A new numerical model for simulating the propagation of and inundation by tsunami waves
A new numerical model for simulating the propagation of and inundation by tsunami waves

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Preface

This thesis is submitted for the degree of Doctor of Philosophy in the Environmental Fluid Mechanics Section of the Civil Engineering and Geoscience Faculty, Delft University of Technology. It was carried out under the supervision of Prof. dr. ir. G.S. Stelling, and Prof. Dr. Julie D. Pietrzak, Environmental Fluid Mechanics.

This thesis has involved the development of an unstructured grid ocean model, H2Ocean, with accurate flooding and drying algorithms for tsunami studies. My PhD is co-funded by the Alfred Wegener Institute (AWI) in Bremerhaven, Germany, as part of their contribution to the German-Indonesian Tsunami Early Warning System, which is installed in Jakarta, Indonesia. The objectives are to improve the capability of the model in handling the flooding and drying, and dispersion. Based on an unstructured mesh finite element model TsunAWI, a finite volume model H2Ocean has been developed. This model is an analogue of the $P_{1}^{NC} - P_{1}$ finite element.
ii  PREFACE
Summary

In the aftermath of the Great Sumatra Earthquake and Indian Ocean Tsunami of December 26, 2004, a German-Indonesian Tsunami Early Warning System (GITEWS), was initiated in 2005 and was completely handed over to Indonesia on March 29th 2011. Under the framework of GITEWS, a finite element model TsunAWI has been developed by the Alfred Wegener Institute (AWI). To further improve its capability in handling flooding and drying and wave dispersion, a joint research project between AWI and TUDelft was launched. As one of the research achievements, a finite volume unstructured grid ocean model, H2Ocean, has been developed. It is an analogue of the $P_1^{NC} - P_1$ finite element discretization, which has the water elevation located on the vertex, and velocity vectors located on the middle of the edge. By interpreting the advective term in a flux sense, not only is the mass conservation guaranteed in a global sense, but it is also explicitly guaranteed in each individual cell. Another feature of the model H2Ocean is its ability to conserve momentum, which provides a better description of bores or hydraulic jumps that mimic breaking waves in depth-integrated flows. By using the upwind water depths in the flux computations, no special flooding and drying procedures need to be implemented. The model is efficient, does not produce non-physical negative water depths. It has been extensively tested against a wide variety of flooding and drying problems and always produces good results. It has also been successfully applied to the simulations of the 2004 Indian Ocean Tsunami and 2011 Japan Tsunami.

Despite the success of Non-linear Shallow Water Equations (NSWE) in tsunami simulations, there are always discussions that the frequency dispersive effects, which are missing in the shallow water models, should be included. By using the fractional step procedure, H2Ocean has been extended to a non-hydrostatic model. However, the original implementation of the non-hydrostatic algorithm results in a large stencil and a sparse matrix with a large number of non-zero elements. Since in tsunami simulations, the efficiency of the model is a major
issue, a lumping scheme has been introduced to improve the efficiency of the model. Instead of calculating the non-hydrostatic pressure gradient by the four neighbouring nodes, the non-hydrostatic pressure gradient is only approximated by three node values within one triangle, when substituting the new velocities into the 3D continuity equation. In so doing, the number of non-zero elements of the matrix is reduced by half, and the execution CPU time is shortened by around 30%. The non-hydrostatic version of H₂Ocean has been validated against several classic test cases, such as a standing wave in a closed basin, the propagation of solitary waves, wave deformation over the Berkhoof shoal and wave run-up onto a conical island. In addition, for a model using a collocated grid, both the water surface elevation and the non-hydrostatic pressure need to be defined at the incoming wave boundary. Based on linear wave theory, a non-hydrostatic pressure boundary has been derived. It can serve as a simple way to introduce short incoming waves in models with collocated grids.

With the increasing demands of performing large scale flow simulations with non-hydrostatic models, improving their efficiency has become a competitive necessity. In this research, a novel numerical technique has been developed to improve the efficiency of non-hydrostatic models. It reduces the degree of the stencil of the pressure Poisson equation near the bottom by approximating the pressure gradient term of the horizontal velocity in the bottom cell with only the pressures at the top face of the bottom cells. By doing so, the stencil of the pressure Poisson equation at the bottom is simplified. By combining the equations in the two bottom layers, the vertical velocity and the pressure at the bottom are even eliminated from the set of equations. This method can be readily applied to any non-hydrostatic models with multiple layers. In the current work, we only implement it in H₂Ocean with two layers. Compared to the single layer models, the new model improves the accuracy of dispersion substantially, but requires the same computational effort. For a monotonic wave, the dispersion relation can be further optimized by choosing the layer thickness to be a function of $kh$ ($k$ is the wave number, and $h$ is the water depth).

To allow easy implementation of the reduced two-layer method in depth-integrated hydrostatic models, the velocity difference has been introduced as an extra variable. This means the method can be easily adopted by depth-averaged models and allows use of momentum conservation schemes. This offers an alternative to include dispersive effects in the depth-integrated models. In the successful applications of the model in simulations of the 2004 Indian Ocean Tsunami and the 2011 Japan Tsunami, excellent agreement have been achieved, when compared to both flooding data and measured run-up. The model is found to be an accurate and efficient model for tsunami simulations.
Samenvatting

In de nasleep van de Sumatra aardbeving en de daaropvolgende verwoestende Indische Oceaan tsunami van 26 December, 2004, is een Duits-Indonesisch Tsunami vroeg waarschuwingssysteem ontwikkeld (GITEWS) ¹, en overgedragen aan Indonesië op 29 Maart 2011. Binnen dit kader is door het Alfred Wegener Instituut (AWI) het eindige elementen model TsuNAWI ontwikkeld. Tezamen met de TU Delft is vervolgens een onderzoeksproject geïnitieerd met als doel een verbeterde weergaven van onderlopen en droogvallen, en het toevoegen van frequentie dispersie aan het model. Als een van de onderzoeksresultaten van dit project is het ongestructureerde, eindige volume model, H₂Ocean ontwikkeld. Dit model is gebaseerd op een element analoog aan het P_{1}^{NC} − P_{1} eindige element, waarin het waterniveau gelokaliseerd is in het hoekpunt terwijl de snelheidsvectoren in het midden van de desbetreffende zijdes zijn geplaatst. Wanneer de advectieve termen in flux vorm beschouwd worden, is het behoud van massa, almede impuls, niet alleen globaal, maar ook lokaal verzekerd. Door het impulsbehoudende karakter van H₂Ocean is het model in staat tot een goede weergave van een watersprong en, als gevolg hiervan, golfbreken aangezien dit proces in dieptegemiddelde modellen analoog aan een watersprong wordt verondersteld. Het gebruik van bovenstroomse waterdieptes zorgt er bovendien voor dat aparte onderloop en droogval procedures overbodig zijn. Het resulterende model is dan ook efficiënt, voorkomt niet-fysische negatieve waterstanden, en is met goed resultaat getest ten aanzien van een groot aantal verschillende onderloop en droogval situaties. Daarnaast is het succesvol toegepast voor de simulatie van zowel de Indische Ocean tsunami, als de Tohoko Tsunami bij Japan van 2011. Ondanks het overweldigende succes van ondiep water modellen in het modelleren van tsunamis, bestaat er nog steeds discussie of de effecten van frequentie dispersie welke ontbraken in ondiep water modellen van belang zijn. Gebruik makend van een fractionele stap methode, is

¹German-Indonesian Tsunami Early Warning System
H₂Ocean daarom uitgebreid om rekening te houden met zulke niet-hydrostatische effecten. Echter, de introductie van zulke niet-hydrostatische effecten leidt tot een groot rekenschema, met als gevolg dat een matrix systeem met een groot aantal niet-nul elementen moet worden opgelost. Omdat model efficiëntie in tsunami modellen van groot belang is, is een zogenaamd lumping algoritme toegepast om de efficiëntie van het model te verbeteren. Wanneer de nieuwe snelheden worden gesubstitueerd in de 3D continuïteitsvergelijking wordt, in plaats van gebruik te maken van de vier omliggende knooppunten, de niet-hydrostatische drukgradient benaderd met behulp van de drie knooppunten in de driehoek. Het aantal niet-nul elementen in de matrix wordt hierdoor gehalveerd, met als gevolg een reductie in CPU tijd van ongeveer 30%. De niet-hydrostatische versie van H₂Ocean is gevalideerd met behulp verscheidene klassieke casussen zoals o.a. een staande golf in een gesloten bassin, de propagatie van een eenling golf, golf vervorming over een ondiepe en golfoploop op een conisch eiland. Omdat in modellen waarin een niet-versprongen (of collocated) rooster wordt gebruikt, zowel de vrije oppervlakte uitwijking als de niet-hydrostatische druk op de rand moet worden gespecificeerd, is - op basis van lineaire golftheorie - daarom een niet-hydrostatische randvoorwaarde afgeleid. Deze randvoorwaarde is zodanig geformuleerd dat korte golven op een eenvoudige manier kunnen worden geïntroduceerd in zulke modellen.

De toenemende vraag naar grootschalige stroming simulaties met niet-hydrostatische modellen heeft ervoor gezorgd dat het verbeteren van de efficiëntie van deze modellen een competitieve noodzakelijkheid is geworden. In dit onderzoek is een nieuwe techniek ontwikkeld om de efficiëntie van niet-hydrostatische modellen te verbeteren. Deze techniek reduceert de graad van het stencil van de Poisson druk vergelijking door bij de bodem de druk gradint van de horizontale snelheid te schatten met alleen de druk ten plaatse van de bovenkant van de bodem cel. Op deze manier wordt de staande golf ter plaatse van de bodem gesimplificeerd. Door de vergelijkingen in de twee bodem lagen te combineren worden de verticale snelheid en druk bij de bodem zelfs gelimineerd uit de vergelijkingen. Deze methode kan direct worden toegepast in elk niet-hydrostatische model met meerdere lagen. In dit werk is deze techniek alleen toegepast in het H₂Ocean twee lagen model. In het nieuwe model is de nauwkeurigheid van de dispersie relatie substantieel verbeterd terwijl het dezelfde rekentijd vergt als het enkele laag model. De dispersie relatie kan verder worden geoptimaliseerd voor een monotone golf door de laagdikte te kiezen als een functie van de relatieve waterdiepte \( kh \).

Om de implementatie van de gereduceerde twee-lagen model te vereenvoudigen is gekozen voor een methode waarin het snelheidsverschil wordt geïntroduceerd als een extra variabele. Dit heeft als voordeel dat de methode gemaakt uit bestaande diepte-gemiddelde hydrostatische modellen kan worden ingevoerd, en het maakt het gebruik van impuls behoudende schemas mogelijk. Het vormt dan ook een aantrekkelijk alternatief om de mogelijkheden van dieptegemiddelde modellen uit te breiden. Dit wordt bevestigd in de succesvolle toepassing van de methode in de simulatie van de tsunami in de Indische Ocean van 2004, en de
Tohoko Tsunami bij Japan van 2011, waarin goede overeenkomsten met metingen van zowel het overstromingsgebied alsmede de maximale golftoeploopt zijn gevonden. Voor het simuleren van tsunamis is dan ook gebleken dat het model H2Ocean accuraat en efficiënt is.
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1.1 Tsunami

Tsunami is a Japanese word, represented by two characters: ‘tsu’, meaning harbor, and ‘nami’, meaning wave. It is said that the term was created by Japanese fishermen, who upon returning to a harbor, found their villages had been devastated. However, when they were at sea, they did not notice anything out of the ordinary. Due to this, they believed that the enormous wave which destroyed their home, originated in the harbor. Therefore, they named it a harbor wave.

Today, the word ‘tsunami’ is commonly used in English, especially after the disaster that devastated Indonesia in 2004. Technically, tsunami is a set of ocean waves which arise from the sudden displacement of water masses generated by any large, violent displacement of the ocean floor or ocean surface. Tsunami is usually associated with seismic activity, landslides or volcanic eruptions (Dudley and Lee, 1998). Offshore earthquakes are by far the most common cause of tsunamis. The uplift and subduction of the seafloor, displaces the entire water column up and down. The displacement can be several meters in the vertical, and can cover thousands of square kilometers in area in the horizontal. The enormous amount of the potential energy of the uplifted water is then transferred to horizontal propagation of the tsunami wave (kinetic energy). The potential energy is given by

$$\text{Potential energy} = \int \int_{0} \rho g \eta(x, y) dz dx dy$$

where $\rho$ is the density of water, $g$ is the gravitational constant, and $\eta(x, y)$ is the displacement of the surface sea level, $(x, y, z)$ are the Cartesian coordinates. The energy can be as much as $10^{16}$ J for large tsunami, equivalent to roughly 10 megatons of TNT (Leveque et al., 2011).
In the deep ocean, tsunamis have relatively small amplitude, typically a meter or so. However, being generated by an event as large as an earthquake, the wavelength of the tsunami is extremely large, typically of several hundred kilometers, far greater than the ocean depth. The wave steepness, which is equal to the wave height divided by the wave length, is so small that the waves often pass by ships in the deep ocean without anyone on board even noticing. As a result of their long wave lengths, tsunamis behave as shallow-water waves that move at a speed \( c = \sqrt{gh} \). As such, tsunamis in the deep ocean travel very fast. For example, in the middle of the Pacific Ocean, where the average water depth is 4,000 m, a tsunami propagates at approximately 720 km/h, which is roughly the speed of a jet airplane. A tsunami can travel across the Pacific Ocean in less than one day.

Another consequence of such a large wave length is that tsunamis can travel transoceanic distances with limited energy loss, because in the deep ocean the rate at which a wave loses its energy is inversely related to its wavelength (Dean and Dalrymple, 1991). The tsunami’s energy flux \( E \) is given by:

\[
E = \int_{-h}^{a} \left( \frac{1}{2} \rho |\mathbf{v}|^2 + \rho g z \right) dz
\]

where \( \rho \) is the fluid density, \( a \) is the wave amplitude \( \mathbf{v} \) is the fluid velocity vector.

As the tsunami waves travel into the shallower water and approach the shore, they begin to lose energy through reflection, bottom friction and turbulence. Despite these losses, the tsunami can still reach the coast with a significant amount of energy. As the water becomes shallow, the wave speed decreases and the wave will compress. The waves can become extremely high when they come ashore. The tsunami wave height in shallow water (\( h_s \)) is given by:

\[
\frac{h_s}{h_d} = \left( \frac{H_d}{H_s} \right)^{1/4}
\]

where \( h_d \) is the wave height at deep water, \( H_s \) and \( H_d \) are the depths of the shallow and deep water. A tsunami wave that was only half a meter high in the deep sea may become a ten meter high giant wave at the shoreline. Tsunami waves only become dangerous and pose a threat to lives in coastal communities, when reaching coastal areas. It is the flooding effects of a tsunami that cause the most damage. There are two terms to describe the extent of tsunami flooding: inundation and run-up (see Fig. 1.1). Inundation is the depth of the water above the normal level, and run-up is the maximum vertical height to which the wave can reach (Helal and Mehanna, 2008).
1.2 Historical Tsunami

Tsunamis have been leaving tragic traces in human history. Since 1990, over ten major tsunami events have impacted on the world’s coastline, causing devastation and loss of life. On the web site of NOAA (http://www.ngdc.noaa.gov/), the global historical tsunami and run-up data are presented back to 2000 year BC. Below we only name the three most recent disasters.

1.2.1 2004 Indian Ocean Tsunami

The Indian Ocean Tsunami on December 26, 2004 was triggered by an underwater megathrust earthquake, which is known by the scientific community as the Great Sumatra-Andaman Earthquake. The earthquake lasted for approximately 8-10 minutes (Vigny et al., 2005) and had a moment magnitude between 9.1 and 9.3 (Ammon et al., 2005; Stein and Okal, 2005). An enormous amount of water was displaced, sending powerful tsunami waves in every direction. Within hours, these deadly waves caused catastrophic destruction in countries around the Indian Ocean basin. They even travelled as far as 5,000 kilometers to Africa and still arrived with sufficient power to kill people and destroy property. It caused an unprecedented disaster and claimed the priceless lives of more than 300,000 persons worldwide.

1.2.2 2010 Chile Tsunami

On the 27th February, 2010, Chile was hit by an earthquake, which had a magnitude of 8.8 on the moment magnitude scale. A significant tsunami was generated
by the earthquake and devastated several coastal towns in south-central Chile and
damaged the port at Talcahuano. Unusually, the destructive waves in Talcahuano
came three hours after the earthquake. One of the explanations is resonance os-
cillations due to the trapping of the radiated tsunami energy over the continental
margin (Yamazaki and Cheung, 2011).

1.2.3 2011 Japan Tsunami

The Mw=9.0 Tohoku-Oki Earthquake occurred on 11 March 2011 at 05:46:23
UTC, off the Pacific coast of Honshu, Japan. It was the largest earthquake ever
measured in Japan and generated a tsunami that reached heights of up to 40 m
inland and traveled up to 10 km inland in the Sendai area (The 2011 Tohoku
Earthquake Tsunami Joint Survey Group, 2011). The tsunami even caused nuc-
lear accidents at three reactors in the Fukushima Daiichi Nuclear Power Plant
complex. As a consequence, today there are still more than 80,000 residents
remaining evacuated across the country. Thanks to the dense national Global
Position System (GPS) network, the real-time tsunami monitoring system DART
(Deep-ocean Assessment and Reporting of Tsunamis) and the bottom pressure
tidal gauges and the 2011 Tohoku Earthquake Tsunami Joint Survey Group, this
tsunami becomes the most well recorded one. This wealthy of data provides us
with an opportunity to better understand the mechanism of the earthquake and
the characteristics of the tsunami propagation and inundation.

1.3 Tsunami Early Warning System

Although a tsunami cannot be prevented, through community preparedness, timely
warnings, and effective response, the impact of a tsunami can be mitigated, in
particular, loss of life and injury. Since earthquakes are often a cause of a tsunami,
the earthquake may be considered as an indication that a tsunami probably will
follow shortly. If the leading edge of the tsunami wave is its trough, as happened
along the coast of Thailand during the Indian Ocean Tsunami, the water along
the shoreline recedes dramatically, exposing normally submerged areas. This can
also serve as an advance warning of the approaching crest of the tsunami. If you
are near the ocean and you feel a large earthquake, or you see the sea water recede
dramatically, you should flee to higher ground immediately.

In regions with a high risk of a tsunami, many countries have contributed to
the development of tsunami warning systems, which are used to detect tsunamis
in the open ocean and to issue warnings. This has a direct humanitarian aim: to
mitigate the effects of tsunami, in order to save lives and property.

NOAA’s Pacific Tsunami Warning Center (PTWC), based in Ewa Beach,
Hawaii, was established in 1949, and is now part of an international tsunami
warning system (TWS) program. It serves as the operational headquarters for
the Pacific Tsunami Warning and Mitigation System (PTWS). The PTWC dis-
seminates tsunami information and warning messages to over 100 locations across the Pacific. After the 2004 Indian Ocean earthquake and tsunami, the PTWC extended its warning guidance to include the Indian Ocean, Caribbean and adjacent regions. The Japan Meteorological Agency (JMA) initiated tsunami warning services in 1952. They can provide a tsunami warning or tsunami advisory within about three minutes after the occurrence of an earthquake.

In the aftermath of the Great Sumatra Earthquake and Indian Ocean Tsunami of December 26, 2004, a German-Indonesian Tsunami Early Warning System (GITEWS) (Behrens, 2008; Rudloff et al., 2009; Lauterjung et al., 2010), was initiated and has been completely handed over to Indonesia on March 29th 2011. The earthquakes in the Indian Ocean off the coast of Indonesia often occur along a subduction zone, the Sunda Trench. Once a tsunami happens, it could reach the coast within 20 minutes. This poses a big challenge to the design of an early warning system.

GITEWS distinguishes itself from other tsunami warning systems by employing many new scientific methods and novel technologies to make it work effectively. GITEWS is an integrated system with the components of a dense network of seismic stations, ocean bottom pressure instruments, a nearly real-time GPS network, satellite observation, pre-calculated tsunami scenarios database, capacity building activities and warning center (Lauterjung et al., 2008). 3470 regional tsunami scenarios for GITEWS and 1780 Indian Ocean wide scenarios in support of Indonesia as a Regional Tsunami Service Provider (RTSP) were computed with TsunAWI (Lauterjung et al., 2008). Once an earthquake happens, the system can locate the most similar scenarios and interpolate the most possible arrival time and run-ups. The warnings can be issued in less than five minutes after the earthquake, followed by updates or cancellation messages. Its full functionality has been demonstrated during several strong earthquakes and tsunamis. An effective tsunami early warning system could have saved thousands of lives that were lost in the devastating tsunami.

One of the subjects of GITEWS is to accurately simulate the propagation and run-up of a tsunami wave. The Alfred Wegener Institute for Polar and Marine Research (AWI) played a leading role among the international effort to develop tsunami warning system capabilities in the Indian Ocean. A tsunami model TsunAWI which is based on nonlinear shallow water equations, has been developed and validated by the mathematical modeling group of AWI (Behrens, 2008). It uses the finite element method, approximates elevation and velocity by using linear conforming shape functions and linear non-conforming shape functions, respectively (Androsov et al., 2008). It has been validated by benchmark experiments as well as by comparing model results to a great amount of real measurement data, such as tide gauge data, satellite altimetry and inundation depth by the 2004 Indian Ocean Tsunami (Harig et al., 2008).
1.4 Tsunami Waves Modeling

The objective of tsunami modeling research is to develop numerical models for fast and reliable forecasts of the propagation of a tsunami through the ocean and the inundation by a tsunami in coastal communities. It is worth noting that tsunami modeling involves a huge chain of other activities, including generating the mesh, collecting geometry and elevation data, reconstructing the initial uplift of the water surface, field survey or laboratory experiments to provide data to validate the model. Any of these activities requires a substantial amount of time and effort.

As tsunamis are generally regarded as long waves, they can be modeled by using nonlinear shallow water equations (NSWE). A number of NSWE models have been developed for tsunami studies and are currently available, for instance COMMIT/MOST (Titov and González, 1997; Titov and Synolakis, 1998), TUNAMI-N2 (Imamura, 1996), TsunAWI (Androsov et al., 2008; Harig et al., 2008), and H2Ocean (Cui et al., 2010). As the advance of computer power and numerical techniques, numerical simulations of tsunami propagation have been greatly improved in the last 30 years. However, there are still abundant challenges. One of the main difficulties is the inherent multi-scale character of this phenomenon. The evolution of the tsunami involves different scales, from propagation in the deep ocean (length scale is of thousands of kilometers), to the run-up and inundation in a coastal area (length scale is of tens of meters). The other difficulty needs to be mentioned is how to accurately simulate the phenomenon of wetting and drying.

In TsunAWI, the wetting and drying is handled by utilizing linear least square extrapolation method. A ‘dry node concept’ is also applied to exclude dry nodes from the solution. However, this extrapolation scheme is neutrally stable; therefore it demands horizontal viscosity to stabilize the model. In the cases of steep wave fronts or strong reflections from jumps of topography, wiggles could be generated. This in general results in the numerical degradation and breakdown in the simulation of wave run-up. To formulate a robust wetting and drying scheme for TsunAWI, cooperation between TUDelft and AWI was initiated. Within the project, TsunAWI has been reformulated in a finite volume formulation, which allows one to implement a simpler, yet more consistent wetting and drying scheme. The finite volume pair can easily be incorporated into the existing TsunAWI code, offering flexibility to the user.

In the NSWE equations, the vertical accelerations are neglected and the vertical momentum equation is reduced to an expression of hydrostatic pressure. The underlying assumption is that the depth of the fluid is small compared to the wave length of the disturbance, which is valid for many tsunamis. However, there is more and more concern in the tsunami community, that the frequency dispersion, which is absent in the NSWE equations, might be important (Walters, 2005; Horrillo et al., 2006; Kulikov, 2006). Kulikov (2006) has demonstrated, on the basis of the analysis of the satellite altimetry, that the propagation of the
2004 Indian Ocean tsunami in the south-western direction was strongly frequency-dispersive. Recent studies also suggest that dispersion has a non-negligible effect on the tsunami evolution and run-up (Walters, 2005; Horrillo et al., 2006).

There are two conventional approaches to model the dispersion: Boussinesq type equations and non-hydrostatic models. The principle behind Boussinesq formulations is to incorporate the effects of non-hydrostatic pressure, while eliminating the vertical coordinate. The high accuracy comes at the expense of a rather complicated system with high-order derivatives, which requires an equally complex numerical scheme and may pose problems for the numerical implementation of the model (Løvholt and Pedersen, 2009).

The other approach is the so-called non-hydrostatic models, which introduce non-hydrostatic pressure and vertical velocity terms in the NSWE equations to resolve weakly dispersive waves (Casulli and Stelling, 1998; Stansby and Zhou, 1998; Casulli, 1999). In the non-hydrostatic models, a Poisson equation for the non-hydrostatic pressure needs to be solved. This accounts for most of the computational cost of the model. To get a more accurate dispersion relation, the model needs more layers, which makes the model computationally expensive. Therefore, efficient non-hydrostatic models are highly desired.

Many efforts have been devoted to improving the efficiency of the non-hydrostatic models (Stelling and Zijlema, 2003; Yuan and Wu, 2004). Reeuwijk (2002) proposed a method in which the number of pressure layers can be chosen independently from the number of horizontal velocity layers. A similar idea was recently presented by Bai and Cheung (2012b). A parameterized non-hydrostatic pressure distribution was proposed to reduce the computational costs. However, in both of Reeuwijk (2002) and Bai and Cheung (2012b)’model, the accuracy of the dispersion only depends on the number of pressure layers. If there is only one pressure unknown, their models require the same computational effort as the depth-integrated non-hydrostatic models, but also produce almost the same linear dispersion relationship as that of depth-integrated non-hydrostatic models. With the increasing demands of performing large scale flow simulations with non-hydrostatic models, it is highly desirable to have a model which can resolve the most significant surf zone physics while maximizing computational efficiency.

1.5 Objectives

The main objective of this research is to develop an accurate and efficient numerical model for tsunami study, with enhanced accuracy in flooding and drying, and the capability to efficiently resolve dispersive effects. This newly developed model with accurate flooding and drying, could act as a numerical tool in the assessment of tsunami inundation hazard and risk. Dispersive effects should be implemented in an efficient way, to enable the models application in large scale tsunami simulations. The new model should be easily incorporated into the existing TsunAWI code, and offer flexibility to the user. These objectives are accomplished via the
development of the unstructured grid, finite volume model, H₂Ocean. It has been extensively validated by a range of laboratory experiment simulations, and has been applied to a series of real tsunami simulations.

1.6 Outline of this thesis

In Chapter 1, background information about the tsunami and tsunami modeling is presented. In Chapter 2, a new unstructured grid, finite volume ocean model is presented, which is derived from the \( P_1^{NC} - P_1 \) finite element model TsuNAWI. By casting the finite element model in a finite volume way, both local and global mass conservations can be guaranteed. A momentum conservative advection scheme is also developed, which is needed in the simulation of flooding and wetting problems. The accuracy of the finite volume model has been extensively validated.

In Chapter 3, a non-hydrostatic extension of the model H₂Ocean is presented. To improve the computational efficiency, a lumping of the pressure gradient has been introduced, which could save the computational time by up to 30%, compared to the model without lumping. A non-hydrostatic pressure boundary condition has also been derived based on linear wave theory, which is accompanied by the regular incoming short wave for depth-integrated models. It can serve as a simple way to introduce incoming short waves in models with collocated grids.

In Chapter 4, a minimized Poisson equations formulation has been developed to improve the efficiency of the non-hydrostatic models. It simplifies the pressure stencil of the horizontal velocity in the bottom layer by approximating the pressure gradient term with only the pressures at the top face of the bottom cells. The continuity equations of the two bottom layers are combined. By doing so, the vertical velocity and the pressure at the bottom are eliminated from the set of equations. This makes the model computational efficient in simulating non-hydrostatic waves.

In Chapter 5, simulations of the 2004 Indian Ocean Tsunami have been conducted as a validation of the model. Detailed run-up and inundation results are compared with the field measurements. Two different initial uplifts are used, inverted from GPS data and established from coseismic vertical deformations and tsunami data. The simulation results based on the inversion of the GPS data shows poor performance in inundation. Scenario studies indicate that this poor performance is due to the small amplitude and the offshore location of the rupture, which is too far to the west.

In Chapter 6, the model H₂Ocean has been applied to the 2011 Japan Tsunami. In cooperation with scientists from the Department of Geoscience and Remote Sensing, the source of the 2011 Tohoku-Oki earthquake has been determined by means of a joint inversion of displacement measurements and seafloor pressure data. The propagation and inundation by this initial uplift is simulated with H₂Ocean. The simulated results show good agreement with the measurements of the tsunami in the ocean and on the land. A more detailed flooding simulation
of the 2011 Japan Tsunami along the Sanriku coast is also presented. In Chapter 7, some issues about tsunami modeling are discussed and possible future research is elaborated.
In this chapter, a finite volume model H₂Ocean is presented with accurate flooding and drying. It guarantees both mass conservation and momentum conservation. By using upwind water depth in the flux computation, negative non-physical water depth can be avoided. Fulfillment of all these properties is crucial for simulating flooding and drying. The model has been tested against a variety of flooding and drying problems, and has achieved good results.

