CONCENTRATION EFFECTS ON SETTLING-TUBE ANALYSIS

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Report 74-1

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Published in a slightly modified form in: Journal of Hydraulic Research, Vol. 12, 1974, No. 3.
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SUMMARY

A mathematical model describing the unsteady settling of sediment samples and nonuniform suspensions has been developed, taking into account the influence of the sediment concentration on the fall velocity. In principle, the resulting equations can be solved using the method of integration along characteristics.

An estimating procedure for the settling-tube size required has been established. The procedure is based on the requirement that the relative error in the settling time of each particle, which is caused by possible concentration effects, viz. hindered settling and settling convection, be less than a prescribed value.

Although a lack of unambiguous experimental data as regards settling convection prevents a positive statement, this phenomenon seems to be more severe than hindered settling. If settling convection occurs, it will apparently cause unacceptable errors in the analysis of the relatively large sample sizes (as compared to the settling-tube dimensions) recommended in the literature.

*stratified
INTRODUCTION

In the literature on settling-tube analysis, two possible sources of errors caused by the sediment concentration, are mentioned:

1. hindered settling, which is the phenomenon that the settling velocity of a particle in an evenly distributed suspension is smaller than the settling velocity of the same particle in the absence of other particles (2), (8), (9), (10), (11), (12), and

2. settling convection, which is the phenomenon that certain particles in a suspension are in close proximity to each other (clusters) and thus fall at a relatively high velocity (5), (6), (7), (13), (15).

In order to ensure that the deviation of the actual fall velocity from the fall velocity of a single particle (the ideal fall velocity) is small, certain restrictions on the sizes of settling-tubes and sediment samples must be made. Published recommendations for the design of settling-tubes and the choice of sample sizes are based on experiments (3), (4). The present paper describes a mathematical model for the settling of a sediment sample, in which the dependence of the fall velocity on particle size and sediment concentration are included. The properties of the model are demonstrated by applying it to the hindered settling of a given sample using the method of integration along characteristics. A similarity solution of the governing equations provides a means for estimating settling-tube dimensions and sizes of (nonuniform) samples in cases where either hindered settling or settling convection occurs.

MATHEMATICAL MODEL

The mathematical model is based on the assumption that the sediment concentration and the fall velocity in the suspension are continuous functions of time and coordinate along the tube axis. This assumption is
certainly not true for the behaviour of the suspension on the scale of
the distances between the particles, or even on the scale of clusters in
the case of settling convection. The general behaviour of the suspen-
sion, however, can be described satisfactorily starting from a continuum
approach (14). Despite this approach, for the sake of convenience terms
like "particles" and "particle trajectory" instead of the correct ex-
pressions "fluid element containing particles" and "trajectory of con-
stant ideal fall velocity" are used hereafter.

**Governing Equations.** - The continuity equation which expresses the
conservation of the volume of the sediment reads

$$\frac{\partial c}{\partial t} + \frac{\partial c w}{\partial x} = 0 \quad (1)$$

in which $c =$ sediment concentration by volume, $w =$ actual settling veloc-
ity of a particle, $t =$ time coordinate, and $x =$ coordinate along tube
axis (positive in fall direction).

The settling equation which states that the ideal fall velocity $w_o$ of a
particle is constant along the particle trajectory, is

$$\frac{Dw_o}{Dt} = 0 \quad (2)$$

which yields

$$\frac{\partial w_o}{\partial t} + w \frac{\partial w_o}{\partial x} = 0 \quad (3)$$

The order of magnitude of the time $t_p$ required for a particle to adjust
its fall velocity to a new situation (a sudden change in the liquid veloc-
ity, for instance) is given by

$$t_p = \frac{w_o}{g} \quad (4)$$
in which \( g \) = acceleration of gravity. Since the order of magnitude of the fall velocity \( w_0 \) is 0.10 m/s, and hence that of time \( t_p \) about 0.01 s, the settling of a sample in a settling-tube, which lasts many seconds, can be considered as a quasisteady process. Consequently, the functional relationship between the fall velocity and sediment concentration determined for steady-state situations can be applied, i.e.

