,\bar{x}} + c_{21}\hat{\hat{\phi}}_3 + c_{22}\hat{\hat{\phi}}_{6,\bar{x}\bar{x}} + c_{23}\hat{\hat{\phi}}_{6,\bar{x}} + c_{24}\hat{\hat{\phi}}_6 + c_{31}\hat{\hat{\phi}}_{9,\bar{x}\bar{x}}$$

$$+ c_{32}\hat{\hat{\phi}}_{9,\bar{x}} + c_{33}\hat{\hat{\phi}}_9 + c_{37}C,_{tt}\} +$$

$$+ \{c_7A^2 + c_8B^2 + c_9[(A,_{\bar{x}})^2 + (B,_{\bar{x}})^2 + A,_{\bar{x}\bar{x}}A + B,_{\bar{x}\bar{x}}B] + c_{13}A\hat{\hat{\phi}}_{1,\bar{x}\bar{x}}$$

$$+ c_{14}A,_{\bar{x}}\hat{\hat{\phi}}_{1,\bar{x}} + c_{15}\hat{\hat{\phi}}_1A,_{\bar{x}\bar{x}} + c_{16}B\hat{\hat{\phi}}_{2,\bar{x}\bar{x}} + c_{17}B,_{\bar{x}}\hat{\hat{\phi}}_{2,\bar{x}} + c_{18}\hat{\hat{\phi}}_2B,_{\bar{x}\bar{x}}$$

$$+ c_{25}A\hat{\hat{\phi}}_{6,\bar{x}\bar{x}} + c_{26}A,_{\bar{x}}\hat{\hat{\phi}}_{6,\bar{x}} + c_{27}A,_{\bar{x}\bar{x}}\hat{\hat{\phi}}_6 + c_{28}B\hat{\hat{\phi}}_{8,\bar{x}\bar{x}} + c_{29}B,_{\bar{x}}\hat{\hat{\phi}}_{8,\bar{x}}$$

$$+ c_{30} B,_{\overline{\overline{xx}}} \hat{\hat{\phi}}_8 + c_{34} A \hat{\hat{\phi}}_{9,\overline{\overline{xx}}} + c_{35} A,_{\bar{x}} \hat{\hat{\phi}}_{9,\bar{x}} + c_{36} \hat{\hat{\phi}}_9 A,_{\overline{\overline{xx}}} \}$$

where

$$a_o = \bar{D}_{xx} + \frac{\bar{Q}_{xx}^2}{\bar{H}_{xx}}$$

$$a_1 = \tilde{a}_{11} + \tilde{a}_{12}$$

$$\tilde{a}_{11} = \frac{1}{a_o} \{- 2 \frac{R}{h} \frac{\bar{Q}_{xx} \bar{H}_{xy}}{\bar{H}_{xx}} \ell^2 + 2 \frac{R}{h} \ell^2 \bar{Q}_{xy} - 4c \left(\frac{R}{h}\right)^2 \}$$

$$\tilde{a}_{12} = \frac{1}{a_o} \delta_{2\ell,n} [-2c \frac{R}{h} n^2 (\hat{W}_1 + A_1)]$$

$$a_2 = \frac{1}{a_0} \delta_{2\ell,n} [- 4c \frac{R}{h} \ell n (A_1 + \hat{W}_1),_{\bar{x}}]$$

$$a_3 = \tilde{a}_{31} + \tilde{a}_{32}$$

$$\tilde{a}_{31} = \frac{1}{a_o} \{2 \frac{R}{h} \frac{\bar{Q}_{xx} \bar{H}_{yy}}{\bar{H}_{xx}} \ell^4 - 2 \frac{R}{h} \ell^4 \bar{Q}_{yy} - 4c \frac{R}{h} \ell^2 (A_o + \hat{W}_o),_{\overline{\overline{xx}}} \}$$

$$\tilde{a}_{32} = \frac{1}{a_o} \delta_{2\ell,n} [-2c \frac{R}{h} \ell^2 (\hat{W}_1 + A_1),_{\overline{\overline{xx}}}]$$

$$a_4 = \tilde{a}_{41} + \tilde{a}_{42} + \tilde{a}_{43}$$

$$\tilde{a}_{41} = \frac{1}{a_o} \{[\bar{D}_{xy} + \frac{\bar{Q}_{xy} \bar{Q}_{xx}}{\bar{H}_{xx}}]\ell^2 + 2c \frac{R}{h} \frac{\bar{Q}_{xx}}{\bar{H}_{xx}} - 4c \frac{R}{h} \lambda \}$$

$$\tilde{a}_{42} = \frac{1}{a_o}\, \delta_{n,\ell}\, [-2c\, \frac{R}{h}\, (2n)^2 f_2]$$

$$\tilde{a}_{43} = \frac{1}{a_o}\, \delta_{2\ell,n}\, [-c\, \frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\, (\hat{W}_1+A_1)n^2 - 2c\, \frac{R}{h}\, n^2 f_1]\}$$

$$a_5 = \tilde{a}_{51} + \tilde{a}_{52}$$

$$\tilde{a}_{51} = \frac{1}{a_o}\, \{\delta_{n,\ell}[-8c\, \frac{R}{h}\, n\ell\, f_{2,\bar{x}}]\}$$

$$\tilde{a}_{52} = \frac{1}{a_o}\, \delta_{2\ell,n}\, [-2c\, \frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\, (\hat{W}_1+A_1)_{,\bar{x}}\, \ell n - 4c\, \frac{R}{h}\, f_{1,\bar{x}}\, \ell n]$$

$$a_6 = \tilde{a}_{61} + \tilde{a}_{62} + \tilde{a}_{63}$$

$$\tilde{a}_{61} = \frac{1}{a_o}\, \{-2c\, \frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\, (\hat{W}_o+A_o)_{,\bar{x}\bar{x}}\, \ell^2 - \frac{\Theta_{xx}\Theta_{yy}}{\bar{H}_{xx}}\, \ell^4 - \bar{D}_{yy}\ell^4 - 4c\, \frac{R}{h}\, f_{o,\bar{x}\bar{x}}\, \ell^2\}$$

$$\tilde{a}_{62} = \frac{1}{a_o}\, \delta_{n,\ell}\, [-2c^2\, \frac{1}{\bar{H}_{xx}}\, (\hat{W}_1+A_1)^2\, \ell^4 - 2c\, \frac{R}{h}\, f_{2,\bar{x}\bar{x}}\, \ell^2]$$

$$\tilde{a}_{63} = \frac{1}{a_o}\, \delta_{2\ell,n}\, [-\, \frac{c\bar{Q}_{xx}}{\bar{H}_{xx}}\, (\hat{W}_1+A_1)_{,\bar{x}\bar{x}}\, \ell^2 - 2c\, \frac{R}{h}\, f_{1,\bar{x}\bar{x}}\, \ell^2]$$

$$a_7 = \frac{1}{a_o}\, \delta_{n,\ell}\, [4\, \frac{c^2}{\bar{H}_{xx}}\, \frac{R}{h}\, (\hat{W}_1+A_1)\ell^2]$$

$$a_8 = \frac{1}{a_o}\, \delta_{n,\ell}\, [-3\, \frac{c^2}{\bar{H}_{xx}}\, (\hat{W}_1+A_1)\ell^4]$$

$$a_9 = \frac{1}{a_o} \delta_{n,\ell} \left[- \frac{c^2}{\bar{H}_{xx}} A_1 n^4 \right]$$

$$a_{10} = \frac{1}{a_o} \left(4 \frac{c^2}{\bar{H}_{xx}} \frac{R}{h} \ell^2 \right)$$

$$a_{11} = \frac{1}{a_o} \left[- \frac{c^2}{\bar{H}_{xx}} \ell^4 \right]$$

$$a_{12} = a_{11}$$

$$a_{13} = \frac{1}{a_o} \{\delta_{n,\ell}[-4c \frac{\bar{Q}_{xx}}{\bar{H}_{xx}} (\hat{W}_1+A_1)\ell^2 - 4cn^2 f_1] + \delta_{\ell,2n} [-4c \frac{R}{h} (2n)^2 f_2]\}$$

$$a_{14} = \frac{1}{a_o} \left(-4c \frac{\bar{Q}_{xx}}{\bar{H}_{xx}} \ell^2 \right)$$

$$a_{15} = \frac{1}{a_o} \left(-4c \frac{R}{h} \ell^2 \right)$$

$$a_{16} = \tilde{a}_{161} + \tilde{a}_{162}$$

$$\tilde{a}_{161} = \frac{1}{a_o} \delta_{n,\ell} \left[-2c \frac{R}{h} (\hat{W}_1+A_1)\ell^2 \right]$$

$$\tilde{a}_{162} = \frac{1}{a_o} \delta_{3\ell,n} \left[-2c \frac{R}{h} (\hat{W}_1+A_1)n^2 \right]$$

$$a_{17} = \tilde{a}_{171} + \tilde{a}_{172}$$

$$\tilde{a}_{171} = \frac{1}{a_o} \delta_{n,\ell} \left[-4c \frac{R}{h} (\hat{W}_1+A_1)_{,x} (2\ell)n \right]$$

$$\tilde{a}_{172} = \frac{1}{a_o} \delta_{3\ell,n} [-4c \frac{R}{h} (\hat{W}_1+A_1),_{\bar{x}}(2\ell)n]$$

$$a_{18} = \tilde{a}_{181} + \tilde{a}_{182}$$

$$\tilde{a}_{181} = \frac{1}{a_o} \delta_{n,\ell} [-2c \frac{R}{h} (\hat{W}_1+A_1),_{\bar{x}\bar{x}}(2\ell)^2]$$

$$\tilde{a}_{182} = \frac{1}{a_o} \delta_{3\ell,n} [-2c \frac{R}{h} (\hat{W}_1+A_1),_{\bar{x}\bar{x}} (2\ell)^2]$$

$$a_{19} = \frac{1}{2} a_{15} \qquad a_{20} = 2a_{15} \qquad a_{21} = a_{20} \qquad a_{22} = a_{19} \qquad a_{23} = a_{20} \qquad a_{24} = a_{20}$$

$$a_{25} = \frac{1}{a_o} [-2c \frac{R}{h} (\hat{W}_1+A_1)n^2]$$

$$a_{26} = \frac{1}{a_o} [-4c \frac{R}{h} (\hat{W}_1+A_1),_{\bar{x}} (\ell+n).n]$$

$$a_{27} = \frac{1}{a_o} [-2c \frac{R}{h} (\hat{W}_1+A_1),_{\bar{x}\bar{x}} (n+\ell)^2]$$

$$a_{28} = \frac{1}{a_o} \{-2c \frac{R}{h} (\hat{W}_1+A_1)n^2 + \delta_{2\ell,n} [-2 \frac{R}{h} \frac{\bar{H}_{xy}\bar{Q}_{xx}}{\bar{H}_{xx}} (\ell-n)^2 + 2 \frac{R}{h} \bar{Q}_{xy} (\ell-n)^2 + 4c(\frac{R}{h})^2]$$

$$+ \delta_{n,\ell} [2c \frac{R}{h} (\hat{W}_1+A_1) n^2]\}$$

$$a_{29} = \frac{1}{a_o} [4c \frac{R}{h} (\hat{W}_1+A_1),_{\bar{x}} (\ell-n)n]$$

$$a_{30} = \tilde{a}_{301} + \tilde{a}_{302}$$

$$\tilde{a}_{301} = \frac{1}{a_o} \{-2c \frac{R}{h} (\hat{W}_1+A_1),_{\bar{x}\bar{x}} (\ell-n)^2\}$$

$$\tilde{a}_{302} = \frac{1}{a_o} \delta_{2\ell,n} \{[2 \frac{R}{h} \frac{\bar{Q}_{xx}\bar{H}_{yy}}{\bar{H}_{xx}} (\ell-n)^4 - 2 \frac{R}{h} \bar{Q}_{yy}(\ell-n)^4 - 4c \frac{R}{h} (A_o+\hat{W}_o),_{\overline{xx}} (\ell-n)^2 ]\}$$

$$a_{31} = \frac{1}{2} \delta_{n,\ell} a_{15}$$

$$a_{32} = \frac{1}{a_o} \delta_{n,\ell} [-4c \frac{R}{h} (n+\ell)\ell]$$

$$a_{33} = \frac{1}{a_o} \delta_{n,\ell} [-2c \frac{R}{h} (n+\ell)^2 ]$$

$$a_{34} = \frac{1}{2} \delta_{3\ell,n} a_{15}$$

$$a_{35} = \frac{1}{a_o} \delta_{3\ell,n} [4c \frac{R}{h} (\ell-n)\ell]$$

$$a_{36} = \frac{1}{a_o} \delta_{3\ell,n} [-2c \frac{R}{h} (n-\ell)^2 ]$$

$$a_{37} = \frac{1}{a_o} \delta_{2\ell,n} [-4c \frac{R}{h} (n-\ell)^2 ]$$

$$a_{38} = a_{31} \qquad a_{39} = a_{32} \qquad a_{40} = a_{33}$$

$$a_{41} = - \frac{1}{2} \delta_{3\ell,n} a_{15}$$

$$a_{42} = -a_{35} \qquad a_{43} = -a_{36}$$

$$a_{44} = \frac{1}{a_o} \{-2 \frac{R}{h} \frac{\bar{Q}_{xx}\bar{H}_{xy}}{\bar{H}_{xx}} n^2 + 2 \frac{R}{h} \bar{Q}_{xy} n^2 + 4c (\frac{R}{h})^2 ] + \delta_{\ell,2n}[-2c \frac{R}{h} (\hat{W}_1+A_1)n^2 ]\}$$

$$a_{45} = \frac{1}{a_o} \{\delta_{\ell,2n} [4c \frac{R}{h} (\hat{W}_1+A_1),_{\bar{x}} n^2 ]\}$$

$$a_{46} = \frac{1}{a_o} \{\delta_{n,\ell} \; [2 \frac{R}{h} \frac{\bar{Q}_{xx}\bar{H}_{yy}}{\bar{H}_{xx}} n^4 \; -2 \frac{R}{h} \bar{Q}_{yy} n^4 \; - 4c \frac{R}{h} (\hat{W}_o + A_o),_{\bar{x}\bar{x}} \; n^2]$$

$$+ \delta_{\ell,2n}[-2c \frac{R}{h}(\hat{W}_1 + A_1),_{\bar{x}\bar{x}} \; n^2]\}$$

$$a_{47} = \frac{1}{2} \delta_{2\ell,n} \; a_{15}$$

$$a_{48} = \frac{1}{a_o} \delta_{2\ell,n} \; [-4c \frac{R}{h} \; \ell n]$$

$$a_{49} = \frac{1}{a_o} \delta_{2\ell,n} \; [-2c \frac{R}{h} \; n^2]$$

$$a_{50} = \frac{1}{a_o} \delta_{n,\ell} \; [-4c \frac{R}{h} \; n^2]$$

$$a_{51} = - \frac{4\bar{\rho} R^4 c^2}{Eh^2} \frac{1}{a_o}$$

$$a_{52} = \frac{4R^4 c^2}{Eh^4} \frac{1}{a_o}$$

$$b_1 = \tilde{a}_{11} - \tilde{a}_{12} \qquad\qquad b_2 = - a_2$$

$$b_3 = \tilde{a}_{31} - \tilde{a}_{32} \qquad\qquad b_4 = \tilde{a}_{41} - \tilde{a}_{42} - \tilde{a}_{43}$$

$$b_5 = - \tilde{a}_{51} + \frac{1}{a_o} \delta_{2\ell,n} \; [-4c \frac{R}{h} \; \ell n \; f_{1,\bar{x}} + 2c \frac{\bar{Q}_{xx}}{\bar{H}_{xx}} \; \ell n \; (\hat{W}_1 + A_1),_{\bar{x}}]$$

$$b_6 = \tilde{a}_{61} + \frac{1}{a_o} \delta_{n,\ell} \; [2c \frac{R}{h} \; n^2 f_{2,\bar{x}\bar{x}}] - \tilde{a}_{63}$$

