Instability of the Flow over a Porous Surface: Linear Stability Analysis

MSc Thesis Report

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Instability of the flow over a porous surface

Linear stability analysis

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Dedicated to:

My parents
“The scientist is not a person who gives the right answers, he’s one who asks the right questions”

Claude Lévi-Strauss
Abstract

Direct numerical simulations of the fully developed turbulent flow through a porous channel by Breugem et al. (2006) show the appearance of short span-wise rollers at the fluid- porous interface along with the streaky structures of wall-bounded turbulence near the upper wall of the channel. The scope of the present research project is to investigate the instabilities in porous channel flow by means of local linear stability analysis in order to elucidate their physical origin and underlying mechanisms.

The numerical approach for the problem consists of framing an eigenvalue problem using primitive variable approach to model the porous channel flows. The Volume Averaged Navier-Stokes (VANS) equations are used for modeling the flow through a porous channel with a porosity distribution function varying along the wall-normal direction separating the homogeneous fluid region and the homogeneous porous region by an interface region with smooth transition of porosity. The linearized VANS equations are discretized using 4th order dispersion-relation preserving finite-differences schemes, in order to reduce the computational resources in terms of memory and time. The modal stability analysis is performed to identify the linearly unstable region based on the critical flow parameters (critical Reynolds number and critical wavenumber). Reynold-Orr analysis is carried out to identify the contribution of different terms of the perturbation kinetic energy in the flow. A scaling approach is done to identify and distinguish between the different physical mechanisms governing the instabilities in the flow at different ranges of porosity. Non-modal analysis is carried out to understand the mechanism responsible for optimal energy amplification in the sub-critical regime and the wavenumbers over which maximum energy growth takes place. Non-modal analysis for turbulent porous channel flow is also carried out to get better insight into the underlying mechanisms governing the instabilities in the flow.

It is seen that the 4th order dispersion relation preserving scheme is well able to capture the physics of the flow through porous channel. Initial analysis about the linearly unstable region shows that the Squire transformation holds for porous channel flows and hence the modal stability analysis is performed using two dimensional perturbations. The eigenspectrum and the shape of the eigenmode shows the evolution of viscid instability from fluid region at low values of porosity. At high values of porosity, the eigenmode shows strong flow reversal at the interface due to weakening of wall-blocking and wall induced viscous effects. Energy analysis shows the dominance of viscous dissipation at lower values of porosity while the porous dissipation becomes significant at higher values of porosity. The physical origin of the instability is identified by the scaling analysis. It is seen that at lower values of porosity, the instability stems from the viscous fluid region (TS instability) while the primary source of instability at higher values of porosity is a TS instability wave modified by the drag from porous region. Non-modal analysis shows the optimal mechanism of energy amplification over different ranges of porosity is the Lift-up effect due to the streaky structures. However the non-normal energy growth becomes insignificant at extremely high values of porosity. The turbulent flow analysis provides a better understanding of the perturbation kinetic energy gain as seen from the direct numerical
simulations. The optimal mechanism responsible for the energy growth is due to the streaky structures for different values of porosity. However at extremely high values of porosity there is energy gain due to the span-wise rollers as well but energy gain due to streaks still dominates.
Acknowledgements

This thesis was an enlightening journey, which was possible only because of the following persons who have contributed and supported in one way or another.

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Nomenclature

Abbreviations

DNS  Direct Numerical Simulations
DRP  Dispersion Relation Preserving
FD   Finite Difference
FODRP Fourth Order Dispersion Relation Preserving
FOFD Fourth Order Finite Difference
GSF  Grid Stretching Function
KH   Kelvin-Helmholtz
LSA  Linear Stability Analysis
PKE  Perturbation Kinetic Energy
SOFD Second Order Finite Difference
TS   Tollmien-Schlichting
VANS Volume-Averaged Navier-Stokes

List of Symbols

\( \alpha \)  Stream-wise wavenumber
\( \alpha_c \)  Critical stream-wise wavenumber
\( \alpha_{opt} \)  Optimum stream-wise wavenumber
\( \beta \)   Span-wise wavenumber
\( \beta_{opt} \)  Optimum span-wise wavenumber
\( \delta \)  Vorticity thickness at the fluid-porous interface
\( \Delta \xi \)  Grid spacing size
\( \delta_i \)  Interface region thickness
\( \epsilon \)  Porosity distribution
\( \epsilon_c \)  Porosity

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<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$\lambda$</td>
<td>Eigenvalue or complex wave frequency</td>
</tr>
<tr>
<td>$\nu$</td>
<td>Kinematic viscosity of fluid</td>
</tr>
<tr>
<td>$\nu_T$</td>
<td>Total turbulent viscosity</td>
</tr>
<tr>
<td>$\nu_t$</td>
<td>Turbulent viscosity of fluid</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Imaginary part of eigenvalue or frequency</td>
</tr>
<tr>
<td>$\mathcal{U}$</td>
<td>Mean velocity field</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density of fluid</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Real part of eigenvalue or Growth rate</td>
</tr>
<tr>
<td>$\tilde{F}$</td>
<td>Forchheimer coefficient</td>
</tr>
<tr>
<td>$d_p$</td>
<td>Mean particle diameter</td>
</tr>
<tr>
<td>$Da$</td>
<td>Darcy number</td>
</tr>
<tr>
<td>$f$</td>
<td>External forcing frequency</td>
</tr>
<tr>
<td>$Fo$</td>
<td>Forchheimer number</td>
</tr>
<tr>
<td>$H$</td>
<td>Height of fluid or porous region</td>
</tr>
<tr>
<td>$K$</td>
<td>Permeability</td>
</tr>
<tr>
<td>$k$</td>
<td>Effective wavenumber</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of grid points for domain discretization</td>
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<tr>
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<td>Perturbation pressure</td>
</tr>
<tr>
<td>$Re_b$</td>
<td>Bulk Reynolds number</td>
</tr>
<tr>
<td>$Re_c$</td>
<td>Critical Reynolds number</td>
</tr>
<tr>
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<td>Time</td>
</tr>
<tr>
<td>$u$</td>
<td>Perturbation velocity field</td>
</tr>
<tr>
<td>$U_b$</td>
<td>Bulk velocity</td>
</tr>
<tr>
<td>$U_{cp}$</td>
<td>Creep velocity in porous region</td>
</tr>
<tr>
<td>$U_s$</td>
<td>Slip velocity at fluid-porous interface</td>
</tr>
<tr>
<td>$x$</td>
<td>Stream-wise direction</td>
</tr>
<tr>
<td>$y$</td>
<td>Wall-normal direction</td>
</tr>
<tr>
<td>$z$</td>
<td>Span-wise direction</td>
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Chapter 1

Introduction

1.1 Report Outline

The report is organized into 6 main chapters followed by Appendices.

1. The first chapter “Introduction” provides a detailed background followed by the definition of the research problem.

2. The second chapter “Problem Formulation” gives an overview into the modeling characteristics of the porous medium in the channel followed by the steps for setting up of the linearized equations for performing the linear stability analysis. The working procedure for two main aspects of linear stability analysis has been discussed in this section namely the Modal and Non-modal analysis. Wherever required, sufficient theoretical explanation has been provided for better understanding of the reader.

3. The third chapter “Numerical Methods” provides the details about the various discretization schemes for grid generation followed by sufficient evidence in support of using a particular kind of discretization scheme over the others.

4. The fourth chapter “Laminar Porous Channel Flow” provides a detailed insight into the results of the linear stability analysis of laminar porous channel flow to identify the critical parameters and understand the physical mechanism of the instabilities originating in the flow.

5. The fifth chapter “Turbulent Porous Channel Flow” provides a detailed insight into the results of the non-modal stability analysis for the turbulent porous channel flow.

6. The sixth chapter “Conclusions and Recommendations” provides a summary of the analytical results followed by recommendations for future work that will be performed.

7. The chapter “Appendices” provides brief details about the mathematical aspects of the problem and the validation of results performed for different flow cases before proceeding into our actual research problem.

1.2 Background

Flows over porous media are encountered across diverse natural and industrial processes such as porous river beds, transpiration, water filtration, catalytic bed reactors, ground oil wells, geological flows etc. The specific motivating applications of the porous media
flows that have been studied are, crossflow filtration in tubular membranes by Damak et al. (2004), convective/transpiration cooling in porous gas turbine blades by Glass et al. (2001), fluid flow over flexible and rigid plant canopies and monami by Ghisalberti and Nepf (2006) and Grizzle et al. (1996), alteration of near-wall turbulence by wall suction by Chung et al. (2002) and biological flow through lungs and kidneys by Miguel (2011) and Liao et al. (2003). Due to the widespread applications of porous media flows, it has become indispensable to study their detailed physical mechanism in order to get a better understanding of the flow phenomenon.

Although a lot of research has already been done in complete modeling of the flow through porous media using different approaches, there has been discrepancies in the transport equations governing the flow as shown by Whitaker (1996), Giorgi (1997) and Lage (1998). The starting point for the empirical relations for modeling flow through porous media was provided by Darcy(1856) but with specific restriction on the range of validity of the relation. The empirical relation was not in good agreement with the experimental results by Wright (1968) and Dybbs and Edwards (1984). Giorgi (1997) came up with the modeling equations for flows through rigid porous media using the matching asymptotic conditions (Oseen approximation) considering the non-linear Forchheimer term along with the Darcy term. The volume-averaged Navier-Stokes equations were first derived by Whitaker (1996) to couple the flow in the porous region to an adjacent channel region. This approach was followed by Tilton and Cortezezzi (2006) to model the flow in the fluid region of the channel using the incompressible Navier-Stokes equations and the flow in the porous region using the volume averaged Navier-Stokes equation as derived by Whitaker (1996) and then coupled them using the continuity of stress conditions at the fluid-porous interface (direct approach). This direct approach also required closure models for the Darcy and Forchheimer drag. Direct numerical simulation results by Breugem and Boersma (2005) provided a good comparison of results between the direct and continuum approach. In the continuum approach, the VANS equations are derived for the entire flow domain where the porosity effects are considered in terms of a porosity distribution function varying smoothly at the fluid-porous interface as shown by Breugem et al. (2006) for turbulent flows.

The porous layer has a profound effect on destabilizing the flow at the fluid-porous interface. Experiments performed by Sparrow et al. (1973) showed for the first time that the transition to turbulence occurs at lower Reynolds number with porous walls as compared to solid walls. They followed a two-domain approach with separate modeling of fluid region and porous region (Darcy’s law) and using the interface matching conditions as proposed by Beavers and Joseph (1967). A detailed modal linear stability analysis for channel surrounded with two adjoining porous walls was investigated by Tilton and Cortezezzi (2006) using the transport equations modeled separately for the fluid and porous regions as derived by Whitaker (1996), with matching stress conditions at the interface in terms of interface coefficient as shown by Ochoa-Tapia and Whitaker (1995). It was found from the work of Tilton and Cortezezzi (2006) that small increment in values of wall permeability can significantly decrease the stability of the channel flow as compared to Poiseuille flow. One of the major assumptions in the linear stability analysis of porous channel by Tilton and Cortezezzi (2006) was the negligence of the inertial effects in the porous regions on account of small magnitude of maximum fluid velocity in the porous layer as compared to the mean velocity in the fluid region. The stability analysis by Tilton and Cortezezzi (2006) considered same permeability values for both the porous walls with fixed porosity,
height of porous layers, interface coefficient. Another work in modal linear stability analysis of Poiseuille flow in a channel with a lower porous layer was performed by Chang et al. (2006). They also considered a two domain approach with Navier-Stokes equation for modeling the Poiseuille flow in channel and Darcy flow equations for modeling the porous layer. The interface conditions used for coupling the channel flow with porous layer used by Chang et al. (2006) was following the approach of Beavers and Joseph (1967) and Jones (1973). Although the interface conditions considered by Chang et al. (2006) were valid for negligible inertial effects in the porous layer but the analysis was done without the restriction of this assumption. The work on modal stability analysis for channel flow surrounded by two porous walls was further extended for specific cases of porous materials like foametal and aloxite by Tilton and Cortelezzi (2008) to see the characteristics of most unstable modes with the introduction of wall permeability as compared to Poiseuille flow and the effect of varying permeability, porosity, height of the porous region and interface coefficient on the stability of the flow. The linear stability analysis for channel with permeable walls performed by Tilton and Cortelezzi (2008) was the first detailed work in this research area. A similar study of instability of Poiseuille flow in a channel over a porous medium was performed by Hill and Straughan (2008), where the porous layer was modeled using Darcy’s law with an intermediate Brinkman porous transition layer adjoining the channel region. In the work of Hill and Straughan (2008), the Brinkman transition layer was modeled using the Brinkman model and the Darcy layer was modeled using Darcy’s law for porous media while the continuity between the adjacent channel and the Brinkman layer, and the Brinkman layer and the Darcy layer were achieved using the continuity of normal stresses. This work was further extended by Hill (2009), where the only difference as compared to the previous work was the consideration of the same approach of three layer configuration but with variable effective viscosity between the upper fluid region and the lower homogeneous porous region. Following the work of Tilton and Cortelezzi (2006) and Tilton and Cortelezzi (2008), a linear stability analysis of a porous layer surrounded by channel on either sides was provided by Li et al. (2014). They consider the same modeling approach as that of Tilton and Cortelezzi (2008) but observed the instability of the flow for varying values of the porous filling ratio with a fixed value of permeability. The main outcome of the work of Li et al. (2014) was that the configuration of a porous layer surrounded by channels was more stable as compared to the configuration of channel surrounded by two porous walls.

From the earlier work of Tilton and Cortelezzi (2008), it has been seen that the stability of a channel with porous walls is governed by different competing physical mechanism at extremely high values of permeability. In the works of Hill and Straughan (2008) and Hill (2009), it has been observed that two modes of instability exist in channel flow with porous walls corresponding to the fluid and the porous regions respectively. Even though the thorough literature studies provide an insight into the existence of different instability in the porous media flows, nothing much has been discussed about their physical mechanism and origin in the flow. A first classical study to classify the two different competing modes in flow through submerged seagrass bed (monami), similar to a porous media flow with an overlying open channel was provided by Singh et al. (2016). From the linear stability analysis, Singh et al. (2016) found two different modes of instability in flow through seagrass bed and classified one of them as Kelvin-Helmholtz instability mode modified by the vegetation drag and the other mode as an instability mode unrelated to Kelvin-Helmholtz mode, which originates from an interaction of the flow between the vegetated and unvegetated regions (also seen from experiments). They identify the instability modes
in the flow through a scaling approach and by comparing specific characteristics of the modes with that of the Tollmien-Schlichting and Kelvin-Helmholtz modes.

In this thesis work, the VANS equations used for modeling the porous channel flow were derived following the work of Whitaker (1996) and the porosity distribution function was considered from the work of Breugem et al. (2006). This approach avoided the specification of the interface conditions at the fluid-porous interface and considered a single domain approach, with the porosity distribution function clearly distinguishing the homogeneous fluid region from the homogeneous porous region. The linear stability analysis was performed using primitive variable approach and the linearized VANS equations were discretized using 4th order dispersion-relation preserving finite-differences schemes, in order to reduce the computational resources in terms of memory and time. The modal stability analysis was initially performed to see the evolution of the instabilities in the linearly unstable region and the results were verified against the results of Tilton and Cortelezzi (2008). Based on the eigenspectrum and eigenmode analysis, the Reynolds-Orr analysis was performed to see the evolution of the perturbation kinetic energy budget corresponding to different terms in the energy equation. As there is no earlier work that has been performed for energy analysis of porous channel flows using linear stability analysis, the results were compared with the turbulent kinetic energy budget from Breugem et al. (2006) to observe the energy growth trend with rising levels of porosity. The energy analysis gave a detailed idea about the driving mechanism of the instabilities in the flow. It was seen that with higher values of porosity, the tangential and the normal velocity of the unstable modes at the fluid-porous interface assumed a non-zero value and the instabilities were more pronounced at the interface. In order to identify the origin of these instabilities in the flow, a similar scaling approach like Singh et al. (2016) was performed. The scaling results were in good agreement with the results of Singh et al. (2016) at both lower and higher values of porosity. However it has been seen that in the case of a simple wall-bounded channel flow, the flow undergoes transition to turbulence at much lower Reynolds number as predicted from that of the classic linear stability analysis as shown by Schmid and Henningson (2001). Following the approach of Schmid (2007) for Non-modal stability analysis, two different mechanisms responsible for perturbation energy amplification in the linearly stable region were analyzed. The transient growth analysis and the resolvent norm analysis gave a good insight into the range of the wavenumbers over which these non-normal growth of perturbation kinetic energy takes place.

A better insight into the instability mechanisms in porous channel flows would be obtained with the direct numerical simulation and experiments of the porous channel flows. From the literature studies, a detailed direct numerical simulation of the turbulent flow over a porous region was performed by Breugem et al. (2006) considering the modeling approach of Whitaker (1996) and modeling the porous region with a packed bed of spheres of characteristic mean particle diameter. In the last part of the thesis, a resolvent norm analysis was performed using the turbulent baseflow profiles from Breugem et al. (2006) to predict the wavenumbers and the physical mechanism of energy amplification in turbulent porous channel flows. Calculating the additional eddy viscosity in turbulent flows as compared to laminar flows is always a cumbersome process. A new approach for modeling turbulent flows in plane channels without calculating the turbulent viscosity was given by Zare et al. (2016), where the Reynolds stress arising from the Reynolds/ensemble averaging of the Navier-Stokes equations, was used as the external harmonic forcing frequency with the linearized NS equations for laminar channel flow. The results from this approach
were in good agreement with the results of Hwang and Cossu (2010), where they used an approximation formula to model the turbulent viscosity for the channel flow. Following the same approach of Zare et al. (2016), the resolvent norm analysis was performed in this thesis using two different approaches and they were in good agreement with each other. These results provided a good support in evidence of the validation of the approach of using the Reynolds stresses (from ensemble averaging of VANS equations) as an external forcing term in the resolvent analysis of porous channel flows as well.

1.3 Objectives

The study of porous channel flows has become essential to understand the detailed physical mechanism. Direct numerical simulations and experiments could prove to be useful in order to elucidate the complete flow features. However the simulations and experiments are very expensive in terms of the resources and time required. From literature survey, it has been seen that linear stability analysis is a very useful tool in identifying the origin of instabilities in the flow and understand the underlying physical mechanism. Therefore the linear stability analysis of porous channel flow is performed. The main objectives of this study are listed below.

1. Setting up the numerical code for modeling the flow through porous channel using VANS equations.

2. Performing the modal stability analysis to identify the critical parameters responsible for flow transition to turbulence.

3. Performing the perturbation energy analysis to see the energy distribution with varying values of porosity.

4. Performing a scaling analysis based on vorticity thickness at interface to identify the physical mechanism and origin of instabilities.

5. Performing the non-modal stability analysis to identify the optimal mechanism responsible for energy gain leading to early flow transition in the sub-critical regime.
Chapter 2

Problem Formulation

In this chapter the modeling of the laminar and turbulent porous channel flows using the Volume-Averaged Navier-Stokes (VANS) equations are presented followed by the discussion of the mathematical tools used to perform the modal and non-modal linear stability analysis as well as the energy analysis.

2.1 Porous Media Flow

The schematic of a porous channel is shown in figure 2.1, where $x$ is the stream-wise direction, $y$ is the wall-normal direction and $z$ is the span-wise direction.

The porous region is a multiphase layer and is modeled as a packed bed of solid spheres with a characteristic value of mean particle diameter. This mimics the solid phase providing obstruction to the flow, dispersed in the fluid phase through which flow takes place. The quantitative value of the fluid volume fraction is expressed in terms of porosity ($\epsilon_c$), which is defined as the ratio of the volume of void space available for flow to the total

Figure 2.1: Schematic diagram of the flow domain.

The porous region is a multiphase layer and is modeled as a packed bed of solid spheres with a characteristic value of mean particle diameter. This mimics the solid phase providing obstruction to the flow, dispersed in the fluid phase through which flow takes place. The quantitative value of the fluid volume fraction is expressed in terms of porosity ($\epsilon_c$), which is defined as the ratio of the volume of void space available for flow to the total
Problem Formulation

The flow domain consists of three different regions namely the homogeneous fluid domain, the interface region and the homogeneous porous domain. The homogeneous fluid region extends from \( y = 0 \) to \( y = H \) with an upper solid wall, where the porosity value is equal to 1. The interface region extends from \( y = -\delta_i \) to \( y = 0 \) and over this region the porosity distribution \( (\epsilon) \) varies smoothly from a constant value of 1 in the homogeneous fluid region to an uniform porosity value of \( \epsilon_c \) in the homogeneous porous region. The homogeneous porous region extends from \( y = -H \) to \( y = -\delta_i \) with a bottom solid wall, where the porosity value is constant and denoted by \( \epsilon_c \). This study is focused on analyzing the effect of the porous region on destabilizing the flow in the channel. Hence in order to neglect the appended effects due to the porosity, the porous layer considered is of the same dimension as that of the fluid region.

2.1.1 Volume Averaged Navier-Stokes

In the porous region, the flow takes place only through the void space which can be modeled by the Navier-Stokes equations (microscopic scale approach). However, in order to consider the effect of the drag by solid spheres on the flow in macroscopic scale approach, equations have to be derived considering the effect of the solid phase in the porous domain. It is practically not feasible to model the flow considering the effect of each of these solid spheres and hence the concept of averaging the flow equations over the volume of the porous medium is considered. The full Volume Averaged Navier-Stokes (VANS) equations for modeling the flow in porous channel using the concept of volume averaging were first derived by Whitaker (1996). Following the approach of Whitaker (1996) using the volume-averaging method, the VANS equations are derived. The detailed derivation for the VANS equations can be seen in Whitaker (1996). The VANS equations are based on the continuum approach where the flow in the porous region is treated as a continuum and coupled to the flow in the fluid region. The dimensionless VANS equations (continuity and momentum) governing the flow in the porous channel are given as,

\[
\nabla \cdot [\epsilon u] = 0 \\
\partial_t u = -\frac{1}{\epsilon} \nabla \cdot [\epsilon u \otimes u] - \nabla p + \frac{1}{Re_b} \nabla^2 u + \frac{1}{\epsilon Re_b} \nabla \epsilon \cdot \nabla u \\
- \frac{1}{Re_b} \frac{Fo}{Da} \epsilon |u| u + \frac{1}{Re_b} \left[ \frac{\nabla^2 \epsilon}{\epsilon} - \frac{\epsilon}{Da} \right] u
\]

where, \( u = u(y) \) is the fluid velocity in the fluid domain and \( u = (1/V_f) \int_V u(y) dV \) is the fluid velocity averaged over the porous region. The averaging volume used in the above expression corresponds to the volume with contribution of solid and fluid phases i.e. \( V = V_f + V_s \) (\( V_f \) is the fluid volume and \( V_s \) is the solid volume). \( \epsilon \) is the porosity distribution along the wall-normal direction of the flow domain. The value of \( \epsilon \) varies from a constant value \( \epsilon_c \) in the homogeneous porous region to 1 in the homogeneous fluid region. In order to maintain the continuity of \( \epsilon \) at the fluid-porous interface, the porosity distribution is modeled as a function of wall-normal distance in the interface region. The porosity distribution is taken from Breugem et al. (2006) and given as,

\[
\epsilon = \begin{cases} 
1, & 0 \leq y \leq H \\
-6(\epsilon_c - 1)(y/\delta_i)^5 - 15(\epsilon_c - 1)(y/\delta_i)^4 - 10(\epsilon_c - 1)(y/\delta_i)^3 + 1, & -\delta_i \leq y \leq 0 \\
\epsilon_c, & -H \leq y \leq -\delta_i 
\end{cases}
\]
The permeability $K$ is defined as:

$$K = \frac{d_p^2 \epsilon^3}{180(1-\epsilon)^2}$$  \hspace{1cm} (2.4)

The value of permeability ($K$) evolves from a constant value in the homogeneous porous region to infinity in the homogeneous fluid region.