\[
w = w_0 f(c, \text{Re}_p) \tag{5}
\]

in which \( f \) = an experimentally determined function, and \( \text{Re}_p = \) particle Reynolds number. Wall effects have been disregarded in Eq. 5. The function \( f \) is less than unity for hindered settling, whereas it is greater than unity for settling convection. Richardson and Zaki (11), (12) did a series of experiments on hindered settling and found that for \( \text{Re}_p \) less than 0.2 and \( \text{Re}_p \) greater than 500 the influence of the particle Reynolds number is negligible. In the intermediate range \( f \) is a slowly varying function of the particle Reynolds number. The result for small Reynolds number was confirmed theoretically by Batchelor (2).

As regards settling convection it should be noted that the knowledge about the dependence of function \( f \) on particle Reynolds number or a "cluster Reynolds number" (and even on concentration) is poor. Since furthermore the mathematical model is intended to provide only estimates for the concentration effects mentioned, the influence of the particle Reynolds number is disregarded. Eq. 5 then simplifies to

\[
w = w_0 f(c) \tag{6}
\]

From a formal view-point, Eqs. 1, 3 and 6 describe the propagation of continuity waves (14) in a nonhomogeneous fluid.
**Characteristics.** - Eqs. 1, 3 and 6 form a quasilinear system of hyperbolic equations. The characteristic directions of this system can be shown to be

\[
\frac{dx}{dt} = w \tag{7}
\]

and

\[
\frac{dx}{dt} = \frac{w \, d(cf)}{f \, dc} \tag{8}
\]

The characteristics given by Eqs. 7 and 8 are referred to as \(C_1\)-characteristics and \(C_2\)-characteristics, respectively. The compatibility relations read

\[
dw_o = 0 \tag{9}
\]

along the \(C_1\)-characteristics, and

\[
dw = 0 \tag{10}
\]

along the \(C_2\)-characteristics. The \(C_1\)-characteristics are seen to represent particle trajectories; the \(C_2\)-characteristics form trajectories of the property: \(w = \text{constant}\). If the concentration and fall velocity are given along an initial curve in the \(x,t\)-plane (e.g. a part of the \(x\)-axis \((t = 0))\), characteristic directions in discrete points on this curve can be determined using Eqs. 7 and 8. At the points of intersection of \(C_1\) and \(C_2\)-characteristics new concentrations and fall velocities can be found applying Eqs. 9 and 10, etc. In this way the method of characteristics provides a simple step by step integration method for the governing equations (see Fig. 1). In principle, the integration method can be continued till the end of the measuring section of the settling-tube.
is reached in order to obtain arrival times for the particles.

Figure 1: Initial curve and characteristics in x,t-plane.
An example. - The method of characteristics has been applied graphically to the hindered settling of a sample of known distribution (see Fig. 2). The particles are arranged according to their ideal fall velocities, which is a requirement of the continuum model. Actually, this need not be true at the moment the experiment is started, but this arrangement takes place within a very short time in a well-functioning settling-tube. The sample size corresponding to the concentration distribution assumed is 166 g/m² (sediment density 2.65 g/cm³), or a sample of about 5.2 g for a settling-tube having a diameter of 0.20 m.

In the case of hindered settling, function f can be approximated by

\[ f(c) = (1 - c)^\beta \] (11)

in which \( \beta \) varies between 2.4 and 4.7, depending on the particle Reynolds number. The influence of this number, however, has been disregarded; therefore, in the example of Fig. 2 a constant value, viz. \( \beta = 4 \), has been assumed.

