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Proceedings of the American Mathematical Society is currently published by American Mathematical Society.
MORE ON M. E. RUDIN'S DOWKER SPACE

KLAAS PIETER HART

ABSTRACT. It is shown that M. E. Rudin's Dowker space is finitely-fully normal and orthocompact, thus answering questions of Mansfield and Scott.

0. Introduction. In [Ma] Mansfield defined the notions of \( \kappa \)-full normality and finite-full normality. One of the questions he raised was, whether there exists a finitely-fully normal space which is not an \( \omega_0 \)-fully normal space.

In [Sc] Scott asked whether M. E. Rudin's Dowker space [Ru] is orthocompact. We answer both questions simultaneously by showing that the above-mentioned space is both finitely-fully normal and orthocompact. Mansfield's question is hereby answered since in [Ma] he showed that almost \( \omega_0 \)-fully normal spaces are countably paracompact. Almost \( \kappa \)-full normality will not be defined here; it suffices to know that it is weaker than \( \kappa \)-full normality.

1. Definitions and preliminaries.

1.0 \( \kappa \)-full normality and orthocompactness. Let \( Y \) be a topological space, \( \mathcal{U} \) an open cover of \( Y \) and \( \kappa \geq 2 \) a cardinal. An open cover \( \mathcal{V} \) is said to be a \( \kappa \)-star (finite-star) refinement of \( \mathcal{U} \) if for all \( \mathcal{V}' \subseteq \mathcal{V} \) with \( |\mathcal{V}'| = \kappa \) (\( \mathcal{V}' \) finite) and \( \cap \mathcal{V}' \neq \emptyset \) there is a \( U \in \mathcal{U} \) with \( \bigcup \mathcal{V}' \subseteq U \), and \( \mathcal{V} \) is a \( Q \)-refinement of \( \mathcal{U} \) if \( \mathcal{V} \) refines \( \mathcal{U} \) and \( \cap \mathcal{V}' \) is open for all \( \mathcal{V}' \subseteq \mathcal{V} \). (Recent practice is to call \( Q \)-refinements interior-preserving open refinements.)

\( Y \) is called \( \kappa \)-fully (finitely-fully) normal [Ma] if every open cover of \( Y \) has a \( \kappa \)-star (finite-star) refinement. \( Y \) is called orthocompact [Sc] if every open cover of \( Y \) has a \( Q \)-refinement.

1.1 M. E. Rudin’s Dowker space. Let \( F = \prod_{n=1}^{\infty} (\omega_n + 1) \) endowed with the box topology. Furthermore let \( X' = \{ f \in F : \forall n \in \mathbb{N} \ cf(f(n)) > \omega_0 \} \) and \( X = \{ f \in X' : \exists i \in \mathbb{N} : \forall n \in \mathbb{N} \ cf(f(n)) < \omega_i \} \). Then \( X \) is M. E. Rudin’s Dowker space [Ru].

We give an alternative description of the canonical base for \( X' \) (and \( X \)). For \( f, g \in F \) we say

\[ f < g \text{ if } f(n) < g(n) \text{ for all } n, \]
\[ f \leq g \text{ if } f(n) \leq g(n) \text{ for all } n. \]

For \( f, g \in F \) with \( f < g \) we let

\[ U'_{f,g} = \{ h \in X' : f < h \leq g \} \]

Received by the editors February 9, 1982.

1980 Mathematics Subject Classification. Primary 54D20, 54G20.

Key words and phrases. \( \kappa \)-fully normal, finitely-fully normal, orthocompact.
and

\[ U_{f,g} = U_{f,g}' \cap X. \]

Then

\[ \{ U_{f,g}' : f, g \in F, f < g \} \]

is a base for the topology of \( X' \). Notice that the basic open sets are convex in the partial order \( \leq \) on \( X \), a fact we will use in the proof of Theorem 2.2.

2. The main result. In this section we prove using the results from [Ru] and [Ha] that the Dowker space \( X \) is finitely-fully normal and orthocompact. First we formulate a lemma, the proof of which can be found (implicitly) in the proof in [Ru] that \( X \) is collectionwise normal.

2.0 Lemma. a. Every open cover of \( X' \) has a disjoint refinement consisting of basic open sets.

b. If \( A, B \subseteq X \) are closed and disjoint then

\[ \text{Cl}_X A \cap \text{Cl}_X B = \emptyset. \]

The next result is from [Ha].

2.1 Lemma. For all \( n \in \mathbb{N} \): \( (X')^n \) is homeomorphic to \( X' \), and the homeomorphism can be chosen to map \( X^n \) onto \( X \).

Now we are ready to prove the main result.