The Forchheimer coefficient $\tilde{F}$ is defined as:

$$\tilde{F} = \frac{\epsilon}{100(1-\epsilon)} \frac{d_p}{\nu}$$  \hspace{1cm} (2.5)

In the case of laminar porous channel flow analysis, the bulk velocity ($U_b$) is averaged over the entire channel height while for the turbulent porous channel flow analysis, $U_b$ is averaged over the fluid region only (chosen by Breugem et al. (2006)). In both cases, $U_b$ is set to a constant value of 1. The bulk velocity ($U_b$), calculated for both the cases is shown below.

$$\frac{1}{2H} \int_{-H}^{H} U(y)dy = U_b \hspace{0.5cm} (Laminar) \hspace{1cm} \frac{1}{H} \int_{0}^{H} U(y)dy = U_b \hspace{0.5cm} (Turbulent)$$  \hspace{1cm} (2.6)

The mean particle diameter considered for modeling the porous domain is $d_p = 0.01H$ and the thickness of the interface between the fluid and the porous domain is considered as $\delta_i = 0.02H$ following the approach of Whitaker (1996).

The bulk Reynolds number $Re_b$ for the flow is defined as:

$$Re_b = \frac{U_bH}{\nu}$$  \hspace{1cm} (2.7)

The scaling for the Reynolds number is different for both cases of laminar and turbulent channel flow based on the scaling of the bulk velocity ($U_b$) mentioned above. The dimensionless Darcy number $Da$ is defined as:

$$Da = \frac{K}{H^2}$$  \hspace{1cm} (2.8)

Substituting the value of $d_p = 0.01H$ in above eqn.2.8, the expression for $Da$ is given as,

$$Da = \frac{(0.01)^2 \epsilon^3}{180(1-\epsilon)^2}$$  \hspace{1cm} (2.9)

The dimensionless Forchheimer number $Fo$ is defined as:

$$Fo = \tilde{F}U_b$$  \hspace{1cm} (2.10)

Substituting the value of $d_p = 0.01H$ and $Re_b = U_bH/\nu$ in above eqn.2.10, the expression for $Fo$ is given as,

$$Fo = \frac{0.01 \epsilon}{100(1-\epsilon)} Re_b$$  \hspace{1cm} (2.11)

The VANS equation 2.1 and 2.2 are normalized with the bulk velocity $U_b$ and the height of the fluid or porous region ($H$).
2.2 Linear Stability Analysis

The traditional local linear stability analysis is performed to determine the lowest possible critical parameters (critical Reynolds number, $Re_c$ and critical wavenumber, $\alpha_c$) above which the baseflow becomes unstable due to the susceptibility to infinitesimal perturbations, whose fate is determined by the time-asymptotic limit.

2.2.1 Linearized VANS equations

Laminar Porous Channel Flow

The continuity equation 2.1 can be written in Einstein convention as,

$$\frac{\partial}{\partial x_i} (\epsilon u_i) = 0 \quad (2.12)$$

The VANS momentum equation 2.2 can be written in Einstein convention as shown below,

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\epsilon} \frac{\partial}{\partial x_j} [\epsilon u_i \otimes u_j] - \frac{\partial p}{\partial x_i} + \frac{1}{Re_b} \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{1}{\epsilon Re_b} \frac{\partial \epsilon}{\partial x_j} \frac{\partial u_i}{\partial x_j}$$

$$- \frac{1}{Re_b Da} \epsilon |u| u_i + \frac{1}{Re_b} \left[ \frac{1}{\epsilon} \frac{\partial^2 \epsilon}{\partial x_j^2} - \frac{\epsilon}{Da} \right] u_i \quad (2.13)$$

For obtaining the linearized VANS equations, the flow variables are decomposed into the mean part and the infinitesimal perturbation part. The decomposition of the flow variables for the VANS equations are performed using the following relations.

$$u_i = \bar{U}_i + u'_i$$

$$p = \bar{P} + p' \quad (2.14)$$

Substituting the values of decomposed variables from equation 2.14 in equation 2.13 and subtracting the mean part of the flow from the resulting equation, neglecting the non-linear terms ($u'_i u'_i$) and replacing $u'_i$, $p'$ by $u_i$ and $p$, and simplifying, the following three-dimensional Cartesian equations are obtained,

$$\dot{u} = -\bar{U} \frac{\partial u}{\partial x} - v \frac{\partial \bar{U}}{\partial y} - \frac{\partial p}{\partial x} + \frac{1}{Re_b} \nabla^2 u + \frac{1}{\epsilon Re_b} \left( \frac{\partial \epsilon}{\partial y} \frac{\partial u}{\partial y} \right)$$

$$- \frac{1}{Re_b Da} 2 \epsilon |\bar{U}| u_i + \frac{1}{Re_b} \left[ \frac{\nabla^2 \epsilon}{\epsilon} - \frac{\epsilon}{Da} \right] u \quad (2.15)$$

$$\dot{v} = -\bar{U} \frac{\partial v}{\partial x} - \frac{\partial p}{\partial y} + \frac{1}{Re_b} \nabla^2 v + \frac{1}{\epsilon Re_b} \left( \frac{\partial \epsilon}{\partial y} \frac{\partial v}{\partial y} \right)$$

$$- \frac{1}{Re_b Da} \epsilon |\bar{U}| v_i + \frac{1}{Re_b} \left[ \frac{\nabla^2 \epsilon}{\epsilon} - \frac{\epsilon}{Da} \right] v \quad (2.16)$$

$$\dot{w} = -\bar{U} \frac{\partial w}{\partial x} - \frac{\partial p}{\partial z} + \frac{1}{Re_b} \nabla^2 w + \frac{1}{\epsilon Re_b} \left( \frac{\partial \epsilon}{\partial y} \frac{\partial w}{\partial y} \right)$$

$$- \frac{1}{Re_b Da} \epsilon |\bar{U}| w_i + \frac{1}{Re_b} \left[ \frac{\nabla^2 \epsilon}{\epsilon} - \frac{\epsilon}{Da} \right] w \quad (2.17)$$

In order to solve these linearized VANS equation 2.15 - 2.17, one can expand the wave-like perturbation velocity in terms of the normal modes given as,

$$u(x, y, z, t) = u(y) e^{(i\alpha x + i\beta z + \lambda t)} \quad (2.18)$$
where $\alpha$ is the real-valued stream-wise wavenumber, $\beta$ is the real-valued span-wise wavenumber and $\lambda$ is the complex wave frequency. The total wavenumber for a specific oblique wave is given by the relation $k^2 = \alpha^2 + \beta^2$. The consideration of a complex wave frequency and real wavenumbers is known as the temporal stability problem where the spatial structure of the wave-like perturbation remains unchanged and the amplitude of the wave grows or decays with time. The complex wave frequency ($\lambda$) can be decomposed into a real and an imaginary part as shown.

$$\lambda = \sigma + i\omega$$ (2.19)

The temporal stability problem in the above case is defined in such a way that the sign of the real part of the complex wave frequency dictates the presence of instability of a specific wave in the flow. A positive value of $\sigma$ represents that the perturbation will increase exponentially with time, responsible for causing instabilities in the flow.

Incorporating the normal mode ansatz from equation 2.18 into equation 2.15, 2.16 and 2.17, we get the following relations for the VANS equations,

$$\lambda u = -i\alpha\bar{U} - \nu\bar{U}' - i\alpha p + \frac{1}{Re_b}(D^2 - \alpha^2 - \beta^2)u + \frac{1}{\epsilon Re_b}(Du')$$

$$- \frac{F_o}{Re_b Da} 2\epsilon |\bar{U}| u + \frac{1}{Re_b} \left[ \nabla^2 \epsilon \epsilon - \frac{\epsilon}{Da} \right] u$$ (2.20)

$$\lambda v = -i\alpha\bar{U} - Dp + \frac{1}{Re_b}(D^2 - \alpha^2 - \beta^2)v + \frac{1}{\epsilon Re_b}(Dv')$$

$$- \frac{F_o}{Re_b Da} \epsilon |\bar{U}| v + \frac{1}{Re_b} \left[ \nabla^2 \epsilon \epsilon - \frac{\epsilon}{Da} \right] v$$ (2.21)

$$\lambda w = -i\alpha\bar{U} - i\beta p + \frac{1}{Re_b}(D^2 - \alpha^2 - \beta^2)w + \frac{1}{\epsilon Re_b}(Dw')$$

$$- \frac{F_o}{Re_b Da} \epsilon |\bar{U}| w + \frac{1}{Re_b} \left[ \nabla^2 \epsilon \epsilon - \frac{\epsilon}{Da} \right] w$$ (2.22)

where, $\epsilon' = \frac{\partial \epsilon}{\partial y}$, $\bar{U}' = \frac{\partial \bar{U}}{\partial y}$ and $D = \frac{\partial}{\partial y}$ are the short hand notation used to indicate the gradient of porosity, mean velocity and the differential operators, respectively.

**Turbulent Porous Channel Flow**

For the turbulent channel flow case, the same procedure is followed as in the case of laminar porous channel flow and the linearized VANS equation 2.20 - 2.22 are modified in terms of the turbulent viscosity and simplified as given below,

$$\lambda u = -i\alpha\bar{U} - \nu\bar{U}' - i\alpha p + \frac{1}{Re_b} \left( 1 + \frac{\nu_t}{\nu} \right) (D^2 - \alpha^2 - \beta^2)u + \frac{1}{\epsilon Re_b} \left( \frac{\nu_t}{\nu} \right)' (Du + i\alpha v)$$

$$+ \frac{1}{\epsilon Re_b} (Du') - \frac{1}{Re_b Da} 2\epsilon |\bar{U}| u + \frac{1}{Re_b} \left[ \nabla^2 \epsilon \epsilon - \frac{\epsilon}{Da} \right] u$$ (2.23)

$$\lambda v = -i\alpha\bar{U} - Dp + \frac{1}{Re_b} \left( 1 + \frac{\nu_t}{\nu} \right) (D^2 - \alpha^2 - \beta^2)v + \frac{1}{\epsilon Re_b} \left( \frac{\nu_t}{\nu} \right)' (2Dv)$$

$$+ \frac{1}{\epsilon Re_b} (Dv') - \frac{1}{Re_b Da} \epsilon |\bar{U}| v + \frac{1}{Re_b} \left[ \nabla^2 \epsilon \epsilon - \frac{\epsilon}{Da} \right] v$$ (2.24)
\( \lambda w = -i\omega \bar{U} - i\beta p + \frac{1}{Re_b} \left( 1 + \frac{\nu_t}{\nu} \right) \left( D^2 - \alpha^2 - \beta^2 \right) w + \frac{1}{Re_b} \left( \frac{\nu_t}{\nu} \right)' (Dw + i\beta v) \\
+ \frac{1}{\epsilon Re_b} (Dw') - \frac{1}{Re_b} \frac{Fo}{D_a} \epsilon |\bar{U}| w + \frac{1}{Re_b} \left[ \nabla^2 \epsilon - \frac{\epsilon}{D_a} \right] w \\
\) 

(2.25)

where, \( \left( \frac{\nu_t}{\nu} \right)' = \frac{\partial}{\partial y} \left( \frac{\nu_t}{\nu} \right) \), \( \epsilon' = \frac{\partial \epsilon}{\partial y} \) and \( D = \frac{\partial}{\partial y} \) are the short hand notations used to indicate the gradient of ratio of eddy viscosity to fluid viscosity, porosity and the differential operators, respectively. \( \nu + \nu_t = \nu_T \), is the total eddy viscosity of the fluid.

The detailed derivation of the linearized VANS equations for the turbulent porous channel flow is presented in Appendix A.1.2.

### 2.3 Modal Stability Analysis

The dynamics of infinitesimally small perturbations are governed by equation 2.32. In the asymptotic time limit \( (t \rightarrow \infty) \), their fate is solely dictated by the eigenspectrum of the linearized VANS equations. Such analysis is known as Modal or Normal mode stability analysis.

#### 2.3.1 Eigenvalue Problem

In this section, the method to formulate the eigenvalue problem is presented which is same for both the laminar and the turbulent porous channel flow case except for the change in the scaling of Reynolds number. The eigenvalue problem is framed using the primitive variable approach for the VANS equation. The linearized VANS equation as mentioned in equation 2.12 and 2.13 can be written in the matrix form as shown below,

\[
Du = 0 \\
\dot{u} = Au - Gp
\]  

(2.26a)  

(2.26b)

where \( D \) and \( G \) are the divergence and gradient operators given by,

\[
D = \left[ \frac{\partial}{\partial x} \frac{\partial}{\partial y} + \frac{\epsilon'}{\epsilon} \frac{\partial}{\partial z} \right] \quad \& \quad G = \left[ \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} \right]
\]

and \( \dot{u} \) represents the time derivative of velocity and \( A \) is the matrix containing the linearized coefficients operated on the velocity vector and is shown for the laminar and turbulent porous channel flow case in Appendices A.1.1 and A.1.2.

On application of the boundary conditions for the velocity at the walls given by no-slip condition, the modified form of equation 2.26(b) for the velocity is given as,

\[
B \dot{u} = BAu - BGp
\]

(2.27)

where, \( B \) is a matrix enforcing the Dirichlet boundary condition \( (u|_{y=\pm 1}) = 0 \)

Taking the divergence of the above equation 2.27 we arrive at the following equation given as,

\[
DB \dot{u} = DBAu - DBGp
\]

(2.28)
The LHS of the above equation 2.28 vanishes on account of the continuity equation 2.26a. Hence the following equality relation arises for the value of pressure given as,

\[ DBGp = DBAu \]  
\[ \Rightarrow p = (DBG)^{-1}DBAu \]

The above equation 2.29(a) is a Poisson equation in pressure in which the boundary conditions have already been incorporated.

Substituting the value of \( p \) from equation 2.29(b) in equation 2.27 and simplifying, we obtain the following equation,

\[ Bu = BAu - BG(DBG)^{-1}DBAu \]
\[ \Rightarrow B\dot{u} = [I - BG(DBG)^{-1}D]BAu \]  
\[ (2.30) \]

The above equation 2.30 can be arranged to form a modified linearized VANS equation in terms of velocity only as the unknown parameter as given below,

\[ \dot{u} = \tilde{A}u \]  
\[ (2.31) \]

where, \( \tilde{A} = B^{-1}[I - BG(DBG)^{-1}D]BA \) is the Jacobian of the linearized VANS equations projected onto a divergence-free vector space.

From equation 2.31, the modified eigenvalue can be framed as given below,

\[ \tilde{A}u = \lambda u \]
\[ (2.32) \]

where, \( \lambda (= \sigma + i\omega) \) represents the set of the eigenvalues and \( u \) represents the set of the eigenvectors corresponding to the eigenvalues respectively.

The stability of the baseflow \( \bar{U} \) is governed by the growth rate \( \sigma \):

\( \Rightarrow \) If \( \sigma < 0 \), the baseflow is locally stable.
\( \Rightarrow \) If \( \sigma > 0 \), the baseflow is locally unstable.

### 2.3.2 Neutral Curves

Neutral curves are drawn for the \( \alpha - Re \) plane which plots the least stable eigenvalue corresponding to each \( \alpha \) and \( Re \) for the porous channel flow. This helps to identify the critical values of Reynolds number \( (Re_c) \) and wavenumber \( (\alpha_c) \) for the flow. The region of the neutral curve corresponding to \( Re < Re_c \) is referred to as the stable region (growth rate of least stable eigenvalue is negative) and the region corresponding to \( Re > Re_c \) is referred to as the unstable region (growth rate of least stable eigenvalue is positive). \( Re = Re_c \) (growth rate of least stable eigenvalue is zero) is the boundary denoting the transition of the flow from stable to unstable region.

### 2.3.3 Reynolds-Orr Energy Analysis

The energy analysis provides another way of analyzing the instabilities in the flow and its underlying mechanism. The energy analysis of the flow helps to identify the perturbation kinetic energy budget terms, enhancing or quenching the instability. The energy analysis is performed by solving the Reynolds-Orr equation and is derived for the laminar porous
Problem Formulation

channel flow as specified below. In the finite difference discretized space, equation 2.15 - 2.17 can be written in the general Einstein convention as given below,

\[
\frac{\partial u_i}{\partial t} = -\frac{1}{\epsilon} \frac{\partial}{\partial x_j}(\epsilon U_i u_j + e u_i U_j) - \frac{p}{\Re} \frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{\epsilon \Re} \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_j} u_i \right) - \frac{1}{\Re \Da} \epsilon \left[ [\bar{U}] u_i + |[\bar{U}]| u_i \right] + \frac{1}{\Re} \left[ \frac{1}{\epsilon} \frac{\partial^2}{\partial x_j^2} - \frac{\epsilon}{\Da} \right] u_i
\]

(2.33)

The hydrodynamic perturbation kinetic energy is obtained by multiplying the complex conjugate of the perturbation velocity with the linearized VANS equation 2.33,

\[
u^* \frac{\partial u_i^*}{\partial t} = -\frac{u_i^*}{\epsilon} \frac{\partial}{\partial x_j}(\epsilon U_i u_j + e u_i U_j) - \frac{p}{\Re} \frac{\partial^2 u_i^*}{\partial x_j^2} + \frac{1}{\epsilon \Re} \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_j} u_i^* \right) - \frac{1}{\Re \Da} \epsilon \left[ [\bar{U}] u_i^* + |[\bar{U}]| u_i u_i^* \right] + \frac{1}{\Re} \left[ \frac{1}{\epsilon} \frac{\partial^2}{\partial x_j^2} - \frac{\epsilon}{\Da} \right] u_i u_i^* \]

(2.34)

Then taking the conjugate of the above equation 2.34 and applying the commutative property for the operations of conjugate derivative as \((\frac{\partial f}{\partial x})^* = \frac{\partial f^*}{\partial x}\) the following relation is obtained,

\[
u_i \frac{\partial u_i^*}{\partial t} = -\frac{u_i}{\epsilon} \frac{\partial}{\partial x_j}(\epsilon U_i u_j + e u_i U_j) - \frac{p}{\Re} \frac{\partial^2 u_i^*}{\partial x_j^2} + \frac{1}{\epsilon \Re} \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_j} u_i^* \right) - \frac{1}{\Re \Da} \epsilon \left[ [\bar{U}] u_i^* + |[\bar{U}]| u_i u_i^* \right] + \frac{1}{\Re} \left[ \frac{1}{\epsilon} \frac{\partial^2}{\partial x_j^2} - \frac{\epsilon}{\Da} \right] u_i^* u_i^* \]

(2.35)

Adding equation 2.34 and equation 2.35 and dividing by 2, the Reynolds-Orr equation for perturbation kinetic energy (PKE) is obtained as given by,

\[
\frac{\partial}{\partial t} \left( \frac{u_i u_i^*}{2} \right) = -\frac{1}{2\epsilon} \frac{u_i^*}{\epsilon} \frac{\partial}{\partial x_j}(\epsilon U_i u_j + e u_i U_j) + \frac{p}{\Re} \frac{\partial^2 u_i^*}{\partial x_j^2} + \frac{1}{\epsilon \Re} \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_j} u_i^* \right) - \frac{1}{2\Re \Da} \epsilon \left[ [\bar{U}] u_i^* + |[\bar{U}]| u_i u_i^* + |[\bar{U}]| u_i^* u_i + |[\bar{U}]| u_i u_i^* \right] + \frac{1}{2\Re} \left[ \frac{1}{\epsilon} \frac{\partial^2}{\partial x_j^2} - \frac{\epsilon}{\Da} \right] (u_i^* u_i + u_i u_i^*) \]

(2.36)

Integrating the above equation 2.36 by parts and using the continuity equation 2.1 to simplify the terms, the following equation is obtained,

\[
\frac{\partial e}{\partial t} = -\frac{\partial}{\partial x_j} \left( \left( \frac{u_i u_i^* U_j}{2} \right) + \left( \frac{u_i^* p + u_i p^*}{2} \delta_{ij} \right) \delta_{ij} + \frac{1}{\Re} \frac{\partial}{\partial x_j} \left( \frac{u_i u_i^*}{2} \right) \right) - \left( \frac{u_i u_j + u_i^* U_j}{2} \right) \delta_{ij} - \frac{1}{\Re} \left( \frac{\partial u_i}{\partial x_j} \right) \left( \frac{\partial u_i^*}{\partial x_j} \right) + \frac{1}{\epsilon \Re} \frac{\partial}{\partial x_j} \left( \frac{\partial}{\partial x_j} \left( \frac{u_i u_i^*}{2} \right) \right) \]

(2.37)

\[
- \frac{1}{\Re \Da} \epsilon \left( [\bar{U}] u_i u_i^* + |[\bar{U}]| u_i u_i^* \right) + \frac{1}{\Re} \left[ \frac{1}{\epsilon} \frac{\partial^2}{\partial x_j^2} - \frac{\epsilon}{\Da} \right] (u_i u_i^*) \]

where \(e(y,t) = \frac{u_i u_i^*}{2}\) is the perturbation kinetic energy.
The terms indicated in curly bracket in the above equation 2.37 correspond to the transport terms of the energy balance which do not produce nor dissipate the energy in the flow, rather just redistribute the energy within the flow domain. Inside the curly bracket, the first term represents the transport term due to perturbations (TRANS), second term represents the velocity-pressure-gradient term (VPG) and third term represents the viscous diffusion (VD) term respectively. The transport terms have no contribution to the energy budget on account of the consideration of no slip Dirichlet boundary condition at the walls ($u = 0$ at $y = \pm 1$).

