BIFURCATION IN FLUID FLOW NEAR A BOUNDARY SURFACE

Bifurcatie in stromingen van vloeistof of gas in de omgeving van een begrenzend oppervlak

PROEFSCHRIJT

ter verkrijging van de graad van doctor aan de Technische Universiteit Delft, op gezag van de Rector Magnificus Prof.dr ir J. Blaauwendraad, in het openbaar te verdedigen ten overstaan van een commissie door het College van Dekanen aangewezen, op maandag 22 september 1997, 16:00 uur precies

door

Roland Johannes Petrus BOON,

wiskundig ingenieur,

geboren te 's Gravenhage.
Dit proefschrift is goed gekeurd door de promotoren:
Prof.dr ir P.G. Bakker en
Prof.dr ir J.W. Reyn

Samenstelling promotiecommissie:
De Rector Magnificus, voorzitter,
Prof.dr ir P.G. Bakker, promotor,
Prof.dr ir J.W. Reyn, promotor,
Prof.dr H.W. Broer,
Prof.dr ir A.J. Hermans,
Prof.dr ir F.T.M. Nieuwstadt,
Dr A. Vanderbauwhede,
Technische Universiteit Delft,
Technische Universiteit Delft,
Rijksuniversiteit Groningen
Technische Universiteit Delft,
Technische Universiteit Delft,
Universiteit Gent, België

This document was prepared in \$\LaTeX\$ and reproduced by Facilitair Bedrijf TU Delft, Beeld en Grafisch Centrum, from a camera-ready copy supplied by the author.
Copyright © 1997 by Roland J.P. Boon

The research was partially sponsored by ‘Onderzoeksprofileringsruimte’,
project: ‘Topologie van gecompliceerde drie-dimensionale loslaat structuren’.
5.3. Rotational invariance

<table>
<thead>
<tr>
<th>$\mathcal{D}_1$</th>
<th>$\mathcal{D}_2$</th>
<th>$\mathcal{D}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1, K_1$</td>
<td>$u_{300}^{(1)} - t_{300}^{(2)} - t_{210}^{(1)}$</td>
<td>$u_{210}^{(2)} + t_{210}^{(1)}$</td>
</tr>
<tr>
<td></td>
<td>$u_{120}^{(1)} - t_{120}^{(2)}$</td>
<td>$u_{030}^{(2)} + t_{030}^{(1)} + t_{120}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td>$2t_{210}^{(1)} - 3t_{030}^{(1)}$</td>
<td>$-t_{210}^{(1)} - t_{120}^{(2)}$</td>
</tr>
<tr>
<td>$C_2, K_2$</td>
<td>$u_{210}^{(2)} - t_{210}^{(1)}$</td>
<td>$u_{120}^{(1)} + t_{120}^{(1)}$</td>
</tr>
<tr>
<td></td>
<td>$+ 3t_{300}^{(1)} - 2t_{120}^{(1)}$</td>
<td>$+ 2t_{210}^{(2)} - 3t_{030}^{(2)} + t_{120}^{(1)} + 3t_{030}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td>$u_{030}^{(1)} - t_{030}^{(2)} + t_{120}^{(1)}$</td>
<td>$u_{300}^{(2)} + t_{300}^{(1)} - t_{210}^{(1)}$</td>
</tr>
<tr>
<td>$C_3, K_3$</td>
<td>$u_{012}^{(1)} - t_{012}^{(2)} - t_{012}^{(1)}$</td>
<td>$u_{012}^{(2)} + t_{012}^{(1)} + t_{012}^{(2)}$</td>
</tr>
<tr>
<td>$C_4, K_4$</td>
<td>$u_{012}^{(1)} - t_{012}^{(2)} + t_{012}^{(1)}$</td>
<td>$u_{102}^{(2)} + t_{102}^{(1)} - t_{012}^{(2)}$</td>
</tr>
<tr>
<td>$I_1$</td>
<td>$u_{201}^{(1)} - t_{201}^{(2)} - t_{111}^{(1)}$</td>
<td>$u_{111}^{(2)} + t_{111}^{(1)}$</td>
</tr>
<tr>
<td></td>
<td>$+ 2t_{201}^{(2)} - 2t_{021}^{(2)}$</td>
<td>$+ 2t_{201}^{(2)} - 2t_{021}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td>$u_{111}^{(1)} - t_{111}^{(2)}$</td>
<td>$u_{021}^{(2)} + t_{021}^{(1)} + t_{111}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td>$+ 2t_{201}^{(1)} - 2t_{021}^{(1)}$</td>
<td>$+ 2t_{201}^{(1)} - 2t_{021}^{(1)}$</td>
</tr>
<tr>
<td></td>
<td>$u_{021}^{(1)} - t_{021}^{(2)} + t_{111}^{(1)}$</td>
<td>$u_{201}^{(1)} + t_{201}^{(1)} - t_{111}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td>$u_{021}^{(1)} - t_{021}^{(2)} + t_{111}^{(1)}$</td>
<td>$u_{201}^{(1)} + t_{201}^{(1)} - t_{111}^{(2)}$</td>
</tr>
<tr>
<td>$I_2$</td>
<td>$u_{003}^{(1)} - t_{003}^{(2)}$</td>
<td>$u_{003}^{(2)} + t_{003}^{(1)}$</td>
</tr>
<tr>
<td>$I_3$</td>
<td>$u_{003}^{(1)} + t_{003}^{(2)}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.3: $u_3(x) + \text{ad} J_4 B(t_3(x))$, with $u_3(x), t_3(x) \in V_3^2 \cap H_3^2$, re-ordering into subspaces.
Table 5.4: $\mu_0(\xi) + \text{ad} J_{4B}(t_0(\xi))$, with $\mu_0(\xi), t_0(\xi) \in \mathcal{V}_\nu^3 \cap H_0^3$

**Versal deformation**

The second step towards Proposition 5.2 is the computation of the versal deformation within $\mathcal{V}_\nu^3$. Table 5.4 and Table 5.5 present the adjoint operation of $J_{4B}$ on $\mathcal{V}_\nu^3 \cap H_0^3$ and $\mathcal{V}_\nu^3 \cap H_1^3$, respectively. Table 5.4 shows that there is no complementary subspace nor kernel at zero order.

The representative terms in the complementary subspaces $C_1$ and $C_2$ in Table 5.5 are

$$G^\nu_1 = \mu_1 (r \partial_r - x_3 \partial_3) + \mu_2 r \partial_\theta,$$

where $\mu_1 = \frac{1}{2}(\mu_{100}^{(1)} + \mu_{010}^{(2)})$, and $\mu_2 = \frac{1}{2}(\mu_{010}^{(1)} - \mu_{100}^{(2)})$. The term $\mu_2 r \partial_\theta$ equals the 1-jet of the vector field and can be omitted if we reparametrize $\omega$.

The above computation of $G^\nu_1$ in (5.28) prove the deformation part of the topological normal form (5.2) in Proposition 5.2.

**Scaling**

All terms in the direction $r \partial_\theta$ other than $\omega r \partial_\theta$ do not change the topological classification. Substituting the coordinate transformation $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, and $x_3 = z$ into the topological normal form (5.2) results in

$$\dot{r} = r(\mu + g_{10} z + g_{01} r^2),$$

$$\dot{z} = -z(\mu + \frac{2}{3} g_{10} z + 2 g_{01} r^2).$$

(5.29)
5.4. Triple zero eigenvalue

\[
\begin{array}{|c|c|c|}
\hline
C_1 & n_{300}^{(2)} & u_{300}^{(2)} \\
\hline
C_2 & n_{210}^{(2)} & u_{210}^{(2)} \\
 & u_{300}^{(1)} & 0 \\
\hline
C_3 & n_{201}^{(2)} & u_{201}^{(2)} \quad -9F_1n_{200}^{(2)} \\
 & u_{210}^{(1)} & -5F_1n_{200}^{(2)} \\
 & u_{120}^{(2)} & 5F_1n_{200}^{(2)} \\
\hline
C_4 & n_{111}^{(2)} & u_{111}^{(2)} \quad 2F_2n_{200}^{(2)} - \frac{7}{2}F_1n_{110}^{(2)} \\
 & u_{201}^{(1)} & -F_2n_{200}^{(2)} - 2F_1n_{110}^{(2)} \\
 & u_{120}^{(1)} & -5F_1n_{110}^{(2)} \\
 & u_{030}^{(2)} & -\frac{5}{2}F_1n_{110}^{(2)} \\
\hline
\end{array}
\]

Table 5.9: The Lie product of \( G_2^\nu \) and \( K_2^\nu \) (fragment). The labels \( C_i, i \in \{1, 2, 3, 4\} \), refer to the subspaces in Table 5.8.

and

\[
B = \sum_{i=1}^{p} \alpha_i M_i.
\]

In the case of \( J_{3C} \) we have

\[
\text{span}\{M_1, \ldots, M_p\} = \text{span}\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \quad (5.40)
\]

which makes the proposed change of coordinates

\[
\mathcal{X} = e^B \bar{x} = \begin{pmatrix} 1 & 0 & \alpha_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bar{x}. \quad (5.41)
\]

This transformation was first proposed by Gamero et al. [GFR+93] in the general case. Note that the change of coordinates (5.41) the conditions stated in Lemma 4.3 to ensure that the resulting vector field is again an element of \( V^3_3 \).

Let \( \tilde{u}(x, \mu) \) denote the vector field after application of (5.39). The normal form coefficient \( n_{101}^{(2)} \) then reads

\[
n_{101}^{(2)} = \tilde{u}_{101}^{(2)} - 2\tilde{u}_{020}^{(2)} - \frac{1}{2}\tilde{u}_{110}^{(2)}, \quad (5.42)
\]
and similar results hold for the other coefficients in $\mathcal{G}_2'$. By computing $\bar{u}_2(x) = e^{-B_y} y_2(e^B x')$ we get

$$
\bar{u}_1^{(2)} = u_1^{(2)} + 2\alpha_1 u_{200}^{(2)}, \\
\bar{u}_2^{(2)} = u_2^{(2)}, \\
\bar{u}_1^{(2)} = u_1^{(2)}.
$$

(5.43)

The parameter $\alpha_1$ enters other coefficients than just $\bar{u}_1^{(2)}$. However, all these coefficients are part in the image of ad $J_{3C}(\cdot)$-operator, see Table 5.6b. Therefore, under the condition $u_{200}^{(2)} \neq 0$, which generally holds, the parameter $\alpha_1$ can be used to remove $n_1^{(2)}$.

**Versal deformation**

The second step towards Proposition 5.3 involves the computation of the bifurcation terms. A transformation $x \mapsto \ell(x, \mu)$ is constructed so as to remove as much of the perturbation terms $\mu_{ijk}$ as possible. The remaining parameters are part of the complementary subspace of the ad $J_{3C}(\cdot)$-operator, see (5.33). In the formal series of $\ell(x, \mu)$ in $\varepsilon$ let the homogeneous polynomial $t_n(x)$ be the terms of order $\varepsilon^{\alpha+n-1}$, $n \geq 0$. As usual, the coefficients of $t_m(x)$ are denoted by $t_{i,j,k}^{(s)}$, $s \in \{1, 2, 3\}$ and $i, j, k \geq 0$ with $i + j + k = m$.

The result of the adjoint of $J_{3C}$ acting on $t_n(x) \in \mathbb{V}_3^3$ for both $n = 0$ and $n = 1$ is given by Table 5.10a and Table 5.11a, respectively. From the reordering into subspaces given by Table 5.10b and Table 5.11b we conclude that the complementary subspace and the kernel are given by

$$
\mathcal{G}_0' = \mu_{000}^{(2)} \partial_2, \quad \mathcal{K}_0' = \mu_{000}^{(1)} \partial_1,
$$

(5.44)

and

$$
\mathcal{G}_i' = \hat{\mu}_{000}^{(2)} (x_2 \partial_2 - \frac{1}{2} x_3 \partial_3) + \hat{\mu}_{100}^{(2)} x_1 \partial_2, \\
\mathcal{K}_i' = F_1 (x_2 \partial_2 + x_3 \partial_3) + \mu_{001}^{(1)} \partial_1,
$$

(5.45)

where $\hat{\mu}_{000}^{(2)} = \mu_{000}^{(2)}$ in $\mathcal{G}_0'$, and $\hat{\mu}_{010}^{(2)} = \mu_{010}^{(2)} + \mu_{100}^{(1)}$ and $\hat{\mu}_{100}^{(2)} = \mu_{100}^{(2)}$ in $\mathcal{G}_1'$.

As shown in Chapter 3, the coefficients in the kernel $\mathcal{K}_i$ allow for the simplification of the terms in the complementary subspaces $\mathcal{G}_{i+1}'$, $i \geq 0$. Essentially
Wie ben ik?
Een kind van de wind ...  
--Jane Elvira
opgedragen aan
Ome Freek en
Opa Entius
Contents

1 Introduction ................................................................. 5
  1.1 Flow topology ....................................................... 5
  1.2 Topology of local flow patterns about hyperbolic critical points ................................................................. 8
    1.2.1 Critical points away from the boundary surface ............................................................... 9
    1.2.2 Critical points on a boundary surface ................................................................. 14
    1.2.3 Skin-friction critical points ................................................................. 17
    1.2.4 Example flow topology ................................................................. 19
  1.3 Topology of fundamental local flow patterns ................................................................. 21
  1.4 About this thesis ................................................................. 30

2 Vector Fields Describing Fluid Flow ......................................................... 31
  2.1 Flow equations ................................................................. 32
  2.2 Interior critical points ................................................................. 33
    2.2.1 Coefficient relations ................................................................. 33
    2.2.2 Number of coefficients and relations ................................................................. 35
  2.3 Boundary critical points ................................................................. 37
    2.3.1 Flat boundary surface ................................................................. 38
    2.3.2 Number of coefficients and relations ................................................................. 41
    2.3.3 Curved boundary surface ................................................................. 41
    2.3.4 Volume preservation ................................................................. 45
    2.3.5 Coefficient relations after projection ................................................................. 46
  2.4 Skin-friction vector field ................................................................. 50
  2.5 Describing the flow near a boundary surface ................................................................. 53
  2.6 Example: Two-dimensional fundamental local flow patterns ................................................................. 55
  2.7 Discussion ................................................................. 59
3 Bifurcation in General 61
3.1 Bifurcation in Vector fields 62
  3.1.1 Topological equivalence 62
  3.1.2 Structural stability 69
  3.1.3 2-D example. Hyperbolic critical points 70
  3.1.4 Vector fields depending on parameters 70
  3.1.5 Topological normal form 74
3.2 Methods for simplification of vector fields 75
  3.2.1 Center manifold 75
  3.2.2 Normal form 77
  3.2.3 2-D example. Normal form 81
  3.2.4 Alternative complementary subspaces 83
  3.2.5 Finding complementary subspaces 85
  3.2.6 Finding kernel subspaces 91
  3.2.7 Transformation theory 92
  3.2.8 2-D example. Normal form (continued) 98
3.3 Versal deformation of vector fields 101
  3.3.1 Versal deformation of families of matrices 102
  3.3.2 Versal deformation of nonlinear vector fields 113
  3.3.3 2-D example. Topological normal form 116
  3.3.4 k-Determinacy and transversality 117
  3.3.5 2-D example. 2-determinacy 119
3.4 Analysis of the dynamics 121
3.5 Discussion 121

4 Bifurcation in Fluid Flow 123
4.1 2D local flow patterns 124
  4.1.1 Jordan normal form 125
  4.1.2 Normal form and co-dimension 125
  4.1.3 Hyperbolicity and versal deformations 129
4.2 Parameter dependent vector field expansions 132
4.3 Jordan normal form 136
  4.3.1 Eigenvalues 137
  4.3.2 Linear coordinate transformations 138
  4.3.3 Decomplexification 139
  4.3.4 Eigenvector basis 140
  4.3.5 2-D example. Jordan normal form 143
4.4 Normal form 144
  4.4.1 Nonlinear transformations 144
  4.4.2 2-D example. Normal form 149
4.5 Miniversal deformations .................................................. 154
4.5.1 Perturbation-parameter reduction scheme ......................... 154
4.5.2 Deformation of matrices .............................................. 156
4.5.3 2-D example. Miniversal deformation of a matrix ............... 157
4.5.4 2-D example. Miniversal deformation of a vector field ....... 159
4.5.5 2-D example. Co-dimension 3 degeneracy ......................... 161
4.5.6 Miniversal deformations within a set of vector fields .......... 161
4.5.7 2-D example. Topological normal form ........................... 162
4.6 Projection of the series onto a polynomial ........................ 163
4.6.1 Congruence: more of the same ..................................... 165
4.6.2 2D-example. Co-dimension 3 degeneracy (continued) .......... 168
4.7 2-D local flow patterns ................................................. 169
4.8 Discussion ..................................................................... 170

5 Topological Normal Forms for Fluid-flow Vector Fields ........... 173
5.1 Main results ..................................................................... 174
5.2 Single zero eigenvalue ...................................................... 176
5.3 Rotational invariance ....................................................... 183
5.4 Triple zero eigenvalue ...................................................... 192
5.5 Mirror symmetry ............................................................. 204
5.6 Higher co-dimensions ....................................................... 212
5.7 Discussion ..................................................................... 212
  5.7.1 About the vorticity transport equation ............................ 212
  5.7.2 About viscosity .......................................................... 215

6 Degenerate Local Flow Patterns with Mirror Symmetry .......... 217
6.1 Skin-friction patterns ....................................................... 218
6.2 Flow patterns in the plane of symmetry .............................. 222
6.3 Third invariant manifold ................................................... 226
6.4 Discussion ..................................................................... 231

7 Summary, Conclusions and Recommendations ...................... 233
7.1 Summary ..................................................................... 233
7.2 Conclusions and Recommendations ................................... 235

A Conjugating Matrices ......................................................... 237
A.1 Jordan normal forms ....................................................... 237
A.2 Conjugating matrices ...................................................... 240
## Contents

### B Some Basic Definitions and Theorems
- B.1 Implicit and Inverse Function Theorems 247
- B.2 Taylor expansion 248
- B.3 A Theorem by Andronov *et al.* 250

### C Third-order coefficient relations
- C.1 Computing the generator 253
- C.2 Computing the coefficient relations 255

Samenvatting 267

Acknowledgements 269

Curriculum Vitae 271
Chapter 1

Introduction

*I'll have to see it first, then maybe I'll believe it.*

1.1 Flow topology

One of the basic things a researcher of the motion of fluids wants to know is: "How does the flow look like?" The data gathered from computations or experiments need to be evaluated, interpreted and understood. Drawing a set of well-chosen streamlines or stream surfaces generally provides a qualitative overview of the geometrical features of a flow and helps to check if these data is conform expectations. However, flows in real life are three-dimensional and unsteady which makes them rather complicated, at least geometrically. Consequently, it is difficult to find those streamlines or stream surfaces which highlights the desired information.

Looking at flows around aerodynamic bodies, one might distinguish five types of regions of particular interest; regions of separation, regions of attachment, stagnation regions, regions with back flow, and regions with recirculation. Such regions can all be found near boundary surfaces and typically have critical points (definition follows later). Near each critical point those streamlines and stream surfaces which characterize the flow locally and those streamlines which connect critical points are relevant information for the flow topology.

How can one identify critical points, characteristic streamlines or stream surfaces? Consider the two oil-streak flow-visualizations in Fig. 1.1. They show the oil-flow pattern after a run in a blown-down wind tunnel on the boundary surface of a hemispheric cylinder at a 5° angle of attack in high subsonic flow,
Figure 1.1: Oil-flow pattern on a hemispheric cylinder, Mach number $M_\infty = 0.85$ and angle of attack $\alpha = 5^\circ$. Courtesy High Speed Aerodynamics Laboratory, Faculty of Aerospace Engineering, Delft University of Technology, photo: Frank Vossen and Frits Donker Duyvis.
$M_\infty = 0.85$. Oil-streak lines form a visualization of the skin-friction lines. These lines are tangent to the skin-friction vector on the boundary surface. The direction of the oil flow on the skin-friction lines gives a strong indication of the direction of the flow above the boundary surface.

The major part of the surface of the hemispheric cylinder is covered with oil-streak lines which are nearly parallel. There are, however, oil-streak lines which seem to intersect. For example, in Fig. 1.1a, there seems to be a cross-like structure with one of its axis in the plane of symmetry. In Fig. 1.1b, there is a region visible with a kind of spiralling motion. Fig. 1.2 presents an sketch of both.

The arrows on the lines in Fig. 1.2 indicate the presumed direction of the flow of oil. Near the transition from the blunt nose and the cylinder, there is a thick white region of oil visible in Fig. 1.1 that is indicated as a thick gray area in Fig. 1.2. Before the experiment the oil was an evenly-spread film covering the surface. Therefore, the oil must have flown towards that region. This fixes the direction of the oil flow in the ‘plane of symmetry’. We consider the velocity
vector field above the boundary surface to be continuous. This assumption helps to find the direction on the other lines in Fig. 1.2.

In a continuous vector field, trajectories can only intersect in points where the vector vanishes. Therefore, skin-friction lines only intersect in points where the skin-friction vector vanishes. Similarly, streamlines only intersect at points where the velocity vector vanishes (the so-called stagnation points). Points of a vector field where the vector vanishes are called critical points.

1.2 Topology of local flow patterns about hyperbolic critical points

It helps the determination of a flow topology considerably if we know the theoretically-possible fundamental local flow patterns about a critical point. These fundamental local flow patterns then form building blocks from which complicated flow patterns are constructed.

Already in 1958, Oswatitsch [Osw58] published the first systematic classification of the three-dimensional fundamental local flow patterns about a critical point on the boundary surface in a steady flow. A more recent classification by Chong, Perry and Cantwell [CPC90] basically still follows his method of analysis. The method of analysis is also useful for other applications. For example, Lugt [Lug87] applied it to viscous free surfaces while Brøns [Brø94] applied it to interfacial flows.

The central idea is that the velocity vector $\mathbf{u}(\mathbf{x})$ is a vector field that assigns a vector to a fluid particle at location $\mathbf{x}$ in some region in three-dimensional space. If $\mathbf{x}(t)$ denotes the position of a fluid particle at time $t$, then $\frac{d\mathbf{x}(t)}{dt}$ is a tangent vector to the path of the particle (i.e., the streamline) at that position. This tangent vector should be along the velocity vector. The following set of three ordinary differential equations then results for the streamlines in a steady flow,

$$\dot{\mathbf{x}}(t) = \mathbf{u}(\mathbf{x}(t)), \quad \mathbf{u} : U \rightarrow \mathbb{R}^3,$$

where the overdot represents the differentiation in time, and $U \subset \mathbb{R}^3$ is some region surrounding a certain point of interest, say $\mathbf{x}_0$. The point $\mathbf{x}_0$ can be located either on or above the boundary surface. Let $\partial B$ denote the boundary surface. In a critical point $\mathbf{x}_0 \notin \partial B$ in the velocity vector field, $\mathbf{u}(\mathbf{x}_0) = 0$. Note that since we assume steady flow, $\mathbf{u}$ does not depend on the time parameter $t$.

For viscous fluids, the no-slip condition holds which requires the velocity vector to vanish everywhere on the boundary surface. This condition makes
every point on the boundary surface a critical point. As a result, the vector field (1.1) does not provide useful information on the boundary surface. Fortunately, the skin-friction vector field does.