$b_7 = a_{10}$

$b_8 = \frac{3}{2} a_8$

$b_9 = a_{12}$

$b_{10} = b_9$

$b_{11} = a_{14}$

$b_{12} = a_{15}$

$b_{13} = \tilde{a}_{161} - \tilde{a}_{162}$

$b_{14} = \tilde{a}_{171} - \tilde{a}_{172}$

$b_{15} = \tilde{a}_{181} - \tilde{a}_{182}$

$b_{16} = - a_{19}$

$b_{17} = - a_{18}$

$b_{18} = -a_{21}$

$b_{19} = a_{22}$

$b_{20} = a_{23}$

$b_{21} = a_{24}$

$b_{22} = a_{25}$

$b_{23} = a_{26}$

$b_{24} = a_{27}$

$b_{25} = -a_{28}$

$b_{26} = a_{29}$

$b_{27} = \tilde{a}_{301} - \tilde{a}_{302}$

$b_{29} = -a_{32}$

$b_{30} = -a_{33}$

$b_{31} = -\frac{1}{2}\,\delta_{3\ell,n}a_{15}$

$b_{32} = -a_{35}$

$b_{33} = -a_{36}$

$b_{34} = -b_{28}$

$b_{35} = -b_{29}$

$b_{36} = -b_{30}$

$b_{37} = a_{41}$

$b_{38} = a_{42}$

$b_{39} = a_{43}$

$b_{40} = -a_{37}$

$b_{41} = -a_{47}$

$b_{42} = -a_{48}$

$b_{43} = -a_{49}$

$b_{44} = a_{51}$

$$c_1 = \frac{1}{a_o} \{4c \frac{R}{h} (\frac{\bar{Q}_{xx}}{\bar{H}_{xx}} - \lambda)\}$$

$$c_2 = \frac{1}{a_o} 4 \frac{c^2}{\bar{H}_{xx}} (\frac{R}{h})^2$$

$$c_3 = \frac{1}{a_o} \{\delta_{n,\ell} [-2c \frac{R}{h} \ell^2 f_{1,\bar{x}\bar{x}} - c \frac{\bar{Q}_{xx}}{\bar{H}_{xx}} \ell^2 (A_1+\hat{W}_1)_{,\bar{x}\bar{x}} + 2 \frac{c^2}{\bar{H}_{xx}} \frac{R}{h} \ell^2 (\hat{W}_1+A_1)] + \delta_{\ell,2n} [-2c \frac{R}{h} \ell^2 f_{2,\bar{x}\bar{x}}]\}$$

$$c_4 = \frac{1}{a_o} \delta_{n,\ell} [-2c^2 \frac{1}{\bar{H}_{xx}} \frac{R}{h} (\hat{W}_1+A_1)\ell^2 ]$$

$$c_5 = \frac{1}{a_o} \{\delta_{n,\ell} [-4c \frac{R}{h} \ell n f_{1,\bar{x}} - 2c \frac{\bar{Q}_{xx}}{\bar{H}_{xx}} (\hat{W}_1+A_1)_{,\bar{x}} \ell n] + \delta_{\ell,2n} [-8c \frac{R}{h} \ell n f_{2,\bar{x}}]\}$$

$$c_6 = \frac{1}{a_o} \{\delta_{n,\ell}[-4c \frac{R}{h} \ell n f_{1,\bar{x}} - 2c \frac{\bar{Q}_{xx}}{\bar{H}_{xx}} (\hat{W}_1+A_1)_{,\bar{x}} \ell n] + \delta_{\ell,2n}[-8c \frac{R}{h} \ell n f_{2,\bar{x}}]\}$$

$$c_7 = \frac{1}{a_o} \frac{c^2}{\bar{H}_{xx}} \frac{R}{h} \ell^2$$

$$c_8 = c_7$$

$$c_9 = \frac{1}{a_o} \frac{c\bar{Q}_{xx}}{\bar{H}_{xx}} \ell^2$$

$$c_{10} = \frac{1}{a_o} \delta_{n,\ell} \left[-2c \frac{R}{h} (\hat{W}_1 + A_1) n^2\right]$$

$$c_{11} = \frac{1}{a_o} \delta_{n,\ell} \left[-4c \frac{R}{h} \ell n \; (\hat{W}_1 + A_1)_{,\bar{x}}\right]$$

$$c_{12} = \frac{1}{a_o} \delta_{n,\ell} \left[-2c \frac{R}{h} n^2 \; (\hat{W}_1 + A_1)_{,\bar{x}\bar{x}}\right]$$

$$c_{13} = \frac{1}{2} a_{15}$$

$$c_{14} = c_{15}$$

$$c_{15} = c_{13}$$

$$c_{16} = c_{13}$$

$$c_{17} = c_{14}$$

$$c_{18} = c_{15}$$

$$c_{19} = \frac{1}{a_o} \delta_{2\ell,n} \left[-2c \frac{R}{h} n^2 \; (\hat{W}_1 + A_1)\right]$$

$$c_{20} = \frac{1}{a_o} \delta_{2\ell,n} \left[-4c \frac{R}{h} (2\ell)\ n\ (\hat{W}_1 + A_1)_{,\bar{x}}\right]$$

$$c_{21} = \frac{1}{a_o} \delta_{2\ell,n} \left[-2c \frac{R}{h} (2\ell)^2\ (\hat{W}_1 + A_1)_{,\bar{x}\bar{x}}\right]$$

$$c_{22} = \frac{1}{a_o} \delta_{\ell,2n} \left[-2c \frac{R}{h} n^2\ (\hat{W}_1 + A_1)\right]$$

$$c_{23} = \frac{1}{a_o} \delta_{\ell,2n} \left[4c \frac{R}{h} \ell(n-\ell)\ (\hat{W}_1 + A_1)_{,\bar{x}}\right]$$

$$c_{24} = \frac{1}{a_o} \delta_{\ell,2n} \left[-2c \frac{R}{h} (n-\ell)^2\ (\hat{W}_1 + A_1)_{,\bar{x}\bar{x}}\right]$$

$$c_{25} = \frac{1}{a_o} \delta_{2\ell,n} \left[-2c \frac{R}{h} \ell^2\right]$$

$$c_{27} = \frac{1}{a_o} \delta_{2\ell,n} \left[4c \frac{R}{h} \ell(\ell-n)\right]$$

$$c_{28} = -c_{25}$$

$$c_{29} = -c_{26}$$

$$c_{30} = -c_{27}$$

$$c_{31} = \frac{1}{a_o} \left\{-2c \frac{R}{h} n^2\ (\hat{W}_1 + A_1)\right\}$$

$$c_{32} = \frac{1}{a_o} \left\{-4c \frac{R}{h} n^2\ (\hat{W}_1 + A_1)_{,\bar{x}}\right\}$$

$$c_{33} = \frac{1}{a_o} \left\{-2c \frac{R}{h} n^2\ (\hat{W}_1 + A_1)_{,\bar{x}\bar{x}}\right\}$$

$$c_{34} = \frac{1}{a_o} \delta_{n,\ell} \left[-2c \frac{R}{h} \ell^2\right]$$

$$c_{35} = \frac{1}{a_o} \delta_{n,\ell} \left[-4c \frac{R}{h} \ell n\right]$$

$$c_{36} = \frac{1}{a_o} \delta_{n,\ell} \left[-2c \frac{R}{h} n^2\right]$$

$$c_{37} = a_{51}$$

6-A.4 COEFFICIENTS AND FUNCTIONS OF EQUATIONS (6-2-15) AND (6-2-16)

$$d_o = \frac{1}{2} \frac{\bar{Q}_{xx}}{\bar{H}_{xx}} \frac{h}{R}$$

$$d_1 = \frac{\bar{H}_{xy}}{\bar{H}_{xx}} \ell^2 + d_o a_1$$

$$d_2 = d_o a_2$$

$$d_3 = -\frac{\bar{H}_{yy}}{\bar{H}_{xx}} \ell^4 + d_o a_3$$

$$d_4 = -\frac{1}{2} \frac{h}{R} \frac{\bar{Q}_{xy}}{\bar{H}_{xx}} \ell^2 - \frac{c}{\bar{H}_{xx}} + d_o a_4$$

$$d_5 = d_o a_5$$

$$d_6 = \frac{1}{\bar{H}_{xx}} \frac{h}{R} c \ell^2 (\hat{W}_o + A_o)_{,\overline{\overline{xx}}} + \frac{1}{2} \frac{h}{R} \frac{\bar{Q}_{yy}}{\bar{H}_{xx}} \ell^2 + d_o a_6$$

$$d_i = d_o a_i \quad (i=7,8,\ldots,13)$$

$$d_{14} = \frac{c}{\bar{H}_{xx}} \frac{h}{R} \ell^2 + d_o a_{14}$$

$$d_j = d_o a_j \quad (j=15,16,\ldots,51)$$

$$e_1 = \frac{\bar{H}_{xy}}{\bar{H}_{xx}} \ell^2 + d_o b_1$$

$$e_2 = d_o b_2$$

$$e_3 = -\frac{\bar{H}_{yy}}{\bar{H}_{xx}} \ell^4 + d_o b_3$$

$$e_4 = -\frac{1}{2} \frac{h}{R} \frac{\bar{Q}_{xy}}{\bar{H}_{xx}} \ell^2 - \frac{c}{\bar{H}_{xx}} + d_o b_4$$

$$e_5 = d_o b_5$$

$$e_6 = \frac{c}{\bar{H}_{xx}} \frac{h}{R} (\hat{W}_o + A_o)_{,\bar{x}\bar{x}} + \frac{1}{2} \frac{h}{R} \frac{\bar{Q}_{yy}}{\bar{H}_{xx}} \ell^4 + d_o b_6$$

$$e_i = d_o b_i \quad (i=7,8,9,10)$$

$$e_{11} = \frac{c}{\bar{H}_{xx}} \frac{h}{R} \ell^2 + d_o b_{11}$$

$$e_j = d_o b_j \quad (j=12,13,\ldots,44)$$

## 6-A.5 COEFFICIENTS OF EQUATIONS (6-2-29) TO (6-2-32)

$\bar{f}_2 = \bar{f}_2$ (Linear part) + $\bar{f}$ (Nonlinear part)

$$
\begin{aligned}
&= \{a_1\bar{\bar{\hat{\phi}}}''_{11} + a_2\bar{\bar{\hat{\phi}}}'_{11} + a_3\bar{\bar{\hat{\phi}}}_{11} + a_4\bar{A}'' + a_5\bar{A}' + a_6\bar{A} + \delta_{n,\ell}[a_7\bar{C}_3 + a_{13}\bar{C}''_3] \\
&\quad + a_{25}\bar{\bar{\hat{\phi}}}''_{51} + a_{26}\bar{\bar{\hat{\phi}}}'_{51} + a_{27}\bar{\bar{\hat{\phi}}}_{51} + a_{28}\bar{\bar{\hat{\phi}}}''_{61} + a_{29}\bar{\bar{\hat{\phi}}}'_{61} + a_{30}\bar{\bar{\hat{\phi}}}_{61} \\
&\quad + a_{44}\bar{\bar{\hat{\phi}}}''_{93} + a_{45}\bar{\bar{\hat{\phi}}}'_{93} + a_{46}\bar{\bar{\hat{\phi}}}_{93}] - \omega^2 a_{51}\bar{A} + a_{52}\bar{Q} + \\
&+ \{a_{10}\bar{A}\bar{C}_1 + \frac{1}{2}\bar{A}\bar{C}_2 + \frac{3}{4}a_{11}\bar{A}^3 + \frac{1}{4}a_{12}\bar{A}\bar{B}^2 + a_{14}\bar{A}\bar{C}''_1 + \frac{1}{2}a_{14}\bar{A}\bar{C}''_2 \\
&\quad + a_{15}\bar{\bar{\hat{\phi}}}_{11}\bar{C}''_1 + \frac{1}{2}a_{15}\bar{\bar{\hat{\phi}}}_{11}\bar{C}''_2 + \frac{1}{2}a_{15}\bar{\bar{\hat{\phi}}}_{12}\bar{C}''_2 + a_{19}\bar{A}\bar{\bar{\hat{\phi}}}''_{31} + \frac{1}{2}a_{19}\bar{A}\bar{\bar{\hat{\phi}}}''_{32} \\
&\quad + a_{20}\bar{A}'\bar{\bar{\hat{\phi}}}'_{31} + \frac{1}{2}a_{20}\bar{A}'\bar{\bar{\hat{\phi}}}'_{32} + a_{21}\bar{A}''\bar{\bar{\hat{\phi}}}_{31} + \frac{1}{2}a_{21}\bar{A}''\bar{\bar{\hat{\phi}}}_{32} + \frac{1}{2}a_{22}\bar{\bar{\hat{\phi}}}''_{41}\bar{B} \\
&\quad + \frac{1}{2}a_{23}\bar{\bar{\hat{\phi}}}'_{41}\bar{B}' + \frac{1}{2}a_{24}\bar{\bar{\hat{\phi}}}_{41}\bar{B}'' + a_{37}\bar{\bar{\hat{\phi}}}_{61}\bar{C}''_1 + \frac{1}{2}a_{37}\bar{\bar{\hat{\phi}}}_{61}\bar{C}''_2 + a_{47}\bar{A}\bar{\bar{\hat{\phi}}}'_{91} \\
&\quad + \frac{1}{2}a_{47}\bar{A}\bar{\bar{\hat{\phi}}}'_{92} + a_{48}\bar{A}'\bar{\bar{\hat{\phi}}}'_{91} + \frac{1}{2}a_{48}\bar{A}'\bar{\bar{\hat{\phi}}}'_{92} + a_{49}\bar{A}''\bar{\bar{\hat{\phi}}}_{91} + \frac{1}{2}a_{49}\bar{A}''\bar{\bar{\hat{\phi}}}_{92} \\
&\quad + \delta_{n,\ell}[\frac{3}{4}a_{15}\bar{\bar{\hat{\phi}}}_{13}\bar{C}''_3 + a_{50}\bar{\bar{\hat{\phi}}}_{91}\bar{C}''_3 + \frac{1}{2}a_{50}\bar{\bar{\hat{\phi}}}_{92}\bar{C}''_3 + a_{50}\bar{\bar{\hat{\phi}}}_{93}\bar{C}''_1 + \frac{1}{2}a_{50}\bar{\bar{\hat{\phi}}}_{93}\bar{C}''_2\}
\end{aligned}
$$

$$
\bar{f}_3 = \{b_1\bar{\bar{\hat{\phi}}}''_{21} + b_2\bar{\bar{\hat{\phi}}}'_{21} + b_3\bar{\bar{\hat{\phi}}}_{21} + b_4\bar{B}'' + b_5\bar{B}' + b_6\bar{B} + b_{22}\bar{\bar{\hat{\phi}}}''_{71} + b_{23}\bar{\bar{\hat{\phi}}}'_{71} + b_{24}\bar{\bar{\hat{\phi}}}_{71}
$$

$$+ b_{25}\bar{\hat{\hat{\phi}}}''_{81} + b_{26}\bar{\hat{\hat{\phi}}}'_{81} + b_{27}\bar{\hat{\hat{\phi}}}_{81} - \omega^2 b_{44}\bar{B}\} +$$

$$+ \{b_7\bar{B}\bar{C}_1 - \frac{1}{2} b_7\bar{B}\bar{C}_2 + \frac{1}{4} b_9\bar{B}\bar{A}^2 + \frac{3}{4} b_{10}\bar{B}^3 + b_{11}\bar{B}\bar{C}''_1 - \frac{1}{2} b_{11}\bar{B}\bar{C}''_2 + b_{12}\bar{\hat{\hat{\phi}}}_{12}\bar{C}''_1$$

$$- \frac{1}{2} b_{12}\bar{\hat{\hat{\phi}}}_{21}\bar{C}''_2 + \frac{1}{2} b_{12}\bar{\hat{\hat{\phi}}}_{22}\bar{C}''_2 + b_{16}\bar{B}\bar{\hat{\hat{\phi}}}''_{31} + b_{17}\bar{B}'\bar{\hat{\hat{\phi}}}'_{31} + b_{18}\bar{B}''\bar{\hat{\hat{\phi}}}_{31}$$