Integrating the above equation 2.37 for the perturbation energy over the entire flow domain ($\Omega$) with $L_x = \frac{2\pi}{\alpha}$ and $L_z = \frac{2\pi}{\beta}$ being the length of the domain in x and z directions, the following time derivative of the perturbation kinetic energy in the (x,z) plane is obtained as given by,

$$\frac{1}{\Omega} \int_{\Omega} \frac{\partial (e)}{\partial t} d\Omega = \frac{1}{2L_x L_z} \int_{-1}^{1} \int_{0}^{L_z} \int_{0}^{L_x} \frac{\partial (e)}{\partial t} d\zeta dx dy$$

$$\Rightarrow \int_{-1}^{1} \partial \frac{e}{\partial t} dy = - \int_{-1}^{1} \left( \frac{u_i^* u_j + u_i u_j^*}{2} \right) \frac{\partial U_i}{\partial x_j} dy - \int_{-1}^{1} \frac{1}{Re_b} \left( \frac{\partial u_i}{\partial x_j} \right) \left( \frac{\partial u_i^*}{\partial x_j} \right) dy$$

$$- \int_{-1}^{1} \frac{1}{Re_b Da} (u_i u_i^*) dy - \int_{-1}^{1} \frac{1}{Re_b} \frac{\partial^2 \epsilon}{\partial x_j^2} (u_i u_i^*) dy + \int_{-1}^{1} \frac{1}{Re_b} \frac{\partial \epsilon}{\partial x_j} \frac{\partial}{\partial x_j} \left( \frac{u_i u_i^*}{2} \right) dy$$

$$\Rightarrow \lambda \hat{e} = WS + DIS + FOR + DAR + POR$$

where $WS = \int_{-1}^{1} \left( \frac{u_i^* u_j + u_i u_j^*}{2} \right) \frac{\partial U_i}{\partial x_j} dy$, $DIS = - \int_{-1}^{1} \frac{1}{Re_b} \left( \frac{\partial u_i}{\partial x_j} \right) \left( \frac{\partial u_i^*}{\partial x_j} \right) dy$

$FOR = - \int_{-1}^{1} \frac{1}{Re_b Da} (U_i |u_i u_i^*|) dy$, $DAR = \int_{-1}^{1} \frac{1}{Re_b} \frac{\partial \epsilon}{\partial x_j} \frac{\partial}{\partial x_j} \left( \frac{u_i u_i^*}{2} \right) dy$

$POR = \int_{-1}^{1} \frac{1}{Re_b} \frac{\partial^2 \epsilon}{\partial x_j^2} (u_i u_i^*) dy$

In equation 2.38, WS represents the production term by deformation work of the shear stress, DIS represents the viscous dissipation of PKE, FOR represents the Forchheimer drag from porous region, DAR represents the Darcy drag from the porous region, and POR represents the dissipation and transport of PKE due to local gradients of porosity. FOR, DAR and POR terms contribute to the PKE only in the porous region.

From equation 2.36, the Reynolds-Orr equation in the short hand linearized notation can be written as,

$$\frac{\partial (e)}{\partial t} = \frac{\partial}{\partial t} \left( \frac{u_i u_i^*}{2} \right) = \frac{u_i A u_i^*}{2} + \frac{u_i^* A u_i}{2}$$
Expanding equation 2.39 as wave-like perturbations in terms of the normal modes as shown in equation 2.18, we get the following notation of the energy equation given as,

\[ \frac{\partial (e)}{\partial t} = \frac{u_i u_i^*}{2} \frac{\partial}{\partial t} (e^{\lambda + \lambda^* t}) \]

\[ \Rightarrow \frac{\partial (e)}{\partial t} = (\lambda + \lambda^*) \frac{u_i u_i^*}{2} (e^{\lambda + \lambda^* t}) \]

\[ \Rightarrow \frac{\partial (e)}{\partial t} = 2 \sigma \frac{u_i u_i^*}{2} (\lambda = \sigma + i\omega) \quad (2.40) \]

Integrating above equation 2.40 on both the sides over the flow domain control volume (Ω) which is time invariant, we obtain,

\[ \frac{1}{\Omega} \int_{\Omega} \frac{\partial (e)}{\partial t} d\Omega = \frac{1}{\Omega} \int_{\Omega} u_i u_i^* d\Omega \]

\[ \Rightarrow \frac{1}{\Omega} \int_{\Omega} \frac{\partial (e)}{\partial t} d\Omega = \sigma \]

(2.41)

From the above equation 2.41, it is evident that the time derivative of the perturbation kinetic energy is equal to the real part of the eigenvalue \( \lambda \), which is the growth rate \( \sigma \) of the least stable eigenvalue.

### 2.4 Non-Modal Stability Analysis

In traditional linear stability analysis, the governing Navier-Stokes (VANS) equations are linearized to formulate an eigenvalue problem. This method determines the critical parameters \((Re_c, \alpha_c)\) above which the flow becomes unstable to infinitesimal perturbations thereby making the flow transition to the turbulence on account of positive growth rate of the eigenvalues. In energy stability analysis, the largest possible critical parameters are determined below which all perturbations decay in energy monotonically. The non-linear terms in the Navier-Stokes equations contribute to the conservation of energy. It is seen from previous experimental as well as numerical results that there is a substantial gain in the kinetic energy due to the perturbations during the flow transition phenomenon. These analyses attribute to the fact that the energy gain is a consequence of the linear mechanism withdrawing energy from the mean flow. The linear stability analysis determines only the unstable eigenmodes for Reynolds number above the critical Reynolds number \((Re_c)\). However it is seen in case of wall-bounded shear flows that there are discrepancies between the linear stability analysis, energy analysis and the experimental results to determine the critical Reynolds number \((Re_c)\) owing to the laminar-turbulent transition. In these cases, the flow becomes turbulent at much lower Reynolds number as compared to the predicted \(Re_c\) from the traditional theoretical linear stability analysis. In order to provide a justification for this energy gain during the transition phenomenon, the mechanism responsible for the perturbation kinetic energy growth has to be analyzed without depending on the unstable eigenmodes. It has been observed that many linear stability operators for wall-bounded shear flows are non-normal and as a consequence evolve with a set of non-orthogonal eigenmodes as seen from Schmid (2006). Analysis of these non-orthonormal eigenmodes is known as Non-modal or Non-normal stability analysis.
The maximum energy amplification corresponding to the initial disturbance imposed to the fluid velocity is measured as,

$$G(t) = \max_{u_0 \neq 0} \frac{E(u(t))}{E(u_0)}$$  \hspace{1cm} (2.42)

where $E(u(t))$ is the kinetic energy of the perturbation $u$.

In order to obtain the optimized energy gain over all permissible initial conditions, the energy norm of the perturbation kinetic energy can be expressed in terms of common $L^2$-norm of the perturbation vector. The relation between the energy norm ($|| \cdot ||_E$) and the Euclidean norm ($|| \cdot ||_2$) as shown in Schmid and Brandt (2014) is given below,

$$E(u(t)) = ||u(t)||_E^2 = \langle u, u \rangle_E = u^HQu = u^HF^HFu = \langle Fu, Fu \rangle_2 = ||Fu||_2^2$$  \hspace{1cm} (2.43)

where $Q$ is the energy weight matrix which incorporates the proper weighting of the perturbation velocity along the wall-normal direction. In order to maintain the positive definiteness of the energy weight matrix $Q$, it is decomposed as $Q = F^HF$ using Cholesky decomposition.

The relation between energy norm ($|| \cdot ||_E$) and $L^2$-norm for matrices as shown in Schmid and Brandt (2014) is given below,

$$||L||_E = \max_{u \neq 0} \frac{||Lu||_E}{||u||_E} = \max_{u \neq 0} \frac{||FAF^{-1}Fu||_2}{||Fu||_2} = ||FAF^{-1}||_2$$  \hspace{1cm} (2.44)

2.4.1 Transient Growth Analysis

In cases of wall-bounded flows, the optimal mechanism responsible for maximum energy amplification of perturbations is lift-up effect due to the transient growth of non-orthogonal eigenmodes over a finite time interval as shown by Brandt (2014). The energy gain during the transient growth can be several times larger than the initial perturbation energy leading to instability before the perturbation energy asymptotically decays to zero as predicted by the wave-like assumption above. The transient growth is an initial value problem which signifies the evolution of the system with time, based on the initial conditions. The initial value problem formulated for transient growth analysis is shown below,

$$\frac{\partial}{\partial t} u = Lu, \hspace{1cm} u(t = 0) = u_0$$  \hspace{1cm} (2.45)

where $u_0$ is the initial condition at time ($t = 0$).

The solution to the initial value problem in equation 2.45 is of the form,

$$u(t) = \exp(tL)u_0$$  \hspace{1cm} (2.46)

The energy growth depends on the initial condition and hence this problem can be optimized over all acceptable initial conditions and denoting the maximum value as the optimal energy growth. In the transient growth analysis an initial disturbance in form of perturbations is imposed on the fluid velocity and the response is recorded in terms of perturbation kinetic energy shown as,

$$E(u(t)) = \frac{1}{2} \int_{\Omega} u^HQud\Omega$$

$$= \frac{1}{2} \int_{\Omega} u^HF^HFud\Omega \hspace{1cm} \text{ (from equation 2.43)}$$  \hspace{1cm} (2.47)

$$= \frac{1}{2} \int_{\Omega} (Fu)^HFud\Omega$$
Following the same approach as shown in equation 2.42 and expressing the energy growth in $L^2$-norm as shown in equation 2.43, the transient growth is mathematically formulated as,

$$G(t) = \max_{u_0 \neq 0} \frac{||u(t)||^2_E}{||u_0||^2_E}$$

$$\Rightarrow G(t) = \max_{u_0 \neq 0} \frac{||\exp(tL)u_0||^2_E}{||u_0||^2_E}$$

$$\Rightarrow G(t) = ||\exp(tL)||^2_E$$  

(2.48)

Using the eigenvalue decomposition of the matrix $L = F\Lambda F^{-1}$ as shown in equation 2.44, the transient growth expression from equation 2.48 can be written as,

$$G(t) = ||F\exp(t\Lambda)F^{-1}||^2_E$$  

(2.49)

The final expression of transient growth as shown in equation 2.48 is calculated over a specific time interval (considering different time scales involved in flow) to determine the ability of the flow to optimally amplify perturbation energy corresponding to the initial conditions, in the linearly stable region.

### 2.4.2 Resolvent Norm Analysis

Another approach to measure the effect of non-normality of the eigenmodes on the energy amplification is Resolvent norm analysis. The approach is very similar to the case of transient growth but with an additional forcing term in the input-output system. This input-output analysis is considered into the system in terms of an external harmonic forcing and is mathematically expressed as,

$$\frac{\partial}{\partial t} u = L u + f$$

$$f = u_f \exp(-i\omega t)$$  

(2.50)

where $f$ is the external harmonic forcing term and the initial condition is obtained by setting $t = 0$, which gives $f(t = 0) = u_f$.

On expanding equation 2.50 in terms of the time exponential, a solution can be obtained to the differential equation and is of the form as shown below,

$$-i\omega u = Lu + u_f$$

$$\Rightarrow (-i\omega I - L)u = u_f$$

$$\Rightarrow u(t) = (-i\omega I - L)^{-1}u_f$$  

(2.51)

The energy amplification depends on the initial forcing condition and hence this problem can also be optimized over a range of initial forcing frequencies and denoting the maximum value as the optimal energy amplification. In a similar way like transient growth, the initial forcing in form of perturbations is imposed on fluid velocity and the response is measured as kinetic energy. Expressing the maximum energy amplification optimized over range of initial frequencies, the term for maximum resolvent norm is given as,

$$R(\omega) = \max_{u_f \neq 0} \frac{||u(t)||_E}{||u_f||_E}$$

$$\Rightarrow R(\omega) = \max_{u_f \neq 0} \frac{||(-i\omega I - L)^{-1}u_f||_E}{||u_f||_E}$$

$$\Rightarrow R(\omega) = ||(-i\omega I - L)^{-1}||_E$$  

(2.52)
The final expression of resolvent norm as shown in equation 2.52 measures the maximum output response due to the harmonic forcing optimized over all range of the initial forcings, in the linearly stable region.

Using eigenvalue decomposition of the matrix $\mathcal{L}$ as $\mathcal{L} = F\Lambda F^{-1}$ as shown in equation 2.44, the resolvent norm expression from equation 2.52 can be written as,

$$R(\omega) = \|F(-i\omega I - \Lambda)^{-1}F^{-1}\|_2$$  \hspace{1cm} (2.53)

The expression $(-i\omega I - \Lambda)^{-1}$ in equation 2.53 represents the resonant limit and is defined as the inverse distance of the external harmonic forcing frequency from the eigenvalues of the linear matrix.
Chapter 3

Numerical Methods

In this chapter the different schemes used for discretization of the grid for porous channel flows are discussed with respect to their accuracy. Also the approach used for grid stretching is presented.

3.1 Dispersion Relation Preserving Schemes

Dispersion relation preserving (DRP) schemes are finite difference (FD) schemes for which the optimized coefficients are evaluated in the Fourier space over a designated scale of wave numbers. Following the approach of Merle et al. (2010), the DRP scheme is employed in the given discretized flow domain. Considering the expansion of the first derivative of the flow variable $u$ along the wall-normal direction of the grid, we have the general form as,

$$
\left( \frac{\partial u}{\partial \xi} \right)_i = \frac{1}{\Delta \xi} \sum_{j=-m}^{m} a_j u_{i+j} = \frac{1}{\Delta \xi} \sum_{j=-m}^{m} a_j u(\xi + j\Delta \xi)
$$

(3.1)

The most preferred way of estimating the errors involved in equation 3.1 is in equivalence of dispersive errors. Fourier analysis is the most efficient method to enumerate the dispersion errors as well as the resolution features of the employed differencing approximation scheme. Taking the Fourier transform of the above equation 3.1 on both sides, we obtain the following relation given as,

$$
\tilde{u}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u e^{-ik\xi} d\xi
$$

(3.2)

Rearranging both sides of equation 3.2, we get the following equation,

$$
u = \int_{-\infty}^{\infty} \tilde{u}(k)e^{ik\xi} dk
$$

(3.3)

Substituting the value of $u$ from above equation 3.3 into equation 3.1, we get the following relation,

$$
i \zeta \tilde{u} = \left( \frac{1}{\Delta \xi} \sum_{j=-m}^{m} a_j e^{jk\Delta \xi} \right) \tilde{u}
$$

(3.4)

In the above equation 3.4, $\zeta = k\Delta \xi$ is the exact reduced wavenumber. Expressing the modified wavenumber of approximation as $\zeta^*$, we have,

$$
i \zeta^* = \sum_{j=-m}^{m} a_j e^{j\zeta^*}
$$

(3.5)
The modified wavenumber ($\zeta^*$) is a periodic function of the actual wavenumber ($\zeta$) with a period of $2\pi$. For the employed differencing scheme to be reasonable, the modified wave numbers should overlap the actual wavenumbers over a wide scale of wave numbers. The deviation of the modified wave numbers from the actual wavenumbers indicates the dispersive errors manifested in the finite differencing approximation scheme. For the Fourier transform to provide a good approximation to the exact first order derivative, the error difference between the actual and the modified wave numbers ($E$) is calculated using the integral equation 3.6(a) and then the minimization principle is used on $E$ in order to reduce the error with respect to the coefficients ($a_j$). The integrated error equation is a function of the variable coefficients ($a_j$) and hence the error minimization is performed with respect to the DRP coefficients as shown in equation 3.6(b).

$$E = \int_{-\pi}^{\pi} |\zeta - \zeta^*|^2 d\zeta = \int_{-\pi}^{\pi} \left| \zeta - (-i) \sum_{j=-m}^{m} a_j e^{ij\zeta} \right|^2 d\zeta \quad (3.6a)$$

$$\frac{\partial E}{\partial a_j} = 0, \quad j = 1, ..., m \quad (3.6b)$$

The order of accuracy considered for both the first and second order derivatives using DRP finite-difference scheme is 4th order thereby resulting in a seven point stencil. However the only exception is that for case of first order derivative, we consider $a_j = -a_{-j}$ while for second order derivative, we consider $a_j = a_{-j}$. The fourth order DRP (FODRP) coefficients are derived using the above conditions specified in equation 3.6(a) and (b) by forming the system of $m$ equations for the first and second order derivatives. The DRP coefficients are optimized globally by considering one of the coefficients as a free parameter. The detailed procedure of the derivation is presented in Appendix B.1.1.

The required 4th order DRP optimized coefficients for first and second order derivatives are calculated following the approach of Bauer et al. (2015) and mentioned in Appendices B.1.2 and B.1.3. The modified wavenumber expressions for the first and second order derivative for FODRP are presented in Appendix B.4.1. The required coefficients for first and second order derivative using fourth order finite-difference (FOFD) are presented in Appendices B.2.1 and B.2.2. The coefficients for first and second order derivative using second order finite-difference (SOFD) are presented in Appendices B.3.1 and B.3.2. The modified wavenumber expressions for FOFD and SODFD schemes are derived and presented in Appendices B.4.2 and B.4.3.

The modified wavenumbers are plotted and compared against the exact wavenumbers for the different schemes of discretization of the domain as shown in figure 3.1. The exact wavenumbers are obtained with the exact Chebyshev spectral scheme while the modified wavenumbers for SOFD, FOFD and FODRP schemes are obtained through Fourier transformation respectively. The modified wavenumber for FODRP scheme is in good approximation with the exact wavenumber over wide range of wavenumbers ($k \leq \pi/2$) as compared to the SOFD and FOFD schemes as seen from figure 3.1. The modified wavenumber using FODRP scheme shows congruence with the exact physical wavenumber in the range of $[0, \pi/2]$, after which the curve tends to deviate from the exact wavenumber representing the dispersive error of the discretization scheme. This curve shows that the FODRP scheme resembles the exact wavenumber with a minimum of 4 mesh grid points per wavelength while the FOFD scheme requires at least 10 mesh grid points per wavelength and the SOFD scheme requires at least 18 mesh grid points per wavelength.
This analysis concludes that the FODRP scheme gives a better resolution of the mesh grid with fewer number of points per wavelength as compared to the FOFD and SOFD schemes respectively.

![Graph showing wave numbers for different schemes](image)

**Figure 3.1**: Plot of modified wave numbers for FODRP scheme with comparison to other FD discretization schemes (SOFD, FOFD) and exact Chebyshev scheme in the interval $[0,\pi]$.

### 3.2 Grid Stretching

In the case of porous channel flows, the most significant gradients of the flow profile is found near the two walls of the channel and the interface separating the channel and the porous medium. A standard FD mesh splits the entire domain into equal spaced grid points while the points have to be clustered around the walls of the channel and at the interface. Therefore the grid has to be stretched in order to gather the grid points around the walls and the interface to increase the accuracy and efficiency of the analysis. The collocation of the grid points in the stretched domain as compared to the original domain is seen in figure 3.2.

![Comparison of two grids](image)

**Figure 3.2**: Comparison of two different grids. The blue one is the original DRP scheme grid and the red one is the stretched DRP scheme grid.
A suitable Grid stretching function (GSF) has to be defined in order to have almost zero gradient at the walls and near the interface region. Therefore the mapping function used for transforming the original grid into the stretched grid is a sigmoid function given as,

$$y = c(f_1 + f_2)$$

where,

$$f_1 = \frac{1}{1+e^{-7\xi +0.5}}, \quad f_2 = \frac{-1}{1+e^{-7(-\xi +0.5)}}, \quad \text{and} \quad c = \max[f_1] + \max[f_2]$$

In the above GSF, 'ξ' are the original DRP points while 'y' represents the stretched DRP points.

The mapping function used for the transformation of the original grid into the stretched domain is shown in figure 3.3. The horizontal coordinate represents the original DRP points on [-1,1] and the vertical coordinate represent the stretched DRP points, which get collocated near the origin and the end points. It is seen that to fulfill the requirement of clustering of points i.e. near the walls and the fluid-porous interface, the function is such chosen that the slope of the GSF is extremely less at these places within the domain. Away from the wall and the interface, the slope of GSF increases significantly in order to achieve large gradients so that the curve becomes steeper and the grid points are placed far apart.

![Figure 3.3: Mapping between two grid points distribution. The abscissa represents the DRP grid points on [-1,1]. The ordinate represents the stretched DRP grid points on [-1,1].](image)

With the change in the distribution of the discretized points in the domain, the derivative matrices need to be modified. The first order derivative in the stretched domain is modified as,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} \frac{\partial}{\partial y}$$

(3.8)

The second order derivative in the stretched domain is modified as,

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial \xi} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial \xi^2} \left( \frac{\partial}{\partial y} \right)^2$$

(3.9)

where, $\frac{\partial}{\partial \xi}$ and $\frac{\partial^2}{\partial \xi^2}$ represent the first and the second order derivatives in the original DRP discretized domain along the wall-normal direction.
3.3 Validation

In this section, the convergence of the eigenvalue for different number of grid points used for discretizing the domain are presented.

Table 3.1: Convergence of the least stable eigenvalue using different number of grid points for discretizing the domain is shown ($Re_b$, $\epsilon_c$, $\alpha$) with $\beta = 0$ & $\delta_i = 0.02$ for both the cases.

<table>
<thead>
<tr>
<th>Number of points (N)</th>
<th>(4000,0.3,2)</th>
<th>(2000,0.95,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>33</td>
<td>0.00878085-1.55014172i</td>
<td>0.08517696 - 1.28807185i</td>
</tr>
<tr>
<td>65</td>
<td>0.00120113-1.55597723i</td>
<td>0.08999718 - 1.28817582i</td>
</tr>
<tr>
<td>129</td>
<td>0.00388665-1.55160105i</td>
<td>0.08965474 - 1.28675826i</td>
</tr>
<tr>
<td>193</td>
<td>0.00376749-1.55129387i</td>
<td>0.08957961 - 1.28639585i</td>
</tr>
<tr>
<td>257</td>
<td>0.00374423-1.55127121i</td>
<td>0.08955174 - 1.28638445i</td>
</tr>
<tr>
<td>321</td>
<td>0.00373853-1.55126331i</td>
<td>0.08953846 - 1.28636223i</td>
</tr>
<tr>
<td>385</td>
<td>0.00373598-1.55123185i</td>
<td>0.08953122 - 1.28633933i</td>
</tr>
<tr>
<td>449</td>
<td>0.00373461-1.55121324i</td>
<td>0.08952686 - 1.28632553i</td>
</tr>
<tr>
<td>513</td>
<td>0.00373380-1.55120128i</td>
<td>0.08952401 - 1.28631658i</td>
</tr>
<tr>
<td>577</td>
<td>0.00373328-1.55121321i</td>
<td>0.08952020 - 1.28631044i</td>
</tr>
<tr>
<td>641</td>
<td>0.00373292-1.55123733i</td>
<td>0.08952064 - 1.28630606i</td>
</tr>
</tbody>
</table>

As seen from table 3.1 that the least unstable eigenvalue converges up to four decimal places with N=193 as the minimum number of points used for mesh discretization. The convergence is achieved until four decimal places for both the real as well as the imaginary part of the least stable eigenvalue for both the cases of porosity, $\epsilon_c = 0.3, 0.95$ respectively.

To evaluate the optimum number of grid points (N) for accurate resolution of the eigenvalue problem, the error analysis is performed for different discretization schemes. The convergence of the problem is validated by considering the twenty least stable eigenvalues. The relative error is calculated for specific value of N using the $L^2$ norm given as,

$$e_N = \frac{||EV_{N+1} - EV_N||_2}{||EV_N||_2}$$

where, $EV_N$ and $EV_{N+1}$ are the arrays containing the twenty least stable eigenvalues computed using N and N + 1 DRP grid points over the entire domain.

