AN EXTENSION OF NEUMANN'S INTEGRALRELATION
FOR GENERALIZED LEGENDRE FUNCTIONS

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In this paper we obtain an integralrelation connecting the two linearly inde-
pendent generalized Legendre functions of Kuipers and Meulenbeld. The
result is a generalization of F. Neumann's relation of 1848 for the two kinds
of Legendre functions

\[ Q_k(z) = \frac{1}{2} \int_{-1}^{1} \frac{P_k(x)}{z - x} \, dx \]

where \( k \) is a nonnegative integer, and \( z \) is not lying on the segment \((-1, 1)\) of
the complex plane.

The main result is in §2; generalizations can be found in §4. E. R. Love's
integralrelations of 1965 for associated Legendre functions follow as special
cases.

1. The generalized Legendre functions \( P_{\alpha, \beta}'(z) \) and \( Q_{\alpha, \beta}'(z) \), two specified
linearly independent solutions of the differential equation

\[ (1 - z^2) \frac{d^2w}{dz^2} - 2z \frac{dw}{dz} + \left\{ k(k + 1) - \frac{m^2}{2(1 - z)^2} - \frac{n^2}{2(1 + z)^2} \right\} w = 0, \]

have been introduced by Kuipers and Meulenbeld \([3]\) as functions of \( z \) for all
points of the \( z \)-plane, in which a cross-cut exists along the real \( x \)-axis from 1 to
\(-\infty\), and for complex values of the parameters \( k \), \( m \) and \( n \). On the segment
\(-1 < x < 1\) of the cross-cut these functions are defined in \([7]\). If \( m = n \),
they reduce to the associated Legendre functions, defined in \([2]\).

For the sake of brevity we put

\[ \alpha = k + \frac{1}{2}(m + n), \quad \beta = k - \frac{1}{2}(m - n), \]
\[ \gamma = k + \frac{1}{2}(m - n), \quad \delta = k - \frac{1}{2}(m + n). \]

Generalized Legendre functions can be written in terms of hypergeometric
functions, such as \([4, (9)]\)

\[ Q_{\alpha, \beta}'(z) = e^{z \cdot im \cdot 2\theta} \frac{\Gamma(\alpha + 1)\Gamma(\gamma + 1)}{\Gamma(2k + 2)} (z + 1)^{-k+i-m-1}(z - 1)^{-k-i-m} \]
\[ \cdot \, \, _2F_1\left(\beta + 1, \delta + 1; 2k + 2; \frac{2}{1 + z} \right) \]

if \( z \) is not lying on the cross-cut.

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2. The extension of Neumann’s integral is contained in the following theorem.

**Theorem 1.** If \( \Re m < 1, \Re \alpha > -1, \Re \gamma > -1, \) and \( z \) is not lying on the cross-cut, then

\[
\frac{(z + 1)^{\alpha + \frac{k+1}{m}}}{(z - 1)^{\frac{k+1}{m}}} e^{-mz} Q_{m,n}(z) = \frac{1}{2} \int_{-1}^{1} \frac{(1 + x)^{\alpha + \frac{k+1}{m}} P_{k,n}(x)}{z - x} dx.
\]

**Proof.** In [8, (13)] one can find the integral transform

\[
\int_{-1}^{1} (1 - x)^{-\frac{k}{m}}(1 + x)^{\alpha + 1} Q_{m,n}(x) dx
\]

\[
= 2^{\alpha + \frac{1}{m} - \frac{k}{m}} \frac{\Gamma(q + \frac{1}{2}n + 1)\Gamma(q - \frac{1}{2}n + 1)}{\Gamma(q - k - \frac{1}{2}m + 1)\Gamma(q + k - \frac{1}{2}m + 2)} (z - 1)^{-\sigma}
\]

\[
\cdot \; _2F_1\left(q + \frac{1}{2}n + 1, \sigma, q - \frac{1}{2}n + 1; \frac{2}{1 - z}\right)
\]

if \( \Re m < 1, \Re q + 1 > \frac{1}{2} \Re m \), \( z \) not lying on the cross-cut.

Choose \( \sigma = 1, q = k + \frac{1}{2}m \) and replace \( z \) by \(-z\) in (3); then

\[
\int_{-1}^{1} (1 + x)^{\alpha + \frac{k+1}{m}} P_{k,n}(x) \frac{dx}{z - x}
\]

\[
= 2^{\alpha + 1} \frac{\Gamma(\alpha + 1)\Gamma(\gamma + 1)}{\Gamma(2k + 2)} (z + 1)^{-1} \; _2F_1\left(\alpha + 1, \gamma + 1; 2k + 2; \frac{2}{1 + z}\right)
\]

if \( \Re m < 1, \Re \alpha > -1, \Re \gamma > -1, \) \( z \) not lying on the cross-cut.

