THEORY OF VISCIOUS FLOW IN CURVED SHALLOW CHANNELS

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SUMMARY

The axisymmetrical, viscous flow in curved channels is considered in the case where the hydraulic radius of the cross-section is small with respect to the average radius of curvature of the bend. First Ananyan's theory on this subject is reconsidered, using a regular perturbation method. The results are applied to a specific channel with a shallow rectangular cross-section. This yields solutions for the tangential velocity-component, the secondary circulation and the influence of the latter on the former.

Second the case of a shallow, rectangular cross-section is treated, using the method of matched asymptotic expansions. This yields solutions of the velocity-components in three separate regions of the cross-section (viz. near each side-wall and near the central axis of the cross-section), which turn out to be sums

\[
\begin{align*}
\text{(constant, depending on)} & \quad \text{(function of the normal-)} \\
\text{(the flow parameters and)} & \times \quad \text{(ized radial and vertical)} \\
\text{(the channel geometry)} & \quad \text{(co-ordinates)}
\end{align*}
\]

Within the limitations of the theory, the functions, which have been tabulated here, are valid for all channels with a shallow, rectangular cross-section. Once these functions being known, it is possible to determine the velocity-components without solving the differential equations.

Finally, in order to find continuous velocity-profiles, the solutions in the three separate regions are composed to solutions valid in the entire cross-section. These composite solutions turn out to agree well with the results of Ananyan's theory.
CHAPTER I

A. INTRODUCTION

In river morphology, the hydrodynamic phenomena caused by bends play an important role. Without pretending a description of physical reality, some insight into these phenomena can be obtained by considering the viscous axisymmetrical curved flow (in which the flow pattern does not change from one cross-section to another) in an open channel with a constant rectangular cross-section and a fixed horizontal bed.

Boussinesq /1/ was the first to give a mathematical description of this flow, for arbitrary radii of curvature, but only in the central region (i.e. far from the side-walls) of a shallow cross-section.

An important and extensive treatment of the problem can be found in Ananyan's book /2/. He studied the axisymmetrical curved flow for large radii of curvature as compared with a linear dimension of the cross-section, which may have an arbitrary shape.

Ananyan's theory has been elaborated by Rozovs'kii /3/ for a shallow rectangular cross-section and large radii of curvature as compared with the channel width.

This report gives a systematical derivation of Ananyan's theory. It is elaborated for a shallow rectangular cross-section and arbitrary radii of curvature. Subsequently, the same problem is solved in a slightly different way, applying the method of matched asymptotic expansions in three separate regions of the cross-section.

![Diagram of investigated problems](image)

**fig. 1. Investigated problems.**
B. THE MATHEMATICAL MODEL

The problem is described mathematically by the Navier-Stokes equations for axisymmetrical incompressible fluid flows, the equation of continuity and the boundary conditions at the bed, at the side-walls and at the surface.

fig. 2. Definitions.

In the cylindrical co-ordinate system, indicated in fig. 2, the Navier-Stokes equations and the equation of continuity are:

\[
\begin{align*}
\text{v}_R \frac{\partial v_r}{\partial R} + v_z \frac{\partial v_r}{\partial z} - \frac{v_r^2}{R} & = \\
- \frac{1}{\rho} \frac{\partial p}{\partial R} + \nu \left( \frac{\partial^2 v_r}{\partial z^2} + \frac{\partial^2 v_r}{\partial R^2} + \frac{1}{R} \frac{\partial v_r}{\partial R} - \frac{v_r}{R^2} \right) & = \\
\text{v}_\phi \frac{\partial v_\phi}{\partial R} + v_z \frac{\partial v_\phi}{\partial z} + \frac{v_\phi}{R} \frac{\partial v_r}{\partial R} & = \\
- \frac{1}{\rho \nu} \frac{\partial p}{\partial \phi} + \nu \left( \frac{\partial^2 v_\phi}{\partial z^2} + \frac{\partial^2 v_\phi}{\partial R^2} + \frac{1}{R} \frac{\partial v_\phi}{\partial R} - \frac{v_\phi}{R^2} \right) & = 
\end{align*}
\]

(1)
\[ v \frac{\partial v}{\partial R} + v \frac{\partial v}{\partial z} + g' = \]

\[ - \frac{1}{\rho} \frac{\partial p}{\partial z} + v \left( \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial R^2} + \frac{1}{R} \frac{\partial v}{\partial R} \right) \]  

\[ \frac{1}{R} \frac{\partial}{\partial R} \left( R v_R \right) + \frac{\partial v}{\partial z} = 0 \]  

(3)

(4)

Here \( v_R \), \( v_\phi \) and \( v_z \) represent the velocity-components\(^*\), \( p \) the pressure, \( g' \) the acceleration of gravity, \( v \) the kinematic viscosity of the fluid and \( \rho \) its density.

The boundary conditions are:

\[ v_R(z=0) = 0 ; \ v_\phi(z=0) = 0 ; \ v_z(z=0) = 0 \]  

\[ v_R(R=R_i) = 0 ; \ v_\phi(R=R_i) = 0 ; \ v_z(R=R_i) = 0 \]  

\[ v_R(R=R_o) = 0 ; \ v_\phi(R=R_o) = 0 ; \ v_z(R=R_o) = 0 \]  

(5)

(6)

(7)

at the bed, at the inner wall and at the outer wall, respectively.

The boundary conditions at the surface are given by the vanishing of the normal velocity and the stresses. Using the real depth of flow, which will be a function of \( R \), these conditions become rather complicated. That is why, provisionally, the cross-slope of the surface is assumed to be so small, that the range of variation of the depth of flow in a cross-section is much smaller than its average value \( d \). This means, that:

\[ \frac{v^2}{gR} \ll d \quad \text{or} \quad Fr \frac{R}{d} \ll 1 \quad \text{in which} \quad Fr = \frac{v^2}{gd} = \text{a Froude number}. \]  

(8)

\(^*\) \( v_\phi \) will be called the main (tangential) velocity-component; \( v_R \) and \( v_z \) will be referred to as the secondary velocity-components.
Accepting this limitation, the boundary conditions at the surface become:

\[ \frac{\partial \nu}{\partial z}(z=d) = 0 ; \frac{\partial \psi}{\partial z}(z=d) = 0 ; \nu_2(z=d) = 0 \]  \tag{9}

Finally it should be pointed out, that this mathematical method is also applicable to turbulent flows with a constant eddy viscosity.

C. THE METHOD OF SOLUTION

Before solving the equations 1-4, they are normalized, i.e. they are brought into a dimensionless form, such that each term is written as the product of:

1. a constant scale-factor, being of the order of magnitude (abbreviated as o.o.m. and indicated by the symbol 0)\(^{*}\) of the relevant term,

2. a variable dimensionless quantity, which, as a consequence of the definition of the scale-factors, is 0(1).

In order to realize this normalization, all independent and dependent variables and their partial derivatives, occurring in eqs. 1-4, are replaced by the product of a constant scale-factor and a variable dimensionless quantity of the o.o.m. 0(1). Choosing the latter scale-factors (which is equivalent with estimating the o.o.m. of the variables and derivatives) is the crucial step in the normalization, since it determines the o.o.m. and thus the mutual proportion of the terms in the differential equations. The free choice of the scale-factors within the relevant o.o.m. (only the o.o.m. is important for the normalization) makes it possible to measure the o.o.m. of each scale-factor by a power of one small parameter \( \varepsilon \), which is characteristic for the flow. Then the o.o.m. of each term in the normalized differential equations is indicated by the exponent of \( \varepsilon \) in the scale-factors.

\(^{*}\) According to Landau's definitions:

\[ f = O \{ (\varepsilon')^i \} \text{ if } \lim_{\varepsilon' \to 0} \left\{ \frac{f}{(\varepsilon')^i} \right\} \text{ exists} \]

\[ f = O_s \{ (\varepsilon')^i \} \text{ if } \lim_{\varepsilon' \to 0} \left\{ \frac{f}{(\varepsilon')^i} \right\} \text{ exists and } \neq 0 \]
The normalized differential equations are solved by means of a perturbation method: the dimensionless dependent variables are expanded in power series of the parameter \( \varepsilon \). So, if \( f \) represents each of these dependent variables and \( \xi \) and \( \zeta \) represent the dimensionless radial and vertical co-ordinate, respectively:

\[
f(\xi, \zeta; \varepsilon') = \sum_{i=0}^{\infty} (\varepsilon')^i f_i(\xi, \zeta; \varepsilon') \text{ such that } f_i(\xi, \zeta; \varepsilon') = O(1) \quad (10)
\]

in which the function \( f_i(\xi, \zeta; \varepsilon') \) is called the \((i)\)th order subfunction of \( f(\xi, \zeta; \varepsilon') \).

N.B. By this definition the functions \( f_i(\xi, \zeta; \varepsilon') \) are not determined uniquely!

A similar reasoning is applied to the differential equations. The normalized tangential equation of motion, for instance, can be written symbolically as:

\[
\nabla^2 f = F(f, \frac{\partial f}{\partial \xi}, \frac{\partial f}{\partial \zeta}, \ldots) \quad \text{in which } \nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2} \quad (11)
\]

Substituting the series expansions as \((10)\) into this equation, it can be written as:

\[
\nabla^2 \left( f_0 + \varepsilon' f_1 + (\varepsilon')^2 f_2 + \ldots \right) = F_0 + \varepsilon' F_1 + (\varepsilon')^2 F_2 + \ldots \quad (12)
\]

in which \((\varepsilon')^i F_i\) contains all terms of \( F \) of the o.o.m. \( O_s \) \((\varepsilon')^i\).

The subfunctions \( f_i \) are now chosen such, that for every \( i \geq 0 \):

\[
\nabla^2 (\varepsilon')^i f_i = (\varepsilon')^i F_i + \nabla^2 f_i = F_i \quad (13)
\]

The boundary conditions to be imposed on \( f_i \) are chosen to be the same as the conditions to be imposed on \( f \).

The subfunctions have been determined uniquely then and it has to be proved,
that they satisfy the condition $f_i(\xi, \zeta; \varepsilon') = O(1)$.
Assuming $\bar{f}_i = O_s(1)$ if $\nabla^2 f_i = O_s(1)$ with the present boundary conditions, it can easily be seen, that $(\varepsilon')^i f_i$ must contain all components of $f$ being of the o.o.m. $O_s((\varepsilon')^i) + f_i = O(1)$ for $i > 0$. This agrees with (10), so defns. (3) are legal.

Thus the system of normalized differential equations is split up into an infinite number of "subsystems", which, generally speaking, can be solved more easily than the original system. The subsystems are solved successively, starting from the "zero order subsystem" $(i=0)$, which yields the "zero order solution". This solution can be used as a set of known functions while solving the "first order subsystem" $(i=1)$, etc.
CHAPTER II

SOLUTION FOR THE ENTIRE CROSS-SECTION

A. DERIVATION OF ANANYAN'S THEORY

In his book "Fluid flow in bends of conduits" /2/, Ananyan gives an extensive treatise on the simplification and the solution of the differential equations, describing turbulent axisymmetrical curved flow. This theory can be translated very simply into a consideration of the viscous flow, treated in this report. The basic outlines of Ananyan's approach are quite similar to the method of solution, described in the foregoing chapter. That is why Ananyan's theory is derived here anew, using this method of solution.

1. Normalization of the differential equations.

In order to choose the scale-factors of the variables and the partial derivatives, the following assumptions are made:

a. the partial derivatives \( \frac{\partial f}{\partial R} \) and \( \frac{\partial f}{\partial z} \) are of the same o.o.m.: \( O\left(\frac{f}{b}\right) \), in which \( f \) represents each of the dependent variables and \( b \) a linear dimension of the cross-section, which is characteristic for the flow (\( b \) should be chosen such, that none of the abovementioned partial derivatives is estimated too small; in a shallow channel, for instance, \( b \) will be the depth of flow; in a deep, narrow channel \( b \) is the channel width, etc.).

N.B. The assumed o.o.m. does not depend on the average radius of curvature \( R_0 \), since the assumption holds in curved as well as in straight channels.

b. the secondary velocity-components \( v_R \) and \( v_z \) are \( O\left(\frac{V_R}{R_0}\right) \), if \( v_\phi = O(V) \) and \( R_0 \) is the average radius of curvature of the bend.

N.B. According to this assumption, the o.o.m. of the secondary velocity-components becomes very small for large radii of curvature which seems to be in agreement with physical reality.

c. in agreement with the uniform viscous flow in a straight open channel:
   \( v_\phi : \frac{A}{b}^2 \) in which \( A' = -\frac{1}{\rho} \frac{3p}{\partial R} \). This means, that the longitudinal gradient \( \frac{1}{\rho} \frac{\partial p}{\partial R} \) must be \( O(V^2b^{-2}) = O(V^2b^{-1}Re^{-1}) \) in which \( Re \) represents a Reynolds-number: \( Vb\nu^{-1} \).

N.B. In axisymmetrical curved flow \( A' \) must be a constant as well as \( \frac{1}{\rho} \frac{\partial p}{\partial s} \) is a constant in uniform straight channel flow.
d. the aforementioned Reynolds number $\overline{v}b^{-1}$ is $O(1)$. So its reciprocal $\alpha = O_s(1)$ or larger.

According to these assumptions, the variables and the partial derivatives are replaced by products of scale-factors and dimensionless quantities as follows:

- independent variables: $R = R_0 + b\zeta$ ; $z = b\zeta$ ; $\phi = \frac{b}{R_0} \phi'$

- dependent variables: $v_{\phi} = \frac{\overline{v}}{}u$ ; $v_{R} = \frac{\overline{v}}{R_0} v$ ; $v_{z} = \frac{\overline{v}}{R_0} w$ ; $p + \rho g' z = \rho \overline{v}^2 \pi$

- partial derivatives: $\frac{\partial}{\partial z} = \frac{1}{b} \frac{\partial}{\partial \zeta}$ ; $\frac{\partial}{\partial R} = \frac{1}{b} \frac{\partial}{\partial \zeta}$ ; $\frac{\partial}{\partial R} \phi = \frac{1}{b} \frac{\partial}{\partial \phi}$

It should be checked if these definitions meet the general requirement of the normalization, mentioned in part C of chapter I: the dimensionless quantities must be $O(1)$.

The only independent variable, appearing explicitly in eqs. 1-4, is $R$.

According to def. 14, the dimensionless quantity $\zeta$, which represents $R$, can be written as:

$$\zeta = \frac{R - R_0}{b}$$

Then, since the cross-section extends from $R_0 - \frac{B}{2}$ to $R_0 + \frac{B}{2}$:

$$|\zeta| \leq \frac{B}{2b}$$

So, if $b = B$ (in channels with a deep, narrow cross-section)*, the o.o.m. of $\zeta$ will be $O(1)$, indeed. If $b = d$ (in channels with a shallow, wide cross-section)*,

* in case of a channel with a "normal" cross-section (depth of flow \(=\) channel width), $b$ can be chosen $B$ as well as $d$. 
however, expr. 18 turns into:

\[ |\xi| \leq \frac{B}{2d} \]  \hspace{1cm} (19)

If \( \frac{b}{R_0} = \varepsilon \ll 1 \), this means, that in shallow channels with \( \frac{d}{B} = O(\varepsilon) \), \( R_0 = O(1) \) and \( \xi = O(\varepsilon^{-1}) \). So then:

\[ |\varepsilon \xi| \leq \frac{B}{2R_0} \leq 1 \]  \hspace{1cm} (20)

Consequently, two different cases have to be distinguished as to the factor \( \frac{1}{R} \) in eqs. 1-4:

1. if \( \frac{B}{b} = O(1) \), the factor \( \frac{1}{R} \) can be written as:

\[ \frac{1}{R} = \frac{1}{R_0} \frac{1}{1+\varepsilon \xi} = \frac{1}{R_0} \{ 1 - \varepsilon \xi + (\varepsilon \xi)^2 - \ldots + (-\varepsilon \xi)^k + O(\varepsilon^{k+1}) \} \]  \hspace{1cm} (21)

2. if \( \frac{B}{b} = O(\varepsilon^{-1}) \), the factor \( \frac{1}{R} \) can not be expanded in a power series like (21) and has to be maintained as:

\[ \frac{1}{R} = \frac{1}{R_0} \frac{1}{1+\varepsilon \xi} \]  \hspace{1cm} (22)

Considering the dependent variables, \( \overline{V} \) must be chosen such, that \( u = O(1) \). That is why \( \overline{V} \) will be defined as the average tangential velocity. Consequently, if \( Q \) represents the total discharge and \( Ar \) the cross-sectional area:

\[ \overline{V} = \frac{Q}{Ar} \]  \hspace{1cm} (23)

and:

\[ \overline{u} = \frac{1}{Ar} \int \int u dAr = 1 \]  \hspace{1cm} (24)
Besides, all dimensionless quantities representing the dependent variables will be $O(1)$.

The dimensionless quantities, representing the partial derivatives, have all been assumed to be $O(1)$.

N.B. The symbol $O(\cdot)$ only indicates an upper limit (see the definitions in the footnote of page 5).

After substitution of def. 14-16 into the differential equations 1-4, with $\frac{b}{R_0} = \epsilon$, the normalized differential equations are found to be:

\[ \epsilon^2 \frac{\partial v}{\partial \xi} + \epsilon^2 \frac{\partial w}{\partial \xi} - \frac{\epsilon}{1+\epsilon \xi} u^2 = -\frac{\partial \pi}{\partial \xi} \]

\[ + \epsilon \alpha \left( \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 w}{\partial \xi^2} + \frac{\epsilon}{1+\epsilon \xi} \frac{\partial v}{\partial \xi} - \frac{\epsilon^2 v}{(1+\epsilon \xi)^2} \right) \]  \hspace{1cm} (25)

\[ \epsilon \frac{\partial u}{\partial \xi} + \epsilon \frac{\partial u}{\partial \xi} + \frac{\epsilon}{1+\epsilon \xi} uv = -\frac{1}{1+\epsilon \xi} \frac{\partial \pi}{\partial \xi} \]

\[ + \alpha \left( \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi^2} + \frac{\epsilon}{1+\epsilon \xi} \frac{\partial u}{\partial \xi} - \frac{\epsilon^2 u}{(1+\epsilon \xi)^2} \right) \]  \hspace{1cm} (26)

\[ \epsilon^2 \frac{\partial w}{\partial \xi} + \epsilon^2 \frac{\partial w}{\partial \xi} = -\frac{\partial \pi}{\partial \xi} \]

\[ + \epsilon \alpha \left( \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \xi^2} + \frac{\epsilon}{1+\epsilon \xi} \frac{\partial w}{\partial \xi} \right) \]  \hspace{1cm} (27)

\[ \frac{\epsilon v}{1+\epsilon \xi} + \frac{\partial v}{\partial \xi} + \frac{\partial w}{\partial \xi} = 0 \]  \hspace{1cm} (28)

in which the factors $\frac{1}{1+\epsilon \xi}$ must be expanded in the power series (21) or maintained as a whole, according as $\frac{b}{b} = O(1)$ or $O(\epsilon^{-1})$.

In both cases the boundary conditions are given by the condition of adherence at the walls and the vanishing of the normal velocity and the viscous stresses at the free surface. For a rectangular cross-section and small values of $F \frac{b}{R_0}$ (see chapter I), these boundary conditions are:

- at the bed ($\xi=0$): \[ u = v = w = 0 \]  \hspace{1cm} (29)
- at the surface \( \xi = \frac{d}{b} \): \[ \frac{\partial u}{\partial \xi} = \frac{\partial v}{\partial \xi} = w = 0 \] (30)

- at the outer wall \( \xi = \frac{B}{b} \): \[ u = v = w = 0 \] (31)

- at the inner wall \( \xi = -\frac{B}{b} \): \[ u = v = w = 0 \] (32)

Eqs. 25-28 can be solved for small values of \( \epsilon \), using the perturbation method, described in chapter I. But first some definitions as to \( \pi \) must be given:

- since the velocity-components do not depend on \( \phi \), the normalized equations 25-28 yield the conclusion, that \( \frac{\partial \pi}{\partial \phi} \) is a constant; this constant is called \( -\Delta \).

- the average value of \( \pi \) in the cross-section \( \phi' = 0 \) is named \( \pi^* \).

The average value of \( \pi \) in any cross-section \( \phi' \) must then be given by:

\[ \bar{\pi} = \pi^* - \Delta \phi \] (33)

2. Application of the perturbation method for \( \frac{B}{b} = O(1) \) and \( \frac{B}{R_0} = O(\epsilon) \).

This is the case of the deep, narrow (and the normal) channel. The perturbation method, described in chapter I, is applied to the normalized differential equations 25-28 with the boundary conditions 29-32. Since \( \frac{B}{b} = O(1) \), the factor \( \frac{1}{1 + \epsilon \xi} \) should be expanded in the power series 21.

Successively, the piezometric head and the tangential velocity-component, the secondary velocity-components and the effect of the latter on the tangential velocity-component are treated.

a. Piezometric head and tangential velocity-component.

Substituting the series expansions of \( u, v, w \) and \( \pi \) into eqs. 19-22 and omitting all terms containing the factor \( \epsilon^n \) (\( n \geq 1 \)), the zero order subsystem is obtained:

\[ 0 = -\frac{\partial \pi}{\partial \xi} \] (34)
\[ 0 = - \frac{3\pi_0}{\phi'} + \alpha \left( \frac{\partial^2 u_0}{\partial \zeta^2} + \frac{\partial^2 u_0}{\partial \zeta^2} \right) \]  
(35)

\[ 0 = - \frac{3\pi_0}{\partial \zeta} \]  
(36)

\[ \frac{\partial v_0}{\partial \zeta} + \frac{3w_0}{\partial \zeta} = 0 \]  
(37)

Herein \( \pi_0 \) is defined such that:

\[ \frac{3\pi_0}{\partial \phi'} = -\Delta \quad \text{and} \quad \pi_0 (\phi' = 0) = \Pi^* \]  
(38)

and consequently eqs. 29-31 yield:

\[ \pi_0 = \Pi^* - \Delta \phi' \]  
(39)

So in the zero order approximation the pressure is hydrostatic. Substituting this expression into eq. 35, the latter turns into:

\[ \frac{\partial^2 u_0}{\partial \zeta^2} + \frac{\partial^2 u_0}{\partial \zeta^2} = -\frac{\Delta}{\alpha} \]  
(40)

which is identical to the equation for the longitudinal velocity in a straight channel. For the present problem, with the boundary conditions:

\[ u_0 (\zeta = 0) = 0 ; \quad \frac{\partial u_0}{\partial \zeta} (\zeta = 1) = 0 ; \quad u_0 (\zeta = \frac{B}{2b}) = 0 \]  
(41)
the solution of this equation is:

\[ u_0 = \frac{\Delta}{\alpha} \sum_{k=1,3,5,...}^{\infty} \frac{16}{k^3 \pi^3} \left( \frac{\cosh \frac{k\pi \xi}{2}}{\cosh \frac{k\pi B}{2b}} \right) \sin \frac{k\pi \xi}{2} \quad (42) \]

From 24: \( \bar{u}=1 \) it can be concluded, that \( \frac{\Delta}{\alpha} \) must be a constant, only depending on the channel geometry.

For the sake of convenience, \( u_0 \) will be split up into two factors:

\[ u_0 = \frac{\Delta}{\alpha} u_0^* (\xi, \zeta) \quad (43) \]

b. Secondary velocity-components.

The zero order subfunctions \( v_0 \) and \( w_0 \), representing the main terms of the secondary velocity-components, can be solved from the zero order equation of continuity:

\[ \frac{\partial v_0}{\partial \xi} + \frac{\partial w_0}{\partial \zeta} = 0 \quad (37) \]

and the radial and vertical equations of motion of the first order subsystem:

\[ -u_0^2 = \frac{\partial v_0}{\partial \xi} + \alpha \left( \frac{\partial^2 v_0}{\partial \xi^2} + \frac{\partial^2 v_0}{\partial \zeta^2} \right) \quad (44) \]

\[ 0 = \frac{\partial^2 w_0}{\partial \xi^2} + \alpha \left( \frac{\partial^2 w_0}{\partial \xi^2} + \frac{\partial^2 w_0}{\partial \zeta^2} \right) \quad (45) \]

The normalized equation of continuity 28 is satisfied by defining a stream function \( g \) such, that:

\[ v = -\frac{\partial g}{\partial \zeta} \quad \text{and} \quad w = \frac{1}{1+\xi \zeta} \frac{\partial}{\partial \zeta} \{(1+\xi \zeta)g\} \quad (46) \]
Accordingly, the stream function for \( v_0 \) and \( w_0 \) is defined by:

\[
 v_0 = -\frac{\partial g_0}{\partial \zeta} \quad \text{and} \quad w_0 = \frac{\partial g_0}{\partial \zeta}
\]  

(47)

so that eq. 37 is satisfied. Substituting this definition into eqs. 44 and 45 and eliminating \( \pi_1 \) yields the equation:

\[
 \frac{\partial}{\partial \zeta} (u_0^2) = \alpha \left( \frac{\partial^4 g_0}{\partial \zeta^4} + 2 \frac{\partial^4 g_0}{\partial \zeta^2 \partial \zeta^2} + \frac{\partial^4 g_0}{\partial \zeta^4} \right)
\]  

(48)

Defining: \( \varepsilon_0^* = \frac{\alpha}{\Delta^2} \varepsilon_0 \)

(49)

\[
 v_0^* = \frac{\alpha}{\Delta^2} \, v_0
\]  

(50)

\[
 w_0^* = \frac{\alpha}{\Delta^2} \, w_0
\]  

(51)

eq. 48 turns into:

\[
 \frac{\partial}{\partial \zeta} \left( u_0^* \right)^2 = \frac{\partial^4 \varepsilon_0^*}{\partial \zeta^4} + 2 \frac{\partial^4 \varepsilon_0^*}{\partial \zeta^2 \partial \zeta^2} + \frac{\partial^4 \varepsilon_0^*}{\partial \zeta^4}
\]  

(52)

The boundary conditions of this problem are derived from the conditions to be imposed on \( v_0 \) and \( w_0 \):

\[
 v_0(\zeta=0) = 0 \quad ; \quad \frac{\partial v_0}{\partial \zeta}(\zeta=1) = 0 \quad ; \quad v_0(\zeta = \pm \frac{B}{2b}) = 0
\]  

(53)

\[
 w_0(\zeta=0) = 0 \quad ; \quad w_0(\zeta=1) = 0 \quad ; \quad w_0(\zeta = \pm \frac{B}{2b}) = 0
\]  

(54)
which yield:

$$g^*_0(\xi=0) = g^*_0(\xi=1) = g^*_0(\xi = \pm \frac{B}{2b}) = 0$$

$$\frac{\partial g^*_0}{\partial \xi}(\xi=0) = 0 ; \frac{\partial^2 g^*_0}{\partial \xi^2}(\xi=1) = 0 ; \frac{\partial g^*_0}{\partial \xi}(\xi = \pm \frac{B}{2b}) = 0$$ (55)

N.B. It can be concluded from eq. 52 and conds. 55, that $$g^*_0$$ is a function of $$\xi$$, $$\zeta$$ and $$\frac{B}{b}$$ alone.

c. First order correction of the tangential velocity-component.