Figure 3.4 shows the relevance of using the FODRP schemes over SOFD and FOFD schemes for the discretization of the domain. Furthermore, all the schemes with the incorporation of the GSF shows the significant reduction in the error for the calculation of the eigenvalues. The DRP scheme with the GSF shows the maximum reduction in error as compared to other schemes of discretization. This provides a clear evidence in support of choosing DRP scheme over other discretization schemes for efficient analysis of the eigenvalue problem.

From the eigenvalue convergence analysis for different values of porosity ($\epsilon_c = 0.3, 0.95$) as shown in Table3.1 and the relative error analysis with different discretization schemes as shown in figure 3.4, it can be well inferred that the maximum number of points to be
used for the grid discretization is 193. With $N = 193$, the accuracy of the resolution of the grid is of the order of $10^{-4}$ (0.01% error) and the convergence is also achieved for the least stable eigenvalue. So 193 grid points are considered for discretizing the domain for carrying out the further analysis of the stability problem for this case.

![Figure 3.4: Log-log plot of the relative error, $e_N$ against the number of polynomials used to discretize the domain, $N$ for different schemes. Parameters considered are $Re_b = 2000$, $\epsilon_c = 0.95$, $\delta_i = 0.02$, $\alpha = 1$, $\beta = 0$.](image)

Even though the number of grid points used for discretizing the domain using DRP schemes is much higher than the Chebyshev spectral domain, the time and resources required for the computation is drastically reduced due to the creation of sparse matrices. Hence DRP schemes are preferred over the spectral scheme since this will have a significant effect on the computation time and memory footprint when our analysis will be extended to a three-dimensional domain used for the porous duct case.
Chapter 4

Laminar Porous Channel Flow

In this chapter, the results of linear stability analysis for porous channel flow in the laminar regime are presented. As a starting point, the numerical code is validated for the Poiseuille flow and Couette flow results which are presented in Appendix C.

4.1 Baseflow

The baseflow, \( \vec{U} = (U(y), 0, 0) \) is obtained numerically by solving the VANS equation given below,

\[
0 = -\frac{\partial P}{\partial x} + \frac{1}{Re_b} \frac{\partial^2 U}{\partial y^2} + \frac{1}{\epsilon Re_b} \frac{\partial \epsilon}{\partial y} \frac{\partial U}{\partial y} - \frac{1}{Re_b} Fo \epsilon |U| U + \frac{1}{Re_b} \left[ \frac{1}{\epsilon \partial y^2} - \frac{\epsilon}{Da} \right] U \tag{4.1}
\]

The baseflow profile for the porous channel flow is shown for \( \epsilon_c = 0, 0.3, 0.6, 0.95 \) at \( Re_b = 2000 \) in Fig.4.1. For \( \epsilon_c \) ranging between 0.3 and 0.6, it can be seen that the velocity magnitude within the porous medium is negligible. On the other hand, for \( \epsilon_c = 0.9 \) and higher, the porous layer becomes permeable enough to allow some flow through it and a non-zero velocity magnitude is observed. As the bulk velocity is set to a constant value of 1 over the entire channel height, a small reduction in the velocity in the upper half of the channel is observed for \( \epsilon_c = 0.95 \) owing to non-zero velocity in the porous layer.

![Baseflow profile for porous channel flow for different values of porosity.](image)

Figure 4.1: Baseflow profile for porous channel flow for different values of porosity. The parameters are: \( Re_b = 2000 \) and \( \delta_i = 0.02 \).
The baseflow evolves on different scales corresponding to different values of porosities as given below.

- $\epsilon_c = 0$: The baseflow is a regular Poiseuille flow in the fluid domain.
- $\epsilon_c \neq 0$: The baseflow shows the existence of an inflection point and a vorticity thickness.
- $\epsilon_c = 1$: The baseflow is the classical Poiseuille flow characterised by a parabolic velocity profile over the entire height of the channel.

### 4.2 Modal Stability Analysis

As a starting point, normal analysis is performed to check the validity of the Squire theorem for the VANS equations. In case of porous channel flow, the flow experiences shear along the wall-normal direction only and hence the assumption of considering the perturbations along the stream-wise and wall-normal directions should be sufficient to evaluate the critical parameters. As shown by Lauga and Cossu (2005), the Squire theorem can be extended to the case of slip channel flows. The plot of the least stable eigenvalues on the $\alpha - \beta$ plane is shown in figure 4.2 for porosity values of $\epsilon_c = 0.3, 0.6, 0.95$ and $0.98$ respectively. It can be seen that for all cases of porosity, the least stable eigenvalues are confined along the $\beta = 0$ axis. Thus, the flow first becomes linearly unstable toward two-dimensional waves as expected due to the Squire theorem. Similar conclusions have been reached by Tilton and Cortelezzi (2008) for similar flow configurations. Hence for the neutral curve calculation, only two-dimensional perturbations in the stream-wise and the wall-normal directions are considered and the perturbations in the span-wise direction are neglected ($\beta = 0$).

![Eigenspectrum contour for different values of porosity](image)

Figure 4.2: Eigenspectrum contour for different values of porosity. The parameters are $Re_b = 4000$ ($\epsilon_c = 0.3, 0.6$), $Re_b = 500$ ($\epsilon_c = 0.95$), $Re_b = 200$ ($\epsilon_c = 0.98$) and $\delta_i = 0.02$ for all cases.
4.2.1 Neutral Curves

Effect of $\delta_i$

Here the effect of the thickness of the porosity transition layer ($\delta_i$) on the critical Reynolds number ($Re_c$) is shown for two different extreme values of porosity i.e. one almost mimicking like a solid wall ($\epsilon_c = 0.3$) and the other behaving as a highly permeable medium ($\epsilon_c = 0.9$).

![Figure 4.3: Neutral curves for different values of thickness of the interface fluid-porous transition layer $\delta_i = 0.02(\cdot), \delta_i = 0.05(\cdot), \delta_i = 0.1(\cdot)$ corresponding to the porosity values of (a) $\epsilon_c = 0.3$ and (b) $\epsilon_c = 0.9$.](image)

Table 4.1: Comparison of the production and dissipation terms of the least stable eigenvalue for different values of $\delta_i$ for porosity value of 0.3 and 0.9 with $\beta = 0$ for both cases.

(a) $\epsilon_c = 0.3$ at $\alpha = 2$ and $Re_b = 4000$

<table>
<thead>
<tr>
<th>$\delta_i$</th>
<th>0.02</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production</td>
<td>0.0464</td>
<td>0.0518</td>
<td>0.0669</td>
</tr>
<tr>
<td>Viscous Dissipation</td>
<td>-0.0398</td>
<td>-0.0383</td>
<td>-0.0363</td>
</tr>
<tr>
<td>Porous Dissipation</td>
<td>-0.0033</td>
<td>-0.0065</td>
<td>-0.0133</td>
</tr>
<tr>
<td>Growth Rate</td>
<td>0.0033</td>
<td>0.0069</td>
<td>0.0173</td>
</tr>
</tbody>
</table>

(b) $\epsilon_c = 0.9$ at $\alpha = 2$ and $Re_b = 2000$

<table>
<thead>
<tr>
<th>$\delta_i$</th>
<th>0.02</th>
<th>0.05</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Production</td>
<td>0.3428</td>
<td>0.3499</td>
<td>0.3861</td>
</tr>
<tr>
<td>Viscous Dissipation</td>
<td>-0.0552</td>
<td>-0.0523</td>
<td>-0.0491</td>
</tr>
<tr>
<td>Porous Dissipation</td>
<td>-0.1176</td>
<td>-0.1228</td>
<td>-0.1381</td>
</tr>
<tr>
<td>Growth Rate</td>
<td>0.1699</td>
<td>0.1747</td>
<td>0.1989</td>
</tr>
</tbody>
</table>

From figure 4.3(a), it is clearly seen that $\delta_i$ has significant effect on the prediction of the critical Reynolds number for lower value of porosity, $\epsilon_c = 0.3$. At lower value of porosity, the gradient of porosity across the interface becomes significant as compared to higher values of porosity. From the comparison of the energy terms as shown in table 4.1(a), it is seen that the production and the porous dissipation terms increases with increasing
The sum of the production, viscous dissipation and porous dissipation determines the growth rate of the instability. It can be very well seen that the increase in instability growth rate is around 110% for variation of $\delta_i = 0.02 - 0.05$. This enhancement of growth rate is 150% for variation of $\delta_i = 0.05 - 0.1$. Hence it can be well accounted that the interface thickness $\delta_i$ has important role to play for predicting the critical parameters at lower values of porosity.

Table 4.2: Comparison of the values of $(Re_c, \alpha_c)$ for different values of $\delta_i$ for the cases of porosities of 0.3 and 0.9 with $\beta = 0$ for both cases.

<table>
<thead>
<tr>
<th>$(\epsilon_c, \delta_i)$</th>
<th>$\delta_i = 0.02$</th>
<th>$\delta_i = 0.05$</th>
<th>$\delta_i = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\epsilon_c = 0.3$</td>
<td>(3848,2)</td>
<td>(3640,2)</td>
<td>(3149,2)</td>
</tr>
<tr>
<td>$\epsilon_c = 0.9$</td>
<td>(980,2,6)</td>
<td>(953,2,6)</td>
<td>(887,2,7)</td>
</tr>
</tbody>
</table>

A similar effect can be seen in figure 4.3(b) for $\epsilon_c = 0.9$, though it is not as pronounced and hardly influences the value of $Re_c$. This can also be seen from the energy terms in table 4.1(b). The production term and the porous dissipation term again increases with increasing $\delta_i$ but the increase is not significant as compared to $\epsilon_c = 0.3$ case. It can be seen that the increase in instability growth rate is around 3% for variation of $\delta_i = 0.02 - 0.05$ and 14% for variation of $\delta_i = 0.05 - 0.1$. Hence it can be concluded that the interface thickness $\delta_i$ do not have a profound impact on the prediction of critical parameters at higher values of porosity. However it can be seen for both $\epsilon_c = 0.3$ and 0.9, the viscous dissipation is not affected much due to the change in $\delta_i$. The interface thickness only affects the dissipation due to the porous region. However the effect of interface region thickness is not the significant one in case of evaluating critical parameters for the flow destabilization. From the work of Ochoa-Tapia and Whitaker (1995), it has been seen that it is a good approximation to consider the interface region thickness twice as the diameter of the spheres used for modeling the porous region. Hence in our analysis, the value of $\delta_i = 0.02$ is considered.

**Effect of $\epsilon_c$**

![Neutral curves corresponding to different values of porosity. The parameters are: $\beta = 0$ and $\delta_i = 0.02$.](image-url)
Here the effect of the variation in porosity on the critical Reynolds number is shown. Neutral curves for different values of porosity ($\epsilon_c = 0 - 0.95$) in the $Re - \alpha$ plane are depicted in figure 4.4 for constant thickness of the interface region ($\delta_i = 0.02$). It can be seen that, as the value of porosity of the medium below the fluid-porous interface increases from $\epsilon_c = 0$ up to $\epsilon_c = 0.95$, the critical Reynolds number ($Re_c$) decreases while the critical stream-wise wavenumber ($\alpha_c$) almost remains constant.

Table 4.3: Critical parameters ($Re_c$, $\alpha_c$) for different values of porosity ($\epsilon_c$) with $\beta = 0$ and $\delta_i = 0.02$ for all cases.

<table>
<thead>
<tr>
<th>$\epsilon_c$</th>
<th>0</th>
<th>0.3</th>
<th>0.6</th>
<th>0.7</th>
</tr>
</thead>
<tbody>
<tr>
<td>($Re_c$, $\alpha_c$)</td>
<td>(3848.17, 2.04)</td>
<td>(3738.13, 2.04)</td>
<td>(3361.20, 2.07)</td>
<td>(2902.34, 2.11)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\epsilon_c$</th>
<th>0.8</th>
<th>0.9</th>
<th>0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>($Re_c$, $\alpha_c$)</td>
<td>(2082.94, 2.20)</td>
<td>(980.78, 2.47)</td>
<td>(435.68, 2.77)</td>
</tr>
</tbody>
</table>

Table 4.3 shows the values of the critical parameters ($Re_c$, $\alpha_c$) for different values of porosity. $\epsilon_c = 0$ represents a Poiseuille flow in the part of the channel above the fluid-porous interface. For $\epsilon_c = 0$ the critical Reynolds number is $Re_c = 3848.17$. As the porosity value increases up to $\epsilon_c = 0.3$, the critical Reynolds number does not change significantly. It can however be observed that, further increasing the porosity eventually results in drastic reduction of the value of critical Reynolds number as shown by Tilton and Cortelezzi (2008) for a similar configuration. The critical stream-wise wavenumber ($\alpha_c$) remains constant as 2.04 for lower values of porosity and it changes from 2.04 for $\epsilon_c = 0.3$ to 2.20 for $\epsilon_c = 0.8$.

In the range of higher values of porosity ($\epsilon_c = 0.9, 0.95$), the flow becomes highly susceptible to the destabilizing effects of porosity. The critical Reynolds number decreases significantly from 980.78 for $\epsilon_c = 0.9$ to 435.68 for $\epsilon_c = 0.95$ while the critical wavenumber changes from 2.47 for $\epsilon_c = 0.9$ to 2.77 for $\epsilon_c = 0.95$. It can be noted that, while the critical Reynolds number seems to be quite dependent on the porosity of the medium, the critical stream-wise wavenumber $\alpha_c$ appears on the other hand to be quite insensitive to the variation of porosity. It is also seen that as the value of porosity increases, the neutral curve becomes more flat and large spanning over a wide range of unstable wavenumbers for a particular value of Reynolds number which is also shown by Tilton and Cortelezzi (2008). However the origin of the least stable mode at different values of porosity cannot be predicted from the neutral curves.

### 4.2.2 Eigenspectrum and Eigenmodes

From the neutral curves, the linearly unstable region in the $Re - \alpha$ plot for different values of porosity is clearly identified. The eigenspectrum and the shape of the eigenmode corresponding to the least stable eigenvalue are presented in this section.
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Figure 4.5: (a) Eigenspectrum, (b) stream-wise and (c) wall-normal velocity component of the least stable wall eigenmode (●), (d) stream-wise and (e) wall-normal velocity component of the 1st center eigenmode (●) for \((Re_b, \alpha, \epsilon_c) = (4000, 2, 0.3)\). The perturbation velocity contours for least stable wall and center eigenmode are shown in (f) and (g).

Figure 4.5(a) represents the eigenspectrum for \(\epsilon_c = 0.3\) at \(Re_b = 4000\), \(\alpha = 2\) and \(\beta = 0\). An interesting observation is that the eigenvalues are distributed over 3 different
branches. The eigenvalues on the uppermost right branch have their eigenfunctions varying near the wall region and they are labeled as wall modes. This is also seen from the shape of the stream-wise and wall-normal components of the wall eigenmode (●) presented in figure 4.5(b) and (c). It is well expected that the porosity will affect the wall modes significantly. The uppermost left branch of eigenvalues have their eigenfunctions varying near the center of the homogeneous fluid region and they are labeled as center modes. This is also very well seen from the stream-wise and wall-normal components of the center eigenmode (●) presented in figure 4.5(d) and (e). The lowermost center branch of eigenvalues are highly damped. The same trend of eigenvalues and eigenmodes have also been reported by Tilton and Cortelezzi (2008). It is very well seen that the least stable eigenmode for $\epsilon_c = 0.3$ is confined within the homogeneous fluid region and the perturbations disappear at the interface as shown in figure 4.5(f). The shape of the eigenmode shows that the instability stems from the homogeneous fluid region of the channel.

Figure 4.6: (a) Eigenspectrum, (b) stream-wise and (c) wall-normal velocity component of least stable wall eigenmode (●) for $(Re_b, \alpha, \epsilon_c) = (4000, 2, 0.6)$. The perturbation velocity contours for least stable wall eigenmode is shown in (d).
Figure 4.6(a) represents the eigenspectrum for $\epsilon_c = 0.6$ at $Re_b = 4000$, $\alpha = 2$ and $\beta = 0$. The eigenspectrum is very similar to the case of $\epsilon_c = 0.3$ except for the fact that the growth rate of the least stable eigenvalue is greater in magnitude. The shape of the stream-wise and wall-normal component of the least stable eigenvalue is presented in figure 4.6(b) and (c) and they are also well confined within the homogeneous fluid region.

Figure 4.7: (a) Eigenspectrum, (b) stream-wise and (c) wall-normal velocity component of least stable wall eigenmode (●) for $(Re_b, \alpha, \epsilon_c) = (2000, 2, 0.9)$. The perturbation velocity contours for least stable wall eigenmode is shown in (d).

Figure 4.7(a) represents the eigenspectrum for high $\epsilon_c = 0.9$ at $Re_b = 2000$, $\alpha = 2$ and $\beta = 0$. The shape of the stream-wise and wall-normal component of the least stable eigenmode is presented in figure 4.7(b) and (c). From the wall-normal component of the unstable wall mode as shown in figure 4.7(d), it can be seen that there is a flow reversal at the interface. This is due to the weakening of the wall-blocking and wall induced viscous effects (Perot and Moin (1995)) near permeable walls, which induces transpiration velocity into the porous region. Figure 4.8(a) represents the eigenspectrum for $\epsilon_c = 0.95$ at $Re_b = 2000$, $\alpha = 2$ and $\beta = 0$. The least stable wall mode has significantly
large growth rate. The stream-wise and wall normal components of least stable wall eigenmode are shown in figure 4.8(b) and (c). The wall-normal component as shown in figure 4.8(c) suggests that the wall mode is strongly influenced by the porous region. From the wall-normal velocity contour of the unstable wall mode shown in figure 4.8(d), it can be seen that the real part of this mode shows strong flow reversal at the interface. A similar feature for higher values of permeability is also shown by Chang et al. (2006) and Hill and Straughan (2008). At such high value of porosity, the porous region becomes highly permeable and the viscous effects disappear due to non-zero slip velocity at the interface. The transpiration velocity into the porous region becomes significant in the wall-normal direction but however these effects decay inside the porous region. The stream-wise perturbation velocity for the unstable wall mode has non-zero contribution at the interface and decays in the porous region.

Figure 4.8: (a) Eigenspectrum, (b) stream-wise and (c) wall-normal velocity component of least stable wall eigenmode (●) for \((Re_b, \alpha, \epsilon_c) = (2000, 2, 0.95)\). The perturbation velocity contours for least stable wall eigenmode is shown in (d).

At extremely high value of \(\epsilon_c = 0.95\), a new branch of eigenvalues appear in the
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eigenspectrum (shown as ▼ and •) as compared to earlier cases. This branch consists of the porous modes as shown by Tilton and Cortelezzi (2008) originating from the porous region of the channel. The stream-wise and wall-normal components of the eigenmode for a pair of eigenvalues from this porous branch are shown in figure 4.9(a)-(d). It is clearly seen that these eigenmodes are highly stable and vary significantly within the porous layer. The porous mode branch evolves on account of extremely high value of porosity, but still they are confined with the porous region of the channel and do not become unstable.

![Graphs showing real and imaginary parts of eigenmodes](image)

Figure 4.9: (a), (c) Stream-wise and (b), (d) wall-normal velocity component of 1\textsuperscript{st} (▼) and 2\textsuperscript{nd} (•) eigenmode of the porous branch for \((Re_b, \alpha, \epsilon_c) = (2000, 2, 0.95)\). The corresponding velocity contours for the 1\textsuperscript{st} and 2\textsuperscript{nd} eigenmodes of porous branch are shown in (e) and (f). The two eigenvalues of the porous branch are shown in figure 4.8(a).

The presence of an eigenvalue related to the wall mode as labeled by Tilton and Cortelezzi (2008), has a positive real part. The value of this positive real part of the eigenvalue, referred to as the growth rate has been presented in the table 4.4. It can be seen from table 4.4, the growth rate of least stable eigenvalue \((\sigma)\) for \(\epsilon_c = 0\) (Poiseuille flow) is in good approximation with that of \(\epsilon_c = 0.3\) until 3 decimal places for \(Re_b = 4000\).
and $\alpha = 2$. A similar comparison has also been shown for lower values of permeability by Tilton and Cortelezzi (2008). The similarity in growth rates for $\epsilon_c = 0$ and 0.3 is again seen for $Re_b = 2000$ and $\alpha = 2$ with same order of accuracy even though the parameters considered are in the linearly stable region (negative growth rate). It is also seen that the growth rate increases significantly as the porosity value increases. Another interesting observation from table 4.4 is that the increase in the growth rate corresponding to increase in Reynolds number, increases with rising values of porosity, also shown by Tilton and Cortelezzi (2008).

<table>
<thead>
<tr>
<th>$Re_b$</th>
<th>$\epsilon_c = 0$</th>
<th>$\epsilon_c = 0.3$</th>
<th>$\epsilon_c = 0.6$</th>
<th>$\epsilon_c = 0.9$</th>
<th>$\epsilon_c = 0.95$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2000</td>
<td>-0.05821859</td>
<td>-0.05856422</td>
<td>-0.05287691</td>
<td>0.18095526</td>
<td>0.68022661</td>
</tr>
<tr>
<td>4000</td>
<td>0.00309088</td>
<td>0.00373292</td>
<td>0.01137622</td>
<td>0.37378663</td>
<td>0.93124466</td>
</tr>
</tbody>
</table>

Table 4.4: Comparison of the growth rate ($\sigma$) of the least stable eigenvalue for different values of porosity ($\epsilon_c$) taking critical wavenumber, $\alpha_c = 2$ for all cases.

Table 4.5: Real and imaginary parts of the least stable eigenvalue for different values of porosity.

<table>
<thead>
<tr>
<th>$(\epsilon_c, Re_b, \alpha)$</th>
<th>(0.3, 4000, 2)</th>
<th>(0.6, 4000, 2)</th>
<th>(0.7, 4000, 2)</th>
<th>(0.9, 4000, 2)</th>
<th>(0.95, 4000, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma$</td>
<td>0.00373379</td>
<td>0.01137724</td>
<td>0.02540163</td>
<td>0.37378663</td>
<td>0.93125720</td>
</tr>
<tr>
<td>$\omega$</td>
<td>-1.55120128</td>
<td>-1.54741077</td>
<td>-1.54321094</td>
<td>-1.76187914</td>
<td>-2.29685211</td>
</tr>
</tbody>
</table>

Another interesting observation from the eigenspectrum is the comparison of the imaginary part (frequency) of the least stable eigenvalue for different values of porosity as shown in table 4.5. It can be seen that as the value of porosity increases from $\epsilon_c = 0$ to 0.7, the frequency of the least stable eigenvalue increases from $\omega = -1.551$ to $-1.543$. While at higher value of porosity, $\epsilon_c = 0.9$, the frequency of least stable eigenvalue is much lower as $\omega = -1.761$. The similar trend is seen for extremely high value of porosity, $\epsilon_c = 0.95$ where the unstable eigenvalue occurs at much lower frequency $\omega = -2.296$. This shows that $\omega$ for the mode derived from fluid region keeps on increasing with the rising values of porosity until $\epsilon_c = 0.7$. At higher values of porosity the frequency of the least stable wall eigenvalue keeps on decreasing with further rising values of porosity. From the eigenspectrum and the shape of the wall eigenmode, it can be very well seen that the least stable eigenmode at lower values of porosity is derived from the fluid region of the channel. For higher values of porosity, the least stable eigenmode comes from the fluid region but influenced by the porous region of the channel at the interface.