This chapter has been published as:

Abstract

A new unstructured grid, finite volume ocean model is described, that is suitable for flooding and drying problems. Here we derive the finite volume analogue of the \( P_{1}^{NC} - P_1 \) finite element, by interpreting the advective term in the continuity equation in a flux sense. A corresponding non-overlapping control volume is then selected for the momentum equation. The resulting model employs the median node-dual control volume for the water elevations, and edge-wise control volume for the velocities. In contrast to the \( P_{1}^{NC} - P_1 \) approach, the finite volume model not only guarantees mass conservation in a global sense, but it also explicitly guarantees mass conservation in each individual cell. Another feature of this new model is its ability to conserve momentum locally, and by applying a correction factor it can preserve constant energy head along a streamline in the case of rapidly
varied flows. By using the upwind water depths in the flux computations, no special flooding and drying procedures need to be implemented. The new model is efficient, does not produce non-physical negative water depths and generates accurate results for a wide variety of flooding and drying problems. Compared with the results obtained from the $P_1^{NC} - P_1$ finite element model, the new model produces better solutions in the simulation of inundations and in capturing shock waves.

2.1 Introduction

The Indian Ocean Tsunami on December 26, 2004 was triggered by the Great Sumatra-Andaman Earthquake. It lasted for approximately 8-10 minutes, (Vigny et al., 2005) and had a moment magnitude between 9.1-9.3, (Ammon et al., 2005; Stein and Okal, 2005). It caused one of the largest tsunamis in recent times (Lay et al., 2005; Titov et al., 2005) and led to widespread devastation and loss of life. It flooded coastal regions throughout the Indian Ocean and beyond without any tsunami warning ever having been issued. It took days for seismologists to get an accurate estimate of the magnitude and length of the rupture. As shown by Pietrzak et al. (2007), available GPS data could have been inverted to provide initial sea surface displacement data within 15-30 minutes of the earthquake. Sobolev et al. (2007) demonstrated that tsunami warnings could be issued within 10 minutes by incorporating GPS arrays in a GPS-shield.

However, one of the worst hit regions was Banda Aceh, located in the northern part of Sumatra, 150km to the west of the rupture zone. It was destroyed within 40 minutes of the earthquake, when the tsunami engulfed Banda Aceh. The observed inundation line was 3-4 km inland and the observed flow depth over the ground was over 9m within the city (Borrero et al., 2006). The resulting devastation and loss of life reflects the urgent need to develop accurate simulation models for tsunami inundations. Consequently, accurate estimates of run up and the extent of inundations are now also considered essential components of tsunami warning systems and are the main focus of the work presented here. Understanding the regions most at risk from tsunamis and devising appropriate evacuation strategies is a vital component in helping to mitigate future disasters.

Since the Indian Ocean Tsunami many countries have contributed to the development of tsunami warning systems in the Indian Ocean. A German-Indonesian Tsunami Early Warning System (GITEWS) (www.gitews.de) (Behrens, 2008), is currently under active development. The work presented here is carried out within the GITEWS framework. One of the aims of this project is the development of accurate numerical models with which to simulate the propagation, flooding and drying, and run-up of a tsunami. This is done not only for forecasting purposes, but also for detailed scenario studies, in order to assess the regions most at risk from future tsunamis. In this context TsunAWI (Behrens, 2008) has been developed by the Alfred Wegener Institute (AWI).
TsunAWI is an explicit, $P_{1}^{NC} - P_{1}$ finite element model. An unstructured grid approach was adopted, which offers of flexibility in resolving the details of coastal regions. The $P_{1}^{NC} - P_{1}$ finite element pair was first studied by Hua and Thomasset (1984), and was recently employed by Hanert et al. (2005). It is staggered in space, using discontinuous linear non-conforming ($P_{1}^{NC}$) basis functions for the velocity and the linear continuous basis functions ($P_{1}$) for the water elevation. The $P_{1}^{NC} - P_{1}$ finite element pair has been proven to be computationally efficient, due to the orthogonal mass matrices, and it is free of spurious pressure modes \(^1\) and hence does not require stabilization (Le Roux et al., 2005, 2007; Rostand and Le Roux, 2008). Another important property of the $P_{1}^{NC} - P_{1}$ finite element pair is that it has no spurious velocity modes, which maybe as dramatic as the spurious pressure modes in a discretisation (Rostand and Le Roux, 2008). An extrapolation method is employed in order to handle flooding and drying, using the concept of ‘semi-dry’ elements, i.e. elements with both dry and wet nodes, as described by Lynett et al. (2002). The idea of this concept is to exclude dry nodes from the solution and then to extrapolate elevation to dry nodes from wet neighbors using a linear least square method to capture the moving boundary. However, this scheme is neutrally stable. In places with large gradients, it requires horizontal viscosity. In addition, when a wet node is surrounded by dry nodes, it will be excluded from the computational domain.

There are many practical examples where this finite element pair has been successfully used in studies of wave generation and propagation (Greenberg et al., 1993; Hanert et al., 2005; Harig et al., 2008; Androsov et al., 2008). Harig et al. (2008) demonstrated that it is an excellent choice for tsunami wave propagation in the Indian Ocean, finding excellent agreement with Jason satellite data and coastal tide gauge stations. However, it may produce non-physical negative water depths when applied to flooding and drying problems. Moreover, in the case of a large free surface gradient, the extrapolation cannot be employed any longer. High viscosity is required to damp out the jump.

In inundations caused by either dam breaks or tsunamis, rapidly varied shallow water flows are the dominant phenomena which must be reproduced by a numerical model if one is to have accurate flooding and drying solutions. There are a lot of flooding and drying techniques (Balzano, 1998; Cea et al., 2007; Gourgue et al., 2009). One of the simplest techniques is masking out elements that are dry or partially wet (Lynch and Gray, 1978). Horritt (2002) evaluated this algorithm together with two other methods in finite element models of shallow water flow. He concluded that the element mask (EM) technique reproduced the free surface profile the best, but with poor mass conservation properties. However, in order to simulate such rapidly varied flows, it is crucial to apply the correct mass and momentum conservation properties (Stelling and Duinmeijer, 2003). Since the governing equations are numerically solved using a least squares variation method, the finite element method does not provide an explicit way to conserve

\(^1\)But it still has two spurious inertial modes.
Here we address these limitations and consider an equivalent finite volume approach which we demonstrate to be mass and momentum conserving. Recently, Postma and Hervouet (2007) demonstrated the compatibility between finite volume and finite elements. Following their method, we show how the $P_1^{NC} - P_1$ finite element pair can be reformulated in a finite-volume sense, which allows one to implement a simpler, yet more consistent wetting and drying scheme. The finite volume approach guarantees mass conservation in both individual control elements and the entire computational domain. The new element is referred to as the median node-dual finite volume pair (Kallinderis and Ahn, 2005). The finite volume pair can easily be incorporated into the existing TsunAWI code, offering flexibility to the user.

The flooding and drying scheme used in this new finite volume approach was the one proposed by Stelling and Duinmeijer (2003). Their method used upwind water depths to calculate the fluxes at the velocity points. This approach is suitable for use in a structured C-grid finite volume approach. By rearranging the terms, they demonstrated that this approach could guarantee positive water depths. Therefore, no special flooding and drying procedures need to be implemented. Owing to the upwind water depth used at the velocity points, the free surface gradient will never oppose the incoming wave. This will lead to a much smoother behavior of the wave during flooding and drying. Numerous flooding and drying cases were tested with this method, such as the classic dam break problem. The results show that it works well and efficiently. It can easily be adopted by finite volume unstructured grid models.

This flooding and drying method has been successfully used by Kramer and Stelling (2008) and by Kleptsova et al. (2009). Both of their codes are based on the method described in Casulli and Walters (2000). However, these models require the grid to be orthogonal. Triangular grids with only acute angle triangles satisfy this condition, however, the generation of such grids is not easy for complicated bathymetry (Bern and Eppstein, 1995). One advantage of finite element method is its geometric flexibility, it does not suffer this orthogonality restrictions. Recently, an unstructured grid, finite volume, three-dimensional primitive equation coastal ocean model (FVCOM) has been developed and successfully applied to a range of flooding and drying problems in estuaries and coastal zones (Chen et al., 2003). In contrast to the method of Casulli and Walters (2000), the horizontal components of the velocity are stored at the centroids and all other quantities are stored at the vertices. Because FVCOM uses centroids, rather than circumcentres, it does not suffer from the same grid restrictions. They use a similar flooding and drying algorithm to Stelling and Duinmeijer (2003).

We present the results of our developments of an unstructured grid numerical model suitable for tsunami studies, where we pay particular attention to accurate flooding and drying algorithms. Here we describe the results obtained using TsunAWI with the $P_1^{NC} - P_1$ finite element, as well as the new finite volume model. In Section 2.2 we introduce the basic equations. In Section 2.3 we de-
Figure 2.1: Vertical plane view of water area, with bottom and free surface

scribe the discretization of \( P_1^{NC} - P_1 \) finite element pair. In Section 2.4 the control volumes of the water level and velocity equivalent to \( P_1^{NC} - P_1 \) are constructed following the method from Postma and Hervouet (2007). In Section 2.5 a proper flooding and drying scheme is described. We compare the finite volume approach to the \( P_1^{NC} - P_1 \) finite element method with some test examples in Section 2.6. Finally in Section 2.7 the results are discussed.

### 2.2 Governing equations

The governing equations used here are the 2-D shallow water equations:

\[
\frac{\partial \eta}{\partial t} + \nabla \cdot [(d + \eta)u] = 0
\]

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + f k \times u = -g \nabla \eta
\]

where \( \eta \) is the water level, \( u \) the depth-average water velocity with components \((u, v)\), \( d \) the reference depth of the fluid, \( f \) the Coriolis parameter, \( k \) a unit vector in the vertical direction, \( g \) the gravitational acceleration, \( \nabla \) the two-dimensional gradient operator. If the bottom does not vary in time, Eq. (2.1) could also be written in the following form:

\[
\frac{\partial h}{\partial t} + \nabla \cdot [hu] = 0
\]

where \( h = \eta + d \) is the total water depth (see Fig. 2.1).
2.3 \( P_{1}^{NC} - P_{1} \) finite element discretization

The finite element spatial discretization is based on the approach by Hanert et al. (2005). Here similar notations are used.

The domain \( \Omega \) consist of \( N_{E} \) disjoint triangle elements \( \Omega_{e} \). The total number of vertices and segments are denoted as \( N_{V} \) and \( N_{S} \), respectively.

\[ \Omega = \bigcup_{e=1}^{N_{E}} \Omega_{e} \quad \text{and} \quad \Omega_{e} \cap \Omega_{f} = \emptyset \quad \text{for} \; e \neq f \]

The solution of the equations is \( \eta(x, t) \in E \) and \( u(x, t) \in U \) such that:

\[
\sum_{e=1}^{N_{E}} \int_{\Omega_{e}} \left( \frac{\partial \eta}{\partial t} + \nabla (h \mathbf{u}) \right) \hat{\eta} d\Omega = 0 \quad \forall \hat{\eta} \in E \tag{2.4}
\]

\[
\sum_{e=1}^{N_{E}} \int_{\Omega_{e}} \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + f(k \times \mathbf{u}) + g \nabla \eta \right) \hat{\mathbf{u}} d\Omega = 0 \quad \forall \hat{\mathbf{u}} \in U \tag{2.5}
\]

where \( \hat{\eta} \) and \( \hat{\mathbf{u}} \) are test functions which belong to \( E \) and \( U \) respectively.

A finite element approximation to the exact solution of Eqs. (2.4) and (2.5) is found by replacing \( \eta \) and \( u \) with finite element approximations \( \eta^{h} \) and \( u^{h} \):

\[
\eta \approx \eta^{h} = \sum_{k=1}^{N_{V}} \phi_{k} \eta_{k} \tag{2.6}
\]

\[
u \approx \mathbf{u}^{h} = \sum_{k=1}^{N_{S}} \psi_{k} \mathbf{u}_{k} \tag{2.7}
\]

where \( \eta_{k} \) and \( \mathbf{u}_{k} \) represent elevation and velocity nodal values, and \( \phi_{k} \) and \( \psi_{k} \) are the \( P_{1} \) and \( P_{1}^{NC} \) shape functions respectively.

In TsunAWI, the elevation and the velocity variables are approximated by continuous linear conforming \( P_{1} \) and discontinuous linear non-conforming \( P_{1}^{NC} \) shape functions respectively (Fig. 2.2). Elevation nodes are thus lying on the vertices of the triangulation and velocity nodes are located at mid-segments. The essential advantage of non-conforming linear functions is their orthogonality on triangles:

\[
\int_{\Omega} \psi_{i} \psi_{j} d\Omega = \frac{A_{i}}{3} \delta_{ij}
\]

where \( A_{i} \) is the area of the support of \( \psi_{i} \), \( \delta_{ij} \) is the Kronecker delta. Thus, the mass matrix of the velocity is diagonal automatically and can be inverted easily. This orthogonal property will increase the computation efficiency of the numerical model since matrix inversion is computationally expensive.

In TsunAWI, if the consistent full non-conforming shape function is used in the advection, flux penalty will be introduced due to the discontinuity of the \( P_{1}^{NC} \)

\footnote{Formally, they share common nodes and edges.}
function. This will generate noisy sinks and sources. High viscosity is required to
smooth the velocity field (Danilov et al., 2008). In practice, $P_1$ projection method
is used. The velocity is projected from the $P_{NC}^1$ to $P_1$ space in order to smooth it.
Then the projected velocity is used to estimate the advection term. The resulting
advection scheme is stable (Androsov et al., 2008).

2.4 The control volume analogue to $P_{NC}^1 - P_1$ element

Postma and Hervouet (2007) demonstrated the compatibility between finite volumes
and finite elements. In this section, by following the method in Postma and
Hervouet (2007), a control volume analogue to the $P_{NC}^1 - P_1$ pair of elements is
derived. Some of the equations in their paper are repeated here for clarity.

2.4.1 The control volume for water levels

If the test function used in Eq. (2.4) is the $P_1$ shape function $\phi$, then the represen-
tation of the continuity Eq. (2.3) in finite element models will be:

$$\int_{t^n}^{t^{n+1}} \left( \int_{\Omega} \left( \frac{\partial \eta}{\partial t} + \nabla \cdot [h \mathbf{u}] \right) \phi_i d\Omega \right) dt = 0 \quad \forall i$$

The resulting Eqn. (2.8) consists of two parts: (a) the time derivative of the
water level and (b) the advection term.

---

1 The compatibility between finite volumes and finite elements is a well-known issue, but
Postma and Hervouet (2007) present a detailed numerical procedure for a specific finite element.
The mass matrix in the continuity equation

We assume the depth \( h \) is approximated with the same base functions as the water elevation according to Eqn. (2.6). The time derivative of \( h \) in (2.8) then transforms into:

\[
h = \sum_i h_i \phi_i \tag{2.9}
\]

\[
\frac{\partial}{\partial t} \sum_i \sum_j \int_\Omega h_i \phi_i \phi_j d\Omega = \frac{\partial}{\partial t} M_E \begin{pmatrix} h_i \\ \vdots \\ h_k \end{pmatrix}, \quad i, j, k \in E \tag{2.10}
\]

The mass matrix \( M_E \) of element \( E \) which contains nodes \( i, j, k \) will be \(^1\):

\[
M_E = \frac{S}{3} \begin{pmatrix} 0.50 & 0.25 & 0.25 \\ 0.25 & 0.50 & 0.25 \\ 0.25 & 0.25 & 0.50 \end{pmatrix} \tag{2.11}
\]

where \( S \) is the surface area of element \( E \).

In FE models of ocean and atmosphere, mass lumping has been used to produce explicit time integration schemes. Otherwise, even if the scheme is explicit, a non-diagonal system of equations needs to be solved (Hinton et al., 1976; Le Roux et al., 2009). With mass lumping, the mass matrix can be approximated by a diagonal matrix. The diagonal terms consist of the sum of all the terms in a row.

\[
M_E = \frac{S}{3} \begin{pmatrix} 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 1.00 \end{pmatrix} \tag{2.12}
\]

The total mass matrix \( M \) of the whole domain consists of the sum of all contributions of the element mass matrices. The value \( \frac{1}{3} \) of the surface area of the triangle \( E \) can be interpreted as the share of the surface area of \( E \) that is attributed to each of its nodes \(^2\). The analogous finite volume for the water levels of this FE (\( P_1^{NC} - P_1 \)) with mass-lumping can be constructed by connecting the centroids of the cells adjacent to the center vertex to the corresponding edge midpoints (Fig. 2.3), which is referred to the median node-dual finite volume (Kallinderis and Ahn, 2005).

The advection vector in the continuity equation

The advective terms of (2.8) for all internal nodes \( k \) are given by Eq. (2.13).

\[
\int_\Omega \nabla h u \phi_k d\Omega = \oint_\Gamma h u \cdot n \phi_k d\Gamma - \int_\Omega h u \nabla \phi_k d\Omega = - \int_\Omega h u \nabla \phi_k d\Omega \quad \forall k \notin \Gamma \tag{2.13}
\]

\(^1\)The integration over the domain is approximated by a quadrature rule.

\(^2\)It stems from the quadrature rule.
The function $\phi_k$ is a $P_1$ shape function, which is zero at the boundary for the internal nodes. Therefore, the boundary term in Eqn. (2.13) vanishes for the internal nodes. While for the nodes at the open boundary, this term represents the flux.

If we take a closer look at $\nabla \phi_k$, it is just a vector with magnitude $1/h_k$ where $h_k$ is the distance from node $k$ to the opposite edge in triangle $E$ (Figure 2.4), as shown by Postma and Hervouet (2007). The length of the opposite edge is denoted as $L_k$. The surface area of the triangle $S_E$ is equal to $0.5h_kL_k$. The vector $\nabla \phi_k$ is perpendicular to the opposite edge in the plane of element $E$ with magnitude $0.5L_k/S_E$. Eqn. (2.13) could be written as:

$$- \int_E h\mathbf{u} \nabla \phi_k dE = - \frac{L_k}{2} \frac{1}{S_E} \int_E h\mathbf{u} \cdot \mathbf{n}_{l_k} dE$$  

Eqn. (2.14) is that the flux within each element is equal to the average of $h\mathbf{u}$, denoted by $\bar{h}\mathbf{u}$, multiplied by half of the length of the opposite edge $k$, in the direction normal to this opposite edge.

The line $0.5L_k$ is as long as the projection $c$ of the gravity lines $\mathbf{a}$ and $\mathbf{b}$ which connect the centroid with the midpoints of the edges in the direction of $l_k$. Then the term $\frac{L_k}{2}\bar{\mathbf{u}} \cdot \mathbf{n}_{l_k}$ could be interpreted as the cross product of $\bar{\mathbf{u}}$ and the projection vector $\mathbf{c}$. Here the over bar stands for the average over the element.

$$\frac{L_k}{2}\bar{\mathbf{u}} \cdot \mathbf{n}_{l_k} = \mathbf{c} \times \mathbf{u}_m = |\mathbf{u}_m| |\mathbf{c}| \sin \theta$$  

Figure 2.3: The control volumes for the water elevation and velocity. For water elevation, the median node-dual finite volume is employed around a vertex, $a_m$ and $b_m$ are the segments connecting the centroid and the mid-point of the edges, $u_m$ is the average velocity at the centroid. For the velocity at the midpoint of one edge, $1/3$ of the area of the surface of the element on each side forms the control volume of it. $e_L$ and $e_R$ are the two adjacent elements.
where $u_m$ is the average velocity at the centroid, and $\theta$ is the angle between this velocity and the vector $c$ (Figure 2.4).

The vector $c$ can be split into two vectors $a$ and $b$, therefore:

$$ u_m \times c = u_m \times (a + b) = u_m \times a + u_m \times b = |u_m| |a| \sin \theta_1 + |u_m| |b| \sin \theta_2 \quad (2.16) $$

$\theta_1$ and $\theta_2$ are the angles between the velocity and the vectors $a$ and $b$ respectively.

A finite volume discretization in the control volume around vertex $k$ then reads:

$$ A_k \frac{dh_k}{dt} + \sum_{m \in NE_k} ((a_m \times u_m)h_{am}^* + (b_m \times u_m)h_{bm}^*) = 0 \quad (2.17) $$

where $A_k$ is the surface area of the median node-dual control volume, $NE_k$ is the neighbor elements of vertex $k$, $u_m$ is the average velocity in element $m$. Instead of the average water depth $\bar{h}$, $h_{am}^*$ and $h_{bm}^*$ are used here to achieve higher accuracy. The water height $h_{am}^*$ and $h_{bm}^*$ are defined as (See Fig. 2.3):

$$ h_{am}^* = \begin{cases} 
\min(d_{ka}, d_k) + \eta_{ka} & \text{if } a_m \times u_m \geq 0 \\
\min(d_{ka}, d_k) + \eta_k & \text{if } a_m \times u_m < 0 
\end{cases} \quad (2.18) $$

$$ h_{bm}^* = \begin{cases} 
\min(d_{kb}, d_k) + \eta_{kb} & \text{if } b_m \times u_m \geq 0 \\
\min(d_{kb}, d_k) + \eta_k & \text{if } b_m \times u_m < 0 
\end{cases} \quad (2.19) $$

where $d_k,d_{ka},d_{kb}$ are the bottom levels at vertex $k,k_a,k_b$ respectively.

### 2.4.2 The control volume for velocities

For the purpose of investigating the individual components of the momentum equation, at first, all the terms are disregarded except the barotropic force terms.

**Figure 2.4:** (a) Graphical representation of $\nabla \phi_k$; (b) $\overrightarrow{hu}$ through line $0.5 * L_k$; (c) Projection of the gravity lines (Redrawn from the paper of Postma and Hervouet (2007)).
2.4 THE CONTROL VOLUME ANALOGUE TO $P^1_{NC} - P_1$ ELEMENT

\[ \sum_{e=1}^{N_e} \int_{\Omega_e} \left( \frac{\partial \hat{u}}{\partial t} + g \nabla \eta \right) \hat{u} d\Omega = 0 \quad \forall \hat{u} \in U \]  

(2.20)

If the test function of $\hat{u}$ is the same as the $P^1_{NC}$ basis function $\psi$, and the time integration is the first-order, forward Euler scheme, after integration over the whole domain, we have:

\[
\frac{S_{eL} + S_{eR}}{3} u_f^{n+1} = \frac{S_{eL} + S_{eR}}{3} u_f^n - \Delta t g \left( \frac{\eta_{x_{eL}}^n S_{eL}}{3} + \frac{\eta_{x_{eR}}^n S_{eR}}{3} \right)
\]

(2.21)

\[
\frac{S_{eL} + S_{eR}}{3} v_f^{n+1} = \frac{S_{eL} + S_{eR}}{3} v_f^n - \Delta t g \left( \frac{\eta_{x_{eL}}^n S_{eL}}{3} + \frac{\eta_{x_{eR}}^n S_{eR}}{3} \right)
\]

(2.22)

where $u_f$ and $v_f$ are the two components of the velocity at the face, $\Delta t$ is the discrete time step, $e_L$ and $e_R$ are the two triangles adjacent to the face $f$, $S_{eL}$ and $S_{eR}$ are the surface area of triangle $e_L$ and $e_R$, $\eta_{x_{eL}}$, $\eta_{x_{eR}}$, $\eta_{y_{eL}}$, and $\eta_{y_{eR}}$ are water level gradients in triangle $e_L$ and $e_R$. The hydrostatic pressure terms are calculated based on the $P_1$ shape function, and are element wise constant.

\[
\eta_{x_{eL}} = \sum_{i \in e_L} \frac{\partial \phi_i}{\partial x} \hat{\eta}_i
\]

(2.23)

\[
\eta_{y_{eL}} = \sum_{i \in e_L} \frac{\partial \phi_i}{\partial y} \hat{\eta}_i
\]

(2.24)

The analogue finite volume for the velocity will be the tetragon formed by joining the centroids and the vertices surrounding the face, as depicted in Fig. (2.3). This control volume has been used by Hwang (1995).

If we introduce the following weighting factors:

\[
\alpha_{eL}^f = \frac{S_{eL}}{S_{eL} + S_{eR}}, \quad \alpha_{eR}^f = \frac{S_{eR}}{S_{eL} + S_{eR}}
\]

(2.25)

Then Eq. (2.21) and (2.22) can be rewritten as:

\[
u_f^{n+1} = u_f^n - \Delta t g \left( \alpha_{eL}^f \cdot \eta_{x_{eL}}^n + \alpha_{eR}^f \cdot \eta_{x_{eR}}^n \right)
\]

(2.26)

\[
u_f^{n+1} = v_f^n - \Delta t g \left( \alpha_{eL}^f \cdot \eta_{y_{eL}}^n + \alpha_{eR}^f \cdot \eta_{y_{eR}}^n \right)
\]

(2.27)

### 2.4.3 Advection

The depth-integrated momentum equation can also be written as:

\[
\frac{\partial q_f}{\partial t} + \nabla \cdot (q_f u_f) + g h_f \nabla \eta = 0
\]

(2.28)
Here \( q_f = h_f u_f \). The water depth at the face is defined by:

\[
h_f = \alpha^e_L \cdot h_{e_L} + \alpha^e_R \cdot h_{e_R} \tag{2.29}
\]

where \( h_{e_L} \) and \( h_{e_R} \) are the water depth at the centroid of triangle \( e_L \) and \( e_R \), i.e.,
the average water depth within that element:

\[
h_{e_L} = \frac{1}{3} \sum_{i \in e_L} h_i, \quad h_{e_R} = \frac{1}{3} \sum_{i \in e_R} h_i \tag{2.30}
\]

Similar to Perot (2000)’s scheme, a cell-based momentum vector \( a_c \) is introduced here. By integrating the advection term \( \nabla \cdot (qu) \) over a cell, we obtain:

\[
\int_{\text{cell}_c} \nabla (qu) d\Omega = \int_{\partial \Gamma} qu \cdot n d\Gamma = \sum_{k \in \text{cell}_c} Q_k u^*_k = S_c a_c \tag{2.31}
\]

where \( Q_k = q_k \cdot n_k \) integrated along face \( k \), i.e., the flux going through the face \( k \), \( Q_k \) is positive when \( q_k \) directs into the cell \( c \), otherwise, it is negative. \( S_c \) is the surface area of the cell, \( a_c \) is the advection vector in this cell. The advection vector at the face then can be constructed by the combination of the two vectors at the adjacent cells:

\[
a_f = \alpha^e_L a_{e_L} + \alpha^e_R a_{e_R} = \alpha^e_L \sum_{k \in e_L} Q_k u^*_k + \alpha^e_R \sum_{k \in e_R} Q_k u^*_k \tag{2.32}
\]

Then the integration of Eq. (2.28) over its control volume is:

\[
\frac{dq_f}{dt} + \alpha^e_L a_{e_L} + \alpha^e_R a_{e_R} + gh_f \left( \alpha^e_L \nabla \eta_{e_L} + \alpha^e_R \nabla \eta_{e_R} \right) = 0 \tag{2.33}
\]

The term \( \frac{dq_f}{dt} \) can be split into two parts:

\[
\frac{dq_f}{dt} = \frac{dh_f u_f}{dt} = h_f \frac{du_f}{dt} + u_f \frac{dh_f}{dt} \tag{2.34}
\]

\[
\frac{dh_f}{dt} = \alpha^e_L \frac{dh_{e_L}}{dt} + \alpha^e_R \frac{dh_{e_R}}{dt} \approx \alpha^e_L \sum_{k \in e_L} \frac{Q_k}{S_{e_L}} + \alpha^e_R \sum_{k \in e_R} \frac{Q_k}{S_{e_R}} \tag{2.35}
\]

Substitute Eqs. (2.32), (2.34) and (2.35) into (2.33) we obtain:

\[
\frac{dh_f}{dt} \frac{du_f}{dt} + \alpha^e_L \sum_{k \in e_L} \frac{Q_k (u_f - u^*_k)}{S_{e_L}} + \alpha^e_R \sum_{k \in e_R} \frac{Q_k (u_f - u^*_k)}{S_{e_R}} + gh_f \left( \alpha^e_L \nabla \eta_{e_L} + \alpha^e_R \nabla \eta_{e_R} \right) = 0 \tag{2.36}
\]

Here if the \( u^*_k \) is defined as:

\[
u^*_k = \begin{cases} 
  u_k, & \text{if } Q_k \geq 0 \\
  u_f, & \text{if } Q_k < 0 
\end{cases} \tag{2.37}
\]
Then the outgoing fluxes can be omitted. We need only to consider the incoming fluxes at the four faces around that face. The resulting advection scheme is similar to the one used by Wenneker et al. (2002).