Let \( \Omega \subset \partial B \) be a smooth region surrounding a point \( \bar{x}_0 \) on the boundary surface \( \partial B \). The skin-friction vector field \( \tau(\bar{x}) \) in

\[
\dot{x}(t) = \tau(\bar{x}(t)), \quad \tau: \Omega \to \mathbb{R}^3,
\]

is defined such that for all \( \bar{x} \in \Omega \), \( \tau(\bar{x}) \) lies in the tangent plane to the boundary surface at \( \bar{x} \). This means that integration of (1.2) leads to trajectories in \( \Omega \). These trajectories are the skin-friction lines. In a (boundary) critical point \( \bar{x}_0 \in \partial B \) in the skin-friction vector field, \( \tau(\bar{x}_0) = 0 \).

The skin-friction vector \( \tau \) is related to the velocity vector \( u \) via shear-stress the precise nature of which we shall elaborated upon in Chapter 2. In this section, we assume a flat boundary surface \( \partial B \). Let \( (x_1 x_2 x_3) \) be a local Cartesian coordinate system with its origin taken in the point \( \bar{x}_0 \). The orientation is such that \( \partial B \) is located at \( x_3 = 0 \) and that the third base vector points into the flow.

Let \( u^{(\ell)}(\bar{x}), \ell = 1, 2, 3 \), denote the three components of the velocity vector \( u(\bar{x}) \). In that case the skin-friction vector is given by

\[
\tau(\bar{x}) = \mu \begin{pmatrix}
\partial_3 u^{(1)}(\bar{x}) \\
\partial_3 u^{(2)}(\bar{x}) \\
0
\end{pmatrix},
\]

where \( \partial_3 \) denotes a differentiation with respect to \( x_3 \), and \( \mu \) denotes the kinematic viscosity.

The velocity vector \( u(\bar{x}) \) and the skin-friction vector \( \tau(\bar{x}) \) have to obey the flow equations. To make an inventory of local flow patterns about critical points, one needs to generate solutions of the flow equations. The question is how.

### 1.2.1 Critical points away from the boundary surface

Oswatitsch’s idea was to concentrate on approximations of solutions. To this end, he represented the velocity vector \( u(\bar{x}) \) in (1.1) about a critical point \( \bar{x}_0 \) by a finite Taylor series. This series is substituted in the flow equations. As a result, a number of coefficients in the series are related.

If the critical point is located away from the boundary surface, the resulting first-order Taylor-polynomial (or linearized) vector field can then be written as

\[
\dot{\xi} = A \xi,
\]

(1.4)
where $\xi = x - x_0$ translates the critical point to the origin, and $A$ is the $3 \times 3$ Jacobian matrix of the velocity vector $u(x)$ in the critical point,

$$A = (A_{ij}) \equiv \frac{\partial u}{\partial x}(x_0).$$

(1.5)

The critical point $\xi = 0$ of the vector field (1.4) is called hyperbolic if every eigenvalue of the Jacobian matrix $A$ has a nonzero real part. The (nonlinear) vector field (1.1) has a hyperbolic critical point in $x_0$ if the linearized vector field about $x_0$ has a hyperbolic critical point.

As a result of the flow equations (and the continuity equation in particular), the coefficients in the matrix $A$ have to satisfy the relation

$$A_{11} + A_{22} + A_{33} = 0.$$ 

(1.6)

Since the sum of the eigenvalues of $A$ equals the trace of that matrix we also find that

$$\lambda_3 = - (\lambda_1 + \lambda_2).$$ 

(1.7)

The notation $\lambda_1$, $\lambda_2$, and $\lambda_3$ for the eigenvalues is chosen such that if two eigenvalues form a complex conjugate pair they are denoted by $\lambda_1$ and $\lambda_2$. The result is that $\lambda_3$ is always real.

Chong, Perry and Cantwell [CPC90] have made a general classification of three-dimensional local flow patterns described by (1.4) using three invariants of the Jacobian matrix. These invariants are given as follows.

The eigenvalues of a matrix $A$ are determined by the characteristic equation,

$$\det [A - \lambda I] = 0,$$ 

(1.8)

where $I$ is the identity matrix. If $A$ is a $3 \times 3$ matrix, (1.8) leads to the following cubic polynomial equation

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0,$$ 

(1.9)

where,

$$P \equiv -\text{tr}A = - (\lambda_1 + \lambda_2 + \lambda_3),$$ 

(1.10)

$$Q \equiv \det A_1 + \det A_2 + \det A_3 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3,$$ 

(1.11)
and

\[ R = -\det A = -\lambda_1 \lambda_2 \lambda_3, \]  

(1.12)

see Reyn [Rey64]. The three \( 2 \times 2 \) submatrices \( A_i, i \in \{1, 2, 3\} \), are found by removing the \( i \)th column and the \( i \)th row from \( A \). The variables \( P, Q \) and \( R \) are three invariants of the matrix \( A \) (under affine transformations).

As a result of the eigenvalue relation (1.7) the first invariant \( P = 0 \). The eigenvalues are given by

\[
\begin{align*}
\lambda_1 &= -\frac{1}{2}(A + B) + i\frac{\sqrt{3}}{2}(A - B), \\
\lambda_2 &= -\frac{1}{2}(A + B) - i\frac{\sqrt{3}}{2}(A - B), \\
\lambda_3 &= A + B,
\end{align*}
\]

(1.13)

e.g. see Abramowitz and Stegun [AS70], where

\[
A = \left(-\frac{1}{2}R + \sqrt{\Delta}\right)^{\frac{1}{2}}, \quad B = \left(-\frac{1}{2}R - \sqrt{\Delta}\right)^{\frac{1}{2}}, \quad \Delta = \frac{1}{4}R^2 + \frac{1}{27}Q^3.
\]

The surface \( \Delta = 0 \) divides the real roots of (1.9) from the complex roots: If \( \Delta > 0 \), there exist one real eigenvalue and a pair of complex conjugate eigenvalues. If \( \Delta = 0 \), all eigenvalues are real and at least two are equal. If \( \Delta < 0 \), all eigenvalues are real.

The \( R-Q \) chart in Fig. 1.3 shows all possible local flow patterns about a hyperbolic critical point. The gray lines represent the streamlines in the \( \lambda_1, \lambda_2 \) plane. This plane contains either the two-dimensional unstable subspace or the two-dimensional stable subspace.

The three-dimensional space \( \mathbb{R}^3 \) can be represented as the direct sum of three subspaces denoted \( E^s \), \( E^u \), and \( E^c \), which are defined as follows:

\[
\begin{align*}
E^s &= \text{span} \{ \xi_1, \cdots, \xi_{n_s} \}, \\
E^u &= \text{span} \{ \xi_{n_s+1}, \cdots, \xi_{n_s+n_u} \}, \quad n_s + n_u + n_c = 3, \\
E^c &= \text{span} \{ \xi_{n_s+n_u+1}, \cdots, \xi_3 \},
\end{align*}
\]

(1.14)

where \( \{ \xi_1, \cdots, \xi_{n_s} \} \) are the (generalized) eigenvectors of \( A \) corresponding to the \( n_s \) eigenvalues having negative real part, \( \{ \xi_{n_s+1}, \cdots, \xi_{n_s+n_u} \} \) are the (generalized) eigenvectors of \( A \) corresponding to the \( n_u \) eigenvalues having positive real part, and \( \{ \xi_{n_s+n_u+1}, \cdots, \xi_3 \} \) are the (generalized) eigenvectors of \( A \) corresponding to the \( n_c \) eigenvalues having zero real part. \( E^s \), \( E^u \), and \( E^c \) are referred to as the stable, unstable and center subspaces, respectively.

If \( \lambda_1 \) and \( \lambda_2 \) are a pair of complex conjugate eigenvalues, \( \rho \pm i\omega, \rho > 0, \omega \neq 0 \), the third eigenvalue is given by \( \lambda_3 = -(\lambda_1 + \lambda_2) = -2\rho < 0 \). Then \( A \)
Figure 1.3: Topology of local flow patterns about a hyperbolic critical point away from a boundary surface.

Figure 1.4: The geometry of $E^u$ and $E^s$. 
has three real eigenvectors $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$, which can be used as the columns of a matrix $M$ in order to transform $A$ as follows

$$J \equiv \begin{pmatrix} \rho & \omega & 0 \\ -\omega & \rho & 0 \\ 0 & 0 & -2\rho \end{pmatrix} = M^{-1}AM.$$  \hspace{1cm} (1.15)$$

Thus, the trajectories of the linear vector field (1.4) through an initial position $\xi_0$ are given by

$$\xi(t) = Me^{Jt}M^{-1}\xi_0$$

$$= M \begin{pmatrix} e^{\rho t} \cos \omega t & e^{\rho t} \sin \omega t & 0 \\ -e^{\rho t} \sin \omega t & e^{\rho t} \cos \omega t & 0 \\ 0 & 0 & e^{-2\rho t} \end{pmatrix} M^{-1}\xi_0.$$  \hspace{1cm} (1.16)$$

It should be clear that $E^u = \text{span} \{\varepsilon_1, \varepsilon_2\}$ is an invariant plane of streamlines that decay exponentially to zero as $t \to -\infty$, and $E^s = \text{span} \{\varepsilon_3\}$ is an invariant plane of streamlines that decay exponentially to zero as $t \to +\infty$ (see Fig. 1.4).

It can be proven that there are at most three invariant planes of streamlines through a hyperbolic critical point. The invariant planes become mutually perpendicular after an affine coordinate transformation, $\xi \mapsto M\xi$, where $M$ is a matrix whose columns are the generalized eigenvectors of the Jacobian matrix $A$. A hyperbolic critical point has, as a result of the eigenvalue relation (1.7), either a stable invariant subspace with dimension 2 and an unstable invariant subspace with dimension 1, or a stable invariant subspace with dimension 1 and an unstable invariant subspace with dimension 2. By definition, hyperbolic critical points do not contain a center subspace.

With these considerations in mind it is easy to understand the local flow patterns in Fig. 1.3. The gray lines represent the streamlines in the $\lambda_1$-$\lambda_2$ plane whereas the black lines represent the streamlines above that plane. Chapter 3 explains how to interpret these local flow patterns of the linearized vector field (1.4) in relation to the original nonlinear vector field (1.1).

Finally, to answer our question on page 5, characteristic streamlines can be approximated by drawing trajectories through points in the neighborhood of a critical point on the stable and unstable invariant subspaces. This procedure is a good approximation because in general the invariant manifolds\(^1\) of a critical point are tangent to its corresponding invariant subspaces. Also, two linearly-independent vector contained in the subspaces can be used to span a characteristic stream surface.

\(^1\)For now, it suffices to think of a manifold as a possibly curved surface or line.
1.2.2 Critical points on a boundary surface

If expansion of the velocity vector \( \mathbf{u}(\mathbf{x}) \) takes place about a point \( \mathbf{x}_0 \) on a flat boundary surface \( x_3 = 0 \), we get

\[
\mathbf{u}(\mathbf{x}_0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.17}
\]

\( \xi \) as a result of the no-slip condition. Because the Jacobian matrix \( \mathbf{A} \) has three zero eigenvalues, \( \mathbf{x}_0 \) is a nonhyperbolic critical point in the velocity vector field.

The coefficients \( A_{13} \) and \( A_{23} \) are the zero-order terms in the Taylor series expansion of the components in the skin-friction vector (1.3). Consequently, in a critical point of the skin-friction vector field, \( A_{13} = A_{23} = 0 \). If we expand the velocity vector field up to second order about a boundary critical point, we get

\[
\dot{\xi} = \xi_3 \tilde{\mathbf{A}}_1 \xi, \tag{1.18}
\]

where

\[
\tilde{\mathbf{A}} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & -\frac{1}{2} (\tilde{A}_{11} + \tilde{A}_{22}) \end{pmatrix}. \tag{1.19}
\]

Notice that the three components of the vector field (1.18) have a common factor \( \xi_3 \). This factor only scales the velocity vector above the boundary surface. For this reason, the nonhyperbolicity can be removed and the linear vector field

\[
\dot{\xi} = \tilde{\mathbf{A}}_1 \xi \tag{1.20}
\]

can be used to study the local flow pattern above the boundary surface.

The relation satisfied by coefficients in the matrix \( \tilde{\mathbf{A}}_1 \) in (1.19) is given by

\[
\lambda_3 = -\frac{1}{2} (\lambda_1 + \lambda_2). \tag{1.21}
\]

We once again employ the notation introduced around (1.7) that \( \lambda_1 \) and \( \lambda_2 \) denoted the possible pair of complex conjugate eigenvalues. It is easy to see that the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) then stem from a \( 2 \times 2 \) submatrix \( \mathbf{B} \) that is the restriction of \( \tilde{\mathbf{A}}_1 \) to the boundary surface,

\[
\mathbf{B} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix}. \tag{1.22}
\]
The corresponding eigenvectors \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) span the boundary surface \( \xi_3 = 0 \). The matrix \( \mathbf{B} \) in effect is the Jacobian matrix of the two-dimensional skin-friction vector field (1.2). Therefore, the vector field (1.20) also enables us to study the local skin-friction pattern on the boundary surface \( \xi_3 = 0 \).

The eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are found through the characteristic equation of

\[
\det(\mathbf{B} - \lambda \mathbf{I}) = \det \begin{pmatrix}
\tilde{A}_{11} - \lambda & \tilde{A}_{12} \\
\tilde{A}_{21} & \tilde{A}_{22} - \lambda
\end{pmatrix} = 0,
\]

which reads

\[
\lambda^2 - (\tilde{A}_{11} + \tilde{A}_{22})\lambda + \tilde{A}_{11}\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{12} = 0.
\]

This quadratic equation (1.23) has two invariants

\[
p = \tilde{A}_{11} + \tilde{A}_{22} = \lambda_1 + \lambda_2, \quad q = \tilde{A}_{11}\tilde{A}_{22} - \tilde{A}_{21}\tilde{A}_{12} = \lambda_1\lambda_2
\]

(compare with the cubic equation (1.9)). Solving the characteristic equation (1.23) produces the first two eigenvalues of \( \mathbf{A} \). The third eigenvalue can simply be read off the third (and last) row of \( \mathbf{A} \) in (1.19). Together they read

\[
\lambda_1 = \frac{1}{2}p - \frac{1}{2}\sqrt{D}, \quad \lambda_2 = \frac{1}{2}p + \frac{1}{2}\sqrt{D}, \quad \lambda_3 = -\frac{1}{2}p,
\]

where \( D \equiv p^2 - 4q \) is the discriminant of the quadratic equation (1.23). Notice that indeed the eigenvalue relation (1.21) holds.

The \( p-q \) chart Fig. 1.5 sketches the topology of local flow patterns about a hyperbolic critical point on a boundary surface. The gray lines in Fig. 1.5 represent the skin-friction lines while the black lines represent the streamlines.

The \( p-q \) chart features two parabolas; \( D = 0 \) on which \( \lambda_1 \) and \( \lambda_2 \) coincide, and \( E \equiv \frac{1}{2}p^2 + q = 0 \) on which \( \lambda_3 \) coincides with \( \lambda_1 \) or \( \lambda_2 \).

From (1.25), it is easy to verify that \( \lambda_1 \) and \( \lambda_2 \) indeed coincide on the parabola \( D = 0 \). The parabola also separates the complex conjugate eigenvalues from the real eigenvalues.

On the parabola \( E = 0 \), \( D = 4p^2 \), with which (1.25) can be written as

\[
\lambda_1 = \frac{1}{2}\sqrt{p^2 - p}, \quad \lambda_2 = \frac{1}{2}\sqrt{p^2 + p}, \quad \lambda_3 = -\frac{1}{2}p.
\]

Therefore, the right branch of the parabola \( E = 0 \) \( (p > 0) \) contains the critical points with \( \lambda_1 = \lambda_3 \) where the left branch \( (p < 0) \) contains the critical points with \( \lambda_2 = \lambda_3 \).

We observe in Fig. 1.5 that for negative \( p = \lambda_1 + \lambda_2 \), i.e., for positive \( \lambda_3 \), fluid moves away from the boundary surface (in the direction of the third eigenvector).
Figure 1.5: Topology of local flow patterns about a hyperbolic critical point on a boundary surface. Black lines represent streamlines whereas gray lines represent skin-friction lines.
In other words, flow separation occurs near the critical point. On the other hand, for positive $p$, i.e., for negative $\lambda_3$, we see fluid moving towards the critical point (in the direction of the third eigenvector). In that case, flow attachment occurs near the critical point.

The nonhyperbolic critical points are found in the $p$-$q$ chart on the $p$-axis and on the $q$-axis. Outside the $p$-axis and the $q$-axis the three eigenvalues are nonzero. On the $p$-axis, $D = p^2$. Hence, we can write the eigenvalues as

$$\lambda_1 = \frac{1}{2}p - \frac{1}{2} \mid p \mid, \quad \lambda_2 = \frac{1}{2}p + \frac{1}{2} \mid p \mid, \quad \text{and} \quad \lambda_3 = -\frac{1}{2}p.$$  

If $p > 0$, $\lambda_1 = 0$, whereas if $p < 0$, $\lambda_2 = 0$. On the other hand, on the $q$-axis, $\lambda_3 = 0$. Finally, in the origin $(p, q) = (0, 0)$, the three eigenvalues vanish.

### 1.2.3 Skin-friction critical points

The flow equations nor the no-slip conditions yield a relation between the coefficients in the matrix $B$ in (1.22). Therefore, the possible hyperbolic critical points in the skin-friction field on the boundary surface are precisely all the possible hyperbolic critical points in the plane (i.e., in two-dimensional space) known from the qualitative theory of ordinary differential equations, e.g., see Andronov et al. [ALGM73]. These well-known critical points are usually classified in a $p$-$q$ chart, see Fig. 1.6. As before $p$ equals the sum of the eigenvalues, while $q$ equals their product.

The third eigenvalue of $\hat{A}$ in (1.19) is linearly related to the first and the second eigenvalue via (1.21). As a result, the three-dimensional local flow pattern about the critical point on the boundary surface follows directly once we know the two eigenvalues $\lambda_1$ and $\lambda_2$ of the matrix $B$.

Three important hyperbolic critical points in the plane are the saddle, the node, and the spiral (or focus). A saddle occurs if the eigenvalues are of opposite sign, a node occurs if the signs are equal, and a spiral occurs if the two eigenvalues are complex conjugate. Nonhyperbolic critical points are located on the lines $p = 0$ ($q > 0$) and $q = 0$.

If the sign of the real part of the two eigenvalues are both positive (negative), the critical points is said to be unstable (stable). All trajectories tend to the critical point of a stable (unstable) spiral or a stable (unstable) node for $t \to +\infty$ ($t \to -\infty$). The trajectory through (almost) any point in the neighborhood of a saddle critical point tends away for both $t \to +\infty$ and $t \to -\infty$. Thus the saddle is also an unstable critical point. Only four trajectories tend to a saddle

---

2 In this field of research, the name phase portrait is used instead of flow pattern.
critical point. These lines are called the *separatrices* and are tangent to the two eigenvectors of the Jacobian matrix in the critical point.

If the eigenvalues of the matrix $B$ coincide, then the critical point is called a *star node* if $B$ has two eigenvectors whereas it is called a *inflected node* if only one eigenvector can be found. These nodes are located in the $p$-$q$ chart on the parabola

\[ D \equiv p^2 - 4q = (\lambda_1 - \lambda_2)^2 = 0. \]

The unstable star and inflected nodes are found on the right branch of the parabola whereas the stable star and inflected nodes are found on the left branch. Fig. 1.6 sketches only unstable stars and inflected node. Stable stars and inflected nodes are found by a reversal of the arrows.

**Bibliographical notes**

Maskell [Mas55] first attempted a rational interpretation of flow separation using oil-flow patterns combined with topological aspects of limiting streamlines. Leg-
1.2. Topology of local flow patterns about hyperbolic critical points

Endre [Leg56] showed the equivalence of limiting streamlines and skin-friction lines. His most significant observations are that separation occurs near a point with vanishing skin friction and that the number of qualitatively different flow separation patterns about such a point is limited; only a node, spiral or saddle point are found on the boundary surface.

Oswatitsch [Osw58] first derived the complete flow topology for the three-dimensional case by studying the qualitative properties of local flow patterns about a point of separation located on a boundary surface based on Taylor series expansion of the velocity vector field. The origin of the coordinate system and of expansion is taken to be the point of separation, i.e., there where the skin-friction vanishes. Lighthill [Lig63] showed that flow patterns on the boundary surface of completely-submerged, simple three-dimensional aerodynamic bodies obey a topological law; the number of nodal points (i.e., nodes and spirals) exceeds the number of saddle points by exactly two. Later, Hunt et al. [HAPW78] introduced similar topological laws for more complicated aerodynamic bodies with possibly surface-mounted obstacles.

1.2.4 Example flow topology

With the classifications of the local skin-friction patterns in Fig. 1.6 and the three-dimensional local flow patterns in Fig. 1.5 in mind, consider the oil-flow pattern in Fig. 1.1 together with our interpretation in Fig. 1.2. In the region labelled 'cross-like structure' there is a saddle whereas the region labelled 'spiral-like motion' seems to contain a (unstable) spiral. We say 'seems to' because spirals and nodes are hard to distinguish in oil-flow visualizations.

Section 1.1 said that flow topology required a set of 'well chosen (later called characteristic) streamlines'. The characteristic streamlines restricted to the boundary surface are those skin-friction lines which are separatrices of the critical points. Fig. 1.2 depicts separatrices as thick black lines. The figure also shows two saddle separatrices emanating from an unstable spiral. Here we have an example of 'streamlines which connect critical points'.

With the aid of Fig. 1.5 it is possible to reconstruct much of the flow pattern above the boundary surface of the hemispheric cylinder. However, in an oil-flow visualization, the sign of the third eigenvalue of a saddle is not immediately clear. More information is necessary to establish whether flow separation or flow attachment takes place.

Our oil-flow pattern interpretation in Fig. 1.2 shows two saddles in the plane of symmetry with a connecting skin-friction line. Similar oil-flow patterns are found by Bippes and Turk [BT83].

However, the flow about a blunt nose is well known. Already in 1955, Maskell
[Mas55] presented a sketch as in Fig. 1.7 of the three-dimensional flow pattern about a saddle of separation, which he called bubble-type separation. Note that all skin-friction lines on the boundary surface $\partial B$ (indicated as gray lines) converge to a line called the separation line. Oil accumulates about such a line as demonstrated by the thick region of oil on the 'nose' on the hemispheric cylinder in Fig. 1.1.

Oil tends to flow towards a point of separation and away from a point of attachment. We conclude from the oil-flow pattern in Fig. 1.1a that the saddle near the transition from the blunt nose and the cylinder is a point of separation and that the other saddle is a point of attachment. Fig. 1.8 shows two suggestions of the flow pattern in the plane of symmetry above the boundary surface. Both flow patterns require an additional spiral. The directions in Fig. 1.8 are consistent with the direction of the oncoming flow.

An important question is whether the separatrix from the saddle of attachment $S_a$ or from the saddle of separation $S_s$ tends to the spiral critical point. In Fig. 1.8a, the spiral $F_u$ is unstable meaning that fluid flows towards the plane of symmetry. In the opposite case, Fig. 1.8b, the spiral $F_s$ is stable and the situation is reverse; fluid flows away from the plane of symmetry.

It is hard to decide which flow pattern actually occurs based solely on the oil-flow pattern visualization in Fig. 1.1. Further experimental or computational investigation can decide which of the two flow patterns in (1.8) actually occurs, e.g. see Bippes and Turk [BT83], and Ying, Schiff and Steger [YSS87].
1.3  Topology of fundamental local flow patterns

If characteristic flow parameters such as the Mach number, the Reynolds number, the angle-of-attack, or geometry parameters are varied, the location and possibly the number of critical points change. Given a certain aerodynamic configuration one would like to predict the boundaries separating regimes in which the flow topology remains fixed under variation of the characteristic flow parameters.