$$- \frac{1}{2} [b_{16}\bar{B}\bar{\hat{\hat{\phi}}}''_{32} + b_{17}\bar{B}'\bar{\hat{\hat{\phi}}}'_{32} + b_{18}\bar{B}''\bar{\hat{\hat{\phi}}}_{32}] + \frac{1}{2} [b_{19}\bar{A}\bar{\hat{\hat{\phi}}}''_{41} + b_{20}\bar{A}'\bar{\hat{\hat{\phi}}}'_{41} + b_{21}\bar{A}''\bar{\hat{\hat{\phi}}}_{41}] +$$

$$+ b_{40}\bar{\hat{\hat{\phi}}}_{81}\bar{C}''_1 - \frac{1}{2} b_{40}\bar{\hat{\hat{\phi}}}_{81}\bar{C}''_2 + b_{41}\bar{B}\bar{\hat{\hat{\phi}}}''_{91} - \frac{1}{2} b_{41}\bar{B}\bar{\hat{\hat{\phi}}}''_{92} + b_{42}\bar{B}'\bar{\hat{\hat{\phi}}}'_{91} - \frac{1}{2} b_{42}\bar{B}'\bar{\hat{\hat{\phi}}}'_{92}$$

$$+ b_{43}\bar{B}''\bar{\hat{\hat{\phi}}}_{91} - \frac{1}{2} b_{43}\bar{\hat{\hat{\phi}}}_{92}\bar{B}'' + \frac{1}{2} \delta_{n,\ell} b_{12}\bar{\hat{\hat{\phi}}}_{23}C''_3\}$$

$$\bar{f}_{41} = \{c_1\bar{C}''_1 + c_2\bar{C}_1 + c_{19}\bar{\hat{\hat{\phi}}}''_{31} + c_{20}\bar{\hat{\hat{\phi}}}'_{31} + c_{21}\bar{\hat{\hat{\phi}}}_{31} + c_{31}\bar{\hat{\hat{\phi}}}''_{91} + c_{32}\bar{\hat{\hat{\phi}}}'_{91} + c_{33}\bar{\hat{\hat{\phi}}}_{91}$$

$$+ \delta_{n,\ell}\,[\frac{1}{2} c_{10}\bar{\hat{\hat{\phi}}}''_{13} + \frac{1}{2} c_{11}\bar{\hat{\hat{\phi}}}'_{13} + \frac{1}{2} c_{12}\bar{\hat{\hat{\phi}}}_{13}]\} +$$

$$+ \{\frac{1}{2} c_7\bar{A}^2 + \frac{1}{2} c_8\bar{B}^2 + \frac{1}{2} c_9[(\bar{A}')^2 + (\bar{B}')^2 + \bar{A}\bar{A}'' + \bar{B}\bar{B}''] + \frac{1}{2} c_{13}\bar{A}\,\bar{\hat{\hat{\phi}}}''_{11}$$

$$+ \frac{1}{2} c_{14}\bar{A}'\,\bar{\hat{\hat{\phi}}}'_{11} + \frac{1}{2} c_{15}\bar{A}''\bar{\hat{\hat{\phi}}} + \frac{1}{2} c_{16}\bar{B}\,\bar{\hat{\hat{\phi}}}''_{21} + \frac{1}{2} c_{17}\bar{B}'\,\bar{\hat{\hat{\phi}}}'_{21} + \frac{1}{2} c_{18}\bar{B}''\bar{\hat{\hat{\phi}}}_{21}$$

$$+ \frac{1}{2} [c_{25}\bar{A}\,\bar{\hat{\hat{\phi}}}''_{61} + c_{26}\bar{A}'\,\bar{\hat{\hat{\phi}}}'_{61} + c_{27}\bar{A}''\bar{\hat{\hat{\phi}}}_{61} + c_{28}\bar{B}\,\bar{\hat{\hat{\phi}}}''_{81} + c_{29}\bar{B}'\,\bar{\hat{\hat{\phi}}}'_{81} + c_{30}\bar{B}''\bar{\hat{\hat{\phi}}}_{81}]$$

$$+ \delta_{n,\ell} \left[\frac{1}{2} c_{34}\bar{A}\, \bar{\hat{\hat{\phi}}}{}''_{93} + \frac{1}{2} c_{35}\bar{A}'\bar{\hat{\hat{\phi}}}{}'_{93} + \frac{1}{2} c_{36}\bar{A}''\, \bar{\hat{\hat{\phi}}}_{93}\right]\}$$

$$\bar{f}_{42} = \{c_1\bar{C}''_2 + c_2\bar{C}_2 + c_{19}\bar{\hat{\hat{\phi}}}{}''_{32} + c_{20}\bar{\hat{\hat{\phi}}}{}'_{32} + c_{21}\bar{\hat{\hat{\phi}}}_{32} + c_{31}\bar{\hat{\hat{\phi}}}{}''_{92} + c_{32}\bar{\hat{\hat{\phi}}}{}'_{92} + c_{33}\bar{\hat{\hat{\phi}}}_{92}$$

$$- 4\, c_{37}\omega^2\bar{C}_2 + \frac{1}{2}\delta_{n,\ell}\left[c_{10}\bar{\hat{\hat{\phi}}}{}''_{13} + c_{11}\bar{\hat{\hat{\phi}}}{}'_{13} + c_{12}\bar{\hat{\hat{\phi}}}_{13}\right]\} +$$

$$+ \frac{1}{2}\{c_7\bar{A}^2 - c_8\bar{B}^2 + \left[(\bar{A}')^2 - (\bar{B}')^2 + \bar{A}\bar{A}'' - \bar{B}\bar{B}''\right] + c_{13}\bar{A}\,\bar{\hat{\hat{\phi}}}{}''_{11} + c_{13}\bar{A}\,\bar{\hat{\hat{\phi}}}{}''_{12}$$

$$+ c_{14}\bar{A}'\,\bar{\hat{\hat{\phi}}}{}'_{11} + c_{14}\bar{A}'\,\bar{\hat{\hat{\phi}}}{}'_{12} + c_{15}\bar{A}''\,\bar{\hat{\hat{\phi}}}_{11} + c_{15}\bar{A}''\,\bar{\hat{\hat{\phi}}}{}'_{12} - c_{16}\bar{B}\,\bar{\hat{\hat{\phi}}}{}''_{21}$$

$$+ c_{16}\bar{B}\,\bar{\hat{\hat{\phi}}}{}''_{22} + c_{17}\bar{B}'\,\bar{\hat{\hat{\phi}}}{}'_{21} + c_{17}\bar{B}'\,\bar{\hat{\hat{\phi}}}{}'_{22} + c_{17}\bar{B}'\,\bar{\hat{\hat{\phi}}}{}'_{22} - c_{18}\bar{B}''\,\bar{\hat{\hat{\phi}}}_{21}$$

$$+ c_{18}\,\bar{B}''\,\bar{\hat{\hat{\phi}}}_{22} + c_{25}\bar{A}\,\bar{\hat{\hat{\phi}}}{}''_{61} + c_{26}\bar{A}'\,\bar{\hat{\hat{\phi}}}{}'_{61} + c_{27}\bar{A}''\,\bar{\hat{\hat{\phi}}}_{61} - c_{28}\bar{B}\,\bar{\hat{\hat{\phi}}}{}''_{81}$$

$$- c_{29}\bar{B}'\,\bar{\hat{\hat{\phi}}}{}'_{81} - c_{30}\bar{B}''\,\bar{\hat{\hat{\phi}}}_{81} + \delta_{n,\ell}\left[c_{34}\bar{A}\,\bar{\hat{\hat{\phi}}}{}''_{93} + c_{35}\bar{A}'\,\bar{\hat{\hat{\phi}}}{}'_{93} + c_{36}\bar{A}''\,\bar{\hat{\hat{\phi}}}_{93}\right]\}$$

$$\bar{f}_{43} = \delta_{n,\ell}\{c_3\bar{A} + c_5\bar{A}' + c_6\bar{A}'' + c_{10}\bar{\hat{\hat{\phi}}}{}''_{11} + c_{11}\bar{\hat{\hat{\phi}}}{}'_{11} + c_{12}\bar{\hat{\hat{\phi}}}_{11} + c_{22}\bar{\hat{\hat{\phi}}}{}''_{61} + c_{23}\bar{\hat{\hat{\phi}}}{}'_{61} + c_{24}\bar{\hat{\hat{\phi}}}_{61}$$

$$+ c_1\bar{C}''_3 + c_2\bar{C}_3 + c_{31}\bar{\hat{\hat{\phi}}}{}''_{93} + c_{32}\bar{\hat{\hat{\phi}}}{}'_{93} + c_{33}\bar{\hat{\hat{\phi}}}_{93} - c_{37}\omega^2\bar{C}_3\} +$$

$$+ \delta_{n,\ell}\{c_{34}\bar{A}\bar{\hat{\hat{\phi}}}{}''_{91} + \frac{1}{2}c_{34}\bar{A}\bar{\hat{\hat{\phi}}}{}''_{92} + c_{35}\bar{A}'\bar{\hat{\hat{\phi}}}{}'_{91} + \frac{1}{2}c_{35}\bar{A}'\bar{\hat{\hat{\phi}}}{}'_{92} + c_{36}\bar{A}''\bar{\hat{\hat{\phi}}}_{91} + \frac{1}{2}c_{36}\bar{A}''\bar{\hat{\hat{\phi}}}_{92}$$

$$+ \frac{3}{4}\,[c_{13}\bar{A}\,\bar{\bar{\hat{\phi}}}''_{13} + c_{14}\bar{A}'\bar{\bar{\hat{\phi}}}'_{13} + c_{15}\bar{A}''\bar{\bar{\hat{\phi}}}_{13}] + \frac{1}{2}\,[c_{18}\bar{B}''\bar{\bar{\hat{\phi}}}_{23} + c_{17}\bar{B}'\bar{\bar{\hat{\phi}}}'_{23}$$

$$+ c_{16}\bar{B}\,\bar{\bar{\hat{\phi}}}''_{23}]\}$$

$$\bar{f}_{11} = \bar{f}_{2}\ (a_i \to d_i)$$

$$\bar{f}_{12} = \{d_1\bar{\bar{\hat{\phi}}}''_{12} + d_2\bar{\bar{\hat{\phi}}}'_{12} + d_3\bar{\bar{\hat{\phi}}}_{12}\} + \{\frac{1}{2}\,d_{10}\bar{A}\,\bar{C}_2 + \frac{1}{4}\,d_{11}\bar{A}^3 - \frac{1}{4}\,d_{12}\,\bar{A}\,\bar{B}^2 + \frac{1}{2}\,d_{14}\bar{A}\,\bar{C}''_2$$

$$+ \frac{1}{2}\,d_{15}\,\bar{\bar{\hat{\phi}}}_{11}\bar{C}''_2 + d_{15}\bar{\bar{\hat{\phi}}}_{12}\bar{C}''_1 + \frac{1}{2}\,d_{19}\bar{A}\,\bar{\bar{\hat{\phi}}}''_{32} + \frac{1}{2}\,d_{20}\bar{A}'\bar{\bar{\hat{\phi}}}'_{32} + \frac{1}{2}\,d_{21}\bar{A}''\bar{\bar{\hat{\phi}}}_{32} - \frac{1}{2}\,d_{22}\bar{B}\bar{\bar{\hat{\phi}}}''_{41}$$

$$- \frac{1}{2}\,d_{23}\bar{\bar{\hat{\phi}}}'_{41}\,\bar{B}' - \frac{1}{2}\,d_{24}\bar{B}''\bar{\bar{\hat{\phi}}}_{41} + \frac{1}{2}\,d_{37}\bar{\bar{\hat{\phi}}}_{61}\bar{C}''_2 + \frac{1}{2}\,d_{47}\bar{A}\bar{\bar{\hat{\phi}}}''_{92} + \frac{1}{2}\,d_{48}\bar{A}'\,\bar{\bar{\hat{\phi}}}'_{92}$$

$$+ \frac{1}{2}\,d_{49}\bar{A}''\bar{\bar{\hat{\phi}}}_{92} + \delta_{n,\ell}\,[\frac{1}{4}\,d_{15}\bar{\bar{\hat{\phi}}}_{13}\bar{C}''_3 + \frac{1}{2}\,d_{50}\bar{\bar{\hat{\phi}}}_{92}\bar{C}''_3 + \frac{1}{2}\,d_{50}\bar{\bar{\hat{\phi}}}_{93}\bar{C}''_2]\}$$

$$\bar{f}_{13} = \delta_{n,\ell}\,\{d_1\bar{\bar{\hat{\phi}}}''_{13} + d_2\bar{\bar{\hat{\phi}}}'_{13} + d_3\bar{\bar{\hat{\phi}}}_{13}\} + \delta_{n,e}\{d_8\bar{A}^2 + d_{31}\bar{A}\bar{\bar{\hat{\phi}}}''_{51} + d_{33}\bar{A}''\bar{\bar{\hat{\phi}}}_{51} + d_{32}\bar{A}'\bar{\bar{\hat{\phi}}}'_{51}$$

$$+ d_{34}\bar{A}\,\bar{\bar{\hat{\phi}}}''_{61} + d_{35}\bar{A}'\bar{\bar{\hat{\phi}}}'_{61} + d_{36}\bar{A}''\bar{\bar{\hat{\phi}}}_{61} + d_{10}\bar{A}\,\bar{C}_3 + d_{14}\bar{A}\,\bar{C}''_3 + d_{15}\bar{\bar{\hat{\phi}}}_{11}\bar{C}''_3$$

$$+ d_{15}\bar{\bar{\hat{\phi}}}_{13}\bar{C}'' + d_{37}\bar{\bar{\hat{\phi}}}_{61}\bar{C}''_3 + d_{47}\bar{A}\,\bar{\bar{\hat{\phi}}}''_{91} + d_{48}\bar{A}'\bar{\bar{\hat{\phi}}}'_{93} + d_{49}\bar{A}''\bar{\bar{\hat{\phi}}}_{93} + d_{50}\bar{\bar{\hat{\phi}}}_{93}\bar{C}''_3$$

$$+ \frac{1}{3}\,[d_9\bar{B}^2 + d_{38}\bar{B}\,\bar{\bar{\hat{\phi}}}''_{71} + d_{39}\bar{B}'\bar{\bar{\hat{\phi}}}'_{71} + d_{40}\bar{B}''\bar{\bar{\hat{\phi}}}_{71} + d_{41}\bar{B}\,\bar{\bar{\hat{\phi}}}''_{81} + d_{42}\bar{B}'\bar{\bar{\hat{\phi}}}'_{81} + d_{43}\bar{B}'\bar{\bar{\hat{\phi}}}'_{81}]$$

$$+ \frac{4}{3} [d_7\bar{C}_1 + d_{13}\bar{C}''_1 + d_{16}\overset{=}{\hat{\phi}}''_{31} + d_{17}\overset{=}{\hat{\phi}}'_{31} + d_{18}\overset{=}{\hat{\phi}}_{31} + d_{44}\overset{=}{\hat{\phi}}''_{91} + d_{45}\overset{=}{\hat{\phi}}'_{91} + d_{46}\overset{=}{\hat{\phi}}_{91}$$

$$+ d_{50}\overset{=}{\hat{\phi}}_{91}\bar{C}''_1 + \frac{1}{2} d_{50}\overset{=}{\hat{\phi}}_{92}\bar{C}''_2 + \frac{1}{4} d_{15} \overset{=}{\hat{\phi}}_{13} \bar{C}''_2] +$$