The first order subfunction of the tangential velocity-component can be solved from the equation:

$$v_0 \frac{\partial u_0}{\partial \xi} + w_0 \frac{\partial u_0}{\partial \zeta} = - \frac{3 \pi_1}{\Delta \phi} + \frac{\partial u_1}{\partial \xi} + \frac{\partial^2 u_1}{\partial \xi^2} \alpha + \frac{\partial^2 u_1}{\partial \zeta^2} \sigma + \frac{\partial \pi_0}{\partial \phi}$$ (56)

As a consequence of the definition of $$\pi_0$$ (see 39), $$\frac{\partial \pi_1}{\partial \phi}$$ must vanish.

$$u_0, v_0, w_0$$ and $$\pi_0$$ being known, $$u_1$$ can be solved from eq. 56, using the boundary conditions:

$$u_1(\xi=0) = 0 ; \frac{\partial u_1}{\partial \xi}(\xi=1) = 0 ; u_1(\xi = \pm \frac{B}{2b}) = 0$$ (57)

Using the definitions of $$u_0^*, v_0^*$$ and $$w_0^*$$, equation 56 can be written as:

$$v^2 u_1 = \frac{\Delta}{3} \left( \frac{\partial u_0^*}{\partial \xi} + \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} \right)$$ (58)

For the sake of convenience, this equation is split up into three parts:

$$v^2 u_{1,0} = \xi$$ (59)
\[ \nabla^2 u_{1,1} = v_0 \frac{\partial u_0}{\partial z} + \omega_0 \frac{\partial u_0}{\partial r} \quad (60) \]

\[ \nabla^2 u_{1,2} = -\frac{\partial u_0}{\partial z} \quad (61) \]

while the boundary conditions for each of these equations are taken the same as conditions 57.

Obviously, the solution of eq. 59 gives the first order part of the effect of the tangential slope, the solution of eq. 60 represents the effect of the secondary flow on the tangential velocity-component and the solution of eq. 61 gives the influence of the "extra" friction-terms, generated by the polar co-ordinate system. The mutual proportion of these effects depends on \( \Delta \) and \( \alpha \), as eq. 58 shows.

d. Synopsis.

The results of the present analysis can be resumed as follows:

\[ v_\phi = \bar{v}(\frac{\Delta}{\alpha} (u_0^* + \varepsilon u_{1,0}^* + \varepsilon u_{1,2}^*) + \varepsilon \frac{\Delta^2}{\alpha} u_{1,1}^*) + \bar{v} O(\varepsilon^2) \quad (62) \]

\[ v_R = \bar{v} \varepsilon \frac{\Delta^2}{\alpha^3} v_0^* + \bar{v} O(\varepsilon^2) \quad (63) \]

\[ v_z = \bar{v} \varepsilon \frac{\Delta^2}{\alpha^3} v_0^* + \bar{v} O(\varepsilon^2) \quad (64) \]

N.B. The ratio of \( u_0^* \) and the effect of the secondary flow depends on \( \varepsilon \), \( \Delta \) and \( \alpha \), but the ratio of \( u_0^* \) and the effect of the "extra" friction-terms only depends on \( \varepsilon \).

3. Application of the perturbation method for \( \frac{B}{R_0} = O(\varepsilon^{-1}) \) and \( \frac{B}{R_0} = O(1) \).

This is the case of the wide, shallow (and the normal) channel. Again, the perturbation method, described in chapter I, is applied to eqs. 25-28 with the boundary conditions 29-32, but now the factor \( \frac{1}{1+\varepsilon \xi} \) can not be expanded.
a. **Piezometric head and tangential velocity-component.**

The zero order subsystem is:

\[ 0 = - \frac{3\pi}{\xi} \]  
\[ 0 = - \frac{1}{1+\varepsilon\xi} \frac{\partial^3 \gamma}{\partial \phi'} + \alpha \left( \frac{3^2 u_0}{\xi^2} + \frac{3^2 u_0}{\zeta^2} \right) \]  
\[ 0 = - \frac{3\pi}{\zeta} \]  
\[ \frac{3\nu}{\xi} + \frac{3\omega}{\zeta} = 0 \]

As far as \( \gamma_0 \) is concerned, there is no difference between this and the foregoing considerations. So:

\[ \gamma_0 = \gamma^* - \Delta \phi' \]

Substituting this expression into eq. 66 and defining \( u_0^* \) by:

\[ u_0^* = \frac{a}{A} u_0 \]

yields:

\[ \frac{3^2 u_0^*}{\xi^2} + \frac{3^2 u_0^*}{\zeta^2} = - \frac{1}{1+\varepsilon\xi} \]

The relevant boundary conditions are:

\[ u_0^*(\zeta=0) = 0 ; \frac{3u_0^*}{\zeta}(\zeta=1) = 0 ; u_0^*(\zeta=\frac{B}{2b}) = 0 \]
b. Secondary velocity-components.

The zero order subfunctions \( v_0 \) and \( w_0 \) are solved from the zero order equation of continuity:

\[
\frac{\partial v_0}{\partial \xi} + \frac{\partial w_0}{\partial \zeta} = 0
\]

and the radial and vertical equations of motion of the first order subsystem:

\[
- \frac{u_0^2}{1 + \varepsilon \zeta} = - \frac{3\pi_1}{\delta \zeta} + \alpha \left( \frac{3^2 v_0}{\delta \zeta^2} + \frac{3^2 w_0}{\delta \zeta^2} \right)
\]

\[
0 = - \frac{3\pi_1}{\delta \zeta} + \alpha \left( \frac{3^2 w_0}{\delta \zeta^2} + \frac{3^2 w_0}{\delta \zeta^2} \right)
\]

Here the definition of the zero order stream function, to be derived from def. 46, becomes:

\[
v_0 = - \frac{3g_0}{\delta \zeta} \quad \text{and} \quad w_0 = + \frac{3g_0}{\delta \zeta}
\]

by which eq. 68 is satisfied.

Substituting this definition into eqs. 73 and 74, eliminating \( \pi_1 \) and defining:

\[
g_0^* = \frac{\alpha^3}{\Delta^2} g_0
\]

\[
v_0^* = \frac{\alpha^3}{\Delta^2} v_0
\]

\[
w_0^* = \frac{\alpha^3}{\Delta^2} w_0
\]

yields the equation:

\[
\frac{\partial u_0^*}{\partial \zeta} = \frac{3^4 g_0^*}{\delta \zeta^4} + \frac{3^4 g_0^*}{\delta \zeta^2 \delta \zeta^2} + \frac{3^4 g_0^*}{\delta \zeta^4}
\]
The boundary conditions of this problem are derived from the boundary conditions to be imposed on \( v_0 \) and \( w_0 \); they are:

\[
\frac{\partial g_0^*}{\partial \xi} (\xi=0) = g_0^* (\xi=1) = g_0^* (\xi=\pm \frac{B}{2b}) = 0
\]

\[
\frac{\partial^2 g_0^*}{\partial \xi^2} (\xi=0) = \frac{\partial^2 g_0^*}{\partial \xi^2} (\xi=1) = \frac{\partial g_0^*}{\partial \xi} (\xi=\pm \frac{B}{2b}) = 0
\]

(80)

c. First order correction of the tangential velocity-component.

The tangential equation of motion of the first order subsystem is:

\[
v_0 \frac{\partial u_0}{\partial \xi} + w_0 \frac{\partial u_0}{\partial \zeta} = \frac{-1}{1+\varepsilon \xi} \frac{\partial \pi}{\partial \phi} + \alpha \left( \frac{\partial^2 u_1}{\partial \xi^2} + \frac{\partial^2 u_1}{\partial \zeta^2} \right) + \alpha \frac{1}{1+\varepsilon \xi} \frac{\partial u_0}{\partial \xi}
\]

(81)

As a consequence of the definition of \( \pi_0 \) (see 69), \( \frac{\partial \pi}{\partial \phi} \) vanishes. Then \( u_1 \) can be solved from this equation, using the boundary conditions:

\[
u_1 (\xi=0) = 0 ; \frac{\partial u_1}{\partial \xi} (\xi=1) = 0 ; u_1 (\xi=\pm \frac{B}{2b}) = 0
\]

(82)

For the sake of convenience, \( u_1 \) is split up into two parts:

\[
u_1 = \frac{\Delta}{\alpha} u_{1,1} + \frac{\Delta}{\alpha} u_{1,2}
\]

(83)

such that:

\[
\nabla^2 u_{1,1} = \frac{\partial u_0}{\partial \xi} + \frac{\partial u_0}{\partial \zeta}
\]

(84)

\[
\nabla^2 u_{1,2} = \frac{-1}{1+\varepsilon \xi} \frac{\partial u_0}{\partial \xi}
\]

(85)

The boundary conditions of each of these equations are assumed to be similar to conditions 82.
Again, \( u^{*}_{1,1} \) represents the influence of the secondary flow and \( u^{*}_{1,2} \) gives the effect of the "extra" friction-terms.

N.B. In this case the effect of the tangential slope has already been accounted for in \( u^{*}_{0} \) (cf. part 2c of this chapter).

d. Synopsis

The results of this part of the chapter can be resumed as:

\[
v_{\phi} = \overline{V} \left( \frac{\Delta}{\alpha} (u^{*}_{0} + \varepsilon u^{*}_{1,2}) + \frac{\varepsilon^{3}}{\alpha^{3}} u^{*}_{1,1,1} \right) + \overline{V} \; 0(\varepsilon^{2})
\]  

(86)

\[
v_{R} = \overline{V} \varepsilon^{2} \frac{\Delta}{\alpha^{3}} v^{*}_{0} + \overline{V} \; 0(\varepsilon^{2})
\]  

(87)

\[
v_{z} = \overline{V} \varepsilon^{2} \frac{\Delta}{\alpha^{3}} v^{*}_{0} + \overline{V} \; 0(\varepsilon^{2})
\]  

(88)

4. Ananyan's theory compared with the results of the perturbation method.

a. Tangential velocity-component.

In his extensive book on curved open channel flow /2/, Ananyan considers axisymmetrical turbulent flow with a variable or constant coefficient of turbulent viscosity (eddy-viscosity) \( \alpha \). He uses the following equation of motion to solve \( \alpha \) from:

\[
\frac{3}{3R} \frac{\partial}{\partial R} (\alpha \frac{\partial \phi}{\partial R}) + \frac{3}{3z} (\alpha \frac{\partial \phi}{\partial z}) = \frac{1}{R} \frac{\partial P}{\partial \phi}
\]  

(89)

in which \( v_{\phi} \) represents the longitudinal velocity in the straight part of the channel before the bend.

For constant \( \alpha \), this equation is equivalent with eq. 66, in which the factor \( \frac{1}{1+\varepsilon \xi} \) has been maintained as a whole. The use of \( v_{\phi}^{*}_{0} \), however, agrees with eq. 40, in which \( \frac{1}{1+\varepsilon \xi} \) has been expanded!
b. Secondary velocity-components.

Ananyan defines a stream function $F$ such that:

$$v_R = -\frac{1}{R} \frac{\partial F}{\partial z} \quad \text{and} \quad v_z = \frac{1}{R} \frac{\partial F}{\partial R}$$  \hspace{1cm} (90)

This definition is substituted into the radial and vertical equations of motion:

$$-\frac{v_R^2}{R} = -\frac{1}{\rho} \frac{\partial p}{\partial R} + v\left(\frac{\partial^2 v_R}{\partial R^2} + \frac{\partial^2 v_R}{\partial z^2}\right)$$  \hspace{1cm} (91)

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - \gamma + v\left(\frac{\partial^2 v_z}{\partial R^2} + \frac{\partial^2 v_z}{\partial z^2}\right)$$  \hspace{1cm} (92)

which, after elimination of $p$ and neglect of terms $O(R^{-2})$ and smaller, leads to the equation:

$$\frac{\partial^4 F}{\partial R^4} + 2\frac{\partial^4 F}{\partial R^2 \partial z^2} + \frac{\partial^4 F}{\partial z^4} = \frac{1}{\nu} \frac{\partial}{\partial z} \left( v_0^2 \right)$$  \hspace{1cm} (93)

N.B. In the right hand part of this equation the factor $\frac{1}{R}$ has not been neglected, but it has disappeared as a consequence of the multiplication of the whole equation by $R$ during the transformations, which yield 93.

Eq. 93 is not consequent in that $v_0$ represents the longitudinal velocity in the straight part of the channel before the bend (which is in agreement with eq. 48), while the factor $\frac{1}{R}$ is maintained, actually (which is in agreement with eq. 79).

Besides, in the author's opinion the definition of the stream function 46 should be preferred in the present theory, regarding the equations of continuity in the higher order subsystems. The differences, however, will lie within the accuracy of the relevant subsystem.
c. First order correction of $v^\phi_0$.

Ananyan defines the real tangential velocity-component $v^\phi$ by:

$$v^\phi = v^\phi_0 + \frac{c(R,z)}{R}$$ (94)

in which $v^\phi_0$ represents the longitudinal velocity in the straight part of the channel before the bend, while $c(R,z)$ is a function of $R$ and $z$, characterizing the redistribution of the longitudinal velocity, caused by the bend. As a first approximation of $c(R,z)$ Ananyan takes the solution of a differential equation, equivalent with eq. 56:

$$- \frac{1}{R_0} \frac{\partial v^\phi_0}{\partial z} + \frac{1}{R_0} \frac{\partial F}{\partial R} \frac{\partial v^\phi_0}{\partial z} - \frac{\partial F}{\partial R} \frac{\partial v^\phi_0}{\partial R} - \frac{v^\phi_0}{R_0} \frac{\partial p}{\partial R} =$$

$$= \frac{v}{R_0} \left( \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 c}{\partial z^2} \right)$$ (95)

in which $F$ is the stream function of the secondary velocity-components, as defined by 90.

Obviously, Ananyan has expanded the factor $\frac{1}{R}$ in the equation. This is in agreement with the use of $v^\phi_0$ as a first approximation of the tangential velocity-component, but not with the use of the "complete" factor $\frac{1}{R}$ in the equations, describing the secondary velocity-components.

5. Conclusions.

From the considerations in this part of the chapter, the following conclusions can be drawn:

- application of the method of solution, described in chapter I, yields differential equations, which are quite similar to the equations, derived by Ananyan. The differences between the solutions may be expected to lie within the accuracy of the relevant equations.

- Ananyan's considerations are not entirely consequent in that the equations for the tangential velocity-component include the expansion of the factor $\frac{1}{R}$ in a power series of $\epsilon$, while the equations for the secondary velocity-components do not.
B. APPLICATION TO A SHALLOW RECTANGULAR CROSS-SECTION.

The theory, developed in the first part of this chapter, will be applied to a shallow rectangular cross-section. In that case the characteristic linear dimension of the flow is \( d \) (the average depth of flow); so \( b := d \).
This implies, that the solutions hold true for small values of \( \varepsilon = \frac{d}{R_0} \).

1. Normalization of the differential equations.

The assumptions, which have to be made for the normalization of the differential equations have been described in general terms in part A of this chapter. For the present case they are:

a. \( \frac{\partial f}{\partial R} \text{ and } \frac{\partial f}{\partial z} \) are \( O\left(\frac{\varepsilon}{d}\right) \); \( f \) represents any dependent variable.

b. \( \nu_R \) and \( \nu_z \) are \( O(\varepsilon \overline{V}) \).

c. if \( \text{Re} = \frac{\overline{V} d}{\nu}, \frac{1}{\rho R} \frac{\partial p}{\partial \phi} \) is \( O\left(\overline{V}^2 d^{-1} \text{Re}^{-1}\right) \).

d. \( \text{Re} \) is \( O(1) \).

Accordingly, the scale-factors and dimensionless quantities are defined by (see 14-16):

- (independent variables) \( R = R_0 \left(1 + \varepsilon \xi\right); z = d \zeta; \phi = \varepsilon \phi' \) \hfill (96)

- (dependent variables) \( v_\phi = \overline{V} u; v_R = \varepsilon \overline{V} v; v_z = \varepsilon \overline{V} w; p + \rho g' z = \rho \overline{V}^2 \pi \) \hfill (97)

- (partial derivatives) \( \frac{\partial}{\partial z} = \frac{1}{d} \frac{\partial}{\partial \zeta}; \frac{\partial}{\partial R} = \frac{1}{d} \frac{\partial}{\partial \xi}; \frac{\partial}{\partial \phi} = \frac{1}{1 + \varepsilon \xi} \frac{1}{d} \frac{\partial}{\partial \phi'} \) \hfill (98)

If the scale-factor \( \overline{V} \) is chosen equal to the average tangential velocity (so \( \overline{V} = \frac{Q}{Ar} \)), then:

\[
\overline{u} = \frac{1}{Ar} \int \int udAr = \frac{d}{B} \int \int \frac{1}{ud\xi d\zeta} = 1 \\
\frac{B}{2d} 0
\] \hfill (99)
Besides, the definitions \( \frac{\partial \pi}{\partial \phi'} = -\Delta \) and \( \pi = \Pi^* - \Delta \phi' \) (see 33) remain unchanged, whatever the channel geometry may be.

The normalized differential equations are given by 25-28, in which the factor \( \frac{1}{1 + \varepsilon \xi} \) may not be expanded, now (shallow cross-section, so \( \frac{d}{B} \) is small). The boundary conditions for a rectangular cross-section are given in a general form by 29-32. Obviously, for \( b = d \) the only geometrical parameter in the boundary conditions is \( \frac{d}{B} \).

The differential equations of the subsystems contain the geometrical parameter \( \frac{d}{R_0} \) and the flow parameters \( \Delta \) and \( \alpha \); the latter can be eliminated rather simply (by defining \( u_0^*, v_0^*, w_0^*, \) etc.), but the former cannot. This implies, that for every new value of \( \frac{d}{B} \) or \( \frac{d}{R_0} \) all differential equations have to be integrated anew!

As an example, in this part of the chapter the theory of part A3 will be elaborated for:

\[
\frac{d}{B} = \varepsilon_1 = .1 \quad \text{and} \quad \frac{d}{R_0} = \varepsilon_2 = .4
\]  \hspace{1cm} (100)

2. Solution of the differential equations.

a. \( u_0^* \) and \( \pi_0 \).

The general considerations as to \( \pi_0 \) (see def. 38 and expr. 39) can be applied to the present problem without any changes. So:

\[
\pi_0 = \Pi^* - \Delta \phi'
\]  \hspace{1cm} (39)

The first approximation of the tangential velocity-component, \( u_0^* \), is solved from eq. 47 with the boundary conditions:

\[
u_0^*(\xi = 0) = 0 ; \quad \frac{\partial u_0^*}{\partial \xi} (\xi = 1) = 0 ; \quad u_0^*(\xi = \pm \frac{B}{2d} = \pm 5) = 0 \]  \hspace{1cm} (101)

This problem has been solved numerically, using an overrelaxation method. The results are illustrated in fig. 3.
Tangential velocity-component.

(a) $u_0^*$ at the surface ($\xi=1$).

(b) $u_0^*$ at the central axis ($\xi=0$).

(c) lines of constant $u_0^*$

b. $g_0^*$, $v_0^*$ and $w_0^*$

The stream function $g_0^*$ of the secondary velocity-components is solved from eq. 79 with the boundary conditions:

$$g_0^*(\xi=0) = g_0^*(\xi=1) = g_0^*(\xi=\frac{B}{2d} = \pm 5) = 0$$

$$\frac{\partial g_0^*}{\partial \xi} (\xi=0) = 0; \quad \frac{\partial^2 g_0^*}{\partial \xi^2} (\xi=1) = 0; \quad \frac{\partial g_0^*}{\partial \xi} (\xi=\pm 5) = 0$$

Eq. 79 is identical to the differential equation, describing the deflections of a flat plate with stiffness 1, which is loaded perpendicularly by a distributed load $\frac{1}{1+\varepsilon \xi} \frac{3}{3} \{(u_0^*)^2\}$. Subsequently, this equation has been solved numerically, using an existing computer program for the computation of the deflections of flat plates, based on the method of finite elements. The results of this computation are given in fig. 4.

It should be noted, that in the present stationary flow field the lines of constant $g_0^*$ are "secondary streamlines" (i.e. in every point of the cross-section the tangent of these lines gives the direction of the secondary flow).
fig. 4

Stream function of the secondary velocity-components.
(a) $g^*_{0}$ at the level $\zeta=0.5$
(b) $g^*_{0}$ at the central axis ($\xi=0$)
(c) lines of constant $g^*_{0}$ (streamlines secondary flow).

The secondary velocity-components $v^*_{0}$ and $w^*_{0}$ are derived from $g^*_{0}$, using def. 73, which yields the results, given in figs. 5 and 6.

fig. 5.

Radial velocity-component.
(a) $v^*_{0}$ at the surface ($\zeta=1$)
(b) $v^*_{0}$ at the central axis ($\xi=0$)
(c) lines of constant $v^*_{0}$. 
fig. 6.
Vertical velocity-component.
(a) $u_0^*$ at the level $\xi=0.5$
(b) $u_0^*$ at $\frac{1}{4}d$ from the inner wall ($\xi=-4.5$)
(c) lines of constant $w_0^*$

$c. u_{1,1}^*$ and $u_{1,2}^*$

The function $u_{1,1}^*$, representing the influence of the secondary velocity-components on the main flow, is solved from eq. 84 with the boundary conditions:

$$u_{1,1}^*(\xi=0) = 0 ; \quad \frac{\partial u_{1,1}^*}{\partial \xi}(\xi=1) = 0 ; \quad u_{1,1}^*(\xi=5) = 0$$

(103)

This problem has been solved numerically by the overrelaxation program, that was used to solve $u_0^*$. Fig. 7 gives the results.

The function $u_{1,2}^*$, representing the influence of the extra friction-terms, is solved from eq. 85 with the boundary conditions:

$$u_{1,2}^*(\xi=0) = 0 ; \quad \frac{\partial u_{1,2}^*}{\partial \xi}(\xi=1) = 0 ; \quad u_{1,2}^*(\xi=5) = 0$$

(104)

The overrelaxation program, used to solve $u_0^*$ and $u_{1,1}^*$, can be applied here, too. It leads to the solution, given in fig. 8.
fig. 7.
Influence of the secondary velocity-components on the main flow.
(a) $u_{1,1}^*$ at the surface ($\xi = 1$)
(b) $u_{1,1}^*$ at $d$ from the inner wall ($\xi = -4$)
(c) lines of constant $u_{1,1}^*$

fig. 8.
Influence of the extra friction terms on the main flow.
(a) $u_{1,2}^*$ at the surface ($\xi = 1$)
(b) $u_{1,2}^*$ at $\frac{1}{4}d$ from the inner wall ($\xi = -4, 5$)
(c) lines of constant $u_{1,2}^*$
3. Conclusions.

From the results, given in figs. 4-8, it can be concluded, that:

- the region of the highest (zero order) tangential velocities is situated near the inner wall. This agrees with the main flow pattern, observed by many investigators in the first part of the bend (see for instance Ananyan /2/ and Rozovs'kii /3/);
- the secondary flow is directed towards the inner wall in the lower part of the cross-section and towards the outer wall in the upper part (which could be expected; it agrees with all experimental observations, too).
- the magnitude of the secondary flow is maximal near the inner wall.
- the vertical velocity-component is directed downward in the entire cross-section, except in a region, extending over a few times the depth of flow from the inner wall. There a relatively high positive peaque appears.
- in the central region of the cross-section the vertical velocity-component is small, when compared with the radial velocity-component.
- as a consequence of the secondary flow, \( v_\phi \) tends to be larger than the zero order tangential velocity-component \( \bar{v}_0 \), except in the region near the inner wall, where the secondary flow causes a decrease of \( v_\phi \) with respect to \( \bar{v}_0 \). This decrease is considerably larger than the increase in the remaining part of the cross-section.
- the "extra" friction terms cause an increase of \( v_\phi \) near the inner wall and a (smaller) decrease in the central region and near the outer wall. In other words: the tendency of \( v_\phi \) to have its maximum near the inner wall is emphasized by the effect of the extra friction terms.
- each of the functions of \( \xi \) and \( \zeta \), given in figs. 4-8, has the following property:
  - the shape of the curve, representing the function for \( \xi = \xi_0 \) = a constant, nearly does not change if \( \xi_0 \) is varied.
  - the shape of the curve, representing the function for \( \zeta = \zeta_0 \) = a constant, nearly does not change if \( \zeta_0 \) is varied.