In the early eighties Peake and Tobak published a series of inventory studies on experimental and computational results concerning changes in the flow topology of three-dimensional separated flow around various types of aerodynamic configurations, see [TP79], [PT80], [TP81], [TP82], [PT82a], [PT82b], [PT82c]. One of their conclusion is that the topology of critical points, discussed in the previous section, coupled with simple topology laws, are notions which "( ... ) enable us to create sequences of plausible flow structures, to deduce mean flow characteristics, expose flow mechanisms, and to aid theory and experiment where lack of resolution in numerical calculations or wind-tunnel observation causes imprecision in diagnosing the three-dimensional flow features." [PT80], page 1. These notions "( ... ) rest on an exceedingly simple theoretical base. If we ask now how the theoretical base might be extended, particularly in the quantitative direction, we are led immediately to the main question - as the parameters of the problem are varied, can one map out in advance the boundary separation regimes within which the ( ... ) flow structure remains fixed?" [PT80], page 15. Peake and Tobak suspected that the ‘main question’ can be attacked best by posing that question within the framework of bifurcation theory.

The extension of the ‘theoretical base’ consisted of finding local approxima-
tions of solutions of the flow equations which describe possibly more than one critical point. A critical point analysis of the resulting polynomial vector fields then produces a topology of fundamental local flow patterns. More important, variation of the parameters in such polynomial vector fields could clarify the underlying principle behind changes from one fundamental local flow pattern to another.

A typical example of a fundamental local flow pattern with more than one critical point is the bubble-type separation found on the front part of the hemispheric cylinder. Another important fundamental local flow pattern, one that does not have any critical points, is free-shear layer separation. A bifurcation analysis that uses local approximations of solutions of the flow equations should be able to produce both.

The most simple fundamental local flow patterns are those found about one hyperbolic critical point. The previous section showed that, from a theoretical point of view, their topology is completely understood. That is far from true for fundamental local flow patterns with more than one critical point.

A description of local flow patterns with more than one critical point requires third or higher order Taylor-polynomial approximations of solutions of the flow equations (and the no-slip condition). Because of the Taylor approximation, these critical points should all be present in an arbitrary small neighborhood of the point of expansion. Also, at third or higher order, we cannot expect a priori that a flat boundary surface is a valid approximation.

With these thoughts in mind, we discuss the results obtained on the subject, more or less in chronological order.

Dallmann [Dal83] [Dal85] [Dal88] was the first to present a topology of three-dimensional flow patterns with a plane of symmetry near a flat boundary surface, see Fig. 1.9. Flow patterns with symmetry are of great importance since most aerodynamic bodies are symmetric. The flow patterns in Fig. 1.9 feature several critical points. Some of these flow patterns are known to occur in computations and experiments [DHR*90], [DG94], and [Roc93].

Dallmann studied several critical points using a Taylor series expansion of the velocity vector truncated at third order which results in a second-order polynomial vector field (compare with (1.18) and (1.20)). However, the critical points found this way should not be taken into account in the analysis of the flow pattern near the point of Taylor expansion. They are a by-product of truncating the (Taylor) series.

Perry [Per84] in co-operation with Hornung [Hor83] presented a special example
Figure 1.9: Flow patterns with mirror symmetry. Possible skin-friction patterns are on the left. Possible streamline patterns in the plane of symmetry are on the top. Circles indicate admissible combinations. Taken from [Dal83]
of a velocity-vector series expansion. Their vector field reads

\[
\begin{align*}
\dot{x}_1 &= \mu - ex_1x_2^2 - \frac{1}{24}\mu f x_3^3 \\
\dot{x}_2 &= fx_2 + gx_3 + ex_1^2x_3 \\
\dot{x}_3 &= -\frac{1}{2}fx_3 - \frac{1}{2}ex_1^2x_3 + \frac{1}{2}ex_2^2x_3
\end{align*}
\]

where \(\mu\) is a small perturbation parameter that induces shearing-effects. The matrix shown in (1.26) is Jacobian matrix of the unperturbed vector field in the origin (i.e., with \(\mu = 0\)). It is necessary according to Hornung to include the perturbation term \(-\frac{1}{24}\mu f x_3^3\) in the \(\dot{x}_1\)-equation in order to balance the inertia term introduced by the superimposed shearing in that equation.

The unperturbed vector field (1.26) has an infinite number of critical points on the \(x_1\)-axis and is symmetric in the plane \(x_1 = 0\). From the Jacobian matrix in the origin we see that that critical point is a saddle in the \(x_2-x_3\) plane, and, more important, that the origin is a nonhyperbolic critical point. Fig. 1.10 sketches the local flow pattern of the unperturbed vector field with \(e = 1.0, f = -2.0\) and \(g = 2.0\) according to Perry [Per84].

For \(\mu \neq 0\), the vector field (1.26) has critical points, but they remain at a fixed distance from the origin as \(\mu \rightarrow 0\). These critical points do not take part in the local flow pattern about the origin.

Fig. 1.11 sketches the three-dimensional local flow pattern about the origin of the vector field (1.26) for \(\mu > 0\). Note the strong convergence of skin-friction
1.3. Topology of fundamental local flow patterns

Figure 1.11: Free-shear layer separation of the vector field (1.26) for $\mu > 0$.

...lines towards the plane denoted as the 'surface of separation'. Maskell [Mas55] classified this flow behavior as free-shear layer separation.

Several questions come to mind about this example:

- Why are there no second order terms in the vector field (1.26), and what is the influence of additional higher order terms?

- What happens if one uses $e = -1.0$ instead of $e = 1.0$ in Fig. 1.10? Does this change the flow topology, and if not, why?

- Are other perturbations of the vector field possible, and if so, do they produce a different flow topology?

- Is the perturbation parameter in the third order term really necessary?

These questions can be answered if the vector field (1.26) is found in a more systematic manner. That way, the vector field would not be an isolated numerical example but represent a class of vector fields.

Perry and Chong [PC86] developed a computer algorithm to generate Taylor-series-expansion approximations to arbitrary order of the solutions of the flow equations and the no-slip condition in the case of a flat surface boundary. Special (but unexplained) boundary conditions were needed to reproduce the Taylor-polynomial expansion of a certain two-dimensional exact-known vector field. The point of expansion was taken to be a noncritical point. The region around that point in which the Taylor expansion approximates the vector field with 99% accuracy did not contain critical points.
The algorithm was then used with other boundary conditions to synthesize two and three-dimensional fundamental local flow patterns with several critical points. These fundamental local flow patterns had previously been classified by Hornung and Perry [HP84] with a so-called ‘vortex-skeleton method’ and visualized by Perry and Hornung [PH84] using an electromagnetic apparatus. Their results seem to confirm some of the flow patterns found by Dallmann [Dal83].

The critical points in ‘synthesized’ vector fields are at fixed distances of the point about which the Taylor expansion takes place. We already mentioned that such critical points should not be taken into account in local flow patterns.

Bakker [Bak88] was the first to examine two-dimensional local flow patterns about a point on a boundary surface in a systematic manner. Here, the velocity vector field is expanded into a Taylor series about a nonhyperbolic boundary critical point. The two-dimensional flow patterns about the nonhyperbolic critical point can be classified using the work of Andronov et al. [ALGM73].

After perturbation, the nonhyperbolic critical point can bifurcate into several hyperbolic critical points. In the limit that the perturbation parameters all vanish, these so-called unfolded critical point return to the origin. Therefore, these points are all contained in a small neighborhood of the point of expansion. Bakker introduced an unfolding technique that aims to produce perturbations of vector fields which satisfy the flow equations and the no-slip condition.

De Winkel and Bakker [WB88] [Win89] [BW90] [Bak90] [Bak91], and Kooij and Bakker [KB89] set up a classification scheme of three-dimensional fundamental local flow patterns about a point on a boundary surface using a bifurcation analysis. One of the vector fields they studied reads:

$$
\begin{align*}
\dot{x}_1 &= \mu_2 x_1 + x_2 + x_1^2 - \frac{2}{3} x_3^2 \\
\dot{x}_2 &= \mu_1 + x_3 + cx_1^2 - \frac{1}{3} c x_3^2 \\
\dot{x}_3 &= -\frac{1}{5} \mu_2 x_3 - x_1 x_3
\end{align*}
$$

The Jacobian matrix of the (unperturbed) vector field has three eigenvalues equal to zero and a one-dimensional eigenvector space. Fig. 1.12 and Fig. 1.13 sketch the local flow patterns of the vector field (1.27) for $c < 0$ and $c > 0$, respectively. Details about the curves which separate the various regions of flow patterns with equal topological structure can be found in [Bak90].

Recently, Bakker [Bak92] presented an interesting interpretation based on the bifurcation diagram Fig. 1.13. The local flow patterns $II$ and $VIII$ feature
Figure 1.12: Bifurcation diagram of the vector field (1.27) for $c < 0$. After [Bak90]
Figure 1.13: Bifurcation diagram of the vector field (1.27) for $c > 0$. After [Bak90]
a saddle and a node on the boundary surface. As \( \mu_1 \) changes sign (and \( \mu \neq 0 \)) the vector field (1.27) undergoes a saddle-node bifurcation. The resulting local flow pattern \( I \) resembles free-shear layer separation as shown in Fig. 1.11. In the terminology of Strogatz [Str94], the remnant of the annihilation of a saddle on a node produces a ghost (or bottleneck) region. Above the boundary surface streamlines are sucked into such a region and greatly delayed (or accelerated) before exiting. In retrospect, De Winkel [Win89] has reported similar bifurcation behavior in the case of a boundary critical point with a Jacobian matrix with a single zero eigenvalue.

The flow pattern in the region \( V I I \) of Fig. 1.13 has a saddle and a spiral. Flow separation occurs near both critical points. The well-known Werlé-Legendre flow-separation pattern results if the separatrix from the spiral (i.e., the vortex core) and the separatrix from the saddle intertwine, see Fig. 1.14. The vector field (1.27) thus is the possible relation between free-shear layer separation with Werlé-Legendre flow separation.

The classification scheme set up by Bakker and co-workers contains elements from nonlinear dynamics and bifurcation theory. As discussed in Chapter 3, a major part of a bifurcation analysis consists of reducing the (unnecessary) complexity of the representation of a vector field. To achieve this goal a systematic reduction process must be followed:- The linear part of the vector field is written
in Jordan form, the nonlinear part is brought into normal form, and perturbation parameters need to be concentrated in a minimum number of bifurcation parameters. The resulting simple polynomial vector fields are called topological normal forms. A critical point analysis of a topological normal form then yields bifurcation diagrams of fundamental local flow patterns, for example, see Fig. 1.12 and Fig. 1.13.

Bakker and co-workers applied their classification scheme to various types of nonhyperbolic critical points. The objective was to find a topological normal form for each type of nonhyperbolic critical point. Although promising candidate topological normal forms were found, questions raised regarding the mathematical foundation and the physical interpretation of the fundamental local flow patterns they produced. As shown in Chapter 4, topological normal forms found in pure mathematical communications do not at all correspond to vector fields describing fluid flows. Also, the unfolding technique introduced by Bakker does not work for every possible nonhyperbolic critical point.

1.4 About this thesis

This thesis sets out to accomplish the following task:

Find a method of analysis to study bifurcation in fluid flows that enables a systematic classification of the possible fundamental local flow patterns.

Unlike before, the emphasis will not be on critical point analyses of topological normal forms. Instead, we focus on the derivation of topological normal forms. The main point will be to assure that these topological normal forms have the correct physical properties; at all times they should describe the flow of a fluid near a possibly curved boundary surface. To this end, we discuss various mathematical techniques such as normal form theory and miniversal deformations.

Because formula manipulation of three-dimensional vector fields is laborious we introduce new techniques, for example, for the transformation of the nonlinear part to normal form, and for the embedding in a parameter space using miniversal deformations. These techniques will greatly simplify the process of finding those vector fields which have the correct physical properties.

The method of analysis set up in this thesis forms a solid mathematical basis for fundamental local flow patterns found in the past. Vortex generation, free-shear layer separation, and flows with mirror symmetry are some of the example/future applications.
Chapter 2

Vector Fields Describing Fluid Flow

But it was a mathematical game and nothing more. He had psychohistory; or, at least, the basis of psychohistory; but only as a mathematical curiosity. Where was the historical knowledge that could perhaps give some meaning to the empty equations? —Isaac Asimov, Prelude to Foundation

Introduction

Chapter 1 discussed two topologies of local flow patterns; about a hyperbolic critical point in the velocity vector field, and about a hyperbolic critical point in the skin-friction vector field. All that was needed was the knowledge of the structure of two linear vector fields.

In this thesis, we are interested in local flow patterns with possibly more than one critical point. An accurate description of the topological and geometrical features of such flow patterns requires third or higher order Taylor-polynomial approximations of the velocity vector field. Because of the nonlinear terms, the curvature of a boundary surface becomes important.

This chapter concentrates on how to describe fluid flow using third or higher order Taylor-polynomial approximations of the velocity vector field. Special attention will be paid to curved boundary surfaces.
2.1 Flow equations

The equations governing the motion (or flow) of a fluid depend on the assumptions made about the flow and the fluid. This thesis considers steady flow of a viscous, incompressible fluid. Such a flow is governed by two conservation laws: conservation of mass and conservation of momentum. The two conservation laws lead to the continuity equation

$$\nabla \cdot \mathbf{u} = 0, \quad (2.1)$$

and the Navier-Stokes equation of motion

$$u \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \frac{\mu}{\rho} \Delta \mathbf{u}, \quad (2.2)$$

respectively. In these equations, $\mathbf{u}(\mathbf{x})$ is the velocity vector, $p(\mathbf{x})$ is the pressure, and $\rho$ and $\mu$ are the constants of density and dynamic viscosity. The vector $\nabla = (\partial_1, \partial_2, \partial_3)^T$, $\partial_i = \partial/\partial x_i$, for $i \in \{1, 2, 3\}$, is the gradient operator while $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ denotes the Laplace operator.

The pressure component in (2.2) can be removed altogether by taking the curl. The result can be written as

$$(\mathbf{u} \cdot \nabla) \mathbf{\omega} = \left( \mathbf{\omega} \cdot \nabla \right) \mathbf{u} + \frac{1}{Re} \Delta \mathbf{\omega}, \quad (2.3)$$

where $\mathbf{\omega}$ is the vorticity vector defined as

$$\mathbf{\omega}(\mathbf{x}) \equiv \nabla \times \mathbf{u}(\mathbf{x}) = \begin{pmatrix} \partial_2 u^{(3)} - \partial_3 u^{(2)} \\ \partial_3 u^{(1)} - \partial_1 u^{(3)} \\ \partial_1 u^{(2)} - \partial_2 u^{(1)} \end{pmatrix}(\mathbf{x}). \quad (2.4)$$

Here, $\nabla \times$ denotes the curl operator and $u^{(i)}$, $i \in \{1, 2, 3\}$, the three component of the velocity vector $\mathbf{u}(\mathbf{x})$ along the axis $x_i$, $i \in \{1, 2, 3\}$, respectively. The dimensionless constant $Re$ in (2.3) is the well-known Reynolds number $\rho U L / \mu$, found by nondimensionalizing with a reference velocity, $U$, and a reference length, $L$. Equation (2.3) is known as the vorticity transport equation.

On the boundary surface, say $\partial B$, of a smooth and rigid aerodynamic body the velocity vector has to satisfy the no-slip condition,

$$\mathbf{u}(\mathbf{x}) = 0, \quad \text{for } \mathbf{x} \in \partial B. \quad (2.5)$$

This condition requires the velocity to vanish everywhere on the boundary surface.
We assume that solutions $u$ of (2.1), (2.3) and (2.5) are analytic, i.e., Taylor series expansions\(^1\) of the three components of $u$ exist and have a positive radius of convergence. However, we will not bother much with convergence: the series is always truncated at a certain order leaving a Taylor polynomial. We assume that Taylor polynomial approximations of solutions $u$ suffice for a local description of the topological and geometrical features of flow patterns in fluid flow.

Let the origin of a locally defined Cartesian coordinate system $(x_1, x_2, x_3)$ be a point of interest in the flow domain. A Taylor series expansion in $x$ about the origin basically is a sum of homogeneous polynomials in $x$. Let $H^m_n$ be the set of all homogeneous polynomials $\mathbb{R}^n \to \mathbb{R}^n$ of degree $m$. We denote a function $u_m(x) \in H^m_n$ as

$$u_m(x) = \sum_{i+j+k=m} \begin{pmatrix} u_{i,j,k}^{(1)} \\ u_{i,j,k}^{(2)} \\ u_{i,j,k}^{(3)} \end{pmatrix} x_1^i x_2^j x_3^k,$$

(2.6)

where $i, j, k$ are nonnegative integers. With this notation, the velocity vector can be written as

$$u(x) = \sum_{m \geq 0} u_m(x),$$

(2.7)

where it is understood that the right-hand side is the Taylor series expansion about the origin. The series will be truncated at a certain order, say $n$, resulting in a Taylor polynomial of index $n$, denoted as $u^{[n]}(x)$.

### 2.2 Interior critical points

A point above the boundary surface is called an interior point. In this section, we investigate the properties of Taylor series expansions of the velocity vector about such points.

#### 2.2.1 Coefficient relations

Substitution of (2.7) into the continuity equation (2.1) results in

$$\nabla \cdot u_m(x) = 0, \quad \forall m \geq 0.$$

\(^1\)Appendix B discusses the definition of the Taylor series expansion.
where the gradient operator is brought inside the summation. Using (2.6), the latter equation yields
\[
\sum_{i+j+k=m} \left( i u_{i,j,k}^{(1)} x_1^{i-1} x_2^j x_3^k + j u_{i,j,k}^{(2)} x_1^i x_2^{j-1} x_3^k + k u_{i,j,k}^{(3)} x_1^i x_2^j x_3^{k-1} \right) = 0.
\] (2.8)

Note that every one of the three terms in (2.8) is of order \(O(\|z\|^{m-1})\). By equating terms with the same monomial we get
\[
(i + 1)u_{i+1,j,k}^{(1)} + (j + 1)u_{i,j+1,k}^{(2)} + (k + 1)u_{i,j,k+1}^{(3)} = 0,
\] (2.9)
\[
\forall i + j + k = m - 1, \ m > 0. \ \text{No coefficient relation is found for} \ m = 0.
\]

Every function in (2.7) and therewith every term in (2.6) is differentiated the same number of times in the continuity equation. Since one differentiation takes place, collecting terms of \(O(\|z\|^{m-1})\) produced relations between \(O(\|z\|^{m})\) coefficients only.

At most three differentiations take place in vorticity transport equation (2.3). Therefore, collecting terms of \(O(\|z\|^{m-3})\) after substitution of (2.7) and (2.6) into (2.3) relates \(O(\|z\|^{m})\) coefficients to lower order coefficients. At \(O(\|z\|^{0})\) we find the following three coefficient relations:
\[
u_{000}^{(1)}(u_{110}^{(1)} - u_{101}^{(2)}) + u_{000}^{(2)}(2u_{020}^{(3)} - u_{011}^{(2)}) + u_{000}^{(3)}(u_{011}^{(3)} - 2u_{002}^{(2)}) =
\]
\[
(u_{010}^{(3)} - u_{001}^{(2)})u_{100}^{(1)} + (u_{001}^{(1)} - u_{100}^{(3)})u_{010}^{(1)} + (u_{100}^{(2)} - u_{010}^{(1)})u_{001}^{(1)} +
\]
\[
+ \frac{2}{Re} \left( u_{210}^{(3)} + 3u_{030}^{(3)} + u_{012}^{(2)} - u_{201}^{(2)} - 3u_{021}^{(2)} \right),
\] (2.10)
\[
u_{000}^{(1)}(u_{101}^{(1)} - 2u_{200}^{(3)}) + u_{000}^{(2)}(u_{111}^{(1)} - u_{110}^{(3)}) + u_{000}^{(3)}(2u_{002}^{(1)} - u_{101}^{(3)}) =
\]
\[
(u_{010}^{(3)} - u_{001}^{(2)})u_{100}^{(2)} + (u_{001}^{(1)} - u_{100}^{(3)})u_{010}^{(2)} + (u_{100}^{(2)} - u_{010}^{(1)})u_{001}^{(2)} +
\]
\[
+ \frac{2}{Re} \left( u_{201}^{(1)} + u_{021}^{(1)} + 3u_{030}^{(1)} - 3u_{300}^{(3)} - u_{120}^{(3)} - u_{012}^{(3)} \right),
\] (2.11)
\[
u_{000}^{(1)}(2u_{200}^{(2)} - u_{110}^{(1)}) + u_{000}^{(2)}(u_{110}^{(2)} - 2u_{020}^{(1)}) + u_{000}^{(3)}(2u_{002}^{(1)} - u_{011}^{(3)}) =
\]
\[
(u_{010}^{(3)} - u_{001}^{(2)})u_{100}^{(3)} + (u_{001}^{(1)} - u_{100}^{(3)})u_{010}^{(3)} + (u_{100}^{(2)} - u_{010}^{(1)})u_{001}^{(3)} +
\]
\[
+ \frac{2}{Re} \left( 3u_{300}^{(2)} + u_{120}^{(2)} + u_{102}^{(2)} - u_{210}^{(1)} - 3u_{030}^{(1)} - u_{012}^{(1)} \right).
\] (2.12)

Higher order coefficient relations become increasing more space consuming, so we will not list them here.
Let the origin of our locally defined Cartesian coordinate system be an interior critical point, i.e.,

$$u(0) = 0.$$  \hfill (2.13)

As a result, the three coefficients $u^{(1)}_{000}$, $u^{(2)}_{000}$ and $u^{(3)}_{000}$ are zero and the Taylor polynomial can be written as

$$u^{[n]}(\bar{x}) = A \bar{x} + \sum_{m=2}^{n} u_m(\bar{x}).$$  \hfill (2.14)

Notice that the first order function $u_1(\bar{x})$ in (2.14) is rewritten as a matrix-vector product $A \bar{x}$. Here, $A$ is the Jacobian matrix of the velocity vector $u(\bar{x})$ at the origin. The coefficients of $A$ are defined as $A_{ij} \equiv \partial_j u^{(i)}(0)$.

The only relation between the coefficients of $A$ stems from continuity equation, and reads

$$A_{11} + A_{22} + A_{33} = 0,$$  \hfill (2.15)

see (2.9). The coefficient relations due to the Navier-Stokes equation first enter (2.14) at third order, that is, in $u_3(\bar{x})$, see (2.10), (2.11), and (2.12). The relation (2.15) between the coefficients of the Jacobian matrix $A$ proves the relation (1.6). Based on that relation, Section 1.2.1 already discussed all possible local flow patterns governed by the vector field

$$\dot{x} = A \bar{x}.$$  \hfill (2.16)

Chapter 3 will explain that in the case of a hyperbolic (interior) critical point in $\bar{x} = 0$, the flow pattern of the linearized vector field (2.16) is in fact topological conjugate to the flow pattern of the vector field (2.14) in a sufficiently small neighborhood of the critical point.

### 2.2.2 Number of coefficients and relations

The question is whether the above method of substituting a Taylor series into the flow equations does lead to approximate solutions. For, it is not inconceivable that at a certain order, more relations are produced than coefficients introduced. Fortunately, this situation never happens.

Each monomial in the homogeneous polynomial of degree $m$ in (2.6) has three recipient locations for the distribution of the power $m$. An equivalent and well-known counting problem is the number of different ways in which $m$
identical balls can be placed in three baskets, see Fig. 2.1. Thus, inclusion of $O(\|x\|^m)$ terms in the Taylor series expansion (2.6) introduces

$$N_c(m) = 3 \left( \frac{3 + m - 1}{m} \right) = \frac{3}{2}(m + 2)(m + 1)$$  \hspace{1cm} (2.17)

new coefficients. Since there are three rows in (2.6), the total number is multiplied by three, yielding (2.17).