In Fig. 2, \( C_1 \) and \( C_2 \)-characteristics have been constructed using Eqs. 7 through 10. They show a peculiarity in the region indicated by the character A, where intersection of two \( C_2 \)-characteristics (starting at \( x = 0.22 \) mm and \( x = 0.24 \) mm) occurs. This is not caused by the method of solution of the equations, but is a property of the mathematical model. Physically, the intersection of \( C_2 \)-characteristics indicates the development of a discontinuity (shock wave) in the sediment concentration. The shock wave propagates in the region indicated by hatching. It is possible to determine the path and the strength of a shock wave by using a shock condition derived from the conservation of sediment volume across the discontinuity (14), but this procedure is beyond the scope of this paper.

It is seen in Fig. 2 that as time elapses, the particle spacing
becomes wider, whereby the fall velocity increases (Eq. 11). This causes the concave shape of the particle trajectories.

ESTIMATION OF REQUIRED SETTLING-TUBE DIMENSIONS

The relative error $\delta$ caused by concentration effects and finite sample size can be chosen as follows (see Fig. 3)

$$\delta = \left| \frac{t_1 - \frac{1}{\omega_{oh}}}{\frac{1}{\omega_{oh}}} \right|$$

(12)

in which $l = $ length of measuring section of settling-tube, $\omega_{oh} = $ ideal fall velocity of largest particle starting at $x = h$, $h = $ initial sample thickness, $t_1 = $ actual time of arrival at $x = l$ of the largest particle, and $1/\omega_{oh} = $ ideal time of arrival of this particle at $x = 1$. The fall velocity of the largest particle has been used in the definition of the error $\delta$, since this particle causes the greatest error.

Figure 3: Ideal and actual trajectories of the largest particle.
The equations describing the settling of the sediment, Eqs. 1, 3 and 6 allow a similarity solution of the form

\[ c(x,t) = \phi(t) \]  \hspace{1cm} (13)

\[ w_o(x,t) = x\psi(t) \]  \hspace{1cm} (14)

Eq. 13 indicates that in this solution the concentration does not depend on coordinate \( x \); Eq. 14 represents a linear relationship between ideal fall velocity and coordinate \( x \), which means that a nonuniform sample is considered. The initial conditions according to Eqs. 13 and 14 read

\[ c(x,0) = c_i = \phi(0), \quad 0 < x < h \]  \hspace{1cm} (15)

\[ w_o(x,0) = \frac{x}{h} w_{oh} = x\psi(0), \quad 0 < x < h \]  \hspace{1cm} (16)

This similarity solution, which is dealt with in App. I, is used to estimate the actual time of arrival \( t_1 \), after which the relative error can be determined from Eq. 12. App. I gives for time \( t_1 \)

\[ t_1 = \frac{hc_i}{\omega_{oh}} \frac{c_i}{h} \int_{c_i}^{c_1} \frac{d\phi}{\phi^2 f(\phi)} \]  \hspace{1cm} (A9)

in which \( \phi \) now is a dummy variable. Substituting Eq. A9 into Eq. 12 yields for the relative error

\[ \delta = \left| \frac{c_1}{c_i} \int_{c_i}^{c_1} \frac{d\phi}{\phi^2 f(\phi)} - 1 \right| \]  \hspace{1cm} (17)

in which \( c_1 = \) sediment concentration by volume, averaged over the whole settling-tube (total particles volume divided by tube volume),
\[ c_1 = \frac{c_i h}{l} \]  \hspace{1cm} (18)

When \( f(\phi) = 1 \) (no effect of concentration on fall velocity) is substituted into Eq. 17, the error caused by the finite sample size only is found (\( \delta = h/l \)).

In order to determine the relative error from Eq. 17, function \( f \) must be known. If the only concentration effect occurring is hindered settling, Eq. 11 could be applied. However, the second effect, viz. settling convection, has also been observed, even at concentrations where other research-workers found hindered settling (13). Hulsey (5) observed an increase in fall velocity in the lower part of a uniform sample, and a decrease in the upper part, which apparently indicates that both effects can occur simultaneously.