This implies, that each of the functions can be approximated by the product of function of \( \xi \) and a function of \( \zeta \).
CHAPTER III

SOLUTION BY MEANS OF THE METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

A. INTRODUCTION

From the results of the foregoing chapter it appears, that there is a considerable difference between the regions near the side-walls (to be called inner and outer wall region) and the central region of the cross-section, if the depth of flow is small with respect to the channel width. This may be expressed by the following theorem:

In the central region of a shallow cross-section the partial derivatives \( \frac{\partial}{\partial R} \) are much smaller (viz. a factor \( O\left(\frac{d}{R}\right) \)) than the partial derivatives \( \frac{\partial}{\partial z} \); in the regions near the side-walls these derivatives have the same order of magnitude.

This phenomenon makes it possible to treat these three regions separately: in each of the regions the differential equations of the mathematical model are normalized, taking advantage of the abovementioned theorem, after which a perturbation method is used to solve them. This perturbation method, called the method of matched asymptotic expansions (see Van Dyke /4/; Lagerstrom and Casten /6/), is rather like the method described in chapter I, extended by a matching of the corresponding solutions in the three regions.
B. SOLUTION IN THE CENTRAL REGION

1. Normalization of the differential equations.

In the central region the differential equations of the mathematical model (eqs. 1-4) are normalized, starting from the assumptions:

a. \( \frac{\partial f}{\partial R} = O\left(\frac{f}{R_0}\right) \) and \( \frac{\partial f}{\partial z} = O\left(\frac{f}{d}\right) \), if \( f \) represents each of the dependent variables.

N.B. \( R_0 \) has been chosen here as the reference length of \( R \) in \( \frac{\partial f}{\partial R} \), since these derivatives vanish in the central region of the cross-section of a shallow, straight channel.

b. \( v_R = \overline{v} O\left(\frac{d}{R_0}\right) \). Then the equation of continuity (eq. 4) yields:

\[
\frac{\partial v_z}{\partial z} = O\left(\frac{v_z}{d}\right) = - \frac{v_R}{R} - \frac{\partial v_R}{\partial R} = - \frac{1}{R_0} O(v_R)
\]

so that:

\[
v_z = O\left(v_R \frac{d}{R_0}\right) = \overline{v} O\left(\frac{d}{R_0}\right)^2
\]

(105)

(106)

c. \( \frac{1}{\rho R} \frac{\partial p}{\partial \phi} = O(\overline{v}d^{-1}Re^{-1}) \). See chapter II.

d. \( Re = O(1) \). See chapter II.

According to these assumptions, the variables and the partial derivatives are replaced by products of scale-factors and dimensionless quantities as follows:

- independent variables: \( R = R_0 r \); \( z = d \zeta \); \( \phi = \frac{d}{R_0} \phi' \)

(107)

- dependent variables:

\[
\nu_\phi = \overline{v} u \; ; \; v_R = \frac{d}{R_0} \overline{v} v \; ; \; v_z = \left(\frac{d}{R_0}\right)^2 \overline{v} w
\]

\[
p + \rho g'z = \rho \overline{v}^2 \pi
\]

(108)

- partial derivatives:

\[
\frac{\partial}{\partial z} = \frac{1}{d} \frac{\partial}{\partial \zeta} \; ; \; \frac{\partial}{\partial R} = \frac{1}{R_0} \frac{\partial}{\partial r} \; ; \; \frac{\partial}{\partial \phi} = \frac{d}{R_0} \frac{\partial}{\partial \phi'}
\]

(109)

but \( \frac{1}{\rho} \frac{\partial p}{\partial R} = \frac{\overline{v}^2}{d} \frac{\partial \pi}{\partial r} \)

(109a)

N.B. Taking the scale-factors this way, the dimensionless independent variables are \( O(1) \), in accordance with the starting-point of the normalization. This is evident as to \( \zeta \) and \( \phi' \) and it can be proved for \( r \) as follows:
\[ R_0 - \frac{B}{2} \leq R \leq R_0 + \frac{B}{2} \]  \hspace{1cm} (110)

so that: \[ 1 - \frac{B}{2R_0} \leq r \leq 1 + \frac{B}{2R_0} \]  \hspace{1cm} (111)

and since \( 0 \leq \frac{B}{2R_0} \leq 1 \), this implies:

\[ 0 \leq r \leq 2 \Rightarrow r = O(1) \]  \hspace{1cm} (112)

Using definitions 107-109 and writing \( \frac{d}{R_0} \) as \( \varepsilon \), again, the normalized differential equations in the central region become:

\[ \varepsilon^3 \frac{\partial^2 v}{\partial r^2} + \varepsilon^3 \frac{\partial v}{\partial \zeta} - \varepsilon \frac{u^2}{r} = - \frac{\partial v}{\partial r} + \alpha \left( \varepsilon \frac{\partial^2 v}{\partial \zeta^2} + \varepsilon \left( \frac{\partial^2 v}{\partial r^2} + \frac{1}{r} \frac{\partial v}{\partial r} - \frac{v}{r^2} \right) \right) \]  \hspace{1cm} (113)

\[ \varepsilon^2 \frac{\partial^2 u}{\partial r^2} + \varepsilon^2 \frac{\partial u}{\partial \zeta} + \varepsilon^2 \frac{uv}{r} = - \frac{\partial u}{\partial r} + \alpha \left( \frac{\partial^2 u}{\partial \zeta^2} + \varepsilon \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) \right) \]  \hspace{1cm} (114)

\[ \varepsilon^5 \frac{\partial^2 w}{\partial r^2} + \varepsilon^5 \frac{\partial w}{\partial \zeta} = - \frac{\partial w}{\partial \zeta} + \alpha \left( \varepsilon \frac{\partial^2 w}{\partial \zeta^2} + \varepsilon \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) \right) \]  \hspace{1cm} (115)

\[ \frac{1}{r} \frac{\partial}{\partial r} (rv) + \frac{\partial w}{\partial \zeta} = 0 \]  \hspace{1cm} (116)

with the boundary conditions:

\[ u(\zeta=0) = 0 \quad ; \quad v(\zeta=0) = 0 \quad ; \quad w(\zeta=0) = 0 \]  \hspace{1cm} (117)

\[ \frac{\partial u}{\partial \zeta}(\zeta=1) = 0 \quad ; \quad \frac{\partial v}{\partial \zeta}(\zeta=1) = 0 \quad ; \quad w(\zeta=1) = 0 \]  \hspace{1cm} (118)

The boundary conditions at the side-walls have disappeared from this system. Mathematically, this is a consequence of neglecting the partial derivatives.
\[ \frac{\partial}{\partial R} \text{ with respect to } \frac{\partial}{\partial z'}; \text{ physically, it means, that the friction at the side-
walls will not influence the flow pattern in the central region.} \]

N.B. The side walls do influence the flow pattern in the central region in
that they cause the curvature of the flow. In the present problem, however,
this curvature has been introduced by the assumption of axial symmetry.

It can be concluded from eqs. 113-115, that \( \frac{\partial \pi}{\partial \phi} \) must be a constant, which will
be named \( -A \), in accordance with chapter II. The average value of \( \pi \) (indicated
by \( \overline{\pi} \)) in the cross-section \( \phi' = 0 \) is called \( \overline{\pi}^x \) again, which yields the well-
known expression for \( \overline{\pi} \) in a cross-section \( \phi' \):

\[
\overline{\pi} = \overline{\pi}^x - A \phi'
\]  \hspace{1cm} (33)

2. Solution in the central region.

The system of differential equations 113-116 with the boundary conditions
117-118 will be solved, using a perturbation method like described in chapter
I. However, dividing the cross-section into three regions as described above,
is only possible in shallow channels (if \( \epsilon_1 = \frac{d}{B} \) is small); but then \( \epsilon = \frac{d}{R_0} \)
will be small, whatever the ratio \( \epsilon_2 = \frac{B}{R_0} \) (always \( \leq 2 \)) may be. So both \( \epsilon_1 \) and
\( \epsilon = \epsilon_1 \epsilon_2 \) can be used as the perturbation parameter. \( \epsilon_1 \) has been preferred here,
since it is the only parameter in the system, that has been assumed to be small;
\( \epsilon_2 \) can be left free then for the moment and can be maintained in the solution.

Just like it has been done in chapter II, the piezometric head, the tangential
velocity-component, the secondary flow and the first order correction of the
tangential velocity are treated successively.

a. Piezometric head and tangential velocity-component

These two quantities are solved from the zero order subsystem of eqs. 113-116:

\[
o = - \frac{\partial \pi_0}{\partial R}
\]  \hspace{1cm} (119)

\[
o = - \frac{\partial \pi_0}{\partial \phi'} + \frac{\partial^2 u_0}{\partial \zeta^2}
\]  \hspace{1cm} (120)
\[ 0 = - \frac{\partial \pi_0}{\partial \zeta} \]  
(121)

\[ \frac{1}{r} \frac{\partial}{\partial r} (r v_0) + \frac{\partial w_0}{\partial \zeta} = 0 \]  
(122)

Again \( \pi_0 \) is defined such, that:

\[ \frac{\partial \pi_0}{\partial \phi} = - \Delta \quad \text{and} \quad \pi_0 (\phi' = 0) = \Pi^* \]  
(38)

and consequently eqs. 119-121 yield:

\[ \pi_0 = \Pi^* - \Delta \phi' \]  
(39)

So the hydrostatic pressure is found here, too.

Defining:

\[ u_0^* = \frac{\alpha}{\Delta} u_0 \]  
(123)

and substituting expr. 39 into eq. 120, this equation turns into:

\[ \frac{\partial^2 u_0^*}{\partial \zeta^2} = - \frac{1}{r} \]  
(124)

The relevant boundary conditions are:

\[ u_0^*(\zeta=0) = 0 ; \quad \frac{\partial u_0^*}{\partial \zeta} (\zeta=1) = 0 \]  
(125)
which yield to solution:

\[ u^* = \frac{1}{r} (-\frac{1}{4} \xi^2 + \xi) = \frac{1}{r} u_0(\xi) \]  

(126)

This solution does not contain the parameter \( \varepsilon_2 \), which means, that it is not influenced by the channel width. This agrees with the assumption of the flow pattern in the central region not to be influenced by the friction at the side-walls.

Expr. 126 can be written in the co-ordinates of chapter II (\( r = 1 + \varepsilon \xi; \xi = \xi \)):

\[ u^* = \frac{1}{1 + \varepsilon \xi} (-\frac{1}{4} \xi^2 + \xi) = \frac{1}{1 + \varepsilon \xi} \sum_{k=1,3,5,...}^{\infty} \frac{16}{3^3} \sin \frac{k\pi \xi}{2} \]  

(127)

In the central axis of the cross-section (\( \xi=0 \)) this solution practically corresponds with expr. 42, while in a certain region around the central axis it agrees very well with the solution of eq. 71 (given in fig. 3 for \( \varepsilon_1 = .1 \) and \( \varepsilon_2 = .4 \)), like fig. 9 indicates.

N.B. The extension of the region of good agreement does depend on \( \varepsilon_2 \), provided that \( \varepsilon_1 \) is small enough (in other words: provided that a central region exists).

-fig. 9.-

Tangential velocity-component in the central region of the cross-section.

--- solution present chapter

---------- solution chapter II (\( \varepsilon_1 = .1; \varepsilon_2 = .4 \))
b. Secondary velocity-components.

The zero order subfunctions of the secondary velocity-components in the central region, \( v_0 \) and \( w_0 \), can be solved from the zero order equation of continuity:

\[
\frac{1}{r} \frac{\partial}{\partial r} (r v_0) + \frac{\partial w_0}{\partial \zeta} = 0
\]  

(122)

and the radial and vertical equations of motion of the first order subsystem:

\[
-\varepsilon_2 \frac{v_0^2}{r} = - \frac{\partial \pi_1}{\partial r} + \alpha \varepsilon_2 \frac{\partial^2 v_0}{\partial \zeta^2}
\]

(128)

\[
0 = - \frac{\partial \pi_1}{\partial \zeta}
\]

(129)

Eliminating \( \pi_1 \) from eqs. 128 and 129 and defining:

\[
v_0^* = \frac{\alpha}{\Delta^2} v_0
\]

(130)

\[
w_0^* = \frac{\alpha}{\Delta^2} w_0
\]

(131)

this system becomes:

\[
\frac{1}{r} \frac{\partial}{\partial r} (r v_0^*) + \frac{\partial w_0^*}{\partial \zeta} = 0
\]

(132)

\[
- \frac{1}{r} \frac{\partial}{\partial \zeta} \left( (u_0^*)^2 \right) = \frac{\partial^2 v_0^*}{\partial \zeta^2}
\]

(133)
The relevant boundary conditions are:

\[ v_0^*(\zeta = 0) = 0 ; \quad \frac{\partial v_0^*}{\partial \zeta} (\zeta = 1) = 0 \]  \hspace{1cm} (134)

\[ w_0^*(\zeta = 0) = 0 ; \quad w_0^*(\zeta = 1) = 0 \]  \hspace{1cm} (135)

It is possible to solve \( v_0^* \) directly from eq. 133, making use of an integrated form of the equation of continuity:

\[ \int_0^1 v_0^* \, d\zeta = 0 \]  \hspace{1cm} (136)

(or: in axisymmetric flow the net radial discharge vanishes). This yields:

\[ v_0^* = -\frac{1}{r^3} \frac{1}{840} \left( 7\zeta^6 - 42\zeta^5 + 70\zeta^4 - 72\zeta^2 + 32\zeta \right) = \frac{1}{r^3} \frac{\nu}{v_0^*} (\zeta) \]  \hspace{1cm} (137)

Subsequently, \( w_0^* \) can be solved from eq. 132:

\[ w_0^* = -\frac{1}{r^4} \frac{1}{840} \left( 2\zeta^7 - 14\zeta^6 + 28\zeta^5 - 48\zeta^3 + 32\zeta^2 \right) = \frac{1}{r^4} \frac{\nu}{w_0^*} (\zeta) \]  \hspace{1cm} (138)

Exprs. 137 and 138 can also be found, using a stream function, like it has been done in chapter II. Defining this stream function \( \phi_0^* \) by:

\[ v_0^* = -\frac{\partial \phi_0^*}{\partial \zeta} \quad \text{and} \quad w_0^* = \frac{1}{r} \frac{\partial}{\partial r} (r \phi_0^*) \]  \hspace{1cm} (139)

and substituting it into eq. 133 yields:

\[ \frac{1}{r} \frac{\partial}{\partial \zeta} \left( w_0^* \right)^2 = \frac{\partial^4 \phi_0^*}{\partial \zeta^4} \]  \hspace{1cm} (140)
with the boundary conditions:

\[ g_0^*(\zeta=0) = 0 ; \frac{\partial g_0^*}{\partial \zeta}(\zeta=0) = 0 ; g_0^*(\zeta=1) = 0 ; \frac{\partial^2 g_0^*}{\partial \zeta^2}(\zeta=1) = 0 \]  

(141)

Then the stream function must be:

\[ g_0^* = \frac{1}{r^3} \frac{1}{840}(\zeta^7 - 7\zeta^6 + 14\zeta^5 - 24\zeta^3 + 16\zeta^2) = \frac{1}{r^3} g_0^*(\zeta) \]  

(142)

from which expr. 137 and 138 can be derived.

Exprs. 142, 137 and 138 can not be compared directly with the results of chapter II. In that chapter the definition of the stream function \( g_0^* \), equivalent with def. 139, would have been:

\[ \nu_0^* = -\frac{\partial g_0^*}{\partial \zeta} \quad \text{and} \quad \nu_0^* = \frac{\partial g_0^*}{\partial \zeta} + \frac{e}{1+\varepsilon \zeta} g_0^* \]  

(143)

Substitution of this definition into eq. 67 would have yielded:

\[ \nu^2 \nu^2 g_0^* + \nu^2 \left( \frac{\partial}{\partial \zeta} \left( \frac{e}{1+\varepsilon \zeta} g_0^* \right) \right) = \frac{1}{1+\varepsilon \zeta} \frac{\partial}{\partial \zeta} \left( \nu_0^* \right)^2 \]  

(144)

in which

\[ \nu^2 = \frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial \zeta^2} \]

However, since the second term in the left hand part of this equation may be expected to be small with respect to the first one, \( g_0^* \) might have been approximated by the solution of:

\[ \nu^2 \nu^2 g_0^* = \frac{1}{1+\varepsilon \zeta} \frac{\partial}{\partial \zeta} \left( \nu_0^* \right)^2 \]  

(79)
So, despite of the abovementioned complication, the stream function given in chapter II, may be considered to be equivalent with the stream function, given by expr. 142. Fig. 10 shows, that this approximation is a rather good one.

![Stream function graph](image)

Fig. 10.

Stream function of the secondary velocity-components in the central region of the cross-section.

------------- solution present chapter.

------------- solution chapter II ($\varepsilon_1 = .1; \varepsilon_2 = .4$).

Since $v_0^*$ is defined in the same way by def. 75 and def. 143, in the central region of the cross-section the curves of $v_0^*$, given in chapter II, must agree with expr. 137. This agreement is good, indeed, like fig. 11 shows.

For two reasons expr. 138 will not agree with the result of chapter II:

- in chapter II the dimensionless vertical velocity-component $w$ is defined by:

$$v_z = \overline{V} \varepsilon w$$  \hspace{1cm} (145)

and in the present chapter by:

$$v_z = \overline{V} \varepsilon^2 w$$  \hspace{1cm} (146)
Radial velocity-component in the central region of the cross-section.

Solution present chapter.
Solution chapter II ($\epsilon_1 = .1; \epsilon_2 = .4$)

b. in chapter II, def. 75 was used for the derivation of $w_0^*$ from the stream function $S_0^*$, while in the present chapter the derivation is based on def. 143.

That is why expr. 138 has to be compared with the results of chapter II by comparing $\star \star \star$ expr. 138 with $w_0^*$, derived from the stream function of chapter II by means of def. 143. Fig. 12 shows, that in a relatively narrow region around the central axis the agreement is good.

Vertical velocity-component in the central region of the cross-section.

Solution present chapter ($\star \star$)
Solution chapter II ($\epsilon_1 = .1; \epsilon_2 = .4$)
c. **First and second order correction of the tangential velocity-component.**

The tangential equation of motion of the first order subsystem is:

$$ 0 = -\frac{\partial \pi_1}{\partial \phi} + \frac{\partial^2 u_1}{\partial \zeta^2} $$

(147)

According to the definition of $\pi_0$, $\frac{\partial \pi_1}{\partial \phi}$, vanishes; so:

$$ \frac{\partial^2 u_1}{\partial \zeta^2} = 0 $$

(148)

with the boundary conditions:

$$ u_1(\zeta=0) = 0 ; \frac{\partial u_1}{\partial \zeta}(\zeta=1) = 0 $$

(149)

This implies, that $u_1$ vanishes, too.

The tangential equation of motion of the second order subsystem is:

$$ \varepsilon^2_2 \left( \nu_0 \frac{\partial u_0}{\partial r} + \omega_0 \frac{\partial u_0}{\partial \zeta} + \frac{u_0 \nu_0}{r} \right) = -\frac{\partial \pi_2}{\partial \phi} + $$

$$ \alpha \left( \frac{\partial^2 u_2}{\partial \zeta^2} + \varepsilon^2_2 \left( \frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r} \frac{\partial u_0}{\partial r} - \frac{u_0}{r^2} \right) \right) $$

(150)

Herein $\frac{\partial \pi_2}{\partial \phi}$ vanishes, as a consequence of the definition of $\pi_0$, $u_0$, $\nu_0$ and $\omega_0$ being known functions, $u_2$ can be solved then from this equation and the boundary conditions:

$$ u_2(\zeta=0) = 0 ; \frac{\partial u_2}{\partial \zeta}(\zeta=1) = 0 $$

(151)
According to expr. 126, $u_0$ is proportional to $\frac{1}{r}$. This implies that the group of terms in eq. 150:

$$\frac{\partial^2 u_0}{\partial r^2} + \frac{1}{r} \frac{\partial u_0}{\partial r} - \frac{u_0}{r^2}$$

vanishes. Substituting the definitions of $u_0^*, v_0^*$ and $w_0^*$ into eq. 150 and defining $u_2^*$ by:

$$u_2^* = \frac{u_2}{\epsilon_2^* \Delta}$$

yields:

$$\frac{\partial^2 u_2^*}{\partial \zeta^2} = v_0^* \frac{\partial u_0^*}{\partial r} + u_0^* \frac{\partial u_0^*}{\partial \zeta} + \frac{u_0^* v_0^*}{r}$$

The relevant boundary conditions are equivalent with conds. 151. The solution of this problem is:

$$u_2^* = \frac{1}{r^5} \left( \frac{1}{840} \frac{1}{45} \zeta^{10} - 10 \zeta^9 + \frac{270}{8} \zeta^8 - 30 \zeta^7 - 72 \zeta^6 + 180 \zeta^5 - 120 \zeta^4 + 32 \zeta \right)$$

This expression can not be compared directly with $u_{1,1}^*$ of chapter II, since:

a. expr. 155 represents a second order subfunction, while in chapter II $u_{1,1}^*$ represents a first order subfunction. Besides, the perturbation parameter used in chapter II was $\epsilon$, while here it is $\epsilon_1$.

b. in expr. 155 the term $(u_0 v_0 / r)$ has been accounted for, while in chapter II it has been neglected.
This implies, that:

\[ \varepsilon_1^2 u_{\varepsilon_2} = \varepsilon_1^2 \Delta_2^3 \] corresponds with \( \varepsilon u'_{1,1} = \varepsilon \Delta_2^3 u'''_{1,1} \) \( (156) \)

or:

\[ \varepsilon u_{\varepsilon_2}' \] corresponds with \( u'^{**}_{1,1} \) \( (157) \)

in which \( u'^{**}_{1,1} \) follows from:

\[ \psi^2 u'^{**}_{1,1} = \varphi_0^*(II) \frac{\partial u_0^*(II)}{\partial \zeta} + \omega_0^*(II) \frac{\partial u_0^*(II)}{\partial \zeta} + \frac{\varepsilon}{1+\varepsilon} u_0^*(II) v_0^*(II) \] \( (158) \)

and the boundary conditions:

\[ u'^{**}_{1,1} (\zeta=0) = 0 ; \frac{\partial u_0^*}_{1,1} (\zeta=1) = 0 ; u'^{**}_{1,1} (\xi= \pm 5) = 0 \] \( (159) \)

In eq. 158 \( u_0^*(II), v_0^*(II) \) and \( \omega_0^*(II) \) represent the functions \( u_0^*, v_0^* \) and \( \omega_0^* \) of chapter II, respectively.

Fig. 13 shows the agreement between expr. 155 and \( u'^{**}_{1,1} \) is rather poor, except in the immediate vicinity of the central axis.

Influence of the secondary flow on the tangential velocity-component in the central region of the cross-section.

\[ \text{solution present chapter (**)} \]
\[ \text{solution chapter II (i.e. } u'^{**}_{1,1} \text{ for } \varepsilon_1 = 1; \varepsilon_2 = 4) \]
d. Synopsis.

The velocity-components in the central region can be written as:

$$v_\phi = \bar{V} \left\{ \frac{\Delta}{a} \frac{1}{r} \tilde{\nu}_0(\zeta) + \frac{\varepsilon_2}{\varepsilon_2} \frac{2\Delta^3}{5} \frac{1}{r^3} \tilde{u}_2(\zeta) + 0(\varepsilon_1^3) \right\}$$  \hspace{1cm} (160)

$$v_R = \bar{V} \left\{ \varepsilon_1 \tilde{e}_2 \frac{\Delta^2}{a} \frac{1}{r^2} \tilde{v}_0(\zeta) + 0(\varepsilon_1^2) \right\}$$  \hspace{1cm} (161)

$$v_z = \bar{V} \left\{ \varepsilon_1 \tilde{e}_2 \frac{\Delta^2}{a} \frac{1}{r^4} \tilde{w}_0(\zeta) + 0(\varepsilon_1^3) \right\}$$  \hspace{1cm} (162)

The functions $u_0(\zeta), u_2(\zeta), v_0(\zeta)$ and $w_0(\zeta)$ represent expr. 126, 155, 137 and 138, respectively. They are tabulated in table I; for the sake of completeness their curves are given in fig. 14.

![Graph](image)

**fig. 14.**

$\zeta$-functions in the central region.


In 1868, already, Boussinesq /1/ solved the problem of the axisymmetrically curved, viscous flow in a wide, shallow channel in the region far from the side-walls. The main outlines of his theory agree with the consideration of the central region, given in this part of the chapter.