$$+ \frac{2}{3} [d_7\bar{C}_2 + d_{13}\bar{C}''_2 + d_{16}\overset{=}{\hat{\phi}}''_{32} + d_{17}\overset{=}{\hat{\phi}}'_{32} + d_{18}\overset{=}{\hat{\phi}}_{32} + d_{44}\overset{=}{\hat{\phi}}''_{92} + d_{45}\overset{=}{\hat{\phi}}'_{92} + d_{46}\overset{=}{\hat{\phi}}_{92}$$

$$+ d_{50}\overset{=}{\hat{\phi}}\ \bar{C}''_1 + d_{54}\overset{=}{\hat{\phi}}_{91}\bar{C}''_2 + \frac{1}{2} d_{15}\overset{=}{\hat{\phi}}_{12}\bar{C}''_3 + \frac{1}{2} d_{15}\overset{=}{\hat{\phi}}_{13}\bar{C}''_2]\}$$

$$\bar{f}_{21} = \bar{f}_3\ (b_i \to e_i)$$

$$\bar{f}_{22} = \{e_1\overset{=}{\hat{\phi}}''_{22} + e_2\overset{=}{\hat{\phi}}'_{22} + e_3\overset{=}{\hat{\phi}}_{22}\} + \{\frac{1}{2} e_7\bar{B}\ \bar{C}_2 + \frac{1}{4} e_9\bar{B}\ \bar{A}^2 - \frac{1}{4} e_{10}\bar{B}^3 + \frac{1}{2} e_{11}\bar{B}\ \bar{C}''_2$$

$$+ \frac{1}{2} e_{12}\overset{=}{\hat{\phi}}_{21}\bar{C}''_2 + e_{12}\overset{=}{\hat{\phi}}_{22}\bar{C}''_1 + \frac{1}{2} [e_{16}\bar{B}\overset{=}{\hat{\phi}}''_{32} + e_{17}\bar{B}'\overset{=}{\hat{\phi}}'_{32} + e_{18}\bar{B}''\overset{=}{\hat{\phi}}_{32}] + e_{19}\bar{A}\ \overset{=}{\hat{\phi}}''_{41}$$

$$+ e_{20}\bar{A}'\overset{=}{\hat{\phi}}'_{41} + e_{21}\bar{A}''\overset{=}{\hat{\phi}}_{41} + e_{40}\overset{=}{\hat{\phi}}_{81}\bar{C}''_2 + e_{41}\overset{=}{\hat{\phi}}''_{92}\bar{B} + e_{42}\overset{=}{\hat{\phi}}'_{92}\bar{B}' + e_{43}\overset{=}{\hat{\phi}}_{92}\bar{B}''$$

$$+ \delta_{n,\ell}\ \frac{1}{2}\ e_{12}\overset{=}{\hat{\phi}}_{23}\bar{C}''_3\}$$

$$\bar{f}_{23} = \delta_{n,\ell}\ \{e_1\overset{=}{\hat{\phi}}''_{23} + e_2\overset{=}{\hat{\phi}}'_{23} + e_3\overset{=}{\hat{\phi}}_{23} + e_{13}\overset{=}{\hat{\phi}}''_{41} + e_{14}\overset{=}{\hat{\phi}}'_{41} + e_{15}\overset{=}{\hat{\phi}}_{41}\} +$$

$$+ \delta_{n,\ell}\{\frac{1}{2}\ [e_8\bar{A}\ \bar{B} + e_{28}\bar{B}\ \overset{=}{\hat{\phi}}''_{51} + e_{29}\ \bar{B}'\overset{=}{\hat{\phi}}'_{51} + e_{30}\bar{B}''\overset{=}{\hat{\phi}}_{51} + e_{31}\overset{=}{\hat{\phi}}''_{61}\ \bar{B} + e_{32}\bar{B}'\overset{=}{\hat{\phi}}'_{61}$$

$$+ e_{33}\bar{\bar{\hat{\phi}}}\,\bar{B}'' + e_{34}\bar{A}\bar{\bar{\hat{\phi}}}''_{71} + e_{35}\bar{A}'\bar{\bar{\hat{\phi}}}'_{71} + e_{36}\bar{A}''\bar{\bar{\hat{\phi}}}_{71} + e_{37}\bar{A}\bar{\bar{\hat{\phi}}}''_{81} + e_{38}\bar{A}'\,\bar{\bar{\hat{\phi}}}'_{81}$$

$$+ e_{39}\bar{A}''\bar{\bar{\hat{\phi}}}_{81} + e_{7}\bar{B}\,\bar{C}_{3} + e_{11}\bar{B}\,\bar{C}''_{3} + e_{12}\bar{\bar{\hat{\phi}}}_{21}\bar{C}''_{3} + e_{12}\bar{\bar{\hat{\phi}}}\,\bar{C}''_{3} + e_{12}\bar{\bar{\hat{\phi}}}_{23}\bar{C}''_{1}$$

$$+ e_{40}\bar{\bar{\hat{\phi}}}_{81}\bar{C}''_{3} + e_{41}\bar{B}\,\bar{\bar{\hat{\phi}}}''_{93} + e_{42}\bar{\bar{\hat{\phi}}}'_{93}\bar{B}' + e_{43}\bar{B}''\bar{\bar{\hat{\phi}}}_{93}]\}$$

$$\bar{f}_{31} = g_{1}\bar{\bar{\hat{\phi}}}''_{31} + g_{2}\bar{\bar{\hat{\phi}}}_{31} - \frac{1}{2}\,g_{3}\,[(\bar{A}')^{2} - (\bar{B}')^{2} - \bar{A}\bar{A}'' + \bar{B}\bar{B}'']$$

$$\bar{f}_{32} = g_{1}\bar{\bar{\hat{\phi}}}''_{32} + g_{2}\bar{\bar{\hat{\phi}}}_{32} - \frac{1}{2}\,g_{3}\,[(\bar{A}')^{2} + (\bar{B}')^{2} - \bar{A}\bar{A}'' - \bar{B}\bar{B}'']$$

$$\bar{f}_{41} = g_{1}\bar{\bar{\hat{\phi}}}''_{41} + g_{2}\bar{\bar{\hat{\phi}}}_{41} - g_{3}\,[\bar{A}'\bar{B}' - \frac{1}{2}\,\bar{A}\,\bar{B}'' - \frac{1}{2}\,\bar{B}\,\bar{A}'']$$

$$\bar{f}_{51} = h_{1}\bar{\bar{\hat{\phi}}}''_{51} + h_{2}\bar{\bar{\hat{\phi}}}_{51} + h_{3}\bar{A}'' - h_{4}\bar{A}' + h_{5}\bar{A}$$

$$\bar{f}_{61} = i_{1}\bar{\bar{\hat{\phi}}}''_{61} + i_{2}\bar{\bar{\hat{\phi}}} + h_{3}\bar{A}'' + h_{4}\bar{A}' + h_{5}\bar{A}$$

$$\bar{f}_{71} = h_{1}\bar{\bar{\hat{\phi}}}''_{71} + h_{2}\bar{\bar{\hat{\phi}}}_{71} + h_{3}\bar{B}'' - h_{4}\bar{B}' + h_{5}\bar{B}$$

$$\bar{f}_{81} = i_{1}\bar{\bar{\hat{\phi}}}''_{81} + i_{2}\bar{\bar{\hat{\phi}}}_{81} + h_{3}\bar{B}'' + h_{4}\bar{B}' + h_{5}\bar{B}$$

$$\bar{f}_{91} = j_{1}\bar{\bar{\hat{\phi}}}''_{91} + j_{2}\bar{\bar{\hat{\phi}}}_{91} + j_{3}\bar{C}''_{1}$$

$$\bar{f}_{92} = j_1 \bar{\bar{\hat{\Phi}}}{}''_{92} + j_2 \bar{\bar{\hat{\Phi}}}_{92} + j_3 \bar{C}''_2$$

$$\bar{f}_{93} = \delta_{n,\ell} \{ j_1 \bar{\bar{\hat{\Phi}}}{}''_{93} + j_2 \bar{\bar{\hat{\Phi}}}_{93} + j_3 \bar{C}''_3 \}$$

Where the coefficients $a_i$, $d_i$ ($i=1,2,\ldots,52$) $b_j$, $e_j$ ($j=1,2,\ldots,44$) and $c_k$($k=1,2,\ldots,37$) are listed in 6-A.3, the coefficients $g_m$ ($m=1,2,3$), $h_n$($n=1,2,\ldots,5$) $i_\ell$($\ell=1,2$) and $j_m$($m=1,2,3$) are listed below:

$$g_1 = \frac{\bar{H}_{xy}}{\bar{H}_{xx}} (2\ell)^2$$

$$g_2 = -\frac{\bar{H}_{yy}}{\bar{H}_{xx}} (2\ell)^4$$

$$g_3 = \frac{1}{2} \frac{c}{\bar{H}_{xx}} \frac{h}{R} \ell^2$$

$$h_1 = \frac{\bar{H}_{xy}}{\bar{H}_{xx}} (n+\ell)^2$$

$$h_2 = -\frac{\bar{H}_{yy}}{\bar{H}_{xx}} (n+\ell)^4$$

$$h_3 = \frac{1}{2} \frac{c}{\bar{H}_{xx}} \frac{h}{R} (\hat{W}_1 + A_1)\ n^2$$

$$h_4 = \frac{c}{\bar{H}_{xx}} \frac{h}{R} (\hat{W}_1 + A_1)'\ \ell n$$

$$h_5 = \frac{1}{2}\,\frac{c}{\bar{H}_{xx}}\,\frac{h}{R}\,(\hat{W}_1 + A_1)''\,\ell^2$$

$$i_1 = \frac{\bar{H}_{xy}}{\bar{H}_{xx}}\,(\ell - n)^2$$

$$i_2 = -\,\frac{\bar{H}_{yy}}{\bar{H}_{xx}}\,(\ell - n)^4$$

$$j_1 = \frac{\bar{H}_{xy}}{\bar{H}_{xx}}\,n^2$$

$$j_2 = -\,\frac{\bar{H}_{yy}}{\bar{H}_{xx}}\,n^4$$

$$j_3 = \frac{c}{\bar{H}_{xx}}\,\frac{h}{R}\,n^2\,(\hat{W}_1 + A_1)$$

6-A.6 COEFFICIENTS OF EQUATIONS (6-2-33) TO (6-2-35)

$$\tilde{f}_2 = a_1\bar{\bar{\hat{\phi}}}''_{11} + a_2\bar{\bar{\hat{\phi}}}'_{11} + a_3\bar{\bar{\hat{\phi}}}_{11} + a_4\bar{A}'' + a_5\bar{A}' + a_6\bar{A} + \frac{3}{2}\,a_{10}\bar{A}\,\bar{C}_1 + \frac{3}{4}\,a_{11}\bar{A}^3$$

$$+ \frac{3}{2}\,a_{14}\bar{A}\,\bar{C}''_1 + \frac{3}{2}\,a_{15}\bar{\bar{\hat{\phi}}}_{11}\bar{C}''_1 + \frac{1}{2}\,a_{15}\bar{\bar{\hat{\phi}}}_{12}\bar{C}''_1 + \frac{3}{2}\,a_{19}\bar{A}\bar{\bar{\hat{\phi}}}''_{31} + \frac{3}{2}\,a_{20}\bar{A}'\bar{\bar{\hat{\phi}}}'_{31} + \frac{3}{2}\,a_{21}\bar{A}''\bar{\bar{\hat{\phi}}}_{31}$$

$$+ a_{25}\bar{\bar{\hat{\phi}}}''_{51} + a_{26}\bar{\bar{\hat{\phi}}}'_{51} + a_{27}\bar{\bar{\hat{\phi}}}_{51} + a_{28}\bar{\bar{\hat{\phi}}}''_{61} + a_{29}\bar{\bar{\hat{\phi}}}'_{61} + a_{30}\bar{\bar{\hat{\phi}}}_{61} + \frac{3}{2}\,a_{37}\bar{\bar{\hat{\phi}}}_{61}\bar{C}''_1$$

$$+ \frac{3}{2}\,a_{47}\bar{A}\bar{\bar{\hat{\phi}}}''_{91} + \frac{3}{2}\,a_{48}\bar{A}'\bar{\bar{\hat{\phi}}}'_{91} + \frac{3}{2}\,a_{49}\bar{A}''\bar{\bar{\hat{\phi}}}_{91} - \omega^2\,a_{51}\bar{A} + a_{52}\bar{Q}$$

$$+ \delta_{n,\ell}\{a_7\bar{C}_3 + a_{13}\bar{C}''_3 + \frac{3}{4}\, a_{15}\bar{\bar{\hat{\hat{\phi}}}}_{13}\bar{C}''_3 + a_{44}\bar{\bar{\hat{\hat{\phi}}}}''_{93} + a_{45}\bar{\bar{\hat{\hat{\phi}}}}'_{93} + a_{46}\bar{\bar{\hat{\hat{\phi}}}}_{93}$$

$$+ \frac{3}{2}\, a_{50}\, (\bar{\bar{\hat{\hat{\phi}}}}_{91}\bar{C}''_3 + \bar{\bar{\hat{\hat{\phi}}}}_{93}\bar{C}''_1)\}$$

$$\tilde{f}_{31} = c_1\bar{C}''_1 + c_2\bar{C}_1 + \frac{1}{2}\,(c_7\bar{A}^2 + c_9(\bar{A}')^2 + c_9\bar{A}\bar{A}'' + c_{13}A\bar{\bar{\hat{\hat{\phi}}}}''_{11} + c_{14}\bar{A}'\bar{\bar{\hat{\hat{\phi}}}}'_{11} + c_{15}\bar{\bar{\hat{\hat{\phi}}}}_{11}\bar{A}''$$

$$+ c_{25}\bar{A}\bar{\bar{\hat{\hat{\phi}}}}''_{61} + c_{26}\bar{A}'\bar{\bar{\hat{\hat{\phi}}}}'_{61} + c_{27}\bar{A}''\bar{\bar{\hat{\hat{\phi}}}}_{61}) + c_{19}\bar{\bar{\hat{\hat{\phi}}}}''_{31} + c_{20}\bar{\bar{\hat{\hat{\phi}}}}'_{31} + c_{21}\bar{\bar{\hat{\hat{\phi}}}}_{31} + c_{31}\bar{\bar{\hat{\hat{\phi}}}}''_{91}$$

$$+ c_{32}\bar{\bar{\hat{\hat{\phi}}}}'_{91} + c_{33}\bar{\bar{\hat{\hat{\phi}}}}_{91} + \frac{1}{2}\,\delta_{n,\ell}\,(c_{10}\bar{\bar{\hat{\hat{\phi}}}}''_{13} + c_{11}\bar{\bar{\hat{\hat{\phi}}}}'_{13} + c_{12}\bar{\bar{\hat{\hat{\phi}}}}_{13} + c_{34}\bar{A}\,\bar{\bar{\hat{\hat{\phi}}}}''_{93}$$

$$+ c_{35}\bar{A}'\bar{\bar{\hat{\hat{\phi}}}}'_{93} + c_{36}\bar{A}''\bar{\bar{\hat{\hat{\phi}}}}_{93}) + \frac{1}{6}\,(c_{13}\bar{A}\bar{\bar{\hat{\hat{\phi}}}}''_{12} + c_{10}\bar{A}'\bar{\bar{\hat{\hat{\phi}}}}'_{12} + c_{15}\bar{A}''\bar{\bar{\hat{\hat{\phi}}}}_{12}) - \frac{4}{3}\,\omega^2\, c_{37}\bar{C}_1$$

$$\tilde{f}_{32} = \delta_{n,\ell}\,\{c_1\bar{C}''_3 + c_2\bar{C}_3 + c_3\bar{A} + c_4\bar{A}' + c_6\bar{A}'' + c_{10}\bar{\bar{\hat{\hat{\phi}}}}''_{11} + c_{11}\bar{\bar{\hat{\hat{\phi}}}}'_{11} + c_{12}\bar{\bar{\hat{\hat{\phi}}}}_{11} + c_{22}\bar{\bar{\hat{\hat{\phi}}}}''_{61}$$