### 2.4.4 Local constancy of the energy head

Rapidly varied flows, such as occur in the two-dimensional dam break problem investigated by Stelling and Duinmeijer (2003), involve sudden transitions. Near such sudden flow transitions the hydrostatic pressure assumption may become invalid (Fig. 2.5). In this situation application of the momentum conservation scheme would lead to an increase of energy, which is physically incorrect. Instead it is better to look at the energy head, as discussed in detail by Stelling and Duinmeijer (2003) and Kramer and Stelling (2008). The pressure term needs to be corrected. Only a small modification to the momentum conservative scheme is required, which is described below. Moreover, it is only applied locally.

![Figure 2.5: 1D uniform flow. In the sudden flow transition from 1 to 2, the pressure is non-hydrostatic.](image)

For simplicity of demonstration, we consider the situation of a 1D uniform flow with a structured triangular mesh (Fig. 2.6). Only the balance between the advection and the hydrostatic pressure for the steady flows needs to be considered.

\[
\frac{1}{h_f} \alpha^e_L \sum_{k \in e_L} Q_k (u_f - u^*_k) + \frac{1}{h_f} \alpha^e_R \sum_{k \in e_R} Q_k (u_f - u^*_k) + g \left( \alpha^e_L \nabla \eta_{e_L} + \alpha^e_R \nabla \eta_{e_R} \right) = 0
\]

(2.38)

For the \( u_f \) shown in the Fig. 2.6, only the left \( u_k \) contributes to the change of the momentum. Eq. (2.38) then becomes:

\[
\frac{1}{h_f} \frac{1}{2} Q_k (u_f - u_k) + g \left( \frac{1}{2} \nabla \eta_{e_L} + \frac{1}{2} \nabla \eta_{e_R} \right) = 0
\]

(2.39)
Here, we introduce a factor, \((h_k + h_f)/(2h_kh_f)\), in front of the advection term:

\[
(h_k + h_f)/(2h_kh_f) \left( \frac{Q_k (u_f - u_k)}{S_{eL}} \right) + g \left( \frac{1}{2} \nabla \eta_{eL} + \frac{1}{2} \nabla \eta_{eR} \right) = 0 \tag{2.40}
\]

For steady flow we have:

\[
Q_k = h_k u_k \delta y = h_f u_f \delta y \tag{2.41}
\]

By substituting Eq. (2.41) into Eq. (2.40), we obtain:

\[
\left( \frac{1}{2h_k} + \frac{1}{2h_f} \right) u_k h_k \delta y \frac{u_f - u_k}{\delta x \delta y} + g \left( \nabla \eta_{eL} + \nabla \eta_{eL} \right) = 0 \tag{2.42}
\]

\[
\frac{(u_k + u_f)(u_f - u_k)}{\delta x} + g \left( \nabla \eta_{eL} + \nabla \eta_{eL} \right) = 0 \tag{2.43}
\]

\[
\frac{(u_f^2 - u_k^2)}{\delta x} + g \left( \nabla \eta_{eL} + \nabla \eta_{eL} \right) = 0 \tag{2.44}
\]

By applying this correction factor, local constancy of the energy head along a streamline can be achieved for steady flow.

## 2.5 Flooding and drying

In the present work the method proposed by Stelling and Duinmeijer (2003) is used. This method accurately tracks the moving shoreline by employing the upwind water depth in the flux computations. However, in the new model the fluxes are not computed at the velocity points, therefore two situations need to
be considered; how the fluxes are computed and how the velocities are masked during flooding and drying.

First we consider the flux computations. We take the flux in element \( m \) as an example, the water depth used for the computation of the flux in the section \( a_m \) (black part in Fig. 2.7) is:

\[
h_{am}^* = \begin{cases} 
\min(d_{ka}, d_k) + \eta_{ka} & \text{if } a \times u_m \geq 0 \\
\min(d_{ka}, d_k) + \eta_k & \text{if } a \times u_m < 0 
\end{cases}
\]  

(2.45)

The wet or dry criterion for this section (black part in Fig 2.7) is:

\[
\begin{cases} 
wet & \text{if } \min(d_k, d_{ka}, d_{kb}) + \max(\eta_k, \eta_{ka}, \eta_{kb}) \geq h_{\min} \\
dry & \text{if } \min(d_k, d_{ka}, d_{kb}) + \max(\eta_k, \eta_{ka}, \eta_{kb}) < h_{\min}
\end{cases}
\]  

(2.46)

where \( h_{\min} \) is the threshold thickness.

![Figure 2.7: The illustration of dry/wet section.](image)

If this section is dry, then the flux through it is zero. Due to the upwind water depth used for the flux computation, the water depth used in the outward fluxes of the control volume of vertex \( k \) is at most \( h_k \). Thus, there will never be a flux out of a dry node. In the case shown in Figure (2.7), if \( \eta_k + d_k < h_{\min} \), which means node \( k \) is dry, then there will be no outward fluxes. If \( \eta_k + d_k \geq h_{\min} \), in the worst case, in which all fluxes are outward directed, the following criterion needs to be satisfied in order to guarantee positive water depths,

\[
\sum_{m \in NE_k} \Delta t(|a_m \times u_m| + |b_m \times u_m|) < A_k
\]  

(2.47)
This condition is the same as the CFL condition, and therefore poses no new time step limitation. In the case of inward fluxes, it is only when the water level $\eta_{ka}$ is higher than the bottom level $d_k$, that there can be a flux from node $k_a$ to $k$.

In scenarios of tsunami propagation, wave celerity needs to be taken into account. A more severe, but empirical time step limitation is employed:

$$\sum_{m \in NE_k} \Delta t(|a_m \times u_m| + |b_m \times u_m| + \sqrt{gh_m(L_{a_m} + L_{b_m})}) < A_k \quad (2.48)$$

Secondly, we consider how the velocities are masked during flooding and drying. The water depth at the side of the cell is defined by:

$$h_u = \min(d_{n1}, d_{n2}, d_{n3}, d_{n4}) + \eta_f \quad (2.49)$$

where $\eta_f = \alpha^{e_L} \cdot \eta_{e_L} + \alpha^{e_R} \cdot \eta_{e_R}$, $\eta_{e_L}$ and $\eta_{e_R}$ are the average water levels at the centroid of triangle $e_L$ and $e_R$. Whether or not a velocity point is wet, is based on a very simple criteria (Fig. (2.8)):

$$\begin{cases} 
    \text{wet} & \text{if} \quad h_u \geq h_{\text{min}} \\
    \text{dry} & \text{if} \quad h_u < h_{\text{min}} 
\end{cases} \quad (2.50)$$

![Figure 2.8: The bathymetry around a velocity point and its surrounding vertexes.](image)

If the velocity point under consideration is marked as dry, then it will be set to zero, and the momentum equations for this point are skipped.
2.6 Test Cases

2.6.1 Dam Break

To test whether the advection scheme is momentum conservative or not, a typical and simple test is the dam break problem. The upstream water level is 1m and the downstream water level is 0.1m for wet bed case and 0.0m for dry bed case. The initial velocities are set to zero. The time step has been chosen sufficiently small for convergence \((dt = 0.001s)\). The basin is \(100m \times 1m\). Two grids are used, which consist of 1987 and 20334 nodes respectively. The corresponding mesh size are \(\Delta x = 0.2m\) and \(\Delta x = 0.05m\).

The case with dry bed was first studied by \((\text{Ritter}, 1892)\), and the solutions were extended to wet bed and with non-zero initial velocities. All the analytical solutions could be found in \(\text{Chanson (2006)}\). Both of these cases were studied by \(\text{Stelling and Duinmeijer (2003)}\), where they showed that their depth averaged model, with a momentum conservative scheme, could accurately reproduce the dam break problem.

Figure 2.9 and 2.10 show that a correct speed and height of the shock wave was reproduced by the finite volume model. The velocity profile is not so sharp at the wave front for the dry bed case. This is due to the upwind scheme used. As the mesh is refined, the velocity profile and the water levels converged to the analytical solution. This convergence is significant in the velocity profile of the dry bed case. Owing to the initial large free surface gradient, the \(P_1^{NC} - P_1\) with extrapolation scheme, cannot handle this problem.

![Figure 2.9: The profile of the free surface for dry bed case (left) and wet bed case (right).](image)

2.6.2 2D Dam Break

\(\text{Stelling and Duinmeijer (2003)}\) carried out a laboratory flooding experiment of the two-dimensional dam break problem. The experimental set-up is shown in Fig. (2.11). It consists of two reservoirs A and B. The upstream reservoir B is initially filled with water of height 0.6m. The downstream reservoir A is initially dry or filled with water of height 0.05m. The opening between these two reservoirs is
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Figure 2.10: The profile of the velocity for dry bed case (left) and wet bed case (right).

0.4m wide with a gate which can be lifted with a speed of 0.16m/s. In the center line at 1.6,9,13,17,21 and 23m away from the opening are located 7 gauges. The approximate area of the elements in the basin is 0.06m², and are refined to 0.01m² in the area surrounding the gate. The mesh consists of 19899 nodes in total. The time step is set to 0.0001s.

After the gate was lifted, the flow expanded rapidly, then a hydraulic jump was formed and propagated outward from the gate. Local constancy of the energy head in the flow contraction of the gate needed to be applied (see Section 2.4.4 ¹), otherwise, inaccurate results were generated. Fig. 2.12 and 2.13 show the wave fronts at t = 1,2,3 and 4 seconds after the gate was lifted for the dry bed case and wet bed case respectively. Here the wave front is defined as the line reached by the shock wave at the same instant, with an initial perturbation more than 0.001m. Because of symmetry, only half of the domain is shown. The propagation of the wave front for the wet bed case agrees well with the experimental results, because the wet bed computation was not so sensitive to the friction value. The computed wave front on the dry bed propagated slower than that as measured in the diagonal direction. In the lab experiment, a piece of plastic was laid in the middle of the basin, and reduced the friction dramatically. The transition from dry to wet is a contact discontinuity, which makes the dry bed computation sensitive to the friction value.

The snapshots of the front on the dry bed and wet bed as computed at t = 4s, and 18s are given in Fig. (2.14) and (2.15).

Again, the $P_{NC}^1 - P_1$ with extrapolation method cannot handle this dam break problem. Therefore, no results are presented.

2.6.3 Outflow from a basin with reservoir

Here we consider a basin of 1200 m wide, 13800 m long and with a uniformly sloping bed. The depth of the basin is 5 m at the left end and zero at the other end. To do the drying test simulation, a reservoir is formed by a relative maximum of the bottom level at $x = 9000m$ (See Fig. 2.17). The initial water

¹When the flow is constrained by the gate, the advection will be multiplied by a factor.
Figure 2.11: Laboratory experiment set-up for the 2-D dam break problem, from Stelling and Duinmeijer (2003).

Figure 2.12: The wave front for the dry bed two-dimensional dam break test at $t = 1, 2, 3$ and $4$ s. The black dot line indicates the experimental data, the red dot line indicates the numerical solution of the FVM model.

Figure 2.13: The wave front for the wet bed two-dimensional dam break test at $t = 1, 2, 3$ and $4$ s. The black dot line indicates the experimental data, the red dot line indicates the numerical solution of the FVM model.
Figure 2.14: Two-dimension dry bed dam break experiment. Snapshots of water levels at $t = 4\text{s}$ and $18\text{s}$, corresponding to the top and bottom panel respectively.

Figure 2.15: Two-dimension wet bed dam break experiment. Snapshots of water levels at $t = 4\text{s}$ and $18\text{s}$, corresponding to the top and bottom panel respectively.
level is 2 m everywhere. The water level at the left boundary changes in a cosine function, \( \eta_L = 2.0 \times \cos\left(\frac{2\pi}{360000} t\right) \). Then the basin is emptied by the depletion of the downstream level in 100 h. The water level in the reservoir will asymptotically reach the relative maximum bottom elevation. This is a simple but challenging test to check if the model can conserve mass during flooding and drying. It has been done by Balzano (1998) with several wetting and drying methods. The basin is discretized by a very coarse mesh (See Fig. 2.16). The time step \( \Delta t = 100 \text{s} \). Fig. 2.17 presents the water level at \( t = 100h \). It shows that the finite volume model can conserve the mass of a wet area enclosed by a dry area \(^1\).

![Figure 2.16: The mesh used in the drying test simulation in a 1D basin with reservoir.](image)

![Figure 2.17: Drying test simulation in a basin with reservoir.](image)

\(^1\)In Balzano (1998)' paper, the asymptotically levels unrealistically higher or lower than the reservoir fill level were yielded by several wetting and drying schemes.
2.6.4 Tsunami run-up

There is a simple setup for tsunami run-up modeling exercise: tsunami run-up and draw-down motions on a uniformly sloping beach. The analytical solution of this problem was first expressed by Carrier and Greenspan (1958) and later revisited by Siden and Lynch (1988). The initial-value-problem technique introduced by Carrier et al. (2003) is used to produce the analytical solution. The beach slope is set to 1/10 and the initial free surface elevation is given as shown in Fig. 2.18.

Fig. 2.19 presents the simulated wave levels at selected time for the finite volume model and the $P_1^{NC} - P_1$ model. Both of these models give almost the same results as the analytical ones at $t = 160s$ and $t = 175s$. However, at $t = 220s$, the $P_1^{NC} - P_1$ approach gives a poor solution which has wiggles at the front, whilst the finite volume method gives a much smoother solution. In the locally enlarged Fig. (2.20), negative water depths during the flooding and drying, can be found in the results of the $P_1^{NC} - P_1$ model. In Fig. 2.21, the location of the shoreline as a function of time computed by the finite volume model is given. It shows that the new finite volume model can accurately simulate the run-up and draw-down motion of a tsunami.

![Initial water level of Tsunami run-up on a plane beach.](image)

Figure 2.18: Initial water level of Tsunami run-up on a plane beach.

2.6.5 Okushiri test case

Okushiri island tsunami of 1993 produced unexpectedly large tsunami run-up heights at the lee side. The record maximum wave run-up height was about 20m (Liu et al., 1995). To study this unusual problem, the Okushiri tsunami was simulated in the Central Research Institute for Electric Power Industry (CRIEPI) in Abiko, Japan (Matsuyama and Tanaka, 2001). It was simulated by a 1/400 scale laboratory experiment in a large-scale tank (205 long, 6 m deep, 3.4m wide). The bathymetry used in the laboratory experiment is shown in Fig. (2.22). Three stations were mounted to provide observation data for comparison, which are marked by stars in the figure. An initial tsunami wave (shown in Fig. 2.23) entered the domain through the left open boundary.

The comparison of numerical result with observation data is given in Fig. (2.24). Both models can predict the amplitude and phase of first wave well.
Figure 2.19: Tsunami run-up on a plane beach. (Top) Run-up after 160 seconds; (Middle) Run-up after 175 seconds; (Bottom) Run-up after 220 seconds. Red, blue and green lines correspond to analytical, FVM, and $P_1^{NC} - P_1$ results, respectively.
Figure 2.20: Zoom in view of Tsunami run-up on a plane beach. Red, blue and green lines correspond to analytical, FVM, and $P_1^{NC} - P_1$ results, respectively. It clearly shows that $P_1^{NC} - P_1$ generates negative water depths and oscillations at the wave front.

Figure 2.21: Location of the shoreline as a function of time computed by FVM.
Figure 2.22: The bathymetry for computation and experiment of Okushiri, 3m × 5m, with stations 5, 7, 9 indicated.

Figure 2.23: The input tsunami wave for Okushiri test case, with duration of 22.5 s.
Figure 2.24: Comparison of experimental data (blue line) with numerical simulation results of FVM (blue line), $P_1^{NC} - P_1$ (green line) for the Okushiri test case.

However, in the $P_1^{NC} - P_1$ model, lots of oscillations were generated when flooding and drying. This can be seen clearly if we plot the whole wave field (see Fig. (2.25)). All these wiggles will propagate back to the deep part, and will degrade the final solution.

2.6.6 Run-up of solitary waves on a Conical island

Another benchmark of the tsunami simulation is the run-up of a solitary wave on a conical island. Experiments were performed in a wave basin at the US Army engineer Waterways Experiment Station, Coastal Engineering Research Center (Liu et al., 1995). A basin of 30-m-wide by 25-m-long was constructed. The center of the circular island was located at $(x = 12.96m, y = 13.80m)$. Fig. (2.26) shows the experimental setup. The vertical height of the island was approximately 62.5 cm, with a slope of 0.25. A total of 27 gauges were used to measure the
surface elevation. From these, 7 gauges (1,4,6,9,16,12) are freely available to use. Different water depths and wave steepnesses were tested in the experiments. However, here only the case with water depth at 32 cm and wave steepness $\zeta_0/L = 0.0063$ was investigated. The ratio between wave height and the initial water depth is $\zeta_0/d_0 = 0.10$. As the flooding and drying happens on the island, large gradients of velocity and water level are expected there. Then higher resolution of the mesh is employed in the region near the island. The mesh size beyond the island is around $\Delta x = 0.3m$, and is gradually refined to $\Delta x = 0.03m$ near the island (See Fig. 2.27). The time step is $\Delta t = 0.001s$. Because the model does not have a moving boundary to produce the solitary wave, an analytical solution of solitary wave was introduced at the left boundary and was calibrated by the wave signals at gauge 4.

In Figure (2.28) the measurements at gauges 6,9,16,22 are compared to the numerical results. The locations of the gauge can be found in Fig. (3.15). For both the $P_1^{NC} - P_1$ and the finite volume model, the overall agreement between the numerical solution and the experimental data are good. At gauge 16 and 22, the arrival times were predicted well, which were of engineering importance in the development of tsunami warning systems. The wave at gauge 22 was much steeper than the experimental data. This is due to the lack of dispersion in shallow water equations. Still, wiggles can be noticed in the $P_1^{NC} - P_1$ model, when the wave arrived at the island, where the flooding and drying happened (Fig. 2.30).

The run-up around the island is also compared with the measured data. On the left panel of Figure (2.29) the maximum inundation position around the island is plotted against the experimental measurements. In the right panel of Figure (2.29) the vertical run-up heights are converted to horizontal run-ups, scaled by the initial water depth. The $P_1^{NC} - P_1$ overestimates the maximum run-up at all positions. This is the consequence of wiggles, which degraded the solution. All
Figure 2.26: Experimental setup of the experiment. (Top) Top view of the wave basin and the island and the locations of the wave gauges. (Bottom) Vertical view of the circular island on the cross-section A-A.

Figure 2.27: The grid used for the conical island test case. The mesh is gradually refined toward the island.
2.7 Discussion

The $P_1^{NC} - P_1$ approach is a promising method among low order finite elements for use in unstructured mesh models. It is free of spurious elevation modes and spurious velocity modes, has good dispersive properties and reasonable computational cost due to the diagonal mass matrices. There are already practical examples of using this discretization in studies of wave generation and propagation (Hua and Thomasset, 1984; Hanert et al., 2005). The $P_1^{NC} - P_1$ has been shown to be a good choice for tsunami wave propagation (Harig et al., 2008).

However, when it comes to the flooding and drying problem, the $P_1^{NC} - P_1$ finite element together with the semi-dry elements suffer from a number of problems. First, because the governing equations are written in a weighted residual sense, mass conservation is not cast in the flux form, which is the source of diffi-
Figure 2.29: Measured (red stars) and Numerical FVM (blue line), $P_{1NC}^1 - P_1$ (green line) results of maximum inundation position and maximum run-up around the conical island. (Top) The maximum inundation position around the island. (Bottom) Maximum relative run-up around the island.

Figure 2.30: The snapshot of conical island test case at 7.8 s. (Left Panel) Result of FVM, shows smooth behavior. (Right Panel) Result of $P_{1NC}^1 - P_1$, generates oscillations when involves flooding and drying.
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2.7

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culties in formulating the wetting and drying algorithm. It may generate wiggles when it involves flooding and drying. This has no effect on the simulation of wave propagation in the deep ocean. However, in the situation of inundation, these wiggles could contaminate the whole flow field and generate inaccurate results. Figs. (2.30) and (2.25) clearly show the wiggles generated by the $P_{1}^{NC} - P_1$ approach. Secondly, the extrapolation method only works well for gentle gradients. In places with large gradients, viscosity is required to smooth out the jumps. In the case of dam break problems, the water level gradient is too large, consequently extrapolation cannot be employed. Hence we have no results for the dam break test case by the $P_{1}^{NC} - P_1$ approach. Moreover, the extrapolation introduces extra mass to the system. Thirdly, it is difficult to conserve momentum in the $P_{1}^{NC} - P_1$ approach. For the $P_{1}^{NC}$ approximation, there is only one common node between two adjacent triangles. So it is discontinuous across triangle boundaries except at edge midpoints. When calculating the advection, flux penalties will be added due to the discontinuity of the base functions. This will introduce noisy sinks and sources (Danilov et al., 2008). It therefore requires more viscosity to stabilize the model. The general practical procedure used, is to project the $P_{1}^{NC}$ to the $P_1$ space in order to smooth it. This results a robust advection scheme, but does not conserve the momentum.

In order to overcome the disadvantages of the extrapolation scheme, an alternative method is to employ a moving computational grid. In this method, an interface line between wet and dry points is tracked and used as the boundary of the computational domain (Lynch and Gray, 1980). Thus the boundary of the computational mesh coincides with the shoreline, where the total water depth and normal transport are equal to zero. Although this method can provide a rigorous numerical solution to the wetting and drying problem, the grid has to be re-generated at every time step in order to track the boundary. This leads to a considerable computational burden.

More recently, Casulli (2009) developed a method which is quite efficient, does not require a threshold value for minimal water depth, or produce non-physical negative water depths. However, this method requires a semi-implicit model, resulting in a nonlinear system. An iteration procedure has to be used to ensure that, for the drying cell, only the precise amount of water within this cell is allowed to flow out.

Both of these two methods degrade the simplicity of the model. In order to take advantage of the explicit approach, a finite volume analogue of the $P_{1}^{NC} - P_1$ finite element was formulated. This new approach allows one to apply a simpler, yet more consistent wetting and drying scheme. Postma and Hervouet (2007) showed how to derive an equivalent finite volume schematization when applied to the continuity equation. The resulting control volume of the node is constructed by connecting the corresponding edge midpoints in the mesh to centroids of the

---

1 When momentum advection contribution is not small, it will contain a noisy component, it appears because the velocity space is too large for $P_{1}^{NC} - P_1$ discretization.
cells adjacent to these points. This control volume is denoted as median node-dual finite volume (Kallinderis and Ahn, 2005). In this paper we extended Postma and Hervouet (2007)'s method to the momentum equations. An edge-based control volume for the velocities was constructed. After reformulation in the finite-volume sense, the wetting and drying scheme proposed by Stelling and Duinmeijer (2003) can be adopted easily. The upwind water depth used in the flux computation does not allow a flux to leave a dry node. This guarantees positive water depth under the CFL condition.

Numerical treatment of advection is still a central issue in unstructured grid ocean modeling. When simulating rapidly varied flows, it is important to conserve momentum. To guarantee momentum conservation, the conservative form of the momentum advection needs to be used. However, in $P_{1}^{NC} - P_{1}$ model, when integrate the advection term over the domain, flux penalties will be added over the internal boundaries due to the discontinuity of the $P_{1}^{NC}$ base functions. This will not only introduce noisy sinks and sources, but also consume lots of time to compute these flux penalties. In the finite volume model we developed, velocities are treated in a finite volume form. This avoids the flux penalties across the edge when computing the momentum flux. By properly defining the advection vectors within the element and the water depth at velocity points, a momentum conservative advection scheme has been constructed. When local constancy of the energy head is necessary, it can be easily achieved by applying a correction factor.

The treatment of the continuity equation in the model described here is identical to that of FVCOM. However, a different velocity space is described here. Although it is a factor 1.5 larger than that used in FVCOM, FVCOM has to implement ghost cells in order to implement boundary conditions. This could lead to energy reflection when the dominant flow direction is normal to the boundary edge. By locating velocities at the edge as presented here, rather than at the center of the triangle, the boundaries are easier to handle. In addition, a least-squares velocity interpolation is not required for the advection. Detailed comparison between these two models is the subject of future work.

Two-dimensional depth integrated shallow water equations are used here. Provided that the wavelength of tsunami is mush longer than the water depth, the hydrostatic pressure balance assumption can be used, and it has been widely used in studying tsunamis (Androsov et al., 2008; Wijetunge, 2009). With this assumption, the momentum equation in the vertical direction is reduced to an expression of hydrostatic pressure. This can significantly simplify the model. Moreover, in this paper, our main objective is flooding and drying. For such problems, the hydrostatic pressure assumption is valid. However, there is growing evidence that dispersive effects maybe important in tsunami propagation (Hanson and Bowman, 2005; Horrillo et al., 2006). The hydrostatic assumption might yield inaccuracies with respect to the propagation and shoaling of the tsunami, depending on the complicated topography and the interaction with structures. Dispersion might be desirable for more accurate modelling of tsunamis in certain cases (Walters, 2005,
A non-hydrostatic version of this model is already under development. An unstructured finite volume method analogue of the $P_1^{NC} - P_1$ finite element has been derived. This model not only guarantees mass conservation locally and globally, but also guarantees momentum conservation, both of which are crucial to the simulation of rapidly varied flows and in flooding and drying problems. The performance of the new model has been tested against analytical solutions (Dam Break, Tsunami run-up) and observational data (the Okushiri test case) and shows good agreement (Fig. 2.9, 2.10, 2.19, 2.24). The results of the new finite volume model have been compared with that of the $P_1^{NC} - P_1$ finite element model for a number of test cases, such as tsunami run-up, Okushiri and the conical island. The comparison shows that the new model can generate more accurate results than the $P_1^{NC} - P_1$ finite element model in flooding and drying.

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Non-hydrostatic H$_2$Ocean

In this chapter, the hydrostatic finite volume model H$_2$Ocean, is extended to non-hydrostatic by following a fractional step procedure. A lumping method is employed in the computation of the non-hydrostatic gradient terms when constructing the Poisson equation. The number of non-zero elements in the sparse matrix of the Poisson equation is reduced by half, while the CPU time is reduced by 30%, compared to that of without lumping.

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Abstract

An efficient depth-integrated non-hydrostatic unstructured grid finite volume model is presented and applied to several test cases, which involve the computation of free surface flows. The model solves the non-linear shallow water equations, with extra non-hydrostatic pressure terms to describe dispersion effects. The efficiency of the model is a major issue when it involves large spatial domains with high resolution meshes. Lumping of the pressure gradient of the horizontal velocities is employed, which reduces the number of non-zero elements in the sparse matrix of the Poisson equation by half. This greatly reduces both the memory requirements and the number of floating point operations required to solve the Poisson equations. In addition, for a model using a collocated grid, both the water surface and the non-hydrostatic pressure need to be defined at the incoming wave boundary. A non-hydrostatic pressure boundary condition has been derived based on linear wave theory, that is accompanied with the regular incoming short wave for
depth-integrated models. It can serve as a simple way to introduce short incoming waves in models with collocated grids. The model has been validated through several test problems including an oscillating basin, propagation of a solitary wave, wave propagation over a submerged bar, wave refraction and diffraction over an elliptical shoal, as well as solitary wave run-up on a conical island. The model gives good results for all test cases. We show that the lumping of the pressure gradient generates identical results to simulations without lumping, while the execution CPU time is reduced by around 30%, demonstrating a good computational efficiency of the model.

3.1 Introduction

The non-linear shallow water equations (NSWE) approach has been extensively used in tsunami modeling, because of its low computational cost and simple formulation. Phenomena such as breaking bores and inundation can be simulated by NSWE with special techniques (Stelling and Zijlema, 2003). A number of NSWE models have been developed for tsunami studies and are currently available, for instance ComMIT/MOST (Titov and González, 1997; Titov and Synolakis, 1998), TUNAMI-N2 (Imamura, 1996), TsunAWI (Androsov et al., 2008; Harig et al., 2008), and H₂Ocean (Cui et al., 2010).

In the NSWE equations, the vertical accelerations are neglected and the vertical momentum equation is reduced to an expression of hydrostatic pressure. The underlying assumption is that the depth of the fluid is small compared to the wave length of the disturbance, which is valid for many tsunamis. For tsunamis generated by sub-marine earthquakes, the typical wave-length is of the order of hundreds of kilometers, which is much larger than the average ocean depth. For example, Pietrzak et al. (2007) compared a hydrostatic and non-hydrostatic simulation of the Indian Ocean tsunami and found only negligible differences when compared with the leading waves recorded by the Jason-1 satellite. They did not investigate the impact of non-hydrostatic effects on flooding. However, Kulikov (2006) demonstrated, on the basis of the analysis of the complete satellite altimetry, that the propagation of the 2004 Indian Ocean tsunami in the south-western direction was strongly frequency-dispersive. Recent studies also suggest that dispersion has a non-negligible effect on the tsunami evolution and run-up (Walters, 2005; Horrillo et al., 2006).