### 4.2.3 Reynolds-Orr Analysis

In order to get a better understanding of the flow instabilities and the underlying physical mechanisms, an analysis of the perturbation kinetic energy budget is conducted for different values of porosity. This is obtained by writing kinetic energy equation 2.37 in expanded
form considering two-dimensional perturbations only as given below,

\[
\frac{\partial}{\partial t} \left( \frac{u^*v^* + vv^*}{2} \right) = - \left( \frac{u^*v^* + vv^*}{2} \right) \frac{\partial U}{\partial y} - \frac{1}{2} \left( u^* \frac{\partial p^*}{\partial x} + u^* \frac{\partial p}{\partial x} + v^* \frac{\partial p}{\partial y} + v^* \frac{\partial p^*}{\partial y} \right) + \frac{1}{\Re_b} \left[ \frac{\partial^2}{\partial x^2} \left( \frac{u^*v^* + vv^*}{2} \right) + \frac{\partial^2}{\partial y^2} \left( \frac{u^*v^* + vv^*}{2} \right) \right] + \frac{1}{\epsilon \Re_b} \left[ \frac{\partial^2}{\partial y^2} \left( \frac{u^*v^* + vv^* + v^*v}{2} \right) \right] + \frac{\epsilon F_0}{\Re_b Da} \left[ \frac{2u^*v^* + 2u^*u + vv^* + v^*v}{2} \right] + \frac{1}{\Re_b Da} \left( \frac{u^*v^* + u^*u + vv^* + v^*v}{2} \right) - \frac{1}{\Re_b} \left[ \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial u^*}{\partial x} \right) + \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial v^*}{\partial x} \right) + \left( \frac{\partial u}{\partial y} \right) \left( \frac{\partial u^*}{\partial y} \right) + \left( \frac{\partial v}{\partial y} \right) \left( \frac{\partial v^*}{\partial y} \right) \right]
\]

\[(4.2)\]

Table 4.6: Notation of different terms in the perturbation kinetic energy for different values of porosity, where D=DIS+POR+FOR+DAR.

<table>
<thead>
<tr>
<th>Term</th>
<th>Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shear production</td>
<td>(\frac{1}{7} \int_O WS d\Omega)</td>
</tr>
<tr>
<td>Porous drag</td>
<td>(\frac{1}{7} \int_O POR d\Omega)</td>
</tr>
<tr>
<td>Forchheimer drag</td>
<td>(\frac{1}{7} \int_O FOR d\Omega)</td>
</tr>
<tr>
<td>Darcy drag</td>
<td>(\frac{1}{7} \int_O DAR d\Omega)</td>
</tr>
<tr>
<td>Viscous dissipation</td>
<td>(\frac{1}{7} \int_O DIS d\Omega)</td>
</tr>
<tr>
<td>Total porous dissipation</td>
<td>(\frac{1}{7} \int_O (POR + FOR + DAR) d\Omega)</td>
</tr>
</tbody>
</table>

The spatial distribution of different terms in the perturbation kinetic energy budget are presented in figure 4.10(a)-(f) for the porosity values of \(\epsilon_c = 0.3, 0.6, 0.9\). All the terms of the perturbation kinetic energy are normalized with the total dissipation. It can be seen from figure 4.10(a) that for \(\epsilon_c = 0.3\), the WS (shear production) term is dominant near the wall and the interface and resembles the energy distribution of a Poiseuille channel flow. DIS (viscous dissipation) term is also dominant near the walls and interface while the VPG (velocity-pressure-gradient) term locally balance the WS term. At the walls and the interface, VD (viscous diffusion) term is in balance with the DIS term which is also seen in case of channel Poiseuille flow by Mansour et al. (1988). The contributions from POR (porous drag), DAR (Darcy drag) and FOR (Forchheimer drag) terms are seen in the porous region only as shown in figure 4.10(b). In the porous layer, the maximum peak is seen for the DAR term followed by POR term and with a small peak for the FOR term near the interface. For the porosity case of \(\epsilon_c = 0.6\) shown in figure 4.10(c), it is seen that the WS term has a similar profile like that of Poiseuille flow but with slight more contribution at the interface as compared to the wall. The VPG term hence increases near the interface to balance the WS term. The VD and DIS terms balance each other near the interface and wall. The DAR term again has a maximum peak near the interface. The VPG term in the porous layer balances the DAR term. The contribution due to the POR term decreases as compared to \(\epsilon_c = 0.3\) case while the FOR term remains almost the same.
as shown in figure 4.10(d).

For high values of porosity, $\epsilon_c = 0.9$ as shown in figure 4.10(e), there is a sharp increase of WS term near the interface and accordingly VPG term increases sufficiently so as to locally balance the WS term. The terms VD and DIS still balance each other locally near the interface. In the porous region, it can be seen from figure 4.10(f) that the DAR term increases significantly and hence the VPG term in the porous layer increases sufficiently in order to provide a local balance. The POR term decreases due to gradual decrease of porosity gradient across the interface and the magnitude of FOR term becomes comparable to the POR term.

The perturbation kinetic energy budget terms integrated along the wall-normal direction and normalized with the total dissipation ($D = \text{POR} + \text{FOR} + \text{DAR} + \text{DIS}$) for $\epsilon_c = 0.3$, 0.6 and 0.9 are shown in figure 4.10(g), (h) and (i) respectively. The corresponding terms are presented in table 4.6. From the energy budget it can be seen that for $\epsilon_c = 0.3$, the production terms ($E_1$) and the viscous dissipation term ($E_5$) almost balance each other with a small difference while the contribution due to the dissipative term ($E_6$) from the porous region is significantly small as shown in figure 4.10(g). For $\epsilon_c = 0.6$, the energy budget is almost similar to the case of Poiseuille flow, but with small contributions from the porous dissipative terms ($E_2, E_3, E_4$) as shown in figure 4.10(h). However the major contribution for the dissipation of the kinetic energy flow from the porous layer comes from the DAR ($E_4$) term. For high porosity case $\epsilon_c = 0.9$, the production term ($E_1$) increases drastically with increased contribution from the DAR term ($E_4$) as seen in figure 4.10(i).

The contribution of POR term ($E_2$) reduces as compared to $\epsilon_c = 0.3$ and 0.6. While the contribution from FOR term ($E_3$) increases with $\epsilon_c$ increasing from 0.3 to 0.9 but the % increase is not so significant as compared to the % increase in DAR term ($E_4$). The numerical values of the contributions from the different terms of the perturbation kinetic energy for different cases of porosity are presented in table 4.7.

The perturbation kinetic energy budget for extremely high value of porosity, $\epsilon_c = 0.95$ is presented in figure 4.11. It can be seen that the WS term increases tremendously near the fluid-porous interface and it is balanced by the VPG term and DIS term to a small extent very close the interface, in the fluid region as shown by Breugem et al. (2006). Due to the added slip velocity at the fluid-porous interface at high porosity, the viscous effects become less significant near the interface. The interesting difference can be seen in the porous region near the interface, where the peak value due to the FOR term becomes comparable with the DAR term. The VPG term increases significantly in the porous region in order to balance the enhanced FOR and DAR terms locally as shown by Breugem et al. (2006).

For extremely high porosity case $\epsilon_c = 0.95$, the production term ($E_1$) increases significantly with increased contribution from the DAR term ($E_4$) as seen in figure 4.11(c). The contribution of POR term ($E_2$) reduces. The notable difference is for the increased contribution from FOR term ($E_3$). The % increase in $E_3$ term from $\epsilon_c = 0.9 – 0.95$ is much higher than $E_4$ term.
Figure 4.10: (a)-(f) shows perturbation kinetic energy distribution and (g)-(i) shows integrated kinetic energy terms normalized by total dissipation with $Re_b = 4000(\epsilon_c = 0.3, 0.6), Re_b = 2000(\epsilon_c = 0.9), \alpha = 2$ and $\beta = 0$ for all cases.
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Figure 4.11: (a), (b) Perturbation kinetic energy distribution and (c) integrated kinetic energy terms normalized by total dissipation with \( Re_b = 2000 \), \( \alpha = 2 \) and \( \beta = 0 \).

The integrated contributions due to the different terms of the perturbation kinetic energy budget normalized with the total dissipation from the fluid and porous regions, for different values of porosity are presented in table 4.7.

Table 4.7: Integrated perturbation kinetic energy terms, normalized with respect to the total dissipative energy given by \( D = \text{DIS} + \text{POR} + \text{FOR} + \text{DAR} \), for different values of porosity. The data are displayed in figures 4.10(g)-(i) and 4.11(c).

<table>
<thead>
<tr>
<th>((\epsilon_c, Re_b, \alpha))</th>
<th>(0.3, 4000, 2)</th>
<th>(0.6, 4000, 2)</th>
<th>(0.9, 2000, 2)</th>
<th>(0.95, 2000, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E_1)</td>
<td>1.0747</td>
<td>1.1792</td>
<td>1.9835</td>
<td>2.8353</td>
</tr>
<tr>
<td>(E_2)</td>
<td>-0.0160</td>
<td>-0.0139</td>
<td>-0.0045</td>
<td>-0.0029</td>
</tr>
<tr>
<td>(E_3)</td>
<td>-0.0077</td>
<td>-0.0129</td>
<td>-0.0211</td>
<td>-0.0697</td>
</tr>
<tr>
<td>(E_4)</td>
<td>-0.0534</td>
<td>-0.1710</td>
<td>-0.6548</td>
<td>-0.7486</td>
</tr>
<tr>
<td>(E_5)</td>
<td>-0.9228</td>
<td>-0.8021</td>
<td>-0.3194</td>
<td>-0.1786</td>
</tr>
<tr>
<td>(E_2 + E_3 + E_4 (E_6))</td>
<td>-0.0771</td>
<td>-0.1978</td>
<td>-0.6805</td>
<td>-0.8213</td>
</tr>
</tbody>
</table>

In order to check the correctness of the perturbation kinetic energy analysis, different terms in the perturbation kinetic energy budget are integrated along the wall-normal direction and added up to see if that equals the growth rate of the least stable eigenvalue and are presented in table 4.8.
Table 4.8: Comparison of total perturbation kinetic energy with the growth rate of the least stable eigenvalue $\sigma$ for different values of porosity.

<table>
<thead>
<tr>
<th>$(\epsilon_c, Re, \alpha)$</th>
<th>(0.3, 4000, 2)</th>
<th>(0.6, 4000, 2)</th>
<th>(0.9, 2000, 2)</th>
<th>(0.95, 2000, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int_{\Omega} WS d\Omega$</td>
<td>0.046</td>
<td>0.059</td>
<td>0.343</td>
<td>1.035</td>
</tr>
<tr>
<td>$\int_{\Omega} POR d\Omega$</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.001</td>
</tr>
<tr>
<td>$\int_{\Omega} FOR d\Omega$</td>
<td>-0.001</td>
<td>-0.001</td>
<td>-0.004</td>
<td>-0.025</td>
</tr>
<tr>
<td>$\int_{\Omega} DAR d\Omega$</td>
<td>-0.002</td>
<td>-0.007</td>
<td>-0.113</td>
<td>-0.265</td>
</tr>
<tr>
<td>$\int_{\Omega} DIS d\Omega$</td>
<td>-0.039</td>
<td>-0.039</td>
<td>-0.045</td>
<td>-0.064</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>0.003</td>
<td>0.011</td>
<td>0.180</td>
<td>0.680</td>
</tr>
<tr>
<td><strong>$\sigma$</strong></td>
<td>0.003</td>
<td>0.011</td>
<td>0.180</td>
<td>0.680</td>
</tr>
</tbody>
</table>

Figure 4.12: Contours of ratio of viscous dissipation to the total dissipation for different values of porosity corresponding to least stable mode. The black line represents the neutral curve. The parameters are $\alpha = \alpha_c$ and $\delta_i = 0.02$.

Figure 4.12 represents the contours of viscous dissipation to the total dissipation for different values of porosity. The black boundary line represents the neutral curve ($Re_c$). It can be seen from figure 4.12 that corresponding to least stable mode, the contribution due to the viscous dissipation is significant at lower values of porosity. At higher values of porosity, the porous dissipation becomes significantly important over the viscous dissipation due to non-zero slip velocity at the fluid-porous interface. This makes the viscous effects insignificant near the permeable wall.

### 4.3 Physical Origin of Flow Instabilities

Even though the modal stability analysis clearly gives a good idea about the presence of unstable eigenmode contributing to the instabilities in the flow. The underlying mechanism governing the fate of the instabilities need to be identified in order to understand their physical origin. From the information of neutral curves and energy analysis, we can see that the least stable mode rely on the base flow gradients to extract their energy. However the physical mechanism of the least stable stable eigenmode is essentially different with the variation in porosity. The scaling analysis provides a good understanding of the physical mechanism for the origin of the instabilities in the flow.

In order to support our assumption, a scaling approach in a very similar way as shown by Singh et al. (2016) for the monami in submerged seagrass bed, is performed. For the
scaling analysis, the critical parameter considered is the thickness of the vorticity layer ($\delta$) at the fluid-porous interface. The definition of vorticity ($\Omega$) is given as,

$$\Omega(\hat{k}) = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = \frac{-\partial U}{\partial y}$$ (4.3)

The least stable mode is characterized based on the vorticity thickness ($\delta$) as the scaling parameter, calculated for different cases of $\epsilon_c$ at the fluid-porous interface. With increasing values of $\epsilon_c$, the vorticity thickness ($\delta$) at the fluid-porous interface increases significantly.

\[\text{Figure 4.13: Schematic of the fluid-porous interface region.}\]

According to Beavers and Joseph interface condition for a permeable wall as shown in Beavers and Joseph (1967),

$$\frac{d\bar{U}}{dy} \bigg|_{y=0} = \gamma \left( \frac{U_s - U_{cp}}{\sqrt{K \epsilon_c}} \right)$$ (4.4)

where $\gamma$ is a coefficient of $O(1)$.

$U_s$ is the slip velocity at the fluid-porous interface which becomes significant as the permeability of porous region increases thus diminishing the wall-blocking and wall induced viscous effects. $U_{cp}$ is the creep velocity in the porous region which is significant at higher values of porosity due to non-zero inertial velocity in the porous region.

Using $(U_s - U_{cp})$ as the characteristic velocity difference, the vorticity thickness ($\delta$) is calculated as,

$$\delta = \gamma \left( \frac{U_s - U_{cp}}{\frac{d\bar{U}}{dy} \bigg|_{y=0}} \right)$$ (4.5)

Comparing equations 4.4 and 4.5, it can be shown that,

$$\delta \sim \sqrt{\frac{K}{\epsilon_c}} \sim \sqrt{\frac{Da}{\epsilon_c}} \sim \frac{\epsilon_c}{1 - \epsilon_c}$$ (4.6)

The vorticity thickness for different values of porosity is calculated using equation 4.6. The vorticity thickness is scaled with $H$ for non-dimensionalization.
Laminar Porous Channel Flow

<table>
<thead>
<tr>
<th>Extent of domain</th>
<th>Low $\epsilon_c$ (0.05 - 0.3)</th>
<th>High $\epsilon_c$ (0.9 - 0.95)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vorticity thickness</td>
<td>$-1 \leq y \leq 1$</td>
<td>Across the fluid region</td>
</tr>
<tr>
<td>Location of mode</td>
<td>$\delta \sim \left(\frac{\epsilon_c}{1-\epsilon_c}\right)$</td>
<td>Around $y = 0$ ($\frac{\partial^2 U}{\partial y^2} = 0$)</td>
</tr>
<tr>
<td>Inflection point</td>
<td>$Re_c \sim \left(\frac{1}{\delta}\right)^0$</td>
<td>$Re_c \sim \left(\frac{1}{\delta}\right)^{1.4}$</td>
</tr>
<tr>
<td>Critical Reynolds number</td>
<td>$\alpha_c \sim \left(\frac{1}{\delta}\right)^0$</td>
<td>$\alpha_c \sim \left(\frac{1}{\delta}\right)^{-0.14}$</td>
</tr>
</tbody>
</table>

Table 4.9: Comparison of the characteristics of least stable mode originating in the flow over different ranges of porosity.

The scaling analysis has been performed for low range of $\epsilon_c = 0.05$ to 0.3 and high range of $\epsilon_c = 0.9$ to 0.95 to clearly distinguish between the two different physical mechanisms governing the flow at different ranges of porosity.

It is seen from the neutral curves that the critical Reynolds number ($Re_c$) and the critical wavenumber ($\alpha_c$) vary over different ranges of porosity. It is also seen from energy analysis that the instability in the flow at low values of porosity is quenched by viscous dissipation. At high values of porosity, the instability in the flow is quenched by viscous dissipation as well as porous dissipation. The scalings for the critical Reynolds number ($Re_c$) and critical wavenumber ($\alpha_c$) for the least stable eigenmode are presented in figures 4.14 and 4.15 respectively. The scaling law for $Re_c$ and $\alpha_c$ for least stable mode in the low and high porosity range are presented in table 4.9.

![Figure 4.14](image-url)

Figure 4.14: Scaling of the critical Reynolds number ($Re_c$) with respect to the vorticity thickness ($\delta$) for the least stable mode.

4.3.1 Low values of porosity

For low values of $\epsilon_c = 0.05 - 0.3$, $Re_c$ and $\alpha_c$, the critical Reynolds number ($Re_c$) and the critical wavenumber ($\alpha_c$) of least stable mode scales as $Re_c \sim (1/\delta)^0$ and $\alpha_c \sim (1/\delta)^0$. Both critical parameters are of the order $O(1)$ as they become invariant with the values of...
porosity \((\epsilon_c = 0.05 - 0.3)\). This is very much evident from the fact that at very low values of \(\epsilon_c\), the porous region almost behaves as a solid wall and hence the flow problem reduces to only Poiseuille channel flow confined in the fluid region. It is also seen from figure 4.4 for neutral curves, that as \(\epsilon\) increases from 0 to 0.3, the critical parameters \(Re_c\) changes from 3852.84 to 3738.13, \(\alpha_c\) remains constant at 2.04. At this range of \(\epsilon_c = 0.05 - 0.3\), the physical mechanism corresponding to instability in the flow comes from the viscous mechanism of Poiseuille flow in fluid region of the channel. It is very clear that the destabilizing mode stems from the viscous Tollmien-Schlichting (TS) wave instability of the fluid region at lower values of porosity.

![Figure 4.15: Scaling of the critical wave number \((\alpha_c)\) with respect to the vorticity thickness \((\delta)\) for the least stable mode.](image)

**4.3.2 High values of porosity**

For high values of porosity \((\epsilon_c = 0.9 - 0.95)\), the critical Reynolds number \((Re_c)\) and the critical wavenumber \((\alpha_c)\) of least stable mode scales as \(Re_c \sim (1/\delta)^{1.4}\) and \(\alpha_c \sim (1/\delta)^{-0.1}\). From the neutral curves, it is seen that the critical Reynolds number \((Re_c)\) changes drastically from 980.78 to 435.68 over the range \(\epsilon_c = 0.9 - 0.95\) while critical wavenumber \((\alpha_c)\) remains almost invariant. The scaling for \(Re_c\) in porosity range \((\epsilon_c = 0.9 - 0.95)\) shows the least stable mode is a kind of modified mode which arises due to interaction of flow in the fluid region and the porous region. The scaling for \(\alpha_c\) is very similar to that of the viscous instability mode which becomes invariant with the change in porosity. A similar characteristic mode has been identified as a mode different from Kelvin-Helmholtz (KH) mode by Singh et al. (2016). The mode originates as the flow in fluid region is influenced by the inflection point near the interface which becomes important as the porosity increases significantly. The characteristic feature of this mode is very different from an inviscid KH mode. It is very clear that the destabilizing mode is derived from fluid region which is significantly modified by the porous drag at the interface and is classified as a modified TS mode.
4.4 Non-modal Analysis

In this section, the result for the kinetic energy gain due to the perturbations are presented in the linearly stable region of the flow. Two different analysis known as resolvent norm and transient growth are discussed for different cases of porosity.

4.4.1 Resolvent Norm

The maximum energy amplification of initial disturbance (harmonic forcing) is measured as resolvent norm, calculated using equation 2.52 and is presented for different values of porosity. The maximum optimal energy gain for $\epsilon_c = 0.3$ and $0.6$ are presented in figure 4.16(a) and (b) at $Re_b = 2000$ respectively. The Reynolds number chosen is in the sub-critical range and well below the critical Reynolds number $Re_c$, for the onset of linear instability. The maximum optimal gain is plotted in logarithmic scale. It can be seen that the kinetic energy of a perturbation can grow by almost 1000 times. The maximum energy gain optimized over all ranges of $\alpha$ and $\beta$ is shown as a black dot in the resolvent norm map.

![Resolvent Norm Maps](image_url)

(a) $\epsilon_c = 0.3$  
(b) $\epsilon_c = 0.6$

(c) $\epsilon_c = 0.95$

Figure 4.16: Maximum energy growth ($\log_{10}(R_{\text{max}})$) for different values of porosity. The parameters are: $Re_b = 2000$ (for $\epsilon_c = 0.3, 0.6$), $Re_b = 150$ (for $\epsilon_c = 0.95$) and $\delta_i = 0.02$.

The maximum optimal energy gain for $\epsilon_c = 0.95$ at $Re_b = 150$ is shown in figure 4.16(c). It can be seen that at such low Reynolds number, the perturbation energy gain is around 4 times the initial harmonic forcing and is not very significant as seen in the case of $\epsilon_c = 0.3$ and $0.6$. Therefore there is no significant perturbation energy amplification due to the non-modality of the eigenmodes at high value of porosity.
The maximum energy amplification as function of the stream-wise and the span-wise wavenumbers are presented in figure 4.17 for different values of porosity. The maximum resolvent norm for purely two-dimensional perturbations, i.e. \( \beta = 0 \) are presented in figure 4.17(a) for \( \epsilon_c = 0.3, 0.6 \) and figure 4.17(b) for \( \epsilon_c = 0.95 \) respectively. The maximum energy growth for zero span-wise wavenumber \( (\beta = 0) \) takes place at \( \alpha_{opt} \approx 2 \) for all cases of porosity. The maximum resolvent energy growth along the span-wise line are presented in figure 4.17(c) for \( \epsilon_c = 0.3, 0.6 \) and figure 4.17(d) for \( \epsilon_c = 0.95 \) respectively. The maximum energy growth for stream-wise invariant perturbation, i.e. \( \alpha = 0 \), takes place at \( \beta_{opt} \approx 3 \) for all cases of porosity. The optimum values of stream-wise and span-wise wavenumber for different values of porosity are presented in table 4.10.

It can be seen from the resolvent norm analysis that zero stream-wise wavenumber corresponding to streaks structure, is the responsible optimal mechanism for energy amplification thereby possibly causing the flow transition.