The number of new coefficient relations due to the continuity equation (2.1) equals

$$E_C(m) = \left( \frac{3 + (m - 1) - 1}{m - 1} \right) = \frac{1}{2}(m + 1)m, \quad m \geq 1,$$  \hspace{1cm} (2.18)

since the highest number of differentiations in that partial differential equation is 1. For $m = 0$, no coefficient relations are found.

The vorticity transport equation (2.3) resulted after taking the curl operator over the Navier-Stokes equation of motion (2.2). Then, the inner product of the gradient and the vorticity transport equation results in the trivial equation $0 = 0$. As we collect the terms of order $O(\|x\|^{m-3})$ (since 3 is the highest number of differentiations), we find that there are some linear dependent coefficient relations from $m = 4$ onwards. If we correct for these dependencies, the number
### 2.3 Boundary critical points

The Taylor polynomial of the velocity vector $\mathbf{u}(x)$ of index $n$ forms an $O(\|x\|^n)$ approximation of that vector in a small enough neighborhood of the origin. What we like to know is if the flow pattern of a Taylor-polynomial vector field forms a sufficiently well approximation of the flow pattern of the velocity vector field. In particular, we need to know if the (approximate) skin-friction lines described by Taylor-polynomial vector fields are found on the (approximate) boundary surface.

#### Table 2.1: The number of coefficients versus the number of relations.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$N_c(m)$</th>
<th>$S_1$</th>
<th>$E_C(m)$</th>
<th>$E_{NS}(m)$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_1 - S_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>12</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>8</td>
<td>11</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>30</td>
<td>3</td>
<td>0</td>
<td>4</td>
<td>15</td>
<td>26</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>60</td>
<td>6</td>
<td>3</td>
<td>13</td>
<td>21</td>
<td>47</td>
</tr>
<tr>
<td>4</td>
<td>45</td>
<td>105</td>
<td>10</td>
<td>8</td>
<td>31</td>
<td>27</td>
<td>74</td>
</tr>
<tr>
<td>5</td>
<td>63</td>
<td>168</td>
<td>15</td>
<td>15</td>
<td>61</td>
<td>33</td>
<td>107</td>
</tr>
<tr>
<td>15</td>
<td>408</td>
<td>2448</td>
<td>120</td>
<td>195</td>
<td>1681</td>
<td>93</td>
<td>767</td>
</tr>
</tbody>
</table>

$S_1 = \sum_m N_c(m), \quad S_2 = \sum_m \{E_C + E_{NS}\}(m), \quad S_3 = \{N_c - E_C - E_{NS}\}(m)$

of new coefficient relations equals

$$E_{NS}(m) = 3 \left( \frac{3 + (m - 3) - 1}{m - 3} \right) - \frac{1}{2} (m - 2)(m - 3)$$

$$= m(m - 2), \quad m \geq 3. \quad (2.19)$$

For $0 \leq m \leq 2$, no coefficient relations are found.

Subtracting (2.18) and (2.19) from (2.17) we get

$$\{N_c - E_C - E_{NS}\}(m) = 6m + 3, \quad m \geq 3, \quad (2.20)$$

meaning that always more new coefficients than new relations are introduced.

Perry and Chong [PC86] also counted the number of new coefficients and new relations. Their results match ours.
2.3.1 Flat boundary surface

The literature on local flow separation patterns almost without exception assumes the boundary surface to be flat, e.g. see Dallmann [Dal83], Perry [Per84], Perry and Chong [PC86], Bakker [Bak88], and De Winkel and Bakker [WB88]. In our notation, their results are as follows.

Let the flat boundary surface \( \partial B \) be located at \( x_3 = 0 \). Substitution of the Taylor series expansion (2.7) in the no-slip condition (2.5) results in (per component \( \ell, \ell \in \{1, 2, 3\} \))

\[
\begin{align*}
    u^{(\ell)}(x_3 = 0) &= u^{(\ell)}_{000} \\
    &\quad + u^{(\ell)}_{010}x_1 + u^{(\ell)}_{011}x_2 \\
    &\quad + u^{(\ell)}_{100}x_2 + u^{(\ell)}_{110}x_1x_2 + u^{(\ell)}_{020}x_2^2 + \ldots = 0,
\end{align*}
\]

where the dots represent terms of third order and higher. These conditions are satisfied if and only if

\[
u^{(\ell)}_{ij0} = 0,
\]

\( \forall i + j = m, m \geq 0 \). With (2.22), the Taylor series expansion of the velocity vector can be written as

\[
u(x) = x_3 \hat{\mu}(x), \quad \hat{\mu}(x) \equiv \sum_{m \geq 0} \hat{\mu}_m(x),
\]

where each \( \hat{\mu}_m(x) \in H^3_m \).

The relation between the coefficients of the vector functions \( \hat{\mu}_m(x) \) and \( \mu_{m+1}(x) \) is given by

\[
\hat{\mu}^{(\ell)}_{i+j+k} = \mu^{(\ell)}_{i+j+k+1},
\]

\( \forall i + j + k = m, m \geq 0 \). With this relation, the coefficient relations for the vector function \( \hat{\mu}(x) \) can be derived from relations between the coefficients in the vector function \( \nu(x) \). Due to the continuity equation, (2.9), and the no-slip boundary condition, (2.22), we get

\[
(i+1)\hat{\mu}^{(1)}_{i+1+j+k} + (j+1)\hat{\mu}^{(2)}_{i+j+1+k} + (k+2)\hat{\mu}^{(3)}_{i+j+k+1} = 0,
\]

\[
\hat{\mu}^{(3)}_{i+j+0} = 0,
\]
\[ \forall i + j + k = m - 1, \ m \geq 1, \ \text{and} \ \forall i + j = m, \ m \geq 0, \ \text{respectively.} \ \text{From the vorticity transport equation we get} \]

\[
\begin{align*}
\dot{u}_{110}^{(1)} + 2\dot{u}_{200}^{(2)} + 4\dot{u}_{020}^{(2)} + 6\dot{u}_{002}^{(2)} &= 0, \\
4\dot{u}_{200}^{(1)} + 2\dot{u}_{020}^{(1)} + 6\dot{u}_{002}^{(1)} + \dot{u}_{110}^{(2)} &= 0, \\
\dot{u}_{101}^{(2)} - \dot{u}_{011}^{(1)} &= 0, \\
\end{align*}
\]

(2.26)

see (2.10), (2.11) and (2.12). Note that by (2.25), the coefficients of \( \dot{u}^{(3)}(x) \) are either zero or linearly related to the coefficients of \( \dot{u}^{(1)}(x) \) and \( \dot{u}^{(2)}(x) \). This makes it possible to rewrite the coefficient relations due to the vorticity equation such that they relate coefficients of \( \dot{u}^{(1)}(x) \) to coefficients of \( \dot{u}^{(2)}(x) \).

To interpret the flow patterns of (2.23) in a physical context it is worthwhile to relate the coefficients of \( \ddot{u}_0(x) \) to (the gradients of) the skin-friction vector. The skin-friction vector \( \tau \) is given by

\[ \tau(x) \equiv \frac{1}{Re} \partial_3 \bar{u}(x) = \frac{1}{Re} \dot{u}(x), \quad \text{on} \ x_3 = 0. \]  

(2.27)

The inverse scaling with the Reynolds number \( Re \) results from nondimensionalizing the skin-friction vector using the reference value \( \rho U^2 \). We get \( \ddot{u}_0 = Re \tau(0) \) and \( \ddot{u}_1(x) = A \dot{x} \), where \( A \equiv Re \nabla \tau(0) \).

Due to the combination of the no-slip boundary condition and the continuity equation, there are no terms without a factor \( x_3 \) in the third component of the vector function \( \ddot{u}(x) \), see (2.25). As a result, the third component \( \tau^{(3)}(x) = 0 \) on the boundary surface \( x_3 = 0 \), which makes the skin-friction vector \( \tau(x) \) a tangent vector to boundary surface. Therefore, we can define a two-dimensional vector field

\[ \dot{x} = \tau(x), \quad \text{on} \ x_3 = 0. \]

(2.28)

The trajectories of this vector field are the so-called skin-friction lines. The skin-friction lines are found on a so-called invariant manifold in the vector field

\[ \dot{x} = \ddot{u}(x). \]

(2.29)

For our purposes it is sufficient to think of a manifold \( M \subset \mathbb{R}^n \) as a set of points in \( \mathbb{R}^n \) that satisfy a system of \( m \) scalar equations

\[ \bar{F}(x) = 0, \]

(2.30)

where \( \bar{F} : \mathbb{R}^n \to \mathbb{R}^m \) for some \( m \leq n \). The manifold \( M \) is smooth (differentiable) if \( \bar{F} \) is smooth and the rank of the Jacobian matrix of \( \bar{F} \) is equal to \( m \) at each
point \( \mathbf{x} \in M \). At each point \( \mathbf{x} \) of a smooth manifold \( M \), there is an \((n-m)\)-dimensional tangent space \( T_{\mathbf{x}}M \) which consists of all vectors \( \mathbf{v} \in \mathbb{R}^n \) that can be represented as \( \mathbf{v} = \dot{\gamma}(0) \), where \( \gamma : \mathbb{R}^1 \to M \) is a smooth curve on the manifold satisfying \( \gamma(0) = \mathbf{x} \). \( M \) is called an invariant manifold of the vector field induced by a vector function \( f : \mathbb{R}^n \to \mathbb{R}^n \) if in every point \( \mathbf{x} \in M \), \( f(\mathbf{x}) \in T_{\mathbf{x}}M \). Geometrically, trajectories of the vector field through a point \( \mathbf{x} \in M \) remain on the invariant manifold \( M \) at all time.

In the present case, the flat boundary surface \( x_3 = 0 \) is an invariant manifold of the vector field (2.29).

Above the boundary surface the common factor \( x_3 \) in (2.23) only influences the magnitude and not the direction of velocity vector. Therefore, the trajectories of the vector field (2.29) above the boundary surface are identical to the streamlines, i.e., the trajectories in the vector field induced by velocity vector,

\[
\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}) = x_3 \dot{\mathbf{u}}(\mathbf{x}). \tag{2.31}
\]

Every point on the boundary surface \( x_3 = 0 \) is a critical point in the vector field (2.31), but not in the vector field (2.29). A critical point in the vector field (2.29) on the boundary surface \( x_3 = 0 \) is a point with vanishing skin-friction. Such points are associated with flow separation and flow attachment. Because we are interested in the flow pattern of both the streamlines as well as the skin-friction lines it is best to work with the vector field (2.29).

The origin is a critical point of the vector field (2.29) if \( \dot{\mathbf{u}}_0 = 0 \), that is, if \( \dot{u}^{(1)}_{000} = v^{(1)}_{001} = 0 \) and \( \dot{u}^{(2)}_{000} = v^{(2)}_{001} = 0 \). The Jacobian matrix of vector function \( \dot{\mathbf{u}}(\mathbf{x}) \) reads

\[
\dot{\mathbf{A}} = \begin{pmatrix}
\dot{u}^{(1)}_{100} & \dot{v}^{(1)}_{010} & \dot{u}^{(1)}_{001} & \dot{u}^{(2)}_{100} \\
\dot{u}^{(2)}_{100} & \dot{v}^{(2)}_{010} & \dot{u}^{(2)}_{001} \\
0 & 0 & -\frac{1}{2}(\dot{u}^{(1)}_{100} + \dot{v}^{(2)}_{010}) \\
\end{pmatrix} \\
\equiv \begin{pmatrix}
u^{(1)}_{101} & u^{(1)}_{011} & \dot{u}^{(1)}_{002} & \dot{u}^{(2)}_{101} \\
u^{(2)}_{101} & u^{(2)}_{011} & \dot{u}^{(2)}_{002} \\
0 & 0 & -\frac{1}{2}(\dot{u}^{(1)}_{101} + \dot{u}^{(2)}_{011}) \\
\end{pmatrix}. \tag{2.32}
\]

Section 1.2.2 already discussed all possible local flow patterns governed by the vector field

\[
\dot{\mathbf{x}} = \dot{\mathbf{A}}\mathbf{x}. \tag{2.33}
\]

Chapter 3 will explain that if \( \mathbf{x} = 0 \) is a hyperbolic critical point of the vector field (2.33), the flow pattern of that vector field is topological conjugate to the
flow pattern of the vector field (2.29) in a sufficiently small neighborhood of the critical point. Thus, we know the topology of all possible local flow patterns about a critical point on a flat boundary surface.

### 2.3.2 Number of coefficients and relations

The number of new coefficient relations in (2.22) equals three times the number of different ways in which \( m \) identical balls can be placed in \( two \) baskets. Thus, the number of coefficient relations due to the no-slip boundary condition equals

\[
E_0(m) = 3 \left( 2 + \frac{m - 1}{m} \right) = 3m + 3, \quad m \geq 0.
\]  

(2.34)

Subtracting (2.18), (2.19), and (2.34) from (2.17) we get

\[
\{N_c - E_C - E_{NS} - E_0\}(m) = 3m, \quad m \geq 3,
\]  

(2.35)

meaning that always more new coefficients than new relations are introduced, see Table 2.2.

### 2.3.3 Curved boundary surface

The common factor \( x_3 \) in the Taylor series approximation (2.23) of the velocity vector about a point on the flat boundary surface \( x_3 = 0 \) insures that the streamlines found using the vector field (2.31) do not intersect the boundary
surface. When the common factor \( x_3 \) was left out, we found that the coordinate plane \( x_3 = 0 \) is an invariant manifold in the resulting vector field (2.29), and that the trajectories of that vector field in that coordinate plane are in fact the skin-friction lines.

The question is if we can find similar results in the case of a curved boundary surface. Another (related) question is whether curvature influences the Jacobian matrix \( A \) in (2.32) and therewith the topology of fundamental local flow patterns.

Consider a \( \varepsilon \)-sphere \( U_\varepsilon \) around a point \( x_0 \) located on the boundary surface \( \partial B \), see Fig. 2.2. The behavior at the closure \( \partial U_\varepsilon \) of \( U_\varepsilon \) is of no interest, as is the part of \( U_\varepsilon \) that is located outside the flow domain (i.e., under the boundary surface). Let the origin of a locally defined Cartesian coordinate system \( (x_1, x_2, x_3) \) be located in \( x_0 \). The two unit base vectors \( e_1 \) and \( e_2 \) are located in the tangential plane, and the third unit base vector \( e_3 \) is pointing into the flow domain and is perpendicular to that plane at the origin.

The boundary surface \( \partial B \) is assumed to be smooth and arbitrarily curved. Let \( x_3 = h(x_1, x_2) \) denote the displacement of the boundary surface with respect to its tangential plane at the origin (locally). Since \( \partial B \) is smooth,

\[
h(0, 0) = h_{,1}(0, 0) = h_{,2}(0, 0) = 0.
\]  

(2.36)

where \( h_{,i} = \partial_i h \), \( i \in \{1, 2\} \). With (2.36), the Taylor series expansion of the function \( h(x_1, x_2) \) about \( x_1 = x_2 = 0 \) can be written as

\[
h(x_1, x_2) = h_{20} x_1^2 + h_{11} x_1 x_2 + h_{02} x_2^2 + O(\|\varepsilon\|^2).
\]  

(2.37)
2.3. Boundary critical points

The no-slip condition (2.5) should be applied at the boundary surface \( x_3 = h(x_1, x_2) \). Substitution of this expression together with (2.37) into (2.5) results in

\[
\begin{align*}
&u^{(\ell)}(x_3 = h(x_1, x_2)) = u_{000}^{(\ell)}, \\
&\quad + u_{100}^{(\ell)} x_1 + u_{010}^{(\ell)} x_2 + u_{001}^{(\ell)} \left( h_{20} x_1^2 + h_{11} x_1 x_2 + h_{02} x_2^2 ight) \\
&\quad + h_{30} x_1^3 + h_{21} x_1^2 x_2 + h_{12} x_1 x_2^2 + h_{03} x_2^3 + \ldots \\
&\quad + u_{200}^{(\ell)} x_1 x_2 + u_{110}^{(\ell)} x_1 + u_{101}^{(\ell)} x_1 \left( h_{20} x_1^2 + h_{11} x_1 x_2 + h_{02} x_2^2 + \ldots \right) \\
&\quad + u_{20}^{(\ell)} x_1^2 + u_{11} x_2 \left( h_{20} x_1^2 + h_{11} x_1 x_2 + h_{02} x_2^2 + \ldots \right) \\
&\quad + u_{10}^{(\ell)} x_1 x_2 + u_{01} x_2 \left( h_{20} x_1^2 + h_{11} x_1 x_2 + h_{02} x_2^2 + \ldots \right) \\
&\quad + u_{00}^{(\ell)} x_2 + u_{\ell}^{(\ell)} + u_{0\ell} x_2 + u_{\ell0} x_1 + u_{\ell1} x_1 x_2 + u_{\ell2} x_2^2 + \ldots = 0,
\end{align*}
\]

for each \( \ell \in \{1, 2, 3\} \). The dots represent terms of fourth order and higher. Note that, for example, the term \( u_{002} x_3^2 \) is one of them. Equating monomials with the same index results in the following list of coefficient relations:

\[
\begin{align*}
&u_{000}^{(\ell)} = 0, \quad u_{100}^{(\ell)} = 0, \quad u_{010}^{(\ell)} = 0, \quad \frac{u_{\ell00}}{u_{000}} = -h_{20} u_{001}, \\
&u_{110}^{(\ell)} = -h_{11} u_{001}, \quad u_{020}^{(\ell)} = -h_{02} u_{001}, \quad (2.38)
\end{align*}
\]

\[
\begin{align*}
&u_{300}^{(\ell)} = -h_{30} u_{001} - h_{20} u_{101}, \\
&u_{210}^{(\ell)} = -h_{21} u_{001} - h_{11} u_{101} - h_{20} u_{011}, \\
&u_{120}^{(\ell)} = -h_{12} u_{001} - h_{02} u_{101} - h_{11} u_{011}, \\
&u_{030}^{(\ell)} = -h_{03} u_{001} - h_{02} u_{011}. \quad (2.39)
\end{align*}
\]

These relations combined with the coefficient relations due to the continuity equation (2.9) lead to

\[
\begin{align*}
&u_{000}^{(3)} = 0, \quad u_{100}^{(3)} = 2 h_{20} u_{001}^{(1)} + h_{11} u_{001}^{(2)}, \\
&u_{010}^{(3)} = h_{11} u_{001}^{(1)} + 2 h_{02} u_{001}^{(2)}, \\
&u_{110}^{(3)} = 2 h_{21} u_{001}^{(1)} + 2 h_{11} u_{101}^{(1)} + 2 h_{20} u_{011}^{(1)} + 2 h_{01} u_{011}^{(2)} + 2 h_{11} u_{011}^{(2)}, \\
&u_{020}^{(3)} = h_{12} u_{001}^{(1)} + h_{02} u_{101}^{(1)} + h_{11} u_{011}^{(1)} + 3 h_{03} u_{001}^{(2)} + 3 h_{02} u_{011}^{(2)}. \quad (2.40)
\end{align*}
\]

We see that the right-hand side in these coefficient relations is possibly nonzero (compare with (2.25)). Note also that the same number of coefficient relations are formed.
By substituting of (2.38) and (2.9) into the coefficient relations (2.10), (2.11), and (2.12) due to the vorticity transport equation, we get

\[
-u_{111}^{(1)} - 2u_{201}^{(2)} - 4u_{021}^{(2)} - 6u_{003}^{(2)} = 2h_{11}(2h_{20}u_{001}^{(1)} + h_{11}u_{001}^{(2)}) + 2(h_{20} + 3h_{02})(h_{11}u_{001}^{(1)} + 2h_{02}u_{001}^{(2)}),
\]

\[
2h_{11}(2h_{20}u_{001}^{(1)} + h_{11}u_{001}^{(2)}) + 2(h_{20} + 3h_{02})(h_{11}u_{001}^{(1)} + 2h_{02}u_{001}^{(2)}),
\]

\[
4u_{201}^{(1)} + 2u_{021}^{(1)} + 6u_{003}^{(1)} + u_{111}^{(2)} = -2(3h_{20} + h_{02})(2h_{20}u_{001}^{(1)} + h_{11}u_{001}^{(2)}) - 2h_{11}(h_{11}u_{001}^{(1)} + 2h_{02}u_{001}^{(2)}),
\]

\[
u_{102}^{(2)} - u_{012}^{(1)} = (3h_{30} + h_{12})(u_{001}^{(2)} + (3h_{20} + h_{02})u_{011}^{(2)}) + h_{11}u_{011}^{(2)} - h_{11}u_{101}^{(1)} - (h_{20} + 3h_{02})u_{011}^{(1)},
\]

respectively. Note that in the aforementioned coefficients relations all terms not multiplied by the factor 2/Re are identical to zero. Again, we see that the right-hand side in these coefficient relations is possibly nonzero (compare with (2.26)).

Substitution of the various coefficient relations into the vector functions \(u_1(\vec{x})\) and \(u_2(\vec{x})\) produces

\[
u_1(\vec{x}) = \begin{pmatrix} u_{001}^{(1)} x_3 \\ u_{001}^{(2)} x_3 \\ 0 \end{pmatrix},
\]

and

\[
u_2(\vec{x}) = \begin{pmatrix} -h_{20}u_{001}^{(1)} x_1^2 - h_{11}u_{001}^{(1)} x_1 x_2 - h_{02}u_{001}^{(1)} x_2^2 \\ + u_{011}^{(1)} x_1 x_3 + u_{011}^{(2)} x_2 x_3 + u_{002}^{(2)} x_3^2 \\ -h_{20}u_{001}^{(2)} x_1^2 - h_{11}u_{001}^{(2)} x_1 x_2 - h_{02}u_{001}^{(2)} x_2^2 \\ + u_{011}^{(2)} x_1 x_3 + u_{011}^{(1)} x_2 x_3 + u_{002}^{(1)} x_3^2 \\ (2h_{20}u_{001}^{(1)} + h_{11}u_{001}^{(2)}) x_1 x_3 \\ + (h_{11}u_{001}^{(1)} + 2h_{02}u_{001}^{(2)}) x_2 x_3 - \frac{1}{2}(u_{101}^{(1)} + u_{101}^{(2)}) x_3^2 \end{pmatrix}.
\]

In a boundary critical point, \(u_{001}^{(1)} = u_{001}^{(2)} = 0\), and all derivatives of the displacement function \(h(x_1, x_2)\) are removed from \(u_2(\vec{x})\). Then \(u_2(\vec{x})\) has a common factor \(x_3\). If we were to use a Taylor-polynomial approximation of the velocity vector field of index 2, the flat surface \(x_3 = 0\) acts as the boundary surface.
Similarly, we can find the Taylor-polynomial approximation of the velocity vector field of index \( m > 2 \). We need at least an expression of the Taylor-polynomial of index 3 for the description of local flow patterns with possibly more than one critical point. Within that expression, we have to look for the (approximation of the) invariant manifold containing the skin-friction lines.

It would be more efficient if the invariant manifold in the Taylor-polynomial vector field is a coordinate plane as it was in the case of the flat boundary surface. At the same time, the approximation of the skin-friction lines found with the Taylor-polynomial vector field would gain in accuracy since the invariant manifold (being the curved boundary surface) does not have to be approximated.