A numerical study of the 2004 Indian Ocean tsunami carried out by Horrillo et al. (2006), suggests that dispersion can produce significant differences in coastal run-up. Their one-dimensional numerical simulation shows that, the run-up from the dispersive model can be up to 60% higher than the run-up obtained by the NSWE models. By simulating the same event with a dispersive Boussinesq model, Grilli et al. (2007) estimate dispersive effects on maximum deep water elevations

\(^1\text{In NSWE equations, it is also assumed that } H \ll L, \text{ and } \eta \ll H\)
to be more than 20% in some areas. Walters (2005) also pointed out that dispersive effects can significantly modify the shape of the tsunami for both the onshore and offshore propagating waves by performing simulations of a tsunami generated by a submarine fault rupture on the New Zealand continent shelf. When a tsunami wave travels from the deep ocean to shallow-water regions, its evolution is dominated by the combined effects of dispersion due to non-hydrostatic effects and wave-wave interaction due to non-linear effects. Studies show that the dispersive features can also be recognized both in the 2010 Chilean tsunami (Saito et al., 2010) and the 2011 Tohoku-Oki Japanese tsunami (Saito et al., 2011).

There are more situations, such as flows over rapidly varying slopes, propagation of non-linear (solitary) waves, wave propagation in deep water that cannot be accurately represented under the hydrostatic assumption. Therefore, it is important to explore non-hydrostatic effects in tsunami simulations, especially in a coastal context.

In the past decades, significant effort has been devoted to developing non-hydrostatic models, in order to simulate relatively short wave propagation, where both frequency dispersion and nonlinear effects play an important role. The traditional approach used to include the non-hydrostatic effects in depth-averaged models is by using Boussinesq-type equations. Boussinesq-type equations were first introduced by Peregrine (1967) and have been extended to describe highly nonlinear processes up to wave breaking (Madsen and Sørensen, 1992; Nwogu, 1993; Gobbi et al., 2000; Madsen et al., 2003). The basic idea with the Boussinesq equations is to eliminate the vertical coordinate from the flow equations, while retaining some of the influence of the vertical structure of the flow. A Taylor expansion is applied to the horizontal and vertical flow velocities, then the incompressible and irrotational conditions are used to replace vertical partial derivatives of quantities in the Taylor expansion with horizontal partial derivatives. As a result, the resulting partial differential equations are only functions of the horizontal coordinates and time. However, this procedure results in high-order correction terms, which make the model complicated and computationally less efficient.

The other approach is to employ the so-called non-hydrostatic models, which introduce non-hydrostatic pressure and vertical velocity terms in the NSWE equations to resolve weakly dispersive waves. The pioneering works of Casulli and Stelling (1998) and Stansby and Zhou (1998) proposed to decompose the pressure into hydrostatic and non-hydrostatic components. In these so-called traditional non-hydrostatic models (Casulli, 1999; Lin and Li, 2002), the vertical grid is such that the non-hydrostatic pressure is located at the center of the computational cell, and the hydrostatic assumption is employed within the top layer. Since this assumption is not physically sound, a large number of vertical layers are needed to resolve wave dispersion to an acceptable accuracy. Stelling and Zijlema (2003) developed an efficient and accurate numerical method utilizing a Keller-box scheme instead, which has an edge-based grid system in the vertical direction. In this

1The high order terms also lead to instability over complex terrain.
way the zero pressure boundary condition can be imposed in a straightforward manner at the free surface, without the need for any approximation. Their results showed that with only two layers their model could handle wave breaking and short wave propagation problems, and produced comparable results to those of extended Boussinesq models (Zijlema and Stelling, 2008; Lynett et al., 2002). This approach has now been adopted in a number of finite difference (Yamazaki et al., 2009), finite element (Walters, 2005) and finite volume (Ai and Jin, 2010) models.

Stelling and Zijlema (2003) used a fractional step procedure, which consists of two steps. In the first step, the continuity equation and momentum equations are solved using the hydrostatic assumption. In the second step, the velocities from the first step are corrected by the non-hydrostatic pressure, and are substituted into the three-dimensional (3D) continuity equation. This results in a Poisson equation for the non-hydrostatic pressure. Once the linear equations are solved, the velocities can be updated by the solution of the non-hydrostatic pressure. By employing this fractional step procedure, non-hydrostatic effects can be easily incorporated into existing hydrostatic models.

Recently, a two-dimensional unstructured grid finite-volume model for tsunami simulations has been presented by Cui et al. (2010). This model has been shown to be mass and momentum conservative. The model produces accurate solutions in the simulation of flooding and drying. Based on this model, we follow the procedure proposed by Stelling and Zijlema (2003) to incorporate the effects of dispersion. In tsunami simulations, the efficiency of the model is a major issue. The original implementation of the non-hydrostatic algorithm results in a large stencil and a sparse matrix with a large number of non-zero elements, which leads to high simulation times. To improve the efficiency of the model, the non-hydrostatic pressure gradient is only approximated by three node values within one triangle when substituting the new velocities into the 3D continuity equation. In so doing, the number of non-zeros elements of the matrix is reduced by half, and the execution CPU time is shortened by around 30%. The new model is validated against several classic test cases, such as a standing wave in a closed basin, the propagation of solitary waves, wave deformation over the Berkhoof shoal, and wave run-up onto a conical island.

This paper proceeds as follows. In Section 3.2, the basic governing equations are introduced. In Section 3.3, we present the formulation of the fractional step method and the efficient implementation of the non-hydrostatic pressure projection method into H2Ocean, together with the boundary conditions. Finally, several test cases are given in Section 3.4 to demonstrate the accuracy and efficiency of the model.
3.2 Governing equations

The initial governing equations are the incompressible Reynolds’s averaged Navier-Stokes equations (RANS). These equations are given by:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + v \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{3.1}
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} + v \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \tag{3.2}
\]

\[
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - g + v \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \tag{3.3}
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.4}
\]

where \((u, v, w)\) are the flow velocities in \(x-, y-\) and \(z-\) direction; \(t\) is the time; \(p\) is the pressure, \(\rho_0\) is constant reference density; \(g\) is the gravitational acceleration; \(v\) is the kinematic viscosity coefficient. The kinematic free surface and bottom boundary conditions are given by:

\[
w = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y}, \text{ at } z = \eta \tag{3.5}
\]

\[
w = -u \frac{\partial d}{\partial x} - v \frac{\partial d}{\partial y}, \text{ at } z = -d \tag{3.6}
\]

where \(\eta\) is surface elevation, \(d\) is the depth of the water.

We follow Stelling and Zijlema (2003) and decompose the pressure into a hydrostatic component \(p_h\) and a non-hydrostatic component \(q\), such that \(p = p_h + q\) and by definition:

\[
\frac{\partial p_h}{\partial z} = -\rho_0 g \quad \text{or} \quad p_h = p_a + \rho_0 g(\eta - z) \tag{3.7}
\]

where \(p_a\) is the atmospheric pressure, and is neglected in the following.

Substituting this definition into Eqs. (3.1)-(3.4), integrating over depth, using the kinematic boundary condition (3.5) and (3.6), and ignoring the vertical acceleration terms, leads to \(^1\):

\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + g \frac{\partial \eta}{\partial x} + \frac{1}{h} \int_{-d}^{\eta} \frac{\partial q}{\partial x} dz + \frac{g \eta^2}{h^{4/3}} U \sqrt{U^2 + V^2} = 0 \tag{3.8}
\]

\(^1\)It is also assumed that there is no surface stress and the external contract force at the bottom is approximated by Manning friction.
\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + g \frac{\partial \eta}{\partial y} + \frac{1}{h} \int_{-d}^{\eta} \frac{\partial q}{\partial y} dz + \frac{g n^2}{h^{4/3}} V \sqrt{U^2 + V^2} = 0 \tag{3.9}
\]
\[
\frac{\partial \eta}{\partial t} + \frac{\partial (hU)}{\partial x} + \frac{\partial (hV)}{\partial y} = 0 \tag{3.10}
\]

where \((U, V)\) is depth averaged velocity; \(h = d + \eta\) is total water depth and \(n\) is Manning’s roughness coefficient.

Special attention is paid to the vertical integration of the non-hydrostatic pressure gradient. By virtue of Leibniz’s rule, we have
\[
\int_{-d}^{\eta} \frac{\partial q}{\partial x} dz = \frac{\partial}{\partial x} \int_{-d}^{\eta} q dz - q \bigg|_{z=-d} \frac{\partial d}{\partial x}
\]

The non-hydrostatic pressure at the free surface is zero, \(q \big|_{z=\eta} = 0\). We only consider the \(q\) at the bottom. The subscript has been dropped for simplicity. Studies show that \(\int_{-d}^{\eta} q dz = \frac{hq}{2}\) gives the best dispersion relation (Stelling and Zijlema, 2003). The integral of the pressure gradient becomes:
\[
\int_{-d}^{\eta} \frac{\partial q}{\partial x} dz = \frac{1}{2} h \frac{\partial q}{\partial x} + \frac{q}{2} \frac{\partial (\eta - d)}{\partial x}
\]

Now the horizontal momentum equation becomes
\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + g \frac{\partial \eta}{\partial y} + \frac{1}{2 h} \frac{q}{\partial x} + \frac{1}{2 h} \frac{\partial (\eta - d)}{\partial x} + \frac{g n^2}{h^{4/3}} U \sqrt{U^2 + V^2} = 0 \tag{3.13}
\]
\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + g \frac{\partial \eta}{\partial y} + \frac{1}{2 h} \frac{q}{\partial y} + \frac{1}{2 h} \frac{\partial (\eta - d)}{\partial y} + \frac{g n^2}{h^{4/3}} V \sqrt{U^2 + V^2} = 0 \tag{3.14}
\]

The vertical momentum equation also needs to be retained. The vertical velocity \(w\) is approximated as a linear function in the vertical. The vertical advective and dissipative terms, which are small compared with the non-hydrostatic pressure term, can be omitted. Then the vertical momentum equation based upon the Keller-box scheme becomes \(^1\):
\[
\frac{Dw}{Dt} = \frac{\partial}{\partial t} \left( \frac{w \eta + w_{-d}}{2} \right) = - \frac{1}{h} \left( \frac{\partial q}{\partial z} \bigg|_{z=\eta} + \frac{\partial q}{\partial z} \bigg|_{z=-d} \right) = \frac{q}{h}
\]

The discretized form of Eqn. (3.15) reads as:

\(^1\)For simplicity, the non-linear terms have been ignored. The pressure is assumed to be linearly distributed in the vertical.
\[
\frac{w_{\eta}^{n+1} - w_{\eta}^n + w_{-d}^{n+1} - w_{-d}^n}{2\Delta t} = \frac{q^{n+1}}{h}
\]

(3.16)

Eqs. (3.10), (3.13), (3.14), (3.15) together with the continuity Eq. (3.4), form the governing equations for depth-integrated, non-hydrostatic free-surface flows.

It is worth mentioning that the \( q \) used in the model is simply a numerical parameter to resolve the dispersion. It is not the real non-hydrostatic pressure at the bottom. If the vertical profile of the non-hydrostatic pressure \( q(z) \) needs to be reconstructed, it should satisfy the following requirements:

\[
q(z) = 0, \quad \text{at} \quad z = \eta
\]

\[
\int_{z=-d}^{z=\eta} q(z) = \frac{1}{2}hq
\]

(3.17)

(3.18)

\[
\frac{\partial q(z)}{\partial z} = -2\frac{q}{h}, \quad \text{at} \quad z = \eta
\]

\[
\frac{\partial q(z)}{\partial z} = 0, \quad \text{at} \quad z = -d
\]

(3.19)

(3.20)

One of the functions that meets all the requirements is:

\[
q(z) = \frac{2q}{3} \left(1 - \left(\frac{z + d}{h}\right)^3\right)
\]

(3.21)

3.3 Numerical Formulation

3.3.1 Fractional-step procedure

Decomposing the pressure term into hydrostatic and non-hydrostatic components is one of the most popular ways to include non-hydrostatic effects into the NSWE equations. Then two methods are typically employed to solve the system, the pressure projection method and the pressure correction method. Both employ the fractional-step procedure, which was first suggested by Harlow and Welch (1965) and Chorin (1968). In the pressure projection method, during the first step, the momentum equations are solved without the non-hydrostatic pressure terms. This yields an approximate velocity field, which is in general not divergence free. In the second step, a correction with the non-hydrostatic pressure is applied to the velocity field. By taking the divergence of the momentum equations and employing the three dimensional continuity equation, a divergence free velocity field is produced. The correction of the velocity field is an orthogonal projection in the sense that it projects the intermediate velocity field onto the divergence free field. The solution of the non-hydrostatic pressure is then used to correct the intermediate velocity field and to enforce continuity. Similar to the projection method, the pressure correction method also solves the momentum equations.
with a fractional step approach. The key difference is that the pressure correction method retains the non-hydrostatic pressure terms in the momentum equation when determining the intermediate velocity field in the first step. In the second step, the Poisson equations of the non-hydrostatic pressure correction ($\Delta q$, rather than $q$) are obtained and solved. Since the pressure correction method requires changes of the existing shallow water solver, we do not consider it here.

### 3.3.2 Pressure projection method

We follow the pressure projection method described in Stelling and Zijlema (2003). In the first step, the horizontal momentum equations are solved without the non-hydrostatic pressure terms, i.e., the horizontal momentum equations from the NSWE equations are solved. This solution, denoted as $\tilde{U}^{n+1}, \tilde{V}^{n+1}$ is used as an input for the second step. In the second step, the horizontal velocities as well as the vertical velocity are corrected by the non-hydrostatic pressures and are substituted into the 3D continuity equation (3.4). This results in Poisson equations for the dynamic pressure $q$. Once the system with unknowns of $q$ is solved, the horizontal velocity and vertical velocity are updated with the solution for $q$. The new water levels are computed by using the depth-integrated continuity Eq. (3.10). The details of the first step can be found in Cui et al. (2010)\(^1\). Here we only present the details of the second step.

The velocity solution from the first step is updated with the non-hydrostatic pressure terms as:

$$
U^{n+1} = \tilde{U}^{n+1} - \Delta t \left( \frac{1}{2} \frac{\partial q^{n+1}}{\partial x} + \frac{1}{2} \frac{q^{n+1}}{h^n} \left( \frac{\partial \eta^n}{\partial x} - \frac{\partial d}{\partial x} \right) \right) \tag{3.22}
$$

$$
V^{n+1} = \tilde{V}^{n+1} - \Delta t \left( \frac{1}{2} \frac{\partial q^{n+1}}{\partial y} + \frac{1}{2} \frac{q^{n+1}}{h^n} \left( \frac{\partial \eta^n}{\partial y} - \frac{\partial d}{\partial y} \right) \right) \tag{3.23}
$$

In the present model, the bottom topography is represented as a staircase. Therefore, the vertical velocity at the bottom equals zero ($w_{-d} = 0$). Since the advective and dissipative terms are omitted in the vertical momentum equation, in the second step, the approximation of the vertical velocity $\tilde{w}^{n+1}_\eta$ is the same as the solution from the last step, i.e., $\tilde{w}^{n+1}_\eta = w^n_\eta$. By using Eq. (3.16), the vertical velocity can also be expressed by the non-hydrostatic pressure:

$$
w^{n+1}_\eta = \tilde{w}^{n+1}_\eta + 2 \Delta t \frac{q^{n+1}}{h} \tag{3.24}
$$

Before substituting the new velocities into the local continuity equation (3.4), we integrate it over the control volume of water levels, following the same procedure in Cui et al. (2010). The depth-integrated form of the continuity equation reads:

---

\(^1\)The details of the first step can also be found in chapter 2
where $A_k$ is the area of the control volume of the water level, $NGB_E_k$ is the total number of the neighbor elements of the node $k$, $a_m$ and $b_m$ are the two sections which connect the midpoint of the edge and the centroid of the element, $\bar{U}_m$ is the average velocity at the centroid of the element $m$. All these definitions can also be found in Cui et al. (2010), (see Fig. 3.1). This equation differs from the integrated continuity equation (17) of Cui et al. (2010) in that the free surface kinematic boundary condition is not used, and here $h_k$ is not up-winded. To calculate the second term in Eqn. (3.25), we need to go through all the neighbor elements of node $k$, and compute the cross product of the average velocity $\bar{U}_m$ and the sections $a_m$ and $b_m$.

$$\bar{U}_m \times a_m = a_{mx} \cdot \bar{U}_m - a_{my} \cdot \bar{V}_m$$ (3.26)

where $a_{mx}$ and $a_{my}$ are the components of vector $a_m$, $\bar{U}_m$ and $\bar{V}_m$ are components of velocity vector $\bar{U}_m$.

Substituting Eqn. (3.22), (3.23) and (3.24) into equation (3.25), gives a system of linear algebraic equations of the non-hydrostatic pressure $q$:

$$Aq = b$$ (3.27)

The matrix $A$ will be solved by the bi-conjugate gradient squared stabilized (BGSTAB) algorithm together with the Incomplete lower upper pre-conditioner with dual truncation strategy (ILUD) from SparseKit, which is developed by Yousef Saad. Once the non-hydrostatic pressure has been determined, the horizontal and vertical velocities can be updated by substituting the non-hydrostatic
pressure $q$ back to the equations (3.22), (3.23) and (3.24). Then the solution of the new water levels can be determined by Eqn. (3.10).

### 3.3.3 Reduce stencil

In the second step, the horizontal and vertical velocities are updated as:

$$U^{n+1}_f = \hat{U}^{n+1}_f - \Delta t \left( \frac{1}{2} \frac{\partial q^{n+1}}{\partial x} \bigg|_f + \frac{1}{2} q^{n+1} \frac{\partial \eta^{n+1}}{\partial x} \bigg|_f - \frac{\partial d}{\partial x} \bigg|_f \right) \quad (3.28)$$

$$V^{n+1}_f = \hat{V}^{n+1}_f - \Delta t \left( \frac{1}{2} \frac{\partial q^{n+1}}{\partial y} \bigg|_f + \frac{1}{2} q^{n+1} \frac{\partial \eta^{n+1}}{\partial y} \bigg|_f - \frac{\partial d}{\partial y} \bigg|_f \right) \quad (3.29)$$

where the subscript $f$ denotes the value evaluated at the edge face.

For each velocity on the edge, the non-hydrostatic pressure gradients are approximated as follow:

$$\frac{\partial q}{\partial x} \bigg|_f = \alpha^e_{fL} q_{x_{eL}} + \alpha^e_{fR} q_{x_{eR}} \quad (3.30)$$

$$\frac{\partial q}{\partial y} \bigg|_f = \alpha^e_{fL} q_{y_{eL}} + \alpha^e_{fR} q_{y_{eR}} \quad (3.31)$$

where $q_{x_{eL}}, q_{x_{eR}}, q_{y_{eL}}, q_{y_{eR}}$ are the gradients of the non-hydrostatic pressure in element $e_L, e_R$ respectively, $\alpha^e_{fL}, \alpha^e_{fR}$ are the two weighting factors based on the areas (see Fig. 3.2):  

$$\alpha^e_{fL} = \frac{S_{eL}}{S_{eL} + S_{eR}}, \quad \alpha^e_{fR} = \frac{S_{eR}}{S_{eL} + S_{eR}} \quad (3.32)$$

where $S_{eL}$ and $S_{eR}$ are the area of the elements $e_L, e_R$ respectively.

The gradients of the non-hydrostatic pressure are assumed to be element wise constant, and can be calculated with line integration, or by using the $P_1$ shape function (more details can be found in Cui et al. (2010)):

$$q_{x_{el}} = \sum_{i \in eL} \frac{\partial \varphi_i}{\partial x} q_i$$

$$q_{y_{el}} = \sum_{i \in eL} \frac{\partial \varphi_i}{\partial y} q_i \quad (3.33)$$

where $\varphi_i$ is the $P_1$ shape function, $\frac{\partial \varphi_i}{\partial x}$ and $\frac{\partial \varphi_i}{\partial y}$ are the derivatives in $x$ and $y$ directions.

The calculation of $\frac{\partial q}{\partial x} \bigg|_f$ and $\frac{\partial q}{\partial y} \bigg|_f$ on an edge will involve four neighbor nodes (see Fig. 3.2). After substituting Eqn. (3.24), (3.28) and (3.29) into (3.25), in
Figure 3.2: The value of the non-hydrostatic pressure gradient on the edge $f$, is evaluated by the four nodes $k_1$ to $k_4$.

the row $k$ of the resulting matrix $A$, there will be $2NGB E_k + 1$ entries. Take the middle blue node in Fig. (3.3) as an example, the stencil of $q$ includes all the red nodes and the blue node. The resulting linear equation is given by:

$$a_0 q_k + \sum_{m=1}^{NE1 V_k} a_m q_m = RHS_k$$  \hspace{1cm} (3.34)

where $NE1 V_k$ is the number of neighboring vertices of node $k$ in stencil 1. These neighboring nodes are marked with a red color in Fig. 3.3.

Because inverting a large sparse matrix for the non-hydrostatic pressure needs significant computing power, we prefer to have a smaller stencil and less non-zero entries in the sparse matrix. To improve the efficiency of the model, when computing the second term of Eqn. (3.25) within element $m$, a lumping scheme is employed when evaluating the gradients of the non-hydrostatic pressure:

$$\frac{\partial q}{\partial x} \bigg|_f = \begin{cases} q_{x e_L}, & \text{if } m = e_L \\ q_{x e_R}, & \text{if } m = e_R \end{cases}$$  \hspace{1cm} (3.35)

$$\frac{\partial q}{\partial y} \bigg|_f = \begin{cases} q_{y e_L}, & \text{if } m = e_L \\ q_{y e_R}, & \text{if } m = e_R \end{cases}$$

By using this lumping scheme, when computing the velocity flux of Eqn. (3.25), the non-hydrostatic pressure gradient is approximated by only three node values. Take the element $m$ as an example, the non-hydrostatic pressure gradients for the three velocities at the edges of this triangle, are all computed by the
three nodes $k_1,k_2,k_3$ (Fig. 3.4). In doing so, the number of the non-zero entries in the row $k$ of the matrix is reduced to $NGB_E k + 1$, which is around half of that number with stencil 1. Once the solution of the non-hydrostatic pressure is found, the velocities are still updated with Eqn. (3.22), (3.23) and (3.24).

### 3.3.4 Boundary conditions

To solve the governing equations, boundary conditions of $q$ must be specified. One of the boundaries, for which we pay more attention here, is the wave incoming boundary. In lots of cases, we need to introduce regular wave to the computational domain through the boundary. For models with collocated grids, together with the water level, the non-hydrostatic pressure also needs to be specified. This can be done by calculating it from the analytical solution.

Basing on linear wave theory, if the incoming wave with an amplitude of $A$ is written in following form:

$$\eta = A \cos (\omega t - kx) \quad (3.36)$$

The velocity potential $\phi$ can be expressed as (Svendsen, 2006):

$$\phi = -A c \frac{\cosh (k(z + d))}{\sinh (kd)} \sin (\omega t - kx) \quad (3.37)$$
where \( c \) is wave celerity.

Since we are studying an infinitesimally small wave, the wave amplitude is much smaller than the water depth. In potential flows, the pressure can be determined from the Bernoulli equation. If we neglect higher order terms and impose \( p = 0 \) at \( z = 0 \), we get

\[
\begin{align*}
gz + \frac{p}{\rho} + \phi_t &= 0 \quad (3.38) \\
p &= -\rho gz - \rho \phi_t \quad (3.39)
\end{align*}
\]

We can see that \( p \) consists of a hydrostatic component \(-\rho gz\) and a dynamic component \( p_D = -\rho \phi_t \).

Substituting for \( \phi \) we then get

\[
\begin{align*}
p_D &= \rho g A \frac{\cosh (k (z + d))}{\cosh (kd)} \cos (\omega t - kx) = \rho g \eta \frac{\cosh (k (z + d))}{\cosh (kd)} \quad (3.40) \\
p &= -\rho gz + p_D = \rho g (\eta - z) + q \quad (3.41) \\
q &= p_D - \rho g \eta = -\rho g \eta \left( 1 - \frac{\cosh (k (z + d))}{\cosh (kd)} \right) \quad (3.42)
\end{align*}
\]
This is the vertical distribution of $q$. In the depth-integrated model, we only use the non-hydrostatic pressure at the bottom $q|_{z=-d}$, denoted as $q_0$ here. The consistency condition requires that the linear distribution of the non-hydrostatic pressure should have the same contribution as the original distribution in the vertical direction:

$$\int_{-d}^{\eta} q \, dz = \frac{hq_0}{2}$$  \hspace{1cm} (3.43)

By substituting Eqs. (3.42) into (3.43) we get:

$$\int_{-d}^{\eta} q \, dz = \int_{-d}^{\eta} -\rho g \eta \left( 1 - \frac{\cosh (k(z+d))}{\cosh (kd)} \right) \, dz = -\rho g \eta \left( h - \frac{\sinh (kh)}{k \cosh (kd)} \right)$$  \hspace{1cm} (3.44)

$$q_0 = -2\rho g \eta \left( 1 - \frac{\sinh (kh)}{k \cosh (kd)} \right)$$  \hspace{1cm} (3.45)

This is the analytical non-hydrostatic pressure which needs to accompany any regular incoming short wave in depth-integrated non-hydrostatic models.

### 3.4 Test Cases

#### 3.4.1 Standing wave in a closed basin

We consider a standing wave in a closed basin of rectangular shape $(10 \times 4 m)$. The basin is discretized with 359 triangular elements. By choosing a relatively small wave length $\lambda$ compared to the depth $d$, the vertical accelerations of the standing wave are of the same order as the horizontal, therefore the hydrostatic pressure assumption does not apply. This test case is commonly used to verify the accuracy of non-hydrostatic models in the computation of wave celerity. Initially, all velocities are set to zeros, and the initial water levels are specified by:

$$\eta(x, 0) = \eta_0 \sin\left(2\pi \frac{x - L/2}{\lambda}\right)$$  \hspace{1cm} (3.46)

where $\eta_0 = 0.01 m$ is the amplitude of the wave. The length of the wave is $\lambda = 2L = 20 m$. The calculation is carried out with a time step $\Delta t = 0.01 s$.

For sufficiently small wave amplitude, the wave celerity $c$ is given by the linear dispersion relation as:

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k} \tanh(kd)} = \sqrt{\frac{g \lambda}{2\pi} \tanh\left(\frac{2\pi d}{\lambda}\right)}$$  \hspace{1cm} (3.47)

where $\omega$ is the angular velocity, $k$ is the wave number. In this case, the wave period should be 3.59 s.
The simulation with the hydrostatic assumption produces an incorrect wave period (see Fig. 3.5). It is improved by including the non-hydrostatic pressure terms, even though the period is still a little larger than the analytical solution. This is the natural consequence of the depth-integrated non-hydrostatic model, which can only capture wave dispersion up to $kd \approx 1$ (Stelling and Zijlema, 2003). Stencil 2 produces identical results to stencil 1, while it takes much less computational time (0.0064 s versus 0.0108 s of CPU time per time step).

A grid convergence study is performed in the standing wave case. The solution of the numerical scheme should converge towards the real solution. Here the numerical results with a very fine grid is used as the real solution. The simulations are performed on five successively finer grids. The time step is adjusted by keeping the Courant Number the same. Figure 3.6 shows that stencil 2 has a similar convergence rate as that of stencil 1.
3.4.2 Solitary wave propagation in a channel

A solitary wave is defined as a single elevation above the surrounding undisturbed water level, producing a definite transport in the direction of wave propagation only. The solitary wave is not a solution of the shallow water equations, and, cannot be produced in models with the hydrostatic assumption. If the bottom friction and viscosity are absent, the solitary wave will travel without changing its shape and velocity. This numerical experiment has become a standard test for non-hydrostatic models (Stelling and Zijlema, 2003; Walters, 2005; Yamazaki et al., 2009). The computational domain is a 1000 m long, 2 m wide and 10 m deep channel with solid wall boundary condition at both ends. A third order Grimshaw solution was used to specify the initial surface elevation and velocity field (Grimshaw, 1971). This solution has also been reproduced in the literature, for example see Lubin and Lemonnier (2003).

Two cases were studied with different wave height to the water depth ratios $\varepsilon = \frac{H_0}{d} = 0.1, 0.2$, where $H_0$ is the wave height of the solitary wave, $d$ is the water depth. In Figs 3.7 to 3.9, the comparisons of simulated surface elevations, horizontal velocity $u$, vertical velocity $w$ and the analytical solution at $t = 20, 50s$ are shown, which demonstrate a good agreement.

3.4.3 The Beji and Battjes experiment

The next test case is the Beji and Battjes experiment in a wave flume with a submerged trapezoidal bar (Beji and Battjes, 1994). This test case is used to examine the capability of the model to simulate the interaction of waves with
3.4 TEST CASES

Figure 3.8: Comparison of the horizontal velocity $u$ of the solitary wave along a channel, at the moment $t = 20s$ and $t = 50s$ respectively, upper: $\varepsilon = 0.1$, lower: $\varepsilon = 0.2$.