2.2 Theorem. The space \( X \) is both 2-fully normal and orthocompact.

Proof. Let \( \mathcal{U} \) be a basic open cover of \( X \). Put \( U = \bigcup \{ 0 \times 0 \times 0 : 0 \in \mathcal{U} \} \); \( U \) is a neighborhood of \( \{ (x, x, x) : x \in X \} \) in \( X^3 \). Using 2.1 and 2.0b find a neighborhood \( U'' \) of \( \{ (x, x, x) : x \in X' \} \) in \( (X')^3 \) such that \( U'' \cap X^3 = U \).

For \( x \in X' \setminus X \), choose \( U, U' \) open such that \( U \cup U' \supseteq \{ (x, x, x) : x \in X' \} \). By 2.0a let \( \Theta' \) be a disjoint basic open refinement of the open cover

\[ \{ X' \setminus \text{Cl}_X (X \setminus 0) \}_{0 \in \mathcal{U}} \cup \{ U_x : x \in X \setminus X' \}. \]

Let \( \emptyset = \{ 0' \cap X : 0' \in \Theta' \} \).

Let \( 0 \in \emptyset \) and \( \{ x, y, z \} \subseteq 0 \).

Then \( \{ x, y, z \} \subseteq \text{some } V \in \mathcal{U} \) or \( \{ x, y, z \} \subseteq \text{some } U_p \), but then \( \{ x, y, z \} \subseteq U_p^3 \cap X^3 \subseteq U \), so \( \{ x, y, z \} \subseteq V' \) for some \( V \in \mathcal{U} \) in any case. This implies that \( \{ x, y, z \} \subseteq V \).

For each \( 0 \in \emptyset \) define \( \mathcal{U}_0 \) as follows: \( 0 = U_{p,q} \) for some \( p, q \in F \), so put \( \mathcal{U}_0 = \{ U_{p,q} : x \in 0 \} \). Let \( \mathcal{W} = \bigcup \{ \mathcal{U}_0 : 0 \in \emptyset \} \). Then \( \mathcal{W} \) is both a 2-star and a \( Q \)-refinement of \( \mathcal{U} \).

First, assume \( U_{p,x} \cup U_{q,y} \neq \emptyset \) for some \( U_{p,x} \) and \( U_{q,y} \) in \( \mathcal{W} \). Then \( x \) and \( y \) are elements of the same \( 0 \in \emptyset \) and hence \( p = q \). Define \( p' \) by \( p'(n) = p(n) + \omega_1 \) (\( n \in \mathbb{N} \)); then \( p < p' \leq x, y \) and \( p' \in X \), so \( p' \in 0 \).

Pick \( u \in \mathcal{U} \) such that \( \{ p', x, y \} \subseteq U \). Since \( U \) is basic (and hence \( \leq \)-convex) and \( U_{p,z} = \{ t : p' \leq t \leq z \} \) for \( z = x, y \), it follows that \( U_{p,x} \cup U_{p,y} \subseteq U \). So \( \mathcal{W} \) is a 2-star refinement of \( \mathcal{U} \). Second, let \( \mathcal{W}' \subseteq \mathcal{W} \) with \( \cap \mathcal{W}' \neq \emptyset \). Then all \( W \in \mathcal{W}' \) are
contained in the same $0 \in \emptyset$, so $\mathcal{U}' = \{ U_{p,q} : x \in A \}$ for some subset $A$ of $0$, where $0 = U_{p,q}$. Define $f$ by $f(n) = \min \{ x(n) : x \in A \}$. Then $\bigcap \mathcal{U}' = U_{p,f}$ is open. So $\mathcal{W}$ is a $Q$-refinement of $\mathcal{U}$. □

It now follows easily that $X$ is finitely-fully normal:

2.3 Corollary. $X$ is finitely-fully normal.

Proof. Let $\mathcal{U}$ be an open cover of $X$. Let $\mathcal{V}_1$ be a 2-star refinement of $\mathcal{U}$, and (inductively) let $\mathcal{V}_{n+1}$ be a 2-star refinement of $\mathcal{V}_n$ ($n \in \mathbb{N}$). Since $X$ is a $P$-space ($G_\delta$'s are open) we can take the common refinement of all $\mathcal{V}_n$; call it $\mathcal{V}$. Let $\mathcal{V}' \subseteq \mathcal{V}$ be finite with $\bigcap \mathcal{V}' \neq \emptyset$. Pick $n \in \mathbb{N}$ such that $2^n \geq |\mathcal{V}'|$. Since $\mathcal{V}$ refines $\mathcal{V}_n$ and since $\mathcal{V}_n$ is a $2^n$-star refinement of $\mathcal{U}$, it follows that $\bigcup \mathcal{V}'$ is contained in some $U \in \mathcal{U}$. □

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