Table 4.10: Optimum values of wave numbers in the stream-wise and the span-wise direction from resolvent norm for different values of porosity. The parameters are: \( Re_b = 2000 \) (for \( \epsilon_c = 0.3, 0.6 \)) and \( Re_b = 150 \) (for \( \epsilon_c = 0.95 \)).

<table>
<thead>
<tr>
<th>( \epsilon_c )</th>
<th>( \alpha_{opt} )</th>
<th>( \beta_{opt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0.6</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>0.95</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
The optimal harmonic amplification for different values of porosity as a function of the forcing frequency is shown in figure 4.18. The resolvent norm over the range of forcing frequencies for different values of $\epsilon_c = 0.3, 0.6$ and 0.95 corresponding to uniform span-wise perturbations ($\beta = 0$) are shown in figure 4.18(a) and uniform stream-wise perturbations ($\alpha = 0$) in figure 4.18(b) respectively. The maximum peak for the resolvent norm is seen again in case of $\alpha = 0$ and it is reached with steady forcing ($\omega = 0$) for all cases of porosity. In case of $\beta = 0$, it is seen for all values of porosity that there are strong peaks as well and these strong peaks correspond to the location of the least stable eigenvalues. In figure 4.18 for all values of porosity, the red line represents the resonant limit. This resonant limit is calculated based on the inverse of the minimal distance of the forcing frequency from the eigenspectrum of the linear system as shown in equation 2.53.

For a normal system matrix $L$, the response to forcing is determined by minimal distance of the forcing frequency to the eigenvalues of $L$. However for a non-normal system
matrix $L$, a large response to forcing can be achieved even though the forcing frequency is far away from the eigenvalues of $L$. This is referred to as pseudo-resonance. The difference between the red line and the line of maximum resolvent norm $((-i\omega I - L)^{-1})$ as shown in equation 2.52 represents the non-normal contribution of the eigenmodes.

As seen from the results in figure 4.17 that the optimal mechanism responsible for the energy amplification is due to uniform stream-wise perturbations i.e. the streaks structure. Hence the dependence of the maximum energy amplification due to the resolvent growth on the span-wise wavenumber ($\beta$) at specified values of stream-wise wavenumber ($\alpha$) is calculated and presented in figure 4.19 for different values of porosity. From figure 4.19, it can be seen that for all cases of $\epsilon_c = 0.3$, 0.6 and 0.95, the maximum energy amplification is for $\alpha = 0$ and then the magnitude of energy amplification keeps on decreasing with increasing $\alpha$. Hence it can be concluded that the flow becomes more stable with the traveling oblique waves as compared to only span-wise waves. It can also be seen that the magnitude of the maximum energy amplification optimized over all ranges of $\alpha$ and $\beta$ decreases significantly at extremely high value of porosity ($\epsilon_c = 0.95$). Hence as the channel becomes highly porous, the flow stability is not susceptible to the resolvent energy growth mechanism.

![Figure 4.19](image)

**Figure 4.19:** Dependence of the maximum energy amplification ($R_{max}$) on span-wise wavenumber ($\beta$) at specified values of stream-wise wavenumber ($\alpha = 0(−), 0.05(−), 0.1(−), 0.5(−)$ and $1(−)$) for different values of porosity. The parameters are: $Re_b = 2000$ (for $\epsilon_c = 0.3, 0.6$) and $Re_b = 150$ (for $\epsilon_c = 0.95$).

### 4.4.2 Transient Growth

The transient growth of initial disturbance as calculated using equation 2.48 is presented for different cases of porosity. The maximum transient growth for $\epsilon_c = 0.3$ and 0.6 are
presented in figure 4.20(a) and (b) at \( Re_b = 2000 \). The Reynolds number chosen is in the sub-critical range and well below the critical Reynolds number \( Re_c \), for the onset of linear instability. The maximum transient growth is plotted in logarithmic scale. It can be seen that the kinetic energy of a perturbation can grow by a factor of roughly 1000 before the perturbation eventually decays due to the consideration of linearly stable regime of the flow. The maximum transient growth over all ranges of \( \alpha \) and \( \beta \) is shown as a black dot in the transient growth map. The maximum transient growth map for \( \epsilon_c = 0.95 \) is shown in figure 4.20(c) at \( Re_b = 300 \) in the sub-critical regime. Here it can be seen that the maximum transient growth is around 40 times the initial perturbation energy. The same phenomenon of limited transient growth is seen for the case of \( \epsilon_c = 0.98 \) in figure 4.20(d) due to the extremely low value of the Reynolds number \( Re_b = 100 \). Hence at very high values of porosity, there cannot be significant energy growth due to the non-normality of the eigenmodes.

Figure 4.20: Maximum transient growth for different values of porosity. The parameters are: \( Re_b = 2000 \) (for \( \epsilon_c = 0.3, 0.6 \), \( Re_b = 300 \) (for \( \epsilon_c = 0.95 \), \( Re_b = 75 \) (for \( \epsilon_c = 0.98 \)) and \( \delta_i = 0.02 \).

The maximum transient growth as function of the stream-wise and the span-wise wavenumbers are presented in figure 4.21 for different values of porosity. Figure 4.21(a) represents the maximum transient growth for purely two-dimensional perturbations, i.e. \( \beta = 0 \). The maximum transient growth for zero span-wise wavenumber (\( \beta = 0 \)) takes place at \( \alpha_{opt} \approx 3 \) for all cases of porosity. Figure 4.21(b) represents the maximum transient growth along the span-wise line. The maximum transient growth for stream-wise invariant perturbation, i.e. \( \alpha = 0 \), takes place at \( \beta_{opt} \approx 4 \) for all cases of porosity. In transient energy growth analysis, zero stream-wise wavenumber corresponds to streaks structure which is the responsible optimal mechanism for energy amplification thereby causing the flow transition.
Figure 4.21: Maximum transient growth for different porosity values $\epsilon_c = 0.3(-)$, $\epsilon_c = 0.6(-)$, $\epsilon_c = 0.95(-)$ and $\epsilon_c = 0.98(-)$. The parameters are: $Re_b = 2000$ (for $\epsilon_c = 0.3, 0.6$), $Re_b = 300$ (for $\epsilon_c = 0.95$) and $Re_b = 75$ (for $\epsilon_c = 0.98$).

The evolution of the optimal gain as a function of time is shown for different cases of porosity in figure 4.21. For case of $\epsilon_c = 0.3, 0.6$ from figure 4.21(a) and (b), it can be seen that again the maximum transient growth takes place for vanishing stream-wise wavenumber and the time taken to reach the maximum value of $G(t)$ is also more for the case of $\alpha = 0$ as compared to the case of $\beta = 0$.

Figure 4.22: Time evolution of the transient growth is shown for different porosity values. The parameters are: $Re_b = 2000$ (for $\epsilon_c = 0.3, 0.6$), $Re_b = 300$ (for $\epsilon_c = 0.95$) and $Re_b = 75$ (for $\epsilon_c = 0.98$).

This reason can be attributed to the fact that zero stream-wise wavenumber corresponds to the streaks structure which take much more time to develop in the flow as
compared to the span-wise rollers (corresponding to zero span-wise wavenumber). The same trend can be seen for the case of $\epsilon_c = 0.95, 0.98$ from figure 4.21(c) and (d) even though the order of time required for transient growth is much smaller than the case of $\epsilon_c = 0.3, 0.6$ respectively.

![Graph showing normalized maximum transient growth versus different values of porosity for different values of stream-wise wavenumber ($\alpha$). The parameters are $Re_b = 2000$, $\delta_i = 0.02$ and $\beta_{opt}$ for all cases.](image-url)

Figure 4.23: Normalized maximum transient growth versus different values of porosity for different values of stream-wise wavenumber ($\alpha$). The parameters are $Re_b = 2000$, $\delta_i = 0.02$ and $\beta_{opt}$ for all cases.

Table 4.11: Maximum transient growth $G_{max}$ for different values of porosity $\epsilon_c$ and stream-wise wavenumber $\alpha$. The parameters considered are $\beta_{opt}$, $\delta_i = 0.02$ and $Re_b = 2000$. The normalized maximum transient growth $\tilde{G}_{max}$ are plotted in figure 4.23.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\epsilon_c = 0.05$</th>
<th>$\epsilon_c = 0.1$</th>
<th>$\epsilon_c = 0.2$</th>
<th>$\epsilon_c = 0.3$</th>
<th>$\epsilon_c = 0.4$</th>
<th>$\epsilon_c = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0$</td>
<td>1763.8407</td>
<td>1763.8186</td>
<td>1763.7691</td>
<td>1763.7108</td>
<td>1763.6473</td>
<td>1763.5911</td>
</tr>
<tr>
<td>$\alpha = 0.05$</td>
<td>1709.5901</td>
<td>1709.5635</td>
<td>1709.4996</td>
<td>1709.4304</td>
<td>1709.3700</td>
<td>1709.3315</td>
</tr>
<tr>
<td>$\alpha = 0.1$</td>
<td>1622.2184</td>
<td>1622.1945</td>
<td>1622.1361</td>
<td>1622.0716</td>
<td>1622.0148</td>
<td>1622.0033</td>
</tr>
<tr>
<td>$\alpha = 0.5$</td>
<td>1029.2172</td>
<td>1029.1316</td>
<td>1028.9425</td>
<td>1028.7424</td>
<td>1028.5595</td>
<td>1028.4977</td>
</tr>
<tr>
<td>$\alpha = 1$</td>
<td>653.4858</td>
<td>653.4233</td>
<td>653.2873</td>
<td>653.1500</td>
<td>653.0432</td>
<td>653.0364</td>
</tr>
<tr>
<td>$\alpha = 2$</td>
<td>335.3133</td>
<td>335.2627</td>
<td>335.1539</td>
<td>335.0434</td>
<td>334.9522</td>
<td>334.9267</td>
</tr>
<tr>
<td>$\alpha = 3$</td>
<td>199.6138</td>
<td>199.5801</td>
<td>199.5077</td>
<td>199.4325</td>
<td>199.3644</td>
<td>199.3251</td>
</tr>
<tr>
<td>$\alpha = 4$</td>
<td>132.3680</td>
<td>132.3341</td>
<td>132.2606</td>
<td>132.1812</td>
<td>132.0997</td>
<td>132.0247</td>
</tr>
</tbody>
</table>

Figure 4.23 shows the effect of porosity ($\epsilon_c$) on the maximum transient growth ($\tilde{G}_{max}$) at different stream-wise wavenumbers ($\alpha$) for optimal value of the span-wise wavenumber ($\beta_{opt}$). The maximum transient growth ($\tilde{G}_{max}$) is obtained by normalizing $G_{max}$ value at different porosity with respect to $G_{max}$ value at porosity ($\epsilon_c = 0.05$) for each stream-wise wavenumber ($\alpha$). The corresponding values of $G_{max}$ at different values of $\epsilon_c$ and $\alpha$ are presented in table 4.11. It can be seen that the value of $G_{max}$ decreases with the increase of the stream-wise wavenumber ($\alpha$) for a specific porosity value. Hence the flow becomes more stable with the traveling oblique waves as compared to span-wise waves for particular porosity case. $\tilde{G}_{max}$ almost remains the same for case of $\alpha = 0$ for $\epsilon_c = 0.05 - 0.5$ showing that the optimal mechanism responsible for maximum transient growth is dominated by the streaks structure in this porosity range. The change in value of $G_{max}$ with the varying porosity values for a specific stream-wise wavenumber is also quite insignificant. The
maximum change in $G_{\text{max}}$ is seen for the case of $\alpha = 4$ which is around 0.3%. This reason can be attributed to the fact that the flow in the porous region over the porosity range of $\epsilon_c = 0.05 - 0.5$, is insignificant as compared to the upper fluid region. Hence the transient energy growth remains almost same as like the case of a Poiseuille channel flow confined in the fluid region of the channel. The higher values of porosity are not considered for this analysis owing to the reason that there is limited transient growth due to extremely low values of Reynolds number in the sub-critical regime.

![Optimal perturbation and response](image)

**Figure 4.24:** Velocity vectors in span-wise plane as optimal initial condition and stream-wise velocity as optimal response for different values of porosity. The parameters considered are $Re_b = 2000$ ($\epsilon_c = 0.3$ and 0.6) and $Re_b = 300$ ($\epsilon_c = 0.95$), $\alpha = 0$ and $\beta = 4$ for all cases.

The optimal initial condition and the response for different values of porosity are shown in figure 4.24. The optimal initial condition shows the stream-wise counter-rotating
vortices spread along the span-wise direction and the optimal response consists of stream-wise velocity perturbations and a pair of streaks along the wall-normal direction. This pair of streaks consists of alternate high and low speed streaks along the wall-normal direction. For the case of porosity $\epsilon_c = 0.3, 0.6$ and $0.95$ shown in figures 4.24(a), (b) and (c), the optimal initial condition is the vortices in the mean shear direction in the fluid region of the channel and the response in form of streaks appear along the mean shear direction. This shows that the lift-up mechanism (shear-tilting) for energy growth holds in the porous channel flow. In case of higher value of porosity $\epsilon_c = 0.95$ shown in figure 4.24(c), the initial optimal condition in form of perturbations are strongly influenced by the permeability of the porous region. Due to a wall-normal transpiration velocity at the interface, the energy gain due to lift-up effect is not significant.

Table 4.12: Optimum values of wave numbers in the stream-wise and the span-wise direction from transient growth for different values of porosity. The parameters are: $Re_b = 2000$ (for $\epsilon_c = 0.3, 0.6$), $Re_b = 300$ (for $\epsilon_c = 0.95$) and $Re_b = 75$ (for $\epsilon_c = 0.98$).

<table>
<thead>
<tr>
<th>$\epsilon_c$</th>
<th>$\alpha_{opt}$</th>
<th>$\beta_{opt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>0.6</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>0.95</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>0.98</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

The optimal values of the stream-wise ($\alpha$) and span-wise ($\beta$) wavenumbers for different cases of porosity are presented in table 4.12. It can be seen that the optimal values of $\alpha$ and $\beta$ over which the transient energy growth takes place remains constant over the different ranges of porosity values.
Chapter 5

Turbulent Porous Channel Flow

In this chapter, the results of non-modal stability analysis for porous channel flow in the turbulent regime has been presented. As a starting point, the numerical code is validated for the turbulent Poiseuille flow results which are presented in Appendix D.

5.1 Baseflow

The baseflow profiles are obtained for different values of porosity from the direct numerical simulations performed by Breugem et al. (2006) and interpolating them on the finite-differences grid using sufficient number of grid points enough to resolve the wall and the interface regions. In this case, the Reynolds number \( \text{Re}_b \) is calculated in terms of \( \bar{U}_b \), which is the bulk velocity averaged over the fluid region and is set to a constant value of 1 as chosen by Breugem et al. (2006). The perturbation velocities are normalized with \( \bar{U}_b \) and the spatial length scales are normalized with the fluid region height (\( H \)).

For \( \epsilon_c = 0 \), the baseflow is a regular Poiseuille flow in the upper half of the channel. For the case of \( \epsilon_c = 0.6 \), it behaves almost like a solid wall in the porous layer and hence the velocity profiles for \( \epsilon_c = 0 \) and 0.6 overlap each other. On the other hand, the velocity profiles for higher values of porosity are more skewed with the maximum velocity shifted...
towards the upper solid wall. For $\epsilon_c = 0.95$, the porous layer becomes permeable enough to allow some flow through it and a non-zero velocity magnitude is observed. As observed by Breugem et al. (2006) that the skin friction at the fluid-porous interface is significantly larger than the upper solid wall and as the bulk velocity is kept constant as 1 over the upper half of the channel, a significant increase in the maximum velocity is seen which is shifted towards the upper solid wall.

### 5.2 Non-modal Analysis

In the turbulent porous channel flow, two different approaches are considered to perform the non-modal analysis. The first approach is based on the estimation of the normalized turbulent viscosity ($\nu_t/\nu$) from the ratio of the Reynolds stress and viscous stress, recorded in DNS simulations performed by Breugem et al. (2006). The normalized turbulent viscosity profile for $\epsilon_c = 0$ is shown in figure 5.2 where the profile has been smoothened using a fitting function at the center of the fluid region, in order to avoid the singularities obtained as a consequence of zero velocity gradient. The linearized VANS equations are then obtained considering the effect of the turbulent viscosity given as,

$$
\frac{\partial u_i}{\partial t} = -\frac{1}{\epsilon} \frac{\partial}{\partial x_j} \left( \epsilon u_i u_j + \nu u_i \right) - \frac{\partial p}{\partial x_i} + \frac{1}{Re_b} \left( 1 + \frac{\nu}{\nu} \right) \frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{Re_b} \frac{\partial}{\partial x_j} \left( \frac{\nu}{\nu} \right) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \\
+ \frac{1}{\epsilon Re_b} \left( \frac{\partial \epsilon}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right) - \frac{1}{Re_b} \frac{Fo}{Da} \epsilon [|U_b|u_i + |U_i|u_i] + \frac{1}{Re_b} \left[ \frac{1}{\epsilon} \frac{\partial^2 \epsilon}{\partial x_j^2} - \frac{\epsilon}{Da} \right] u_i
$$

(5.1)

The linearized equations for each velocity component are presented in equation 2.23-2.25. The resolvent analysis for the above equation can then be performed in a similar way taking into account the external harmonic forcing as shown in equation 2.50.

![Figure 5.2: Turbulent viscosity profile for turbulent porous channel flow for $\epsilon_c = 0$ (Poiseuille flow in fluid region) extracted from DNS data by Breugem et al. (2006).](image-url)
Following the same approach, the VANS equations given in equations 2.1 and 2.2 are averaged for the turbulent porous channel flows. The following equations are obtained for the mean velocity fields which satisfies the averaged VANS equations,

\[ \frac{\partial}{\partial x_i} [\epsilon U_i] = 0 \]  

(5.2)

\[ \frac{\partial U_i}{\partial t} = -\frac{1}{\epsilon} \frac{\partial}{\partial x_j} [\epsilon U_i U_j] + \frac{1}{\epsilon Re_b} \frac{\partial^2 U_i}{\partial x_j^2} + \frac{1}{\epsilon Re_b} \frac{\partial \epsilon}{\partial x_j} U_i - \frac{1}{Re_b} \frac{F_o}{Da} \epsilon U_i \]  

(5.3)

where \( \frac{1}{\epsilon} \frac{\partial}{\partial x_j} [\epsilon u_i' u_j'] \) is the Reynolds stress tensor that originates from the velocity perturbations in turbulent porous channel flows.

Linearizing VANS equations 2.1 and 2.2 about the turbulent mean flow velocity (\( \bar{U}_i \)), and then subtracting the turbulent mean flow equation 5.3 from it, the following relation is obtained,

\[ \frac{\partial u_i}{\partial t} = -\frac{1}{\epsilon} \frac{\partial}{\partial x_j} (\epsilon U_i u_j + \epsilon u_i \bar{U}_j) - \frac{\partial p}{\partial x_i} + \frac{1}{\epsilon Re_b} \frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{\epsilon Re_b} \frac{\partial \epsilon}{\partial x_j} \frac{\partial u_i}{\partial x_j} \]  

(5.4)

where \( \frac{1}{\epsilon} \frac{\partial}{\partial x_j} [\epsilon u_i' u_j'] \) is the zero-mean harmonic forcing term which comes from the Reynolds stress. In case of turbulent porous channel flow, this Reynolds stress is responsible for producing the fluctuations in the flow and can be considered as a forcing input for resolvent analysis.

For turbulent channel flows, the resolvent norm is considered as a significant analysis to determine the optimal mechanism responsible for energy gain as shown by Moarref et al. (2014). Hence the resolvent analysis of the turbulent porous channel flow is performed to see the effect of the non-normality of the eigenmodes.

5.2.1 Resolvent Norm

The maximum energy amplification of initial disturbance (harmonic forcing), measured as resolvent norm using equation 2.52 is presented for turbulent porous channel flow for \( \epsilon_c = 0 \) in figure 5.3 which is the regular Poiseuille flow over the fluid region. The resolvent norm is shown in figure 5.3(a) using the turbulent viscosity approach and figure 5.3(b) using the linearized NS equation approach. The maximum optimal energy gain is plotted in logarithmic scale for both cases at \( Re_b = 5500 \) respectively. It can be seen that the kinetic energy of a perturbation can grow by almost 1000 times using the turbulent viscosity approach while the growth is almost 10000 times using the linearized NS equation approach. The maximum energy gain optimized over all ranges of \( \alpha \) and \( \beta \) is shown as a black dot in the resolvent norm map.
The maximum energy amplification as function of the stream-wise and the span-wise wavenumbers are compared in figure 5.4 for \( \epsilon_c = 0 \) at \( Re_b = 5500 \) using two different approaches. The maximum resolvent norm for purely two-dimensional perturbations, i.e. \( \beta = 0 \) is presented in figure 5.4(a). The maximum energy growth for zero span-wise wavenumber \( (\beta = 0) \) takes place at \( \alpha_{\text{opt}} \approx 4 \) for both turbulent viscosity approach and linearized NS equation approach. The maximum resolvent energy growth along the span-wise line is presented in figure 5.4(b). The maximum energy growth for stream-wise invariant perturbations, i.e. \( \alpha = 0 \), takes place at \( \beta_{\text{opt}} \approx 2 \) using both approaches. It can be also seen from the resolvent norm analysis that zero stream-wise wavenumber corresponding to streaks structure, is the responsible optimal mechanism for energy amplification as compared to two-dimensional perturbations \( (\beta = 0) \) corresponding to the span-wise rollers.

Since the optimal mechanism for resolvent energy growth corresponds to uniform stream-wise perturbations (streaks structure), the dependence of the maximum energy amplification due to the resolvent growth on the span-wise wavenumber \( (\beta) \) at specified values of stream-wise wavenumber \( (\alpha) \) is calculated and presented in figure 5.5 for \( \epsilon_c = 0 \). From figure 5.5(a) and (b), it can be seen that for both the approaches, the maximum energy amplification is for \( \alpha = 0 \) and then the magnitude of energy amplification keeps on decreasing with increasing \( \alpha \). Hence it can be seen that the flow becomes more stable with
the traveling oblique waves as compared to only span-wise waves. However the magnitude of the maximum energy amplification following both the approaches are significantly different, but the prediction of the optimal wavenumbers over which the energy amplification takes place from both the approaches are approximately equal.

(a) Using turbulent viscosity

(b) Using linearized NS equation

Figure 5.5: Dependence of maximum energy amplification \( R_{\text{max}} \) on span-wise wavenumber \( \beta \) at specific values of stream-wise wavenumber \( \alpha \) for porosity, \( \epsilon_c = 0 \). The parameters are: \( Re_b = 5500 \) and \( \delta_i = 0.02 \). Here \( \alpha = 0, 0.05, 0.1, 0.5, 1, 2 \) in the direction of arrow.

Figure 5.6: Resolvent norm \( \| (-i\omega I - L)^{-1} \| \) comparison for \( \epsilon_c = 0 \). Top panel shows results using turbulent viscosity and bottom panel shows results using linearized NS equation. The parameters are: \( Re_b = 5500 \) and \( \delta_i = 0.02 \). The resonant limit is shown as red line in both cases.