Figure 3.9: Comparison of the vertical velocity $w$ at the free surface of the solitary wave along a channel, at the moment $t = 20s$ and $t = 50s$ respectively, upper: $\varepsilon = 0.1$, lower: $\varepsilon = 0.2$. 
uneven bottoms. The flume is 30 m long, 0.4 m deep. In the middle of the flume, there is a submerged bar, with a 1:20 slope at the front of the bar and a 1:10 slope at the back of the bar. The water depth is reduced to 0.1 m on top of the bar. On the right of the flume, there is a beach with 1:25 slope, where the water depth is limited to 0.2 m in order to apply a radiation condition at the right boundary. The bottom geometry of the experiment and the locations of the wave gauges are shown in Fig. 4.9. The flume is discretized by 19200 right-angled triangles with a right-angle side length of 0.0125 m. The time step is 0.005 s.

A wave with period of 2.02 s and amplitude of 1.0 cm is imposed at the left boundary. The corresponding water depth parameter $kh$ is 0.67, where $k$ is the wave number. As the wave shoals, higher harmonics are generated on the upward slope of the bar, and are released on the downward slope, resulting in irregular wave forms. Previous numerical studies have shown that the hydrostatic model generates unrealistic free-surface elevations that are totally different from measured data (Chen et al., 2003).

Comparisons of the free-surface elevation at the eight experimental stations between numerical results and measured data are plotted in figure 3.11. The model correctly simulates the shoaling phenomenon at gauge 4, and is capable of resolving the dispersion of high-frequency harmonics at gauge 5-8. Small discrepancies arise at gauge 9-11, located between the back of the bar and the beach. The same discrepancies have also been found by other depth-integrated non-hydrostatic models (Stelling and Zijlema, 2003; Walters, 2005; Yamazaki et al., 2009). Despite those small discrepancies, the overall agreement with the experimental data is good.
Figure 3.11: The numerical results of the non-hydrostatic model (red dash dot line for stencil 1, green dash line for stencil 2) and observations (circles) for the Beji and Battjes experiment at 8 gauges, which are located at x=10.5, 12.5, 13.5, 14.5, 15.7, 17.3, 19, 21m respectively.
3.4.4 Wave deformation by an elliptical shoal on a sloped bottom

The Berkhoff Shoal is a classic test case conducted by Berkhoff et al. (1982). It has been widely used to test the capability of models to simulate wave refraction and diffraction over a 3D uneven bottom (Stelling and Zijlema, 2003; Cea et al., 2009; Ai et al., 2011).

Figure 3.12 shows the experimental setup for the bathymetry of the elliptic shoal and the location of the transects along which the measurements were recorded. The spatial domain is a rectangle of 35 m in the x-direction and 20 m in y-direction with a 1/50 bed slope. Let \((x', y')\) be the slope-oriented coordinates, which are defined by a 20\(^\circ\) clockwise rotation of the x,y coordinates:

\[
x' = x \cos 20^\circ + y \sin 20^\circ \\
y' = x \sin 20^\circ - y \cos 20^\circ
\]

(3.48)

The water depth without the shoal is given by:

\[
d_0 = 0.45 \\
d_0 = \max \left(0.10, 0.45 - \frac{5.484 + x'}{50}\right) \quad x' \geq -5.484
\]

(3.49)

The shoal has an elliptical shape and is bound by:

\[
\left(\frac{x'}{3}\right)^2 + \left(\frac{y'}{4}\right)^2 = 1
\]

(3.50)

and the thickness of the shoal is defined by:

\[
d_s = -0.3 + 0.5 \sqrt{1 - \left(\frac{y'}{5}\right)^2 - \left(\frac{x'}{3.75}\right)^2}
\]

(3.51)

The water depth becomes:

\[
d = d_0 - d_s = \min \left[0.45, \max \left(0.10, 0.45 - \frac{5.484 + y'}{50}\right)\right] - d_s
\]

(3.52)

A regular wave with a wave height of \(H_0 = 4.64cm\) and a wave period of \(T_0 = 1s\) enters the domain from the left boundary located at \(x = -10m\). Here both the free surface and the non-hydrostatic pressure calculated from Eqn. (3.45) are specified. A radiation boundary condition is employed at right boundary.

Figure 3.13 shows the computed relative wave height, where the relative wave height is obtained by averaging over five wave periods (i.e. from t=30s to 35s), once the steady solution is achieved, divided by the incident wave height \((H_0 = 0.0464m)\). The effect of wave refraction, diffraction and shoaling can be clearly observed. Due to refraction, the incoming waves are focused behind the shoal, with a maximum wave height of approximately two times the incident wave height.
Figure 3.12: Bottom configuration of the Berkhoff experiment and the location of the transects along which measurements were recorded.

Figure 3.13: The relative wave height for the 2D elliptical shoal test case.
Figure 3.14 gives the comparisons between the numerical results and the experimental data at the eight transects indicated in Figure 3.12. The model slightly under-predicts the peak wave height at Section 3 and 5. However, the focusing effect behind the shoal (transect 3-5), as well as the interference patterns caused by diffraction (transect 6-8), are well captured.

### 3.4.5 Run-up of solitary waves on a Conical island

Motivated by the tragedy of 1992 Flores Island tsunami in Babi Island, Indonesia (Yeh et al., 1994), Briggs et al. (1995) conducted a laboratory experiment for solitary waves climbing up a conical island in a 30 m-wide, 25 m-long, and 60 cm-deep wave basin at the US Army Engineer Waterways Experiment Station, Coastal Engineering Research Center. A detailed description of the experiment can be found elsewhere (Liu et al., 1995; Briggs et al., 1995). Here we use the same set up and same mesh as used in Cui et al. (2010). The schematic view of the experiment is reproduced in Fig. 3.15.

In the laboratory, a series of experiments were conducted with waves of different incident height-to-depth ratios. The present study only covers two of them, which have the wave height-to-depth ratio $H_0/d = 0.096, 0.181$ respectively, and water depth of 32 cm. The incoming solitary wave amplitude is calibrated by the measured data, to ensure that the wave has the same amplitude as that at gauge 4.

Figure 3.16 presents comparisons between the wave gauge data and numerical results for both cases ($H_0/d = 0.096, 0.181$). For the case with $H_0/d = 0.096$, the overall agreement between experimental data and numerical solutions (both hydrostatic and non-hydrostatic) at all gauges is good, although there is a small deviation in the phase of the peak at gauge 22. The wave amplitude at gauge 22 located behind the island is over-predicted by the hydrostatic model while it is correctly reproduced by the non-hydrostatic model. For the case with $H_0/d = 0.181$, the wave appears to have broken at gauge 22. Both the hydrostatic and non-hydrostatic approaches over-estimate the wave amplitude at this gauge. Fig. 3.17 shows snapshots of the surface when the edge waves collided, and 2s afterward, from which the collision of the two edge waves at the lee side of the island can be seen. The non-hydrostatic model generates a much smoother free surface and diffracted wave pattern.

The comparisons of the measured and computed inundation positions and run-up around the conical island are presented in Fig. 3.18. For both cases, the non-hydrostatic model gives good predictions of the inundation position all around the island. The hydrostatic model over-predicts the run-up at the front of the island. The over-prediction by the hydrostatic model illustrates the known property of NSWE waves getting steeper when shoaling, and is found in other models as well (Zijlema et al., 2011; Yamazaki et al., 2009).
Figure 3.14: The numerical results of non-hydrostatic model (red line for stencil 1, green line for stencil 2) and observations (symbols) for the Berkhoff shoal experiment.
3.5 Discussion

Despite the successful applications of the non-linear shallow water (NSWE) equations in tsunami simulations, there exists a demand to include the dispersive effects in such models. In this paper, the hydrostatic version of H2Ocean developed by Cui et al. (2010) is extended using the method of Stelling and Zijlema (2003). It has been applied to the simulation of solitary wave propagation, wave shoaling, refraction, and diffraction, in order to demonstrate the capability of the model in resolving the effects of dispersion.

Stelling and Zijlema’s method ¹ has been adopted in a structured finite difference model (Yamazaki et al., 2009), a finite element model (Walters, 2005), a staggered finite volume model, SUNTANS (Fringer et al., 2006) and a non-staggered finite volume model FVCOM (Lai et al., 2010). This is the first time this method has been employed in an unstructured grid model with a median node-dual control volume for wave elevation and an edge-based non-overlapping control volume for the full velocity vector. The present formulation differs from other models, such as SUNTANS and FVCOM, in the discretization and grid stencils used. Cui et al. (2010) have also shown that the present model is an

Case B: $H_0/d = 0.096$

Case C: $H_0/d = 0.181$

Figure 3.16: Comparison of the time histories of surface elevation between experimental data (black dash line) and results from hydrostatic model (thin blue line), non-hydrostatic model with stencil 1 (red dash dot line), and stencil 2 (green dash line).
**Figure 3.17:** Wave transformation on the lee side of the conical island for the case with $H_0/d = 0.181$, left: hydrostatic, right: non-hydrostatic.

**Figure 3.18:** Comparison of the maximum run-up around the island of measured data (circles), hydrostatic results (blue line), non-hydrostatic with stencil 1 (red line) and non-hydrostatic with stencil 2 (green line): $H_0/d = 0.096$ (left panel), $H_0/d = 0.181$ (right panel).
analogue of the $P_{1}^{NC} – P_{1}$ finite element. So the procedure of extending the model from hydrostatic to non-hydrostatic described in this paper, can be implemented in depth-integrated models employing the $P_{1}^{NC} – P_{1}$ finite element in a straightforward manner.

Placing the full velocity vector on the edge, has the advantage of having more degrees of freedom and easy implementation of the Coriolis force and solid wall boundary condition. The water level gradient and the non-hydrostatic pressure gradient are approximated by the four neighboring nodal values supported by the average factors based on the area. However, when constructing the Poisson equation, a large stencil of the non-hydrostatic pressure results (stencil 1). Therefore, a sparse matrix with a large number of non-zero entries needs to be solved, which requires extreme computational effort, compared with that paid in the explicit hydrostatic step.

In this paper, a lumping technique is introduced to reduce the stencil. When computing the mass flux within a neighboring element of one node, the gradient of the non-hydrostatic pressure of all three velocity vectors is calculated with the three nodal values on the nodes of that element. When updating the velocity with the solution of the non-hydrostatic pressure, the original averaging scheme is still used. This lumping method reduces the number of non-zero elements in the sparse matrix by half but still produces almost identical results to stencil 1. The computation time saved was up to 30% with stencil 2 (See Table 3.1). This benefit would be even higher if this method was implemented in a parallel code, having a small stencil is critical for efficiency.

It is worth noting that, since bottom gradient terms are present in matrix $A$, the system could generate unstable solutions if the local bathymetry changes abruptly. To avoid instability, Yamazaki et al. (2009) were compelled to smooth the bathymetry when conducting real tsunami simulations. We have tried a number of simple 1D test cases and only find the instability to be a problem in the presence of a vertical wall. By completing the vertical momentum equations with wall friction, advection and viscosity, the instability will disappear. Smoothing the bathymetry also provides a solution to minimize the instability issue.

Table 3.1: Comparison of the CPU time per time step between model with stencil 1 and stencil 2.

<table>
<thead>
<tr>
<th></th>
<th>node number</th>
<th>cpu time per time step</th>
<th>cpu time per time step</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standing wave</td>
<td>359</td>
<td>0.0108</td>
<td>0.0064</td>
</tr>
<tr>
<td>Solitary Wave</td>
<td>9193</td>
<td>0.391</td>
<td>0.276</td>
</tr>
<tr>
<td>Baji Battjes</td>
<td>12005</td>
<td>0.391</td>
<td>0.24</td>
</tr>
<tr>
<td>Berkhoff</td>
<td>281101</td>
<td>10.904</td>
<td>7.835</td>
</tr>
<tr>
<td>Conical Island</td>
<td>85583</td>
<td>3.105</td>
<td>2.134</td>
</tr>
</tbody>
</table>

1The CPU time was based on a Intel Quad Core 2.4GHz processor.
A non-hydrostatic pressure boundary condition is also introduced in the present paper. The wave incoming boundary can be easily implemented in a staggered grid model by specifying the velocity at the boundary. In this case no momentum equations are solved at the boundary, and there is no need to prescribe the water surface and pressure at the boundary. For a non-staggered grid model, both the wave elevation and the non-hydrostatic pressure have to be defined at the boundary. For long incoming waves, the non-hydrostatic pressure can be set to zero. For short waves, this approach is problematic. The non-hydrostatic pressure for the depth-integrated model is then derived based on linear wave theory. All the test cases have demonstrated that this boundary can serve as an easy and accurate way to introduce short incoming waves in depth-averaged non-hydrostatic models with non-staggered grids.

In summary, the present paper outlines the extension of the hydrostatic version of H2Ocean to non-hydrostatic. The newly developed non-hydrostatic model is accurate and efficient and has been validated by several test cases, including an oscillating basin, solitary wave propagation, Beji-Battjes experiment, wave deformation over an elliptical shoal and the run-up of solitary waves on a conical island. All validation experiments suggest that the newly developed model is capable of resolving linear dispersion reasonably and can accurately reproduce essential wave phenomena, such as wave shoaling, refraction and diffraction.
In this chapter, a minimized Poisson equations formulation is presented and has been implemented in H$_2$Ocean. The objective is to improve the efficiency and accuracy in resolving the dispersive effects. When applying this method in a two-layer non-hydrostatic model, the computational effort to solve the Poisson equation is reduced to the same as that of a single layer non-hydrostatic model, but the dispersion accuracy remains high. This makes the model computational efficient in simulating non-hydrostatic waves.

This paper has been submitted as:

Abstract
A novel numerical technique has been developed to improve the efficiency of non-hydrostatic models. It reduces the degree of the stencil of the pressure Poisson equation near the bottom by approximating the pressure gradient term of the horizontal velocity in the bottom cell with only the pressures at the top face of the bottom cells. By doing so, the stencil of the pressure Poisson equation at the bottom is simplified. By combining the equations in the two bottom layers, the vertical velocity and the pressure at the bottom are eliminated from the set of equations. The bottom two layers are reduced to a one-layer like system. We refer
to this new technique as the minimized Poisson equations formulation. In the new system, the size of the pressure Poisson equation has been reduced by one in the vertical column. This improves the efficiency of the non-hydrostatic models. As a demonstration of the new method, it has been implemented in a non-hydrostatic model with only two layers. The new model consists of five equations and only one non-hydrostatic pressure unknown. It requires the same computational effort as the depth-integrated non-hydrostatic models, but can provide a much better description of dispersive waves. For a monotonic wave, the dispersion property can be further optimized by setting the bottom layer thickness to be a function of $kh$.

To allow easy implementation of the new method in depth-integrated models, the governing equations are transformed into a depth-integrated system, in which the velocity difference serves as an extra variable. The minimized Poisson equations formulation produces good results in a series of numerical experiments, including a standing wave in a basin, a non-linear wave test, a shoaling wave test and a wave propagation over a submerged bar.

### 4.1 Introduction

In fluid dynamics, dispersion means that waves of different wavelengths travel at different phase speeds. It plays an important role in wave transformation from deep water to intermediate water and in wave interactions with an uneven bottom. In the past decades, significant effort has been devoted to the development of models that can accurately and efficiently predict free surface wave propagation in a wide range of water depths (Marshall et al., 1997; Casulli and Stelling, 1998; Stansby and Zhou, 1998; Nwogu, 1993).

There are two conventional approaches for modelling the dispersive effects: the Boussinesq-type approach and non-hydrostatic models. Boussinesq-type equations (Peregrine, 1967) have provided a general framework to extend the applicability of depth-integrated equations into deep water. They are typically based on shallow water equations and they utilize an expansion in $kh$ ($k$ is the wave number and $h$ is the water depth).

The range of applicability of the conventional Boussinesq equations is limited to $kh < 0.75$, as stated in Madsen et al. (2002, 2003). Substantial effort has been devoted to extending the linear and nonlinear range of applicability of Boussinesq-type models. As a result, a number of enhanced and higher-order Boussinesq models have been developed. For instance, Nwogu (1993) used the velocity at an arbitrary distance from the still water level as the velocity variable and made the model applicable up to $kh \approx 4$. Using a fourth-order polynomial, Gobbi et al. (2000) developed a model which has good linear dispersive accuracy up to $kh = 6$. A considerable improvement by Madsen and Sørensen (1992) has resulted in a formulation including fifth-derivative operators, accurate to extremely deep water ($kh \approx 40$). Recently, Lynett and Liu (2004a) have also proposed a two-layer Boussinesq approach, with good linear wave characteristics up to $kh \approx 6$. 
They also extended this approach to multiple layers, and have achieved accurate linear dispersive properties up to $kh \approx 17$ with three layers, up to $kh \approx 30$ with four layers, including only third-order spatial derivatives (Lynett and Liu, 2004b). The principle behind Boussinesq formulations is to incorporate the effects of non-hydrostatic pressure, while eliminating the vertical coordinate. The high accuracy is at the expense of the simplicity and efficiency of the model. It results in a rather complicated system with high-order derivatives, which requires an equally complex numerical scheme and leads to instability over complex terrain. The high-order derivatives may also pose problems for the numerical implementation of the model (Løvholt and Pedersen, 2009).

The second approach is the non-hydrostatic models. In the original work of Chorin (1968), the so called ‘projection method’ was developed to solve the Navier-Stokes equations. The solution of the problem is split into two steps. In the first step, the velocity field is calculated by using the momentum equations without taking the pressure gradient into account. In the second step, the projection step, the resulting intermediate velocity field is then projected onto a divergence-free space by solving a Poisson equation of the full pressure, which is computationally expensive. To improve the efficiency of this method in the non-hydrostatic models, Marshall et al. (1997) and Casulli and Stelling (1998) independently proposed an alternative approach in which the pressure is decomposed into a hydrostatic component and a non-hydrostatic component. This pressure decomposition method has been successfully used in many non-hydrostatic models (such as MITgcm (Marshall et al., 1997), Delft3D (Bijvelds, 2003), SUNTANS (Fringer et al., 2006), non-hydrostatic ROMS (Kanarska et al., 2007), SWASH (Zijlema et al., 2011)).

However, the implementation of the zero pressure boundary condition at the free-surface is difficult for a staggered grid where the pressure is located at the center of the cell. It has been recognized that 10-20 vertical layers are normally required in a staggered grid model to describe wave dispersion characteristics up to an acceptable level if the hydrostatic assumption is employed at the top layer (Casulli, 1999). To address the issue mentioned above, Stelling and Zijlema (2003) developed an efficient and accurate numerical method which utilizes a Keller-box scheme and an edge-based grid system in the vertical direction. This enables the non-hydrostatic pressure to be located at the cell faces rather than at the cell centers. Therefore, the top-layer pressure boundary condition can be assigned exactly without any approximation. Their model can resolve the frequency dispersion up to an acceptable level of accuracy with a few number of vertical layers.

Studies show that the dispersion property of non-hydrostatic models can be improved further by using more layers, without increasing the order of the spatial derivatives (Stelling and Zijlema, 2003; Yuan and Wu, 2006). However, the price to pay for such an improvement is a significant increase in computational cost, due to the large size of the resulting matrix system that needs to be solved. With the increasing demands of performing large scale flow simulations with non-
hydrostatic models, improving their efficiency has become a competitive necessity. One of the advances in non-hydrostatic modeling is the use of a small number of vertical layers to efficiently and accurately model free-surface waves (Stelling and Zijlema, 2003; Yuan and Wu, 2004). Already in 2002, Reeuwijk (2002) proposed a method to improve the efficiency of non-hydrostatic models in which the number of pressure layers can be chosen independently from the number of horizontal velocity layers. A few free parameters were introduced to express the pressure at the place where the value of pressure is missing. However, if we examine the Poisson equation of the non-hydrostatic pressure in his model, we would find that it is constructed by summing up all local continuity equations within one pressure layer. In the case of only one pressure layer, the summation of the horizontal velocities appears both in the integrated continuity equation and in the summed local continuity equation used for constructing the Poisson equation. If all the horizontal momentum equations are summed up as well, the summation of the horizontal velocities also appears. It means that in his model, the horizontal velocities are not independent. It is the summation of the horizontal velocities that is used as an independent variable to determine the dispersion relation. Increasing the number of horizontal momentum layers does not increase the degree of freedom of the independent variables in determining the dispersion relation, therefore does not increase the dispersion accuracy. The accuracy of dispersion only depends on the number of pressure layers. This also explains why the wave propagation speed in his computations is only determined by the number of pressure layers. If the summation of the horizontal velocity is expressed as one variable, the system is degraded back to a single layer system.

A similar idea to that described in Reeuwijk (2002) was recently presented, Bai and Cheung (2012b) proposed a parameterized non-hydrostatic pressure distribution to reduce the computational costs. A free parameter is introduced to express the non-hydrostatic pressure at the mid flow depth in terms of the bottom pressure. With this approximation, the two-layer flow system is reduced to a hybrid system with a free parameter. The free parameter is optimized against the exact linear dispersion relation in the range of $0 < kd < 3$. The computational cost is reduced to the same cost as that of a one-layer system. However, the accuracy of the dispersion is only improved slightly. The reason for this is the same as stated above for Reeuwijk (2002)’s model. Compared to a one-layer model, the hybrid system does not increase the freedom of the independent variables in determining the dispersion relation.

In this paper, an alternative approach is introduced to improve the efficiency of non-hydrostatic models. A simpler stencil of the horizontal velocity in the bottom layer is proposed to reduce the degree of the stencil of the pressure Poisson equation near the bottom. The pressure gradients of the horizontal velocity in the bottom layer are approximated only by the pressures at the top of the bottom layer. With this simplified stencil and manipulations of the equations in the two bottom layers, it turns out that the calculations of vertical velocities and pressures at the bottom are no longer needed. The size of the Poisson equation is reduced
by one. Since most of the computational effort is devoted to inverting the Poisson equation, reducing the number of unknowns improves the efficiency of the model. It is worth noting that this procedure only changes the degree of the stencil of the pressure Poisson equation, but does not change the freedom of independent variables. When needed, the vertical velocities and pressures at the bottom can still be calculated by using the kinematic boundary conditions.

The new method is referred to as the minimized Poisson equations formulation. It brings a considerable improvement to the method described in Stelling and Zijlema (2003) and can be incorporated into any existing non-hydrostatic models to improve their efficiency in modeling the dispersive effects. We demonstrate the new method by implementing it in a two-layer non-hydrostatic model. For the same computational cost, the new model can achieve much more accurate linear dispersion than one-layer models. To improve the dispersion property of the model further, the thickness of the layer is chosen to be a function of \( kh \). The governing equations have also been transformed into an equivalent, depth-integrated system, with the velocity difference as a correction to the depth-integrated flow. This allows easy implementation of the method in depth-integrated models. The new formulation is implemented in the newly developed, two-dimensional, unstructured, non-hydrostatic finite volume model, H2Ocean (Cui et al., 2010, 2012). Here several classic test cases are used to validate the model. It is demonstrated that the new method leads to a significant reduction of computational effort, while maintaining high linear dispersion accuracy.

This paper proceeds as follows. In Section 4.2, the basic governing equations are presented. In Section 4.3, the minimized Poisson equations formulation is introduced. In Section 4.4, the method is applied to a two-layer model and is referred to as the reduced two-layer model. In Section 4.5, the reduced two-layer model has been transformed to a depth-integrated system to allow easy implementation in one-layer non-hydrostatic models. Several test cases are given in Section 4.6 to demonstrate the accuracy and efficiency of the new formulation. Finally, in section 4.7 the method is discussed.

\subsection*{4.2 Governing equations}

The governing equations used here are derived from the incompressible Reynolds’s averaged Navier-Stokes equations (RANS) with the pressure decomposed into hydrostatic and non-hydrostatic components \((q)\) using the notation in Casulli and Stelling (1998). In order to highlight the new method, all the non-linear terms have been ignored and only the two-dimensional \((x, z)\) plane is examined.

\begin{align}
\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \quad (4.1) \\
\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} + \frac{\partial q}{\partial x} &= 0 \quad (4.2)
\end{align}
where $\eta$ is the free surface elevation, $u$ and $w$ are the velocities in Cartesian coordinate system $(x, z)$, $q$ is the non-hydrostatic pressure (see Fig. 4.1). The kinematic boundary conditions at the free surface and at the bottom are given by

$$w_\eta = \frac{\partial \eta}{\partial t} + u_\eta \frac{\partial \eta}{\partial x}$$

(4.4)

$$w - d = -u - d \frac{\partial d}{\partial x}$$

(4.5)

where $d$ is the still water depth, $h$ is the total water depth, $u - d$ and $u_\eta$ are horizontal velocity at the bottom and the surface respectively. At the water surface, the non-hydrostatic pressure vanishes, $q|_{z=\eta} = 0$. Integrating Eqn. (4.1) over depth and using the kinematic boundary condition (4.4) and (4.5) lead to the integrated continuity equation:

$$\frac{\partial \eta}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$$

(4.6)

In the vertical direction, a sigma coordinate system is employed. The sigma transformation was first introduced by Phillips (1957) and it introduces a transformation of the vertical $z$-axis.

$$\sigma = \frac{z - \eta}{h}, \quad \sigma = [0, 1]$$

(4.7)

### 4.3 Minimized Poisson equations formulation

Assume there are $N$ layers vertically. Here we only pay attention to the lowest two layers, whose thicknesses are $\Delta z_1 = \Delta \sigma_1 h$ and $\Delta z_2 = \Delta \sigma_2 h$ respectively (see Fig. 4.1).
4.3 MINIMIZED POISSON EQUATIONS FORMULATION

The vertical velocity is assumed to be distributed linearly within one layer. By means of integration of Eqn. (4.1)-(4.3) within each layer, and subsequent the use of the Leibniz’s rule, we obtain:

\[ \frac{\partial \Delta z_1 u_1}{\partial x} - u_{z_1} \frac{\partial z_1}{\partial x} + w_1 = 0 \]  

\[ \frac{\partial \Delta z_1 u_1}{\partial t} + g \Delta z_1 \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x} \int_{-d}^{z_1} qdz - q_1 \frac{\partial z_1}{\partial x} - q_0 \frac{\partial d}{\partial x} = 0 \]  

\[ \frac{\partial \Delta z_1 \bar{w}_{01}}{\partial t} + q_1 - q_0 = 0 \]  

\[ \frac{\partial \Delta z_2 u_2}{\partial x} - u_{z_2} \frac{\partial z_2}{\partial x} + u_{z_1} \frac{\partial z_1}{\partial x} + w_2 - w_1 = 0 \]  

\[ \frac{\partial \Delta z_2 u_2}{\partial t} + g \Delta z_2 \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial x} \int_{z_1}^{z_2} qdz - q_2 \frac{\partial z_2}{\partial x} + q_1 \frac{\partial z_1}{\partial x} = 0 \]  

\[ \frac{\partial \Delta z_2 \bar{w}_{12}}{\partial t} - q_1 = 0 \]

where \( \bar{w}_{01} = \frac{w_0 + w_1}{2}, \bar{w}_{12} = \frac{w_1 + w_2}{2} \) and \( w_0 \) is computed by Eqn (4.5).

The computation procedure is as follows. First compute \( w_0 \) and substitute it into Eqn. (4.10) to express \( w_1 \) as a function of the non-hydrostatic pressure. Secondly, substitute both Eqn. (4.10) and (4.9) into Eqn. (4.8). Repeat step 1 and step 2 for other layers to construct the Poisson equation of the non-hydrostatic pressure. Once the non-hydrostatic pressure has been determined by inverting the matrix of Poisson equation, the horizontal and vertical velocities can be updated. Then the solution of the new water levels can be computed from Eqn. (4.6). The dispersion accuracy of the non-hydrostatic models can be improved by employing more layers (Casulli, 1999; Stelling and Zijlema, 2003; Yuan and Wu, 2006). However, it results in a sparse matrix with large number of non-zero elements, which lead to high simulation times.