### 2.3.4 Volume preservation

To ensure that the curved boundary surface is a coordinate plane, we shall project that surface onto its tangent plane in the origin.

Let \((y_1, y_2, y_3)\) be the coordinate system found by projection of \((x_1, x_2, x_3)\) onto the boundary surface by the transformation,

\[
\begin{align*}
x_1 &= y_1, \\
x_2 &= y_2, \\
x_3 &= y_3 + h(y_1, y_2),
\end{align*}
\]  

(2.47)

see Fig. 2.2. This paragraph discusses the consequences of the change from the coordinate system \((x_1, x_2, x_3)\) to \((y_1, y_2, y_3)\). We do this to compute the coefficients in the Taylor series expansion of the vector field in the \(y\)-coordinates.

The projection (2.47) is an volume-preserving change of coordinates; for, the determinant of the Jacobian of the right-hand side equals unity everywhere. The result is that the continuity equation also holds in the new coordinates. The proof runs as follows.

The velocity vector field in the original coordinates is given by

\[
\dot{x} = u(x).
\]

(2.48)

The inverse transformation of (2.47) is easily stated:

\[
\begin{align*}
y_1 &= x_1, \\
y_2 &= x_2, \\
y_3 &= x_3 - h(x_1, x_2).
\end{align*}
\]  

(2.49)

Differentiate (2.49) with respect to time, use (2.48), to obtain

\[
\begin{align*}
\dot{y}_1 &= v^{(1)}(y) \equiv \{u^{(1)}\} (y_1, y_2, y_3 + h(y_1, y_2)), \\
\dot{y}_2 &= v^{(2)}(y) \equiv \{u^{(2)}\} (y_1, y_2, y_3 + h(y_1, y_2)), \\
\dot{y}_3 &= v^{(3)}(y) \equiv \{u^{(3)} - h,1 u^{(1)} - h,2 u^{(2)}\} (y_1, y_2, y_3 + h(y_1, y_2)).
\end{align*}
\]  

(2.50)
These three equations define the vector function \( \mathbf{z}(y) \) in the coordinate system \((y_1, y_2, y_3)\).

The partial derivatives with respect to the new coordinates are found using the chain-rule,

\[
\frac{\partial}{\partial x_i} = \frac{\partial y_1}{\partial x_i} \frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_i} \frac{\partial}{\partial y_2} + \frac{\partial y_3}{\partial x_i} \frac{\partial}{\partial y_3},
\]

for \(i = 1, 2, 3\), and (2.49). The result reads

\[
\begin{align*}
\frac{\partial}{\partial x_1} &= \frac{\partial}{\partial y_1} - h,1|_{(y_1, y_2)} \frac{\partial}{\partial y_3}, \\
\frac{\partial}{\partial x_2} &= \frac{\partial}{\partial y_2} - h,2|_{(y_1, y_2)} \frac{\partial}{\partial y_3}, \\
\frac{\partial}{\partial x_3} &= \frac{\partial}{\partial y_3}.
\end{align*}
\]

(2.51)

Rewrite the conservation of mass equation,

\[
\nabla \cdot \mathbf{u} = \frac{\partial u^{(1)}}{\partial x_1} + \frac{\partial u^{(2)}}{\partial x_2} + \frac{\partial u^{(3)}}{\partial x_3} = 0,
\]

(2.52)

in the new coordinates using (2.51),

\[
\frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} + \frac{\partial}{\partial y_3} \left(u_3 - h,1|_{(y_1, y_2)} u_1 - h,2|_{(y_1, y_2)} u_2\right) = 0,
\]

(2.53)

where \(u_i\) is short for \(u^{(i)}(y_1, y_2, y_3 + h(y_1, y_2))\). Notice that a re-ordering has taken place of the partial derivatives with respect to \(y_3\) (actually, this trick makes the proof work). Using (2.50), (2.53) is equivalent with

\[
\frac{\partial v^{(1)}}{\partial y_1} + \frac{\partial v^{(2)}}{\partial y_2} + \frac{\partial v^{(3)}}{\partial y_3} = \nabla \cdot \mathbf{v} = 0.
\]

(2.54)

This result means that the continuity equation again holds in the new coordinates.

### 2.3.5 Coefficient relations after projection

The vector function \(\mathbf{v}(y)\) in (2.50) that resulted after the projection of the boundary surface onto its tangential plane in the origin can also be expanded in a series

\[
\mathbf{v}(y) = \sum_{m \geq 0} \mathbf{v}_m(y),
\]

(2.55)
where each \( u_m(y) \in H_m^3 \). Since \((y_1, y_2, y_3)\) is now a Cartesian coordinate system, the right-hand side of (2.55) is once again a Taylor series expansion.

The previous paragraph concluded that since \( u(x) \) satisfies the continuity equation (2.1) in the \((x_1, x_2, x_3)\) coordinate system, \( v(y) \) satisfies the continuity equation (2.54) in the \((y_1, y_2, y_3)\) coordinate system. It is, therefore, possible to find relations between the coefficients in the series expansion of \( v(y) \) directly by substituting the series in the continuity equation (2.54).

No-slip boundary condition

The no-slip boundary condition (2.5) for the vector field \( v(y) \) reads

\[
v(y) = 0, \quad \text{at } y_3 = 0.
\]

This condition is satisfied if and only if (per component \( \ell, \ell \in \{1, 2, 3\} \))

\[
v^{(\ell)}_{i,j,0} = 0, \quad \forall i + j = m, \; m \geq 0.
\]

These coefficient relations are much simpler than their counterpart in (2.38). They are identical to those derived for a flat boundary surface (2.22). Condition (2.57) is equivalent to a common factor \( y_3 \) in all the vector functions \( \hat{v}_m(y) \),

\[
\hat{v}_{m+1}(y) = y_3 \hat{v}_m(y), \quad m \geq 0.
\]

with each \( \hat{v}_m(y) \in H_m^3 \). Note that the substitution proposed in (2.58) excludes the case \( m = 0 \). This case could be excluded because \( v_0(y) \equiv 0 \).

Continuity equation

Precisely as the continuity equation (2.1) for \( u(x) \) lead to coefficient relations (2.9), so does the continuity equation (2.54) for \( v(y) \) lead to

\[
(i + 1)v^{(1)}_{i+1,j,k} + (j + 1)v^{(2)}_{i,j+1,k} + (k + 1)v^{(3)}_{i,j,k+1} = 0,
\]

\( \forall i + j + k = m - 1, \; m \geq 1 \). If the coefficient relation (2.57) is inserted into (2.59), one also finds that

\[
v^{(3)}_{i,j,1} = 0, \quad \forall i + j = m - 1, \; m \geq 1.
\]

Using the substitution proposed in (2.58), the polynomial expansion of index \( n \) of the vector function \( v(y) \) expanded near a point on the boundary surface can be written as

\[
v^{[n]}(y) = y_3 \hat{v}^{[n-1]}(y), \quad \hat{v}^{[n-1]}(y) = \sum_{m=0}^{n-1} \hat{v}_m(y),
\]
where the following coefficient relations hold,

\[
(i + 1)\hat{a}_{i+1,j,k} + (j + 1)\hat{a}_{i,j+1,k} + (k + 2)\hat{a}_{i,j,k+1} = 0, \\
\hat{a}_{i,j,0} = 0,
\]

(2.62)

\[\forall i + j + k = m, \text{ and } \forall i + j = m, \text{ respectively, with } m \geq 0. \]

The coefficient relations in (2.62) show that all the coefficients of the velocity vector component function \(\hat{a}^{(3)}(y)\) are either zero or directly related to the coefficients of the velocity vector component functions \(\hat{a}^{(1)}(y)\) and \(\hat{a}^{(2)}(y)\).

**Vorticity transport equation**

Although \((y_1, y_2, y_3)\) is a Cartesian coordinate system, we cannot simply use the Cartesian vorticity transport equation representation (2.3). For the Navier-Stokes equation of motion are only invariant under the Euclidean group \(\mathbb{E}_3\) of all translations, rotations and inflections (i.e., mirroring) of space, according to Sattinger and Weaver [SW86] and Langford (in Golubitsky et al. [GSS88]). Since we have an expression for coordinates \(y\) in terms of the coordinates \(x\), it is possible to express the coefficients in the series expansion of \(\nu(y)\) in terms of the coefficients in the Taylor series expansion of \(\nu(x)\). This way it should be possible, in principle, to find relations between the coefficients in the series expansion of \(\nu(y)\).

A coordinate transformation that has a Jacobian matrix which equals identity in the origin is called a near-identity transformation. The volume-preserving projection (2.47) of the boundary surface to its tangential plane in the origin thus also is a near-identity transformation. The next chapter will explain how a near-identity transformation interacts with a vector field. Since (2.42), (2.43) and (2.44) relate coefficients of \(u_3(x)\) to one another, we will, after application of (2.47), find that the coefficients of \(\nu_3(y) = y_3\hat{u}_2(y)\) are related to one another. As a result, the vorticity transport equation leads to relations between the coefficients in \(\hat{u}_2(y)\).

At this point, not having read the next chapter, such a statement may not be satisfactory. Therefore, we will write out some of the computations involved.

Section 3.2.7 will explain how the series of one vector function is related to the series of another vector function if they are mapped to another by near-identity transformation \(x = y + t_3(y) + \ldots \) like (2.47). Up to second order, the relations are

\[
\nu_1(y) = y_3\hat{u}_0(y) = \left\{ u_1(x) \right\}(y), \quad (2.63)
\]
and

\[ v_2(y) = y_3 \hat{u}_1(y) = \left\{ y_2(x) + \frac{\partial u_1(x)}{\partial x} t_2(x) - \frac{\partial t_2(x)}{\partial x} u_1(x) \right\}(y). \] (2.64)

As can be seen from the latter expression, \( u_3(x) \) does not enter the expression of \( \hat{u}_1(y) \).

The reader may wonder if (2.63) and (2.64) actually produce expressions such that the coefficient relations previously derived in (2.62) are satisfied. From (2.47), we get

\[ t_2(y) = \begin{pmatrix} 0 \\ 0 \\ h_{20}y_1^2 + h_{11}y_1y_2 + h_{02}y_2^2 \end{pmatrix} \] (2.65)

If (2.65), (2.45) and (2.46) are substituted into (2.63) and (2.64), we get

\[ y_3 \hat{u}_0(y) = y_3 \begin{pmatrix} u^{(1)}_{001} \\ u^{(2)}_{001} \\ 0 \end{pmatrix}, \] (2.66)

and

\[ y_3 \hat{u}_1(y) \equiv y_3 \hat{u}_1(y) = y_3 \begin{pmatrix} u^{(1)}_{101} & u^{(1)}_{011} & u^{(1)}_{002} \\ u^{(2)}_{101} & u^{(2)}_{011} & u^{(2)}_{002} \\ 0 & 0 & -\frac{1}{2}(u^{(1)}_{101} + u^{(2)}_{011}) \end{pmatrix} y. \] (2.67)

Note that none of the coefficients in \( A \) contains derivatives of the displacement function \( h(x_1, x_2) \). These expressions show that there is an almost trivial relation between the coefficients of \( \hat{v}(y) \) and the coefficients of \( \hat{u}(x) \) up to first order.

The Jacobian matrix \( \hat{A} \) of \( \hat{v}(y) \) in (2.67) precisely equals the Jacobian matrix \( \hat{A} \) of \( \hat{u}(x) \) in (2.32). Matrix \( \hat{A} \) thus governs the local flow patterns near a critical point on a flat as well as a curved boundary surface.

Wu, Gu and Wu [WGW88] used curvilinear coordinates aligned to the boundary surface. Curvilinear coordinates have unit base vectors which are mutually perpendicular everywhere. This requirement is too stringent for our application. Their results, however, support the findings of the above analysis.
2.4 Skin-friction vector field

In the case of a flat boundary surface \( x_3 = 0 \), we found that the trajectories on and above the boundary surface of vector field (2.29),

\[
\dot{x} = \hat{u}(x),
\]

(2.68)

describe the skin-friction lines and the streamlines, respectively, of the original vector field (2.31),

\[
\dot{x} = u(x) = x_3 \hat{u}(x).
\]

(2.69)

We want to know if the vector fields

\[
\dot{y} = \hat{v}(y)
\]

(2.70)

and

\[
\dot{y} = v(y) = y_3 \hat{v}(y)
\]

(2.71)

have the same relation. To this end, we need to take a closer look at the definition of skin-friction.

The \textit{deviatoric stress tensor}, \( d_{ij} \), for a viscous, incompressible fluid is given by

\[
d_{ij} = \frac{1}{Re} \left( \partial_j u^{(i)} + \partial_i u^{(j)} \right),
\]

(2.72)

see [Bat67]. (2.72) was nondimensionalized using the reference value \( \rho U^2 \), resulting in the inverse scaling with the Reynolds number. For every point on the boundary surface \( \partial B \) the deviatoric stress in the direction \textit{normal} to the boundary surface equals

\[
\tau(x) = d_{ij}(x) n_j(x), \quad \text{for } x \in \partial B.
\]

(2.73)

This vector is known as the \textit{skin-friction} vector. A \textit{skin-friction line} is a line on the boundary surface whose tangent is along to the skin-friction vector of the flow.

This section shows that for every point \( a \in \partial B \) integration of

\[
\dot{x} = \tau(x), \quad x(t_0) = a,
\]

(2.74)

leads to trajectories with \( x(t) \in \partial B \) for all \( t \in U \), where \( U \) is a small enough open subset of \( \mathbb{R} \) around \( t_0 \). Note that this statement is equivalent with the nontrivial requirement that \( \tau_i n_i = d_{ij} n_j n_i = 0 \).
Flat boundary surface

For a flat boundary surface, $x_3 = 0$, the normal $n_j$ is given by $(0,0,1)^t$. From definition (2.73) follows that

$$\tau(x) = d_{ij}(x)n_j = \frac{1}{Re} \begin{pmatrix} \partial_3 u^{(1)} + \partial_1 u^{(3)} \\ \partial_3 u^{(2)} + \partial_2 u^{(3)} \\ 2\partial_3 u^{(3)} \end{pmatrix}, \quad \text{on } x_3 = 0. \quad (2.75)$$

Due to the no-slip condition and the continuity equation, the coefficients $u_{i,j0}^{(x)}$ and $u_{i,j1}^{(3)}$ are zero, see (2.57) and (2.60). Therefore, the following expression results:

$$\tau(x) = \frac{1}{Re} \begin{pmatrix} \partial_3 u^{(1)}(x) \\ \partial_3 u^{(2)}(x) \\ 0 \end{pmatrix}, \quad \text{on } x_3 = 0. \quad (2.76)$$

The requirement $\tau_i n_i = 0$ is indeed satisfied since the third component equals zero. The right-hand side $\hat{u}(x)$ in the vector field (2.68) is always along the skin-friction vector $\tau(x)$ on the boundary surface, since $\partial_3 u^{(x)}(x) = x_3 \partial_3 \hat{u}^{(x)}(x) + \hat{u}^{(x)}(x) = \hat{u}^{(x)}(x)$ on $x_3 = 0$ and $\hat{u}^{(3)}(x) = 0$ on $x_3 = 0$. Hence, the trajectories on the boundary surface of vector field (2.68) are identical to the skin-friction lines.

Curved boundary surface

First, a proof is given that

$$n_i = (-h,1,-h,2,1)^t. \quad (2.77)$$

is the normal at the boundary surface. Second, we assert that $\tau_i n_i = 0$.

The vector function $F(x) = x_3 - h(x_1, x_2)$ equals zero for every point on the boundary surface. Let $\xi(t) = (x_1(t), x_2(t), x_3(t))$ be a parameterized curve on the surface through a point $a = x(t_0)$, and let $U$ be a open subset of $\mathbb{R}$ around $t_0$, i.e.,

$$F(\xi(t)) = 0, \quad \forall t \in U. \quad (2.78)$$

Differentiate (2.78) with respect to $t$ and evaluate at $t_0$ to obtain

$$\sum_{i=1}^{3} F_{,i} a_i \cdot \xi_i(t_0) = 0.$$
In general, neither \( \dot{z}_i(t_0) \neq 0 \) nor \( \nabla F(\mathbf{a}) \neq 0 \). Since \( \dot{z}_i(t_0) \) is a vector tangent to \( \partial B \) at \( \mathbf{a} \) and \( \mathbf{z}(t) \) is an arbitrary curve in \( \partial B \), \( \nabla F(\mathbf{a}) \) must be the normal to \( \partial B \) at \( \mathbf{a} \). Then, because \( \nabla F = (-h_1, -h_2, 1)^t \), the first objective is proven (excerpt from §6.11.10 in [ABG86]). Finally note that, in some way, this normal vector was already present in the expression for the vector field in the projected coordinates, see (2.50).

Proving the second objective is more laborious. The first component of the skin-friction vector, \( \tau^{(1)} \), is found using its definition (2.74), which is the product of the deviatoric stress tensor (2.72) with the normal (2.77).

\[
Re \tau^{(1)} = Re d_{1j} \quad n_j = 2(\partial_{x_1} u^{(1)})(-h_1) + \left( (\partial_{x_2} u^{(1)} + \partial_{x_1} u^{(2)})(-h_2) + \partial_{x_3} u^{(1)} + \partial_{x_1} u^{(3)} \right), \quad \text{for } \mathbf{x} \in \partial B. \tag{2.79}
\]

Note that (2.79) is stated in local Cartesian coordinates. Rewrite (2.79) by switching to the projected coordinates,

\[
Re \tau^{(1)} = 2 \left[ (\partial_{y_1} - h_1 \partial_{y_3})u^{(1)} \right] (-h_1) + \left[ (\partial_{y_2} - h_2 \partial_{y_3})u^{(1)} + (\partial_{y_1} - h_1 \partial_{y_3})u^{(2)} \right] (-h_2) + \partial_{y_3} u^{(1)} + (\partial_{y_1} - h_1 \partial_{y_3})u^{(3)}, \quad \text{on } y_3 = 0. \tag{2.80}
\]

In (2.80), the combination \( \partial_{y_3} (u^{(3)} - h_1 u^{(1)} - h_2 u^{(2)}) \) can be replaced by \( \partial_3 v^{(3)} \) using (2.50). Similar, \( \partial_{y_1} u^{(3)} = \partial_1 v^{(3)} + (\partial_{y_1} - h_1 \partial_{y_3})u^{(1)} + h_2 u^{(2)} \).

\[
Re \tau^{(1)} = \left( 1 + (h_1)^2 + (h_2)^2 \right) \partial_3 v^{(1)} - 2h_1 \partial_1 v^{(1)} - h_1 \partial_3 v^{(3)} + h_1 \partial_2 v^{(2)} - h_2 \partial_2 v^{(1)} - h_2 \partial_3 v^{(2)}, \quad \text{on } y_3 = 0. \tag{2.81}
\]

Most of the terms in (2.81) vanish due to the coefficient relations (2.57) and (2.60). The resulting expression reads

\[
\tau^{(1)} = \frac{1}{Re} \left( 1 + (h_1)^2 + (h_2)^2 \right) \partial_3 v^{(1)}, \quad \text{on } y_3 = 0. \tag{2.82}
\]

Similar it can be proven that

\[
\tau^{(2)} = \frac{1}{Re} \left( 1 + (h_1)^2 + (h_2)^2 \right) \partial_3 v^{(2)}, \quad \text{on } y_3 = 0. \tag{2.83}
\]

The computation of the third component of the skin-friction vector, \( \tau^{(3)} \), runs slightly different. As before, use the definition and projected coordinates
to obtain

\[ Re \tau^{(3)} = Re \partial_{y_3} n_j = \left[ \left( \partial_{y_3} - h_1 \partial_{y_1} \right) u^{(3)} + \partial_{y_3} u^{(1)} \right] (-h_1) + \\
+ \left[ \left( \partial_{y_3} - h_2 \partial_{y_2} \right) u^{(3)} + \partial_{y_3} u^{(2)} \right] (-h_2) + \\
+ 2 \partial_{y_3} u^{(3)}, \quad \text{on } y_3 = 0. \]  

(2.84)

Again \( \partial_3 v^{(3)} \) is found as a combination of terms. However, the last term in (2.84) reads \( 2 \partial_{y_3} u^{(3)} \), with the emphasis on \( 2 \). This time use \( \partial_{y_3} u^{(3)} = \partial_3 v^{(3)} + h_1 \partial_3 v^{(1)} + h_2 \partial_3 v^{(2)} \). Collecting all terms, and application of (2.57) and (2.60) leads to

\[ \tau^{(3)} = \frac{1}{Re} \left( 1 + (h_1)^2 + (h_2)^2 \right) \left( h_1 \partial_3 v^{(1)} + h_2 \partial_3 v^{(2)} \right), \quad \text{on } y_3 = 0. \]  

(2.85)

It is easy to see that indeed, \( \tau_i n_i = 0 \).

We shall, however, use (2.82), (2.83) and (2.85) in another form. With the aid of (2.50), we obtain

\[ \dot{y} = \tau(y) \equiv \frac{1}{Re} \left( 1 + (h_1)^2 + (h_2)^2 \right) \begin{pmatrix} \partial_3 v^{(1)}(y) \\ \partial_3 v^{(2)}(y) \\ 0 \end{pmatrix}, \quad \text{on } y_3 = 0. \]  

(2.86)

From this expression it is easy to verify that also in the projected coordinates, the trajectories (skin-friction lines) of (2.86) remain on the boundary surface. The right-hand side \( \dot{y}(x) \) in the vector field (2.70) is always along the skin-friction vector \( \tau(y) \) on the boundary surface, since \( \partial_3 v^{(\ell)}(y) = y_3 \partial_3 \dot{v}^{(\ell)}(y) + \dot{v}^{(\ell)}(y) = \dot{\nu}^{(\ell)}(y) \) on \( y_3 = 0 \) and \( \dot{v}^{(3)}(y) = 0 \) on \( y_3 = 0 \). The (always positive) scale factor \( Re^{-1} \left( 1 + (h_1)^2 + (h_2)^2 \right) \) in (2.86) only affects the magnitude and not the direction along a trajectory. Hence, the trajectories on the boundary surface of vector field (2.70) are identical to the skin-friction lines.

This concludes our proof of the second objective.

### 2.5 Describing the flow near a boundary surface

The vector field \( \hat{v}(y) \) in (2.70) enables us to make a clear distinction between the streamlines and the skin-friction lines. The former are found in the half-space \( y_3 > 0 \) whereas the latter are found on the coordinate-plane \( y_3 = 0 \). We effectively simplified the representation of the geometry of the boundary surface from the curved surface \( x_3 = h(x_1, x_2) \) to the flat surface \( y_3 = 0 \).

If the vector function \( \hat{v}(y) \) in the right-hand side of the vector field (2.70) is expanded in a series, and that series is truncated, then trajectories of the
resulting vector field with an initial position on the boundary surface \( y_3 = 0 \) never leave that surface. The \( O(\|y\|^{k+1}) \) error of the polynomial expansion will not prevail in the \( y_3 \) direction in any way. The same cannot be said of the trajectories with an initial position on \( x_3 = h(x_1, x_2) \) of the vector field that results after truncation of the Taylor series expansion of the velocity vector \( \mathbf{u}(x) \). Therefore, for an accurate description of the pattern of skin-friction lines, it is not an luxury, but a necessity, to use the projection of the boundary surface to its tangential plane in the origin, and therewith the vector field (2.70).