The optimal harmonic amplification for \( \epsilon_c = 0 \) as a function of the forcing frequency
is shown in figure 5.6 for both the approaches. The resolvent norm over the range of forcing frequencies corresponding to uniform span-wise perturbations ($\beta = 0$) are shown in figure 5.6(a) and (c) and uniform stream-wise perturbations ($\alpha = 0$) in figure 5.6(b) and (d) respectively. The maximum peak for the resolvent norm is seen again in case of $\alpha = 0$ and it is reached with steady forcing ($\omega = 0$) for both the cases. In case of $\beta = 0$, it is seen that there are strong peaks as well corresponding to the location of the least stable eigenvalues. The resonant limit calculated using equation 2.53 is also shown. The non-normality of the system matrix $L$ shows large response due to the pseudo-resonance phenomenon. The difference between the resonant limit and the corresponding line of maximum resolvent norm $\left(( -i\omega I - L)^{-1}\right)$ represents the non-normal energy contribution due to the non-orthogonality of the eigenmodes. Again it can be seen that the peak frequencies calculated for two-dimensional perturbations as $\omega = 4$ and uniform stream-wise perturbations as $\omega = 0$ match very well for the turbulent viscosity approach as well as the linearized NS equation approach. Henceforth, the resolvent norm for the different values of porosity $\epsilon_c = 0.6, 0.8$ and 0.95 will be calculated using the linearized NS equation approach only since it avoids the difficulty of calculating the turbulent viscosity.

Figure 5.7: Maximum energy amplification ($\log(R_{\text{max}})$) for different values of porosity computed using linearized NS equation. The parameters are: $Re_b = 5500$ and $\delta_i = 0.02$. The maximum energy gain of initial disturbance (harmonic forcing) for turbulent porous channel flow calculated using linearized NS equation approach is presented for $\epsilon_c = 0.6, 0.8$ and 0.95 in figure 5.7. The maximum optimal energy gain is plotted in logarithmic scale for all cases at $Re_b = 5500$ respectively. For cases of $\epsilon_c = 0.6$ and 0.8 as show in figure 5.7(a) and (b), it can be seen that the kinetic energy of a perturbation grow by almost 2000 times. For $\epsilon_c = 0.95$, the kinetic energy gain increases and is almost 10000 times as shown in figure 5.7(c). The maximum energy gain optimized over all ranges of

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\( \alpha \) and \( \beta \) is shown as a black dot in the resolvent norm map.

Figure 5.8: Optimal initial condition and optimal response for porosity, \( \epsilon_c = 0.95 \). The parameters are: \( Re_b = 5500 \) and \( \delta_i = 0.02 \).

For the case of \( \epsilon_c = 0.95 \), the maximum energy amplification is seen along \( \alpha = 0 \) and small energy gain is also seen along \( \beta = 0 \). The optimal initial condition and the response for \( \epsilon_c = 0.95 \) is shown in figure 5.8. Figure 5.8(a) shows the optimal initial condition in form of counter-rotating vortices spread along span-wise direction and the optimal response consists of stream-wise streaks in the fluid region of the channel. This corresponds to the energy gain due to streaks by the lift-up mechanism (shear-tilting) for \( \alpha = 0 \) and \( \beta = 3 \). Figure 5.8(b) represents the initial condition in form of vortices along the stream-wise direction and the optimal response in form of rollers along the stream-wise direction. This corresponds to the energy gain due to rollers by the Orr-mechanism (tilting of the vortices along the mean shear of the flow) for \( \beta = 0 \) and \( \alpha = 0.6 \). The optimal energy gain mechanism is due to the streaks only with small contribution from the rollers as well as seen from figure 5.7(c).

The optimal harmonic amplification for different values of porosity as a function of the forcing frequency is shown in figure 5.9. From the resolvent norm over the range of forcing frequencies corresponding to uniform span-wise perturbations (\( \beta = 0 \)) for \( \epsilon_c = 0.6, 0.8 \) and 0.95 shown in figure 5.9(a), (c) and (e), it can be seen that there are strong peaks of pseudo-resonances due to the non-normal system matrix \( \mathcal{L} \). However for \( \epsilon_c = 0.95 \), the resolvent growth averaged over the range of forcing frequencies is much higher as compared to the cases of \( \epsilon_c = 0.6 \) and 0.8. The resonant limit due to the normal energy gain also has
several peaks for $\epsilon_c = 0.95$ as compared to $\epsilon_c = 0.6$ and 0.8. This shows that the energy gain due to the rollers ($\beta = 0$) is very significant for $\epsilon_c = 0.95$. The resolvent norm for the uniform stream-wise perturbations ($\alpha = 0$) for different values of porosity are shown in figure 5.9(b), (d) and (f) respectively. The maximum peak for the resolvent norm in case of $\alpha = 0$ is reached with steady forcing ($\omega = 0$) for all cases of porosity. The energy gain due to the non-normality of eigenmodes (difference between the resolvent norm and resonant limit) is higher for $\epsilon_c = 0.95$ as compared to $\epsilon_c = 0.6$ and 0.8. It is also seen that for $\epsilon_c = 0.6$, 0.8 and 0.95, the optimal mechanism for maximum energy amplification is achieved due to the streaks structures ($\alpha = 0$).

Figure 5.9: Resolvent norm $\left( (-i\omega I - L)^{-1} \right)$ for different values of porosity computed using linearized NS equation. The parameters are: $Re_b = 5500$ and $\delta_i = 0.02$. The resonant limit is shown as red line in all cases.

The dependence of the maximum energy amplification due to the resolvent growth on the span-wise wavenumber ($\beta$) at specified values of stream-wise wavenumber ($\alpha$) is presented in figure 5.10 for $\epsilon_c = 0.6$, 0.8 and 0.95. For $\epsilon_c = 0.6$ and 0.8, it is seen that
the value of $\beta_{opt} \approx 4$ corresponding to the line $\alpha = 0$. For $\epsilon_c = 0.95$, it is seen that the value of $\beta_{opt} \approx 3$ corresponding to the line $\alpha = 0$. From figure 5.10(a), (b) and (c), it can be seen that the maximum energy amplification is for $\alpha = 0$ and the magnitude of energy amplification keeps on decreasing with increasing $\alpha$. So the flow becomes more stable with the traveling oblique waves as compared to only span-wise waves. In case of $\epsilon_c = 0.95$, there is sudden increase in the energy gain over the span-wise wavenumber range ($\beta = 1 - 7$) which appear as sudden bumps as shown in figure 5.10(c). This is related to the small defect observed in the baseflow velocity profile near the fluid-porous interface for $\epsilon_c = 0.95$ as shown in figure 5.1. The baseflow for $\epsilon_c = 0.95$ therefore need to be averaged over more time steps in order to obtain a smooth velocity profile at the interface. This would avoid these sudden peaks in the energy amplification curve.

![Graphs of Maximum Energy Amplification for different values of porosity](image)

Figure 5.10: (a)-(c) Maximum energy amplification ($R_{max}$) for different values of porosity computed using linearized NS equation. Here $\alpha = 0, 0.1, 0.5, 1, 2, 3$ in the direction of arrow. (d) Comparison of $R_{max}$ at $\alpha = 0$ for $\epsilon_c = 0.6$ ($-$), $\epsilon_c = 0.8$ ($-$) and $\epsilon_c = 0.95$ ($-$). The parameters are: $Re_b = 5500$ and $\delta_i = 0.02$.

The maximum energy amplification due to the resolvent growth for uniform stream-wise perturbations ($\alpha = 0$) for $\epsilon_c = 0.6, 0.8$ and 0.95 is shown in figure 5.10(d). It can be seen that the maximum energy gain increases with $\epsilon_c$ changing from 0.6 to 0.95. From the comparison of all cases of porosity for turbulent channel flow corresponding to streaks structures ($\alpha = 0$), it can be seen that the energy gain is maximum at $\epsilon_c = 0$ and then it decreases for $\epsilon_c = 0.6$ and 0.8, and subsequently again increases for $\epsilon_c = 0.95$. 

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Chapter 6

Conclusions

6.1 Conclusions

From the analysis, it can be seen that the VANS equations with the porosity distribution function used for modeling porous flows very well predict the onset of instabilities in the flow. The 4th order dispersion relation preserving scheme with the implementation of the grid stretching function provides good resolution of the flow along the wall-normal direction with sufficient number of grid points. This saves a lot of computer memory and computation time as compared to using the Chebyshev spectral discretization. The Squire transformation holds for the porous channel flows and hence it simplifies the stability analysis with the consideration of only two dimensional perturbations to determine the linearly unstable region on the neutral curve map.

The eigenspectrum at lower values of porosity shows the occurrence of the least stable mode originating from the fluid region of the channel as seen from the shape of the eigenmode. At higher values of porosity, the least stable mode comes from the fluid region of the channel which undergoes a strong flow reversal at the interface. The energy analysis shows that perturbation kinetic energy distribution for lower values of porosity is very much similar to that of a Poiseuille flow. At higher values of porosity, the dissipation from the porous region becomes significant over the viscous dissipation due to attainment of non-zero slip velocity at the fluid-porous interface. The scaling analysis shows good congruence of the scaling of the critical Reynolds number and wavenumber of the least stable mode at lower values of porosity with that of the viscid Tollmien-Schlichting instability wave characteristics. While at higher values of porosity, the scaling of critical Reynolds number and wavenumber of least stable mode shows the evolution of a modified mode which originates from fluid region but strongly influenced by porous drag at the interface. From the studies, it can be concluded that the primary source of instability at lower values of porosity originates from the viscid fluid region and is classified as TS mode. At higher values of porosity, the least stable mode still comes from the fluid region but is strongly modified by the drag from the porous region at the interface and is classified as a modified TS mode.

Non-modal analysis for laminar flow shows the optimal mechanism responsible for the maximum energy amplification in the sub-critical regime over different ranges of porosity is due to streaky structures. The resolvent norm analysis shows significant energy growth due to the non-normality of the eigenmodes at lower values of porosity. At higher values of porosity, the non-normal growth is significantly reduced. Transient growth also shows
Conclusions

the energy growth over all ranges of porosity is due to the streaky structures for which the optimal mechanism is Lift-up effect. The non-normal energy growth becomes insignificant at extremely higher values of porosity but still the minimal energy growth takes place due to the streaks.

The resolvent norm analysis for turbulent flow using the linearized NS equation shows that the approach is a good approximation and hence the problem of calculating the turbulent viscosity can be avoided. From the resolvent maps, it can be concluded that for turbulent baseflow the perturbations attain maximum energy in the sub-critical regime due to the streaky structures. However at extremely high value of porosity, the energy gain takes place due to the streaky structures as well as span-wise rollers but the dominant energy gain mechanism is due to the streaks.

6.2 Perspectives for Future Work

The linear stability analysis is very well able to predict the origin and the physical mechanism of the governing instabilities in porous channel flow. However more insight into the topic can be attained by calculating the time variation of the perturbation energy growth from the direct numerical simulations (DNS) to see if it is in agreement with the results predicted from linear stability analysis. The snapshots of proper orthogonal decomposition (POD) of the flow perturbations from DNS would give an idea about the non-linear evolution of the perturbation energy optimized over the POD modes.

The scaling analysis provides the characteristics of origin of the least stable eigenmode. The comparison of the viscous length scale \( \nu/u_\tau \) and the permeability length scale \( \sqrt{K/\epsilon_c} \) would give a better insight about the profound influence of the permeable wall on the flow near the interface.

The non-modal resolvent norm analysis for maximum energy amplification in this research work has been carried out considering only harmonic forcing frequencies. However in real situations most of the internal and external sources of disturbances in the flow are stochastic in nature as shown by Schmid (2007). Hence the resolvent norm analysis for turbulent porous channel flows can be performed using external stochastic forcing to find the optimal mechanism for energy growth.

The turbulent porous channel flow can be analyzed in details to see the validity of linear stability analysis. The resolvent energy growth can be evaluated considering the effect of turbulent viscosity to see if the optimal mechanism for energy growth remains the same as seen from linearized VANS equations approach. In this case, the turbulent viscosity for different values of porosity has to be calculated by using some smoothing function at the point of discontinuities.

This analysis on porous channel flows can be extended to perform the linear stability analysis of porous ducts, where the secondary flow structures will have a profound impact on the instabilities in the flow as shown by Samanta et al. (2015). The analysis for porous ducts can be carried out to see the effect of the varying aspect ratios on the instabilities in the flow and after a certain critical value of the aspect ratio, it is anticipated that the porous duct will behave like a porous channel where the secondary flow instabilities will have no effect on the baseflow.
Appendices
Appendix A

Eigenvalue Problem

A.1 Linearized Matrix

A is the linearized operator matrix on the velocity vector as shown in equation 2.26 and is presented for both the laminar and turbulent porous channel flow case.

A.1.1 Laminar Porous Channel Flow

For the laminar porous channel flow, A is given as,

\[
A = \begin{bmatrix}
-\tilde{U} \frac{\partial}{\partial x} + \frac{1}{Re_b} \nabla^2 + \frac{1}{\epsilon Re_b} \left( \frac{\partial \epsilon}{\partial y} \frac{\partial \tilde{U}}{\partial y} \right) & 0 \\
-\frac{1}{Re_b} \frac{F_o}{Da} |\tilde{U}| + \frac{1}{Re_b} \left[ \nabla^2 \frac{\epsilon}{\epsilon} - \frac{\epsilon}{Da} \right] & -\frac{\partial \tilde{U}}{\partial y} \\
0 & -\frac{U}{\epsilon} \frac{\partial}{\partial x} + \frac{1}{Re_b} \nabla^2 + \frac{1}{\epsilon Re_b} \left( \frac{\partial \epsilon}{\partial y} \frac{\partial \tilde{U}}{\partial y} \right) \\
0 & -\frac{1}{Re_b} \frac{F_o}{Da} |\tilde{U}| + \frac{1}{Re_b} \left[ \nabla^2 \frac{\epsilon}{\epsilon} - \frac{\epsilon}{Da} \right] \\
0 & 0 & -\tilde{U} \frac{\partial}{\partial x} + \frac{1}{Re_b} \nabla^2 + \frac{1}{\epsilon Re_b} \left( \frac{\partial \epsilon}{\partial y} \frac{\partial \tilde{U}}{\partial y} \right) \\
0 & 0 & -\frac{1}{Re_b} \frac{F_o}{Da} |\tilde{U}| + \frac{1}{Re_b} \left[ \nabla^2 \frac{\epsilon}{\epsilon} - \frac{\epsilon}{Da} \right]
\end{bmatrix}
\]

(A.1)

A.1.2 Turbulent Porous Channel Flow

The VANS equations for turbulent channel flow considering the turbulent viscosity is given as,

\[
\nabla \cdot [\epsilon u] = 0 \quad (A.2)
\]

\[
\partial_t u = -\frac{1}{\epsilon} \nabla \cdot [\epsilon u \otimes u] - \nabla p + \frac{1}{Re_b} \nabla \cdot \left[ \frac{\nu_T}{\nu} \left( \nabla u + \nabla u^T \right) \right] + \frac{1}{\epsilon Re_b} \nabla \epsilon \cdot \nabla u \\
-\frac{1}{Re_b} \frac{F_o}{Da} |u| u + \frac{1}{Re_b} \left[ \nabla^2 \frac{\epsilon}{\epsilon} - \frac{\epsilon}{Da} \right] u \\
\]

(A.3)

Linearizing the VANS equations and subtracting from the turbulent baseflow for each of the coordinate directions yields,
\[ \dot{u} = - \bar{U} \frac{\partial u}{\partial x} - v \frac{\partial U}{\partial y} - \frac{\partial p}{\partial x} + \frac{1}{Re_b} \left( 1 + \frac{\nu_t}{\nu} \right) \nabla^2 u + \frac{1}{Re_b} \left( \frac{\nu_t}{\nu} \right)' \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \frac{1}{\epsilon Re_b} \frac{\partial \epsilon}{\partial y} \frac{\partial u}{\partial y} \]
\[ - \frac{1}{Re_b Da} 2 \epsilon |\bar{U}| u + \frac{1}{Re_b} \left[ \nabla^2 \epsilon \frac{\epsilon}{\epsilon} - \epsilon \frac{Da}{Da} \right] u \]  
\[ (A.4) \]

\[ \dot{v} = - \bar{U} \frac{\partial v}{\partial x} - \frac{\partial p}{\partial y} + \frac{1}{Re_b} \left( 1 + \frac{\nu_t}{\nu} \right) \nabla^2 v + \frac{1}{Re_b} \left( \frac{\nu_t}{\nu} \right)' \left( \frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) + \frac{1}{\epsilon Re_b} \frac{\partial \epsilon}{\partial y} \frac{\partial v}{\partial y} \]
\[ - \frac{1}{Re_b Da} 2 \epsilon |\bar{U}| v + \frac{1}{Re_b} \left[ \frac{\nabla^2 \epsilon}{\epsilon} - \frac{\epsilon}{Da} \right] v \]  
\[ (A.5) \]

\[ \dot{w} = - \bar{U} \frac{\partial w}{\partial x} - \frac{\partial p}{\partial z} + \frac{1}{Re_b} \left( 1 + \frac{\nu_t}{\nu} \right) \nabla^2 w + \frac{1}{Re_b} \left( \frac{\nu_t}{\nu} \right)' \left( \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} \right) + \frac{1}{\epsilon Re_b} \frac{\partial \epsilon}{\partial y} \frac{\partial w}{\partial y} \]
\[ - \frac{1}{Re_b Da} 2 \epsilon |\bar{U}| w + \frac{1}{Re_b} \left[ \nabla^2 \epsilon \frac{\epsilon}{\epsilon} - \epsilon \frac{Da}{Da} \right] w \]  
\[ (A.6) \]

where \( \nu_T = \nu_t + \nu \) is the total viscosity.

For the turbulent porous channel flow, \( A \) is given as,
\[
A = \begin{bmatrix}
- \bar{U} \frac{\partial}{\partial x} + \frac{1}{Re_b} \left( 1 + \frac{\nu_t}{\nu} \right) \nabla^2 \\
+ \frac{1}{Re_b} \left( \frac{\nu_t}{\nu} \right)' \left( \frac{\partial}{\partial y} \right) \\
+ \frac{1}{\epsilon Re_b} \frac{\partial \epsilon}{\partial y} \\
- \frac{1}{Re_b Da} 2 \epsilon |\bar{U}| \\
+ \frac{1}{Re_b} \left[ \nabla^2 \epsilon \frac{\epsilon}{\epsilon} - \epsilon \frac{Da}{Da} \right]
\end{bmatrix}
\]
\[
- \bar{U} \frac{\partial}{\partial x} + \frac{1}{Re_b} \left( 1 + \frac{\nu_t}{\nu} \right) \nabla^2 \\
+ \frac{2}{Re_b} \left( \frac{\nu_t}{\nu} \right)' \left( \frac{\partial}{\partial y} \right) \\
+ \frac{1}{\epsilon Re_b} \frac{\partial \epsilon}{\partial y} \\
- \frac{1}{Re_b Da} 2 \epsilon |\bar{U}| \\
+ \frac{1}{Re_b} \left[ \nabla^2 \epsilon \frac{\epsilon}{\epsilon} - \epsilon \frac{Da}{Da} \right]
\end{bmatrix}
\]
\[ (A.7) \]
Appendix B

Discretization Schemes and Coefficients

B.1 FODRP Scheme

B.1.1 Derivation of FODRP Coefficients

The 4th order dispersion relation preserving scheme is computed as following. The spatial derivative for finite-difference scheme is given in the following form,

$$\left( \frac{\partial u}{\partial \xi} \right)_i = \frac{1}{\Delta \xi} \sum_{j=-3}^{3} a_j u_{i+j} \quad (B.1)$$

A polynomial approach is followed to determine the optimized coefficients. Using the polynomial approach, the function \(u\) for 4th order accuracy can be written in the following form,

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4 \quad (B.2)$$

where \(x\) is the distance from the point of derivative.

For DRP scheme, 6 points are required for expressing the derivative. Expanding equation B.2 for different points in the domain,

\[
\begin{align*}
    u_{-3} &= u_0 - 3B\Delta\xi + 9C\Delta\xi^2 - 27D\Delta\xi^3 + 81E\Delta\xi^4 \\
    u_{-2} &= u_0 - 2B\Delta\xi + 4C\Delta\xi^2 - 8D\Delta\xi^3 + 16E\Delta\xi^4 \\
    u_{-1} &= u_0 - B\Delta\xi + C\Delta\xi^2 - D\Delta\xi^3 + E\Delta\xi^4 \\
    u_1 &= u_0 + B\Delta\xi + C\Delta\xi^2 + D\Delta\xi^3 + E\Delta\xi^4 \\
    u_2 &= u_0 + 2B\Delta\xi + 4C\Delta\xi^2 + 8D\Delta\xi^3 + 16E\Delta\xi^4 \\
    u_3 &= u_0 + 3B\Delta\xi + 9C\Delta\xi^2 + 27D\Delta\xi^3 + 81E\Delta\xi^4
\end{align*}
\]

Subtracting equation B.4 from equation B.7 and equation B.3 from equation B.8, we obtain,

\[
\begin{align*}
    u_2 - u_{-2} &= 4B\Delta\xi + 16D\Delta\xi^3 \quad (B.9) \\
    u_3 - u_{-3} &= 6B\Delta\xi + 54D\Delta\xi^3 \quad (B.10)
\end{align*}
\]

Multiplying equation B.9 by 27 and equation B.10 by 8 and subtracting, we get,

$$27(u_2 - u_{-2}) - 8(u_3 - u_{-3}) = 60B\Delta\xi \quad (B.11)$$
Rearranging equation B.11, the finite-difference equation is given as,
\[
\left( \frac{\partial u}{\partial \xi} \right)_i = \frac{27(u_2 - u_{-2}) - 8(u_3 - u_{-3})}{60\Delta \xi}
\]  
(B.12)

Subtracting equation B.5 from equation B.6, we obtain,
\[
u_1 - u_{-1} = 2B\Delta \xi + 2D\Delta \xi^3
\]  
(B.13)

Multiplying equation B.13 by 8 and subtracting equation B.9, we get,
\[
8(u_1 - u_{-1}) - (u_2 - u_{-2}) = 12B\Delta \xi
\]  
(B.14)

Rearranging equation B.14, the finite-difference equation is given as,
\[
\left( \frac{\partial u}{\partial \xi} \right)_i = \frac{8(u_1 - u_{-1}) - (u_2 - u_{-2})}{12\Delta \xi}
\]  
(B.15)

Equating equation B.12 and equation B.15, we obtain,
\[
8(u_1 - u_{-1}) - (u_2 - u_{-2}) = \frac{27(u_2 - u_{-2}) - 8(u_3 - u_{-3})}{5}
\]  
(B.16)

Rearranging equation B.16, we can write it in the following form,
\[
(u_1 - u_{-1}) = \frac{4}{5}(u_2 - u_{-2}) - \frac{1}{5}(u_3 - u_{-3})
\Rightarrow a_1(u_1 - u_{-1}) = \frac{4a_1}{5}(u_2 - u_{-2}) - \frac{a_1}{5}(u_3 - u_{-3})
\]  
(B.17)

where \( a_1 \) is the free parameter used to minimize the error. As there are more points in the finite-difference scheme than required for obtaining 4th order accuracy, one free parameter is required to minimize error.