By using the method of Stelling and Zijlema (2003), the pressure is located at the cell faces and the pressure gradient in the horizontal momentum equation is approximated by both the pressure at the top and bottom faces of the cell. It results in a large stencil of the pressure Poisson equation near the bottom (see Fig. 4.2). A simplified stencil is proposed by approximating the pressure gradient of the horizontal velocity in the bottom layer only by the pressures at the top face of the bottom cell (see Fig. 4.2). This simplified stencil has two-fold advantages. First, the computation of the horizontal velocity in the bottom layer is simplified. The pressure gradient term of \( u_1 \) can be calculated only from \( q_1 \):

\[ \frac{\partial}{\partial x} \int_{-d}^{z_1} qdz = \frac{\partial}{\partial x} (q_1 \Delta z_1) \]  

(4.14)
Secondly, it gives the potential to eliminate the vertical velocity and the pressure at the bottom from the set of equations. It works as follows. First we multiply Eqn. (4.8) by 2, then add it to Eqn. (4.11):

\[
2 \frac{\partial \Delta z_1 u_1}{\partial x} + \frac{\partial \Delta z_2 u_2}{\partial x} - u_{z_1} \frac{\partial \sigma_1}{\partial x} - u_{\sigma_2} \frac{\partial \sigma_2}{\partial x} + \bar{w}_{12} = 0 \quad (4.15)
\]

The variable \( \bar{w}_{12} \) in the equation above also appears in Eqn. (4.13), therefore \( \bar{w}_{12} \) can be treated as a new variable. The system is closed with Eqn. 4.9, 4.12, 4.13 and 4.15. The computation of Eqn. 4.10 and the non-hydrostatic pressure \( q_0 \) at the bottom are not needed any more. The governing equations at the two bottom layers are reduced to:

\[
2 \frac{\partial \Delta z_1 u_1}{\partial x} + \frac{\partial \Delta z_2 u_2}{\partial x} - u_{z_1} \frac{\partial \sigma_1}{\partial x} - u_{\sigma_2} \frac{\partial \sigma_2}{\partial x} + \bar{w}_{12} = 0 \quad (4.16)
\]

\[
\frac{\partial \Delta z_1 u_1}{\partial t} + g \Delta z_1 \frac{\partial \eta}{\partial x} + \frac{\partial (\Delta z_1 q_1)}{\partial x} - q_1 \frac{\partial (\sigma_1 h)}{\partial x} = 0 \quad (4.17)
\]

\[
\frac{\partial \Delta z_2 u_2}{\partial t} + g \Delta z_2 \frac{\partial \eta}{\partial x} + \frac{\partial (\Delta z_2 q_1 + q_2)}{\partial x} - q_2 \frac{\partial \sigma_2}{\partial x} + q_1 \frac{\partial \sigma_1}{\partial x} = 0 \quad (4.18)
\]

\[
\frac{\partial \Delta z_2 \bar{w}_{12}}{\partial t} - q_1 = 0 \quad (4.19)
\]

Compared to the original formulation, the computation of the vertical velocity and pressure at the bottom has been eliminated. The size of the pressure Poisson
equation is reduced by one. The number of non-zero elements in the matrix of Poisson equation is reduced by $1/N$, where $N$ is the number of vertical layers. In computing a matrix inverse, the amount of computational effort decreases as the size of the matrix decreases. Therefore, the reduction of the number of non-hydrostatic pressure unknowns makes the model more efficient. However, as the number of vertical layers increases, the percentage of the reduction of the computational effort will decrease. The computational gain of the method reaches its maximum when implemented in a two-layer model.

### 4.4 Minimized Poisson equations formulation in a two-layer model

In this section, more details of the implementation of the new method in two-layer model are given to demonstrate its efficiency and accuracy. The thickness of each layer is assumed to be $\Delta z_1 = \alpha h$, $\Delta z_2 = (1 - \alpha) h$. By using the minimized Poisson equations formulation, the governing equations of the two-layer model are reduced to:

\begin{align*}
\frac{\partial \eta}{\partial t} + \alpha \frac{\partial h u_1}{\partial x} &+ (1 - \alpha) \frac{\partial h u_2}{\partial x} = 0 \tag{4.20} \\
\frac{\partial h u_1}{\partial t} + g h_1 \frac{\partial \eta}{\partial x} + \frac{\partial (q h_1)}{\partial x} - q \frac{\partial (\alpha h)}{\partial x} & = 0 \tag{4.21} \\
\frac{\partial h u_2}{\partial t} + g h_2 \frac{\partial \eta}{\partial x} + \frac{1}{2} \frac{\partial (q h_2)}{\partial x} + q \frac{\partial (\alpha h - d)}{\partial x} & = 0 \tag{4.22} \\
2 \frac{\partial h u_1}{\partial x} + \frac{\partial h u_2}{\partial x} + w_{12} - \left( u_1 \frac{\partial \eta}{\partial x} + u_{z\alpha} \frac{\partial z_{\alpha}}{\partial x} \right) & = 0 \tag{4.23} \\
\frac{\partial \bar{w}_{12}}{\partial t} + 2 \frac{-q}{(1 - \alpha) h} & = 0 \tag{4.24}
\end{align*}

The new system, referred to as the reduced two-layer model, has 5 equations and 5 unknowns ($\eta$, $u_1$, $u_2$, $\bar{w}_{12}$, and $q$), since $u_1$ and $u_{z\alpha}$ can be expressed as a function of $u_1$ and $u_2$. Compared to one-layer models, the reduced two-layer model has the same number of non-hydrostatic pressure unknowns. After dropping the nonlinear terms and linearizing the equations, the dispersion relationship of this system is given by (a detailed derivation can be found in Appendix 4.7):

\begin{equation}
\begin{aligned}
c^2 &= gh \frac{1 + \frac{\alpha - 2\alpha^2 + \alpha^3}{4}(kh)^2}{1 + \frac{1 + 2\alpha - 3\alpha^2}{4}(kh)^2} \tag{4.25}
\end{aligned}
\end{equation}

where $c$ is the wave celerity, $g$ is the gravitational acceleration, $k$ is the wave number.

The dispersion relations of the one-layer, two-layer non-hydrostatic models, and the exact dispersion relation from the Airy wave theory are given as:
\[ c_1^2 = gh \frac{1}{1 + \frac{1}{4}(kh)^2} \]  
(4.26)

\[ c_2^2 = gh \frac{1 + \frac{1}{16}(kh)^2}{1 + \frac{3}{8}(kh)^2 + \frac{1}{256}(kh)^4} \]  
(4.27)

\[ c^2 = gh \frac{\tanh (kh)}{kh} \]  
(4.28)

The optimal linear dispersion relation of the Boussinesq model of Nwogu (1993) is:

\[ c_N^2 = gh \frac{1 - (\alpha_N + \frac{1}{7})(kh)^2}{1 - \alpha_N(kh)^2} \]  
(4.29)

where \( \alpha_N \) is a parameter to determine the location of the velocity variable.

And the linear dispersion relation of the hybrid system of Bai and Cheung (2012b) has the following form,

\[ c_\beta^2 = \frac{gd \left(1 + \frac{1}{16}k^2d^2\right)}{1 + \left(\frac{3}{16} + \frac{1}{4}\beta\right)k^2d^2} \]  
(4.30)

where \( \beta \) is a free parameter.

The comparison of the exact linear dispersion relation with that of one-layer model, two-layer model, reduced two-layer model, Boussinesq model of Nwogu (1993) and hybrid model of Bai and Cheung (2012b) is shown in Figure 4.3. To quantify the accuracy of the dispersion of each model, we use a relative error of the wave speed, i.e. \( \varepsilon_c = (c - c_{\text{linear}})/c_{\text{linear}} \), with \( c \) the phase speed predicted by the model and \( c_{\text{linear}} \) the phase speed from the linear dispersion theory.

With an optimum value \( \alpha_N = -0.39 \), the Boussinesq model of Nwogu (1993) accurately approximates the exact linear dispersion relation up to \( kh \approx 4.5 \). The hybrid system of Bai and Cheung (2012b) which has very similar coefficients as the linear dispersion relation derived by Nwogu (1993), generates accurate dispersion up to \( kd \approx 4.3 \) when setting \( \beta = 0.85442 \). When \( \beta \) is set to 0.5, the hybrid system even performs worse than the one-layer model.

With the minimized Poisson equations formulation, the dispersion relation is accurately reproduced up to \( kd \approx 7 \), which is still less accurate than that of two-layer non-hydrostatic models. The deterioration of the frequency dispersion accuracy is due to the simplified stencil. However, compared to the one-layer formulation, the reduced two-layer model produces a much more accurate dispersion relation without increasing the size of the Poisson equation. The Poisson equations that need to be solved keep the same structure, only the coefficients are different. The computational effort required to invert the matrix does not change. The extra computational effort comes from solving one more momentum equation, which is almost nothing compared to the effort paid in solving the Poisson equation. The accuracy of the model is improved with an extremely low extra
Figure 4.3: (Upper panel) Comparison of normalized wave celerity as a function of relative depth ($c_0 = \sqrt{gh}$). (Lower panel) The relative error of the wave celerity. Exact dispersion (thick solid line), one-layer system (circle), two-layer system (cross), minimized Poisson equations formulation with $\alpha = 0.33$ (plus), classical Boussinesq model of Nwogu ($\alpha_N = -0.39$) (star), Bai et al, 2012 ($\beta = 0.85442$) (square), Bai et al, 2012 ($\beta = 0.5$)(diamond).
computational cost. This minimized Poisson equations formulation provides an efficient and accurate way to extend the applicability of the shallow water models. For a monotonic wave, the accuracy of the dispersion can be further improved by setting the bottom layer thickness to be function of $kh$:

$$\alpha = \max \left(1 - \frac{2}{kh}, 0.33\right)$$  \hspace{1cm} (4.31)

By using function above, the reduced two-layer model can exactly reproduce the linear dispersion relation in all the range of $kd > 3$ (see Fig. 4.4).

Figure 4.4: (Upper panel) Comparison of normalized wave celerity as a function of relative depth ($c_0 = \sqrt{gh}$). (Lower panel) The relative error of the wave celerity. Exact dispersion (thick solid line), minimized Poisson equations formulation with Eqn. 4.31(circle).

### 4.5 Transformation to depth-integrated system

To allow easy implementation of this newly developed method in depth-integrated non-hydrostatic models, we follow the procedure proposed by Bai and Cheung (2012a) and introduce two variables to express the flow in a depth-integrated form:

$$u = \alpha u_1 + (1 - \alpha) u_2$$  \hspace{1cm} (4.32)

$$\Delta u = \alpha u_1 - (1 - \alpha) u_2$$  \hspace{1cm} (4.33)

where $u$ represents the depth-integrated horizontal velocity, $\Delta u$ represents the velocity difference between the bottom and top layer. The Eqns. (4.20) to (4.24) are transformed to:

$$\frac{\partial \eta}{\partial t} + \frac{\partial h u}{\partial x} = 0$$  \hspace{1cm} (4.34)
\[ \frac{\partial h u}{\partial t} + g h \frac{\partial \eta}{\partial x} + \left( \frac{1 + \alpha}{2} \right) \frac{\partial h q}{\partial x} - q \frac{\partial d}{\partial x} = 0 \] (4.35)

\[ \frac{\partial h \Delta u}{\partial t} + (2\alpha - 1) g h \frac{\partial \eta}{\partial x} + \left( \frac{3\alpha - 1}{2} \right) \frac{\partial h q}{\partial x} - q \frac{\partial (2\alpha h - d)}{\partial x} = 0 \] (4.36)

\[ \frac{\partial \left( \frac{3}{2} u + \frac{1}{2} \Delta u \right) h}{\partial x} + \bar{w}_{12} - \left( \alpha_1 u + \alpha_2 \Delta u \right) \frac{\partial \eta}{\partial x} + \left( \alpha_3 u + \alpha_4 \Delta u \right) \frac{\partial h}{\partial x} \right) = 0 \] (4.37)

\[ \frac{\partial \bar{w}_{12}}{\partial t} + 2 \frac{-q}{(1 - \alpha) h} = 0 \] (4.38)

where

\[ \alpha_1 = \frac{1}{1 - \alpha} \]
\[ \alpha_2 = \frac{1}{1 - \alpha} \]
\[ \alpha_3 = -\frac{2\alpha^2 + 2\alpha - 1}{2\alpha} \]
\[ \alpha_4 = \frac{2\alpha - 2\alpha^2}{2\alpha} \] (4.39)

The reduced two-layer model has been recast as a depth-integrated system with the velocity difference as an extra variable. Without this velocity difference term, and let \( \alpha = 0 \), the system will become the same as the traditional one-layer non-hydrostatic model. When implementing the simplified stencil formulation in the existing depth-integrated non-hydrostatic model, only the equation of \( \Delta u \) needs to be introduced and few extra terms need to be taken into account when constructing the Poisson equation. The structure of the Poisson equation keeps the same, only coefficients are different. The advection can be ignored in the momentum equation of the velocity difference. In the momentum equation of the depth-averaged velocity, the advection can be kept the same as that of the existing depth-integrated non-hydrostatic model. This requires minimum change to the model.

4.6 Model Validation

The minimized Poisson equations formulation has been implemented in the newly developed, two-dimensional, unstructured grid, finite volume model \( H_2O \) (Cui et al., 2010). The model has been applied to several test cases including an oscillating basin, non-linear wave, wave shoaling and short wave propagation over a submerged bar. All the numerical results have been compared with either analytical solutions or with experimental data, and have achieved a very satisfactory agreement in all cases.

4.6.1 Standing wave in a closed basin

The effects of frequency dispersion on the new scheme has been examined by testing a standing wave in a rectangular basin of length \( L = 10m \) and width of \( W = 4m \). The sloshing motion is set by an initial free-surface displacement:
where \( k = \pi / L \) is the wave number, \( \eta_0 \) is the wave amplitude. Here \( \eta_0 \) is set to 0.01 m, yielding \( \eta_0 k = 0.0031 \) to ensure a linear wave condition. An extreme deep water test case is chosen, \( h = 1000 \) m, with corresponding \( kh = 314 \). The basin is discretized by 359 triangular elements. The time step is determined by setting the Courant number \( Cr = 0.5 \).

Figure 4.5 shows the comparison between computed and exact time series of surface elevation. The model results are in good agreement with theoretical predictions, supporting the theoretical analysis.

### 4.6.2 Finite amplitude sloshing motions

To validate the nonlinear behavior predicted by the model, another test including a finite amplitude short wave in an intermediate water depth is performed. In this case we consider a basin with a length of \( L = 20m \) and a still water depth of \( h = 10m \). The initial water levels is given by:

\[
\eta(x, 0) = A \cos(k_2x)
\]  

where \( A = 0.5m \) is the wave amplitude, \( k_2 = 2\pi / L \) is the wave number. The corresponding wave steepness is \( Ak_2 = 0.157 \). For such a large wave steepness, the wave motion has to be described by the non-linear second-order solution (Wu and Eatock Taylor, 1994; Yuan and Wu, 2004):
\[ \eta(x,t) = A \left( \cos(k_2 x) \cos(\omega_2 t) + \frac{A \omega_2^2}{g} \left( \frac{1}{8} \frac{\omega_2^4 + g^2 k_2^2}{\omega_2^2} \right) \cos(2\omega_2 t) \right. \\
\left. + \left( \frac{1}{8} \frac{\omega_2^4 - g^2 k_2^2}{\omega_2^2} - \frac{3}{2} \frac{\omega_2^4 - g^2 k_2^2}{\omega_2^2 (4\omega_2^2 - \omega_2^4)} \right) \cos(2\omega_2 t) \right) \cos(2k_2 x) \] (4.42)

where \( k_4 = 4\pi/L, \omega_2 = \sqrt{g k_2 \tanh(k_2 h)} \) and \( \omega_4 = \sqrt{g k_4 \tanh(k_4 h)} \).

The domain is discretized by 1701 triangular elements. The time step is taken as 0.01 s. The comparison of the free-surface elevations at the middle of the basin, i.e. \( x = L/2 \), among the model results and the second-order analytical solution is shown in Figure 4.4. It shows that the model with a single layer predicts a larger phase speed and a lower crest than those in the second-order analytical solution. Data obtained from the reduced two-layer model are in good agreement with the analytical solution, except for wave crests and troughs, where small discrepancies are found. Figure 4.7 further shows the comparison of the wave profiles across the basin at different times. Model with single layer generates incorrect results, while the model using the reduced two-layer model, generates results much similar to that of analytical solution, indicating the new method’s capability to simulate nonlinear waves. Comparing with other non-hydrostatic model (Yuan and Wu, 2004) that used four vertical layers to accurately model large amplitude dispersive waves, the good results obtained from our model using only one pressure layer demonstrate the model’s efficiency and accuracy.

### 4.6.3 Wave shoaling

Shoaling is a crucial feature when waves propagate to the coast. As the waves enter shallow water, the wave speed and wave length decrease, therefore the wave height increases. From linear wave theory, the wave height in shallow water is given by:

\[ H_1 = \sqrt{\frac{c_{g0}}{c_{g1}}} H_0 = \sqrt{\frac{1}{2n c_1}} H_0 = K_s H_0 \] (4.43)

where

\[ n = \frac{1}{2} \left[ 1 + \frac{2kh}{\sinh(2kh)} \right] \] (4.44)

The wave height remains the same at deep water, then reaches its minimum at \( kh = 0.9425 \) and increases without limit as the water depth reduces further. To assess the linear shoaling property of the new model, an example considers the linear propagation of a monochromatic \( T = 3s \) wave train over a steep
Figure 4.6: Comparison between computed and second-order analytical time series of surface elevation at $x = L/2$ for the finite amplitude sloshing motions, exact solution (circle), single layer model (blue dashed line), reduced two layer model (red solid line).

Figure 4.7: Comparison between computed and second-order analytical spatial profile for the finite amplitude sloshing motions at three different times, exact solution (circle), single layer model (blue dashed line), reduced two layer model (red solid line).
4.6 Model Validation

4.6.4 Wave propagation over a 2D submerged bar

The newly developed model is applied to the numerical simulation of wave propagation over a submerged bar. The laboratory experiment was conducted by Beji and Battjes (1994). The configuration of a submerged trapezoidal bar used in the experiments is illustrated in Fig. 4.9. A sinusoidal wave with height of $A = 0.01m$ and period of $T = 2.02s$ propagates towards the submerged bar. It has been found that shoaling would occur when the wave train passes over the upward slope. Higher harmonics would be generated due to nonlinearity when the wave was released on the downward slope. There are eight stations, which are located at $x=10.5, 12.5, 13.5, 14.5, 15.7, 17.3, 19, 21$ m respectively, to measure the free-surface elevations.

In the model, the flume is discretized by 19200 triangles. The time step is 0.005 s. Comparisons of the wave elevations at gauges between the observations...
and the model results with the single layer and reduced two-layer model are plotted in Figure 4.10. The wave forms at gauge 4-8 are well reproduced by both schemes. Discrepancies in the amplitude and phase predicted by model with single layer arise at gauge 9-11. However, the overall results at gauge 9-11 predicted by the reduced two-layer model are in good agreement with experimental data. Compared to the published results of Bai and Cheung (2012b), the reduced model performs better than their hybrid system, with the same computational efficiency. The quality of the results of the model with the reduced two-layer, are even comparable with the results obtained with a two layer non-hydrostatic model (Stelling and Zijlema, 2003). The improved dispersion for the higher-frequency components occurring behind the bar, demonstrates the method’s capability of simulating dispersive and nonlinear wave characteristics over uneven bottoms.

4.7 Discussion

An efficient and accurate minimized Poisson equations formulation has been proposed in this paper. It simplifies the computation of the pressure gradient in the horizontal momentum equation in the most bottom layer by approximating it with only the pressure at the top face of the bottom cell. By combining the equations in the bottom two layers, the computation of the vertical velocity and pressure at the bottom \((w_0, q_0)\) can be eliminated (see Fig. 4.2). This reduces the size
Figure 4.10: Comparison of surface elevations at several stations between numerical results and experimental data. Experiment (circle), reduced two-layer model (red solid line), single layer model (blue dashed line).
of the pressure Poisson equation. Since most of the computational effort in non-hydrostatic models is paid to solve the Poisson equation, the reduction of the size of the Poisson equation improves the efficiency of the model. This methodology could be readily incorporated into any existing multi-layer non-hydrostatic models in the two bottom layers. When implemented in depth-integrated models, the computational gain reaches its maximum. The reduced two-layer model requires the same computational effort as the original depth-integrated non-hydrostatic models, but can reproduce the linear dispersion relationship accurately up to $kh \approx 7$. This makes the model applicable to many onshore and offshore wave simulations. By varying the thickness parameter, the dispersion relation can be reproduced perfectly for monotonic wave. For groups of waves, the dominant component can be chosen for optimization.

In tsunami modelling, models using the non-linear shallow water equations are still the most popular (Titov and González, 1997; Titov and Synolakis, 1998; Imamura, 1996; Androsov et al., 2008; Harig et al., 2008). However, depending on the conditions, the hydrostatic assumption might yield inaccuracies with respect to the propagation and shoaling of a tsunami. As the tsunami propagates over large distances, the phase error may be accumulated. Moreover, if tsunamis are generated by non-seismic sources, they are often too short to be adequately described by shallow water theory. All these situations require an efficient dispersive model in order to make larger scale tsunami applications feasible.

Dispersive effects in tsunami calculations are usually modelled using the Boussinesq equation (Gobbi et al., 2000; Madsen et al., 2003). However, Boussinesq models usually include high-order derivatives which make the model rather complicated and pose problems for the numerical implementation (Løvholt and Pedersen, 2009). The non-hydrostatic models using the pressure decomposition method provides an easier way to resolve the dispersion and has been successfully used in many models (Marshall et al., 1997; Casulli and Stelling, 1998; Walters, 2005; Fringer et al., 2006; Yamazaki et al., 2009; Zijlema et al., 2011). By using the Keller-box scheme and the edge-based grid system proposed by Stelling and Zijlema (2003), the non-hydrostatic models using a single layer can produce similar results as that of standard Boussinesq models. More accurate dispersion can be achieved by using more vertical layers without increasing the complexity of the model. However, the price to pay for such an improvement is a significant increase in computational cost.

Tsunami simulations often involve a large spatial domain with high resolution. Therefore, the efficiency of the model is a major issue. In practice, the dispersive effects due to a tsunami are usually simulated with non-hydrostatic models using a single layer, which is only accurate up to $kh \approx 3$ (Walters, 2005; Yamazaki et al., 2009). By using the minimized Poisson equations formulation, the dispersion can be accurately reproduced up to $kh \approx 7$ without increasing the computational effort. This improve the accuracy of the depth-integrated non-hydrostatic models considerably and extends their applicability.
The governing equations have been transformed into a depth-integrated form, with the velocity difference as a correction to the depth-integrated flow. This allows the use of momentum conservation advection scheme, which is crucial in simulating flooding and drying problems (Stelling and Duinmeijer, 2003; Cui et al., 2010).

The reduced two-layer model has been applied to several test cases including a standing wave in a basin, a non-linear wave test, shoaling test and wave propagation over a submerged bar. All the numerical results have been compared either with analytical solutions or with experimental data and good agreement has been found in all cases. The results show that the model can effectively and accurately resolve wave dispersion, non-linearity and shoaling, while maintaining the same computational cost as that of a single layer model. It offers a very interesting method to extend the capability of depth-averaged models.

**Appendix A: Numerical Dispersion Relation**

To derive the dispersion relation, we consider a system with flat bottom and with small amplitude periodic waves in the form:

\[
\eta(x,t) = \hat{\eta}e^{i(kx-\omega t)} \\
u(x,t) = \hat{u}e^{i(kx-\omega t)} \\
\Delta u(x,t) = \hat{\Delta}ue^{i(kx-\omega t)} \\
w_{12}(x,t) = \hat{w}_{12}e^{i(kx-\omega t)} \\
q(x,t) = \hat{q}e^{i(kx-\omega t)}
\]

After substituting the Fourier modes into Eqn. (4.20) to (4.24), they read as:

\[ -i\omega \eta + \alpha hiku_1 + (1 - \alpha) hiku_2 = 0 \]
\[ -i\omega u_1 + igk\eta + ikq = 0 \]
\[ -i\omega u_2 + igk\eta + \frac{1}{2} ikq = 0 \]
\[ 2\alphaiku_1 + (1 - \alpha) iku_2 + \frac{1}{h} w_{12} = 0 \]
\[ -\omega \bar{w}_{12} - \frac{2}{(1 - \alpha) h} q = 0 \]

These equations can be written in matrix form \( (5 \times 5) \),
\[
\begin{pmatrix}
-i\omega & \alpha ikh & (1 - \alpha) ikh \\
iki & -i\omega & -i\omega \\
iki & 2\alpha ik & (1 - \alpha) ik \\
\end{pmatrix}
\begin{pmatrix}
\frac{k}{2} \\
\frac{1}{2} i k \\
\frac{1}{2} \frac{1}{(1-\alpha)h} \\
\end{pmatrix}
\begin{pmatrix}
e^{i(kx-\omega t)} \\
\hat{\eta} \\
\hat{u}_1 \\
\hat{w}_2 \\
\hat{w}_{12} \\
\hat{q} \\
\end{pmatrix} = 0 \quad (4.56)
\]

where

\[
A = \begin{pmatrix}
-i\omega & \alpha ikh & (1 - \alpha) ikh \\
iki & -i\omega & -i\omega \\
iki & 2\alpha ik & (1 - \alpha) ik \\
\end{pmatrix}
\begin{pmatrix}
\frac{k}{2} \\
\frac{1}{2} i k \\
\frac{1}{2} \frac{1}{(1-\alpha)h} \\
\end{pmatrix}
\begin{pmatrix}
\hat{\eta} \\
\hat{u}_1 \\
\Delta \hat{u} \\
\hat{w}_2 \\
\hat{w}_{12} \\
\hat{q} \\
\end{pmatrix} = e^{i(kx-\omega t)} \quad (4.57)
\]

If the system of equations has non-trivial solution (\(x \neq 0\), \(A\) should be invertible, i.e., the determinant of the matrix \(A\) has to be zero:

\[
\text{Det}(A) = \begin{vmatrix}
-i\omega & \alpha ikh & (1 - \alpha) ikh \\
iki & -i\omega & -i\omega \\
iki & 2\alpha ik & (1 - \alpha) ik \\
\end{vmatrix} = 0 \quad (4.58)
\]

A cofactor expansion of the determinant of the matrix yields:

\[
\text{Det}(A) = \frac{\omega}{2h^2(\alpha-1)} \left[ 4\omega^2 + (1 + 2\alpha - 3\alpha^2) h^2 k^2 \omega^2 - (\alpha + 2\alpha^2 + \alpha^3) gh^3 k^4 - 4ghk^2 \right] = 0 \quad (4.59)
\]

which results in the dispersion relation of the system:

\[
c^2 = gh \frac{1 + \frac{\alpha - 2\alpha^2 + \alpha^3}{4}(kh)^2}{1 + \frac{1 + 2\alpha - 3\alpha^2}{4}(kh)^2} \quad (4.60)
\]
In this chapter, the H₂Ocean model is validated by conducting real large scale simulations of the 2004 Indian Ocean Tsunami. Numerical results were compared with data from field surveys such as inundation depth, maximum wave run-up and flooding area, and achieved excellent agreement. Compared to the results of TsunAWI, H₂Ocean performs better in handing flooding and drying. It turns out that H₂Ocean is efficient and accurate, and can be used for practical tsunami propagation and inundation studies. A few scenarios have been simulated to gain useful insights for the impacts of real tsunami.

5.1 Introduction

An earthquake with a magnitude of 9.3 occurred on 26th December 2004 at 00:58 UTC, with an epicenter off the west coast of Sumatra, Indonesia (Ammon et al., 2005; Stein and Okal, 2005). The earthquake triggered a tremendously powerful tsunami, which reached the north of Sumatra within only 15-30 minutes after the initial earthquake and destroyed the regions of Banda Aceh in 40 minutes (Borrero et al., 2006). In tsunami wave modeling, the rupture process needs to be estimated to generate the initial uplift. Different methods have been used to invert the rupture process (Vigny et al., 2005; Ammon et al., 2005; Tanioka et al., 2006; Hoechner et al., 2008). Once the initial condition is known, the propagation and coastal behavior can be modeled by computer simulation.

In this study, two different source models are used. The first one (referred to as Source 1) is established by using coastal coseismic vertical deformations and tsunami wave forms at tide gauges (Tanioka et al., 2006) (see Fig. 5.1). The simulation of Source 1 has been conducted with a finite element model TsunAWI
at AWI (Harig et al., 2008) and generates accurate results when compared to tide gauge data, to satellite altimetry and to field measurements of inundation depth. Source 1 is also simulated with the newly developed model H2Ocean to validate the accuracy of H2Ocean in handling flooding and drying. The numerical results are compared to inundation depth as well as max wave run-up, which are challenging to reproduce with numerical models.

The second one (referred to as Source 2) is from the GPS inversion (Hoechner et al., 2008) (see Fig. 5.1). Using Source 2 as the initial uplift, numerical model generates quite good results in sea surface elevation along the satellite track compared to observations (Hoechner et al., 2008). However, it results almost no flooding in Banda Aceh region (Kleptsova et al., 2012). It implies that either the magnitude of the predicted bottom deformation is too small or local rupture is missing. To get an insight of Source 2, two scenarios have been created by doubling the amplitude in the southern patch of Source 2, and by merging the southern patch of Source 1 and Source 2.