**Boussinesq** simplifies the equations of motion by estimating the order of magnitude of the velocity-components (from the number of times they may be
expected to vanish, looking along a vertical line) and eliminating the smaller
terms. This way, he finds solutions, which agree entirely with the zero order
subfunctions, resulting from the theory described herein.


- At the central axis, the velocity-components resulting from the separate
  consideration of the central region, agree well with the relevant results
  of the analysis of the entire cross-section as described in chapter II.
  Besides, in a region around the central axis the agreement uses to be rather
good, too. The extension of this region depends on the subfunction considered
(the region is larger for \( u_0^* \) and \( v_0^* \) than for \( w_0^* \) and \( u_{1,1}^* \)) as well as on the
  channel geometry (viz. \( \varepsilon_2 \)).
  N.B. Nevertheless \( w_0^* \) and \( u_2^* \) are good approximations of the solution in the
  central region, since the accuracy of \( w_0^*(II) \) and \( u_{1,1}^* \) lies within the o.o.m.
  of \( w_0^* \) and \( u_2^* \) (viz. \( O(\varepsilon) \)).
- The zero-order approximation of the velocity-components, resulting from the
  separate consideration of the central region, are identical to the solution
given by Boussinesq.
- The contribution of a subfunction to the relevant velocity-component can be
  written as:

\[
\text{constant} \times \frac{1}{r^k} \times \hat{f}_m(\zeta)
\]  

(163)

in which \( k \) and \( m \) represent integers; the constant depends on \( \overline{V}, \varepsilon_1 \varepsilon_2, \Delta \) and \( \alpha \)
The functions \( \hat{f}(\zeta) \) have been tabulated in table I.
table I

ζ-functions in the central region.

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<th>ζ</th>
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<th>( \tilde{w}_0(\zeta) \times 10^3 )</th>
<th>( \tilde{v}_0(\zeta) \times 10^3 )</th>
<th>( \tilde{w}_0(\zeta) \times 10^3 )</th>
<th>( \tilde{u}_2(\zeta) \times 10^3 )</th>
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C. SOLUTION IN THE INNER WALL REGION.

1. Normalization of the differential equations.

In the region near the inner wall, the friction at this wall and the vanishing of \( v_R \) there will influence the flow pattern, indeed. This implies, that the partial derivatives \( \frac{\partial f}{\partial R} \) must be of the same o.o.m. as the partial derivatives \( \frac{\partial f}{\partial z} \) there. That is why the normalization in this region will be based on the following assumptions:

a. \( \frac{\partial f}{\partial R} = O\left(\frac{f}{R}\right) \) and \( \frac{\partial f}{\partial z} = O\left(\frac{f}{d}\right) \), if \( f \) represents each of the dependent variables.

N.B. The o.o.m. of the derivatives does not depend on the radius of curvature of the bend, here, since the assumption holds true in a straight channel as well as in a bend.

b. \( v_R = \frac{d}{R_0} O\left(\frac{d}{R_0}\right) \). The equation of continuity (eq. 4) yields then:

\[
\frac{\partial v}{\partial z} = O\left(\frac{v}{d^2}\right) = - v_R - \frac{\partial v}{\partial R} = - \frac{1}{R_0} O(v_R) - \frac{1}{d} O(v_R) + \frac{1}{d} O(v_R)
\]  

so:

\[
v_z = O(v_R) = \frac{d}{R_0} O\left(\frac{d}{R_0}\right)
\]  

(164)
\[ \frac{1}{R} \frac{\partial p}{\partial \phi} = O(\nu^2 d^{-1} \Re^{-1}). \text{ See chapter II.} \]

\[ d \Re = O(1). \text{ See chapter II.} \]

So far, there is no difference from the normalization, described in chapter II. According to the abovementioned assumptions, the variables and partial derivatives are replaced by products of scale-factors and dimensionless quantities as follows:

- **Independent variables:** \( R_i = \frac{R}{R_0}, \quad z = Z, \quad \phi = \frac{d}{R_0} \theta \)

- **Dependent variables:** \( v_\phi = \frac{\nu}{R_0} U, \quad v_R = \frac{\nu}{R_0} d \frac{d}{Z} V, \quad v_z = \frac{\nu}{R_0} d \frac{d}{Z} W; \)

\[ p + \rho g' z = \rho \nu^2 \pi \]

- **Partial derivatives:**

\[ \frac{\partial}{\partial z} = \frac{1}{\gamma} \frac{\partial}{\partial Z} ; \quad \frac{\partial}{\partial R} = \frac{1}{\gamma} \frac{\partial}{\partial X} ; \quad \frac{\partial}{\partial \phi} = \frac{1}{\gamma} \frac{1}{R_i + \epsilon X} \frac{\partial}{\partial \theta} \]  

(168)

in which \( R_i = \frac{R_0 - B}{2} = \) the radius of curvature of the inner wall and \( r_i = \frac{R}{R_0} = 1 - \frac{1}{2} \epsilon \).

The factor \( \frac{1}{R} \), appearing in the equations of motion and continuity 1-4, can now be written as:

\[ \frac{1}{R} = \frac{1}{R_0} \frac{1}{r_i + \epsilon X} = \frac{1}{R_0} \frac{1}{r_i} \frac{1}{1 + \epsilon X} \]

(169)

In the present case \( r_i = 1 - \frac{1}{2} \epsilon \) \( = O(1) \), since \( \epsilon \) has been assumed to be \( O(1) \).

This implies that \( \frac{1}{r_i} = O(1) \) or larger. The inner wall region is assumed to extend over a distance \( \beta d \), in which \( \beta = O(1) \).

Then:

\[ 0 \leq R - R_i \leq \beta d \]

(170)

\[ 0 \leq X \leq \beta \text{ and } X = O(1) \]

(171)
Two different cases have to be discerned, now:

1. if \( \frac{1}{r_i} = O(1) \), then \( \frac{\varepsilon X}{r_i} = O(\varepsilon) \) and the factor \( \frac{1}{R} \) can be written as:

\[
\frac{1}{R} = \frac{1}{R_0} \left( \frac{1}{r_i} + \varepsilon \frac{X}{r_i} \right)^k \left( 1 - \frac{\varepsilon X}{r_i} + \left( \frac{\varepsilon X}{r_i} \right)^2 - \left( \frac{\varepsilon X}{r_i} \right)^3 + \ldots \right) + O\left( \varepsilon^{k+1} \right)
\]

(172)

2. if \( \frac{1}{r_i} = O\left( \frac{1}{\varepsilon} \right) \) or larger, then \( \frac{\varepsilon X}{r_i} = O(1) \) or larger and the factor \( \frac{1}{R} \) has to be maintained as:

\[
\frac{1}{R} = \frac{1}{R_0} \frac{1}{r_i + \varepsilon X}
\]

(169)

The latter case represents a bend with a very small radius of curvature, which will not further be considered herein. In other words: the solution in the inner wall region given in this part of the chapter, is limited to bends with \( \frac{B}{R_i} = O(1) \).

Using defns. 166-168, the normalized differential equations become:

\[
\varepsilon^3 \frac{\partial^3 V}{\partial X^3} + \varepsilon^3 \frac{\partial^3 V}{\partial Z^3} - \varepsilon^2 \frac{U^2}{r_i + \varepsilon X} = - \frac{\partial \Pi}{\partial X} + \varepsilon^3 \frac{\partial^2 V}{\partial X^2} + \varepsilon^2 \frac{\partial^2 V}{\partial Z^2} + \varepsilon \frac{\partial V}{r_i + \varepsilon X} \frac{\partial V}{\partial X} - \frac{\varepsilon^2 V}{(r_i + \varepsilon X)^2}
\]

(173)

\[
\varepsilon \frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Z^2} + \varepsilon^2 \frac{U}{r_i + \varepsilon X} = \frac{-1}{r_i + \varepsilon X} \frac{\partial \Pi}{\partial Z}
\]

(174)
\[ \varepsilon \frac{3W}{3X} + \varepsilon^2 \frac{3W}{3Z} = -\frac{3\Pi}{3Z} + \alpha \varepsilon \frac{2W}{3Z} + \frac{\varepsilon}{r_i + \varepsilon X} \frac{3W}{3X} \]  
\[ \frac{\varepsilon V}{r_i + \varepsilon X} + \frac{3V}{3X} + \frac{3W}{3Z} = 0 \]  
\[ \text{with the boundary conditions:} \]
\[ U(Z=0) = 0 ; V(Z=0) = 0 ; W(Z=0) = 0 \]  
\[ \frac{3U}{3Z}(Z=1) = 0 ; \frac{3V}{3Z}(Z=1) = 0 ; W(Z=1) = 0 \]  
\[ U(X=0) = 0 ; V(X=0) = 0 ; W(X=0) = 0 \]

N.B. In eqs. 173-176 the factor \( \frac{1}{r_i + \varepsilon X} \) represents the series expansion, given by 172.

Just like in the foregoing part of this chapter, the normalized differential equations show, that \( \frac{3\Pi}{3\phi} \) must be a constant, which will be named -\( \Delta \) again. So, if \( \Pi^* \) represents the average value of \( \Pi \) in the cross-section \( \phi = 0 \):

\[ \bar{\Pi} = \Pi^* - \Delta \phi \]  

2. Matching conditions.

The boundary conditions 177-179 are not sufficient to determine the solution of eqs. 173-176 or any of its subsystems completely. Therefore a matching of the solution in the inner wall region and the solution in the central region of the cross-section is needed. This is realized by matching the relevant subfunctions in each of the regions, using the so-called "asymptotic matching principle" (see Van Dyke /4/, chapter V; Lagerstrom and Casten /6/).

Following Van Dyke's terminology, the central region is called the "outer
region" and the inner wall region is called the "inner region". Accordingly, 
the co-ordinates of the central region are "outer co-ordinates" and the co- 
ordinates of the inner wall region are "inner co-ordinates"; the series expan-
sions, valid in the central region are "outer expansions" and the series expan-
sions, valid in the inner wall region are "inner expansions".
The asymptotic matching principle says:

\[ \text{the } m \text{-term inner expansion of (the } n \text{-term outer expansion)} \]
\[ = \]
\[ \text{the } n \text{-term outer expansion of (the } m \text{-term inner expansion)} \]

(181)

where \( m \) and \( n \) are integers).

The \( m \)-term inner expansion of the \( n \)-term outer expansion is found by:
- taking the first \( n \) terms of the outer expansion
- rewriting it in inner co-ordinates (viz. \( X, \phi \) and \( Z \))
- expanding the result in a power series of \( \epsilon_1 \)
- taking the first \( m \) terms of this expansion.

The \( n \)-term outer expansion of the \( m \)-term inner expansion is found conversely.

The asymptotic matching principle can not be used in the form of 181, but it 
appears to be possible to translate it into conditions, which can be used to 
solve the present problem. This translation is given in Appendix I.

3. Solution in the inner wall region.

The system of differential equations 173-176 with the boundary conditions 
177-179 (plus the matching conditions derived in appendix I) can be solved 
now, using a perturbation method as described in chapter I. Just like in the 
central region, \( \epsilon_1 \) is chosen as the perturbation parameter.

* This matching principle can only be used if there is an "overlap domain" be-
tween the inner and the outer region, where both the inner and the outer expan-
expansion are valid. The existence of such a domain will not be proved here.
a. Piezometric head and tangential velocity-component.

These two quantities are solved from the zero order subsystem of eqs. 173-176:

\[ 0 = - \frac{3 \Pi}{3 x} \]  
\[ (182) \]

\[ 0 = - \frac{1}{r_i} \frac{3 \Pi}{3 \theta} + \alpha \left( \frac{3^2 u_0}{3 x^2} + \frac{3^2 u_0}{3 z^2} \right) \]  
\[ (183) \]

\[ 0 = - \frac{3 \Pi}{3 z} \]  
\[ (184) \]

\[ 0 = \frac{3 v_0}{3 z} + \frac{3 w_0}{3 z} \]  
\[ (185) \]

\( \Pi_0 \) is defined such, that:

\[ \frac{3 \Pi}{3 \theta} = - \Delta \quad \text{and} \quad \Pi_0 (\theta = 0) = \Pi^* \]  
\[ (186) \]

and consequently eqs. 182-184 yield:

\[ \Pi_0 = \Pi^* - \Delta \theta \]  
\[ (187) \]

This agrees with the solution in the central region and the solution in the entire cross-section.

Defining:

\[ u_0^* = \frac{\alpha}{\Delta} u_0 \]  
\[ (188) \]
and substituting eqn. 187 into eq. 183, this equation turns into:

\[
\frac{\partial^2 U^*_0}{\partial x^2} + \frac{\partial^2 U^*_0}{\partial z^2} = -\frac{1}{r_i}
\]  

(189)

The boundary conditions are:

\[
U^*_0(z=0) = 0 \quad ; \quad \frac{\partial U^*_0}{\partial z}(z=1) = 0 \quad ; \quad U^*_0(x=0) = 0
\]  

(190)

and the relevant matching condition is:

\[
\lim_{x \to \infty} U^*_0 = \lim_{r \to r_i} U^*_0 = -\frac{1}{r_i} (4z^2 - z)
\]  

(191)

The solution of this problem is found by separation of variables:

\[
U^*_0 = \frac{1}{r_i} \sum_{k=1,3,5,..}^{\infty} \frac{16}{3\pi^3} (1 - e^{-\frac{k\pi x}{2\pi}}) \sin \frac{k\pi x}{2} = \frac{1}{r} \nu \cdot U_0(X,Z)
\]  

(192)

All matching conditions of \( U^*_0 \) are satisfied by this solution. According to appendix I, these conditions can be written in the general form:

\[
\lim_{r \to r_i} (r-r_i)^{\delta} U^*_0 = \lim_{X \to \infty} \frac{1}{r} (-e^{2x^2} \frac{\partial}{\partial x})^{\delta} U^*_0
\]  

(193)

\[
= \lim_{X \to \infty} (-e^{2x^2})^{\delta} \left( \frac{X}{r} \frac{\partial U^*_0}{\partial X} \right) + \left( \frac{X^{\delta-1} - 1}{(\delta-1)!} \right) \frac{\partial^{\delta-1} U^*_0}{\partial X^{\delta-1}} + \ldots
\]

\[
+ \left( \frac{\delta}{\delta-2} \right) \frac{X^2}{2} \frac{\partial^2 U^*_0}{\partial X^2} + \frac{\partial U^*_0}{\partial X}
\]  

(194)

For \( \delta = 0 \), cond. 191 is found, which has already been taken account of in the solution of \( U^*_0 \).
For \( \varepsilon \neq 0 \), the right hand part of 194 contains only terms of the form:

\[
\lim_{X \to \infty} \text{constant} * X^{2k-1} \frac{3^{k-i} u_0^*}{3^{k-i}}
\]

(195)

in which \( i \) represents an integer smaller than 1. According to 192, all these terms must vanish.

From the normalized differential equations (113-116) and the zero order solution in the central region it can be foreseen, that \( u_0^* \) must have the form:

\[
\sum_r \sum_j \left( \frac{1}{n_j} \right) u_{k,j}(\zeta)
\]

(196)

in which \( n_j \) and \( j \) represent integers. This implies, that for \( \varepsilon \neq 0 \) the left hand part of 193 and 194 vanishes, too.

The function \( \tilde{u}_0(X,Z) \) has been outlined in fig. 15.

Comparing expr. 192 for \( r_1 - 1 \) \( \varepsilon_2 = .8 \) with the corresponding result of chapter II (viz. \( u_0^*(II) \)), the agreement appears to be rather good near the wall, but it grows worse when the distance to the wall increases (see fig. 16).

**fig. 15.**

Tangential velocity-component in the inner wall region.

(a) \( \tilde{u}_0(X;Z=1) \)

(b) \( \tilde{u}_0(X;Z) \)

(c) lines of constant \( \tilde{u}_0(X,Z) \)
fig. 16.
Tangential velocity-component in the inner wall region.
---------- solution present chapter \( r_1 = 0.8 \)
---------- solution chapter II \( \varepsilon_1 = 0.1; \varepsilon_2 = 0.4 \)

b. Secondary velocity-components.

The zero order subfunctions of the secondary velocity-components in the inner wall region, \( V_0 \) and \( W_0 \), are solved from the zero order equation of continuity:

\[
\frac{\partial V_0}{\partial X} + \frac{\partial W_0}{\partial Z} = 0
\]  

(185)

and the radial and vertical equations of motion of the second order subsystem:

\[
- \alpha^2 \frac{U_0^2}{r_1^2} = - \frac{3\Pi_2}{\partial X^2} + \alpha \varepsilon_2 \left\{ \frac{\partial^2 V_0}{\partial X^2} + \frac{\partial^2 V_0}{\partial Z^2} \right\}
\]  

(197)

\[
0 = - \frac{3\Pi_2}{\partial Z^2} + \alpha \varepsilon_2 \left\{ \frac{\partial^2 W_0}{\partial X^2} + \frac{\partial^2 W_0}{\partial Z^2} \right\}
\]  

(198)

Eliminating \( \Pi_2 \) from these equations and introducing a stream function \( G_0 \) such, that:

\[
V_0 = \frac{\Delta}{\alpha} \frac{\partial^2}{\partial Z^2} G_0 = - \frac{3G_0}{\alpha} = - \frac{\Delta}{\alpha} \frac{\partial G_0}{\partial Z}
\]  

(199)
\[ W_0 = \Delta^2 \frac{\omega}{\alpha} \omega^* = + \frac{\partial G_0}{\partial x} = + \Delta^2 \frac{\partial G_0}{\partial x} \]  
\[ (200) \]

This system becomes:

\[ \frac{\partial^2}{\partial Z^2} \left( \frac{U_0}{r_i^3} \right) = \frac{\partial^4 G_0^*}{\partial x^4} + 2 \frac{\partial^4 G_0^*}{\partial x^2 \partial Z^2} + \frac{\partial^4 G_0^*}{\partial Z^4} \]  
\[ (201) \]

This equation has to be solved with the boundary conditions:

\[ G_0^*(Z=0) = 0 ; \quad G_0^*(X=0) = 0 ; \quad G_0^*(Z=1) = 0 \]  
\[ (202) \]

\[ \frac{\partial G_0^*}{\partial Z} (Z=0) = 0 ; \quad \frac{\partial G_0^*}{\partial X} (X=0) = 0 ; \quad \frac{\partial^2 G_0^*}{\partial Z^2} (Z=1) = 0 \]  
\[ (203) \]

and the matching conditions:

\[ \lim_{X \to \infty} \frac{\partial G_0^*}{\partial x} = 0 \]  
\[ (204) \]

and:

\[ \lim_{X \to \infty} \frac{\partial G_0^*}{\partial Z} = \lim_{r \to r_i} (-\nu_0^*) = \lim_{r \to r_i} \frac{\partial G_0^*}{\partial Z} \]  
\[ (205) \]

or:

\[ \lim_{X \to \infty} G_0^* = \frac{1}{3} \frac{1}{840} \left( 7Z^7 - 7Z^6 + 14Z^5 - 24Z^3 + 16Z^2 \right) \]  
\[ (206) \]

This problem has been solved numerically, using the aforementioned computer-program for the computation of the deflections of flat plates (see chapter II). From eq. 201 and the conditions 202, 203, 204 and 205, it can be concluded, that:

\[ G_0^* = \frac{1}{3} \frac{2}{r_i^3} G_0(X,Z) \]  
\[ (207) \]
This function $\hat{C}_0(X,Z)$ has been outlined in fig. 17.

The exponentially damping character of $\hat{C}_0(X,Z)$ for $X \to \infty$ cannot be proved, since no analytical solution is available. However, since the left hand part of eq. 201 has this character, $\hat{C}_0^x$ may be expected to have it, too. None of the boundary or matching conditions prohibits it and, moreover, neither fig. 17 nor fig. 20 (which represents $\frac{\partial \hat{C}_0}{\partial X}$) contradict it. That is why $\hat{C}_0^x$ is assumed to have the aforementioned character, which implies, that the considerations as to the satisfaction of the matching conditions to be imposed on $\hat{U}_0^x$ are applicable to $\hat{G}_0^x$ (and consequently to $\hat{V}_0^x$ and $\hat{W}_0^x$). So: all matching conditions concerning $\hat{G}_0^x$, $\hat{V}_0^x$ and $\hat{W}_0^x$ are assumed to be satisfied then.

![Diagram](image)

**fig. 17**

Stream function of the secondary flow in the inner wall region.

(a) $\hat{G}_0(X,Z=\frac{1}{2})$

(b) $\hat{G}_0(X=\infty,Z)$

(c) lines of constant $\hat{G}_0(X,Z)$; streamlines.

Comparing expr. 207 for $r_1 = .8$ with the relevant result of chapter II (viz. $\hat{g}_0^x$(II)), the agreement appears to be rather poor (see fig. 18).
Stream function of the secondary flow in the inner wall region.

--- solution present chapter ($r_1 = .8$)

--- solution chapter II ($\varepsilon_1 = .1; \varepsilon_2 = .4$)

From def. 199 and 200 and expr. 206, it may be clear, that:

\[
V_0^* = \frac{1}{3} \bar{V}_0(X,Z) \tag{208}
\]

\[
\bar{W}_0^* = \frac{1}{3} \bar{W}_0(X,Z) \tag{209}
\]

Figs. 19 and 20 give the functions $\bar{V}_0(X,Z)$ and $\bar{W}_0(X,Z)$, respectively, while in figs. 21 and 22 $V_0$ and $W_0$ are compared with the relevant results of chapter II: $V_0^*(II)$ and $W_0^*(II)$. Here the agreement is at least as poor as in fig. 18.
fig. 19.
Radial velocity-component in the inner wall region.
(a) $V_0(X; Z=1)$
(b) $V_0(X=\infty; Z)$
(c) lines of constant $V_0(X, Z)$

fig. 20.
Vertical velocity-component in the inner wall region.
(a) $W_0(X, Z=\frac{1}{2})$
(b) $W_0(X=1; Z)$
(c) lines of constant $W_0(X, Z)$
fig. 21.
Radial velocity-component in the inner wall region.

--- solution present chapter \((r_1 = 0.8)\)
--- solution chapter II \((\epsilon_1 = 1; \epsilon_2 = 0.4)\)

fig. 22.
Vertical velocity-component in the inner wall region.

--- solution present chapter \((r_1 = 0.8)\)
--- solution chapter II \((\epsilon_1 = 1; \epsilon_2 = 0.4)\)

c. First order correction of the tangential velocity-component.

The tangential equation of motion of the first order subsystem is:

\[
\epsilon_2 \left\{ V_0 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_0}{\partial z} \right\} = \frac{\epsilon_2 x}{r_1^2} \frac{\partial \Pi}{\partial \phi} - \frac{1}{r_1} \frac{\partial \Pi}{\partial \phi} + \\
+a \left\{ \frac{\partial^2 U_1}{\partial z^2} + \frac{\partial^2 U_1}{\partial x^2} + \frac{\epsilon_2}{r_1} \frac{\partial U_0}{\partial x} \right\}
\]

(210)
Regarding the definition of $\Pi_0^2$, $\frac{\partial \Pi_0^2}{\partial r}$ must vanish. Then, $U_0$, $V_0$, $W_0$ and $\Pi_0$ being known functions, $U_1$ can be solved from eq. 210 with the boundary conditions:

$$U_1(Z=0) = 0; \quad \frac{\partial U_1}{\partial Z}(Z=1) = 0; \quad U_1(X=0) = 0$$ (211)

The matching condition presents more complications than in the zero order subsystem. The general form of the conditions to be imposed on $U_1$ is:

$$\lim_{r \rightarrow r_1} \frac{3}{\partial r} \left\{ (r-r_1)^{\delta} u_{\chi} \right\} = \lim_{X \rightarrow \infty} \frac{1}{\kappa} \left( -\epsilon_2 X^2 \frac{\partial}{\partial X} \right)^{\lambda} \left\{ \frac{U_1}{\epsilon_2 X} \right\}$$ (212)

If $\lambda = 0$:

$$\lim_{X \rightarrow \infty} \frac{U_1}{2 X} = \lim_{r \rightarrow r_1} \frac{3 u_0}{\partial r} = \frac{\Delta}{\alpha} \frac{1}{r_1^2} \left( \frac{1}{2} \frac{Z^2}{Z} \right)$$ (213)

This condition, however, is not sufficient to determine the solution of eq. 210 completely. It says: if $X \rightarrow \infty$, then $U_1$ approaches the function

$$-\frac{\epsilon_2 X}{2 r_1^2} \left( \frac{1}{2} Z^2 - Z \right) + c(Z)$$ (214)

in which $c(Z)$ represents any function of $Z$.