Using (1) and the relation [5, (6)]

\[
Q_{m,n}(z) = e^{-2mz} z^{\alpha - \frac{k+1}{m}} \frac{\Gamma(\beta + 1)\Gamma(\delta + 1)}{\Gamma(\alpha + 1)\Gamma(\gamma + 1)} Q_{m,n}(z)
\]

we obtain

\[
F(\alpha + 1, \gamma + 1; 2k + 2; \frac{2}{1 + z})
\]

\[
= e^{-mz} z^{-\beta} \frac{\Gamma(2k + 2)}{\Gamma(\alpha + 1)\Gamma(\gamma + 1)} (z + 1)^{\frac{k+1}{m}} (z - 1)^{-\frac{1}{m}} Q_{m,n}(z).
\]

By setting (6) in (4) we complete the proof of the theorem.

**Remark 1.** The above theorem is a generalization of results of Meulenbeld and Robin [9, (30) and (31)]. After some simplifications Neumann’s integral follows from Theorem 1 for \( m = n = 0 \) and nonnegative integer \( k \).

3. In this section we derive two lemmas, in order to generalize Theorem 1.
**Lemma 1.** If $\Re m < 1$ and $j$ is an integer satisfying

$$0 \leq j < \min(\Re \alpha, \Re \gamma),$$

then

$$\int_{-1}^{1} \frac{(1 + x)^{k + \frac{1}{2}m - j - 1}}{(1 - x)^{\frac{1}{2}m}} P_{k}^{m-n}(x) \, dx = 0. \tag{7}$$

**Proof.** In [1, (8.1)] we find the integral transform

$$\int_{-1}^{1} (1 - x)^{\alpha}(1 + x)^{\beta} P_{k}^{m-n}(x) \, dx \quad \text{if } \Re(p - \frac{1}{2}m) > -1, \Re q + 1 > \frac{1}{2} |\Re n|. \text{ Substitute } p = -\frac{1}{2}m, q = k + \frac{1}{2}m - j - 1 \text{ in (8); then}

$$\int_{-1}^{1} (1 - x)^{-\frac{1}{2}m}(1 + x)^{k + \frac{1}{2}m - j - 1} P_{k}^{m-n}(x) \, dx \quad \text{if } \Re m < 1, \Re j < \min(\Re \alpha, \Re \gamma).$$

Using [2, 2.8 (46)] the right-hand side of (9) reduces to

$$2^{\alpha - j} \frac{\Gamma(\alpha - j)}{\Gamma(-j + 2k + 1)} \frac{\Gamma(\gamma - j)}{\Gamma(-j + 2k + 1)} \text{ which is equal to 0 for } j \text{ satisfying the conditions of Lemma 1.}$$

**Lemma 2.** If $\Re m < 1$, $j$ is any integer satisfying

$$0 \leq j < \min(\Re \alpha, \Re \gamma)$$

and $p(x)$ is any polynomial of degree $j$ or less, then

$$\int_{-1}^{1} \frac{(1 + x)^{k + \frac{1}{2}m - j - 1}}{(1 - x)^{\frac{1}{2}m}} P_{k}^{m-n}(x)p(x) \, dx = 0. \tag{10}$$

**Proof.** Let $p(x)$ be a polynomial of degree $j$ or less. Then $p(x)$ can be written in the form

$$p(x) = \sum_{i=0}^{j} p_i (1 + x)^{i - j}. \tag{11}$$

This yields

$$\int_{-1}^{1} \frac{(1 + x)^{k + \frac{1}{2}m - j - 1}}{(1 - x)^{\frac{1}{2}m}} p(x) P_{k}^{m-n}(x) \, dx = \sum_{i=0}^{j} p_i \int_{-1}^{1} \frac{(1 + x)^{k + \frac{1}{2}m - j - 1}}{(1 - x)^{\frac{1}{2}m}} P_{k}^{m-n}(x) \, dx. \tag{12}$$

By virtue of Lemma 1 the right-hand side vanishes.
4. This section contains two generalizations of Theorem 1. The proofs presented are similar to those in [6].