In this study, the same mesh as that of Kleptsova et al. (2012) is used, which has 317569 nodes, and 627788 cells. The simulation time is set to 2 hours, and the time step is based on setting the Courant number to 0.8.
5.2 Simulation results of Source 1

In early 2005, several survey teams went to Aceh region and conducted a field survey of earthquake and tsunami effects (Borrero et al., 2006; Jaffe et al., 2006). The actual wave run-up and inundation flow depth that occurred in Aceh region were investigated. All these data provide us not only a new insight into tsunami dynamics and its characteristics, but also an opportunity to validate and access the accuracy of current tsunami models.

5.2.1 Inundation depth

![Figure 5.2: The positions where field measurements of the inundation depth.](image)

Table 5.1 and Fig. 5.2 show the locations of field measurements of the maximum flow depth in Aceh region. The comparison between the measurements and the computational results are shown in figure 5.3. Although a very coarse grid size was adopted, numerical results match the field survey data fairly well. In this study, the Manning coefficient is set to 0.035, which is recommended in the study of 2004 Indian Ocean Tsunami with TsunAWI (Harig et al., 2008). Compared to the published results of TsunAWI (Harig et al., 2008), H2Ocean generates similar results at location 1-7, but generates much better results at location 8 and 9. The root mean square (RSM) error is only 5.7, indicating a smaller variation than that of TsunAWI.
### 5.2 Wave run-up in Aceh region

#### Table 5.2: Measured runup

<table>
<thead>
<tr>
<th>Station</th>
<th>Location</th>
<th>Latitude(°)</th>
<th>Longitude(°)</th>
<th>Runup(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Panteraja</td>
<td>5.2611</td>
<td>96.18528</td>
<td>4.2-4.7</td>
</tr>
<tr>
<td>2</td>
<td>Lhoknga</td>
<td>5.45777</td>
<td>95.24516</td>
<td>31.0</td>
</tr>
<tr>
<td>3</td>
<td>E. of Banda Aceh</td>
<td>5.65125</td>
<td>95.4231</td>
<td>8.0</td>
</tr>
<tr>
<td>4</td>
<td>Breuh Island</td>
<td>5.67</td>
<td>95.1296</td>
<td>20.1</td>
</tr>
<tr>
<td>5</td>
<td>Deudap Island</td>
<td>5.602</td>
<td>95.14</td>
<td>10.7</td>
</tr>
<tr>
<td>6</td>
<td>Sigli</td>
<td>5.3842</td>
<td>95.9680</td>
<td>4.82</td>
</tr>
<tr>
<td>7</td>
<td>Center of Banda Aceh</td>
<td>5.5494</td>
<td>95.3110</td>
<td>5.96</td>
</tr>
<tr>
<td>8</td>
<td>West Coast of Banda Aceh 1</td>
<td>5.4181</td>
<td>95.2475</td>
<td>24.86</td>
</tr>
<tr>
<td>9</td>
<td>West Coast of Banda Aceh 2</td>
<td>5.4517</td>
<td>95.2424</td>
<td>30.40</td>
</tr>
<tr>
<td>10</td>
<td>West Coast of Banda Aceh 3</td>
<td>5.45</td>
<td>95.2431</td>
<td>20.07</td>
</tr>
</tbody>
</table>

The run-up is the vertical height between the elevation of maximum tsunami penetration and the sea level at the time when tsunami happened. As tsunami waves penetrate inland, the front always propagates as a bore. The accurate simulation of these shock waves has always been a challenge to tsunami models. The collected run-up data and the locations are listed in Table 5.2 (also see Fig. 5.4). The data from station 1-5 refers to Borrero et al. (2006)’s paper, while the data from station 6-10 refers to Tsuji et al. (2005b). The simulated maximum tsunami wave run-up heights at these locations were compared with the survey data. Fig. 5.5 shows the comparison between the simulated maximum tsunami wave run-up heights and the observations. The numerical results are in good agreements with the observations. At Lhoknga, the maximum wave run-up was measured at 31 m (Borrero et al., 2006), while the simulated run-up was 26 m. The maximum wave height there was found on a steep hill at the shoreline in front of a cement factory (Borrero et al., 2006). The grid is not so fine as to resolve
the local feature there. Station 9 and 10 are too close to each other (only 222 meters) that the local bathymetry cannot be distinguished in the current grid. As a result, the simulated wave run-ups at these two stations are identical. The tsunami run-ups on Breuh and Deudap, two small islands lying to the north of Banda Aceh, were of 10-20 m. But the simulated run-ups at these two sites are almost the same (15 m). There is no run-up data from TsunAWI to be compared with.

5.2.3 Flooding in Banda Aceh

Figure 5.6 shows several snapshots of the tsunami waves approaching the western shoreline of Banda Aceh. The color scale gives the free-surface elevation with respect to the MSL (mean sea level). In frames (b) and (c), it shows that the tsunami waves arrived Lhoknga, the west coast of Banda Aceh, between 25 and 28 minutes after the earthquake. This agrees reasonably well with the arrival time of 26 minutes reported in Lavigne et al. (2009). In frame (d) we see the high run-up wave at Lhoknga got diffracted and propagated northward and enhanced the inundation in Banda Aceh. The omission of this run-up calculations in the simulations will have a substantial effect on the resulting wave heights. We also see in frame (e) and (f) the flood flows coming from northwest and northern Sumatra together, and met each other in the lowlands between Banda Aceh and Lhoknga. An eyewitness interviewed reported that waves from both the north and southwest (Borrero, 2005). The simulation reproduced the observed feature.
Figure 5.4: The positions where field measurements of the max wave run-up.

Figure 5.5: Runup results of Source 1 in comparison of field measurements.
Figure 5.6: Snapshots of the propagation of and inundation by tsunami wave in the Banda Aceh area.
The figure 5.7 shows comparison between the computed flooded area in the coastal region of Banda Aceh and the satellite image, which was acquired by the Centre for Remote Imaging, Sensing and Processing at the National University of Singapore via the satellite IKONOS on 2004-12-29 03:55:41.2. H2Ocean reproduces the flooding areas extremely well.

Figure 5.7: The tsunami affected area in the region of Banda Aceh. Left: the satellite image. Right: the simulated results based on Source 1.

5.3 Scenarios study

Numerical study has shown that Source 2 embeds too strong underestimation of the tsunami wave (Kleptsova et al., 2012). In order to better understand the cause of the flooding, variations of Source 2 have been studied. The Source 2, has two separate patches of high slip, divided at about 5°N. Doubling the northern patch of initial uplift Source 2, does not improve the flooding much (Kleptsova et al., 2012). Two more scenarios are set up in this study. In the first one (scenario 1), the amplitude of the southern patch is increased by a factor of 2 (see Fig. 5.8). In the second one (scenario 2), the high uplift of Source 1 in the range of 4-6 degree is merged into Source 2.
5.3.1 Scenarios results

The comparison between the computed and observed tsunami run-ups and inundation depths is shown in Fig. 5.9. In general, the computed tsunami run-ups of scenario 2 are in good agreement with the measured data, except at station 7 located at the south coast of Banda Aceh. The inundation depths at station 5-10 are also well reproduced. No or few waves reach stations 1-4, all located along the south coast. In scenario 1, even more amount of water is displaced by doubling the whole southern patch of Source 2, the computed run-up heights are still underestimated compared to the field survey data. Scenario 2 introduced a smaller uplift patch with the same amplitude as that of scenario 1, and the indentation results have been improved dramatically. This indicates that the flooding of the wave is not only affected by the wave amplitude, but also by its coming direction.

5.4 Discussion

In tsunami studies, accurate simulation of inundation and wave run-up is always a challenge. There are relatively few models available for practical tsunami propagation and inundation studies. Within the framework of GITEWS, TsunAWI has been developed, which is very accurate and efficient in simulating the wave propagation. However, when simulating the inundation and wave run-up, it gives less accurate results. The extrapolation method employed in TsuanAWI is neutrally stable. In place of large water gradient, wiggles in the wave front are generated.
Figure 5.9: The initial uplift fields used for the tsunami simulation. (a) Runup (b) Inundation depth.

Figure 5.10: The flooded area in the region of Banda Aceh.
and the accuracy of the results is degraded. To improve its capability in handling flooding and drying problems, a finite volume analogue of TsunAWI has been developed, named H$_2$Ocean. H$_2$Ocean guarantees both mass and momentum conservation. These characteristics are very important in simulating the convection dominated flows, which happens quite often during the flooding and drying processes.

In this study, the model H$_2$Ocean is validated by simulating the flooding and inundation of 2004 Indian Ocean tsunami. The results are promising. Comparison between the observed and modeled inundation depth at 9 sites shows reasonable agreement. Compared to the published data of TsunAWI, the inundation is better described with H$_2$Ocean. Furthermore, good agreement between the max wave run-up and simulated results have been achieved with H$_2$Ocean. All the encouraging results indicate that H$_2$Ocean is accurate for practical tsunami propagation and inundation studies.

In tsunami simulation, the initial seafloor motion is crucial for tsunami simulations. The onshore flooding of the tsunami waves is affected by the approximation of the tsunami source. However, the source mechanism is often unknown, and there are lots of uncertainties in the source description. In most cases, the source models are validated by comparing to tidal gauge data, and sometimes to satellite track data. However, when validated by comparing to inundation or run-up data, different approaches often give quantitatively different results, even though they have achieved satisfying results compared with tidal gauge data and satellite track data. Two scenarios have been simulated by H$_2$Ocean to gain better understanding of the impact of the initial tsunami source on flooding and drying. It is found that the flooding of tsunami wave is not affected by the amplitude of the initial uplift, but also affected by the location of the initial uplift.
Chapter 6

2011 Tohoku-Oki Tsunami Simulation

Most of this chapter is based on:


which is included here in the Appendix. A more detailed flooding simulation of the Japanese Tsunami along the Sanriku coast is also presented. This study was carried out in collaboration with Dr. Takenori Shimozono from Tokyo University of Marine Science and Technology and a journal paper is in preparation. The tsunami simulations in this chapter are carried out with H2Ocean. They provide further validation of the model, as well as providing insights into the characteristics of tsunami propagation and inundation.

6.1 Introduction

The Tohoku-Oki Earthquake occurred on 11 March 2011 at 05:46:23 UTC, off the Pacific coast of Honshu, Japan, with a moment magnitude of 9.0. It was the largest earthquake ever to hit Japan. The earthquake generated a tremendously powerful tsunami which traveled up to 10 km inland in the Sendai area and reached heights up to 40 m at around latitudes 39°N (The 2011 Tohoku Earthquake Tsunami Joint Survey Group, 2011). Thanks to the wealth of available data, such as the geodetic data, teleseismic data, strong motion data, tsunami
data, there was an excellent opportunity to investigate the details of the earthquake characteristics. Consequently, many source models have been generated (Ide et al., 2011; Lay et al., 2011; Lee et al., 2011; Miyazaki et al., 2011; Ozawa et al., 2011; Simons et al., 2011; Yue and Lay, 2011). Numerous studies indicate that the earthquake ruptured the subduction interface all the way to the Japan Trench. But how much slip occurs at the trench is difficult to determine. Different results have been produced depending on the data used in the inversion.

To gain a better understanding of this question, joint research has been conducted with scientists from multiple disciplines. This research has been published in Hooper et al. (2013). An accurate slip solution is gained by means of a joint inversion of displacement measurements and seafloor pressure data. Their source model is validated by comparison to measurements of seafloor displacements close to the trench and measurements of the tsunami in the ocean and on-land, which all give good agreement.

However, the maximum run-up between latitudes $39^\circ$ and $40^\circ$ is under-predicted. To study the extreme run-up in this area, in collaborating with Dr. Takenori Shimozono from Tokyo University of Marine Science and Technology, a more detailed flooding simulation has been carried out using a grid with very high resolution. These simulations provide a good opportunity to further validate the flooding capability of the model $H_2$Ocean.

### 6.2 Source Model

Here we use the uplift and subsidence fields of the 2011 Tohoku earthquake inverted by Hooper et al. (2013) to generate the initial conditions for the tsunami simulations. A brief description of the method used in this paper is presented here. The fault is divided into a series of rectangular patches. For each patch, the response of unit slip at each GPS and seafloor geodetic benchmark is calculated and is used as the initial sea surface perturbation. A 1 : 1 relationship between the seafloor and sea-surface displacements is assumed. The wave forms at the selected pressure gauges are simulated with $H_2$Ocean. The slip on the fault and the associated displacement of the water column is then reconstructed by applying a Bayesian inversion method using both the geodetic data and the seafloor pressure gauge data. The initial sea surface height is shown in Fig. 6.1. The peak uplift is estimated to be over 10 meters.

### 6.3 Tsunami Data

The locations of seafloor pressure and GPS wave gauges used in this study are indicated by rectangles in Fig. 6.2. The seafloor pressure data are from the Pacific Tsunami Warning Center (Titov et al., 2005b) and are referred to as the DART stations below. We used 7 gauges (21401, 21413, 21414, 21415, 21418, 21419,
52402) in the inversion. The leading waves of the tsunami arrived at all of these gauges within 5 h. The other gauges (46402, 46403, 51407, 52403) that recorded the later arrival of the tsunami were used for independent validation of the initial uplift.

In addition to these DART stations, we used the records obtained at the bottom-pressure and GPS wave gauges offshore to validate the initial uplift. The data was processed by Kawai et al. (2012) and provided by Dr. Takenori Shimo-zono. These stations are referred to as Japanese stations hereafter. Stations KPG1, KPG2, MPG1 and MPG2 are operated by the Japan Agency for Marine-Earth Science and Technology (JAMSTEC, http://www.jamstec.go.jp/). Stations VCM1 and VCM3 are operated by the National Research Institute for Earth Science and Disaster Prevention (NIED, http://www.bosai.go.jp). The stations 801, 802, 803, 804, and 806 are the GPS stations deployed by the Port and Airport Research Institute (PARI, http://www.pari.go.jp). The readers could also see Saito et al. (2011), where the data were first presented.

With the participation of many Japanese researchers, the 2011 Tohoku Tsunami Joint Survey Group conducted a nationwide post-tsunami survey, generating the largest tsunami survey dataset in the world (Mori et al., 2011). The measured inundation and run-up data are used for flooding validation.

### 6.4 Tsunami Simulation

To study both the propagation of and the inundation by the tsunami waves, two sets of grids have been generated, a Pacific wide grid and a flooding grid with high resolution along the eastern coast of Japan (see Fig. 6.2). The Pacific wide mesh
Figure 6.2: The areas covered by the grids used in the Japan Tsunami simulations.

has 1 million nodes with a 500 m resolution near the earthquake source, down to 20 km off the coast of South America. The flooding grid has 3 million nodes with 500 m resolution near the epicenter, increasing to 80 m near the coast and on land. The underlying bathymetry data are from the the 1 arc-minute global relief model, ETOPO1 (Amante and Eakins, March 2009) and the underlying elevation data are from the 30 m ASTER Global Digital Elevation Model.
Figure 6.3: Middle panel shows the maximum modelled tsunami wave height and the locations of DART stations. Surrounding panels show comparisons of the tsunami waveforms for the 2011 Tohoku-Oki earthquake recorded by the DART stations.
Figure 6.4: Comparison of the tsunami waveforms for the 2011 Tohoku-Oki earthquake recorded by the Japanese stations.
First, the propagation of the tsunami due to the uplift of the sea surface shown in Fig. 6.1 is simulated by H2Ocean, with the Pacific wide grid. We run the model for 10 hours and used a time step of 2 s. The tsunami simulation results are presented in Fig. 6.3 and are compared to the DART stations. Station 46402, 46403, 51407 and 52403 are independent data, while the others were used in the inversion. Good agreement is observed. Moreover, Fig. 6.4 shows the comparison to the Japanese stations all located to the west of the trench. The predicted wave heights at these stations fit the observations quite well.

![Figure 6.5: The flooded area in the region of Sendai. Left: Field survey data. Right: Numerical simulation data.](image)

The same uplift is then simulated with the flooding grid. The model is run for 2 hours with a time step of 1 s. The solution is compared against tsunami inundation height and maximum run-up along the coast of Japan obtained by the 2011 Tohoku Tsunami Joint Survey Group. Fig. 6.5 shows the comparison of the flooding area from the tsunami near Sendai between the numerical result and the field survey result. The model H2Ocean reproduces the flooding area of the tsunami quite well. Good agreement with the inundation heights has been achieved based on 914 data points between latitudes 34.5 and 42.5 (see Fig. 6.6). The underestimation of the maximum run-up is likely due to the 80 m grid, which is still too coarse to resolve the local steep and narrow coastal features. It is not computationally feasible to run a model with such high resolution along the whole Japanese coast. To study the effect of the grid resolution on the maximum wave run-up, in the following study, a detailed flooding simulation was carried out using a 20 m resolution grid but in a smaller study area.
Figure 6.6: Tsunami flooding and timing. (a) Location of measurements, indicated by red diamonds. (b) Comparison of the tsunami maximum run-up. (c) Comparison of the inundation height. Measurements are in blue and simulated results are in red. (d)–(g) the snapshots of the propagation of the tsunami wave at 15, 30, 45 and 60 min after initial rupture. Numbers next to locations give the observed arrival time of the leading waves which is measured in minutes after the rupture.
6.5 Detailed Flooding Simulation

The survey found that maximum run-up heights of greater than 20 m are distributed along a 290 km stretch of the Pacific coast. A strong regional dependence of tsunami characteristics is indicated by the survey data (Shimozono et al., 2012). Along the Sanriku coast from about 50 to 200 km north of Sendai, the measured tsunami heights are remarkably large due to the narrow bays where the incoming tsunami waves are focused and amplified. To study the extreme run-up in this area, a more detailed flooding simulation is carried out together with Dr. Takenori Shimozono. The study area is about 42 km by 22 km extending from GPS-buoys 802 and 804 to the coast (see Fig 6.7). The grid has a 50 m resolution offshore and is refined to 20 m around the coastline. The topography and bathymetry are created from 10 m grid topographic data of the Geospatial Information Authority of Japan and digital nautical charts of the Japan Hydrographic Association, respectively. The manning coefficient is set to $0.02m^{1/3}$ in the sea area and set to $0.05m^{1/3}$ in the land area. The water elevation at the offshore boundary is linearly interpolated from the GPS buoy 802 and 804 (see Fig. 6.9).

![Figure 6.7: The study area of the detailed flooding simulation of 2011 Japanese Tsunami and the locations of the GPS-buoys.](image)

Fig. 6.9 shows the comparison of the computed and measure tsunami heights. The non-hydrostatic simulation shows no observable improvement in the simulated extreme run-up. This is most likely because the traveling distance is so short that any effects from dispersion do not affect the inundation and run-up heights. Overall, the results of the numerical simulation with the fine grid fit the survey data quite well. Except for a few extreme data points, the inundation and
run-up heights are accurately reproduced. A possible reason for the underestimation of the extreme run-up (over 30 m) is the use of only two GPS-buoys to derive the boundary condition.

Several grids have been used in the simulations. The results show that as the resolution of the grid increases, the simulated inundation and run-up heights approach the observed values (compare Fig. 6.9 and 6.10).
6.6 Conclusions

Using the seabed deformation inverted by using both the geodetic and seafloor pressure gauge data, H$_2$Ocean can generate results consistent with existing observational data. However, the flooding simulation underestimates the inundation and run-up of the tsunami waves in the area where local topographic features are not well resolved by the low resolution grid. A detailed flooding simulation was conducted to examine the dynamic process of the local amplification and produces accurate inundation and run-up results. It can be concluded that in addition to accurate initial field, it is necessary to have a fine resolution grid with which the local topographic features are well represented in order to accurately predict the inundation and extreme run-up of the tsunami waves.

This study has demonstrated the efficiency and accuracy of the tsunami model H$_2$Ocean, which is sufficiently accurate to provide useful insights into tsunami propagation and inundation. The modelled results can reflect the characteristics of tsunami propagation and inundation.
Discussions and future work

7.1 Discussions

In tsunami risk assessment, inundation studies always require the simulation of the propagation and the inundation by tsunami waves. Wetting and drying, which happens during the inundation process, has always been a well-known challenge in numerical modeling. As the water depth approaches zero in the dry part, the governing equations are not valid anymore. If not handled properly, it may result in problems with mass conservation, negative water depths or artificially generated wiggles. To properly handle flooding and drying problems, the model needs to guarantee both mass and momentum conservation (Stelling and Duinmeijer, 2003). The Alfred Wegener Institute, within the framework of GITEWS, has developed and validated a finite element model, TsunAWI. It employs an unstructured grid to take advantage of its ability to accurately represent the coastlines, and to cover multi-scale spatial phenomena in one grid system. TsunAWI employs a conforming linear finite element ($P_1$) for water elevation and a non-conforming finite element ($P_1^{NC}$) for the velocity. The model has been successfully used in studies of tsunami wave generation and propagation (Androsov et al., 2008; Harig et al., 2008). However, it encounters several problems when handling flooding and drying. First, the conservative formulation of the advective terms in the momentum equations leads to flux penalties in the computation of momentum. AWIs experience has shown that when the contribution of the advection is relatively large, spatial noise components will be generated and will lead to instability (Androsov et al., 2008; Danilov et al., 2008). A $P_1$ projection method is adopted in TsunAWI to smooth the velocity fields and to remove the flux penalty terms. The second problem with TsunAWI is that its variational formulation automatically leads to a centered approximation of the advective flux in the computation of the equation for elevation. This brings difficulties in the simulation of wetting and
drying where one needs an upwind differencing scheme to ensure the positivity of the water depth. Another issue with TsunAWI is the use of a linear least square extrapolation method in handling flooding and drying. In the case of large water level gradients at the wave front, this method may generate artificial wiggles and degrade the accuracy of the simulation results.

To address the issues mentioned above and to formulate a robust wetting and drying scheme for TsunAWI, cooperation between TUDelft and AWI was initiated. Within the project, TsunAWI has been reformulated in a finite volume formulation, which allows one to implement a simpler, yet more consistent wetting and drying scheme. A conservative momentum advection scheme can also be implemented easily. Since the finite volume pair employs the same variable placements as $P^1_{NC} - P_1$, it can be incorporated into the existing TsunAWI code, offering flexibility to the user.

Compared to the original implementation, the finite volume analogue, named H$_2$Ocean, has obvious advantages when implementing a simple conservative momentum advection scheme and an up-wind differencing scheme. A momentum conservative advection scheme has been developed for H$_2$Ocean facilitating the numerical shock capturing capabilities of the model. Shock waves, are frequently observed during tsunami flooding. Up-winded water depths are used in the computation of the advective flux in the equation of elevation to ensure the positivity of the computed water depth. All these advantages make H$_2$Ocean more accurate and robust in handling flooding and drying problems. Despite the successful application of non-linear shallow water equations in tsunami modeling, non-hydrostatic pressure models that can account for wave dispersive are in high demand. In fluid dynamics, dispersion means that waves of different wavelengths travel at different phase speeds. It plays an important role in wave transformation from deep water to intermediate water and in wave interactions with an uneven bottom. When the wave propagates over large distances in shallow seas, the phase error may be accumulated. When the bottom topography varies sharply, the hydrostatic assumption is not valid anymore. All these situations require the dispersive effects to be accurately resolved. In this thesis, the model H$_2$Ocean has been extended to non-hydrostatic by following the fractional step procedure. As there is no diagnostic equation for non-hydrostatic pressures, they have to be solved by inverting a Poisson equation which requires massive computational effort. The efficiency of the model becomes a major issue when it involves large spatial domains. With the increasing demands of performing large scale flow simulations with non-hydrostatic models, improving their efficiency has become a competitive necessity. In H$_2$Ocean, a lumping of the pressure gradient in the horizontal velocities is employed to improve the efficiency of the model. The number of non-zero elements in the sparse matrix of the Poisson equation has been reduced by half. This greatly reduces both the memory requirements and the number of floating point operations required to solve the Poisson equation. When carrying out the non-hydrostatic wave simulations, the incoming wave boundary condition is often required. It can be implemented easily in a staggered grid
model by specifying the velocity at the boundary. However, for a model with a non-staggered variable placement as $P_{i}^{NC} - P_{i}$, it is problematic. Both the wave elevation and the non-hydrostatic pressure have to be specified at the boundary. For long incoming waves, the non-hydrostatic pressure can be set to zero. For short waves, setting the non-hydrostatic pressure to zero will generate unwanted extra energy in the computational domain, and makes the incoming wave different from that expected. In this research, the non-hydrostatic pressure function for the depth-integrated models is derived based on linear wave theory. Once the period and amplitude of the incoming wave is known, the non-hydrostatic pressure can be determined by this function. All the test cases have demonstrated that this boundary can serve as an easy and accurate way to introduce short incoming waves in depth-averaged non-hydrostatic models.

To further improve the efficiency of non-hydrostatic models, a minimized Poisson equations formulation has been proposed. It first simplifies the pressure stencil of the horizontal velocity in the bottom layer by approximating the pressure gradient only with the pressure at the top face of the bottom cell. Then it combines the equations in the two bottom layers to eliminate the vertical velocity and the pressure at the bottom from the set of equations. Without affecting the number of degrees of the independent variables, it reduces the size of the Poisson equation by one in each single vertical column. Since most of the computational effort is paid in inverting the Poisson equation, minimizing the Poisson equation can improve the efficiency of the model. This method can be applied to any non-hydrostatic model with multiple layers. When adopted by depth-integrated non-hydrostatic models, the method can produce accurate dispersion up to $kh \approx 7$, while requiring the same computational effort. The minimized Poisson equations formulation provides an efficient way to model the dispersion effects.

$H_{2}$Ocean has been extensively tested against analytical solutions or laboratory experiments and successfully applied in the study of 2004 Indian Ocean Tsunami and 2011 Japan Tsunami. It has been shown to be an efficient and accurate numerical model for simulating the propagation of and inundation by tsunami waves. It can act as a base both for further numerical research and as a tool for investigating the propagation and inundation of tsunami waves.

7.2 Suggestions for future work

The possible future directions for this research fall into two broad categories, improving the capability of the model and broadening its applications in tsunami study.

First, the stability and convergence properties of $H_{2}$Ocean need to be further investigated to gain a better understanding of the numerical schemes used. There are also potential improvements in the advection scheme and viscosity schemes. Second, the feasibility of combining the minimized Poisson equations formulation with Reeuwijk (2002)'s method needs to be explored. If we combine the minim-
ized Poisson equations formulation with Reeuwijk (2002) ’s method, a promising model might be formulated to extend the capability of the model. The model can employ multiple layers for momentum equations, but only keep one pressure unknown at the middle of the vertical water column. An interpolation function of the pressure can be defined. By using the minimized Poisson equations formulation, a Poisson equation with only one pressure unknown needs to be solved. The pressure at the bottom can be retained by using the kinematic boundary condition, while the pressure at the layers where the dynamic pressure is missing can be interpolated by using the interpolation function. An accurate velocity field still can be achieved, but the computational cost paid in solving the pressure field would be extremely low. The capability of the model can be extended with minimum computational cost.

The results of this dissertation point to several interesting directions for future work in applying the model H₂Ocean in tsunami studies:

- At the moment, the rupture speed is not taken into account when simulating tsunamis in H₂Ocean. All ruptures are transformed to sea-surface uplift and depression as an instantaneous movement. This could be improved by decomposing the rupture area into sub-areas, with each sub-area being allowed to uplift and subside in a prescribed sequence in time.

- The role of the non-hydrostatic effects in tsunami wave propagation and inundation needs to be further studied with H₂Ocean.

- The subgrid method can be used in H₂Ocean to improve its accuracy and efficiency in tsunami flooding simulation.

- The H₂Ocean model needs to be well documented and published as an open source model, offering a new tsunami study tool to the public.

- The H₂Ocean model needs to be incorporated into the existing code of TsunAWI. More detailed comparison between the two models needs to be investigated.
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Importance of horizontal seafloor motion on tsunami height for the 2011 $M_w=9.0$ Tohoku-Oki earthquake


Abstract

It is now clear that the 2011 Tohoku-Oki earthquake ruptured the subduction interface all the way to the Japan Trench. However, there is significant disagreement about just how much slip occurred at the trench, with most geodetic studies locating only a small fraction of the maximum slip there, whereas broadband seismic studies put the majority of the slip near the trench. Measurements of seafloor displacement near the trench also imply more slip there than is estimated by the geodetic studies. Here, by means of a joint inversion of displacement measurements and seafloor pressure data, we show that it is possible to reconcile geodetic and seismic studies and that a considerable amount of slip indeed occurred at the trench, with slip magnitudes reaching $57 - 74\%$ of the maximum slip. The seafloor displacement predicted by our model agrees very well with independent measurements made close to the trench. We also find good agreement between our tsunami model and independent measurements of tsunami height in the ocean and on land. Re-running of the inversion without the near-field pressure gauge data, however, leads to underprediction of the flooding on land, even when the
seafloor geodetic data are still included. Thus, even if the seafloor geodetic measurements had been available in real time, they would still not have allowed reliable prediction of near-field tsunami inundation.