$$+ c_{23}\bar{\bar{\hat{\hat{\phi}}}}'_{61} + c_{24}\bar{\bar{\hat{\hat{\phi}}}}_{61} + c_{31}\bar{\bar{\hat{\hat{\phi}}}}''_{93} + c_{32}\bar{\bar{\hat{\hat{\phi}}}}'_{93} + c_{33}\bar{\bar{\hat{\hat{\phi}}}}_{93} - \omega^2\, c_{37}\bar{C}_3$$

$$+ \frac{3}{4}\,[2c_{34}\bar{A}\,\bar{\bar{\hat{\hat{\phi}}}}''_{91} + 2c_{35}\bar{A}'\,\bar{\bar{\hat{\hat{\phi}}}}'_{91} + 2c_{36}\bar{A}''\,\bar{\bar{\hat{\hat{\phi}}}}_{91} + c_{13}\bar{A}\,\bar{\bar{\hat{\hat{\phi}}}}''_{13} + c_{14}\bar{A}'\,\bar{\bar{\hat{\hat{\phi}}}}'_{13}$$

$$+ c_{15}\bar{A}''\,\bar{\bar{\hat{\hat{\phi}}}}_{13}]\}$$

$$\tilde{f}_{11} = \tilde{f}_2\ (a_i \rightarrow d_i)$$

$$\tilde{f}_{12} = d_1\bar{\bar{\hat{\hat{\phi}}}}''_{12} + d_2\bar{\bar{\hat{\hat{\phi}}}}'_{12} + d_3\bar{\bar{\hat{\hat{\phi}}}}_{12} + \frac{1}{2}\, d_{10}\bar{A}\ \bar{C}'_1 + \frac{1}{4}\, d_{11}\bar{A}'^{2} + \frac{1}{2}\, d_{14}\bar{A}\ \bar{C}''_1 + \frac{1}{2}\, d_{15}\bar{\bar{\hat{\hat{\phi}}}}_{11}\bar{C}''_1$$

$$+ d_{15}\bar{\bar{\hat{\hat{\phi}}}}_{12}\bar{C}''_1 + \frac{1}{2}\left(d_{19}\bar{A}\ \bar{\bar{\hat{\hat{\phi}}}}''_{31} + d_{20}\bar{A}'\ \bar{\bar{\hat{\hat{\phi}}}}'_{31} + d_{21}\bar{A}''\bar{\bar{\hat{\hat{\phi}}}}\right) + \frac{1}{2}\, d_{37}\bar{\bar{\hat{\hat{\phi}}}}_{61}\bar{C}''_1$$

$$+ \frac{1}{2}\left(d_{47}\bar{A}\bar{\bar{\hat{\hat{\phi}}}}''_{91} + d_{48}\bar{A}'\bar{\bar{\hat{\hat{\phi}}}}'_{91} + d_{49}\bar{A}''\bar{\bar{\hat{\hat{\phi}}}}_{91}\right) + \delta_{n,\ell}\Big(\frac{1}{4}\, d_{15}\bar{\bar{\hat{\hat{\phi}}}}_{13}\bar{C}''_3 + \frac{1}{2}\, d_{50}\bar{\bar{\hat{\hat{\phi}}}}_{91}\bar{C}''_3$$

$$+ \frac{1}{2}\,\bar{\bar{\hat{\hat{\phi}}}}_{93}\bar{C}''_2\Big)$$

$$\tilde{f}_{13} = \delta_{n,\ell}\Big(d_1\bar{\bar{\hat{\hat{\phi}}}}''_{13} + d_2\bar{\bar{\hat{\hat{\phi}}}}'_{13} + d_3\bar{\bar{\hat{\hat{\phi}}}}_{13} + d_8\bar{A}^2 + d_{31}\bar{A}\ \bar{\bar{\hat{\hat{\phi}}}}''_{51} + d_{32}\bar{A}'\ \bar{\bar{\hat{\hat{\phi}}}}'_{51} + d_{33}\bar{A}''\ \bar{\bar{\hat{\hat{\phi}}}}_{51}$$

$$+ d_{34}\bar{A}\ \bar{\bar{\hat{\hat{\phi}}}}''_{61} + d_{35}\bar{A}'\ \bar{\bar{\hat{\hat{\phi}}}}'_{61} + d_{36}\bar{A}''\ \bar{\bar{\hat{\hat{\phi}}}}_{61} + d_{10}\bar{A}\ \bar{C}_3 + d_{14}\bar{A}\ \bar{C}''_3 + d_{15}\bar{\bar{\hat{\hat{\phi}}}}_{11}\bar{C}''_3$$

$$+ d_{15}\bar{\bar{\hat{\hat{\phi}}}}_{13}\bar{C}''_1 + d_{37}\bar{\bar{\hat{\hat{\phi}}}}_{61}\bar{C}''_3 + d_{47}\bar{A}\ \bar{\bar{\hat{\hat{\phi}}}}''_{93} + d_{48}\bar{A}'\ \bar{\bar{\hat{\hat{\phi}}}}'_{93} + d_{49}\bar{A}''\ \bar{\bar{\hat{\hat{\phi}}}}_{93}$$

$$+ d_{50}\bar{\bar{\hat{\hat{\phi}}}}_{93}\bar{C}''_3 + 2\big(d_7\bar{C}_1 + d_{13}\bar{C}''_2 + d_{16}\bar{\bar{\hat{\hat{\phi}}}}''_{31} + d_{17}\bar{\bar{\hat{\hat{\phi}}}}'_{31} + d_{18}\bar{\bar{\hat{\hat{\phi}}}}_{31}$$

$$+ d_{44}\bar{\bar{\hat{\hat{\phi}}}}''_{91} + d_{45}\bar{\bar{\hat{\hat{\phi}}}}'_{91} + d_{46}\bar{\bar{\hat{\hat{\phi}}}}_{91} + \frac{5}{3}\, d_{50}\bar{\bar{\hat{\hat{\phi}}}}_{91}\bar{C}''_1 + \frac{1}{3}\, d_{15}\bar{\bar{\hat{\hat{\phi}}}}_{13}\ \bar{C}''_1 + \frac{1}{6}\, d_{15}\bar{\bar{\hat{\hat{\phi}}}}_{12}C''_3\big)\Big)$$

$$\tilde{f}_{31} = g_1\bar{\bar{\hat{\hat{\phi}}}}''_{31} + g_2\bar{\bar{\hat{\hat{\phi}}}}_{31} - \frac{1}{2}\, g_3(\bar{A}')^2 + \frac{1}{2}\, g_3\ \bar{A}\bar{A}''$$

$$\tilde{f}_{51} = \bar{f}_{51}$$

$$\tilde{f}_{61} = \bar{f}_{61}$$

$$\tilde{f}_{91} = \bar{f}_{91}$$

$$\tilde{f}_{93} = \bar{f}_{93}$$

## APPENDIX 6-B DERIVATION OF THE PERIODICITY CONDITION

To satisfy the periodicity condition for the dynamic state we must have

$$\int_{0}^{2\pi R} \frac{\partial \hat{\hat{v}}}{\partial y}\, dy = 0 \tag{6-B-1}$$

where

$$\frac{\partial \hat{\hat{v}}}{\partial y} = \hat{\beta}\,\frac{1-\nu^2}{Eh}\,[-\nu\,\hat{\hat{\Phi}},_{yy} + (1+\mu_1)\,\hat{\hat{\Phi}},_{xx}] - \nu\hat{\beta}x\;{}_1\hat{\hat{W}},_{xx} + (1+\mu_1)\hat{\beta}x\;{}_2\hat{\hat{W}},_{yy}$$

$$+ \frac{1}{R}\,\hat{\hat{W}} - \frac{1}{2}\,(\hat{\hat{W}},_y)^2 - \hat{\hat{W}},_y(\hat{W},_y + \bar{W},_y) \tag{6-B-2}$$

Substituting the expressions for $\hat{\hat{\Phi}}$, $\hat{\hat{W}}$, $\hat{W}$ and $\bar{W}$ into Eq. (6-B-1), carrying out the y-integration and introducing the nondimensional quantities defined in Appendix 1-A.1 reduces Eq. (6-B-1) to:

$$\hat{\hat{\Phi}}_{0,\overline{\overline{xx}}} = \frac{1}{2}\,\frac{h}{R}\,\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\,C,_{xx} - \frac{c}{\bar{H}_{xx}}\,C + \frac{1}{4}\,\frac{c}{\bar{H}_{xx}}\,\frac{h}{R}\,\ell^2\;(A^2+B^2)$$

$$+ \delta_{n,\ell}\{-\hat{\hat{\Phi}}_{6,\overline{\overline{xx}}} + \frac{1}{2}\,\frac{c}{\bar{H}_{xx}}\,\frac{h}{R}\,(\hat{W}_1 + A_1)\;n^2 A\} \tag{6-B-3}$$

Eq. (6-B-3) is the periodicity condition which should be satisfied in the analysis.

## APPENDIX 6-C DERIVATION OF THE REDUCED BOUNDARY CONDITIONS

It is necessary to express the different boundary conditions of the dynamic state in terms of the average values of $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{\hat{\hat{\Phi}}}_i$. The individual boundary conditions can be transformed as follows.

### SS1 Boundary Condition

$$\hat{\hat{W}} = \hat{\hat{N}}_{xy} = \hat{\hat{M}}_x = \hat{\hat{N}}_x = 0 \qquad \text{at } \bar{x} = 0, \frac{L}{R}$$

$\hat{\hat{W}} = 0$ becomes upon substituting for $\hat{\hat{W}}$

$$\hat{\hat{W}}(\bar{x},\bar{y},t) = A(\bar{x},t)h \cos(\ell\bar{y}) + B(\bar{x},t)h \sin(\ell\bar{y}) + C(\bar{x},t)h = 0 \tag{6-C-1}$$

This must be true for all values $\bar{y}$, therefore

$$A(\bar{x},t) = B(\bar{x},t) = C(\bar{x},t) = 0 \quad \text{at } \bar{x} = 0, \frac{L}{R} \tag{6-C-2}$$

Applying the method of averaging to (6-C-2) yields

$$\bar{A}(\bar{x}) = \bar{B}(\bar{x}) = \bar{C}_1(\bar{x}) = \bar{C}_2(\bar{x}) = \bar{C}_3(\bar{x}) = 0 \tag{6-C-3}$$

$\hat{\hat{N}}_{xy} = 0$ becomes, upon substitution

$$\hat{\hat{N}}_{xy} = -\hat{\hat{\Phi}}_{,xy} = -\frac{ERh^2}{c}\left[-\hat{\hat{\Phi}}_{1,\bar{x}}\,\ell \sin(\ell\bar{y}) + \hat{\hat{\Phi}}_{2,\bar{x}}\,\ell\cos(\ell\bar{y}) - \hat{\hat{\Phi}}_{3,\bar{x}}(2\ell)\sin(2\ell\bar{y})\right.$$

$$\left. + \hat{\hat{\Phi}}_{4,\bar{x}}(2\ell)\cos(2\ell\bar{y}) - \hat{\hat{\Phi}}_{5,\bar{x}}(n+\ell)\sin(n+\ell)\bar{y} = 0\right.$$

$$- \hat{\hat{\phi}}_{6,\bar{x}} (\ell-n)\sin(\ell-n)\bar{y} + \hat{\hat{\phi}}_{7,\bar{x}} (n+\ell)\cos(n+\ell)\bar{y}$$

$$+ \hat{\hat{\phi}}_{8,\bar{x}} (\ell-n)\cos(\ell-n)\bar{y} - \hat{\hat{\phi}}_{9,\bar{x}}\, n \sin(n\bar{y})] \qquad (6\text{-C-}4)$$

This must also be true for all values $\bar{y}$, therefore

$$\hat{\hat{\phi}}_{i,\bar{x}} = 0 \qquad (i=1,2,\ldots,9) \qquad (6\text{-C-}5)$$

Substituting Eqs. (6-2-20) to (6-2-28) into Eq. (6-C-5) and then applying the averaging technique to the resulting equations yields

$$\bar{\hat{\hat{\phi}}}'_j = 0 \quad (j=11,12,13,21,22,23,31,32,41,51,61,71,81,91,92,93) \qquad (6\text{-C-}6)$$

$\hat{\hat{M}}_x = 0$ becomes upon substituting and intriduction of the usual nondimensional parameters

$$\tilde{\alpha}_o + \sum_{i=1}^{5} \tilde{\alpha}_i \cos(m_i y) + \sum_{i=1}^{5} \tilde{\beta}_j \sin(n_j y) = 0 \quad (i=1,2,\ldots,5, \quad j=1,2,\ldots,4) \qquad (6\text{-C-}7)$$

where

$$\tilde{\alpha}_1 = \bar{D}_{xx} A,_{\bar{x}\bar{x}} + 2\frac{R}{h}\bar{Q}_{xx}\hat{\hat{\phi}}_{1,\bar{x}\bar{x}} + 2\frac{R}{h}\frac{1+\mu_2}{\nu}\bar{Q}_{xx}\,\ell^2\,\hat{\hat{\phi}}_1$$

$$\tilde{\alpha}_2 = -2\frac{R}{h}\bar{Q}_{xx}\hat{\hat{\phi}}_{3,\bar{x}\bar{x}} - 2\frac{R}{h}\frac{1+\mu_2}{\nu}\bar{Q}_{xx}(2\ell)^2\,\hat{\hat{\phi}}_3$$

$$\tilde{\alpha}_3 = -2\frac{R}{h}\bar{Q}_{xx}\hat{\hat{\phi}}_{5,\bar{x}\bar{x}} - 2\frac{R}{h}\frac{1+\mu_2}{\nu}\bar{Q}_{xx}(n+\ell)^2\,\hat{\hat{\phi}}_5$$

$$\tilde{\alpha}_4 = -2\,\frac{R}{h}\,\bar{Q}_{xx}\,\hat{\hat{\phi}}_{6,\overline{\overline{xx}}} - 2\,\frac{R}{h}\,\frac{1+\mu_2}{\nu}\,\bar{Q}_{xx}\,(n-\ell)^2\,\hat{\hat{\phi}}_6$$

$$\tilde{\alpha}_5 = -2\,\frac{R}{h}\,\bar{Q}_{xx}\,\hat{\hat{\phi}}_{9,\overline{\overline{xx}}} - 2\,\frac{R}{h}\,\frac{1+\mu_2}{\nu}\,\bar{Q}_{xx}\,n^2\,\hat{\hat{\phi}}_9$$

$$\tilde{\alpha}_0 = \left(\bar{D}_{xx} + \frac{\bar{Q}^2_{xx}}{\bar{H}_{xx}}\right) C,_{\overline{\overline{xx}}} - 2\left(\frac{R}{h}\right)\bar{Q}_{xx}\,\delta_{n,\ell}\,\hat{\hat{\phi}}_{6,\overline{\overline{xx}}}$$

$$\tilde{\beta}_1 = \bar{D}_{xx}\,B,_{xx} - 2\,\frac{R}{h}\,\bar{Q}_{xx}\,\hat{\hat{\phi}}_{2,\overline{\overline{xx}}} + 2\,\frac{R}{h}\,\frac{1+\mu_2}{\nu}\,\bar{Q}_{xx}\,\ell^2\,\hat{\hat{\phi}}_2$$

$$\tilde{\beta}_2 = -2\,\frac{R}{h}\,\bar{Q}_{xx}\,\hat{\hat{\phi}}_{4,\overline{\overline{xx}}} - 2\,\frac{R}{h}\,\bar{Q}_{xx}\,\frac{1+\mu_2}{\nu}\,(2\ell)^2\,\hat{\hat{\phi}}_4$$

$$\tilde{\beta}_3 = -2\,\frac{R}{h}\,\bar{Q}_{xx}\,\hat{\hat{\phi}}_{7,\overline{\overline{xx}}} - 2\,\frac{R}{h}\,\bar{Q}_{xx}\,\frac{1+\mu_2}{\nu}\,(n+\ell)^2\,\hat{\hat{\phi}}_7$$