In this connection, the method of introducing the sample in the settling-tube is important: an introduction method causing eddies in the liquid of the settling-tube leads probably to settling convection, whereas the development of this effect can be delayed if the sample is introduced carefully. In the latter case hindered settling would dominate.

Since it does not seem possible to determine which effect will occur in a particular case with the present state of knowledge, the influence of each effect will be examined separately.

**Hindered settling.** - Substituting Eq. 11 into Eq. 17 and introducing some approximations based on the fact that small concentrations are dominant, yields the relative error for hindered settling,

\[ \delta = \beta c_1 \ln \frac{c_i}{c_1} \]  \hspace{1cm} (19)

This relationship is illustrated in Fig. 4 for \( \beta = 3 \) and \( \beta = 5 \). On the
left-hand side of the dotted curves, the error caused by the finite sample size is more important than that owing to hindered settling.

Figure 4: Relative error caused by hindered settling

**Settling convection.** - The data in the literature on this phenomenon do not seem to be in close agreement to each other. Stenhouse (13) mentions experiments in which an increase in fall velocity up to 130% was observed in a uniform suspension (ballatini beads in liquid paraffin); Hulsey (5) found a 5% increase for small samples consisting of relatively large particles (glass spheres in water) to a 60% increase for larger samples of small particles. In his experiments the settling of the sample caused turbulence. Settling convection appears to occur within a certain range of the concentration, possibly from a concentration \( c_a = 0.001 \) to a concentration \( c_b = 0.05 \) for nonuniform suspensions (13).

In view of the great uncertainties involved in this phenomenon, the effect on the fall velocity is simply assumed to be constant in the interval \( c_a < c < c_b \) and to be absent at other concentrations. The function \( f \) (Eq. 6) then reads
\[ f(c) = \begin{cases} f_s & \text{if } c_a < c < c_b, \quad f_s > 1 \\ 1 & \text{otherwise} \end{cases} \] (20)

Substituting Eq. 20 into Eq. 17, and disregarding \( c_a/c_b \) with respect to unity yields a relative error for settling convection,

\[ \delta = \begin{cases} \left(1 - \frac{1}{f_s}\right) \frac{c_1}{c_a} & \text{if } \frac{c_1}{c_a} < 1 \\ 1 - \frac{1}{f_s} & \text{if } \frac{c_1}{c_a} > 1 \end{cases} \] (21)

Settling convection would occur in the whole settling-tube if \( c_1/c_a \) is greater than unity. Eq. 21 is illustrated in Fig. 5 in the case where \( c_1/c_a \) is less than unity. Concentration \( c_a \) has been given the values 0.001 and 0.002. Comparison of Figs. 5 and 4 shows that settling convection will probably require a considerably smaller \( c_1 \) to obtain a certain accuracy than does hindered settling, if \( f_s \) is not less than 1.2. A smaller \( c_1 \) means a larger settling-tube for a given sample (see Eq. 18). Since settling convection appears to be a much stronger effect, it is not likely that it can be completely compensated for by hindered settling.

**Existing Settling-Tube Designs.** - The estimating method developed can be applied to settling-tubes described in the literature. Application, for instance, to the recommendations of the Committee on Sedimentation of the ASCE (1) for visual accumulation tubes (Table 3-F.3 in (1), additional data are given in (3)) yields the following: if only hindered settling occurs, errors from 1% for the smallest sample size up to 10% for the largest sample size (with respect to the tube size) recommended will be found. If, on the other hand, settling convection also takes
Figure 5: Relative error caused by settling convection.

place, errors from 5% to about 20% will be found, assuming a 20% increase of the fall velocity over the ideal fall velocity ($f_s = 1.2$). This value of $f_s$ is mentioned by Hulsey (5).