Close to the trench, the dip of the plate interface is shallow, which serves to decrease the vertical motion of the sea floor for any given slip. However, the horizontal motion is conversely larger. Because the ocean floor slopes steeply near the trench, it acted like a giant wedge, converting horizontal motion of the ground into uplift of the water column and causing the peak initial tsunami wave height to be twice as high as it would have been from the vertical displacement alone.

A.1 Introduction

The Mw=9.0 Tohoku-Oki Earthquake occurred on 11 March 2011, at 05:46:23 UTC, off the Pacific coast of Honshu, Japan, along the subduction interface between the Pacific Sea plate and the Okhotsk extension of the North American plate (Fig. A.1). Here the two plates are converging at 8.3 cm/yr (DeMets et al., 2010). It was the largest earthquake ever measured in Japan and generated a tsunami that reached heights of up to 40 m inland (The 2011 Tohoku Earthquake Tsunami Joint Survey Group, 2011).

Numerous studies indicate that the earthquake ruptured the subduction interface all the way to the Japan Trench (Ide et al., 2011; Lay et al., 2011; Lee et al., 2011; Miyazaki et al., 2011; Ozawa et al., 2011; Simons et al., 2011; Yue and Lay, 2011). However, just how much slip occurred at the trench is disputed. Studies including static GPS displacements locate only a small fraction of the maximum slip at the trench (Miyazaki et al., 2011; Ozawa et al., 2011; Simons et al., 2011; Wright et al., 2012), whereas most broadband seismic studies locate large portions of the slip near the trench (Ide et al., 2011; Lay et al., 2011; Lee et al., 2011; Simons et al., 2011). More recently, geodetic studies that also include measurements of seafloor displacement put a larger proportion of slip at the trench, but still produce differing results, depending on which data are used in the inversion (Gusman et al., 2012; Iinuma et al., 2011; Romano et al., 2012).

When an earthquake occurs, any displacement of the water column leading to a tsunami is primarily due to the vertical displacement of the seafloor. However, when the seafloor has a significant incline, horizontal motion of the substrate can also play a role (Tanioka and Satake, 1996). The first manner in which horizontal motion can induce a tsunami is by imparting a horizontal impulse to the water, but this effect is usually considered negligible for slopes less than 1/3 (Iwasaki, 1982) and we do not consider it here. The second manner is through the vertical displacement of the water column caused by the horizontal displacement of the bathymetry beneath the overlying water mass. In this study we analyse the relative contribution of this additional induced vertical motion for Tohoku-Oki. We also explore its consequence for the evolution of the tsunami and the magnitude of the flooding.
Figure A.1: Coseismic deformation. (a) Tectonic setting and coseismic deformation field recorded by the regional IGS GPS network. Plate motion rates in cm/yr (DeMets et al., 2010) are depicted by black arrows. (b) Northern Honshu. White and green vectors indicate the horizontal component of GPS displacements from the Japanese GEONET array and seafloor displacements (Sato et al., 2011), respectively. Land colour represents interpolated vertical displacements from GPS. A white star in both panels indicates the earthquake epicentre from the Japanese Meteorological Agency.
We use both geodetic and seafloor pressure gauge data to constrain slip on the fault and the associated displacement of the water column. We then validate our model by comparison to independent data from measurements of seafloor displacement, tsunami inundation height and maximum run-up, as well as data from far-field pressure gauges and satellite altimetry that captured the progress of the tsunami across the Pacific Ocean.

A.2 Data and Methodology

A.2.1 Geodetic data

We included GPS measurements from the Japanese GEONET array and International GNSS Service (IGS) stations (Dow et al., 2009) within 5000 km of Japan. To calculate coseismic displacements for IGS stations > 500 km from the epicentre, we used a subset of the IGS network consisting of ~ 60 stations, including the 35 IGS stations within 5000 km of Japan. We processed data spanning 10 days before and after the main shock and obtained a set of independent daily solutions, which we combined into two campaign-like averaged solutions, before and after the earthquake. The typical standard deviations for the averaged solutions were within 2 and 5 mm for the horizontal and vertical positions respectively. The first 10 days were used to determine the reference (non-deformed) network solution, while the following 10 days were used to separate the initial co-seismic deformation field from the early post-seismic signal. The resulting co-seismic displacements are shown in Figure 1a and listed in Table S1 in Supplementary Material, while processing details are described in the Appendix. For the GEONET stations and the near-field IGS stations (< 500 km from the epicentre), we carried out kinematic positioning every 30 s and averaged the solutions for time windows of up to 14 min. We used the weighted root mean square error of the individual position offsets in each time averaged window to re-compute the co-seismic displacement standard error estimates, which were typically below 1 and 2 cm for the horizontal and vertical displacements respectively. The resulting co-seismic displacements are shown in Figure 1b and processing details are described in the Appendix.

We also included seafloor geodetic measurements (Sato et al., 2011), which we corrected for postseismic motion on the assumption that the postseismic motion scaled in time with the co-seismic motion in the same fashion as for the GPS. To estimate this postseismic evolution, we used a subset of ~ 400 GPS stations with relatively large initial co-seismic displacement vectors and excluded stations displaced by the Mw=7.9 aftershock at 06:15:39 UTC.

A.2.2 Seafloor Pressure Gauges

The seafloor pressure data that we included in our model inversion come from seven gauges operated by the Pacific Tsunami Warning Center (Titov et al.,
that were reached by the tsunami within 5 h (five gauges that recorded the later arrival of the tsunami were used for independent validation) and two gauges (TM1 and TM2) closer to the shore, operated by the University of Tokyo and Tohoku University (Maeda et al., 2011). We carried out a discrete Fourier transform of the time series and corrected for tidal fluctuations by high pass filtering, with a cut-off frequency of 1/9600 Hz. Gauge locations and filtered time-series are shown in Figure A.2 and A.3. The variance for each pressure gauge used in the inversion was estimated as the variance of the pressure for the 40 min prior to the earthquake, after filtering (Table S2, Supplementary Material).

A.2.3 Satellite Altimetry

Measurements of sea surface height from satellite altimetry were processed using the Delft University of Technology/NOAA Radar Altimeter Database System (Naeije et al., 2007). Models and corrections detailed in Trisirisatayawong et al. (2011) were applied. We used Pacific altimetry data acquired after the earthquake by the three currently that were operating at the time, Envisat, Jason-1, and Jason-2. In order to reduce the influence of long term changes in ocean dynamics, we referenced the measured water level along the satellite tracks to the water level of an average water level grid for the month before. This reference sea level grid was constructed using all available altimetry data in the 30 days centred around 24 February, 15 days before the earthquake. In order to merge data from different satellite platforms we applied reference frame biases between the different satellite missions that reflect differences in the orbits, as well as some other geographical differences in the altimeter-dependent models, such as sea state bias. As a basis we adopted the NASA Goddard consistently reprocessed TOPEX/Jason reference frame and orbits (Beckley et al., 2007). After minimizing crossover differences between ENVISAT and Jason data, all data were merged and resampled to a rectangular mesh matching the average track spacing of ENVISAT, with block size 0.2°, e-folding of 0.9° and cut-off at 1.8°. The resulting sea-surface height profiles along the three available satellite tracks are shown in Figure A.2.

A.2.4 Model

We modelled the displacement of the Earth’s surface as the elastic response to slip on a fault plane constrained by seismic reflection data (Fujie et al., 2006), with a dip of 5° up to 80 km from the trench and 15° beyond that. We divided the fault into a series of rectangular patches measuring 25 km along strike and 20 km down dip (Table S3, Supplementary Material). For each patch we calculated the response to unit slip at each GPS and seafloor geodetic benchmark. For those in the far field we used an analytical normal mode code for a spherical Earth (Riva et al., 2007), but to avoid Gibbs effects in the near field (<800km), we used a half-space Earth model (Okada, 1985). For each patch we also calculated the sea
surface height change due to unit slip. We calculated the vertical displacement of water at the sea floor \( d \) as

\[
d = u_x \frac{\partial H}{\partial x} + u_y \frac{\partial H}{\partial y} + u_z
\]

where \( u_x, u_y, u_z \) are the components of the displacement in the east, north and up directions respectively and \( H \) is the height of the sea floor, downloaded from the General Bathymetric Chart of the Oceans (http://www.gebco.net) and sampled at a resolution of 1 km. We then calculated the displacement of the sea surface by applying a two-dimensional \( 1/cosh(m) \) filter (Kajiura, 1963) to \( d \), to account for attenuation through the water column, where \( m \) is the wavenumber normalized by depth.

We assumed a mean delay between the initial rupture and the sea-surface uplift of 65 s, based on the occurrence of the peak global moment release rate at 60 s (Yue and Lay, 2011) and allowing 5 s for the motion to propagate to the sea surface. We then calculated the time evolution of sea surface height due to this initial condition, using the hydrostatic non-linear H2Ocean unstructured mesh model (Cui et al., 2010). We used two set-ups with this model: a Pacific wide tsunami model and a local flooding model with high resolution along the coast of Japan. The Pacific wide mesh has 1 million nodes with a 500 m resolution near the earthquake source, down to 20 km off the coast of South America, with the depth derived from the General Bathymetric Chart of the Oceans (http://www.gebco.net). We ran the model for 10 hours and used a time step of 2 s. The flooding model has 3 million nodes with 500 m resolution near the epicentre increasing to 80 m near the coast and on land. We used the 1.5 arc-second ASTER Global Digital Elevation Model (http://www.gdem.aster.ersdac.or.jp) to derive the height for the subaerial nodes. We ran the model for two hours with time steps of 0.1 s. The flooding algorithm is accurate and robust and has been validated in many scenarios including dam break and laboratory tsunami test cases (Cui et al., 2010; Stelling and Zijlema, 2003). Our model ignores the effects of dispersion due to non-hydrostatic physics but we do not expect these effects to play a major role, as the dominant wavelengths in the sea surface displacement are much greater than the depth (Pietrzak et al., 2007).

### A.2.5 Bayesian Inversion

We used the geodetic displacements and the seafloor pressure data from gauges reached by the tsunami within five hours to constrain our model. We applied a Markov chain Monte Carlo algorithm (Mosegaard and Tarantola, 1995; Hooper et al., 2011) to find the posterior probability distribution of slip on each patch. We assumed that the measurement errors are drawn from a zero-mean Gaussian distribution. In order to restrict the range of solutions to distributions of slip that are physically reasonable, we applied an additional smoothing constraint; we assumed that the probability distribution of the Laplacian of the slip is Gaussian
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with zero mean, and include the variance of the distribution as a model parameter (Fukuda and Johnson, 2008). We also assumed that errors in the physical model scale up the effective measurement error. Applying Bayes’ theorem gives the a posteriori probability distribution for the slip and smoothness as

\[ p(s, \sigma^2, \alpha^2 | d) = K [\sigma^2]^{-N/2} [\alpha^2]^{-M/2} e^{-\frac{1}{2\sigma^2} (d - Gs)^T \Sigma^{-1} (d - Gs) - \frac{1}{2\alpha^2} (Ls)^T (Ls)} p(s) \]

where \( s \) is the vector of rake and slip for each patch, \( d \) is the vector of measurements, \( G \) is a matrix of Greens functions mapping slip to displacements and sea surface height changes, \( -d \) is the variance-covariance matrix for the measurements, \( L \) is the Laplace operator for the slip, \( \sigma^2 \) is the scaling factor due to model error, \( N \) is the number of measurements, \( \alpha^2 \) is the variance of the Laplacians, \( M \) is the number of Laplacians, \( K \) is a normalizing constant and \( p(s) \) is the a priori probability of the slip and rake. We assumed the a priori probability to be constant for slip, constant for rakes within 10 degrees of the direction of plate convergence and zero for rakes outside this range.

We sampled the a posteriori distribution using the Metropolis-Hastings algorithm (Hastings, 1970). This involves selecting an initial value for each of the model parameters from \( p(s) \) and calculating the likelihood function, which is the right-hand side of Eq. 2 excluding \( p(s) \). A trial random step is then taken within \( p(s) \), and the new likelihood value is calculated. If the new likelihood value is greater, the step is taken and the trial model values are retained. If it is less, there is a chance that the step is taken, which is calculated as the ratio of the new likelihood over the old likelihood. Otherwise the old model values are retained. A new trial random step is taken, and the process is repeated until a representative sampling of the whole a posteriori distribution is built. The efficiency of this algorithm in reaching this goal depends on the maximum size of the random step that may be taken within \( p(s) \). In order to ensure fast convergence, we perform a sensitivity test for each model parameter after every 1000 iterations, and adjust the maximum step size such that all parameters contribute approximately equally to the change in likelihood and, as a whole, the mean chance of acceptance is approximately 50%.

A.3 Results

A.3.1 Slip distribution

The result of our inversion is not a single model, but rather a distribution of possible slip models. We show the mean and standard deviation of this distribution for each patch in Fig. A.4. The geodetic displacements predicted by the maximum a posteriori probability (MAP) model are also given in Fig. A.4, while the wave height evolution predicted at each of the pressure gauges is shown in Fig. A.2 and A.3. The MAP solution explains 99.2% of the GPS data variance and 98.9% of the pressure gauge variance.
Figure A.2: Open ocean tsunami heights. Middle panel shows the maximum modelled tsunami height and locations of pressure gauges and altimetry tracks. Surrounding panels show the evolution of tsunami height at the seafloor pressure gauges (note the variable scales on the axes). Measurements used to constrain the inversion are in blue while those used for validation only are in black. Modelled values using the MAP slip solution are in red, and the green dashed line gives the modelled values if only the vertical motion of the seafloor is considered. Bottom panels show altimeter tracks from Jason-1 (T147) and Envisat (T417/T419), all of which were acquired from south to north.
Our results show that the area of high slip extends all the way to the trench (Fig. A.4), with slip values at the trench reaching between 57% and 74% of the maximum value (all ranges given are 95% confidence intervals). This is in contrast to other studies that include static GPS (Miyazaki et al., 2011; Ozawa et al., 2011; Simons et al., 2011), for which slip at the trench is limited, but is in broad agreement with seismic waveform studies (Ide et al., 2011; Lay et al., 2011; Lee et al., 2011; Simons et al., 2011) and tsunami-only studies (Fujii et al., 2011; Saito et al., 2011). Two other studies that include both seafloor geodesy and pressure gauge data also have a large fraction of maximum slip at the trench, although the absolute values are significantly smaller. We estimate maximum slip to be between 73 and 81 m and all of our slip solutions with 95% confidence have an equivalent moment magnitude of $M_w = 9.0$.

### A.3.2 Model validation

For independent validation of our slip model we compare measurements of seafloor displacement with the displacement predicted by our MAP solution. Ito et al. (2011) measured displacements of two clusters of ocean-bottom instruments near the trench (Figure 4). For the cluster closest to the trench, they detected $58 \pm 20$ m of horizontal displacement and 5 m of uplift while our MAP solution predicts 46 m and 5.8 m, respectively. For the cluster $\sim 10$ km farther from the trench they were able to measure $74 \pm 20$ m of horizontal displacement only, while our model predicts 53 m. In other words our model agrees very well in the vertical and is at the lower end of the measurement range in the horizontal. A second set of measurements comes from Fujiwara et al. (2011) who compared repeated multibeam surveys from 1999, 2004 and 2011 and calculated average displacements along a strip.
Figure A.4: Fault slip. (a) Yellowed colouring and black vectors indicate mean modelled slip where slip is significant at 95% confidence. White and green vectors indicate the modelled horizontal component of GPS and seafloor displacements, respectively, for the MAP solution. Land colour represents modelled vertical displacements. Grey contours show the degree of estimated interseismic strain accumulation (Hashimoto et al., 2009). (b) Slip standard deviation. Grey contour lines enclose the region where slip is significant at 95% confidence. A white star in both panels indicates the earthquake epicentre from the Japanese Meteorological Agency. The white hexagons in (a) indicate the position of seafloor displacement measurements (Ito et al., 2011) and the white line indicates the strip where seafloor displacement was estimated from repeated bathymetric surveys (Fujiwara et al., 2011).
from the trench axis to 40 km to the west (Figure A.4). They estimated 50 m of horizontal displacement and 7-10 m of uplift, while our model predicts 49 m and 7 m respectively, indicating excellent agreement.

For validation of our tsunami model we ran the sea surface height change predicted by our MAP slip solution through both the Pacific wide tsunami model and the Japanese coastal flooding model. We then compared these results with inundation height and maximum run-up data measured along the coast of Japan (The 2011 Tohoku Earthquake Tsunami Joint Survey Group, 2011) (Figure A.5), tsunami arrival times from tidal gauges (IOC/UNESCO Bulletin, 2011) (Figure 5), measurements of seafloor pressure gauge data more than five hours distant (Figure 2), and satellite altimetry data (Figure A.2).

We find good agreement with the inundation heights based on 914 data points between latitudes 34.5 and 42.5. This implies that the location of the slip and corresponding uplift of the water column in Figure 4 are well resolved by the joint inversion. However, we find that we under-predict the maximum run-up. Data at over 5300 locations show that maximum run up exceeded 20 m for over 290 km of coast (Mori et al., 2011). While this finding is reproduced by our flooding model (Figure A.5) we generally under-predict maximum run-up over 25 m. This is likely due to the steep and narrow local coastal features, which amplify the run-up in some places, and are not resolved by the 80 m grid.

Measurements of tsunami arrival times in coastal waters at coastal tide gauges are imprecise due to the difficulty in picking the maximum point of a broad wave, and here our model arrival times match the data within a few minutes. This highlights the difficulty of using coastal tide gauge data to reconstruct tsunami as they are more sensitive to local processes such as reflection, refraction and diffraction of the incoming tsunami due to local bathymetry and coastline features. In the Pacific, good agreement is found with both distal pressure gauge data and the altimetry data. We did not need to include a time shift to fit the distal pressure gauge data, as in some studies (Simons et al., 2011; Romano et al., 2012) demonstrating that the non-linear shallow water approximation might still be valid for long distance tsunami propagation in the deep ocean. It should be noted, however, that a perfect match with the altimetry data is highly unlikely, because the satellite records many phases in the evolution of the tsunami. Where the satellite first records the edges of the tsunami it records the leading waves of elevation and depression and the signal is quite clear. However, the altimeter also passes directly across the region through which the tsunami has already propagated and for which tsunami generated sea surface displacements are the result of wave interference, refraction, diffraction and reflection. Additionally, altimetry also detects contributions from other high frequency ocean processes such as current meanders, eddies, and storm surges. Moreover, the model grid coarsens away from the source region and errors due to poorly known bathymetry will accumulate at the location of the altimetry tracks and at the far-field pressure gauges. Nevertheless the broad scale features in the data are reflected in our model.
Figure A.5: Tsunami flooding and timing. (a) Location of measurements, indicated by red diamonds. The background is as in Fig. 1b. (b) Tsunami maximum run-up. (c) Inundation height. Measurements are in blue and model values for our MAP solution are in red. (d) (g) open ocean heights at 15, 30, 45 and 60 min after initial rupture. Numbers next to locations on land give the number of minutes after rupture at which the tsunami was first detected (white squares) or at which the maximum tsunami height was measured (black squares).
A.4 Discussion and conclusions

In contrast to other published slip solutions, our model is validated by comparison to measurements of seafloor displacement close to the trench and measurements of the tsunami in the ocean and on land. If we invert GPS data alone we get similar results to those published (Miyazaki et al., 2011; Ozawa et al., 2011), but the pressure gauge data do not then match. If we also include the Pacific-wide pressure gauge data and assume no time delay between the onset of rupture and the uplift of the water column, our result is similar to that of Simons et al. (2011) but the tsunami arrives too early at TM1 and TM2 (Fig. A.3). The addition of a delay between initial rupture and sea-surface uplift plus the extra data from TM1 and TM2 force the region of maximum slip to be farther east, close to the trench, and not in the area of high estimated interseismic strain accumulation (Fig. A.4). The resulting model is then consistent with broadband seismic studies.

The pattern of slip is approximately similar to those of Gusman et al. (2012) and Romano et al. (2012) who use a similar combination of data sets. However, the magnitude of the slip values in our model are significantly higher than those of the other studies, with maximum values of 73 – 81 m versus 44 – 47 m (Romano et al., 2012) and 37 – 49 m (Gusman et al., 2012). Given that seafloor displacements predicted by our slip model agree well with average displacement measurements along a 40 km bathymetry profile perpendicular to the trench (Fujiwara et al., 2011) and are at the lower end of the range measured for the ocean-bottom instruments (Ito et al., 2011), the implication is that the other two studies would underpredict the measured displacements.

The seafloor geodetic data do not significantly affect the inversion and we get a very similar slip distribution if we leave them out altogether (Fig. A.6). This is partly due to their large measurement uncertainties, which leads to them having a low weight in the inversion, but even if we increase their weight by artificially reducing their error variance by a factor of 100, the slip distribution remains approximately the same (Fig. A.6). The implication is that seafloor geodetic measurements are not necessary to constrain the slip distribution if near-field pressure gauge data are available. The converse, however, is not true. We also inverted for the slip distribution using only geodetic data, with the smoothness parameter, $\alpha^2$, fixed to the MAP value from the full inversion. To ensure that the seafloor measurements contributed significantly, we again decreased their error variance by a factor of 100. The resulting slip distribution is significantly different and the peak tsunami amplitudes at the near-field pressure gauges, as well as the flooding on land are underestimated (Fig. A.7). Underestimation of peak tsunami amplitudes was also noted in other studies where only GPS and seafloor geodetic data were used to constrain the slip distribution, using either the same data set as this study (Romano et al., 2012) or an enhanced data set (Iinuma et al., 2011). Thus, even if these seafloor geodetic measurements had been available in real
time, they would still not have allowed reliable prediction of near-field tsunami inundation.

Figure A.6: Comparison of slip on each fault patch between the MAP model with standard weights, and the MAP model with the variance of the seafloor geodetic data reduced by a factor of 100 (red, correlation coefficient 0.97) or with the seafloor data omitted completely (blue, correlation coefficient 0.98).

The coastline with its complex and intricate topography shows significant variation in the height of the maximum run-up. This highlights the strong regional dependence, as discussed by Shimozono et al. (2012). While the inundation height is well represented by the flood model, the extreme maximum run-up values are not, highlighting the need for fine resolution of order 10 m or less for steep embayments. The availability of high quality data from this earthquake is, however, unique, with altimeter, pressure gauge, inundation height and tide gauge data in good agreement.

Models that put slip further from the trench generate a tsunami large enough to match measured tsunami amplitudes by vertical motion of the seafloor alone. If the slip is further towards the trench, however, as the data used in our inversion insist, then the vertical motion is reduced due to the lower dip of the subduction interface there. However, the horizontal motion is conversely larger. This, combined with the increasing steepness of the seafloor close to the trench, serves to increase the contribution to the uplift of the water column from the horizontal motion of the seafloor. We find that the peak uplift from the horizontal motion is $8.9 - 10.0$ m compared to $8.6 - 11.6$ m from the vertical deformation (Fig. A.8).
Figure A.7: Inversion results using geodetic data only. (a) Mean slip. Yellow-red colouring and black vectors indicate mean modelled slip where slip is significant at 95% confidence. White and green vectors indicate the modelled horizontal component of GPS and seafloor displacements, respectively, for the geodetic-only MAP solution. Land colour represents modelled vertical displacements. Grey contours show the degree of estimated interseismic strain accumulation (Hashimoto et al., 2009). (b) Tsunami height evolution at two inshore seafloor pressure gauges (see Fig. 5d for location of TM1 and TM2). Measurements are in blue and red indicates the modelled values for the geodetic-only MAP slip solution. (c) Tsunami maximum run-up and inundation heights. Measurements are in blue and model values for the geodetic-only MAP solution are in red.
Figure A.8: Initial sea surface height change. Redblue colouring represents sea surface height change for the MAP solution considering (a) all motion of the seafloor, (b) vertical motion of the seafloor only and (c) horizontal motion of the seafloor only. White and green vectors in (a) indicate modelled horizontal displacements for the MAP solution and land colour represents vertical modelled displacements. A white star in all panels indicates the earthquake epicentre from the Japanese Meteorological Agency.
The additional contribution of the horizontal motion increases the peak initial sea surface uplift by $77 - 116\%$ and increases the volume of uplifted water by $53 - 56\%$. Support for this conclusion comes from a simplified 2-D model based on measured displacements of ocean-bottom instruments near the trench by Ito et al. (2011). They estimated that the horizontal motion increases the initial wave height locally by $120 - 140\%$, although measurement errors for the displacements are large ($\pm 20$ m). The peak amplitude reaching the pressure gauges is also significantly increased compared to that which would result from only the vertical motion of the seafloor (Fig. A.2 and A.3).

While the vertical uplift of the water column due to horizontal motion has been shown to be non-negligible in terms of peak amplitude for some previous tsunami (Tanioka and Satake, 1996), this is the first time that it is estimated to be of similar importance as the vertical contribution. Furthermore, the dip of $5$ that we use for the subduction interface is actually an upper bound for the region close to the trench (Fujie et al., 2006) and a lesser dip would increase the horizontal contribution still further. Models of the interface that include a more gradual transition from low to high dip may have the dip increasing above $5$ at a point closer to the trench than our model. However, this would not significantly change the horizontal contribution, as it is only in the region within about $40$ km of the trench that the bathymetry is steep enough to generate large uplift of the water column through horizontal motion (Figure 8), and all fault models have a low dip there. The maximum slip of $73 - 81$ m that we find would represent the accumulated plate motion for $\sim 900 - 1000$ years if due entirely to elastic strain release. It is not known when the last earthquake to rupture this segment occurred, but the previous tsunami of a similar scale was caused by the Jogan earthquake in 869 AD (Minoura et al., 2001). However, two mechanisms other than elastic strain release have been proposed that could contribute at least partially to the high slip at shallow depths, which are dynamic overshoot (Ide et al., 2011) and the release of gravitational potential energy (McKenzie and Jackson, 2012). In support of the latter hypothesis, we note a general rotation of the slip vectors in our model from the plate convergence direction at deeper depths, to a direction that is perpendicular to the trench axis at shallower depths, which would be the expected direction for gravitational sliding.

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Appendix: GPS data processing

I. IGS stations \( \leq 500 \text{ km from the epicentre} \)

We applied the GIPSY-OASIS II software using the latest update of the Precise Point Positioning methodology (Zumberge et al., 1997), incorporating absolute IGS antenna phase centre variations and ambiguity resolution using Ambizap (Blewitt, 2008) for the entire network. We mapped the pre-earthquake solution onto the International Terrestrial Reference Frame (ITRF) 2005 (Altamimi et al., 2007), using a subset of 14 well-determined IGS global reference stations. This establishes a non-deformed reference solution in the ITRF on 10 March 2011. We then projected the post-earthquake solution onto this reference solution using only 30 GPS stations that were unaffected by co-seismic motions. In the post-earthquake solution we averaged 3, 7 or 10 days for those stations that were co-seismically deformed in respectively the near-to-mid-field (\(< 1000\text{km}\)), far-field (\(> 1000 \text{ km and } < 2000 \text{ km}\)) and very far-field (\(> 2000 \text{ km}\)). This provides a good trade-off between reduction of noise by averaging and reduction of accumulated post-seismic motion in the co-seismic position estimates. To remove the remaining post-seismic motion component in the 3-, 7- and 10-day averaged post-earthquake station positions, we estimated pure co-seismic displacements for the near- and mid-field stations from 30 s kinematic positioning results (see below) to derive post-seismic correction factors for the averaged static positions of the far- and very far-field IGS stations. The co-seismic displacements of the stations in the earthquake region then follow from coordinate differences between the two solutions, with relative \(1 - \sigma\) horizontal and vertical accuracies of 2 and 5 mm respectively.

II. GEONET and near-field IGS stations

We computed absolute IGS station positions every 30 s with the GIPSY Precise Point Positioning methodology by using high-rate GPS satellite clock solutions. These already provide sufficiently small time steps to accurately detect and estimate co-seismic jumps. We converted the 30 s kinematic time series from our own IGS analysis and the GEONET network analysis by the ARIA team at JPL and Caltech into UTC time, so they could be aligned with the predicted seismic surface wave arrival times at each station from USGS earthquake data. We used the surface wave arrival time as a reference to set-up a time window to average kinematic positions both before and after the main shock and first large aftershock \((M_w=7.9)\). This reduced the high-frequency noise, typically \(\sim 1 \text{ cm}\) in the horizontal, obtained in 30 s kinematic positioning. The first six minutes
after the main shock and three minutes after the aftershock were excluded to allow the station dynamic motions to decay. To avoid inclusion of station motion response to surface waves from other aftershocks (depth < 45 km and $M_w \geq 6.0$) their significance was evaluated using a straightforward magnitude/distance to epicentre test and, if necessary, an additional two minute exclusion window was applied.

To assess the post-seismic motion already accumulating in the first 18 h we similarly computed multiple 30 minute averaged positions. For the following days we computed daily averaged positions from the IGS position results and the 30 minute position time series provided by ARIA.

Appendix A: Supporting information

Supplementary data associated with this article can be found in the online version at http://dx.doi.org/10.1016/j.epsl.2012.11.013
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