We found that identical coefficient relations resulted due to the no-slip condition and conservation of mass for the Taylor series expansion of the vector fields \( \hat{\mathbf{u}}(x) \) and \( \hat{\mathbf{v}}(y) \), see (2.25) and (2.62), respectively. Thus, vector fields which describe fluid flow near a boundary surface are part of the following set:

\[
\mathcal{V}_\nu^3 = \{ \mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3 \mid \nabla \cdot (x_3 \mathbf{u}(x)) = 0 \}.
\]  

(2.87)

The subscript \( \nu \) refers to the well-known kinematic viscosity defined as \( \nu = \mu/\rho \) and indicates the application of the no-slip boundary condition for a viscous fluid.

We also found that the vorticity transport equation does not lead to relations between the coefficients in the Jacobian matrix of the vector field \( \hat{\mathbf{u}}(x) \) in the case of the flat boundary surface \( x_3 = 0 \), nor of the vector field \( \hat{\mathbf{v}}(y) \) in the case of the curved boundary surface \( y_3 = 0 \), see (2.46) and (2.67), respectively. Hence, the vorticity transport equation does not lead to relations between the coefficients in the Jacobian matrix of all vector fields in the set \( \mathcal{V}_\nu^3 \).

It was said that since we have an expression of the coordinate transformation, it is possible to obtain the relations between the second and higher order coefficients of the vector field \( \hat{\mathbf{u}}(y) \) based on the relations between the third and higher order coefficients of the vector field \( \mathbf{v}(x) \), see (2.42), (2.43) and (2.44). No such coefficient relations were, however, given. We still need to address this issue in the chapters to come.

The above results were obtained with the aid of an invertible coordinate transformation. To study the nonlinear dynamics of a certain vector field, we need to use (locally) invertible transformations acting on the flow, i.e., the set of all the trajectories, of that vector field. Also, we need to know how stable the topological structure of that vector field is under a certain class of perturbations. Such an analysis is known as a bifurcation analysis.

The following chapter will discuss bifurcation in general vector fields. The next chapter then sets up a bifurcation analysis for vector fields describing fluid
flow. This bifurcation analysis will enable us to construct a topology of fundamental local flow patterns near a boundary surface.

2.6 Example: Two-dimensional fundamental local flow patterns

The vector fields discussed in this chapter so far are defined in three-dimensional space. This section, however, discusses a bifurcation analysis of two-dimensional local flow pattern near a boundary surface found in §4.1 from [Bak88] by Bakker, using this thesis’ notation. The analysis assumes a flat boundary surface which becomes a straight line in two-dimensional space. Throughout this thesis we will use two-dimensional examples because it allows us to to demonstrate the advantages of the techniques under discussion in a more convenient manner.

The analysis starts with the derivation of the Taylor series expansion of the velocity vector \( \bar{u}(\bar{x}) \) about a boundary critical point (in the origin). A substitution similar to (2.61) is used,

\[
\bar{u}(\bar{x}) = x_2 \bar{\bar{u}}(\bar{x}), \quad \bar{\bar{u}}(\bar{x}) = \sum_{m \geq 0} \bar{\bar{u}}_m(\bar{x}),
\]

with each \( \bar{\bar{u}}_m(\bar{x}) \in H^2_m \). The common factor \( x_2 \) is removed leaving a vector field induced by the vector function \( \bar{\bar{u}}(\bar{x}) \),

\[
\begin{align*}
\dot{x}_1 & = \bar{\bar{u}}_1(\bar{x}) = \bar{\bar{u}}_{10} x_1 + \bar{\bar{u}}_{01} x_2 + \bar{\bar{u}}_{20} x_2^2 + \bar{\bar{u}}_{11} x_1 x_2 - \frac{2}{3} \bar{\bar{u}}_{20} x_2^3 + \mathcal{O}(\|\bar{x}\|^3), \\
\dot{x}_2 & = \bar{\bar{u}}_2(\bar{x}) = -\frac{1}{2} \bar{\bar{u}}_{10} x_2 - \bar{\bar{u}}_{20} x_1 x_2 - \frac{1}{3} \bar{\bar{u}}_{11} x_2^2 + \mathcal{O}(\|\bar{x}\|^3). \quad (2.88)
\end{align*}
\]

As can be seen, \( \bar{\bar{u}}_2(\bar{x}) = 0 \) on the line \( x_2 = 0 \). Note that the coefficients are written without the superscripts (1) and (2). They were dropped since all occurring coefficients have superscript (1).

Hyperbolic critical points

For nonzero \( \bar{\bar{u}}_{10} \), the origin is a hyperbolic critical point and the nonlinear terms can be neglected. In that case, the local flow pattern of (2.88) is as depicted in Fig. 2.3.

Nonhyperbolic critical points

Assume \( \bar{\bar{u}}_{10} = 0 \), in which case a double (or repeated) zero eigenvalue results in the Jacobian matrix of (2.88). For nonzero \( \bar{\bar{u}}_{01} \), the vector field (2.88) can be
rewritten as

\[
\begin{align*}
\dot{x}_1 &= \tilde{u}_1(x) = x_2 + \tilde{u}_{20} x_1^2 + \tilde{u}_{11} x_1 x_2 - \frac{2}{3} \tilde{u}_{20} x_2^3 + \mathcal{O}(\|x\|^3), \\
\dot{x}_2 &= \tilde{u}_2(x) = -\tilde{u}_{20} x_1 x_2 - \frac{1}{3} \tilde{u}_{11} x_2^2 + \mathcal{O}(\|x\|^3),
\end{align*}
\]

where the ‘tilde’ indicate a transformation to Jordan canonical form of the linear part through

\[x_1 \rightarrow \tilde{u}_{01} x_1, \quad x_2 \rightarrow x_2.\]  \hfill (2.90)

The notation in [Bak88], however, suggests a transformation of the time parameter; \(t \rightarrow \tilde{u}_{01} t\). The use of either one of these transformations results in a change in the stability of the critical point in the origin if \(\tilde{u}_{01} < 0\).

Andronov et al. [ALGM73] have presented a complete classification of the local flow patterns of all two-dimensional vector fields of the form

\[\dot{x} = y + P(x, y), \quad \dot{y} = Q(x, y),\]  \hfill (2.91)

where the functions \(P(x, y)\) and \(Q(x, y)\) are \(\mathcal{O}(\|x\|^3)\), and \(x = (x, y)^t\). Application of their results learns that (2.89) is a so-called third order topological saddle point if \(\tilde{u}_{20} \neq 0\) and a fifth order topological saddle point if \(\tilde{u}_{20} = 0\) and \(\tilde{u}_{30} \neq 0\), see Fig. 2.4 and Fig. 2.5.

**Unfolding of the third order topological saddle point**

The flow pattern near a nonhyperbolic critical point is sensitive to perturbations in the vector field. Bakker [Bak88] proposes an embedding in a parameter family through a so-called physical unfolding: All zero coefficients are replaced by small parameters. The result reads

\[
\begin{align*}
\dot{x}_1 &= \tilde{u}_1(x; \mu) = \mu_{00} + \mu_{10} x_1 + x_2 + \tilde{u}_{20} x_1^2 + \tilde{u}_{11} x_1 x_2 - \frac{2}{3} \tilde{u}_{20} x_2^3, \\
\dot{x}_2 &= \tilde{u}_2(x; \mu) = -\frac{1}{2} \mu_{10} x_2 - \tilde{u}_{20} x_1 x_2 - \frac{1}{3} \tilde{u}_{11} x_2^2.
\end{align*}
\]  \hfill (2.92)

Only those solutions \(x(\mu) = (x_1(\mu), x_2(\mu))\) which satisfy

\[\tilde{u}_1(x(\mu), \mu) = \tilde{u}_2(x(\mu), \mu) = 0, \quad \lim_{\mu \to 0} x(\mu) = 0,\]  \hfill (2.93)

are used in the bifurcation analysis. To find the critical points, we equate the right-hand side of the vector field (2.92) to zero, and find

\[
\begin{align*}
\tilde{x}_S(\mu) &= \left(-\mu_{10} \pm \sqrt{\mu_{10}^2 - 4 \mu_{00} \tilde{u}_{20}} \right), 2\tilde{u}_{20}, 0),
\end{align*}
\]  \hfill (2.94)
2.6. **Example: Two-dimensional fundamental local flow patterns**

![Diagram](image)

(a) $\hat{u}_{10} < 0$, separation  
(b) $\hat{u}_{10} > 0$, attachment

Figure 2.3: The local flow patterns with a hyperbolic, boundary critical point.

![Diagram](image)

(a) $\hat{u}_{20} < 0$  
(b) $\hat{u}_{20} > 0$

Figure 2.4: Third order topological saddle points.

![Diagram](image)

(a) $\hat{u}_{30} < 0$, separation  
(b) $\hat{u}_{30} > 0$, attachment

Figure 2.5: Fifth order topological saddle points.
and

\[
x_C(\mu) = \left( \frac{-3\mu_{10} + 2\bar{u}_{11}x_C^2}{6\bar{u}_{20}}, x_C \right),
\]

where \(x_C^2\) is a solution of

\[
\mu_{00}\bar{u}_{20} - \frac{1}{2}\mu_{10}^2 + (\bar{u}_{20} - \frac{1}{2}\mu_{10}\bar{u}_{11})x_C^2 - \frac{2}{9}(\bar{u}_{11}^2 + 3\bar{u}_{20}^2)(x_C^2)^2 = 0.
\]

These equations yield at most three critical points which satisfy condition (2.93), since the latter equation results in a solution \(x_C^2 = \frac{3}{4}\bar{u}_{20}/(\bar{u}_{11}^2 + 3\bar{u}_{20}^2) \neq 0\) for \(\mu = 0\) which is undesirable. A so-called parameterization technique, decides that that solution does not influence the local flow pattern. Let \(x_1 = \mu_{10}\bar{x}_1(\mu)\), \(x_2 = \mu_{20}\bar{x}_2(\mu)\), and \(\mu_{00} = k\mu_{10}^2\). With this scaling, we once again equate the right-hand side of the vector field (2.92) to zero, and find

\[
\begin{aligned}
&\left( k + \bar{x}_1 + \bar{x}_2 + \bar{u}_{20}\bar{x}_2^2 \right) \mu_{10}^2 + \mathcal{O}(\mu_{10}^3) = 0, \\
&\left( -\frac{1}{2}\bar{x}_2 - \bar{u}_{20}\bar{x}_1\bar{x}_2 \right) \mu_{10}^3 + \mathcal{O}(\mu_{10}^4) = 0,
\end{aligned}
\]

which according to Bakker [Bak88] shows that the leading terms in \(x_1(\mu)\) and \(x_2(\mu)\) may be obtained from

\[
\begin{aligned}
&\mu_{00} + \mu_{10}x_1 + x_2 + \bar{u}_{20}x_1^2 = 0, \\
&-\frac{1}{2}\mu_{10}x_2 - \bar{u}_{20}x_1x_2 = 0.
\end{aligned}
\]

[Bak88] presents no reason why these particular orders in \(\mu_{10}\) should be used in the parameterization.

If these equations in (2.98) are solved, the following three (approximative) critical points are found: \(x_C(\mu)\) given by (2.94), and

\[
x_C(\mu) = \left( -\frac{\mu_{10}}{2\bar{u}_{20}}, \mu_{10} - 4\mu_{00}\bar{u}_{20} \right).
\]

It turns out that the local flow patterns near the critical points only depend on the sign of \(\bar{u}_{20}\), and the value of \(\mu_c = \mu_{10}^2 - 4\mu_{00}\bar{u}_{20}\). Fig. 2.6 shows the bifurcation diagrams of the vector field (2.89) for the cases \(\bar{u}_{20} < 0\) and \(\bar{u}_{20} > 0\).

Note that the critical point above the boundary surface, \(x_C(\mu)\) is center if \(\bar{u}_{20} > 0 \cap \mu_c > 0\), and a saddle if \(\bar{u}_{20} < 0 \cap \mu_c < 0\). Two-dimensional vector fields describing fluid flow can be written as a Hamiltonian system, see [Bak88]. Within that class, the center is structurally stable.
Figure 2.6: Bifurcation diagrams of the vector field (2.89).

Several questions come to mind about the validity of the above analysis. For example: Why is it valid to truncate the vector field (2.92) at second order? The physical unfolding does include the bifurcation parameter $\mu_c$ but why as the subset of a higher dimensional $\mu_1-\mu_2$ space? Is it necessary to use a physical unfolding? Does the parameterization technique work in three-dimensional space as well? And, how does one find the parameterization powers in general?

These questions attack the validity of the Taylor-polynomial approximation of vector fields. What we really want to know is whether all possible vector fields which have a double zero eigenvalue and nonzero coefficients $u_{01}$ and $u_{20}$ are represented in the bifurcation diagram Fig. 2.6.

The next chapter will discuss techniques from bifurcation theory (in a general setting) to analyze vector fields depending on small parameters in a systematic manner.

2.7 Discussion

This chapter presented the assumptions made about the flow, the fluid, and the corresponding flow equations. A Taylor series expansion of the velocity vector was substituted into the flow equations and we counted the number of relations produced and the number of coefficients introduced at each order in
the expansion. We found that the latter always exceeds the former.

We expected to find that the skin-friction lines lie on an invariant manifold in the vector field, the invariant manifold, of course, being the boundary surface. However, we encountered some difficulties when we used a Taylor-polynomial vector field in the case of a curved boundary surface. These difficulties were resolved when the boundary surface was projected on to the tangent plane in the point of expansion, which became a coordinate-plane in the new coordinates system. We were able to prove that the skin-friction lines lie in that coordinate-plane.

We introduced the set of vector fields \( \mathbb{V}_x^3 \) in (2.87) which describes the three-dimensional flow in the neighborhood of a point located on a boundary surface. It was shown that the vorticity transport equation does not produce any relations between the coefficients in the Jacobian matrix of these vector fields.

At the end of the chapter, we discussed a bifurcation analysis of vector fields describing the fluid flow near a boundary surface in two-dimensional space, and found a topology of fundamental local flow patterns. However, several \textit{ad hoc} techniques were needed to analyze the nonlinear dynamics of the vector fields. This made us question the validity of the Taylor-polynomial approximation of the vector fields.

The next chapter will discuss techniques from bifurcation theory to analyze vector fields depending on small parameters in a systematic manner.
Chapter 3

Bifurcation in General

*All in all it's just another brick in the wall...*

—Pink Floyd, The Wall

**Introduction**

In the previous chapter, we computed the Taylor series expansion of the velocity vector about a point on a curved boundary surface. When the series was truncated, we found that the resulting Taylor-polynomial vector field did not contain the expected invariant manifold (i.e., the curved boundary surface) filled with skin-friction lines. Through a change of coordinates, which projected the boundary surface onto the tangential plane in the point of expansion, we found another vector field with a series expansion that did contain the desired invariant manifold in a coordinate plane. As a result, the invariant manifold remains after truncation of the series.

Because the two vector fields are linked by a so-called homeomorphism, they have topological equivalent local flow patterns. Topological equivalence is a key theme in nonlinear dynamics and bifurcation theory. Other key themes are structural stability, normal form, versal deformation, and co-dimension.

This chapter discusses these and other related concepts in a general setting. Throughout the chapter, two-dimensional vector fields will be used as examples.

Bifurcation theory is a vast and still rapidly expanding field of research and as such, it is impossible to give full credit to each of its contributors. During the last few years several good textbooks on nonlinear dynamics have appeared for graduate students in applied mathematics. Excellent introductions into nonlin-
ear dynamics and bifurcation theory which I enjoyed reading very much are: (from popular to literature, no disrespect intended) Strogatz [Str94], Kuznetsov [Kuz95], Argyris, Faust and Haase [AFH94], Hale and Koçak [HK91], Wiggins [Wig90], Guckenheimer and Holmes [GH83], and Chow and Hale [CH82]. It is from these books that I will shamelessly lend many notations, definitions, and theorems, and then only in a concise manner. For a more consistent treatment of the field, I warmly recommend the novice and/or interested reader to look up any one of these books.

3.1 Bifurcation in Vector fields

3.1.1 Topological equivalence

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be $C^r$ if it is at least $r$ times differentiable and each derivative is continuous; if $r = 0$, the functions is said to be continuous. A $C^r$ diffeomorphism is an invertible function such that both the function and the inverse are $C^r$. A homeomorphism is a $C^0$ diffeomorphism.

Consider two $C^r$ vector fields

$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}), \quad f : U \rightarrow \mathbb{R}^n,$$

$$\frac{d\mathbf{y}}{d\tau} = g(y), \quad g : V \rightarrow \mathbb{R}^n,$$

where $U$ and $V$ are two regions in $\mathbb{R}^n$. Recall that a region in $\mathbb{R}^n$ is an open connected set in $\mathbb{R}^n$. Let $\phi(t, \mathbf{x}) : I \times U \rightarrow \mathbb{R}^n$ and $\psi(\tau, y) : I^* \times V \rightarrow \mathbb{R}^n$ be the flow generated by the vector fields induced by $f$ and $g$, respectively. Thus, $\phi(t, \mathbf{x})$ is a smooth function defined for all $\mathbf{x}$ in $U$ and $t$ in some interval $I \subseteq \mathbb{R}$, and $\phi$ satisfies the $n$ ordinary differential equations (3.1), i.e.,

$$\frac{d\phi}{dt}(t_0, \mathbf{x}) = f(\phi(t_0, \mathbf{x})), \quad$$

Figure 3.1: Topological equivalence
for all \( \bar{x} \in U \) and \( t_0 \in I \) (similarly for \( \psi(\tau, y) \)).

**Definition 3.1**

The dynamics of the vector field \( f \) in \( U \) is said to be \( C^k \) equivalent \((k \leq r)\) to the dynamics generated by the vector field \( g \) in \( V \) if there exists a \( C^k \) diffeomorphism \( h : \mathbb{R}^n \to \mathbb{R}^n \), \( h(U) = V \), mapping the trajectories of the flow generated by \( f \), \( \phi(t, \bar{x}) \), onto trajectories of the flow generated by \( g \), \( \psi(\tau, y) \) preserving orientation but not necessarily parameterization by time, i.e.,

\[
h \circ \phi(t, \bar{x}) = \psi(\alpha(\bar{x}, t), h(\bar{x})),
\]

where \( \alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is some increasing function of time along the trajectories (note: \( \alpha \) must be increasing to preserve orientation of trajectories). The map \( h \) is called the conjugating \( C^k \) diffeomorphism. If \( k = 0 \), \( f \) and \( g \) are said to be topological equivalent. If \( h \) does preserve parameterization by time, then the dynamics generated by \( f \) and \( g \) are said to be \( C^k \) conjugate.

There is a big difference between working with \( C^0 \) diffeomorphisms (i.e., homeomorphisms) and \( C^k \), \( k \geq 1 \), diffeomorphisms. \( C^k \) equivalence implies that \( \bar{x} \in U \) and \( y \in V \) are related by a change of coordinates \( h, y = h(\bar{x}) \). By substitution of the change of coordinates directly into the vector field \( g \) in (3.1), we get

\[
Dh(\bar{x}) \frac{d\tau}{dt} = g(h(\bar{x})), \quad Dh = \frac{\partial h}{\partial \bar{x}}.
\]

Since \( \tau = \alpha(\bar{x}, t) \) is an increasing function of time along trajectories, the last term in the left-hand side of (3.3) is always positive and nonzero. The first term in the left-hand side of (3.3) is the Jacobian matrix of \( h \). Thus, for the substitution to be valid, the function \( h \) must at least be differentiable once. Working with the flow generated by both vector fields avoids this difficulty.

If \( h \) is \( C^k \), \( k \geq 1 \), diffeomorphism, \( h^{-1} \) exists and is differentiable. Also, by the inverse function theorem, we have \( Dh^{-1} = (Dh)^{-1} \). By using (3.1) once again we get

\[
f(\bar{x}) = \frac{d\tau}{dt} (Dh)^{-1} g(h(\bar{x})).
\]

Let \( f : U \to \mathbb{R}^n \) and \( g : V \to \mathbb{R}^n \) be two \( C^r \), \( r \geq 1 \), vector fields which are \( C^k \), \( k \leq r \), equivalent and let \( h : U \to V \) be a conjugating \( C^k \) diffeomorphism.

**Proposition 3.2**

The following properties hold.
1. critical points of $f$ are mapped to critical points of $g$;

2. periodic trajectories of $f$ are mapped to periodic trajectories of $g$, but the periods need not be equal;

3. let $1 \leq k \leq r$ and $f(x_0) = 0$; then the eigenvalues of $Df(x_0)$ and the eigenvalues of $Dg(h(x_0))$ differ by a positive multiplicative constant.

$\Rightarrow$

**Proof of Proposition 3.2**

Let $x_0$, be a critical point of $f$, i.e., for all $t$,

$$\phi(t, x_0) = x_0. \quad (3.5)$$

Denote by $y_0 = h(x_0)$. We have to prove that for all $\tau$,

$$\psi(\tau, y_0) = y_0. \quad (3.6)$$

If $f$ and $g$ are $C^k$ equivalent, (3.2) holds, and thus for all $t$,

$$h \circ \phi(t, x_0) = y_0 = \psi(\alpha(x_0, t), y_0).$$

Since $\alpha$ is an increasing function of time, there are some $\tau$ for which $\psi(\tau, y_0) = y_0$. The counterpart of (3.2),

$$\phi(\beta(y, \tau), h^{-1}(y)) = h^{-1} \circ \psi(\tau, y), \quad (3.7)$$

also holds for some function $\beta(y, \tau)$. Since $x_0 = h^{-1}(y_0)$, and because (3.5) holds for all $t$, insertion of $y_0$ in (3.7) results in

$$x_0 = h^{-1} \circ \psi(\tau, y_0).$$

Taking $h$ on both sides, results in (3.6). Hence, 1) is proven.

2) is trivial since $C^k$ diffeomorphisms map closed curves to closed curves. (If this were not true, then the inverse would not be continuous.)

To prove 3), differentiate (3.4) to obtain,

$$Df(x) = \{D\frac{d}{dt}\big|_{\mathbb{E}}(Dh)^{-1} + \frac{d}{dt}\big|_{\mathbb{E}} D(Dh)^{-1}\}g(h(x))$$

$$+ \frac{d}{dt}\big|_{\mathbb{E}} (Dh)^{-1} Dg\big|_{h(x)} Dh(x) \quad (3.8)$$
Substitution of the critical points \( x_0 \) and \( y_0 = h(x_0) \), leads to
\[
D_f(x_0) = \frac{d}{dt}(x_0) M^{-1} D_g(y_0) M,
\]
where \( M \equiv D_h(x_0) \) is a nonsingular matrix since \( h \) is a diffeomorphism. Relation (3.9) shows that the matrices \( D_f(x_0) \) and \( D_g(y_0) \) are similar and thus that their eigenvalues are identical (apart from a positive multiplicative constant).

Note: 3) needs more sophistication in the case \( k = 1 \) to avoid the entry \( D(Dh)^{-1} \) in (3.8). The above proof follows Guckenheimer and Holmes [GH83] page 42, and Wiggins [Wig90] page 233, which have the same flaw.