**Theorem 2.** If \( \Re m < 1, \Re \alpha > -1, \Re \gamma > -1, z \) is not lying on the cross-cut, \( s \) is an integer satisfying

\[
0 \leq s < \min (\Re \alpha + 1, \Re \gamma + 1)
\]

and \( p(x) \) is a polynomial of degree \( s \) or less, then

\[
\frac{(z + 1)^{k + 1 + m - s}}{(z - 1)^{k + 1}} e^{-\pi x} Q^{m, n}_k (z) p(z) = \frac{1}{2} \int_{-1}^{1} \frac{(1 + x)^{k + 1 + m - s}}{(1 - x)^{k + 1} z - x} P^{m, n}_k (x) p(x) \, dx.
\]

**Proof.** Using Theorem 1 and subtracting the right-hand side from the left-hand side in (11) we obtain

\[
\frac{p(z)}{(z + 1)^{s + 1}} \frac{1}{2} \int_{-1}^{1} \frac{(1 + x)^{k + 1 + m - s}}{(1 - x)^{k + 1} z - x} P^{m, n}_k (x) \, dx - \frac{1}{2} \int_{-1}^{1} \frac{(1 + x)^{k + 1 + m - s}}{(1 - x)^{k + 1} z - x} p(x) \, dx
\]

\[
= \frac{1}{2} \int_{-1}^{1} \frac{(1 + x)^{k + 1 + m - s}}{(1 - x)^{k + 1} P^{m, n}_k (x)} \left\{ \frac{p(z)}{(z + 1)^{s + 1}} - \frac{(z + 1)^{s + 1}}{(z + 1)^{s + 1}} + \frac{p(z) - p(x)}{z - x} \right\} \, dx
\]

because for \( 1 \leq s < \min (\Re \alpha + 1, \Re \gamma + 1) \) the expression between braces is a polynomial in \( x \) of degree \( s - 1 \), so that we can apply Lemma 2 by putting \( j = s - 1 \). For \( s = 0 \) the polynomial \( p(x) \) reduces to a constant, and (11) follows from Theorem 1.

**Remark 2.** The above theorem is a generalization of results of Meulenbeld and Robin [9, (53) and (55)].

**Theorem 3.** If \( \Re m > -1, \Re \beta > -1, \Re \delta > -1, \alpha \) is not an integer, \( z \) is not lying on the cross-cut, \( s \) is an integer satisfying

\[
0 \leq s < \min (\Re \beta + 1, \Re \delta + 1)
\]

and \( p(x) \) is any polynomial of degree \( s \) or less, then

\[
\frac{(z + 1)^{k + 1 + m - s}}{(z - 1)^{k + 1}} e^{-\pi x} Q^{m, n}_k (z) p(z)
\]

\[
= \frac{1}{2 \sin \alpha \pi} \left\{ \sin \beta \pi \int_{-1}^{1} \frac{(1 + x)^{k + 1 + m - s}}{(1 - x)^{k + 1} z - x} p(x) \, dx
\]

\[
+ 2^{s - m} \sin m\pi \frac{\Gamma(\gamma + 1)}{\Gamma(\beta + 1)} \Gamma(\delta + 1) \int_{-1}^{1} \frac{(1 + x)^{k + 1 + m - s}}{(1 + x)^{k + 1} z + x} P^{m, n}_k (x) p(x) \, dx\right\}
\]

**Proof.** By combining [7, (8)] and [7, (10)] we obtain, if \( \alpha \) is not integer

\[
\frac{\Gamma(\alpha + 1)\Gamma(\gamma + 1)}{\Gamma(\beta + 1)\Gamma(\delta + 1)} P^{m, n}_k (z)
\]

\[
= \frac{1}{\sin \alpha \pi} \left\{ 2^{s - m} \sin \beta \pi P^{m, n}_k (z) + \sin m\pi \frac{\Gamma(\gamma + 1)}{\Gamma(\beta + 1)} P^{m, n}_k (-z) \right\}.
\]
In Theorem 2 replace $m$ by $-m$ and $n$ by $-n$, use (5) for the left-hand side and (13) for the right-hand side, split up the integral in the right-hand side (this is allowed because both integrals exist under given conditions) and in the integral involving $P_n^m(-x)$ change $x$ into $-x$. This completes the proof.

Remark 3. All results given in this paper reduce to E. R. Love's results [6] by setting $m = n$. For references to special cases in the case of associated Legendre functions, we refer to [6].

References


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