That is why the matching condition for $\lambda = 1$ is added to the system:

$$\lim_{X \rightarrow \infty} \left\{ -X^2 \frac{\partial}{\partial X} \left( \frac{U_1}{X} \right) \right\} = \lim_{r \rightarrow r_1} \frac{3}{\partial r} \left\{ (r-r_1) u_{\chi} \right\} = 0$$ (215)

or:

$$\lim_{X \rightarrow \infty} (U_1 - X \frac{2 U_1}{3 X}) = 0$$ (216)

Then:

$$\lim_{X \rightarrow \infty} \left( \frac{\Delta}{\alpha} \frac{\epsilon_2 X}{2 r_1^2} \left( \frac{1}{2} Z^2 - Z \right) + c(Z) - \frac{\Delta}{\alpha} \frac{\epsilon_2 X}{2 r_1^2} \left( \frac{1}{2} Z^2 - Z \right) \right) = 0$$ (217)

or:

$$c(Z) = 0$$ (218)
Both matching conditions (viz. 213 and 215) can be combined then to:

\[
\lim_{X \to \infty} \left\{ U_1 - \frac{\Delta}{\alpha} r_1^2 \left( \frac{1}{2} Z^2 - Z \right) \right\} = 0
\]  

(219)

Elaborating 212 for \( \lambda > 1 \) yields:

\[
\lim_{X \to \infty} \frac{\alpha}{2} \frac{\partial^{2} U}{\partial X^2} = \lim_{r \to r_1} \frac{\partial}{\partial r} \left\{ (r - r_1)^2 u_2 \right\} = 0
\]  

(220)

\[
\lim_{X \to \infty} \frac{2}{3} \frac{\partial^{2} U}{\partial X^2} \left\{ 3X^4 \frac{\partial^2 U}{\partial X^2} + X^5 \frac{\partial^3 U}{\partial X^3} \right\} = \lim_{r \to r_1} \frac{\partial}{\partial r} \left\{ (r - r_1)^3 u_3 \right\} = 0
\]  

(221)

etcetera. Generally speaking, the lowest order derivative on the left hand side is \( \frac{\partial^2 U}{\partial X^2} \); the right hand side always vanishes.

\( U_1 \) is solved from eq. 210 and conds. 211 and 219. For the sake of convenience, it is split up into three parts:

\[
U_1 = \varepsilon_2 \left\{ \frac{\Delta}{\alpha} U_{1,0}^{*} + \frac{\Delta^3}{\alpha^3} U_{1,1}^{*} + \frac{\Delta}{\alpha} U_{1,2}^{*} \right\}
\]  

(222)

such that:

1. \[
\nabla^2 U_{1,0}^{*} = \frac{X}{r_1^2}
\]  

(223)

with the boundary conditions:

\[
U_{1,0}^{*}(Z=0) = 0 \; ; \; \frac{\partial U_{1,0}^{*}}{\partial Z}(Z=1) = 0 \; ; \; U_{1,0}^{*}(X=0) = 0
\]  

(224)
and the matching condition:

\[
\lim_{X \to \infty} \frac{U^*_{1,0}}{r_i^2} - \frac{X}{r_i^2} (12^2 - Z) = 0
\] (225)

2. \[
\frac{v^2 U^*_{1,1}}{r_i^2} = \frac{1}{r_i^2} \left\{ v_0 \frac{\partial u^*_{1,1}}{\partial x} + u_0 \frac{\partial u^*_{1,1}}{\partial z} \right\}
\] (226)

with the boundary conditions:

\[
U^*_{1,1}(Z=0) = 0 ; \quad \frac{\partial U^*_{1,1}}{\partial z}(Z=1) = 0 ; \quad U^*_{1,1}(X=0) = 0
\] (227)

and the matching condition:

\[
\lim_{X \to \infty} U^*_{1,1} = 0
\] (228)

3. \[
\frac{v^2 U^*_{1,2}}{r_i^2} = \frac{1}{r_i^2} (-\frac{\partial u^*_{1,2}}{\partial x})
\] (229)

with the boundary conditions:

\[
U^*_{1,2}(Z=0) = 0 ; \quad \frac{\partial U^*_{1,2}}{\partial z}(Z=1) = 0 ; \quad U^*_{1,2}(X=0) = 0
\] (230)

and the matching condition:

\[
\lim_{X \to \infty} U^*_{1,2} = 0
\] (231)
Problem 1 is solved by means of separation of variables, which yields:

\[
U_{1,0}^* = -\frac{X}{r_1^2} \sum_{k=1,3,5,\ldots}^\infty \frac{16}{3^3} \sin \frac{k\pi Z}{2} = \frac{X}{r_1^2} (1 - Z) = \frac{U_{1,0}}{r_1^2}
\]  

(232)

Since the second and higher order derivatives with respect to X vanish, this solution satisfies all matching conditions.

The function \( U_{1,0} \) has been outlined in fig. 23. It represents the first order effect of the longitudinal slope.

It is impossible to compare \( U_{1,0}^* \) with a result of chapter IIB, since its existence is due to the expansion of the factor \( \frac{1}{R} \), which has been executed here, but not in chapter IIB.

![Graph](image)

fig. 23.

First order correction of the tangential velocity-component in the inner wall region.

1. Effect of the longitudinal slope.
   (a) \( \sim U_{1,0}(X;Z=1) \)
   (b) \( \frac{1}{X} \sim U_{1,0}(X;Z) \)
   (c) lines of constant \( \sim U_{1,0} \)

Problem 2 is solved, using the overrelaxation-program, which has i.a. been used for the solution of \( u_0^*(II) \). The result can be written as:

\[
U_{1,1}^* = \frac{1}{r_1^4} \frac{\sim}{U_{1,1}}(X,Z)
\]  

(233)
This function \( \tilde{U}_{1,1} \) has been outlined in fig. 24. It represents the first order effect of the secondary circulation.

![Diagram](image)

**fig. 24.**

First order correction of the tangential velocity-component in the inner wall region.

2. Effect of the secondary circulation.
   
   (a) \( \tilde{U}_{1,1}(X;Z=1) \)
   
   (b) \( \tilde{U}_{1,1}(X=1;Z) \)
   
   (c) lines of constant \( \tilde{U}_{1,1}(X,Z) \)

Since \( \tilde{U}_{1,1}^* \) is not known in an analytical form, its exponentially damping character has to be assumed. Arguments for this assumption are the exponentially damping character of the right hand side of eq. 226 (regarding the apparent character of \( \frac{\partial U}{\partial Z} \) and \( \frac{\partial U}{\partial X} \) and the assumed character of \( V_0 \) and \( W_0 \)), the boundary and matching conditions used (which do not prohibit the assumption) and fig. 24.

So: all matching conditions to be imposed on \( U_{1,1}^* \) are assumed to be satisfied by the present solution.

Expression 233 can be compared with the solution of chapter II:

\[ \tilde{U}_{1,1}^* \text{ must correspond with } \tilde{U}_{1,1}^*(II) \]

Fig. 25 shows the agreement is even worse than the agreement between the zero order subfunctions.
First order correction of the tangential velocity-component in the inner wall region.

2. Effect of the secondary circulation.

---------- solution present chapter \( r_i = .8 \)
---------- solution chapter II \( \varepsilon_1 = .1; \varepsilon_2 = .4 \)

Problem 3 is solved by separation of variables, again:

\[
U_{1,2}^* = \frac{X}{r_i^2} \sum_{k=1,3,5,...}^{\infty} \frac{8}{3 \pi} e^{-\frac{k\pi X}{2}} \sin \frac{k\pi Z}{2} = \frac{1}{\varepsilon_i^2} U_{1,2}^*(X,Z) \tag{234}
\]

This solution has an exponentially damping character, so all matching conditions to be imposed on \( U_{1,2}^* \) are satisfied.

The function \( \overset{\sim}{U}_{1,2} \) has been outlined in fig. 26.

Comparing \( U_{1,2}^* \) with the result of chapter II \( U_{1,2}^* \) must correspond with \( u_{1,2}^*(II) \), the agreement appears to be poor, again (see fig. 27).
First order correction of the tangential velocity-component in the inner wall region.
3. Effect of the "extra" friction-term.

(a) $U_{1,2}(X;Z=1)$
(b) $U_{1,2}(X=1;Z)$
(c) lines of constant $\tilde{U}_{1,2}(X,Z)$

fig. 27.
First order correction of the tangential velocity-component in the inner wall region.
3. Effect of the "extra" friction-term.

--- solution present chapter ($r_1 = .8$)
--- solution chapter II ($\varepsilon_1 = .1; \varepsilon_2 = .4$)
d. Synopsis.

The velocity-components, resulting from the separate consideration of the inner wall region, can be written as:

\[ v_\phi = \frac{\nu}{\alpha r_i^2} \frac{\alpha}{\partial x_i} U_0(X,Z) + \varepsilon_1 \varepsilon_2 \frac{\Delta \alpha}{r_i^4} U_{1,1}(X,Z) + \varepsilon_1 \varepsilon_2 \frac{\alpha^3 \alpha^4}{r_i^4} U_{1,1}(X,Z) + \varepsilon_1 \varepsilon_2 \frac{\alpha^3}{r_i^4} U_{1,1}(X,Z) + O(\varepsilon_1^2) \]  

(235)

\[ v_R = \nu \varepsilon_1 \varepsilon_2 \frac{\Delta^2}{r_i^3} v_0(X,Z) + O(\varepsilon_1^2) \]  

(236)

\[ v_z = \nu \varepsilon_1 \varepsilon_2 \frac{\Delta^2}{r_i^3} \tilde{w}_0(X,Z) + O(\varepsilon_1^2) \]  

(237)

The functions \( U_0(X,Z), \tilde{v}_0(X,Z), \tilde{w}_0(X,Z), U_{1,0}(X,Z), U_{1,1}(X,Z) \) and \( U_{1,2}(X,Z) \) have been outlined in figs. 15, 19, 20, 23, 24 and 26, respectively. Besides, they have been tabulated in tables II-VIII.

4. Rozovski's results.

Rozovski /3/ elaborates Ananyan's theory for bends with large radii of curvature, such that \( R = R_0 \) in any point of the cross-section. He assumes the tangential velocity in the entire channel to be equal to the tangential velocity at the central axis (which, according to Ananyan, is identical to the longitudinal velocity in the straight part of the channel before the bend) and solves Ananyan's differential equation:

\[ \nu^2 v_\phi^2 F = \frac{\partial}{\partial z}(v_\phi^2) \]  

(93)

In this equation \( F \) represents the stream function of the secondary flow, such that:

\[ v_R = -\frac{1}{R_0} \frac{\partial F}{\partial z} ; \quad v_z = \frac{1}{R_0} \frac{\partial F}{\partial R} \]  

(238)
The solution of this problem shows the existence of a region around the central axis, where the flow pattern does not depend on $R$ (i.e. on the friction at the side-walls). Roughly speaking, the influence of the friction at the side-walls extends over no more than a few times the depth of flow from those walls. In the "central region" the solution appears to be identical to Boussinesq's results for large radii of curvature.

In his further analysis Rozovs'kii considers one of the side-wall regions of a very shallow channel with a central region, covering the greater part of the cross-section. There he solves eq. 93, again, still assuming the tangential velocity to be equal to the tangential velocity at the central axis. Doing so, he neglects the influence of the friction at the side-walls on the tangential velocity, even in the regions near those walls. One of the boundary conditions of $F$ at the wall, however, is the condition of adherence of the vertical velocity-component; obviously, the friction has not been neglected at that point!

Rozovs'kii's argument for this reasoning is the following assumption: taking $v_\phi$ in the side-wall region equal to $v_\phi$ at the central axis has little influence on the secondary flow pattern near the side-walls.

This assumption has been checked by comparing Rozovs'kii's solution with the relevant results of the present analysis in the inner wall region: $V_0(X,Z)$ and $W_0(X,Z)$. Fig. 28 shows the assumption is rather rough if viscous flow is considered.

![Fig. 28](image_url)  

Radial and vertical velocity-components in the inner wall region compared with Rozovs'kii's solution.
5. Conclusions.

- Near the inner wall the velocity-components resulting from the separate consideration of the inner wall region do not agree very well with the relevant results of the analysis of the entire cross-section as described in chapter II. The difference increases with the distance to the wall and with the order of the subfunction considered.

- The contribution of a subfunction or, considering $U_1$, a part of a subfunction to the relevant velocity-component can be written as:

$$\text{constant} \times \frac{1}{k} x \frac{\eta}{r_i} X_m(X,Z)$$  \hspace{1cm} (239)$$

in which $k$ and $m$ represent integers, while the constant depends on $\eta$, $\epsilon_1 \times \epsilon_2$, $\Lambda$ and $\alpha$. The functions $\tilde{\eta}(X,Z)$ have been tabulated in tables II-VIII.

- As far as viscous flow is considered, Rozovs'kii's assumption: "taking $v_\phi$ in the side-wall regions equal to $v_\phi$ at the central axis has little influence on the secondary flow pattern near the side-walls" is a rather rough one. In tables III-V the results of the analysis, based on this assumption, are given in parentheses.

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Table III.
Function $\tilde{G}_0(x, z)$,
compared with Rozos'kii's results (in parentheses)
(all values have been multiplied by $10^6$

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Tabel IV.

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Tabel V.

Function $\tilde{V}_i(X,Z)$,

compared with Rozovskii's results (in parentheses)

(all values have been multiplied by $10^6$)

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<tr>
<td>(\infty)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

D. SOLUTION IN THE OUTER WALL REGION

In the outer wall region, the problem is treated in exactly the same way as in the inner wall region. The only differences lie in the radius of curvature of the wall \(r_o\) instead of \(r_i\) and the radial co-ordinate:

\[
X = \frac{r_o - r}{\varepsilon} \quad \text{instead of} \quad X = \frac{r - r_i}{\varepsilon}
\]

The solution in the outer wall region is then found to be:

\[
\psi = \bar{v} \left\{ \frac{\Delta}{\alpha} \left( \frac{1}{r_o} U_0(X,Z) - \varepsilon_1 \varepsilon_2 \frac{\Delta}{\alpha \varepsilon_2} \hat{U}_{1,1}(X,Z) - \varepsilon_1 \varepsilon_2 \frac{\Delta^3}{\alpha^3} \frac{1}{\varepsilon_2^4} \hat{U}_{1,1}(X,Z) - \varepsilon_1 \varepsilon_2 \frac{\Delta^3}{\alpha^3} \frac{1}{\varepsilon_2^4} U_{1,2}(X,Z) + O(\varepsilon_1^2) \right\}
\]

(240)
\[ v_R = \overline{V} \left\{ \varepsilon_1 \varepsilon_2 \frac{\Delta^2}{a^3} \frac{1}{r_0^3} \tilde{v}_0(X,Z) + O(\varepsilon_1^2) \right\} \] (241)

\[ v_Z = \overline{V} \left\{ \varepsilon_1 \varepsilon_2 \frac{\Delta^2}{a^3} \frac{-1}{r_0^3} \tilde{w}_0(X,Z) + O(\varepsilon_1^2) \right\} \] (242)

The functions \( \tilde{v}_0(X,Z) \), \( \tilde{w}_0(X,Z) \), \( \tilde{w}_1(X,Z) \), \( \tilde{u}_{1,0}(X,Z) \), \( \tilde{u}_{1,1}(X,Z) \) and \( \tilde{u}_{1,2}(X,Z) \) have been outlined in figs. 15, 19, 20, 23, 24 and 26, respectively. Besides, they have been tabulated in tables II-VIII.

Obviously, the essential functions for the description of the velocity-components in the outer wall region are identical to the relevant functions in the inner wall region.

E. CONSTRUCTION OF A COMPOSITE SOLUTION.

1. Introduction.

Considering the solutions in the three separate regions of the cross-section, it appears, that the three curves of one subfunction intersect or even do not meet (see for instance fig. 29). So smooth curves are not found this way.

![Diagram showing tangential velocity-component solutions in three regions](image)

fig. 29.

Tangential velocity-component: solutions in the three regions.

It is possible, however, to smoothen these curves by constructing a "composite solution" (see Van Dyke /4/, chapter V). This is not a solution of a set of
differential equations, but it is constructed from the solutions in the three regions, making use of a condition, which is necessary for the existence of matched asymptotic expansions. In Van Dyke's terminology, this condition is the existence of an "overlap domain" between an inner and an outer region, where both the inner and the outer solution hold.

A composite solution must satisfy only one condition:

attracting the inner region, the composite solution must change into the inner solution; approaching the outer region, the composite solution must change into the outer solution.

This implies, that many composite solutions are possible, two of which are mentioned by Van Dyke as being the most usual ones: the "additively" and the "multiplicatively" composite solution.

2. The additive composition rule

Be $f^{(n)}$ the n-term outer expansion of a function $f$ and $f^{(m)}$ the m-term inner expansion of that function, such that:

$$f^{(n)} = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \ldots + \varepsilon^{n-1} f_{n-1}$$

and:

$$f^{(m)} = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \ldots + \varepsilon^{m-1} f_{m-1}$$

then the composite expansion $f_{c}^{(m,n)}$ is defined by the following rule:

$$f_{c}^{(m,n)} = f^{(n)} + f^{(m)} - \{f^{(n)}\}^{(m)}$$

in which $\{f^{(n)}\}^{(m)}$ represents the m-term inner expansion of the n-term outer expansion. This is the additive composition rule.

The outer region is approached by taking the n-term outer expansion of expression 246:

$$\{f_{c}^{(m,n)}\}^{(n)} = f^{(n)} + \{f^{(m)}\}^{(n)} - \{f^{(n)}\}$$
According to the asymptotic matching principle:

$$\{f(m)\}_n = \{f(n)\}_m$$

(248)

so:

$$\{f_{c(m)}\}_n = \{f(n)\}_m$$

(249)

Similarly, the inner region is approached by taking the $m$-term inner expansion of expr. 246:

$$\{f_{c(m,n)}\}_m = \{f(n)\}_m + P(m) - \{f(n)\}_m$$

(250)

so:

$$\{f_{c(m,n)}\}_m = P(m)$$

(251)

Exprs. 249 and 251 show the additively composite solution satisfies condition 243.

N.B. This does not imply, that for every $m$ and $n$ $f_{c(m,n)}$ changes into $f(n)$ if $X$ goes to infinity or into $P(m)$ if $r$ goes to $r_i$ or $r_o$.

The additive composition rule 246 has been elaborated in appendix II for $m=1,2$ and $n=1,2,3$. It will be applied to the velocity-components, found in the foregoing parts of this chapter, for $\epsilon_1=.1$ and $\epsilon_2=.4$. The results will be compared with the relevant solutions of chapter II.

a. Zero order tangential velocity-component.

According to appendix II, the additively composite equivalent of the zero order tangential velocity-component is defined by:

$$uc_0 = u_0 + U_0 - \lim_{r \to r_i} u_0$$

$$= \frac{\Delta}{\alpha} \left\{ \frac{1}{r} u_0(\zeta) + \frac{1}{r_i} \frac{\partial}{\partial r} u_0(X,Z) - \frac{1}{r_i} u_0(\zeta) \right\} = \frac{\Delta}{\alpha} uc_0^*$$

(252)
in the inner wall half of the cross-section. Similarly, in the outer wall half of the cross-section it is defined by:

\[
uc_0 = u_0 + u_0' - \lim_{r \to r_o} u_0
\]

\[
= \frac{\Delta}{\alpha} \left\{ \frac{1}{r} \tilde{u}_0(\xi) + \frac{1}{r_o} \tilde{u}_0(X,Z) - \frac{1}{r_o} \tilde{u}_0(\xi) \right\} = \frac{\Delta}{\alpha} uc^*_0
\]

(253)

The functions \( \frac{1}{r} \tilde{u}_0(\xi) \) and \( \tilde{u}_0(X,Z) \) being known, \( uc^*_0 \) can be determined easily. According to fig. 30, this composite solution agrees very well with \( u_0^*(II) \), the relevant result of chapter II.

fig. 30.

Tangential velocity-component at the surface.

--- solutions in the three separate regions

--- additively composite solution

--- multiplicatively composite solution

--- solution chapter II: \( u_0^*(II) \)

b. Secondary velocity-components.

The stream function of the secondary flow is constructed from the three separate solutions, using the formula:

\[
gc_0 = g_0 + g_0' - \lim_{r \to r_i} g_0
\]

\[
= \frac{\Delta^2}{\alpha^3} \left\{ \frac{1}{r} \tilde{g}_0(\xi) + \frac{1}{r_i} \tilde{g}_0(X,Z) - \frac{1}{r_i} \tilde{g}_0(\xi) \right\} = \frac{\Delta^2}{\alpha^3} gc^*_0
\]

(254)
in the inner wall half of the cross-section and

\[ g_{c0} = g_0 + G_0 - \lim_{r \to r_o} g_0 = \frac{a^2}{3} \left\{ \frac{1}{r} \frac{\nu}{3} g_0(\zeta) + \frac{1}{r^3} G_0(X,Z) - \frac{1}{r^3} \frac{\nu}{3} g_0(\zeta) \right\} = \frac{a^2}{3} g_{c0}^* \]  

(255)

in the outer wall half of the cross-section.

This composite solution has been compared with the stream function, found in chapter II. The agreement is good, as fig. 31 shows.

![Graph showing stream function of the secondary flow at \( \zeta = 0.5 \)]

---

**Fig. 31**

Stream function of the secondary flow at \( \zeta = 0.5 \)

---

solutions in the three separate regions

additively composite solution

multiplicatively composite solution

solution chapter II: \( g_{c0}^*(II) \)

---

The radial velocity-component can be derived from \( g_{c0} \) as well as directly from the separate solutions:

\[ v_{c0} = -\frac{3}{a^3} (g_{c0}) = -\frac{3g_0}{a^2} - \frac{3G_0}{a^2} + \lim_{r \to r_i} \frac{3g_0}{a^2} \]

\[ = v_0 + V_0 - \lim_{r \to r_i} v_0 = \frac{a^2}{3} \left\{ \frac{1}{r^3} \frac{\nu}{3} v_0(\zeta) + \frac{1}{r^3} V_0(X,Z) - \frac{1}{r^3} \frac{\nu}{3} v_0(\zeta) \right\} = \frac{a^2}{3} v_{c0}^* \]  

(256)
in the inner wall half and, similarly:

\[
vc_0 = \frac{\Delta^2}{\alpha^3} \left( \frac{1}{r_1^3} v_0(\zeta) + \frac{1}{r_0^3} v_0^*_{0}(X,Z) - \frac{1}{r_0^3} v_0^*(\zeta) \right) = \frac{\Delta^2}{\alpha^3} vc_0^* \tag{257}
\]

in the outer wall half of the cross-section.

Fig. 32 shows the agreement between \(vc_0^*\) and \(v_0^*(II)\) may be named good.

![fig. 32. Radial velocity-component at the surface.](image)

solutions in the three separate regions

--- additively composite solution

--- multiplicatively composite solution

--- solution chapter II: \(v_0^*(II)\)

The vertical velocity-component gives some more complications. Considering the three regions, it appears that in both side-wall regions \(v_z'(x) = w_0\) is a first order function, while in the central region \(v_z'(x) = w_0\) is a second order function. According to appendix II, the additively composite solution \(wc_0\) is defined then by:

\[
wc_0 = wc^{(1,2)} = 0 + \varepsilon_1 w_0 + W_0 - \lim_{r \to r_i} \frac{1}{\varepsilon_2^x} \lim_{r \to r_i} (r-r_i)w_0
\]

\[
= \varepsilon_1 w_0 + W_0
\]

\[
= \frac{\Delta^2}{\alpha^3} \left( \frac{\varepsilon_1}{r_i} w_0(\zeta) + \frac{1}{r_i^3} W_0(X,Z) \right) = \frac{\Delta^2}{\alpha^3} wc^*_0 \tag{258}
\]
in the inner wall half and similarly

\[
\omega_c^0 = \frac{\Delta^2}{a^3} \left\{ \frac{\varepsilon}{r^4} \omega_0(\zeta) - \frac{1}{r_0^3} \omega_0(X,Z) \right\} = \frac{\Delta^2}{a^3} \omega_c^* 
\]

(259)

in the outer wall half of the cross-section.

Besides, the vertical velocity-component can be derived from \(g_c^0\):

\[
\omega_c^0' = \frac{\varepsilon}{r} \frac{\partial}{\partial r} (r \ g_c^0) \\
= \frac{\Delta^2}{a^3} \left[ - \frac{2 \varepsilon}{r^3} g_0(\zeta) + \frac{1}{r_1^3} \frac{\partial}{\partial X} g_0 + \frac{\varepsilon}{r_1^3} \frac{\partial}{\partial r} \left\{ g_0(X,Z) - g_0(\zeta) \right\} \right] \\
= \frac{\Delta^2}{a^3} \left[ \omega_c^* + \frac{\varepsilon}{r_0^3 r} \left\{ g_0(X,Z) - g_0(\zeta) \right\} \right] = \frac{\Delta^2}{a^3} \omega_c^* 
\]

(260)

in the inner wall half and:

\[
\omega_c^0' = \frac{\Delta^2}{a^3} \left[ \omega_c^* + \frac{\varepsilon}{r_0^3 r} \left\{ g_0(X,Z) - g_0(\zeta) \right\} \right] = \frac{\Delta^2}{a^3} \omega_c^* 
\]

(261)

in the outer wall half of the cross-section.