To demonstrate the difference between \( C^k, k \geq 1 \), equivalence and topological equivalence consider the following example from Kuznetsov [Kuz95]. The critical point in the origin \( x = (x_1, x_2)^t = 0 \) of the vector field
\[
\begin{align*}
\dot{x}_1 &= -x_1 \\
\dot{x}_2 &= -x_2
\end{align*}
\]
(3.10)
is a stable star node, whereas the critical point in the origin \( y = (y_1, y_2)^t = 0 \) of the vector field
\[
\begin{align*}
\dot{y}_1 &= -y_1 - y_2 \\
\dot{y}_2 &= y_1 - y_2
\end{align*}
\]
(3.11)
is a stable spiral, see Fig. 3.2. The two vector fields cannot be \( C^k \) equivalent, \( k \geq 1 \), because the eigenvalues of the first vector field \( (\lambda_1 = \lambda_2 = -1) \) differ from those of the second vector field \( (\lambda_1, \lambda_2 = -1 \pm i) \). Nevertheless, the two vector fields are topological equivalent, for example, in open unit discs centered around the origin,
\[
U = \{(x_1, x_2)^t : x_1^2 + x_2^2 < 1\}, \quad V = \{(y_1, y_2)^t : y_1^2 + y_2^2 < 1\}.
\]
(3.12)
The construction of the homeomorphism \( h : U \to V \) runs as follows. In polar coordinates \( (\rho, \theta) \) the two vector fields can be written as
\[
\begin{align*}
\dot{\rho} &= -\rho \\
\dot{\theta} &= 0
\end{align*}
\]
and
\[
\begin{align*}
\dot{\rho} &= -\rho \\
\dot{\theta} &= 1,
\end{align*}
\]
respectively. Thus, by straightforward integration we get

\begin{align}
\rho(t) &= \rho_0 e^{-t} \\
\theta(t) &= \theta_0, \tag{3.13}
\end{align}

for the first vector field, and

\begin{align}
\rho(t) &= \rho_0 e^{-t} \\
\theta(t) &= \theta_0 + t, \tag{3.14}
\end{align}

for the second vector field.

Take a point \( \bar{x} \neq 0 \) in \( U \) with polar coordinates \((\rho_0, \theta_0)\) and consider the time \( \tau \) required to move, along a trajectory of vector field (3.10), from the point \((1, \theta_0)\) on the disc’s boundary to the point \( \bar{x} \). This time only depends on \( \rho_0 \) and equals

\[ \tau(\rho_0) = -\ln \rho_0. \]

The trajectory starting from \((1, \theta_0)\) on boundary of \( V \) runs through the point \( y \) with polar coordinates \((\rho_1, \theta_1)\) at time \( \tau(\rho_0) \). Thus, the map \( y = h(\bar{x}) \) that transforms \( \bar{x} = (\rho_0, \theta_0) \neq 0 \) to \( y = (\rho_1, \theta_1) \) is given by

\[ h : \begin{cases} 
\rho_1 = \rho_0, \\
\theta_1 = \theta_0 - \ln \rho_0
\end{cases} \tag{3.15} \]

see (3.13) and (3.14). Remember that a critical point in \( U \) needs to be mapped to a critical point in \( V \). Consequently, for \( \bar{x} = 0 \), set \( y = 0 \), that is, \( h(0) = 0 \).
3.1 Bifurcation in Vector fields

Figure 3.3: The construction of the homeomorphism

Geometrically, the constructed map transforms $U$ to $V$ by rotating each circle $\rho_0 = \text{constant}$ by a $\rho_0$-dependent angle. This angle increases as $\rho_0 \to 0$, but is, by definition, zero at $\rho_0 = 0$. The map is continuous and invertible, and maps trajectories of (3.10) onto trajectories of (3.11) while preserving time direction. Thus, the map is a homeomorphism and the two vector fields are topological equivalent. From the construction it should be obvious that the map is not differentiable in the origin.

Linearized vector field

Consider a $C^r$, $r \geq 1$, vector field

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad (3.16)$$

where $f$ is defined on a sufficiently large open set of $\mathbb{R}^n$. Let $x_0$ be an hyperbolic critical point of (3.16), i.e.,

$$f(x_0) = 0,$$

and $Df(x_0)$ has no eigenvalues on the imaginary axis. The next theorem holds for the linearized vector field of (3.16) near $\xi = 0$,

$$\dot{\xi} = Df(x_0)\xi, \quad \xi \in \mathbb{R}^n. \quad (3.17)$$

**Theorem 3.3** Hartman-Grobman theorem

$\blacktriangleright$ The flow generated by (3.17) is $C^0$ conjugate to the flow generated by (3.16) in a neighborhood of the critical point $x_0$. $\blacktriangleright$
Proof: see Arnol’d [Arn73].

The advantage of a $C^k$, $k \geq 1$, diffeomorphism over a homeomorphism is that the former enables an explicit calculation of the coefficients in the series expansion of the target vector field in terms of coefficients of the series expansion of the original vector field, of course, up to $k$th order. Proofs of topological equivalence using homeomorphisms are usually given in an indirect manner using implicit function theorem. Also, the target vector field has to be known in advance. In the case of Hartman-Grobman theorem, for example, the target vector field is the linearized vector field. However, the target vector field is usually unknown. $C^k$ equivalence provides a good idea about the possible outcome.

**Linear vector fields**

The flow of a linear vector field

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n,$$

(3.18)

near a point $x(0) = x_0$ is given by

$$\phi(x_0, t) = e^{tA}x_0,$$

(3.19)

where $e^{tA}$ is a $n \times n$ matrix defined by the convergent series

$$e^{tA} \equiv \sum_{i \geq 0} \frac{t^i}{i!}A^i = [I + tA + \frac{t^2}{2!}A^2 + \ldots].$$

(3.20)

The flow of the linear vector field (3.18) and the flow generated by the vector field induced by the Jordan normal form,

$$\dot{y} = Jy \equiv M^{-1}AMy$$

(3.21)

are related as

$$e^{tA} = Me^{tJM^{-1}}.$$  

(3.22)

The invertible matrix $M$ consists of the (generalized) eigenvectors of $A$. Note that the flow of the vector fields (3.18) and (3.21) is topological equivalent through the changes of coordinates $x = My$. For more details see [GHS83], page 8, f.f.

The flow $e^{tA} : \mathbb{R}^n \to \mathbb{R}^n$ has linear subspaces spanned by the eigenvectors. The matrix $M$ can be rearranged in such a manner that the eigenvalues with negative real part are followed by the eigenvalues with positive real part which
in turn are followed by the eigenvalues with zero real part. Hence, the spectrum of \( A \) divides into three parts with the names \( \sigma_s \), \( \sigma_u \), and \( \sigma_c \), respectively. The corresponding (generalized) eigenvectors are denoted as \( y^i, i \in \{1, \ldots, n_s\} \), \( u^i, i \in \{1, \ldots, n_u\} \), and \( w^i, i \in \{1, \ldots, n_c\} \), respectively, with \( n_s + n_u + n_c = n \). The subspaces spanned by the eigenvectors are divided into three classes:

- the stable subspace, \( E^s = \text{span}\{y^1, \ldots, y^{n_s}\} \),
- the unstable subspace, \( E^u = \text{span}\{u^1, \ldots, u^{n_u}\} \),
- the center subspace, \( E^c = \text{span}\{w^1, \ldots, w^{n_c}\} \).

The names reflect the fact that the trajectories lying in \( E^s \) are characterized by exponential decay, those lying in \( E^u \) by exponential growth, and those lying in \( E^c \) by neither. The latter trajectories either oscillate at constant amplitude (if \( \lambda, \bar{\lambda} = \pm i \beta \)) or remain constant (if \( \lambda = 0 \)).

### 3.1.2 Structural stability

Denote by \( C^r(M, M) \) the space of \( C^r \) functions on a compact, boundaryless \( n \)-dimensional manifold \( M \subset \mathbb{R}^n \) into \( M \). Two elements of \( C^r(M, M) \) are said to be \( C^k \) \( \varepsilon \)-close \( (k \leq r) \), or just \( C^k \) close, if they, along with their first \( k \) derivatives, are with in \( \varepsilon \) as measured in some norm. The topology induced on \( C^r(M, M) \) by this measure of distance is called the \( C^k \) topology. In this topology the following property is defined.

**Definition 3.4 Structural stability**

Let \( f : M \to M \) be a \( C^r \) diffeomorphism (resp. a \( C^r \) vector field in \( C^r(M, M) \)); then \( f \) is called structurally stable if there exists a neighborhood \( \mathcal{N} \) of \( f \) in the \( C^k \) topology such that \( f \) is \( C^0 \) conjugate (resp. \( C^0 \) equivalent) to every function (resp. vector field) in \( \mathcal{N} \).

**Theorem 3.5**

\(< \) If the \( n \times n \) matrix \( A \in \mathbb{R}^{n^2} \) has \( k \) eigenvalues with negative real part \( (1 \leq k \leq n) \) and \( n - k \) eigenvalues with positive real part, then the dynamical system \( \psi(x, t) = e^{tA}x \) is globally topological equivalent to the dynamical system \( \psi(x, t) = (e^{-t}l_k y, e^{t(l_{n-k}-k)}z) \), \( (3.23) \)

where \( x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \), and \( l_k \) and \( l_{n-k} \) are the identity matrices in \( \mathbb{R}^{k^2} \) and \( \mathbb{R}^{(n-k)^2} \), respectively. \( > \)

Proof: see Medved' [Med92], page 121, ff.

To see the ramifications of this theorem, we consider the flow of planar vector fields.
3.1.3 2-D example. Hyperbolic critical points

Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be \( C^1 \), and let \( \mathbf{x}_0 \) be a hyperbolic critical point, i.e., \( f(\mathbf{x}_0) = 0 \), and the eigenvalues \( \lambda_1, \lambda_2 \) of \( A = Df(\mathbf{x}_0) \) have nonzero real parts. Hartman-Grobman theorem states that the phase portrait of the vector field generated by \( f \) near \( \mathbf{x}_0 \) is topological equivalent with

\[
\dot{\mathbf{\xi}} = A \mathbf{\xi} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}
\]  

(3.24)

near \( \mathbf{\xi} = 0 \). The well known classification of (3.24) is usually depicted in the so-called \( p-q \) chart, see Fig. 1.6. The two variables \( p \) and \( q \) are defined as

\[
p = \text{tr} A = a + d, \quad q = \text{det} A = ad - bc,
\]  

(3.25)

and related to the eigenvalues of \( A \) as \( p = \lambda_1 + \lambda_2 \) and \( q = \lambda_1 \lambda_2 \) or

\[
\lambda_1 = \frac{1}{2} p - \sqrt{p^2 - 4q}, \quad \lambda_2 = \frac{1}{2} p + \sqrt{p^2 - 4q}.
\]  

(3.26)

The heavy dots in Fig. 1.6 represent the critical point and the arrows indicate the direction of the vector field on the trajectories. The phase portraits \( I \) and \( II \) of the unstable star node and inflected node shown separately also have stable counterparts on left-hand side of the parabola \( p^2 - 4q = 0 \).

Remember from the discussion in Section 3.1.1 that a star node and a spiral are topological equivalent but not \( C^k, \ k \geq 1 \), equivalent. Actually, the \( p-q \) chart has only three regions of topological equivalent, structurally stable phase portraits: 1) \( p > 0 \) & \( q > 0 \), with unstable spirals, star nodes, inflected nodes, and nodes, 2) \( p < 0 \) & \( q > 0 \), with stable spirals, star nodes, inflected nodes, and nodes, and 3) \( q < 0 \), with saddles.

3.1.4 Vector fields depending on parameters

So far we looked at vector fields which have the same local phase portrait. In what follows we investigate under which conditions vector fields have different local phase portraits. We start our discussion with two examples.

Consider the following one-dimensional vector field depending on a parameter:

\[
\dot{x} = f(x; \mu) = \mu - x^2,
\]  

(3.27)

where \( \mu \) is the (control) parameter. The critical points of (3.27) are

\[
x_0^\pm = \pm \sqrt{\mu}, \quad \mu \geq 0.
\]  

(3.28)
The Jacobian of (3.27) in the critical points

$$J(x_0^\pm) = \left. \frac{\partial f}{\partial x} \right|_{x_0^\pm} = -2x|_{x_0^\pm} = \mp 2\sqrt{\mu},$$

(3.29)

shows that for $\mu > 0$ the critical point $x_0^+$ is stable whereas the critical point $x_0^-$ is unstable. As $\mu \downarrow 0$, the stable and unstable critical point approach one another until they merge at $\mu = 0$, and neutralise one another: The critical point at $\mu = 0$ is neither stable nor unstable. For $\mu < 0$, no critical points are found, see Fig. 3.4.

The following two-dimensional vector field has a similar bifurcation behavior:

$$\dot{x} = f^{(1)}(x; \mu) = \mu - x^2, \quad \dot{y} = f^{(2)}(x; \mu) = -y.$$  

(3.30)

where $x = (x, y)^t$. The critical points of (3.30) are $x_0^\pm = (\pm \sqrt{\mu}, 0)^t$. The Jacobian matrix

$$\left. \frac{\partial f}{\partial x} \right|_{x_0^\pm} = \begin{pmatrix} \mp 2\sqrt{\mu} & 0 \\ 0 & -1 \end{pmatrix}$$

(3.31)

with $f = (f^{(1)}, f^{(2)})^t$, has the eigenvalues $\lambda_1 = \mp 2\sqrt{\mu}$ and $\lambda_2 = -1$. For $\mu > 0$, $x_0^+$ is a stable node (both eigenvalues are negative) whereas $x_0^-$ is a saddle (the eigenvalues have opposite sign). At $\mu = 0$ the two critical point merge. Half of the phase portrait is characterized by the phase portrait of the saddle, the other half by that of the node. For $\mu < 0$ no critical points are found. One and another is depicted in Fig. 3.5.
Figure 3.5: Two-dimensional saddle-node bifurcation. From [AFH94]
The behavior of the above two vector fields is known as bifurcation. As the parameter $\mu$ crosses the critical value $\mu = 0$ the local phase portrait changes in such a way that it is not topological equivalent for $\mu < 0$ and $\mu > 0$.

According to Guckenheimer and Holmes [GH83], the term bifurcation was originally used by Poincaré to describe the ‘splitting’ of critical points in a family of differential equations. A vector field depending on parameters is a family of differential equations. Let

$$
\dot{x} = f(x; \mu), \quad x \in U \subset \mathbb{R}^n, \quad \mu \in \mathcal{V} \subset \mathbb{R}^p,
$$

be such a vector field, where $U$ is a region in $\mathbb{R}^n$, and $\mathcal{V}$ is a region in a $p$-dimensional parameter space. The critical points of (3.32) are given by the solutions of the equation $f(x; \mu) = 0$. As $\mu$ varies, the implicit function theorem implies that these critical points are described by smooth functions of $\mu$. That is, as long as $\mu$ does not belong to those points at which the Jacobian matrix of $f(x; \mu)$ with respect to $x$, $D_x f$ does not have any eigenvalues with zero real part. The graph in $\mathbb{R}^n \times \mathbb{R}^p$ space of each of the functions describing the critical points forms a so-called branch. At a critical point $(x_0, \mu_0)$ where $D_x f$ has on or more eigenvalues with zero real part, several branches come together. The point $(x_0, \mu_0)$ is said to be a point of bifurcation. For example, $(x_0, \mu_0) = (0, 0)$ is a point of bifurcation in Fig. 3.4. Fig. 3.4 is referred to as a bifurcation diagram.

Structural stability helps to find points of bifurcation.

**Definition 3.6**

The value $\mu_0$ is said to be bifurcation value if no region $N \in \mathcal{V}$ surrounding $\mu_0$ can be found such that for all $\mu \in N$, (3.32) is structurally stable.

Because the definition of structurally stability depends on topological equivalence we need to extend that definition with dependence on parameters.

Consider two vector fields

$$
\dot{x} = f(x; \alpha), \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^m
$$

and

$$
\dot{y} = g(y; \beta), \quad y \in \mathbb{R}^n, \quad \beta \in \mathbb{R}^m.
$$

Let $U_{\alpha}$ and $V_{\beta}$ be two parameter-dependent phase-space regions with coordinates $x$ and $y$.

**Definition 3.7**

Vector field (3.33) is called topological equivalent in $U_{\alpha} \subset \mathbb{R}^n$ to vector field (3.34) in $V_{\beta} \subset \mathbb{R}^n$ if there is
1. a homeomorphism of the parameter space, \( p : \mathbb{R}^m \to \mathbb{R}^m, p(0) = 0 \), and

2. a parameter-dependent homeomorphism of the phase space, \( h_{\alpha} : \mathbb{R}^n \to \mathbb{R}^n, h_{\alpha}(U_{\alpha}) = V_{p(\alpha)}, \)

such that, for all \( \alpha \), \( h_{\alpha} \) maps trajectories of vector field (3.33) in \( U_{\alpha} \) onto trajectories of vector field (3.34) with \( \beta = p(\alpha) \) in \( V_{p(\alpha)} \), preserving the direction in time.

Note that we do not require the homeomorphism \( h_{\alpha} \) to depend continuously on \( \alpha \). This requirement would imply that the map \((\varepsilon, \alpha) \mapsto (h_{p(\alpha)}(\varepsilon), p(\alpha))\) is a homeomorphism on \( \mathbb{R}^n \times \mathbb{R}^m \).

### 3.1.5 Topological normal form

In a local bifurcation analysis, the right-hand sides of the vector field (3.32) is usually given as a Taylor series expansion. This prohibits the construction of a bifurcation diagram because series have an infinite number of parameters. If we truncate the series at a certain order we get a Taylor polynomial that has a finite number of coefficients. This number is usually still too large.

The way we circumvent this problem is by constructing a topological normal form. The topological normal form for the vector field (3.32) is a simple polynomial vector field

\[
\dot{\xi} = p(\xi; \alpha), \quad \xi \in \mathbb{R}^m, \quad \alpha \in \mathbb{R}^l, \tag{3.35}
\]

that has the same dynamical behavior. The dimension of the phase space \( m \) is as low as possible as is the dimension of the parameter space \( l \). Also, the finite number of coefficients in the polynomial is as low as can be, possibly in the form of simple constants like 1 or ±1. For the topological normal form to have the same dynamical behavior near \( \xi = 0 \) and \( \alpha = 0 \), as the vector field (3.32) near \( \varepsilon = x_0 \in U \) and \( \mu = \mu_0 \in \mathcal{V} \), they need to be topological equivalent in small regions surrounding these points.

In the case that the vector field has a hyperbolic critical point, Theorem 3.5 showed that the right-hand side of the topological normal form is simply a linear vector field without any parameter dependence and with a matrix that has all its nonzero entries on the diagonal being either 1 and −1. In what follows, we will discuss techniques that enable a calculation of topological normal forms near nonhyperbolic critical points.
3.2 Methods for simplification of vector fields

3.2.1 Center manifold

Some vector fields are such that their dynamics can be represented by studying them in a lower dimensional phase space.

As before, consider the vector fields (3.27) and (3.30) together with their bifurcation diagrams Fig. 3.4 and Fig. 3.5. The dynamical behavior on the x-axis in Fig. 3.5 is precisely as depicted in Fig. 3.4; for $\mu < 0$ there are no critical points, at $\mu = 0$ there is one critical point, and for $\mu > 0$ there are two critical points. The dynamics outside the x-axis in Fig. 3.5 is not influenced by the control parameter $\mu$ in the sense that no critical points are formed and the direction of the trajectories is always towards the x-axis.

In a way, the bifurcation behavior in the vector field (3.30) is confined to the x-axis.

The following theorem shows how we can foresee such behavior and use it to reduce the dimension of the phase space.

**Theorem 3.8**

Let $f$ be a $C^r$ vector field on $\mathbb{R}^n$ vanishing at the origin ($\underline{f}(0) = 0$) and let $A = \underline{D}f(0)$ denote the Jacobian matrix. Divide the spectrum of $A$ into three parts, $\sigma_s$, $\sigma_c$, $\sigma_u$ with

\[
\Re \lambda \begin{cases} 
< 0 & \text{if } \lambda \in \sigma_s, \\
= 0 & \text{if } \lambda \in \sigma_c, \\
> 0 & \text{if } \lambda \in \sigma_u.
\end{cases}
\]

Let the (generalized) eigenspaces of $\sigma_s$, $\sigma_c$, and $\sigma_u$ be $E_s$, and $E_u$, respectively. Then there exist $C^r$ stable and unstable manifolds $W^s$ and $W^u$ tangent to $E_s$ and $E_u$ at $0$ and a $C^{r-1}$ center manifold $W^c$ tangent to $E^c$ at $0$. The manifolds $W^s$, $W^c$ and $W^u$ are all invariant for the flow of $f$. The stable and unstable manifolds are unique, but $W^c$ need not be. If, however, $f$ is $C^\infty$, then we can find a $C^r$ center manifold for any $r < \infty$. □

Fig. 3.6 shows a geometrical picture of the various manifolds mentioned in Theorem 3.8.

See Guckenheimer and Holmes [GH83] for references to proofs of this theorem. Also, Kirchgraber and Palmer [KP90] present a thorough discussion of all proofs concerned.
Figure 3.6: Stable, unstable, and center manifolds

An example application

The geometry of the manifolds shown in Fig. 3.6 also applies to vector fields with the following Jacobian matrix

\[
J = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & -\lambda & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

(3.36)

where \( \lambda > 0 \). In this case, the invariant subspaces are spanned by the vectors \( \partial_1, \partial_2, \) and \( \partial_3 \), corresponding to \( E^u \), \( E^s \), and \( E^c \), respectively. Vector fields with a Jacobian matrix identical to \( J \) can be written as

\[
\begin{align*}
\dot{x}_1 &= \lambda x_1 + f_1(x), \\
\dot{x}_2 &= -\lambda x_2 + f_2(x), \\
\dot{x}_3 &= f_3(x),
\end{align*}
\]

(3.37)

where each \( f_i(x) = O(\|x\|^2) \).

Since a center manifold is tangent to \( E^c \) we can represent \( W^c \) locally as a curve,

\[
W^c = \{ x = (x_1, x_2, x_3)^t \mid x_1 = h_1(x_3), x_2 = h_2(x_3) \}
\]

(3.38)

where \( h_1 \) and \( h_2 \) are defined in some neighborhood of the origin and \( h_i(0) = Dh_i(0) = 0, i = \{1, 2\} \). Consider the projection of the vector field on a center manifold \( W^c \) onto \( E^c \),

\[
\dot{x}_3 = f_3(h_1(x_3), h_2(x_3), x_3).
\]

(3.39)
Information about the solutions of (3.39) provides a good approximation of the flow of $\dot{x}_3 = f_3(x)$ restricted to $W^c$.

Substituting $x_1 = h_1(x_3)$, $x_2 = h_2(x_3)$ in the first and second row of (3.37) and using the chain rule, we obtain

\[
\begin{align*}
\dot{x}_1 &= D h_1(x_3) \dot{x}_3 = D h_1(x_3) f_3(h_1(x_3), h_2(x_3), x_3)) \\
&= \lambda h_1(x_3) + f_1(h_1(x_3), h_2(x_3), x_3), \\
\dot{x}_2 &= D h_2(x_3) \dot{x}_3 = D h_2(x_3) f_3(h_1(x_3), h_2(x_3), x_3)) \\
&= -\lambda h_2(x_3) + f_2(h_1(x_3), h_2(x_3), x_3).
\end{align*}
\]

(3.40)

Usually, the functions $f_s(x)$, $s \in \{1, 2, 3\}$, are given as series expansion in $x_1$, $x_2$, and $x_3$. As before, let $f_{ij}^{(s)}$ be the coefficients in these series. Also, by setting $h_3(x_3) = \sum_{n \geq 2} h_n(x_3)$, $\ell = \{1, 2\}$, and equating terms with equal monomial in (3.40), we find approximate solutions for the functions $h_1(x_3)$ and $h_2(x_3)$. Up to $O(\|x\|^3)$ from (3.40) we get

\[ h_2^{(1)} = -f^{(1)}/\lambda, \quad h_2^{(2)} = f^{(2)}/\lambda, \]

whereas up to $O(\|x\|^2)$ from (3.39) we get

\[ \dot{x}_3 = f_{002}^{(3)} x_3^2. \]

Analysis of the dynamical behavior of the one-dimensional flow near the critical point $x_3 = 0$ yields the desired information about the dynamical behavior of the three-dimensional flow near $x = 0$.