$$\tilde{\beta}_4 = -2\,\frac{R}{h}\,\bar{Q}_{xx}\,\hat{\hat{\phi}}_{8,\overline{\overline{xx}}} - 2\,\frac{R}{h}\,\bar{Q}_{xx}\,\frac{1+\mu_2}{\nu}\,(n-\ell)^2\,\hat{\hat{\phi}}_8 \qquad (6\text{-C-}8)$$

$$m_1 = \ell$$

$$m_2 = 2\ell$$

$$m_3 = n+\ell$$

$$m_4 = \ell-n$$

$$m_5 = n$$

$$n_1 = m_1$$

$$n_2 = m_2$$

$$n_3 = m_3$$

$$n_4 = m_4$$

Notice that in Egs. (6-C-8) the conditions A=B=C=0 have been used.
The $\bar{y}$-dependence is eliminated by using the following Galerkin integrals

$$\int_0^{2\pi} \{ \quad \} \, dy = 0$$

$$\int_0^{2\pi} \{ \quad \} \cos \ell y \, dy = 0 \tag{6-C-9}$$

$$\int_0^{2\pi} \{ \quad \} \sin \ell y \, dy = 0$$

which lead to the following expressions

$$A_{,\bar{x}\bar{x}} = -2 \frac{R}{h} \frac{\bar{Q}_{xx}}{\bar{D}_{xx}} \hat{\hat{\phi}}_{1,\bar{x}\bar{x}} - 2 \frac{R}{h} \frac{\bar{Q}_{xx}}{\bar{D}_{xx}} \frac{1+\mu_2}{\nu} \ell^2 \hat{\hat{\phi}}_1 \tag{6-C-10}$$

$$B_{,\bar{x}\bar{x}} = -2 \frac{R}{h} \frac{\bar{Q}_{xx}}{\bar{D}_{xx}} \hat{\hat{\phi}}_{2,\bar{x}\bar{x}} - 2 \frac{R}{h} \frac{\bar{Q}_{xx}}{\bar{D}_{xx}} \frac{1+\mu_2}{\nu} \ell^2 \hat{\hat{\phi}}_2 \tag{6-C-11}$$

$$C_{,\bar{x}\bar{x}} = -2 \frac{R}{h} \frac{\bar{Q}_{xx}}{\bar{D}_{xx}} \delta_{n,\ell} \left( \frac{1}{1 + \frac{\bar{Q}^2_{xx}}{\bar{H}_{xx} \bar{D}_{xx}}} \right) \hat{\hat{\phi}}_{6,\bar{x}\bar{x}} \tag{6-C-12}$$

Recalling the assumptions for A,B,C and $\hat{\hat{\phi}}_1$, $\hat{\hat{\phi}}_2$ and $\hat{\hat{\phi}}_6$ described in equations (6-2-17) to (6-2-28) and applying the method of averaging yields:

$$\bar{A}'' = -2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,\hat{\hat{\bar{\phi}}}''_{11} - 2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,\frac{1+\mu_2}{\nu}\,\ell^2\,\hat{\hat{\bar{\phi}}}_{11} \tag{6-C-13}$$

$$\bar{B}'' = -2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,\hat{\hat{\bar{\phi}}}''_{21} - 2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,\frac{1+\mu_2}{\nu}\,\ell^2\,\hat{\hat{\bar{\phi}}}_{12} \tag{6-C-14}$$

$$\bar{C}''_3 = \delta_{n,\ell}\left[-2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,\frac{1}{1+\dfrac{\bar{Q}^2_{xx}}{\bar{H}_{xx}\,\bar{D}_{xx}}}\right]\hat{\hat{\bar{\phi}}}''_{61} \tag{6-C-15}$$

$$\bar{C}''_1 = \bar{C}''_2 = 0 \tag{6-C-16}$$

$\hat{\hat{N}}_x = 0$ becomes upon substitution and application of method of average

$$\hat{\hat{\bar{\phi}}}_j = 0 \quad (j=11,12,13,21,22,23,31,32,41,51,61,71,81,91,92,93) \tag{6-C-17}$$

<u>SS2 Boundary Condition</u>

$$\hat{\hat{W}} = \hat{\hat{u}} = \hat{\hat{N}}_{xy} = \hat{\hat{M}}_x = 0$$

Here we must express the condition $\hat{\hat{u}} = 0$ in terms of $\hat{\hat{W}}$ and $\hat{\hat{\Phi}}$. By definition

$$\hat{\hat{N}}_{xy} = -\frac{1}{R^2}\,\hat{\hat{\Phi}}_{,\bar{x}\bar{y}} = \frac{Eh}{2(1+\nu)}\,\gamma_{xy} = \frac{Eh}{2(1+\nu)}\,\frac{1}{R}\,\Big\{\hat{\hat{u}}_{,\bar{y}} + \hat{\hat{v}}_{,\bar{x}} + \frac{1}{R}\,\big[\hat{\hat{W}}_{,\bar{x}}\,(\hat{\hat{W}}_{,\bar{y}} + \hat{W}_{,\bar{y}} + \bar{W}_{,\bar{y}})$$

$$+ \frac{1}{R}\,\hat{\hat{W}}_{,\bar{y}}\,(\hat{W}_{,\bar{x}} + \bar{W}_{,\bar{x}})\big]\Big\} = 0 \tag{6-C-18}$$

Recalling the procedure described in Ref. [76] when specializing Eq. (6-C-18) to the shell edges results in the following simplifications:

$$\hat{\hat{u}}_{,\bar{y}} = 0 \quad \text{Since } \hat{\hat{u}} = 0 \quad \text{at } \bar{x} = 0, \frac{L}{R}$$

$$\hat{\hat{W}}_{,\bar{y}} = 0 \quad \text{Since } \hat{\hat{W}} = 0 \quad \text{at } \bar{x} = 0, \frac{L}{R}$$

Thus Eq. (6-C-18) reduces to

$$-\frac{1}{R^2}\hat{\hat{\Phi}}_{,\bar{x}\bar{y}} = \frac{Eh}{2(1+\nu)}\frac{1}{R}\{\hat{\hat{v}}_{,\bar{x}} + \frac{1}{R}\hat{\hat{W}}_{,\bar{x}}\bar{W}_{,\bar{y}}\} \tag{6-C-19}$$

This equation is valid at any point of the shell, thus the relation obtained by taking its derivatives with respect to $\bar{y}$ must also be valid at any point of the shell.

$$\hat{\hat{v}}_{,\bar{x}\bar{y}} = -\frac{1}{R}\{\frac{2(1+\nu)}{Eh}\hat{\hat{\Phi}}_{,\bar{x}\bar{y}\bar{y}} - \bar{W}_{,\bar{y}}\hat{\hat{W}}_{,\bar{x}\bar{y}} - \hat{\hat{W}}_{,\bar{x}}\hat{\hat{W}}_{,\bar{y}\bar{y}}\} \tag{6-C-20}$$

Considering further

$$\hat{\hat{\varepsilon}}_{\bar{y}} = \frac{1}{R}\hat{\hat{v}}_{,\bar{y}} - \frac{1}{R}\hat{\hat{W}} + \frac{1}{2}\frac{1}{R^2}(\hat{\hat{W}}^{2}_{,\bar{y}} + 2\hat{\hat{W}}_{,\bar{y}}\hat{W}_{,\bar{y}}) + \frac{1}{R^2}\hat{\hat{W}}_{,\bar{y}}\bar{W}_{,\bar{y}} \tag{6-C-21}$$

This equation is also valid at any point of the shell, thus the relation obtained by taking derivatives with respect to $\bar{x}$ must also be valid at any point of the shell.

$$\hat{\hat{v}}_{,\bar{x}\bar{y}} = \hat{\hat{W}}_{,\bar{x}} - \frac{1}{R}\hat{\hat{W}}_{,\bar{x}\bar{y}}\bar{W}_{,\bar{y}} + \hat{\beta}\frac{1-\nu^2}{Eh}\frac{1}{R}[(1+\mu_1)\hat{\hat{\Phi}}_{,\bar{x}\bar{x}\bar{x}} - \nu\hat{\hat{\Phi}}_{,\bar{y}\bar{y}\bar{x}}]$$

$$+\hat{\beta}(1+\mu_1)\chi_2\frac{1}{R}\hat{\hat{W}}_{,\bar{y}\bar{y}\bar{x}} - \frac{1}{R}\nu\hat{\beta}\chi_1\hat{\hat{W}}_{,\bar{x}\bar{x}\bar{x}}$$

$$-\frac{1}{R}[\hat{\hat{W}}_{,\bar{y}}\hat{\hat{W}}_{,\bar{x}\bar{y}} + \hat{W}_{,\bar{y}}\hat{\hat{W}}_{,\bar{y}\bar{x}} + \hat{\hat{W}}_{,\bar{y}}\bar{W}_{,\bar{x}\bar{y}}] \tag{6-C-22}$$

Specializing it to the shell edges $\hat{\hat{W}}_{,\bar{y}} = 0$. Thus at $\bar{x} = 0, \frac{L}{R}$

$$\hat{\hat{v}}_{,\bar{x}\bar{y}} = \hat{\hat{W}}_{,\bar{x}} - \frac{1}{R}\hat{\hat{W}}_{,\bar{x}\bar{y}}\bar{W}_{,\bar{y}} + \hat{\beta}\frac{1-\nu^2}{Eh}\frac{1}{R}[(1+\mu_1)\hat{\hat{\Phi}}_{,\bar{x}\bar{x}\bar{x}} - \nu\hat{\hat{\Phi}}_{,\bar{y}\bar{y}\bar{x}}]$$

$$+ \hat{\beta}(1+\mu_1)\chi_2\frac{1}{R}\hat{\hat{W}}_{,\bar{y}\bar{y}\bar{x}} - \frac{1}{R}\nu\hat{\beta}\chi_1\hat{\hat{W}}_{,\bar{x}\bar{x}\bar{x}} \qquad (6\text{-C-}23)$$

Eliminating $\hat{\hat{v}}_{,\bar{x}\bar{y}}$ between Egs. (6-C-23) and (6-C-20) and substituting for $\hat{\hat{W}}$ and $\hat{\hat{\Phi}}$, and then equating the coefficients of like terms yields:

$$\hat{\hat{\phi}}_{1,\bar{x}\bar{x}\bar{x}} = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}]\ell^2\hat{\hat{\phi}}_{1,\bar{x}} - \frac{c}{\bar{H}_{xx}}[1 - \frac{1}{2}\frac{h}{R}\frac{\ell^2}{c}\frac{1+\mu_1}{\nu}\bar{Q}_{yy}]P_{,\bar{x}} + \frac{1}{2}\frac{h}{R}\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}P_{,\bar{x}\bar{x}\bar{x}}$$

(if i=1, p=A; if i=2, p=B) (6-C-24)

$$\hat{\hat{\phi}}_{j,\bar{x}\bar{x}\bar{x}} = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}](2\ell)^2\hat{\hat{\phi}}_{j,\bar{x}} \qquad (j=3,4) \qquad (6\text{-C-}25)$$

$$\hat{\hat{\phi}}_{k,\bar{x}\bar{x}\bar{x}} = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}]\alpha_p^2\hat{\hat{\phi}}_{k,\bar{x}} + \frac{1}{2}\frac{h}{R}\frac{c}{\bar{H}_{xx}}n^2 A_1 P_{,\bar{x}}$$

(if k=5, $\alpha_p=(n+\ell)$, p=A; if k=6, $\alpha_p=(\ell-n)$, p=A)

(if k=7, $\alpha_p=(n+\ell)$, p=B; if k=8, $\alpha_p=(\ell-n)$, p=B) (6-C-26)

$$\hat{\hat{\phi}}_{9,\bar{x}\bar{x}\bar{x}} = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}]n^2\hat{\hat{\phi}}_{9,\bar{x}} + \frac{c}{\bar{H}_{xx}}\frac{h}{R}n^2 A_1 C_{,\bar{x}} \qquad (6\text{-C-}27)$$

Applying the method of averaging to equations (6-C-24) - (6-C-27) yields:

$$\hat{\bar{\bar{\phi}}}_i''' = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}]\ell^2 \hat{\bar{\bar{\phi}}}_i' - \frac{c}{\bar{H}_{xx}} [1 - \frac{1}{2}\frac{h}{R}\frac{\ell^2}{c}\frac{1+\mu_1}{\nu}\bar{Q}_{yy}]\bar{P}' + \frac{1}{2}\frac{h}{R}\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\bar{P}'''$$

(if i=1, P=A; if i=2, P=B) (6-C-28)

$$\hat{\bar{\bar{\phi}}}_j''' = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}]\alpha_q^2 \hat{\bar{\bar{\phi}}}_j' \quad (\text{if } j=12,13,22,23,\ \alpha_q=\ell;\ \text{if } j=31,32,41,\ \alpha_q=2\ell)$$

(6-C-29)

$$\hat{\bar{\bar{\phi}}}_k''' = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}]\alpha_p^2 \hat{\bar{\bar{\phi}}}_k' + \frac{1}{2}\frac{h}{R}\frac{c}{\bar{H}_{xx}} n^2 A_1 \bar{P}'$$

(if k=51, $\alpha_p$=(n+$\ell$), P=A; if k=61, $\alpha_p$=($\ell$-n), P=A)

(if k=71, $\alpha_p$=(n+$\ell$), P=B; if k=81, $\alpha_p$=($\ell$-n), P=B) (6-C-30)

$$\hat{\bar{\bar{\phi}}}_{9g}''' = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}]n^2 \hat{\bar{\bar{\phi}}}_{9g}' + \frac{c}{\bar{H}_{xx}}\frac{h}{R} n^2 A_1 \bar{C}_g' \qquad (g=1,2,3) \qquad (6\text{-C-31})$$

The other conditions are the same as for the SS1 boundary condition.

SS3 Boundary Condition

$$\hat{W} = \hat{v} = \hat{M}_x = \hat{N}_x = 0 \quad \text{at} \quad \bar{x} = 0, \frac{L}{R}$$

Here we must express the condition $\hat{v} = 0$ in terms of $\hat{W}$ and $\hat{\Phi}$. Applying the same procedure as the one used with Eq. (6-C-21) yields

$$\hat{v}_{,y} = \frac{h}{c}\hat{\beta}(1-\nu^2)\ [(1+\mu_1)\ \hat{\Phi}_{,\bar{x}\bar{x}} - \nu\ \hat{\Phi}_{,\bar{y}\bar{y}}] - \nu\hat{\beta}\chi_1 \frac{1}{R}\hat{W}_{,\bar{x}\bar{x}} \quad \text{at } \bar{x}=0, \frac{L}{R} \qquad (6\text{-C-32})$$

Substituting for $\hat{W}$ and $\hat{\Phi}$ and eliminate the $\bar{y}$-dependence by using the Galerkin integrals defined previously (see Eqs. 6-C-9) yields

$$\hat{\hat{\phi}}_{i,\bar{x}\bar{x}} = \frac{1}{\bar{H}_{xx}} [\frac{1}{2} \frac{h}{R} \bar{Q}_{xx} P,_{\bar{x}\bar{x}} - \bar{H}_{yy} \frac{\nu}{1+\mu_2} \ell^2 \hat{\hat{\phi}}_i] \text{ (if } i=1, P=A; \text{ if } i=2, P=B)$$

(6-C-33)

$$\hat{\hat{\phi}}_{j,\bar{x}\bar{x}} = \alpha_i^2 \frac{\bar{H}_{yy}}{\bar{H}_{xx}} \frac{\nu}{1+\mu_2} \hat{\hat{\phi}}_i$$

(if $i=3,4$, $\alpha_i=2\ell$; if $i=5,7$, $\alpha_i=(n+\ell)$; if $i=6,8$, $\alpha_i=(\ell-n)$; if $i=9$, $\alpha_9=n$) (6-C-34)

Substituting the time expressions for $\hat{\hat{W}}$ and $\hat{\hat{\Phi}}$ and applying the method of averaging to the resulting equations yields:

$$\bar{\hat{\hat{\phi}}}''_{i1} = \frac{1}{\bar{H}_{xx}} [\frac{1}{2} \frac{h}{R} \bar{Q}_{xx} \bar{P}'' - \frac{\nu}{1+\mu_2} \bar{H}_{yy} \ell^2 \bar{\hat{\hat{\phi}}}_{i1}] \text{ (i=1, P=A; i=2, P=B)} \qquad (6\text{-C-}35)$$

$$\bar{\hat{\hat{\phi}}}''_j = \alpha_j^2 \frac{\bar{H}_{yy}}{\bar{H}_{xx}} \frac{\nu}{1+\mu_2} \bar{\hat{\hat{\phi}}}_j$$

(if $j=12,13,22,23$, $\alpha_j=\ell$; if $j=31,32,41$, $\alpha_j=2\ell$; if $j=51,71$, $\alpha_j=(n+\ell)$; if $j=61,81$, $\alpha_j=(\ell-n)$, if $j=91,92,93$, $\alpha_j=n$) (6-C-36)

The other conditions have been reduced previously.