Exprs. 258-261 have been compared with \(\omega_{low}^*\), derived from \(g_c^*(II)\) by means of the definition:

\[
\omega_{low}^* = \frac{1}{1+\varepsilon\xi} \frac{\partial}{\partial \xi} \left\{ (1+\varepsilon\xi) g_c^*(II) \right\} 
\]

(143)

Fig. 33 shows the agreement is good, except near the side-walls. Nevertheless the deviations there have the o.o.m. \(O(\varepsilon)\):

for expression 258: \(\frac{\varepsilon}{r^4} \omega_0(\zeta)\)

(262)
for expression 259: \( \frac{c}{r_0^4} w_0(\zeta) \)  
(263)

for expression 260: \( -\frac{3c}{4} r_i^{-1} w_0(\zeta) = \frac{3}{2} \frac{c}{r_i^4} w_0(\zeta) \)  
(264)

for expression 261: \( -\frac{3c}{4} r_i^{-1} \zeta = \frac{3}{2} \frac{c}{r_i^4} w_0(\zeta) \)  
(265)

fig. 33.  
Vertical velocity-component at \( \zeta = 0.5 \)

- - - - solutions in the three separate regions
- - - - additively composite solution
- - - - solution derived from the additively composite stream function
- - - - solution derived from the multiplicatively composite stream function
- - - - solution chapter II: \( w_0^\infty \)

\[ c. \text{ First correction of the tangential velocity-component.} \]

Considering the first non-zero correction of \( u_0 \) in the three separate regions, it appears to be a first order subfunction in the side-wall regions and a second order subfunction in the central region. As far as the additively composite solution is concerned, this implies the first correction of \( u_0 \), symbolized by \( u_{c_1} \), is defined by:
\[ uc_1 = \frac{uc(2,3) - uc(1,1)}{\varepsilon_1} \]

\[ = \varepsilon_1 u_2 + U_1 - \lim_{r \to r_i} \frac{\partial u_0}{\partial r} \]

\[ = \varepsilon_2 \frac{\Delta}{\alpha} \left\{ \frac{1}{r_i} U_{1,0}(X,Z) + \frac{1}{r_i^2} U_{1,2}(X,Z) + \frac{x}{r_i^2} u_0(\xi) \right\} + \varepsilon_2 \frac{\Delta}{\alpha} \left\{ \frac{1}{r_i^4} U_{1,1}(X,Z) + \frac{x}{r_i^3} u_2(\xi) \right\} \]

(266)

in the inner wall half of the cross-section.

Since \( \tilde{U}_{1,0}(X,Z) = X^{1/2} Z^{1/2} = \tilde{X} u_0(\xi) \), this expression becomes:

\[ uc_1 = \varepsilon_2 \frac{\Delta}{\alpha} \frac{1}{r_i^2} U_{1,2}(X,Z) + \varepsilon_2 \frac{\Delta}{\alpha} \frac{3}{5} \left\{ \frac{1}{r_i^4} U_{1,1}(X,Z) + \frac{x}{r_i^3} u_2(\xi) \right\} \]

\[ = \varepsilon_2 \frac{\Delta}{\alpha} uc_{1,2}^{*} + \varepsilon_2 \frac{\Delta}{\alpha} \frac{3}{5} uc_{1,1}^{*} \]

(267)

Similarly, in the outer wall half of the cross-section:

\[ uc_1 = \varepsilon_2 \frac{\Delta}{\alpha} \frac{-1}{r_i^2} U_{1,2}(X,Z) + \varepsilon_2 \frac{\Delta}{\alpha} \frac{3}{5} \left\{ \frac{-1}{r_0^4} U_{1,1}(X,Z) + \frac{x}{r_i^3} u_2(\xi) \right\} \]

\[ = \varepsilon_2 \frac{\Delta}{\alpha} uc_{1,2}^{*} + \varepsilon_2 \frac{\Delta}{\alpha} \frac{3}{5} uc_{1,1}^{*} \]

(268)

In fig. 34 \( uc_{1,1}^{*} \) has been compared with \( u_{1,1}^{*}(II) \) and \( u_{1,1}^{**} \) (see eq. 158).

\( uc_{1,1}^{*} \) appears to agree better with \( u_{1,1}^{**} \) than with \( u_{1,1}^{*} \), but nevertheless the agreement is rather poor: the difference between \( uc_{1,1}^{*} \) and \( u_{1,1}^{**} \) at the peak near the inner wall is not smaller than about \( \frac{1}{2} \) time difference between \( u_{1,1}^{*} \) and \( u_{1,1}^{*} \), there.
At the side-walls $u_{c_{1,1}}^*$ does not vanish, but its value remains within $O(\varepsilon)$:

$$\frac{\varepsilon}{r_1} u_2(\zeta) \quad \text{and} \quad \frac{\varepsilon}{r_0} u_2(\zeta)$$

at the inner and outer wall, respectively.

---

![Graph showing the comparison of solutions](image)

**fig. 34.**

First correction of the tangential velocity-component.

1. Influence of the secondary circulation.

- solutions in the three separate regions
- additively composite solution
- multiplicatively composite solution
- solution chapter II: $u_{c_{1,1}}^*$
- solution chapter II: $u_{c_{1,2}}^*$

Besides, $u_{c_{1,2}}^*$ has been compared with $u_{c_{1,2}}^*$ (II) in fig. 35. Since $u_{c_{1,2}}^*$ is identical to the solutions in the side-wall regions, the agreement is just as poor as it appeared to be in fig. 27. However, the differences must still lie within $O(\varepsilon)$. 
First correction of the tangential velocity-component.

2. Influence of the "extra" friction term.

--- solutions in the separate regions (identical to the additively composite solution)

--- multiplicatively composite solution

--- solution chapter II: \( u_{1,2}^*(II) \)

d. Synopsis.

The velocity-components, resulting from the additive composition rule are:

\[
\begin{align*}
\nu_\phi &= \bar{\nu} \left\{ \frac{\Delta}{\alpha} \nu c_{0}^* + \varepsilon \frac{\Delta^2}{\alpha^3} \nu c_{1,1}^* + \varepsilon \frac{\Delta}{\alpha} \nu c_{1,2}^* + O(\varepsilon^2) \right\} \quad (270) \\
\nu_R &= \bar{\nu} \left\{ \varepsilon \frac{\Delta^2}{\alpha^3} \nu c_{0}^* + O(\varepsilon^2) \right\} \quad (271) \\
\nu_z &= \bar{\nu} \left\{ \varepsilon \frac{\Delta^2}{\alpha^3} \nu c_{0}^* + O(\varepsilon^2) \right\} \quad (272)
\end{align*}
\]

in which the functions \( u_{c_{0}}^*, u_{c_{1,1}}^*, u_{c_{1,2}}^*, v_{c_{0}}^* \) and \( w_{c_{0}}^* \) are found from simple linear combinations of the tabulated functions \( u_0(\zeta), u_2(\zeta), g_0(\zeta), v_0(\zeta), w_0(\zeta), U_0(X,Z), U_{1,1}(X,Z), U_{1,2}(X,Z), G_0(X,Z), V_0(X,Z) \) and \( W_0(X,Z) \).

3. The multiplicative composition-rule.

An alternative way to define the composite expansion \( fc^{(m,n)} \) of the n-term outer
expansion \( f^{(n)} \) and the \( m \)-term inner expansion \( F^{(m)} \) is:

\[
fc^{(m,n)} = F^{(m)} \times \frac{f^{(n)}}{\{f^{(n)}\}^{(m)}} \quad (273)
\]

This is the multiplicative composition rule.

If the outer region is approached by taking the \( n \)-term outer expansion of expression 273:

\[
\{fc^{(m,n)}\}^{(n)} = \{F^{(m)}\}^{(n)} \times \frac{f^{(n)}}{\{f^{(n)}\}^{(m)}} \quad (274)
\]

Then, according to the asymptotic matching principle:

\[
\{fc^{(m,n)}\}^{(m)} = f^{(n)} \quad (275)
\]

If the inner region is approached by taking the \( m \)-term inner expansion of expression 273:

\[
\{fc^{(m,n)}\}^{(m)} = F^{(m)} \times \frac{\{f^{(n)}\}^{(m)}}{\{f^{(n)}\}^{(m)}} = F^{(m)} \quad (276)
\]

Exprs. 275 and 276 show the multiplicatively composite solution satisfies condition 243.

N.B. This goes not imply, that for every \( m \) and \( n \) \( fc^{(m,n)} \) changes into \( f^{(n)} \)
if \( X \) goes to infinity or into \( F^{(m)} \) if \( r \) goes to \( r_1 \) or \( r_0 \).

In appendix II the multiplicative composition rule has been elaborated for \( m=1,2 \) and \( n=1,2,3 \). For \( \varepsilon_1 = .4 \) and \( \varepsilon_2 = .4 \), it will be applied now to the velocity-components, found by considering the three separate regions. The results are compared with the relevant results of chapter II.
a. Zero order tangential velocity-component.

According to appendix II, the application of formula 273 to the zero order tangential velocity-components \((m=n=1)\) yields:

\[
uc_0 = U_0 \times \lim_{r \to r_i} \frac{u_0}{r} = \frac{\Delta}{\alpha} \frac{1}{r_i} U_0(X,Z) \times \frac{\frac{\Delta}{\alpha} \frac{1}{r} u_0(\zeta)}{\frac{\Delta}{\alpha} \frac{1}{r_i} u_0(\zeta)} = \frac{\Delta}{\alpha} \frac{1}{r} U_0(X,Z) = \frac{\Delta}{\alpha} uc_0^\star
\]

(277)

Since \(uc_0^\star\) does not depend on \(r_i\), it can be seen immediately, that in the outer wall half of the cross-section:

\[
uc_0^\star = \frac{1}{r} U_0(X,Z)
\]

(278)

In fig. 30, this expression has been compared with the relevant solution of chapter II: \(u_0^\star(\Pi)\) and the result of the additive composition rule. Obviously, there is no practical difference between these three curves.

b. Secondary velocity-components.

Application of the multiplicative composition rule to the stream function of the secondary flow yields:

\[
gc_0 = G_0 \times \lim_{r \to r_i} \frac{g}{r} = \frac{\Delta^2}{\alpha^2} \frac{1}{r_i} G_0(X,Z) \times \frac{\frac{\Delta^2}{\alpha^2} \frac{1}{r} g_0(\zeta)}{\frac{\Delta^2}{\alpha^2} \frac{1}{r_i} g_0(\zeta)} = \frac{\Delta^2}{\alpha^2} \frac{1}{r} G_0(X,Z) = \frac{\Delta^2}{\alpha^2} gc_0^\star
\]

(279)

So in the entire cross-section:

\[
gc_0^\star = \frac{1}{r^3} G_0(X,Z)
\]

(280)
Fig. 31 shows this result differs not much from the solution of chapter II: 
$g_0^*(II)$ and the result of the additive composition rule.
The radial velocity-component follows from $v_0$ and $V_0$ in the same way.
So in the entire cross-section:

$$v_0^* = \frac{1}{r^3} v_0(X,Z)$$  \hspace{1cm} (281)

This result has been illustrated in fig. 32 and it shows only slight differences from $v_0^*(II)$ and the additively composite radial velocity-component.

Considering the vertical velocity-component, the multiplicative composition rule appears to fail:

$$\omega_c = \omega_0 \times \frac{0 + \varepsilon_i \omega_0}{0 + \varepsilon_i \frac{X}{X} \lim_{r \to r_i} (r-r_i) \omega_0} = \omega_0 \times \frac{\varepsilon_i \omega_0}{0}$$  \hspace{1cm} (282)

However, it is possible indeed to derive $\omega_c$ from the multiplicatively composite stream function (see expr. 280):

$$\omega_c = \frac{\varepsilon}{r} \frac{3}{r} \left( r \text{gc}_0 \right) = \frac{\Delta^2}{a^3} \left( \frac{1}{r} \frac{3g_0}{2X} - \frac{2\varepsilon}{r} \frac{\gamma}{4} c_0^*(X,Z) \right) = \frac{\Delta^2}{a^3} \omega_c^*$$  \hspace{1cm} (283)

This expression is valid in the entire cross-section, again.

Fig. 33 shows this expression is a rather good approximation of $\omega_0^{**}$, though near the inner wall it is not as good as the other composite vertical velocity-components.

c. First correction of the tangential velocity-component.

According to the multiplicative composition rule the first correction of the tangential velocity-component is given by:
\[
uc_1 = \frac{uc(2,3) - uc(1,1)}{\varepsilon_1}
\]

\[
= \frac{1}{\varepsilon_1} \left( (U_0 + \varepsilon_1 U_1) \times \frac{u_0 + \varepsilon_1 u_2}{\lim_{r \to r_i} \varepsilon_1 \varepsilon_2 \lim_{r \to r_i} u_0} \right) - U_0 \times \frac{u_0}{\lim_{r \to r_i} u_0} \]  
(284)

After some transformations and neglecting the terms \(O(\varepsilon_1^2)\), this expression becomes:

\[
uc_1 = \varepsilon_2 \frac{\Delta}{\alpha} uc_{1,0}^* + \varepsilon_2 \frac{\Delta^3}{\alpha} uc_{1,1}^* + \varepsilon_2 \frac{\Delta}{\alpha} uc_{1,2}^* 
\]  
(285)

in which:

\[
uc_{1,0}^* = \frac{\nabla u_0(X,Z) - u_0(\zeta)}{r} \frac{X}{r_i} \left( 1 - \frac{\varepsilon X}{r_i} \right) 
\]  
(286)

\[
uc_{1,1}^* = \frac{1}{l - \frac{\varepsilon X}{r_i}} \left[ - \frac{\nabla u_1(X,Z)}{r^3 r} + \frac{\nabla u_0(X,Z)}{u_0^2(\zeta)} \frac{\varepsilon X}{r_i} \frac{u_2(\zeta)}{r^5} \right] 
\]  
(287)

\[
uc_{1,2}^* = \frac{1}{l - \frac{\varepsilon X}{r_i}} \frac{\nabla u_1(X,Z)}{r_i r} 
\]  
(288)

In the outer wall half of the cross-section similar expressions hold true:

\[
uc_{1,0}^* = \frac{\nabla u_0(X,Z) - u_0(\zeta)}{r} \frac{-X}{r_o} \left( 1 + \frac{\varepsilon X}{r_o} \right) 
\]  
(289)

\[
uc_{1,1}^* = \frac{1}{l + \frac{\varepsilon X}{r_o}} \left[ - \frac{\nabla u_1(X,Z)}{r^3 r} + \frac{\nabla u_0(X,Z)}{u_0^2(\zeta)} \frac{\varepsilon X}{r_o} \frac{u_2(\zeta)}{r^5} \right] 
\]  
(290)

\[
uc_{1,2}^* = \frac{1}{l + \frac{\varepsilon X}{r_o}} \frac{\nabla u_1(X,Z)}{r_o r} 
\]  
(291)

The function \(uc_{1,0}^*\) cannot be compared with any of the results of chapter II. 
It has been outlined in fig. 36.

Adding \(uc_{1,0}^*\) to the multiplicatively composite \(uc_0^*\), however, a slightly improved approximation of \(u_0^*(\text{II})\) is obtained: the curve of \(uc_0^* + suc_{1,0}^*\).
practically coincides with the curve of the additively composite \( u_{c0}^* \) (see fig. 30).

![Graph showing the curve of \( u_{c0}^* \) and \( u_{c1,1}^* \)](image)

**fig. 36**

Tangential velocity-component

Multiplicatively composite \( u_{c1,0}^* \)

The function \( u_{c1,1}^* \) has been outlined in fig. 34, which shows the agreement with \( u_{1,1}^* \) and \( u_{1,1}^* \) is not quite convincing, though it is a better approximation than the three separate solutions.

Fig. 35 shows \( u_{c1,2}^* \) practically coincides with its additive equivalent and \( U_{1,2}^* \).

d. Synopsis.

The velocity-components, resulting from the multiplicative composition rule, are:

\[
v_\phi = \nabla \left[ \frac{\Delta}{\alpha} u_{c0}^* + \frac{\Delta}{\alpha} u_{c1,0}^* + \frac{\Delta}{\alpha} u_{c1,1}^* + \frac{\Delta}{\alpha} u_{c1,2}^* + O(\epsilon_1^2) \right]
\]

\[
v_R = \nabla \left[ \epsilon \frac{\Delta^2}{\alpha^3} v_0^* + O(\epsilon_1^2) \right]
\]

(292)

(293)

in which the functions \( u_{c0}^* \), \( u_{c1,0}^* \), \( u_{c1,1}^* \), \( u_{c1,2}^* \) and \( v_0^* \) are found from rather simple, but, as far as the higher order subfunctions are concerned, not always linear combinations of the tabulated functions \( u_0(\xi) \), \( u_2(\xi) \), \( v_0(\xi) \), \( U_0(X,Z) \), \( U_{1,1}(X,Z) \), \( U_{1,2}(X,Z) \) and \( V_0(X,Z) \).

A multiplicatively composite vertical velocity-component can not be constructed directly from \( \tilde{w}_0(\xi) \) and \( \tilde{W}_0(X,Z) \), but only indirectly from the multiplicatively
composite stream function of the secondary flow:

\[ v_z = \overline{V} \left\{ \varepsilon \frac{A^2}{\alpha^3} w_0^\ast + O(\varepsilon_1^2) \right\} \quad (294) \]

in which \( w_0^\ast \) follows from expr. 283.


- The solutions, resulting from the separate consideration of the three regions, described in parts B, C and D of this chapter, can be composed to solutions which agree better with the results of chapter IIB (i.e. consideration of the entire cross-section) than the three separate solutions themselves. This composition can be realized by means of simple rules.
- The additive composition rule yields "subfunctions" which are found from simple linear combinations of the functions, given in tables I-VIII.
- The multiplicative composition rule yields "subfunctions" which are found from simple linear combinations of the tabulated functions, but only as far as the zero order subfunctions are concerned. The higher order "subfunctions" are found from more complicated, mostly non-linear combinations of the tabulated functions.
- Sometimes the multiplicative composition rule fails where the additive composition rule gives good results, indeed. This is the case if \( \{ f^{(n)} \}^{(m)} \) vanishes while \( f^{(n)} \) does not.
- Generally speaking, the additive composition rule gives slightly better approximations of the solutions in the entire cross-section than the multiplicative composition rule, except near the side-walls, where the additively composite higher order subfunctions mostly do not vanish. Resumingly, the additive composition rule should be preferred to the multiplicative composition rule.
CHAPTER IV

A. CONCLUSIONS

The conclusions to be drawn from this report can be summarized as follows:

1. The method of asymptotic expansions is applicable to the problem of viscous, axisymmetrical curved flow in a shallow, rectangular open channel. It results in differential equations which are quite similar to the equations derived by Ananyan. At some points, however, Ananyan’s analysis proves to be not entirely consequent.

2. The region of the most intensive flow (i.e. with maximal values of the three velocity-components) is situated within a few times the depth of flow from the inner side-wall.

3. The effect of the secondary circulation on the main flow pattern is a decrease of the main velocity-component near the inner wall and a (much smaller) increase of it in the rest of the cross-section. This may explain the existence of a region of low tangential velocities at the surface near the inner wall, which is reported by several experimental investigators (see for instance Francis and Asfari /5/).

4. Probably, the subfunctions of the normalized velocity-components can be approximated by the product of a function of the normalized radial co-ordinate $\xi$ and a function of the normalized vertical co-ordinate $\zeta$.

5. If the depth of flow is sufficiently small with respect to the channel-width (for instance $d_B = .1$), there is a region around the central axis, where the flow pattern practically does not depend on the friction and the vanishing of $v_R$ at the side-walls. The extent of this region depends on $d_B$ and the subfunction considered.

The same conclusion has been drawn by Rozovs'kii as to the zero order subfunctions of the velocity-components in a shallow channel with a large radius of curvature with respect to the channel width.

6. If $d_B$ is so small, that the abovementioned region around the central axis (central region) exists, it is possible to apply the method of matched asymptotic expansions to the problem. This implies the separate consideration of the central (i.e. regular) region and the regions near the side-walls (i.e. the singular regions).

7. In the central region the problem can be solved directly, without taking into account the conditions at the side-walls. The velocity-components
agree rather well with the relevant results of the consideration of the entire cross-section. They can be written as:

\[ \text{constant} \times \frac{1}{r^k} \times f_m(\zeta) \]  

(163)

in which \( r \) represents the normalized radial co-ordinate \( (r = \frac{R}{R_0}) \) and \( \zeta \) the normalized vertical co-ordinate \( (\zeta = \frac{Z}{d}) \). \( k \) and \( m \) represent integers; the constant depends on the average tangential velocity \( \bar{V} \), the ratio \( \frac{d}{R_0} \), the longitudinal slope and the Reynolds number \( \bar{V}d/\nu \).

Since the functions \( f_m(\zeta) \) do not depend on the channel geometry or the flow parameters, they have to be computed only once. They have been tabulated in table I.

8. The expressions, describing the zero order subfunctions of the velocity-components in the central region, are identical to those derived by Boussinesq.

9. In the regions near the side-walls the problem cannot be solved without referring to the solution in the central region. The matching is realized by means of matching conditions, which complete the solution. The resulting velocity-components do not agree very well with the results obtained for the entire cross-section, though they show a similar behaviour. The velocity-components in the side-wall regions can be written as the sum of terms of the form:

\[ \text{constant} \times \frac{1}{r_{\text{wall}}} \times F_m(X,Z) \]  

(239)

in which \( r_{\text{wall}} = \frac{R_{\text{wall}}}{R_0} \), \( X \) represents the normalized singular radial co-ordinate, such that \( X = \left| \frac{R - R_{\text{wall}}}{d} \right| \), and \( Z \) represents the normalized vertical co-ordinate, such that \( Z = \frac{Z}{d} \). Besides, \( k \) and \( m \) are integers and the constant depends on the average tangential velocity \( \bar{V} \), the ratio \( \frac{d}{R_0} \), the longitudinal slope and the Reynolds number \( \bar{V}d/\nu \).

Since the functions \( F_m(X,Z) \) do not depend on the channel geometry, they have to be computed only once. The results of these computations have been given in tables II-VIII.

10. The differential equation the stream function of the secondary flow is solved from looks very much like the equation Rozovskii used for this
function. However, as far as viscous flow is concerned, his assumptions:
"taking $v_\phi$ near the side-walls equal to $v_\phi$ at the central
axis has little influence on the secondary flow pattern near
the side-walls"

turns out to bring about considerable differences in the results.

11. The agreement between the velocity-components found from the consideration
of the entire cross-section and the results of the method of matched
asymptotic expansions can be improved considerably by combining the latter
to a "composite solution", using a simple composition rule. The subfunc-
tions of the composite solution consist of simple combinations of the
aforementioned tabulated functions $f(\xi)$ and $F(X,Z)$.

This implies a good approximation of the velocity-components derived for
the entire cross-section can be determined quickly and simply for every
channel geometry and every set of flow conditions, provided that they are
compatible with the assumption of a wide, shallow channel and viscous flow
at low Reynolds-numbers.

B. SUGGESTIONS FOR FURTHER RESEARCH

The most important limitations of the present theory are:
1. the axisymmetrical flow pattern
2. the viscous flow character
3. the wide, shallow cross-section
4. the large radius of curvature with respect to the average depth of flow
5. the assumption of the surface to lie at its average level.

Elimination of assumption of axial symmetry.

The assumption of axial symmetry is fundamental for the mathematical model.
If it is dropped, the differential equations will contain several extra terms,
which make them much more difficult to be solved, even if a perturbation method
is applied:

$$
\frac{\partial v}{\partial R} + \frac{\partial \phi}{\partial \phi} + \frac{\partial v}{\partial z} - \frac{v^2}{R} =
$$

$$
- \frac{1}{\rho} \frac{\partial p'}{\partial R} + \frac{\partial^2 v}{\partial R^2} + \frac{\partial^2 v}{\partial z^2} + \frac{1}{R} \frac{\partial^2 v}{\partial \phi^2} + \frac{1}{R} \frac{\partial v}{\partial R} - \frac{v}{R} - \frac{2}{R^2} \frac{\partial v}{\partial \phi}
$$

(295)
\[
\begin{align*}
\frac{\partial v}{\partial R} + \frac{v}{R} \frac{\partial v}{\partial \phi} + \frac{\partial ^2 v}{\partial z^2} + \frac{\partial v}{\partial z} &= 0 \\
- \frac{1}{\rho R} \frac{\partial p}{\partial \phi} \frac{\partial ^2 v}{\partial \phi^2} + \frac{\partial ^2 v}{\partial R^2} + \frac{1}{R} \frac{\partial ^2 v}{\partial \phi^2} + \frac{1}{R} \frac{\partial v}{\partial R} - \frac{v}{R} + \frac{2}{R} \frac{\partial v}{\partial \phi} \frac{\partial ^2 v}{\partial \phi \partial R} \\
\frac{\partial v}{\partial R} + \frac{v}{R} \frac{\partial v}{\partial \phi} + \frac{\partial v}{\partial z} &= 0 \\
- \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial ^2 v}{\partial R^2} + \frac{\partial ^2 v}{\partial \phi^2} + \frac{1}{R} \frac{\partial ^2 v}{\partial \phi^2} + \frac{1}{R} \frac{\partial v}{\partial R} \right)
\end{align*}
\] (296)

(297)

(298)

Even though it may be the most difficult one, it is definitely the most important continuation of the present theory: the flow pattern in a bend with a finite angle of deviation will not be axisymmetrical; the regions of growing and decreasing secondary circulation will occupy at least the greater part of the bend.

**Extension of the theory to turbulent flow.**

A second important point of consideration in curved flow is the flow character, which can be laminar or turbulent. This will have a considerable influence on the flow pattern, since the vertical derivatives of the tangential velocity-component, which are essential for the intensity of the secondary circulation, depend on the character of the flow to a large extent.

The turbulent problem is much more complicated than the present one. Using a variable, scalar eddy-viscosity \( A \), there is one more unknown variable and the differential equations will contain extra terms with the partial derivatives of \( A \). But one more unknown variable implies, that the four differential equations of the present mathematical model are not sufficient to solve the problem. That is why most investigators assume the tangential velocity-component to be identical to the longitudinal velocity in the straight part of the channel upstream. The tangential equation of motion is then used for the solution of \( A \), after which the other equations are used to determine the secondary velocity-components.
The flow in a curved channel with an arbitrary cross-section.