Obviously, the above results have to be handled with care; the estimating method is rather tentative owing to the lack of sufficient experimental data. Nevertheless, the mathematical model described seems to be a useful aid for estimating the required dimensions of settling-tubes.
APPENDIX I. - SIMILARITY SOLUTION

Eliminating the actual fall velocity \( w \), the governing equations, Eqs. 1, 3 and 6, can be written

\[
\frac{\partial c}{\partial t} + \frac{\partial (cw)f}{\partial x} = 0 \quad (A1)
\]

\[
\frac{\partial w}{\partial t} + wo \frac{\partial w}{\partial x} = 0 \quad (A2)
\]

Substituting Eqs. 13 and 14, which represent the similarity, yields

\[
\dot{\phi} + \phi f(\phi) = 0 \quad (A3)
\]

\[
\dot{\psi} + \psi^2 f(\phi) = 0 \quad (A4)
\]

in which the dots indicate differentiation with respect to time. Eliminating \( f(\phi) \) from Eqs. A3 and A4, and integrating yields

\[
\psi = A \phi \quad (A5)
\]

Integration constant \( A \) can be determined by applying the initial conditions, Eqs. 15 and 16, which changes Eq. A5 to

\[
\psi = \frac{wo}{hc_i} \phi \quad (A6)
\]

Substituting Eq. A6 into Eq. A3, and using Eqs. 13 and 15 yields an expression for the concentration \( c \) as a function of time,

\[
\int_{c}^{ci} \frac{d\phi}{\phi^2 f(\phi)} = \frac{wo}{hc_i} t \quad (A7)
\]
The particle trajectories, along which the ideal fall velocity $w_o$ is constant, are found by substituting Eq. A6, together with Eqs. 13 and 14, into Eq. A7,

\[
\int_{c_i}^{\infty} \frac{d\phi}{\phi^2 f(\phi)} = \frac{w_{oh}}{hc_i} t
\]

Eq. A8 yields the position $(x)$ of a particle as a function of time $t$, its ideal fall velocity $w_o$, and initial conditions $(c_i, h, w_{oh})$. The largest particle ($w_o = w_{oh}$) arrives at the end of the measuring section of the settling-tube $(x = 1)$ after elapse of a time $t_1$, which follows from Eq. A8,

\[
t_1 = \frac{hc_i}{w_{oh}} \int_{c_i}^{\infty} \frac{d\phi}{\phi^2 f(\phi)}
\]

It can be shown that the finite sample size ($0 < x < h$ at $t = 0$) influences the settling process in the case of hindered settling, as a result of which the similarity in the lower part of the sample is disturbed. Nevertheless, the order of magnitude of settling time $t_1$ is given by Eq. A9, which is sufficient since it is used merely in the estimating procedure.
APPENDIX II. - REFERENCES


APPENDIX III. - NOTATION

A = integration constant

\( c = \) sediment concentration by volume

\( c_a, c_b = \) sediment concentrations between which settling convection occurs

\( c_i = \) initial sediment concentration

\( c_s = \) particles volume divided by settling-tube volume

\( d = \) particle diameter

\( f = \) function of sediment concentration

\( f_s = \) constant related to settling convection

\( g = \) gravitational constant

\( h = \) initial sample thickness

\( l = \) length of measuring section of settling tube

\( Re_p = \) particle Reynolds number = \( wd/\nu \)

\( t = \) time

\( t_p = \) characteristic time interval a particle requires to adjust its fall velocity to a new situation

\( t_i = \) time of arrival of the largest particle at the end of the measuring section of the settling-tube

\( w = \) actual fall velocity of a particle

\( wo = \) fall velocity of a particle in the absence of concentration effects ("ideal fall velocity")

\( w_{oh} = \) ideal fall velocity of the largest particle

\( x = \) axial coordinate, positive in downward direction

\( \beta = \) parameter related to hindered settling

\( \delta = \) relative error caused by concentration effects

\( \nu = \) kinematic viscosity of the liquid

\( \phi, \psi = \) time functions in the similarity solution