<u>SS4 Boundary Condition</u>

$$\hat{\hat{W}} = \hat{\hat{u}} = \hat{\hat{v}} = \hat{\hat{M}}_x = 0 \qquad \text{at } \bar{x}=0, \frac{L}{R}$$

These have been reduced previously.

<u>C1 Boundary Condition</u>

$$\hat{\hat{W}} = \hat{\hat{W}},_{\bar{x}} = \hat{\hat{N}}_{xy} = \hat{\hat{N}}_x = 0 \qquad \text{at } \bar{x} = 0, \frac{L}{R}$$

$\hat{\hat{W}},_{\bar{x}} = 0$ becomes upon substituting for $\hat{\hat{W}}$

$$\hat{\hat{W}}_{,\bar{x}} = h(A_{,\bar{x}} \cos \ell y + B_{,\bar{x}} \sin \ell y + C_{,\bar{x}}) = 0 \tag{6-C-40}$$

This must be true for all values of $\bar{y}$, therefore

$$A_{,\bar{x}} = B_{,\bar{x}} = C_{,\bar{x}} = 0 \tag{6-C-41}$$

This equation becomes after applying the method of averaging:

$$\begin{aligned} &\bar{A}' = \bar{B}' = 0 \\ &\bar{C}_i' = 0 \qquad (i=1,2,3) \end{aligned} \tag{6-C-42}$$

All the other conditions have been reduced previously.

Symmetry Condition at $\bar{x} = 0, \frac{L}{R}$

$$\hat{\hat{W}}_{,\bar{x}} = \hat{\hat{N}}_{xy} = \hat{\hat{u}} = \hat{\hat{H}} = 0$$

The condition $\hat{\hat{H}} = 0$ can be transformed as follows:

$$\hat{\hat{H}} = \hat{\hat{M}}_{x,x} + (\hat{\hat{M}}_{xy} + \hat{\hat{M}}_{yx})_{,\bar{y}} + \hat{\hat{N}}_x (\hat{\hat{W}}_{,x} + \bar{W}_{,x}) + \hat{\hat{N}}_{xy} (\hat{\hat{W}}_{,\bar{y}} + \bar{W}_{,\bar{y}}) \tag{6-C-43}$$

where

$$\hat{\hat{M}}_x = D \left[ (1+\eta_{01}) \hat{\hat{\kappa}}_x + \nu \hat{\hat{\kappa}}_y + \zeta_1 \hat{\hat{\varepsilon}}_x \right]$$

$$\hat{\hat{M}}_{xy} = D[(1-\nu) + \eta_{t1}] \hat{\hat{\kappa}}_{xy}$$

$$\hat{\hat{M}}_{yx} = D[(1-\nu) + \eta_{t2}] \hat{\hat{\kappa}}_{xy}$$

Substituting for $\hat{\hat{M}}_x$ and $\hat{\hat{M}}_{yx}$ and applying the symmetric conditions to the resulted equation, and then employing the procedure used previously yields:

$$-\bar{D}_{xx} P_{,\overline{xxx}} = 2\frac{R}{h}\bar{Q}_{xx}\hat{\hat{\phi}}_{i,\overline{xxx}} \qquad \text{(if i=1, P=A; if i=2, P=B)} \tag{4-C-44}$$

$$\hat{\hat{\phi}}_{j,\overline{xxx}} = 0 \qquad (j=1,2,\ldots,9) \tag{4-C-45}$$

$$C_{,\overline{xxx}} = 0 \tag{4-C-46}$$

These equations then become after applying the method of averaging:

$$\bar{P}''' = -2\frac{R}{h}\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\bar{\bar{\hat{\phi}}}'''_{i1} \qquad \text{(if i=1, P=A; if i=2, P=B)} \tag{4-C-47}$$

$$\bar{C}'''_j \qquad (j=1,2,3) \tag{4-C-48}$$

$$\bar{\bar{\hat{\phi}}}'''_k = 0 \qquad (k=12,13,22,23,31,32,41,51,61,71,81,91,92,93) \tag{4-C-49}$$

<u>SUMMARY OF THE REDUCED BOUNDARY CONDITIONS</u>

<u>SS1</u>

$\hat{\hat{W}} = 0 \qquad \bar{A} = \bar{B} = \bar{C}_i = 0$

$\hat{\hat{N}}_x = 0 \qquad \bar{\bar{\hat{\phi}}}_j = 0$

$\hat{\hat{N}}_{xy} = 0 \qquad \bar{\bar{\hat{\phi}}}'_j = 0$

$\hat{\hat{M}}_x = 0$ $\qquad \bar{A}'' = -2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,\bar{\hat{\hat{\phi}}}''_{11}$

$$\bar{B}'' = -2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,\bar{\hat{\hat{\phi}}}''_{21}$$

$$\bar{C}''_i = 0$$

<u>SS2</u>

$\hat{\hat{W}} = 0$ $\qquad \bar{A} = \bar{B} = \bar{C}_i = 0$

$\hat{\hat{N}}_{xy} = 0$ $\qquad \bar{\hat{\hat{\phi}}}'_j = 0$

$\hat{\hat{M}}_x = 0$ $\qquad \bar{A}'' = -2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,[\bar{\hat{\hat{\phi}}}''_{11,\bar{\bar{xx}}} + \frac{1+\mu_1}{\nu}\,\ell^2\,\bar{\hat{\hat{\phi}}}_{11}]$

$$\bar{B}'' = -2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,[\bar{\hat{\hat{\phi}}}''_{21} + \frac{1+\mu_1}{\nu}\,\ell^2\,\bar{\hat{\hat{\phi}}}_{21}]$$

$$\bar{C}''_1 = \bar{C}''_2 = 0$$

$$\bar{C}''_3 = -2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,\delta_{n,\ell}\,\left(\frac{1}{1+\frac{\bar{Q}_{xx}^{\,2}}{\bar{H}_{xx}\,\bar{D}_{xx}}}\right)\,\bar{\hat{\hat{\phi}}}''_6$$

$\hat{\hat{u}} = 0$ $\qquad \bar{\hat{\hat{\phi}}}'''_{k1} = \frac{1}{2}\,\frac{h}{R}\,\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\,\bar{P}''' - \frac{c}{\bar{H}_{xx}}\,[1 - \frac{1}{2}\,\frac{h}{R}\,\frac{\ell^2}{c}\,\frac{1+\mu_1}{\nu}\,\bar{Q}_{xx}]\,\bar{P}'$

(if k=1, P=A; if k=2, P=B)

$$\hat{\hat{\phi}}'''_g = 0$$

$$\hat{\hat{\bar{\phi}}}''{}'_{h} = \frac{1}{2}\,\frac{h}{R}\,\frac{c}{\bar{H}_{xx}}\,n^2\,A_1\,\bar{P}' \quad \text{(if k=51,71, P=A; if k=61,81, P=B)}$$

$$\hat{\hat{\bar{\phi}}}''{}'_{9i} = \frac{c}{\bar{H}_{xx}}\,\frac{h}{R}\,n^2\,A_1\,\bar{C}'_i$$

SS3

$$\hat{\hat{W}} = 0 \qquad \bar{A} = \bar{B} = \bar{C}_i = 0$$

$$\hat{\hat{N}}_x = 0 \qquad \hat{\hat{\bar{\phi}}}_j = 0$$

$$\hat{\hat{v}} = 0 \qquad \hat{\hat{\bar{\phi}}}''_j = 0$$

$$\hat{\hat{M}}_x = 0 \qquad \bar{A}'' = \bar{B}'' = \bar{C}''_i = 0$$

SS4

$$\hat{\hat{W}} = 0 \qquad \bar{A} = \bar{B} = \bar{C}_i = 0$$

$$\hat{\hat{M}}_x = 0 \qquad * \;\bar{A}'' = -\,2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,[\hat{\hat{\bar{\phi}}}''_{11} + \frac{1+\mu_1}{\nu}\,\ell^2\,\hat{\hat{\bar{\phi}}}_{11}]$$

$$* \;\bar{B}'' = -\,2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,[\hat{\hat{\bar{\phi}}}''_{21} + \frac{1+\mu_1}{\nu}\,\ell^2\,\hat{\hat{\bar{\phi}}}_{21}]$$

$$\bar{C}_1 = \bar{C}_2 = 0$$

$$\bar{C}_3 = -\,2\,\frac{R}{h}\,\frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\,\delta_{n,\ell}\,\left(\frac{1}{1+\dfrac{\bar{Q}_{xx}^{\,2}}{\bar{H}_{xx}\,\bar{D}_{xx}}}\right)\,\hat{\hat{\bar{\phi}}}''_6$$

$\hat{\hat{v}} = 0$ $\qquad \hat{\hat{\bar{\phi}}}''_j = \alpha_j \dfrac{\bar{H}_{yy}}{\bar{H}_{xx}} \dfrac{\nu}{1+\mu_2} \hat{\hat{\bar{\phi}}}_j$ (if j=12,13,22,23, $\alpha_j=\ell$; if j=31,32,41, $\alpha_j=2\ell$; if j=51,71, $\alpha_j=n+\ell$; if j=61,81, $\alpha_j=\ell-n$; if j=91,92,93, $\alpha_j=n$)

$$* \; \hat{\hat{\bar{\phi}}}''_{11} = \frac{1}{\bar{H}_{xx}} [\frac{1}{2} \frac{h}{R} \bar{Q}_{xx} \bar{A}'' - \frac{\nu}{1+\mu_2} \bar{H}_{yy} \ell^2 \hat{\hat{\bar{\phi}}}_{11}]$$

$$* \; \hat{\hat{\bar{\phi}}}''_{21} = \frac{1}{\bar{H}_{xx}} [\frac{1}{2} \frac{h}{R} \bar{Q}_{xx} \bar{B}'' - \frac{\nu}{1+\mu_2} \bar{H}_{yy} \ell^2 \hat{\hat{\bar{\phi}}}_{21}]$$

$\hat{\hat{u}} = 0$

$$\hat{\hat{\bar{\phi}}}'''_{11} = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}] \ell^2 \hat{\hat{\bar{\phi}}}'_{11} - \frac{c}{\bar{H}_{xx}} [1 - \frac{1}{2} \frac{h}{R} \frac{\ell^2}{c} \frac{1+\mu_1}{\nu} \bar{Q}_{yy}]\bar{A}'$$

$$+ \frac{1}{2} \frac{h}{R} \frac{\bar{Q}_{xx}}{\bar{H}_{xx}} \bar{A}'''$$

$$\hat{\hat{\bar{\phi}}}'''_{21} = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}] \ell^2 \hat{\hat{\bar{\phi}}}'_{21} - \frac{c}{\bar{H}_{xx}} [1 - \frac{1}{2} \frac{h}{R} \frac{\ell^2}{c} \frac{1+\mu_1}{\nu} \bar{Q}_{yy}]\bar{B}'$$

$$+ \frac{1}{2} \frac{h}{R} \frac{\bar{Q}_{xx}}{\bar{H}_{xx}} \bar{B}'''$$

$$\hat{\hat{\bar{\phi}}}'''_g = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}] (\alpha_q^2) \hat{\hat{\bar{\phi}}}_g$$

(if g=12,13,22,23, $\alpha_q=\ell$; if g=31,32,41, $\alpha_q=2\ell$)

$$\hat{\hat{\bar{\phi}}}'''_h = [\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}]\alpha_p^2 \hat{\hat{\bar{\phi}}}'_h + \frac{1}{2} \frac{h}{R} \frac{c}{\bar{H}_{xx}} n^2 A_1\bar{P}'$$

(if h=51, $\alpha_p$=n+ℓ, P=A; if h=61, $\alpha_p$=n−ℓ, P=A,

if h=71, $\alpha_p$=n+ℓ, P=B; if h=81,, $\alpha_p$=ℓ−n, P=B)

$$\bar{\hat{\hat{\phi}}}{}''_{9i} = \left[\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}\right]n^2\ \bar{\hat{\hat{\phi}}}{}'_{91} + \frac{c}{\bar{H}_{xx}}\frac{h}{R}\, n^2\ A_1\ \bar{C}'_i$$

* These conditions imply

$$\bar{A}'' = -\ 2\ \frac{R}{h}\ \frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\ \ell^2\ \left[\frac{1+\mu_2}{\nu} - \frac{\nu}{1+\mu_1}\right]\bar{\hat{\hat{\phi}}}_{11}/\left(1+\frac{\bar{Q}^2_{xx}}{\bar{H}_{xx}\bar{D}_{xx}}\right)$$

$$\bar{B}'' = -\ 2\ \frac{R}{h}\ \frac{\bar{Q}_{xx}}{\bar{D}_{xx}}\ \ell^2\ \left[\frac{1+\mu_2}{\nu} - \frac{\nu}{1+\mu_1}\right]\bar{\hat{\hat{\phi}}}_{21}/\left(1+\frac{\bar{Q}^2_{xx}}{\bar{H}_{xx}\bar{D}_{xx}}\right)$$

$$\bar{\hat{\hat{\phi}}}{}''_{11} = -\ \left[\frac{\bar{Q}^2_{xx}}{\bar{H}_{xx}\bar{D}_{xx}}\ \frac{1+\mu_2}{\nu} - \frac{\nu}{1+\mu_1}\right]\ \ell^2\ \bar{\hat{\hat{\phi}}}_{11}/\left(1+\frac{\bar{Q}^2_{xx}}{\bar{H}_{xx}\bar{D}_{xx}}\right)$$

$$\bar{\hat{\hat{\phi}}}{}''_{21} = -\ \left[\frac{\bar{Q}^2_{xx}}{\bar{H}_{xx}\bar{D}_{xx}}\ \frac{1+\mu_2}{\nu} - \frac{\nu}{1+\mu_1}\right]\ \ell^2\ \bar{\hat{\hat{\phi}}}_{21}/\left(1+\frac{\bar{Q}^2_{xx}}{\bar{H}_{xx}\bar{D}_{xx}}\right)$$