The present theory is applicable to a channel with an arbitrary cross-section, only as far as the consideration of the entire cross-section is concerned. So the method of matched asymptotic expansions can no longer be used then, which implies that for every new geometry of the cross-section the system of differential equations must be solved anew.

The flow in a curved channel for small radii of curvature.

If none of the dimensions of the cross-section is small with respect to the radius of curvature of the centre-line, the perturbation method described here can no longer be used. Then the complete differential equations have to be solved at once (which is practically impossible), unless another schematization can be found.

Axisymmetrical flow with a free surface.

Actually, the solutions derived in this report describe the flow pattern in an infinitely coiling tube with a shallow rectangular cross-section $B \times 2d$ and a longitudinal slope of the piezometrical head:

$$\frac{\zeta (ph)}{R \zeta} = \frac{\nu^2}{g' d} \frac{-\Delta}{1+\epsilon \zeta}$$

(299)

It may be interesting to solve the problem for an open channel, in order to obtain an impression of the cross-slope and the shape of the surface. That is why a curved open channel is considered in this section; the slope of its bed is taken equal to the slope in expr. 299.

This slope, called $I_b$, may be supposed to be so small, that:

$$\sin I_b = I_b \text{ and } \cos I_b = 1$$

(300)

Then the differential equations, describing the problem, are identical to eqs. 1-4, but now the $z$-axis is not vertical, but perpendicular to the bed,
while:

\[ p' = p + \rho g'z + \rho g'h_b \quad (301) \]

in which \( h_b \) represents the level of the bed with respect to some horizontal reference-level.

The boundary conditions at the surface slightly differ from eqs. 9:

\[ v_z - v_R \frac{\partial h}{\partial R} = 0 \quad (302) \]

\[ \frac{\partial v}{\partial z} = 0 \quad (303) \]

\[ \frac{\partial v_R}{\partial z} = \frac{\partial v_z}{\partial R} = 0 \quad (304) \]

\[ p = 0 \quad (305) \]

N.B. Since the velocity-components have been assumed not to depend on \( \phi \), the depth of flow may not depend on \( \phi \) either.

Normalizing the system like it has been done in chapter II, but now with:

\[ p' = p + \rho g'z + \rho g'h_b = \rho \bar{V}^2 \pi \quad (306) \]

and:

\[ h = d\eta \quad (307) \]

yields differential equations, which are identical to eqs. 25-28.

The boundary conditions at the surface become:

\[ v(\xi=\eta) \frac{\partial \eta}{\partial \xi} - w(\xi=\eta) = 0 \quad (308) \]
\begin{align*}
\frac{\partial u}{\partial \zeta}(\zeta=\eta) &= 0 \\
\frac{\partial v}{\partial \zeta}(\zeta=\eta) + \frac{\partial w}{\partial \zeta}(\zeta=\eta) &= 0 \\
\pi &= \frac{g'd}{V^2} \eta + \frac{g'h}{V^2} b
\end{align*}

(309)

(310)

(311)

The boundary conditions at the bed and at the side-walls are the same as cons. 29, 31 and 32, respectively.

As to \(\frac{\partial \pi}{\partial \phi'}\) turns out to be a constant, again. Regarding 311:

\begin{equation}
\frac{\partial \pi}{\partial \phi'}(\zeta=\eta) = \frac{g'}{V^2} \frac{\partial h}{\partial \phi'} = -\Delta
\end{equation}

(312)

so:

\begin{equation}
\pi = \pi(\phi'=0) - \Delta \phi'
\end{equation}

(313)

As a consequence of def. 307, in which \(d\) represents the average depth of flow, the average value of \(\eta\) must be unity. So:

\begin{equation}
\bar{\eta} = 1
\end{equation}

(314)

Regarding the boundary condition at the surface 311, this implies:

\begin{equation}
\bar{\pi}(\phi'=0) = \frac{g'd}{V^2} + \frac{g'h}{V^2} h_b(\phi'=0)
\end{equation}

(315)

or, taking the horizontal reference-level at \(h_b(\phi'=0)\)

\begin{equation}
\bar{\pi}(\phi'=0) = \frac{g'd}{V^2}
\end{equation}

(316)
Just like the other dependent variables, \( \eta \) is expanded in a power series of \( \varepsilon \):

\[
\eta = \eta_0 + \varepsilon \eta_1 + \varepsilon^2 \eta_2 + \varepsilon^3 \eta_3 + \ldots
\]  

(317)

Substituting the series expansions into the differential equations, the zero order subsystem is found to be identical to eqs. 65-68. The relevant boundary conditions at the surface are:

\[
\frac{\partial u_0}{\partial \xi} (\xi = \eta_0) = 0
\]  

(318)

\[
\pi_0 (\xi = \eta_0) = \frac{g'd}{V^2} \eta_0 + \frac{g'h_b}{V^2}
\]  

(319)

The equations of the zero order subsystem show that \( \frac{\partial \pi_0}{\partial \phi'} \) must be a constant (which is taken equal to \(-\Delta\)) as well as \( \pi_0 (\phi' = 0) \). According to cond. 319, then:

\[
\pi_0 (\phi' = 0) = \frac{g'd}{V^2} \eta_0 = \text{a constant}
\]  

(320)

which implies that \( \eta_0 \) is a constant. Then, defining \( \eta_0 \) such, that its average value equals 1:

\[
\eta_0 = 1
\]  

(321)

and:

\[
\pi_0 (\phi' = 0) = \frac{g'd}{V^2}
\]  

(322)

\[
\pi_0 = \frac{g'd}{V^2} - \Delta \phi'
\]  

(323)
while: \[ \frac{\partial \pi_i}{\partial \zeta} = 0 \]

\[ \pi_i = 0 \quad \text{for } i=1,2,3,\ldots \]  

\[ \eta_i = 0 \]  

Then the problem \( u_0 \) has to be solved from is exactly the same as in chapter II A3 (see eqs. 70 and 71 and the boundary conditions 72).

The differential equations of the first order subsystem are identical to the equations of chapter II A3, again (see eqs. 73, 74 and 81). The boundary conditions at the surface \( \zeta = \eta_0 + \epsilon \eta_1 \) are:

\[ v_0(\zeta=\eta_0 + \epsilon \eta_1) \left( \frac{\partial \eta_0}{\partial \zeta} + \epsilon \frac{\partial \eta_1}{\partial \zeta} \right) - w_0(\zeta=\eta_0 + \epsilon \eta_1) = 0 \]  

\[ \frac{\partial u_0}{\partial \zeta} (\zeta=\eta_0 + \epsilon \eta_1) + \epsilon \frac{\partial u_1}{\partial \zeta} (\zeta=\eta_0 + \epsilon \eta_1) = 0 \]  

\[ \frac{\partial v_0}{\partial \zeta} (\zeta=\eta_0 + \epsilon \eta_1) + \frac{\partial w_0}{\partial \zeta} (\zeta=\eta_0 + \epsilon \eta_1) = 0 \]  

\[ \pi_0 (\zeta=\eta_0 + \epsilon \eta_1) + \epsilon \pi_1 (\zeta=\eta_0 + \epsilon \eta_1) = \frac{g'd}{V^2} (\eta_0 + \epsilon \eta_1) + \frac{g'h_b}{V^2} \]  

Using a Taylor-series to expand the terms like \( v_0(\zeta=\eta_0 + \epsilon \eta_1) \):

\[ v_0(\zeta=\eta_0 + \epsilon \eta_1) = v_0(\zeta=\eta_0) + \epsilon \eta_1 \frac{\partial v_0}{\partial \zeta} (\zeta=\eta_0) + O(\epsilon^2) \]  

and neglecting terms \( O(\epsilon) \) in 327 and 329 and terms \( O(\epsilon^2) \) in 328 and 330, these boundary conditions become:

\[ v_0(\zeta=\eta_0) \frac{\partial \eta_0}{\partial \zeta} - w_0(\zeta=\eta_0) = 0 \]
\frac{\partial u_0}{\partial \zeta} (\zeta=\eta_0) + \varepsilon \frac{\partial u_1}{\partial \zeta} (\zeta=\eta_0) + \eta_1 \frac{\partial^2 u_0}{\partial \zeta^2} (\zeta=\eta_0) = 0 \quad (333)

\frac{\partial v_0}{\partial \zeta} (\zeta=\eta_0) + \frac{\partial w_0}{\partial \zeta} (\zeta=\eta_0) = 0 \quad (334)

\pi_0 (\zeta=\eta_0) + \varepsilon \pi_1 (\zeta=\eta_0) + \eta_1 \frac{\partial \pi_0}{\partial \zeta} (\zeta=\eta_0) = \frac{g'd}{v^2} (\eta_0 + \varepsilon \eta_1) + \frac{g'\eta}{v^2} \quad (335)

or, regarding 318 and 321-323:

w_0 (\zeta=1) = 0 \quad (336)

\frac{\partial u_1}{\partial \zeta} (\zeta=1) + \eta_1 \frac{\partial^2 u_0}{\partial \zeta^2} (\zeta=1) = 0 \quad (337)

\frac{\partial v_0}{\partial \zeta} (\zeta=1) = 0 \quad (338)

\pi_1 (\zeta=1) = \frac{g'd}{v^2} \eta_1 \quad (339)

Then the system of equations \(v_0\) and \(w_0\) have to be solved from is the same as in chapter II A3.

Besides, elimination of \(v_0\) and \(w_0\) from eqs. 73 and 74 yields:

\frac{\partial^2 \pi_1}{\partial \zeta^2} + \frac{\partial^2 \pi_1}{\partial \xi^2} = \frac{\partial^2 u_0^2}{\partial \xi^2} \left( \frac{u_0^2}{1+\varepsilon \xi} \right) = \frac{1}{1+\varepsilon \xi} \frac{\partial}{\partial \xi} (u_0^2) \quad (340)

The relevant boundary conditions are:

\frac{\partial \pi_1}{\partial \zeta} (\zeta=0) = \alpha \frac{\partial^2 w_0}{\partial \zeta^2} (\zeta=0) \quad (341)
\begin{equation}
\frac{\partial \pi_1}{\partial \xi} (\xi = b) = 0
\end{equation}

\begin{equation}
\frac{\partial \pi_1}{\partial \xi} (\xi = \pm \frac{b}{2d}) = \alpha \frac{\partial^2 v_0}{\partial \xi^2} (\xi = \pm \frac{b}{2d})
\end{equation}

The solution of this Neumann-problem is unique except a constant, which follows from the condition:

\begin{equation}
\bar{\pi}_1 = 0
\end{equation}

This solution of \( \pi_1 \) can be written as:

\begin{equation}
\pi_1 = \frac{\Delta^2}{\alpha^2} \pi_1^* (\xi; \xi = 1) \tag{345}
\end{equation}

Then, according to cond. 339:

\begin{equation}
\eta_1 = \frac{\bar{V}^2}{g'd} \frac{\Delta^2}{\alpha^2} \pi_1^* (\xi; \xi = 1) = \frac{\bar{V}^2}{g'd} \frac{\Delta^2}{\alpha} \eta_1^* (\xi) \tag{346}
\end{equation}

from which the first order surface \( \eta_0 + \epsilon \eta_1 \) can be derived.

Subsequently, \( u_1 \) can be solved from:

\begin{equation}
\bar{u}_1 = u_{1,1} + u_{1,2} \tag{347}
\end{equation}

in which \( u_{1,1} \) follows from:

\begin{equation}
\alpha \left( \frac{\partial^2 u_{1,1}}{\partial \xi^2} + \frac{\partial^2 u_{1,1}}{\partial \eta^2} \right) = v_0 \frac{\partial u_0}{\partial \xi} + w_0 \frac{\partial u_0}{\partial \eta} \tag{348}
\end{equation}
with the boundary conditions:

\[ u_{1,1}(\xi=0) = 0 ; \quad \frac{\partial u_{1,1}}{\partial \xi}(\xi=1) = -n_1 \frac{\partial^2 u_0}{\partial \xi^2}(\xi=1) ; \quad u_{1,1}(\xi= \frac{B}{2d}) = 0 \] (349)

while \( u_{1,2} \) follows from:

\[ \frac{\partial^2 u_{1,2}}{\partial \xi^2} + \frac{\partial^2 u_{1,2}}{\partial \xi^2} = -\frac{1}{1+\varepsilon \xi} \frac{\partial u_0}{\partial \xi} \] (350)

with the boundary conditions:

\[ u_{1,2}(\xi=0) = 0 ; \quad \frac{\partial u_{1,2}}{\partial \xi}(\xi=1) = 0 ; \quad u_{1,2}(\xi= \frac{B}{2d}) = 0 \] (351)

So \( u_1 \) can be written as:

\[ u_1 = \frac{\Delta^3}{a^5} u^{*}_{1,1}(\xi, \zeta; \alpha^2 \frac{V^2}{g d}) + \frac{\Delta}{a} u^{*}_{1,2}(\xi, \zeta) \] (352)

in which \( u^{*}_{1,2} \) is identical to the relevant result of chapter II A3, but \( u^{*}_{1,1} \) is not, as a consequence of the boundary condition at the free surface. So \( u^{*}_{1,1} \) is seen to depend on the ratio \( Fr^2/Re^2 \) and has to be recomputed for every new value of this ratio!
APPENDIX I

ELABORATION OF THE ASYMPTOTIC MATCHING PRINCIPLE

Data: the asymptotic solution of a problem reads:

\[ g(x; \varepsilon) = g_0(x) + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \ldots \] (353)

in the outer region \((x\) represents the outer co-ordinate)

\[ F(X; \varepsilon) = F_0(X) + \varepsilon F_1(X) + \varepsilon^2 F_2(X) + \ldots \] (354)

in the inner region \((X = \frac{x}{\varepsilon}\) represents the inner co-ordinate)

Asked: to derive the matching conditions for each pair of subfunctions \(g_i(x)\) and \(F_j(X)\) from the asymptotic matching principle:

the m-term inner expansion of (the n-term outer expansion)

\[ = \] (181)

the n-term outer expansion of (the m-term inner expansion)

Algorithm: a. take the first \(n\) terms of the outer expansion and rewrite them in inner variables (introducing new \(\varepsilon\)'s)

b. expand this rewritten expression in a power series of \(\varepsilon\) and take the first \(m\) terms

c. take the first \(m\) terms of the inner expansion and rewrite them in outer variables (introducing new \(\varepsilon\)'s)

*) The symbols \(e, E, g\) and \(F\) merely represent functions. They do not correspond with any of the symbols used outside this appendix.
d. expand this rewritten expression in a power series of $\varepsilon$ and take the first $n$ terms

e. rewrite this expansion in inner variables, again
f. equate the results of b and e.

Expansion of a rewritten expression in a power series of $\varepsilon$.

Rewriting an expression $e(x)$ in inner variables yields: $e(\varepsilon X)$. This rewritten expression has to be expanded in a power series of $\varepsilon$, now. To this end the Taylor-expansion is used, for $X$ fixed and $\varepsilon$ variable:

$$e(\varepsilon X) = \lim_{\varepsilon \to 0} e + \varepsilon \lim_{\varepsilon \to 0} \frac{de}{d\varepsilon} + \frac{\varepsilon^2}{2!} \lim_{\varepsilon \to 0} \frac{d^2 e}{d\varepsilon^2} + \ldots \quad (355)$$

Since:

$$\frac{d^k e}{d\varepsilon^k} = (X^k \frac{d^k e}{d\varepsilon^k}) = X^k \frac{d^k e}{dx^k} \quad \text{for } k=0,1,2,\ldots \quad (356)$$

and $\lim_{\varepsilon \to 0}$ corresponds with $\lim_{x \to 0}$, this expansion reads:

$$e(\varepsilon X) = \lim_{x \to 0} e + \varepsilon \lim_{x \to 0} X \frac{de}{dx} + \varepsilon^2 \lim_{x \to 0} \frac{X^2}{2!} \frac{d^2 e}{dx^2} + \ldots \quad (357)$$

or:

$$e(x) = e(\varepsilon X) = \sum_{k=0}^{\infty} \varepsilon^k \lim_{x \to 0} \frac{X^k \frac{d^k e}{dx^k}}{k!} \quad X \text{ fixed} \quad (358)$$

Rewriting an expansion $E(X)$ in outer variables yields $E\left(\frac{X}{\varepsilon}\right)$. Expanding this rewritten expression in a power series of $\varepsilon$ is realized by means of the Taylor-expansion (for $x$ fixed and $\varepsilon$ variable):

$$E\left(\frac{X}{\varepsilon}\right) = \lim_{\varepsilon \to 0} E + \varepsilon \lim_{\varepsilon \to 0} \frac{dE}{d\varepsilon} + \varepsilon^2 \lim_{\varepsilon \to 0} \frac{d^2 E}{d\varepsilon^2} + \ldots \quad (359)$$

Since:

$$\frac{d^k E}{d\varepsilon^k} = \frac{(-\frac{X^2}{\varepsilon} \frac{d}{dx})^k}{d\varepsilon} = \left(-\frac{X^2}{\varepsilon} \frac{d}{dx}\right)^k E \quad (360)$$
and \( \lim \) corresponds with \( \lim \), this expansion reads:
\[
\frac{E(\varepsilon^{\frac{X}{x}})}{x + \varepsilon} = \lim_{X \to \infty} E + \varepsilon \lim_{X \to \infty} \left( -\frac{X^2}{2} \frac{dE}{dX} \right) + \varepsilon^2 \lim_{X \to \infty} \left( \frac{X^4}{2x^2} \frac{d^2E}{dx^2} + \frac{X^3}{2x^2} \frac{dE}{dx} \right) + \ldots (361)
\]
\[
x \text{ fixed} \quad x \text{ fixed}
\]

or:
\[
E(X) = E(\varepsilon^{\frac{X}{x}}) = \sum_{k=0}^{\infty} \varepsilon^k \lim_{X \to \infty} \left( \frac{1}{k!} \left( -\frac{X^2}{2x} \frac{d}{dx} \right)^k E \right) \quad \text{(362)}
\]

Elaboration:

1. \( m = n = 1 \)

a. the first term of the outer expansion: \( g_0(x) \)
   rewritten in inner variables: \( g_0(\varepsilon x) \)

b. expansion in a power series of \( \varepsilon \):
\[
g_0(\varepsilon x) = \lim_{X \to \infty} g_0 + \varepsilon \lim_{X \to \infty} \left( X \frac{d g_0}{dx} \right) + \varepsilon^2 \lim_{X \to \infty} \left( \frac{X^2}{2X} \frac{d^2 g_0}{dx^2} \right) + \ldots . (363)
\]
\[
x \text{ fixed} \quad X \text{ fixed}
\]

the first term of this expansion: \( \lim_{X \to \infty} g_0 \)

b. the first term of the inner expansion: \( F_0(X) \)
   rewritten in outer variables: \( F_0(\frac{X}{x}) \)

d. expansion in a power series of \( \varepsilon \):
\[
F_0(\varepsilon x) = \lim_{X \to \infty} F_0 + \varepsilon \lim_{X \to \infty} \left( -\frac{X^2}{2x} \frac{dF_0}{dx} \right) + \varepsilon^2 \lim_{X \to \infty} \left( \frac{X^4}{2X^2} \frac{d^2 F_0}{dx^2} + \frac{X^3}{X^2} \frac{dF_0}{dx} \right) + \ldots . (364)
\]
\[
x \text{ fixed} \quad X \text{ fixed}
\]

the first term of this expansion: \( \lim_{X \to \infty} F_0 \)

e. rewritten in inner variables: \( \lim_{X \to \infty} F_0 \)

f. equating the results of b and e yields:
\[
\lim_{X \to \infty} g_0(x) = \lim_{X \to \infty} F_0(X) \quad \text{(365)}
\]
This agrees with the result of the so-called "limit matching principle" (see Van Dyke /4/, chapter V).

2. \( m = 1 ; n = 2 \)
   a. the first two terms of the outer expansion: \( g_0(x) + \varepsilon g_1(x) \)
      rewritten in inner variables: \( g_0(x) + \varepsilon g_1(\varepsilon x) \)
   b. expansion in a power series of \( \varepsilon \):

   \[
   g_0(\varepsilon x) + \varepsilon g_1(\varepsilon x) = \lim_{x \to 0} g_0 + \varepsilon \lim_{x \to 0} \left( x \frac{dg_0}{dx} \right) + \varepsilon^2 \lim_{x \to 0} \left( \frac{x^2}{2!} \frac{d^2g_0}{dx^2} \right) + \ldots
   \]

   \[
   + \lim_{x \to 0} \left( \frac{x}{X} g_1 \right) + \varepsilon \lim_{x \to 0} \left( x \frac{d}{dx} \left( \frac{x}{X} g_1 \right) \right) + \ldots
   \]

   the first term of this expansion: \( \lim_{x \to 0} g_0 + \frac{1}{X} \lim_{x \to 0} (x g_1) \)

   c. see 1c.
   d. see 1d.
   the first two terms of this expansion: \( \lim_{X \to \infty} F_0 + \frac{\varepsilon}{X} \lim_{X \to \infty} (-X^2 \frac{dF_0}{dx}) \)
   e. rewritten in inner variables: \( \lim_{X \to \infty} F_0 + \frac{1}{X} \lim_{X \to \infty} (-X^2 \frac{dF_0}{dx}) \)
   f. equating the results of b and e yields, regarding 365:

   \[
   \lim_{x \to 0} \{xg_1(x)\} = -\lim_{X \to \infty} \{X^2 \frac{dF_0}{dx}\} \quad (367)
   \]

3. \( m = 2 ; n = 1 \)
   a. see 1a.
   b. see 1b.
   the first two terms of this expansion: \( \lim_{x \to 0} g_0 + \varepsilon x \lim_{x \to 0} \left( \frac{dg_0}{dx} \right) \)
   c. the first two terms of the inner expansion: \( F_0(X) + \varepsilon F_1(X) \)
      rewritten in outer variables: \( F_0 \left( \frac{X}{\varepsilon} \right) + \varepsilon F_1 \left( \frac{X}{\varepsilon} \right) \)
d. expansion in a power series of $\varepsilon$:

$$F_0(\varepsilon) + \varepsilon F_1(\varepsilon) = \lim_{X \to \infty} F_0 + \varepsilon \lim_{X \to \infty} \left(- \frac{x^2}{x} \frac{dF_0}{dx}\right) + \ldots$$

$$x \text{ fixed}$$

$$+ \lim_{X \to \infty} \frac{x}{X} F_1 + \varepsilon \lim_{X \to \infty} \left(- \frac{x^2}{x} \frac{d}{dx} \left(\frac{x}{X} F_1\right)\right)$$

(368)

the first term of this expansion: $\lim_{X \to \infty} F_0 + x \lim_{X \to \infty} \frac{F_1}{X}$

f. rewritten in inner variables: $\lim_{X \to \infty} F_0 + \varepsilon X \lim_{X \to \infty} \left(\frac{1}{X}\right)$

f. equating the results of b and e yields, regarding 365:

$$\lim_{x \to 0} \frac{d\theta_0}{dx} = \lim_{X \to \infty} \frac{1}{X} F_1(\lambda)$$

(369)

4. $m = n = 2$

a. see 2a.

b. see 2b.

the first two terms of this expansion:

$$\lim_{x \to 0} g_0 + \varepsilon x \lim_{x \to 0} \frac{d\theta_0}{dx} + \frac{1}{x} \lim_{x \to 0} (x g_1) + \varepsilon \lim_{x \to 0} \left(\frac{d}{dx} (xg_1)\right)$$

c. see 3c.

d. see 3d.

the first two terms of this expansion:

$$\lim_{X \to \infty} F_0 + \frac{x}{x} \lim_{X \to \infty} \left(- x^2 \frac{dF_0}{dx}\right) + \varepsilon \lim_{X \to \infty} \left(\frac{1}{X}\right) + \varepsilon \lim_{X \to \infty} \left(- x^2 \frac{d}{dx} \left(\frac{1}{X}\right)\right)$$
e. rewritten in inner variables:

\[ \lim_{X \to \infty} F_0 + \frac{1}{X} \lim_{X \to \infty} (-X^2 \frac{dF_0}{dX}) + \epsilon X \lim_{X \to \infty} \left( \frac{F_1}{X} \right) + \epsilon \lim_{X \to \infty} \left( -X^2 \frac{d}{dX} \left( \frac{F_1}{X} \right) \right) \]

f. equating the results of b and e yields, regarding 365, 367 and 369:

\[ \lim_{x \to 0} \left( \frac{d}{dx} (xg_1) \right) = \lim_{X \to \infty} \left( -X^2 \frac{d}{dX} \left( \frac{F_1}{X} \right) \right) \]  
(370)

Proceeding this way yields:

5. \( m = 1 \); \( n = 3 \)

\[ \lim_{x \to 0} \{x^2 g_2(x)\} = \lim_{X \to \infty} \left\{ \frac{x^4 d^2F_0}{dx^2} + x^3 \frac{dF_0}{dx} \right\} \]  
(371)

6. \( m = 3 \); \( n = 1 \)

\[ \lim_{x \to 0} \left\{ \frac{1}{2^2} \frac{d^2g_0}{dx^2} \right\} = \lim_{X \to \infty} \left\{ \frac{F_1}{X^2} \right\} \]  
(372)

7. \( m = 1 \); \( n = 4 \)

\[ \lim_{x \to 0} \{x^3 g_3(x)\} = \lim_{X \to \infty} \left\{ -\frac{x^6 d^3F_0}{dx^3} - x^5 \frac{d^2F_0}{dx^2} - x^4 \frac{dF_0}{dx} \right\} \]  
(373)

8. \( m = 2 \); \( n = 3 \)

\[ \lim_{x \to 0} \left( \frac{d}{dx} (x^2 g_2) \right) = \lim_{X \to \infty} \left\{ \frac{x^4 d^2F_1}{dx^2} + x^3 \frac{dF_1}{dx} \right\} \]  
(374)
9. \( m = 3 ; n = 2 \)

\[
\lim_{x \to 0} \left( \frac{1}{2!} \frac{d^2}{dx^2}(x g_1) \right) = \lim_{x \to \infty} \left( -x^2 \frac{d^2}{dx^2} \frac{F_2}{x^2} \right)
\]

(375)

10. \( m = 4 ; n = 1 \)

\[
\lim_{x \to 0} \left( \frac{1}{3!} \frac{d^3 g_0}{dx^3} \right) = \lim_{x \to \infty} \left( \frac{1}{3} F_3(X) \right)
\]

(376)

etcetera.