Equation B.12 for finite-difference derivative can be expanded in terms of 6 points as shown below,
\[
\left( \frac{\partial u}{\partial \xi} \right)_i = \frac{1}{\Delta \xi} \left[ \frac{27}{60}(u_2 - u_{-2}) - \frac{8}{60}(u_3 - u_{-3}) \right]
= \frac{1}{\Delta \xi} \left[ \left( \frac{27}{60} - \frac{4a_1}{5} \right)(u_2 - u_{-2}) - \left( \frac{8}{60} - \frac{a_1}{5} \right)(u_3 - u_{-3}) + \frac{4a_1}{5}(u_2 - u_{-2}) - \frac{a_1}{5}(u_3 - u_{-3}) \right]
\]  
(B.18)

Substituting the value from equation B.17 in equation B.18, we obtain,
\[
\left( \frac{\partial u}{\partial \xi} \right)_i = \frac{1}{\Delta \xi} \left[ \left( \frac{27}{60} - \frac{4a_1}{5} \right)(u_2 - u_{-2}) - \left( \frac{8}{60} - \frac{a_1}{5} \right)(u_3 - u_{-3}) + a_1(u_1 - u_{-1}) \right]
\]  
(B.19)

Using the condition \( a_j = -a_{-j} \) for first order derivative, we get,
\[
a_1 = -a_{-1} = a_1
\]  
(B.20)

\[
a_2 = -a_{-2} = \left( \frac{27}{60} - \frac{4a_1}{5} \right)
\]  
(B.21)

\[
a_3 = -a_{-3} = -\left( \frac{8}{60} - \frac{a_1}{5} \right)
\]  
(B.22)
For optimizing the coefficients, the error minimization equation B.17 is used as shown below,

\[
E = \int_{-\pi/2}^{\pi/2} |i k \Delta \xi - \sum_{j=-3}^{3} a_j e^{i j k \Delta \xi}|^2 dk \Delta \xi
\]

\[
= \int_{-\pi/2}^{\pi/2} |i k \Delta \xi - a_1(e^{i k \Delta \xi} - e^{-i k \Delta \xi}) - a_2(e^{i 2k \Delta \xi} - e^{-i 2k \Delta \xi}) - a_3(e^{i 3k \Delta \xi} - e^{-i 3k \Delta \xi})|^2 dk \Delta \xi
\]

(B.23)

Substituting the value of \(a_2\) and \(a_3\) from equations B.21 and B.22 in equation B.23, we obtain,

\[
E = \int_{-\pi/2}^{\pi/2} |i k \Delta \xi - a_1(e^{i k \Delta \xi} - e^{-i k \Delta \xi}) - \left(\frac{27}{60} - \frac{4a_1}{5}\right)(e^{i 2k \Delta \xi} - e^{-i 2k \Delta \xi}) + \left(\frac{8}{60} - \frac{a_1}{5}\right)(e^{i 3k \Delta \xi} - e^{-i 3k \Delta \xi})|^2 dk \Delta \xi
\]

(B.24)

Computing the definite integration using MAPLE (Release 2015.2), we obtain,

\[
E = 0.998417987a_1^2 - 1.596003948a_1 + 0.6379505815
\]

(B.25)

For the error to be minimum, we perform,

\[
\frac{\partial E}{\partial a_1} = 0
\]

\[
\Rightarrow 1.996835974a_1 - 1.596003948 = 0
\]

\[
\Rightarrow a_1 = 0.7992664269742
\]

(B.26)

The values of coefficients \(a_2\) and \(a_3\) are found using equation B.21 and equation B.22 and given below.

\[
a_2 = -0.1894131415793 \quad a_3 = 0.0265199520615
\]

(B.27)

Similarly the coefficients for the second order derivatives are derived for the DRP finite-difference scheme with the condition \(a_j = a_{-j}\).

### B.1.2 Coefficients for First Order Derivatives

\[
\begin{align*}
    a_{44}(-3) &= -0.0265199520615 \\
    a_{44}(-2) &= 0.1894131415793 \\
    a_{44}(-1) &= -0.7992664269742 \\
    a_{44}(0) &= 0 \\
    a_{44}(1) &= 0.7992664269742 \\
    a_{44}(2) &= -0.1894131415793 \\
    a_{44}(3) &= 0.0265199520615
\end{align*}
\]

\[
\begin{align*}
    a_{33}(-2) &= 0.0833333333333 \quad a_{33}(-1) = -0.6666666666667 \\
    a_{33}(0) &= 0 \\
    a_{33}(1) &= 0.6666666666667 \\
    a_{33}(2) &= -0.0833333333333
\end{align*}
\]

(B.28a)

\[
\begin{align*}
    a_{11}(0) &= -2.0833333333333 \quad a_{11}(1) = 4 \\
    a_{11}(2) &= -3 \\
    a_{11}(3) &= 1.3333333333333 \\
    a_{11}(4) &= -0.25
\end{align*}
\]

(B.28b)
B.1.3 Coefficients for Second Order Derivatives

\[
\begin{align*}
\begin{cases}
    b_{44}(-3) &= 0.0157364441107 \\
    b_{44}(-2) &= -0.1777519979975 \\
    b_{44}(-1) &= 1.5693799949938 \\
    b_{44}(0) &= -2.814728822140 \\
    b_{44}(1) &= 1.5693799949938 \\
    b_{44}(2) &= -0.1777519979975 \\
    b_{44}(3) &= 0.0157364441107
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    b_{33}(-2) &= -0.0833333333333 \\
    b_{33}(-1) &= 1.3333333333333 \\
    b_{33}(0) &= -2.5 \\
    b_{33}(1) &= 1.3333333333333 \\
    b_{33}(2) &= -0.0833333333333
\end{cases}
(B.29a)
\end{align*}
\]

B.2 FOFD Scheme

B.2.1 Coefficients for First Order Derivatives

\[
\begin{align*}
\begin{cases}
    a_{33}(-2) &= 0.0833333333333 \\
    a_{33}(-1) &= -0.6666666666667 \\
    a_{33}(0) &= 0 \\
    a_{33}(1) &= 0.6666666666667 \\
    a_{33}(2) &= -0.0833333333333
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    a_{22}(-1) &= -0.25 \\
    a_{22}(0) &= -0.8333333333333 \\
    a_{22}(1) &= 1.5 \\
    a_{22}(2) &= -0.5 \\
    a_{22}(3) &= 0.0833333333333
\end{cases}
(B.30a)
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    a_{11}(0) &= -2.0833333333333 \\
    a_{11}(1) &= 4 \\
    a_{11}(2) &= -3 \\
    a_{11}(3) &= 1.3333333333333 \\
    a_{11}(4) &= -0.25
\end{cases}
(B.30b)
\end{align*}
\]

B.2.2 Coefficients for Second Order Derivatives

\[
\begin{align*}
\begin{cases}
    b_{44}(-2) &= -0.0833333333333 \\
    b_{44}(-1) &= 1.3333333333333 \\
    b_{44}(0) &= -2.5 \\
    b_{44}(1) &= 1.3333333333333 \\
    b_{44}(2) &= -0.0833333333333
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    b_{33}(-1) &= 0.8333333333333 \\
    b_{33}(0) &= -1.25 \\
    b_{33}(1) &= 17.8333333333333 \\
    b_{33}(2) &= -0.8333333333333 \\
    b_{33}(3) &= 5.0833333333333 \\
    b_{33}(4) &= 0.0833333333333
\end{cases}
(B.31a)
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
    b_{22}(0) &= 3.75 \\
    b_{22}(1) &= -12.8333333333333 \\
    b_{22}(2) &= 17.8333333333333 \\
    b_{22}(3) &= -13 \\
    b_{22}(4) &= 5.0833333333333 \\
    b_{22}(-1) &= -0.8333333333333
\end{cases}
(B.31b)
\end{align*}
\]
Appendices

B.3 SOFD Scheme

B.3.1 Coefficients for First Order Derivatives

\[
\begin{align*}
    a_{22}(-1) &= -0.5 \\
    a_{22}(0) &= 0 \\
    a_{22}(1) &= 0.5
\end{align*}
\]

\[
\begin{align*}
    a_{11}(0) &= -1.5 \\
    a_{11}(1) &= 2 \\
    a_{11}(2) &= -0.5
\end{align*}
\] (B.32a)

B.3.2 Coefficients for Second Order Derivatives

\[
\begin{align*}
    b_{22}(-1) &= 1 \\
    b_{22}(0) &= -2 \\
    b_{22}(1) &= 1
\end{align*}
\]

\[
\begin{align*}
    b_{11}(0) &= 2 \\
    b_{11}(1) &= -5 \\
    b_{11}(2) &= 4 \\
    b_{11}(3) &= -1
\end{align*}
\] (B.33a)

B.4 Modified Wavenumber Expressions

B.4.1 FODRP scheme

1st order derivative

\[
\left( \frac{\partial u}{\partial \xi} \right)_i = \frac{1}{\Delta \xi} [a_{-3}u(\xi + (i-3)\Delta \xi) + a_{-2}u(\xi + (i-2)\Delta \xi) + a_{-1}u(\xi + (i-1)\Delta \xi) + a_0u(\xi + \Delta \xi) + a_1u(\xi + i\Delta \xi) + a_2u(\xi + (i+1)\Delta \xi) + a_3u(\xi + (i+2)\Delta \xi) + a_4u(\xi + (i+3)\Delta \xi)]
\]

\[\tilde{k}\Delta \xi = -i\{a_{-3}e^{-3i\Delta \xi} + a_{-2}e^{-2i\Delta \xi} + a_{-1}e^{-i\Delta \xi} + a_0 + a_1e^{i\Delta \xi} + a_2e^{2i\Delta \xi} + a_3e^{3i\Delta \xi}\} \] (B.34)

2nd order derivative

\[
\left( \frac{\partial^2 u}{\partial \xi^2} \right)_i = \frac{1}{\Delta \xi^2} [b_{-3}u(\xi + (i-3)\Delta \xi) + b_{-2}u(\xi + (i-2)\Delta \xi) + b_{-1}u(\xi + (i-1)\Delta \xi) + b_0u(\xi + i\Delta \xi) + b_1u(\xi + (i+1)\Delta \xi) + b_2u(\xi + (i+2)\Delta \xi) + b_3u(\xi + (i+3)\Delta \xi)]
\]

\[\tilde{k}^2\Delta \xi^2 = -\{b_{-3}e^{-3i\Delta \xi} + b_{-2}e^{-2i\Delta \xi} + b_{-1}e^{-i\Delta \xi} + b_0 + b_1e^{i\Delta \xi} + b_2e^{2i\Delta \xi} + b_3e^{3i\Delta \xi}\} \] (B.36)

B.4.2 FOFD scheme

1st order derivative

\[
\left( \frac{\partial u}{\partial \xi} \right)_i = \frac{1}{\Delta \xi} [a_{-2}u(\xi + (i-2)\Delta \xi) + a_{-1}u(\xi + (i-1)\Delta \xi) + a_0u(\xi + \Delta \xi) + a_1u(\xi + (i+1)\Delta \xi) + a_2u(\xi + (i+2)\Delta \xi)]
\]

\[\tilde{k}\Delta \xi = -i\{a_{-2}e^{-2i\Delta \xi} + a_{-1}e^{-i\Delta \xi} + a_0 + a_1e^{i\Delta \xi} + a_2e^{2i\Delta \xi}\} \] (B.38)
2\textsuperscript{nd} order derivative

\[
\left( \frac{\partial^2 u}{\partial \xi^2} \right)_i = \frac{1}{\Delta \xi^2} \left[ b_{-2} u(\xi + (i-2)\Delta \xi) + b_{-1} u(\xi + (i-1)\Delta \xi) + b_0 u(\xi + i\Delta \xi) \right. \\
\left. + b_1 u(\xi + (i+1)\Delta \xi) + b_2 u(\xi + (i+2)\Delta \xi) \right] \\
\tilde{k}^2 \Delta \xi^2 = -\{ b_{-2} e^{-2ik\Delta \xi} + b_{-1} e^{-ik\Delta \xi} + b_0 e^0 + b_1 e^{ik\Delta \xi} + b_2 e^{2ik\Delta \xi} \} \tag{B.40}
\]

B.4.3 SOFD scheme

1\textsuperscript{st} order derivative

\[
\left( \frac{\partial u}{\partial \xi} \right)_i = \frac{1}{\Delta \xi} \left[ a_{-1} u(\xi + (i-1)\Delta \xi) + a_0 u(\xi + \Delta \xi) + a_1 u(\xi + (i+1)\Delta \xi) \right] \\
\tilde{k} \Delta \xi = -i \{ a_{-1} e^{-ik\Delta \xi} + a_0 e^0 + a_1 e^{ik\Delta \xi} \} \tag{B.42}
\]

2\textsuperscript{nd} order derivative

\[
\left( \frac{\partial^2 u}{\partial \xi^2} \right)_i = \frac{1}{\Delta \xi^2} \left[ b_{-1} u(\xi + (i-1)\Delta \xi) + b_0 u(\xi + i\Delta \xi) + b_1 u(\xi + (i+1)\Delta \xi) \right] \\
\tilde{k}^2 \Delta \xi^2 = -\{ b_{-1} e^{-ik\Delta \xi} + b_0 e^0 + b_1 e^{ik\Delta \xi} \} \tag{B.44}
\]
Appendix C

Laminar Channel Flow

This section presents the linear stability analysis of laminar Poiseuille and Couette flow using the FODRP scheme numerical code employed for the stability analysis of porous channel flows and validation of results from Schmid and Brandt (2014).

C.1 Governing Equations

The governing Navier-Stokes equations for Poiseuille or Couette flow are given as,

\[ \nabla \cdot u = 0 \]  
\[ \frac{\partial u}{\partial t} + \nabla u \cdot U + \nabla U \cdot u = -\nabla p + \frac{1}{Re} \nabla^2 u + f \]

For Poiseuille flow i.e. the pressure-driven flow between two resting plates yielded a parabolic baseflow velocity profile given by,

\[ U(y) = 1 - y^2 \]  

The walls for Poiseuille flow are considered as \( y = \pm 1 \). For Couette flow i.e. the flow induced by two plates moving in-plane in opposite directions by the same speed produced a linear baseflow velocity profile given by,

\[ U(y) = y \]

The bottom and top plates for both Poiseuille and Couette flow are maintained at \( y = \pm 1 \). The linearized incompressible Navier-Stokes equations are derived about the baseflow \( U(y) \) considering the normal wave-like perturbations for velocity and pressure and are given as,

\[ i\alpha u + Dv + i\beta w = 0 \]  
\[ \lambda u = -i\alpha U_b u - U_b^\prime v - i\alpha p + \frac{1}{Re} (D^2 - \alpha^2 - \beta^2) u + f_u \]  
\[ \lambda v = -i\alpha U_b v - Dp + \frac{1}{Re} (D^2 - \alpha^2 - \beta^2) v + f_v \]  
\[ \lambda w = -i\alpha U_b w - i\beta p + \frac{1}{Re} (D^2 - \alpha^2 - \beta^2) w + f_w \]
C.2 Spectrum and Transient Energy Growth

Figure C.1: Eigen spectrum and Transient energy growth for plane Poiseuille flow. The parameters are $\alpha = 1$, $\beta = 0.25$, $Re = 2000$.

The eigenspectrum for Poiseuille is shown in figure C.1(a) at $Re = 2000$, $\alpha = 1$ and $\beta = 0.25$ and Couette flow is shown in figure C.2(a) at $Re = 1000$, $\alpha = 1$ and $\beta = 0.25$ respectively. There is no unstable eigenvalue for both cases and both flows are considered to be stable with regards to the normal mode analysis. The eigenvalues are real and negative and thus decay exponentially to zero over time in both cases.

Figure C.2: Eigen spectrum and Transient energy growth for plane Couette flow. The parameters are $\alpha = 1$, $\beta = 0.25$, $Re = 1000$.

The transient growth of perturbation energy is shown for Poiseuille flow in figure C.1(b) and Couette flow is shown in figure C.2(b) at the same values of $Re$, $\alpha$ and $\beta$ as considered for the normal eigenspectrum analysis. It can be seen for both cases that even though the system matrix has stable eigenvalues, it can lead to transient growth of the perturbation energy over short time due to the non-normality of the exponentially decaying eigenvalues. The degree of non-orthogonality also increases exponentially with the Reynolds number ($Re$) considered for the analysis which is very much evident from the transient energy growth curves. From the transient energy growth curves, it can be very well seen that significant perturbation energy amplification is seen for least stable eigenvalue before the time-asymptotic decay due to the non-normality of the eigenmodes. The magnitude of the perturbation energy amplification is 10 times more than the initial perturbation energy over a time period of $t \approx 13$ as also shown by Schmid and Brandt (2014).
C.3 Resolvent Norm

Figure C.3: Resolvent norm (−), \(|(−i\omega I - L)^{-1}|\) and Resonant limit (−), the inverse of the minimal distance of the forcing frequency (\(\omega\)) to the spectrum for (a) plane Poiseuille flow and (b) plane Couette flow. The parameters are \((Re, \alpha, \beta) = (2000, 1, 0.25)\) for plane Poiseuille flow and \((Re, \alpha, \beta) = (1000, 1, 0.25)\) for plane Couette flow.

The other mechanism responsible for non-normal energy growth, Resolvent norms are shown for Poiseuille flow at \(Re = 2000\), \(\alpha = 1\) and \(\beta = 0.25\), and Couette flow at \(Re = 1000\), \(\alpha = 1\) and \(\beta = 0.25\) in figure C.3(a) and (b) respectively. The resolvent norm is shown for harmonic forcing frequencies. In both cases, the red curve represents the inverse distance of the external forcing frequency with the eigenvalues of the system (resonance frequency). The blue curve represents the resolvent norm obtained from the eigenvalue decomposition of the system matrix considering the non-normality of the eigenmodes. The difference between the red and blue curve represents the non-normal growth taking place in the flow owing to the non-orthogonality of the eigenmodes in the same way as shown by Schmid and Brandt (2014).

The results for the eigenspectrum analysis, transient growth analysis and resolvent norm analysis are in excellent agreement with the results of Schmid and Brandt (2014) for both cases of Poiseuille flow and Couette flow.
Appendix D

Turbulent Channel Flow

This section presents the non-normal resolvent analysis of the turbulent Poiseuille flow using the FODRP numerical scheme employed for turbulent porous channel flow analysis and the validation of the results from Hwang and Cossu (2010).

D.1 Governing Equations

The linearized Navier-Stokes equation about the baseflow ($U$) for the turbulent porous channel flow considering the total eddy viscosity $\nu_T(y) = \nu + \nu_t(y)$ is given as,

$$\nabla \cdot u = 0 \quad \text{(D.1a)}$$

$$\frac{\partial u}{\partial t} + \nabla u \cdot U + \nabla U \cdot u = -\frac{1}{\rho} \nabla p + \nabla \cdot [\nu_T(\nabla u + \nabla u^T)] + f \quad \text{(D.1b)}$$

The total eddy viscosity ($\nu_T$) has been considered using the semi-empirical relation with $\eta = y/h$, $Re_\tau = u_\tau h/\nu$ as the friction Reynolds number, von Kármán constant as $\kappa = 0.426$ and constant $A = 25.4$ as shown by Hwang and Cossu (2010):

$$\nu_T(\eta) = \frac{\nu}{2} \left\{ 1 + \kappa^2 Re_\tau^2 \left( 1 - \eta^2 \right)^2 \left( 1 + 2\eta^2 \right)^2 \times \left\{ 1 - \exp\left[ (|\eta| - 1) Re_\tau / A \right] \right\}^2 \right\}^{1/2} + \frac{\nu}{2} \quad \text{(D.2)}$$

The plot of the total eddy viscosity as a function of the channel height has been shown in figure D.1. The wall conditions are specified as $u |_{\eta = \pm 1} = 0$.

![Figure D.1: Ratio of total eddy viscosity to fluid viscosity as a function of dimensionless channel height ($\eta$) at $Re_\tau = 10000$.](image_url)
D.2 Energy Amplification of Optimal Harmonic Forcing

The turbulent fluid system is forced with an external harmonic forcing (initial disturbance) and the maximum energy amplification optimized over all range of initial forcings is recorded as response. The resolvent norm is calculated in a similar way as shown in equation 2.53. The friction Reynolds number is set to a value of $Re_\tau = 10000$ as shown by Hwang and Cossu (2010). The maximum value of the resolvent norm ($R_{\text{max}}$) is measured over a range of selected stream-wise and span-wise wavenumbers as shown in figure D.2. It can be seen that the maximum energy amplification takes place for uniform stream-wise perturbations i.e. $\alpha = 0$. The value of $R_{\text{max}}$ reaches maximum at $\beta h = 1.25$ ($\lambda_z/h = 5.5$) and thereafter the value of $R_{\text{max}}$ decreases monotonously with subsequent increase in $\beta h$, with a significant change of slope near $\lambda_z^+ = 80$. The maximum amplification takes place only for stream-wise elongated perturbations i.e. $\alpha h \leq \beta h$. There is a power-law dependence of $R_{\text{max}}$ on span-wise wavenumber ($\beta h$) over the range of $\beta h = 15 - 400$, which is shown as a linear relation in the log–log plot. In this range of $\beta h = 15 - 400$, $R_{\text{max}}$ scales as $\beta h^{-2}$, as shown by Hwang and Cossu (2010).

![Figure D.2: Energy amplifications of optimal harmonic forcing at $Re_\tau = 10000$: $\alpha = 0.0(-), \alpha = 0.1(-), \alpha = 0.2(-), \alpha = 0.5(-), \alpha = 1.0(---), \alpha = 2.0(---), \alpha = 5.0(---), \alpha = 10.0(---)$](image)

The pre-multiplied energy amplification ($\beta h^2$) is shown in figure D.3 to see the deviations of $R_{\text{max}}$ from the $\beta h^{-2}$ scaling. The pre-multiplied energy curves has two peaks for corresponding values of span-wise wavenumber $\beta h$. The first peak is seen at $\beta h \equiv 1.8$ ($\lambda_z/h \equiv 3.5$) which corresponds to the uniform large scale structures in the stream-wise direction while the second peak is seen at $\beta h^+ \equiv 0.079$ ($\lambda_z^+ \equiv 80$) which corresponds to the near wall streaks. The power-law dependence of $R_{\text{max}}$ on $\beta h$ holds between these two values of span-wise wavenumber. The large value of friction Reynolds number $Re_\tau = 10000$ has a significant effect on the inner and outer scales over the intermediate range of wavenumbers as shown by Hwang and Cossu (2010).
The results for the energy amplification and the pre-multiplied energy amplification are in excellent agreement with the results of Hwang and Cossu (2010) for the turbulent Poiseuille flow.
Bibliography


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