<u>C1</u>

$\bar{\hat{\hat{W}}} = 0$ $\qquad \bar{A} = \bar{B} = \bar{C}_i = 0$

$\hat{\hat{W}}_{,x} = 0$ $\qquad \bar{A}' = \bar{B}' = \bar{C}'_i = 0$

$\hat{\hat{N}}_x = 0$ $\qquad \bar{\hat{\hat{\phi}}}_j = 0$

$\hat{\hat{N}}_{xy} = 0$ $\qquad \bar{\hat{\hat{\phi}}}{}'_j = 0$

C2

$\hat{\hat{W}} = 0 \qquad \bar{A} = \bar{B} = \bar{C}_i = 0$

$\hat{\hat{W}},_x = 0 \qquad \bar{A}' = \bar{B}' = \bar{C}_i' = 0$

$\hat{\hat{N}}_{xy} = 0 \qquad \bar{\hat{\hat{\phi}}}_j' = 0$

$\hat{\hat{u}} = 0 \qquad \bar{\hat{\hat{\phi}}}_{11}''' = \frac{1}{2}\frac{h}{R}\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\bar{A}'''$

$$\bar{\hat{\hat{\phi}}}_{21}''' = \frac{1}{2}\frac{h}{R}\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\bar{B}'''$$

$$\bar{\hat{\hat{\phi}}}_g''' = 0$$

$$\bar{\hat{\hat{\phi}}}_h''' = 0$$

$$\bar{\hat{\hat{\phi}}}_{9i}''' = 0$$

C3

$\hat{\hat{W}} = 0 \qquad \bar{A} = \bar{B} = \bar{C}_i = 0$

$\hat{\hat{W}},_x = 0 \qquad \bar{A}' = \bar{B}' = \bar{C}_i' = 0$

$\hat{\hat{v}} = 0 \qquad \bar{\hat{\hat{\phi}}}_{11}'' = \frac{1}{2}\frac{h}{R}\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\bar{A}''$

$$\bar{\hat{\hat{\phi}}}_{21}'' = \frac{1}{2}\frac{h}{R}\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\bar{B}''$$

$$\bar{\hat{\hat{\phi}}}''_g = 0$$

$$\bar{\hat{\hat{\phi}}}''_h = 0$$

$$\bar{\hat{\hat{\phi}}}''_{9i} = 0$$

$$\hat{\hat{N}}_x = 0 \qquad \bar{\hat{\hat{\phi}}}_j = 0$$

C4

$$\hat{\hat{W}} = 0 \qquad \bar{A} = \bar{B} = \bar{C}_i = 0$$

$$\hat{\hat{W}}_{,x} = 0 \qquad \bar{A}' = \bar{B}' = \bar{C}_i' = 0$$

$$\hat{\hat{v}} = 0 \qquad \bar{\hat{\hat{\phi}}}''_{11} = \frac{1}{\bar{H}_{xx}} \left[\frac{1}{2}\,\frac{h}{R}\,\bar{Q}_{xx}\,\bar{A}'' - \frac{\nu}{1+\mu_2}\,\bar{H}_{yy}\,\ell^2\,\bar{\hat{\hat{\phi}}}_{11}\right]$$

$$\bar{\hat{\hat{\phi}}}''_{21} = \frac{1}{\bar{H}_{xx}} \left[\frac{1}{2}\,\frac{h}{R}\,\bar{Q}_{xx}\,\bar{B}'' - \frac{\nu}{1+\mu_2}\,\bar{H}_{yy}\,\ell^2\,\bar{\hat{\hat{\phi}}}_{21}\right]$$

$$\bar{\hat{\hat{\phi}}}''_g = -\,\alpha_q^2\,\frac{\bar{H}_{xy}}{\bar{H}_{xx}}\,\frac{\nu}{1+\mu_2}\,\bar{\hat{\hat{\phi}}}_g \qquad (\text{if } g=12,13,22,23,\ \alpha_q=\ell;\ \text{if } g=31,32,41,\ \alpha_q=2\ell)$$

$$\bar{\hat{\hat{\phi}}}''_h = -\,\alpha_p^2\,\frac{\bar{H}_{xy}}{\bar{H}_{xx}}\,\frac{\nu}{1+\mu_2}\,\bar{\hat{\hat{\phi}}}_h \quad (\text{if } h=51,71,\ \alpha_p=n+\ell;\ \text{if } h=61,81,\ \alpha_p=\ell-n)$$

$$\bar{\hat{\hat{\phi}}}''_{9i} = -\,\ell^2\,\frac{\bar{H}_{xy}}{\bar{H}_{xx}}\,\frac{\nu}{1+\mu_2}\,\bar{\hat{\hat{\phi}}}_{9i}$$

$$\hat{\hat{u}} = 0 \qquad \hat{\hat{\bar{\phi}}}{}''_{11} = \left[\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}\right] \ell^2 \, \hat{\hat{\bar{\phi}}}{}'_{11} + \frac{1}{2}\,\frac{h}{R}\,\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\,\bar{A}''$$

$$\hat{\hat{\bar{\phi}}}{}''_{21} = \left[\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}\right] \ell^2 \, \hat{\hat{\bar{\phi}}}{}'_{21} + \frac{1}{2}\,\frac{h}{R}\,\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\,\bar{B}''$$

$$\hat{\hat{\bar{\phi}}}{}''_{g} = \left[\frac{2(1+\nu)}{\bar{H}_{xx}} - \frac{\nu}{1+\mu_1}\right] \alpha_q^2 \, \hat{\hat{\bar{\phi}}}{}'_{g} \qquad \text{(if } g=12,13,22,23,\ \alpha_q=\ell, \text{ if } g=31,32,41,\ \alpha_q=2\ell)$$

$$\hat{\hat{\bar{\phi}}}{}''_{h} = \left[\frac{2(1+\nu)}{\bar{H}_{xx}} + \frac{\nu}{1+\mu_1}\right] \alpha_q^2 \, \hat{\hat{\bar{\phi}}}{}'_{h} \qquad \text{(if } h=51,71,\ \alpha_p=n+\ell; \text{ if } h=61,81,\ \alpha_p=\ell-n)$$

$$\hat{\hat{\phi}}{}''_{9i} = \left[\frac{2(1+\nu)}{\bar{H}_{xx}} + \frac{\nu}{1+\mu_1}\right] n^2 \, \hat{\hat{\bar{\phi}}}{}'_{9i}$$

SUMMETRY CONDITION

$$\hat{\hat{W}}_{,x} = 0 \qquad \bar{A}' = \bar{B}' = \bar{C}'_i = 0$$

$$\hat{\hat{N}}_{xy} = 0 \qquad \hat{\hat{\bar{\phi}}}{}'_j = 0$$

$$\hat{\hat{u}} = 0 \qquad * \ \hat{\hat{\bar{\phi}}}{}''_{11} = \frac{1}{2}\,\frac{h}{R}\,\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\,\bar{A}''$$

$$* \ \hat{\hat{\bar{\phi}}}{}''_{21} = \frac{1}{2}\,\frac{h}{R}\,\frac{\bar{Q}_{xx}}{\bar{H}_{xx}}\,\bar{B}''$$

$$\hat{\hat{\bar{\phi}}}{}''_{g} = 0$$

$$\hat{\bar{\bar{\phi}}}''''_{h} = 0$$

$$\hat{\bar{\bar{\phi}}}''''_{9i} = 0$$

$\hat{H} = 0$

$$* \quad \bar{A}'''' = -2 \frac{R}{h} \frac{\bar{Q}_{xx}}{\bar{D}_{xx}} \hat{\bar{\bar{\phi}}}''''_{11}$$

$$* \quad \bar{B}'''' = -2 \frac{R}{h} \frac{\bar{Q}_{xx}}{\bar{D}_{xx}} \hat{\bar{\bar{\phi}}}''''_{21}$$

$$\bar{C}''''_{i} = 0$$

* The solutions of these equations are

$$\bar{A}'''' = \bar{B}'''' = \hat{\bar{\bar{\phi}}}''''_{11} = \hat{\bar{\bar{\phi}}}''''_{21} = 0$$

where i=1,2,3

j=11,12,13,21,22,23,31,32,41,51,61,71,81,91,92,93

g=12,13,22,23,31,32,41

h=51,61,71,81

APPENDIX 6-D COEFFICIENTS OF EQUATION (6-3-8)

$$\tilde{e}_o = -\frac{R}{L} \{\alpha_1 \bar{C}''_1 - \frac{1}{2} \alpha_2 [(\bar{C}'_1)^2 + \frac{1}{2} (\bar{C}'_2)^2 + \frac{1}{2} \delta_{n,\ell} (\bar{C}'_3)^2] - \alpha_3 \bar{C}'_1$$
$$+ \alpha_4 \bar{C}_1 - \frac{1}{2} \alpha_5 \bar{A}^2 - \frac{1}{8} \alpha_2 (\bar{A}')^2\}$$

$$\tilde{e}_1 = -\frac{R}{L} \delta_{n,\ell} \{\alpha_1 \bar{C}''_3 - \alpha_2 \bar{C}'_1 \bar{C}'_3 - \frac{1}{2} \alpha_2 \bar{C}'_2 \bar{C}'_3 - \alpha_3 \bar{C}'_3 + \alpha_4 \bar{C}_3 + \alpha_6 \bar{A} - \alpha_7 \bar{A}'\}$$

$$\tilde{e}_2 = -\frac{R}{L} \{\alpha_1 \bar{C}''_2 - \frac{1}{4} \alpha_2 \delta_{n,\ell} (\bar{C}'_3)^2 - \alpha_2 \bar{C}'_1 \bar{C}'_2 - \alpha_3 \bar{C}'_2 + \alpha_4 \bar{C}'_2 - \frac{1}{2} \alpha_5 \bar{A}^2 - \frac{1}{8} \alpha_2 (\bar{A}')^2\}$$

$$\tilde{e}_3 = -\frac{R}{L}\{-\frac{1}{2}\delta_{n,\ell}\,\alpha_2\,\bar{C}_2'\,\bar{C}_3'\}$$

$$\tilde{e}_4 = -\frac{R}{L}\{-\frac{1}{4}\alpha_2\,(\bar{C}_2')^2\}$$

where the coefficients $\alpha_i$ (i=1,2,...,7) are

$$\alpha_1 = \frac{1}{2c}\,\frac{h}{R}\,[\frac{1+\mu_2}{\nu} - \frac{\nu}{1+\mu_1}]\,\bar{Q}_{xx}$$

$$\alpha_2 = \frac{h}{R}$$

$$\alpha_3 = \frac{h}{R}\,A_o'$$

$$\alpha_4 = \frac{\nu}{1+\mu_1}$$

$$\alpha_5 = \frac{1}{4}\,\alpha_2\,\alpha_4\,\ell^2$$

$$\alpha_6 = -\frac{1}{2}\,\alpha_2\,\alpha_4\,n\,\ell\,A_1$$

$$\alpha_7 = -\frac{1}{2}\,\frac{h}{R}\,A_1'$$

## SUMMARY

The main purpose of this thesis is to investigate the influence of the initial geometric imperfections and of the boundary conditions on the nonlinear vibration characteristics of thin cylindrical shells. The thesis consists of two parts.

In the first part the nonlinear vibrations of imperfect thin-walled stiffened cylindrical shells with SS3 boundary conditions at both ends is considered, subjected to axial compression $N_o$ and lateral excitation q. Both single and two modes initial geometric imperfection models are considered. One of the objectives of this part is aimed at the study of the discrepancies existing in the previous investigations and to obtain a reasonable explanation for them. Next the influence of geometric imperfections on the coupled mode response is studied where up to now no solutions are available. The Donnell type nonlinear differential equations for axially compressed stiffened shells with simply supported boundary conditions at the two ends are used. The 'smeared' theory is applied to treat stiffeners and rings. Galerkin's method and the method of averaging are employed in sequence to obtain a set of coupled nonlinear algebraic equations, from which the frequency-amplitude relationship can be obtained for various dampings, amplitudes of excitations and imperfections. The stability of solutions is studied using the so-called method of slowly varying parameters.

The second part of the thesis deals with the investigation of the influence of various boundary conditions on the nonlinear vibrations of imperfect cylindrical shells, which is the first step of the effort to study the effect of elastic boundary conditions on the nonlinear vibration of shells. The problem of determining the effects of elastic boundary conditions on the dynamic response cannot be avoided because in practical applications 'perfect' boundary conditions, for example the simply supported one, do not usually exist. In reality the boundary conditions are elastic or intermediate between the extreme of fixed and free. Once again the Donnell type nonlinear partial differential equations are employed. The procedure used in this part is an extension of the one used by Arbocz for the buckling problems in Ref. [76]. Proceeding as in part one the Donnell type equations are reduced to a set of nonlinear first order ordinary differential equations with two sets of boundary conditions. The resulting 2-point nonlinear boundary value problem is solved by the numerical integration procedure called 'shooting method' yielding the frequency-amplitude relationships and vibration modes for various boundary conditions.

SAMENVATTING

Het hoofddoel van dit proefschrift is de invloed van initiele geometrische imperfecties en de randvoorwaarden op de niet-lineaire trillings-karakteristieken van dunwandige cylindrische schalen te onderzoeken.Het proefschrift bestaat uit 2 delen.

In het eerste deel worden de niet-lineaire trillingen beschouwd van imperfecte dunwandige verstijfde cylindrische schalen met SS3 randvoorwaarden op beide uiteinden, die aan een axiale drukbelasting $N_0$ onderworpen zijn en lateraal geexiteerd worden door een belasting q. Zowel enkele als gecombineerde initiele geometrische imperfectie vormen zijn beschouwd. Een van de doelstellingen van dit gedeelte is gericht op het onderzoek naar de verschillen die er bestaan in voorgaande onderzoeken en hiervoor een verklaring te vinden. Hierna wordt de invloed van geometrische imperfecties en de gekoppelde vorm responsie onderzocht. Hiervoor waren nog geen oplossingen gevonden. De niet-lineaire Donnell differentiaal vergelijkingen voor axiaal belaste verstijfde schalen met scharnierend opgelegde randen bij beide uiteinden zijn gebruikt. De "uitgesmeerde" theorie is toegepast om de verstijvers en ringen te behandelen. De Galerkin methode en de middelwaarde (averaging) methode worden na elkaar gebruikt om een stel gekoppelde niet-lineaire algebraische vergelijkingen te krijgen, waaruit de frequency-amplitude relatie voor verschillende dempingen, excitatie amplitudes en imperfecties zijn te verkrijgen. De stabiliteit van de oplossing is bestudeerd door gebruik te maken van de zogenoemde methode van langzaam varieerende parameters.

Het tweede deel van dit proefschrift is het onderzoek naar de invloed van verschillende randvoorwaarden op de niet-lineaire trillingen van imperfecte cylindrische schalen, hetgeen de eerste stap is van een poging om het effect van elastische randvoorwaarden op niet-lineare trillingen van schalen te onderzoeken. Het probleem van het bepalen van de effekten van elastische randvoorwaarden op dynamische responsie kan niet voorkomen worden omdat in praktische toepassingen perfecte randvoorwaarden, b.v. scharnierende opleggingen, normaal niet voorkomen. In de praktijk zijn de randvoorwaarden elastisch of zitten ze tussen het uiterste van ingeklemd en vrij in. Opnieuw worden Donnell's niet-lineaire partiële differentiaal vergelijkingen gebruikt. De oplossingprocedure welke gebruikt wordt is een uitbreiding van die welke Arbocz gebruikt voor de knikproblemen in Ref.(76). Op dezelfde wijze als in deel 1 zijn de Donnell vergelijkingen te vereenvoudigen tot een set niet-lineaire eerste orde differentiaal vergelijkingen met twee stelsels randvoorwaarden. Het resulterende niet-lineaire 2-punt randwaarde probleem is opgelost door gebruik te maken van de numerieke integratie methode genaamd "shooting methode', waardoor de frequency-amplitude relaties en de trillings vormen voor verschillende randcondities verkregen worden.

CURRICULUM VITAE

The author of this thesis was born in the Shanxi Province of China on September 2, 1947. In 1967 he graduated from Taiyan 6th High School after a six-year study. From 1968 to 1971 he worked as an electronic technician at the Shanxi Radio Factory. After that he studied Airplane Design at the 5th Department of the Beijing Institute of Aeronautics and Astronautics. In 1975 he graduated (B.S.) and was asked to work as an research fellow at the Vibration Group of the Department of Solid Mechanics of the Institute of Mechanics of the Chinese Academy of Sciences. In 1978 he transferred to the Department of Satellite Structural Design and Testing of the Space Science and Technology Center of the Chinese Academy of Sciences, where he jointed a space project and was responsible for the structural design, the statics and dynamics analysis and the static and dynamic tests of the structure. He was conferred the title of Engineer (M.Sc.) in May 1983.
Since September 1983, he is working as a scientist at the Solid Mechanics Group of the Faculty of Aerospace Engineering of the Delft University of Technology for his Doctor Degree on the subject of 'Nonlinear vibrations of thin-walled shells' under the supervision of Prof.Dr. J. Arbocz. The results of this study lead to this thesis.