Conclusion.

For the present case (viz. \( X = \frac{X}{\epsilon} \)), the asymptotic matching principle:

the \( m \)-term inner expansion of the \( n \)-term outer expansion

= 

the \( n \)-term outer expansion of the \( m \)-term inner expansion

(181)

can be translated into the conditions:

\[
\lim_{x \to 0} \frac{1}{k!} \frac{d^k}{dx^k} \{ x^{\frac{k}{2}} g_k(x) \} = \lim_{x \to \infty} \left( -x^2 \frac{d^k}{dx^k} \right) \frac{F_k(X)}{x^k}
\]

(377)

in which \( k \) and \( \ell \) correspond with \( m-1 \) and \( n-1 \), respectively.
APPENDIX II

COMPOSITION RULES

Be \( f^{(n)} \) the \( n \)-term outer expansion of a function and \( F^{(m)} \) the \( m \)-term inner expansion of that function, such that:

\[
f^{(n)}(x; \varepsilon) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \ldots + \varepsilon^{n-1} f_{n-1}(x) \quad (378)
\]

\[
F^{(m)}(X; \varepsilon) = F_0(X) + \varepsilon F_1(X) + \varepsilon^2 F_2(X) + \ldots + \varepsilon^{m-1} F_{m-1}(X) \quad (379)
\]

where \( X = \frac{x}{\varepsilon} \). According to appendix I, the \( m \)-term inner expansion of the \( n \)-term outer expansion is then represented by:

\[
\{f^{(n)}\}^{(m)} = \lim_{x \to 0} f_0 + \ldots + \varepsilon^{m-1} \frac{x^{m-1}}{(m-1)!} \lim_{x \to 0} \frac{\partial^{m-1} f_0}{\partial x^{m-1}} + \frac{1}{\varepsilon} \lim_{x \to 0} x f_1 + \ldots + \varepsilon^{m-1} \frac{x^{m-2}}{(m-1)!} \lim_{x \to 0} \frac{\partial^{m-1} (xf_1)}{\partial x^{m-1}} + \frac{1}{\varepsilon^2} \lim_{x \to 0} x^2 f_2 + \ldots + \varepsilon^{m-1} \frac{x^{m-3}}{(m-1)!} \lim_{x \to 0} \frac{\partial^{m-1} (x^2 f_2)}{\partial x^{m-1}} + \ldots + \frac{1}{\varepsilon^{n-1}} \lim_{x \to 0} x^{n-1} f_{n-1} + \ldots + \varepsilon^{m-1} \frac{x^{m-n}}{(m-1)!} \lim_{x \to 0} \frac{\partial^{m-1} (x^{n-1} f_{n-1})}{\partial x^{m-1}} \quad (380)
\]

The composition rules can be elaborated, now. Since the inner solution of the present problem includes a 2-term expansion (viz. \( U \)), while the outer solution includes a 3-term expansion (viz. \( u \)), this elaboration will be limited to \( m=1,2 \) and \( n=1,2,3 \).
A. The additive composition rule.

According to the additive composition rule, the composite solution \( f_a^{(m,n)} \) is defined by:

\[
f_a^{(m,n)} = f(n) + F(m) - \{f(n)\}(m)
\]

(246)

Elaboration:

1. \( m = n = 1 \)

\[
f_a^{(1,1)} = f_0 + F_0 - \lim_{x \to 0} f_0
\]

(381)

2. \( m = 1 ; n = 2 \)

\[
f_a^{(1,2)} = f_0 + \varepsilon f_1 + F_0 - \lim_{x \to 0} f_0 - \frac{1}{X} \lim_{x \to 0} x f_1
\]

(382)

3. \( m = 1 ; n = 3 \)

\[
f_a^{(1,3)} = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + F_0 - \lim_{x \to 0} f_0 - \frac{1}{X} \lim_{x \to 0} x f_1 - \frac{1}{X^2} \lim_{x \to 0} x^2 f_2
\]

(383)

4. \( m = 2 ; n = 1 \)

\[
f_a^{(2,1)} = f_0 + F_0 + \varepsilon F_1 - \lim_{x \to 0} f_0 - \varepsilon X \lim_{x \to 0} \frac{\partial f_0}{\partial x}
\]

(384)

5. \( m = 2 ; n = 2 \)

\[
f_a^{(2,2)} = f_0 + \varepsilon f_1 + F_0 + \varepsilon F_1 - \lim_{x \to 0} f_0 - \frac{1}{X} \lim_{x \to 0} x f_1
\]

\[
- \varepsilon X \lim_{x \to 0} \frac{\partial f_0}{\partial x} - \varepsilon \lim_{x \to 0} \frac{\partial}{\partial (xf_1)}
\]

(385)
6. \( m = 2 \); \( n = 3 \)

\[
\begin{align*}
    f_{c}^{(2,3)} &= f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + F_0 + \varepsilon F_1 \\
    &= \lim_{x \to 0} f_0 - \frac{1}{X} \lim_{x \to 0} x f_1 - \frac{1}{X^2} \lim_{x \to 0} x^2 f_2 \\
    &\quad - \varepsilon X \lim_{x \to 0} \left( \frac{3 f_0}{3X} - \varepsilon \lim_{x \to 0} \frac{3}{3X}(xf_1) - \frac{\varepsilon}{X} \lim_{x \to 0} \frac{3}{3X}(x^2 f_2) \right)
\end{align*}
\]

(386)

It can be proved that nowhere the o.o.m. of the error in the additively composite solution will be greater than the o.o.m. of the error in its components:

In the overlap domain:

\[
    f^{(n)} = f + O(\varepsilon^n)
\]

(387)

\[
    f^{(m)} = f + O(\varepsilon^m)
\]

(388)

\[
    \{f^{(n)}\}^{(m)} = f + O(\varepsilon^\min(m,n))
\]

(389)

So:

\[
    f_{c}^{(m,n)} = f + O(\varepsilon^n) + f + O(\varepsilon^m) - f + O(\varepsilon^\min(m,n))
\]

(390)

or:

\[
    f_{c}^{(m,n)} = f + O(\varepsilon^\min(m,n))
\]

(391)

Besides, approaching the inner region by taking the \( m \)-term inner expansion of \( f_{c}^{(m,n)} \), this solution changes into the inner solution.

So there:

\[
    f_{c}^{(m,n)} = f + O(\varepsilon^m)
\]

(392)
Approaching the outer region by taking the n-term outer expansion of \( f_{c}^{(m,n)} \), this solution changes into the outer solution. So there:

\[
f_{c_{a}}^{(m,n)} = f + O(\varepsilon^n)
\]  \( (393) \)

**B. The multiplicative composition rule.**

According to the multiplicative composition rule, the composite solution \( f_{c_{m}}^{(m,n)} \) is defined by:

\[
f_{c_{m}}^{(m,n)} = f_{m} \times \frac{f(n)}{\{f(n)\}_{m}(m)}
\]  \( (273) \)

**Elaboration:**

1. \( m = n = 1 \)

\[
f_{c_{m}}^{(1,1)} = f_{0} \times \frac{f_{0}}{\lim_{x \to 0} f_{0}}
\]  \( (394) \)

2. \( m = 1 ; n = 2 \)

\[
f_{c_{m}}^{(1,2)} = f_{0} \times \frac{f_{0} + \varepsilon f_{1}}{\lim_{x \to 0} f_{0} + 1} \lim_{x \to 0} \frac{xf_{1}}{x}
\]  \( (395) \)

3. \( m = 1 ; n = 3 \)

\[
f_{c_{m}}^{(1,3)} = f_{0} \times \frac{f_{0} + \varepsilon f_{1} + \varepsilon^{2} f_{2}}{\lim_{x \to 0} f_{0} + 1} \lim_{x \to 0} \frac{xf_{1}}{x} \lim_{x \to 0} \frac{x^{2} f_{2}}{x^{2}}
\]  \( (396) \)

4. \( m = 2 ; n = 1 \)

\[
f_{c_{m}}^{(2,1)} = (f_{0} + \varepsilon f_{1}) \times \frac{f_{0}}{\lim_{x \to 0} f_{0} + \varepsilon x \lim_{x \to 0} \frac{f_{0}^{3}}{x^{3}}}
\]  \( (397) \)
5. $m = 2 ; n = 2$

\[
f_{cm}^{(2,2)} = (F_0 + \varepsilon F_1) \times \frac{f_0 + \varepsilon f_1}{\lim_{x \to 0} f_0 + \frac{1}{X} \lim_{x \to 0} xf_1 + \varepsilon X \lim_{x \to 0} \frac{\partial f_0}{\partial x} + \varepsilon \lim_{x \to 0} \frac{\partial}{\partial x}(xf_1)}
\]

(398)

6. $m = 2 ; n = 3$

\[
f_{cm}^{(2,3)} = (F_0 + \varepsilon F_1) \times \frac{f_0 + \varepsilon f_1 + \varepsilon^2 f_2}{\{f(3)\}_{(2)}}
\]

(399)

in which:

\[
\{f(3)\}_{(2)} = \lim_{x \to 0} f_0 + \frac{1}{X} \lim_{x \to 0} xf_1 + \frac{1}{X^2} \lim_{x \to 0} x^2 f_2
\]

\[
= \varepsilon X \lim_{x \to 0} \frac{\partial f_0}{\partial x} + \varepsilon \lim_{x \to 0} \frac{\partial}{\partial x}(xf_1) + \frac{\varepsilon}{X} \lim_{x \to 0} \frac{\partial}{\partial x}(x^2 f_2)
\]

(400)

The o.o.m. of the error in the multiplicatively composite solution will not be greater than the o.o.m. of the error in its components:

In the overlap domain:

\[
f^{(n)} = f + O(\varepsilon^n)
\]

(387)

\[
p^{(m)} = f + O(\varepsilon^m)
\]

(388)

\[
\{f^{(n)}\}^{(m)} = f + O(\varepsilon^{\min(m,n)})
\]

(389)

Then:

\[
f_{cm}^{(m,n)} = \{f + O(\varepsilon^m)\} \times \frac{f + O(\varepsilon^n)}{f + O(\varepsilon^{\min(m,n)})}
\]

(401)
This can be written as:

\[ f_c^{(m,n)} = \{ f + O(\epsilon^m) \} x \{ f + O(\epsilon^n) \} x \frac{1}{f} x \{ 1 - \frac{O(\epsilon^{\min(m,n)})}{f} \} + \ldots \} \]

\[ = \{ f + O(\epsilon^m) \} x \{ 1 + \frac{O(\epsilon^n)}{f} + \frac{O(\epsilon^{\min(m,n)})}{f} \} + \text{higher order terms} \]

Then: \[ f_c^{(m,n)} = f + O(\epsilon^{\min(m,n)}) \] (402)

Approaching the inner region by taking the m-term inner expansion of \( f_c^{(m,n)} \), this solution changes into the inner solution. So there:

\[ f_c^{(m,n)} = f + O(\epsilon^m) \] (403)

Approaching the outer region by taking the n-term outer expansion of \( f_c^{(m,n)} \), this solution changes into the outer solution. So there:

\[ f_c^{(m,n)} = f + O(\epsilon^n) \] (404)
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LIST OF SYMBOLS

A \hspace{1cm} \text{coefficient of turbulent viscosity (eddy viscosity)}

Ar \hspace{1cm} \text{cross-sectional area}

b \hspace{1cm} \text{linear dimension of the cross-section, which is characteristic for the flow (symbolical). In a shallow channel } b = d.

B \hspace{1cm} \text{channel-width}

c(R,z) \hspace{1cm} \text{function, which characterizes the redistribution of the longitudinal velocity, caused by the bend}

d \hspace{1cm} \text{average depth of flow}

e \hspace{1cm} 2.71828...

f \hspace{1cm} \text{dependent variable (symbolical)}

f_0, f_1, etc. \hspace{1cm} \text{subfunctions of } f

\gamma_t ( \zeta ) f_m ( \zeta ) \hspace{1cm} \zeta \text{-functions in the central region (symbolical)}

f(n) \hspace{1cm} \text{n-term outer (regular) expansion of } f

F \hspace{1cm} \text{Ananyan's stream function of the secondary flow}

\tilde{F}(X,Z); \tilde{F}_m (X,Z) \hspace{1cm} X,Z \text{-functions in the side-wall regions (symbolical)}

F(m) \hspace{1cm} \text{m-term inner (singular) expansion of } f

\{f^{(n)} \} (m) \hspace{1cm} \text{m-term inner expansion of the n-term outer expansion of } f

\{F^{(m)} \} (n) \hspace{1cm} \text{n-term outer expansion of the m-term inner expansion of } f

\{f^{(n)} \} (m) \hspace{1cm} \text{composite function, built up from } f^{(n)}, F^{(m)} \text{ and } \{f^{(n)} \} (m)

Fr = \frac{v^2}{g'd} \hspace{1cm} \text{Froude number}

g \hspace{1cm} \text{normalized stream function of the secondary flow}

g' \hspace{1cm} \text{acceleration of gravity}

\gamma_0, \gamma_1, etc. \hspace{1cm} \text{subfunctions of } g

\gamma_0^* = \frac{a^3}{\Delta^2} g_0 \hspace{1cm} \text{in the entire cross-section as well as in the central region alone}

\gamma_0^*(II) \hspace{1cm} \text{according to chapter IIB (consideration of the entire cross-section)}

\gamma_0(\zeta) \hspace{1cm} \zeta \text{-function of } g_0 \text{ in the central region}

G \hspace{1cm} \text{normalized stream function of the secondary flow in the side-wall regions}

G_0, G_1, etc. \hspace{1cm} \text{subfunctions of } G

G_0^* = \frac{a^3}{\Delta^2} G_0 \hspace{1cm} \text{X,Z-function of } G_0

\gamma_0(X,Z) \hspace{1cm} \text{zero order composite subfunction of } g

\gamma_0^{(1,1)} \hspace{1cm} \text{zero order composite subfunction of } g

\gamma_0^{(2)} \hspace{1cm} \text{zero order composite subfunction of } g

\gamma_0^{(3)} \hspace{1cm} \text{zero order composite subfunction of } g
$h$ actual depth of flow

$h_b$ level of the channel-bed with respect to some reference-level

$i$ integer (symbolical)

$I_b$ longitudinal slope of the channel-bed

$j$ integer (symbolical)

$k$ integer (symbolical)

$l$ integer (symbolical)

$m$ integer (symbolical)

$n$ integer (symbolical)

$n_j$ integer (symbolical)

C.O.M. abbreviation of: order of magnitude

$O(...)$ symbol, indicating the order of magnitude

$p$ pressure

$p' = p + \rho g'z + \rho g'h_b$ "total" pressure

$ph = p'/\rho g'$ piezometric head

$Q$ discharge

$r = R/R_0$ normalized radial co-ordinate in the central region

$r_i = R_i/R_0$ normalized radius of curvature of the inner wall

$r_o = R_o/R_0$ normalized radius of curvature of the outer wall

$R$ radial co-ordinate in the cylindrical co-ordinate-system

$R_0$ radius of curvature of the central axis of the cross-section (= average of curvature)

$R_i$ radius of curvature of the inner wall

$R_o$ radius of curvature of the outer wall

$Re = \bar{V}d/\nu$ Reynolds number

$u = \nu_\phi/\bar{V}$ normalized tangential velocity-component

$u_0, u_1$, etc. subfunctions of $u$

$u_0 = \frac{a}{\Delta} u_0$ in the entire cross-section as well as in the central region alone

$u_{1,0}$ part of $u_1$, representing the effect of the longitudinal slope

$u_{1,1}$ part of $u_1$, representing the effect of the secondary flow

$u_{1,2}$ part of $u_1$, representing the effect of the "extra" friction-term

$u_1 = u_{1,0} + u_{1,1} + u_{1,2}$

$u_1^{*} = \frac{a}{\Delta^2} u_1$ in the entire cross-section

$u_{1,2}^{*} = \frac{a}{\Delta} u_{1,2}$ in the entire cross-section

$u_{1,1}^{**}$ equivalent of $u_{1,1}^{*}$, corresponding with $\epsilon u_{2}^{*}$ in the central region
\[ u^*_2 = \frac{\alpha^5}{\varepsilon^2 \Delta} u_2 \] in the central region

\[ \tilde{u}_0(\zeta) \] \( \zeta \)-function of \( u_0 \) in the central region

\[ \tilde{u}_2(\zeta) \] \( \zeta \)-function of \( u_2 \) in the central region

\[ \nu_0^{(\text{II})} \] \( u_0^* \) according to chapter II (consideration of the entire cross-section)

\[ \nu_{1,1}^{(\text{II})} \] \( u_{1,1}^* \) according to chapter II (consideration of the entire cross-section)

\[ \nu_{1,2}^{(\text{II})} \] \( u_{1,2}^* \) according to chapter II (consideration of the entire cross-section)

\[ \nu = \frac{v \phi}{V} \] normalized tangential velocity-component in the side-wall regions

\[ U_0, U_1, \text{etc.} \] subfunctions of \( U \)

\[ U_0 = \frac{\alpha}{\Delta} U_0 \] part of \( U_1 \), representing the effect of the longitudinal slope

\[ U_{1,0} \] part of \( U_1 \), representing the effect of the secondary flow

\[ U_{1,1} \] part of \( U_1 \), representing the effect of the "extra" friction-term

\[ U_1 = U_{1,0} + U_{1,1} + U_{1,2} \]

\[ U^*_1 = \frac{\alpha}{\varepsilon^2 \Delta} U_{1,0} \]

\[ U^*_{1,1} = \frac{\alpha^5}{\varepsilon^2 \Delta^3} U_{1,1} \]

\[ U^*_{1,2} = \frac{\alpha}{\varepsilon^2 \Delta} U_{1,2} \]

\[ \tilde{U}_0(X, Z) \] \( X,Z \)-function of \( U_0 \)

\[ \tilde{U}_{1,0}(X, Z) \] \( X,Z \)-function of \( U_{1,0} \)

\[ \tilde{U}_{1,1}(X, Z) \] \( X,Z \)-function of \( U_{1,1} \)

\[ \tilde{U}_{1,2}(X, Z) \] \( X,Z \)-function of \( U_{1,2} \)

\[ uc_0 = uc^{(1,1)} \] zero-order composite subfunction of \( u \)

\[ uc^*_0 = \frac{\alpha}{\Delta} uc_0 \]

\[ uc_1 = \frac{uc^{(2,3)} - uc^{(1,1)}}{\varepsilon_1} \] first order composite subfunction of \( u \)

\[ uc_{1,0} \] part of \( uc_1 \), representing the effect of the longitudinal slope

\[ uc_{1,1} \] part of \( uc_1 \), representing the effect of the secondary flow

\[ uc_{1,2} \] part of \( uc_1 \), representing the effect of the "extra" friction-term
\[ u_{c_1} = u_{c_1,0} + u_{c_1,1} + u_{c_1,2} \]
\[ u_{c_1,0} = \frac{a}{\varepsilon^2 \Delta} u_{c_1,0} \]
\[ u_{c_1,1} = \frac{a^5}{\varepsilon^2 \Delta^3} u_{c_1,1} \]
\[ u_{c_1,2} = \frac{a}{\varepsilon^2 \Delta} u_{c_1,2} \]
\[ v = \frac{R}{d} \frac{v_R}{\overline{V}} \]

normalized radial velocity-component

\[ v_0, v_1, \text{etc.} \]

subfunctions of \( v \)

\[ \frac{3}{\Delta^2} v_0 \]
in the entire cross-section as well as in the central region alone

\[ v_0^{*} \text{(II)} \]

\( v_0^* \) according to chapter IIB

\[ \tilde{v}_0(\zeta) \]
\( \zeta \)-function of \( v_0 \) in the central region

\( v_R \)
radial velocity-component

\( v_\phi \)
tangential (main, longitudinal) velocity-component

\( v_\phi^* \)
longitudinal velocity-component, if the channel would have been straight

\( v_z \)
vertical (axial) velocity-component

\[ \overline{V} = \frac{Q}{\Delta R} \]
average tangential velocity-component

\[ V = \frac{R}{d} \frac{v_R}{\overline{V}} \]
normalized radial velocity-component in the side-wall regions

\[ V_0, V_1, \text{etc.} \]
subfunctions of \( V \)

\[ \frac{3}{\Delta^2} V_0 \]

\[ \tilde{V}_0(X,Z) \]

\( X,Z \)-function of \( V_0 \)

\[ v_{c_0} = v_c(1,1) \]
zero order composite subfunction of \( v \)

\[ v_{c_0}^* = \frac{3}{\Delta^2} v_{c_0} \]

\[ w = \frac{R}{b} \frac{v_z}{\overline{V}} \]
normalized vertical velocity-component in the entire cross-section

\[ w = \frac{R}{d} (\frac{v_z}{\overline{V}})^2 \]
normalized vertical velocity-component in the central region

\[ w_0, w_1, \text{etc.} \]
subfunctions of \( w \)

\[ \frac{3}{\Delta^2} w_0 \]
in the entire cross-section as well as in the central region alone

\[ w_0^{**} \]
equivalent of \( w_0^* \) in the entire cross-section, corresponding with \( \varepsilon w_0^* \) in the central region
\( w_0(II) \quad v_0^* \) according to chapter IIB (consideration of the entire cross-section).

\( \bar{w}_0(\zeta) \) \( \bar{v}_0 \) -function of \( w_0 \) in the central region

\( W = \frac{R_0}{d} v_z/\sqrt{V} \) normalized vertical velocity-component in the side-wall regions

\( W_0, W_1, \text{ etc.} \) subfunctions of \( W \)

\( \chi = \frac{a^2}{\Delta^2} W_0 \)

\( W_0(XZ) \) \( X, Z \)-function of \( W_0 \)

\( w_c^* \) zero order composite subfunction of \( w \)

\( w_c = \frac{a^3}{\Delta^2} w_c^* \)

\( X = \left| \frac{R-R_{wall}}{d} \right| \) normalized radial co-ordinate in the side-wall regions

\( = \frac{R-R_i}{d} \) in the inner wall region

\( = \frac{R_o-R}{d} \) in the outer wall region

\( z \) vertical (axial) co-ordinate in the cylindrical co-ordinate-system

\( Z = \frac{z}{b} \) normalized vertical (axial) co-ordinate in the side-wall regions

\( \alpha \) reciprocate of the Reynoldsnumber \( Re \)

\( \beta \) factor with the o.o.m. \( O(1) \) (symbolical)

\( \Delta = - \frac{3\pi}{3\Phi} \) "longitudinal slope" of \( \pi \)

\( \frac{\Delta \gamma}{R} = - \frac{1}{\rho R} \frac{3\rho}{3\Phi} \) longitudinal slope of the pressure-line

\( \varepsilon = \frac{b}{R_0} \) perturbation parameter \( (\ll 1) \)

\( \varepsilon' \) small parameter, characteristic for the flow (symbolical)

\( \varepsilon_1 = \frac{d}{B} \) aspect-ratio of the cross-section

\( \varepsilon_2 = \frac{B}{R_0} \) ratio of the channel-width and the average radius of curvature

\( \zeta = \frac{z}{b} \) normalized vertical (axial) co-ordinate in the entire cross-section as well as in the central region alone

\( \eta = \frac{h}{d} \) normalized actual depth of flow

\( \eta_0, \eta_1, \text{ etc.} \) subfunctions of \( \eta \)

\( \eta^*_1 = \frac{\alpha^2}{\Delta^2} \frac{g'd}{\sqrt{c}} \frac{\eta_1}{\eta} \) 

\( \nu \) kinematic viscosity of the fluid
\[ \xi = \frac{R - R_0}{b} \]

normalized radial co-ordinate in the entire cross-section

\[ \pi = \frac{p'}{\rho V^2} \]

normalized total pressure

\[ \pi_0, \pi_1, \text{etc.} \]

subfunctions of \( \pi \)

\[ \pi_1 = \frac{\alpha^2}{\Delta^2} \pi_1 \]

\[ \Pi = \frac{p'}{\rho V^2} \]

normalized total pressure in the side-wall regions

\[ \Pi_0, \Pi_1, \text{etc.} \]

subfunctions of \( \Pi \)

\[ \Pi^* \]

average normalized total pressure at \( \phi = 0 \)

\[ \rho \]

density of the fluid

\[ \phi \]

polar angle in the cylindrical co-ordinate-system

\[ \phi' = \frac{R_0}{b} \phi \]

normalized polar angle in the entire cross-section as well as in the central region alone

\[ \phi = \frac{R_0}{d} \phi \]

normalized polar angle in the side-wall regions