### 3.2.2 Normal form

The dynamics of a vector field near a nonhyperbolic critical point heavily depends on the nonlinear part of that vector field. $C^k$ equivalence, $k > 1$, can be used to simplify the nonlinear part.

As a motivation for local topological equivalence consider the example from Argyris, Faust and Haase [AFH94], page 289, f.f. Fig. 3.7 sketches the phase portraits of following two vector fields,

\[
\begin{align*}
\dot{x} &= y + a_1 x^2 + a_2 xy + a_3 y^2, \\
\dot{y} &= b_1 x^2 + b_2 xy + b_3 y^2,
\end{align*}
\]

(3.41)

and

\[
\begin{align*}
\dot{x} &= y + ax^2, \\
\dot{y} &= bx^2 + \frac{1}{2} a xy.
\end{align*}
\]

(3.42)
Figure 3.7: Trajectories for the vector fields (3.41) and (3.42) at different scales. From [AFH94]
at two different scales. At the large scale \(|\|x\|| < 25\) there is no resemblance whatsoever, but at the small scale \(|\|\bar{x}\|| < .5\) the resemblance is striking. The mechanism behind this resemblance is the computation of a normal form.

Consider the vector field

\[
\dot{\bar{x}} = f(x) = A \bar{x} + \sum_{m \geq 2} f_m(\bar{x}), \quad f : \mathbb{R}^n \to \mathbb{R}^n, \quad f_m \in H^n.
\]  

(3.43)

Substitution of the near-identity transformation

\[
\bar{x} = t(y), \quad t(y) = y + t_2(y), \quad t_2 \in H^n,
\]

(3.44)

into (3.43) leads to

\[
\dot{\bar{y}} = (D_y t(y))^{-1} f(t(y)).
\]  

(3.45)

(compare with (3.4)). The transformation is called near-identity because it can be written as \(\bar{x} = y + \ldots\), where the dots denote terms of \(O(||y||^3)\). Writting out (3.45) results in

\[
\dot{\bar{y}} = (1 + D_y t_2(y))^{-1} f(y + t_2(y)),
\]

(3.46)

where \(I\) is the \(n \times n\) identity matrix.

Expansion about \(x = 0\) of (3.46) gives

\[
\dot{\bar{y}} = A \bar{y} + f_2(y) + A t_2(y) - D_y t_2(y) A y + O(||y||^3)
\]

(3.47)

The additional term of \(O(||y||^2)\) besides \(f_2\) in (3.47) compared to (3.43) produced by the near-identity transformation is the term \(A t_2(y) - D_y t_2(y) A y\).

Define the adjoint operation of the \(n \times n\) matrix \(A\) acting on a function \(t_k(y) \in H^n\) as

\[
\text{ad} A(t_k(y)) = [A, t_k(y)] = A t_k(y) - D_y t_k(y) A y,
\]

(3.48)

and the Lie bracket operator \([,]\) as

\[
[X, Y](y) = \frac{\partial X}{\partial y} Y(y) - \frac{\partial Y}{\partial y} X(y) = D_y X Y(y) - D_y Y X(y).
\]

(3.49)

The above operation can be applied repeatedly for each order \(m\) by substitution of \(y \mapsto y + t_m(y), \quad t_m(y) \in H^n, \quad m \geq 2\). The resulting vector field is greatly
simplified compared to (3.43) if a specific \( t_m(x) \) can be found which satisfies the so-called homological equation;

\[
\tilde{f}_m(y) + \text{ad} A(t_m(y)) = 0,
\]

where \( \tilde{f}_m(y) \) equals \( f_m(y) \) plus additional terms of order \( \mathcal{O}(|y|^m) \) formed by transformations which previously simplified the lower orders 2, \ldots, m - 1.

In general it is not possible to solve (3.50) completely. One could expect that as a consequence of Hartman-Grobman’s theorem all higher degree terms can be removed if the eigenvalues of \( A \) all have a nonzero real part (hyperbolic critical point). That theorem, however, works through a homeomorphism acting on trajectories. In this case, we are working with \( \mathbb{C}^m, m \geq 2 \), diffeomorphisms acting on a vector field. If the eigenvalues of a hyperbolic critical point are not resonant it is indeed possible to remove the nonlinear part completely.

The \( n \)-tuple \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of eigenvalues is said to be resonant if among the eigenvalues there exists a relation of the form

\[
\lambda_s = (m, \lambda) = \sum_{k=1}^{n} m_k \lambda_k,
\]

where \( m = (m_1, \ldots, m_n) \) consists of \( n \) nonnegative integers which satisfy \( |m| \equiv \sum m_k \geq 2 \). Such a relation is called a resonance. The number \( |m| \) is called the order of the resonance.

Let \( A \) be a diagonal matrix and let \( \lambda_1, \ldots, \lambda_n \) be the eigenvalues. (note: for the more general case of repeated eigenvalues, see Arnol’d [Arn88]). Let \( \partial_i, \quad 1 \leq i \leq n \), be the standard basis of \( \mathbb{R}^n \). As a basis for \( \mathbb{H}_m^n \) we take the set of elements

\[
x^m \partial_i \equiv y_1^{m_1} \cdot y_n^{m_n} \partial_i, \quad \sum_{j=1}^{n} m_j = m,
\]

for every \( m \), where \( m \) consists of \( n \) nonnegative integers.

Consider the action of \( \text{ad} A(\cdot) \) on each of the basis elements of \( \mathbb{H}_m^n \). By denoting \( p_m^{(i)}(y) = y^m \partial_i \), we get

\[
\text{ad} A(p_m^{(i)}(y)) = A p_m^{(i)}(y) - D p_m^{(i)}(y) A y = [\lambda_i - \sum_{j=1}^{n} m_j \lambda_j] p_m^{(i)}(y).
\]

Thus, each basis element is mapped to itself. If the eigenvalues do not form a resonance, the expression with the square brackets will never vanish. Consequently, the nonlinear part is completely removed.
Let us now deal with general matrices $A$, and concentrate on finding a vector $t_m(x)$ which satisfies the homological equation (3.50) as good as possible, leaving only a minimal number of nonlinear terms. The operator $\text{ad} \, A(\cdot)$ in (3.50) is a linear map from $H^m_n$ into $H^m_n$. Let $G_m$ the (nonunique) complementary space of $\text{ad} \, A(H^m_n)$ such that
\begin{equation}
H^m_n = \text{ad} \, A(H^m_n) \oplus G_m.
\end{equation}

By using successive near-identity transformations, $y \mapsto y + t_m(y), t_m(y) \in H^m_n$, many of the nonresonant coefficients in the higher degree terms of (3.43) can be removed. The resulting simplified vector field
\begin{equation}
\tilde{y} = g(y) = A \cdot y + \sum_{m \geq 2} g_m(y), \quad g : \mathbb{R}^n \to \mathbb{R}^n, \quad g_m \in H^m_n.
\end{equation}
is said to be a normal form up to degree $k$ if all the terms of degree $m$,
\begin{equation}
g_m(y) \equiv \tilde{t}_m(y) + \text{ad} \, A(t_m(y)),
\end{equation}
are located in the complementary subspace $G_m, m \in \{2, \ldots, k\}$.

### 3.2.3 2-D example. Normal form

To illustrate the idea that $\text{ad} \, A(\cdot)$ is a linear mapping from $H^m_n$ into $H^m_n$, let the vector field (3.43) be defined in two-dimensional space, and let its Jacobian matrix (in Jordan normal form) be given by
\begin{equation}
J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\end{equation}

A basis for the second degree monomials in two-dimensional space is given by
\begin{equation}
H^2_2 = \text{span}\{ \begin{pmatrix} y_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ y_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_2^2 \end{pmatrix} \}.
\end{equation}

We want to compute $\text{ad} \, J(H^2_2)$. To this end, we compute the action of $\text{ad} \, J(\cdot)$ on each basis element on $H^2_2$ using the definition (3.48),
\begin{align}
\text{ad} \, J\left( \begin{pmatrix} y_1^2 \\ 0 \end{pmatrix} \right) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1^2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2y_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -2y_1 y_2 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix},
\end{align}

(3.58)
and similarly,

\[
\text{ad } J \left( \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} -y_2^2 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix},
\]

\[
\text{ad } J \left( \begin{pmatrix} y_1^2 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

\[
\text{ad } J \left( \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix} \right) = \begin{pmatrix} y_1^2 \\ -2y_1 y_2 \end{pmatrix} = \begin{pmatrix} y_1^2 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix},
\]

\[
\text{ad } J \left( \begin{pmatrix} y_1 y_2 \\ -y_2^2 \end{pmatrix} \right) = \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ y_2^2 \end{pmatrix},
\]

\[
\text{ad } J \left( \begin{pmatrix} 0 \\ y_2^2 \end{pmatrix} \right) = \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix}.
\]

The result of the computations can be represented in a matrix:

\[
\begin{pmatrix}
\begin{pmatrix} y_1^2 \\ 0 \end{pmatrix} & \begin{pmatrix} y_1 y_2 \\ 0 \end{pmatrix} & \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix} & \begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix} & \begin{pmatrix} 0 \\ y_2^2 \end{pmatrix} \\
0 & 0 & 0 & 1 & 0 & 0 \\
-2 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
\end{pmatrix}
\end{pmatrix}
\]

\[
\begin{cases}
y_1^2 \\
y_1 y_2 \\
y_2^2 \\
y_1^2 \\
y_1 y_2 \\
y_2^2
\end{cases}
\] (3.59)

The vectors in \( H_2 \) on which the operator \( \text{ad } J(\cdot) \) acts are written above the matrix from left to right. The action of the operator on each of these vectors is a column of (3.59) written from top to bottom.

The image of \( \text{ad } J(H_2) \) has dimension 4,

\[
\text{ad } J(H_2) = \text{span} \left\{ \begin{pmatrix} -2y_1 y_2 \\ 0 \end{pmatrix}, \begin{pmatrix} y_1^2 \\ -2y_1 y_2 \end{pmatrix}, \begin{pmatrix} y_1 y_2 \\ -y_2^2 \end{pmatrix}, \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix} \right\}.
\] (3.60)

\( H_2 \) has dimension 6. Thus, the complementary subspace \( G_2 \) is two dimensional.

A representation of the complementary subspace is found by looking for left eigenvectors corresponding to a zero eigenvalue of (3.59). In this case, most of the entries of (3.59) are zero, which makes it easy to verify that two such vectors are

\[
\begin{pmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.
\] (3.61)
Hence, the complementary subspace \( G_2 \) of \( \text{ad} J(H_2^2) \) is given by

\[
G_2 = \text{span}\left\{ \left( \begin{array}{c} y_1^2 \\ \frac{1}{2} y_1 y_2 \end{array} \right), \left( \begin{array}{c} 0 \\ y_1^2 \end{array} \right) \right\}. \tag{3.62}
\]

This implies that a normal form up to second-order is given by

\[
\begin{align*}
\dot{y}_1 &= y_2 + g_1 y_1^2 + O(3), \\
\dot{y}_2 &= g_2 y_1^2 + \frac{1}{2} g_1 y_1 y_2 + O(3),
\end{align*} \tag{3.63}
\]

where \( g_1 \) and \( g_2 \) are constants that depend on the coefficients of \( f_2(x) \) in (3.43). This vector field is identical to vector field (3.42) in the example taken from [AFH94], see Fig. 3.7.

### 3.2.4 Alternative complementary subspaces

Vector field (3.63) is a normal form; other possible representations exist. For example, the basis element

\[
\left( \begin{array}{c} 0 \\ y_1 y_2 \end{array} \right) \tag{3.64}
\]

is contained in one of the two vectors in (3.62), i.e., in

\[
\left( \begin{array}{c} y_1^2 \\ \frac{1}{2} y_1 y_2 \end{array} \right) \tag{3.65}
\]

but is also present in the image of the \( \text{ad} J(\cdot) \) operator, i.e., in

\[
\text{ad} J \left( \begin{array}{c} 0 \\ y_1^2 \end{array} \right) = \left( \begin{array}{c} y_1^2 \\ 0 \end{array} \right) - 2 \left( \begin{array}{c} 0 \\ y_1 y_2 \end{array} \right). \tag{3.66}
\]

It is therefore possible to use

\[
G_2 = \text{span}\left\{ \left( \begin{array}{c} y_1^2 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ y_1^2 \end{array} \right) \right\}, \tag{3.67}
\]

as representation of the complementary subspace. With this choice of \( G_2 \) the normal form becomes

\[
\begin{align*}
\dot{y}_1 &= y_2 + \hat{g}_1 y_1^2 + O(3), \\
\dot{y}_2 &= \hat{g}_2 y_1^2 + O(3),
\end{align*} \tag{3.68}
\]

Takens [Tak74] used this normal form to study the dynamics of planar vector fields with linear part given by (3.56). Fig. 3.8 illustrates the trajectories near
Figure 3.8: Trajectories for the vector field (3.68) with $\hat{g}_1 = \hat{g}_2 = 1$. From [AFH94]

the critical point in the origin for the vector field (3.68).

Another possibility for $G_2$ is given by

$$G_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix}, \begin{pmatrix} 0 \\ y_1 y_2 \end{pmatrix} \right\}. \quad (3.69)$$

The following proposition shows that it is possible to find a similar representation to $G_2$ for the complementary subspaces $G_m$, $m > 2$.

**Theorem 3.9**

A representation of complementary subspace $G_m$, $m \geq 2$, of ad $J(\cdot)$, with $J$ being the double-zero Jordan normal form (3.56) is given by

$$G_m = \text{span}\left\{ \begin{pmatrix} 0 \\ y_1^m \end{pmatrix}, \begin{pmatrix} 0 \\ y_1^{m-1} y_2 \end{pmatrix} \right\}. \quad (3.70)$$

For a proof, see Takens [Tak74], page 56, f.f.

With this choice of $G_2$ the normal form becomes

$$\dot{y}_1 = y_2 + O(3), \quad \dot{y}_2 = \hat{g}_1 y_1^2 + \hat{g}_2 y_1 y_2 + O(3), \quad (3.71)$$

Bogdanov [Bog75] [Bog81] used this normal form to study the dynamics of planar vector fields with linear part given by (3.56). Fig. 3.9 displays the trajectories near the critical point in the origin of the vector field (3.71).
3.2.5 Finding complementary subspaces

There are three reasons which make finding a complementary subspace for a given matrix $J$ cumbersome.

1. To know the dimension of the complementary subspace $G_2$, and therewith the number of left eigenvectors needed, one has to compute the dimension of the image of $\text{ad } J(H^2_2)$. This can be difficult in general;

2. To find the left eigenvectors of the matrix representation of $\text{ad } J(H^2_2)$ takes 'experience and some flair' ([AFH94]);

3. The complementary subspace found that way may not be to most suitable.

Also, the way the coefficients in a normal form depend on the coefficients in the original vector field becomes unclear. For these reasons we present a more algorithmic procedure to find all possible representations of a complementary space.

Let $t_m(y) \in H^2_m$. We write this vector function as

$$ t_m(y) = \sum_{i+j=m} \begin{pmatrix} t^{(1)}_{i,j} \\ t^{(2)}_{i,j} \end{pmatrix} y_1^i y_2^j, \quad (3.72) $$
where $i$ and $j$ are nonnegative integers. The adjoint operation of the matrix $J$ in (3.56) acting on $t_{m}(y)$ equals

$$
\text{ad } J(t_{m}(y)) \equiv J(t_{m}(y)) - D t_{m}(y) J y
$$

$$
= \sum_{i+j=m} \begin{pmatrix}
    t_{i,j}^{(2)} y_i^1 y_j^2 - t_{i,j}^{(1)} i y_i^{i-1} y_j^{j+1} \\
    -t_{i,j}^{(2)} i y_i^{i-1} y_j^{j+1}
\end{pmatrix}.
$$

(3.73)

Since no simplification of the nonlinear terms has taken place, yet, we will start at $m = 2$.

First, we write out (3.55) for $m = 2$,

$$
g_2(y) = f_2(y) + \text{ad } J(t_2(y)),
$$

We get

$$
g_2^{(1)}(y) = (f_{20}^{(1)} + t_{20}^{(2)}) y_1^2 + (f_{11}^{(1)} + t_{11}^{(2)} - 2 t_{20}^{(1)}) y_1 y_2 + (f_{02}^{(1)} + t_{02}^{(2)} - t_{11}^{(1)}) y_2^2,
$$

$$
g_2^{(2)}(y) = (f_{20}^{(2)}) y_1^2 + (f_{11}^{(2)} - 2 t_{20}^{(2)}) y_1 y_2 + (f_{02}^{(2)} - t_{11}^{(2)}) y_2^2.
$$

(3.74)

We then place the right-hand side in a table, see Table 3.1. The notation $y_i^j \partial_i^j$ in combination with $\partial_s$ denotes a vector. For example, $y_1^2 \partial_1^2 = (y_1^2, 0)^t$.

Second, we group entries in the table with common coefficients $t_{i,j}^{(s)}$ from top-left to bottom-right. The top-left entry $y_i^j \partial_1^j$ has only one coefficient, viz., $t_{20}^{(2)}$. Another entry in the table that contains $t_{20}^{(2)}$ is $y_1 y_2 \partial_2$. Table 3.1 shows both entries shaded. There are no further links with other entries in that table, so the locations $y_1^2 \partial_1$ and $y_1 y_2 \partial_2$ form a linearly independent subspace in $\text{ad } J(H_2^2)$. These entries are marked by adding a certain label, say $L_1$, see Table 3.2. This procedure is applied repeatedly until all locations are visited and all coefficients are placed into linear independent subspaces.

Since the table quickly runs full with symbols, the subspaces are re-ordered in such a way that all entries in a linearly independent subspace are directly
### 3.2. Methods for simplification of vector fields

<table>
<thead>
<tr>
<th></th>
<th>( y_1^2 )</th>
<th>( y_1y_2 )</th>
<th>( y_2^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \partial_1 )</td>
<td>( L_1 )</td>
<td>( f_{20}^{(1)} + t_{20}^{(2)} )</td>
<td>( L_2 )</td>
</tr>
<tr>
<td>( \partial_2 )</td>
<td>( L_4 )</td>
<td>( f_{20}^{(2)} )</td>
<td>( L_1 )</td>
</tr>
</tbody>
</table>

Table 3.2: \( f_2(y) + \text{ad} J(t_2(y)) \) with subspaces.

<table>
<thead>
<tr>
<th>( \partial_1 )</th>
<th>( \partial_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_1 )</td>
<td>( C_1 )</td>
</tr>
<tr>
<td>( L_2 )</td>
<td>( I_1 )</td>
</tr>
<tr>
<td>( L_3 )</td>
<td>( K_1 )</td>
</tr>
<tr>
<td>( L_4 )</td>
<td>( C_2 )</td>
</tr>
<tr>
<td>( L_0 )</td>
<td>( K_2 )</td>
</tr>
</tbody>
</table>

Table 3.3: Re-ordering of \( f_2(y) + \text{ad} J(t_2(y)) \) into subspaces plus unused coefficients.
adjacent, see Table 3.3. At the bottom a ‘subspace’ \( L_0 \) is added that lists all coefficients \( t_{ij}^{(e)} \) which are projected on the origin of \( H_2 \).

Every subspace produces a small set of algebraic equations which enables a quick computation of the number of coefficients in \( g_2 \) that can be set to zero. For example, in the subspace \( L_1 \), there are 2 algebraic equations but only 1 coefficient from \( t_2 \). We are missing \( 2 - 1 = 1 \) coefficient(s) from \( t_2 \) to remove the subspace \( L_1 \) altogether. The right-most column in Table 3.3 depicts that number.

Subspaces that have a shortage of coefficients from \( t_2 \) are part of the complementary subspace of the \( \text{adj}(\cdot) \) operator. As an indicator, we replace their labels with \( C_\ell \). On the other hand, if the number of linear independent equations in a subspace is less than or equal to the number of coefficients from \( t_2 \), then that subspace is part of kernel or image, and we replace their label by \( K_\ell \) or \( I_\ell \), respectively.

From each subspace \( C_\ell \) in the complementary subspace it is possible to select a representative vector. For example, \( C_1 \) results in

\[
\begin{pmatrix}
(f_2^{(1)} + t_{20}^{(2)} y_1^2) \\
(f_1^{(2)} - 2t_{20}^{(2)} y_1y_2)
\end{pmatrix}
\]

By setting \( t_{20}^{(2)} = -\frac{1}{5}f_2^{(1)} + \frac{2}{5}f_1^{(2)} \), we get

\[
\left( \frac{4}{5}f_2^{(1)} + \frac{2}{5}f_1^{(2)} \right) \begin{pmatrix} y_1^2 \\ \frac{1}{2}y_1y_2 \end{pmatrix}
\]  

(3.75)

This vector was previously found using the left eigenvectors, see (3.63).

Another choice is \( t_{20}^{(2)} = \frac{1}{2}f_1^{(2)} \). Then, we get

\[
\left( f_2^{(1)} + \frac{1}{2}f_1^{(2)} \right) \begin{pmatrix} y_1^2 \\ 0 \end{pmatrix}
\]  

(3.76)

which is the vector in the normal form used by Takens, see (3.68). Finally, if we set \( t_{20}^{(2)} = -f_2^{(1)} \), we get the vector

\[
\left( f_1^{(2)} + 2f_2^{(1)} \right) \begin{pmatrix} 0 \\ y_1y_2 \end{pmatrix}
\]  

(3.77)

used by Bogdanov, see (3.71). These examples show that by a proper choice of the coefficients from \( t_2 \) leads to the desired representation of the complementary subspace. Note that every time the compound coefficients are a combination of all the coefficients from the subspace that they represent.
Let $G_2 = \sum \ell C_\ell$ be the complementary subspace at second order. The underlined coefficients in Table 3.3 are the ones we selected to be part of a normal form;

$$G_2 = g_{10}^{(2)} y_1^2 \partial_2 + g_{11}^{(2)} y_1 y_2 \partial_2,$$

where $g_{10}^{(2)} = f_{10}^{(2)}$ and $g_{11}^{(2)} = f_{11}^{(2)} + 2f_{20}^{(1)}$.

**Coefficients in a normal form**

The above procedure yields explicit expressions for the relations between the coefficients in a normal form and the coefficients in the original vector field. The example taken from [AFH94] shown in Fig. 3.7 illustrated visually the topological equivalence of the vector fields (3.41) and (3.42). The choice for the coefficients $a_1 = 2$ and $b_1 = b_2 = 1$ in (3.41), and $a = 2$ and $b = 1$ in (3.42) is not entirely accidental since

$$C_1; \quad a = \frac{4}{5} f_{20}^{(1)} + \frac{2}{5} f_{11}^{(2)} = \frac{4}{5} a_1 + \frac{2}{5} b_2 = 2, \quad \text{and}$$

$$C_2; \quad b = f_{20}^{(2)} = b_1 = 1,$$

using the above coefficient relations. The same correspondence is not found for Takens’ normal form,

$$C_1; \quad f_{20}^{(1)} + \frac{1}{2} f_{11}^{(2)} = 2 + \frac{1}{2} \cdot 1 = \frac{5}{2} \neq \hat{g}_1 = 1,$$

see (3.68), nor for Bogdanov’s normal form,

$$C_1; \quad f_{10}^{(2)} + 2 f_{20}^{(1)} = 1 + 2 \cdot 2 = 5 \neq \hat{g}_1 = 1,$$

see (3.71).

Notice that in all three cases the coefficient of the representative term from $C_1$ is a factor times $(f_{20}^{(1)} + \frac{1}{2} f_{11}^{(2)})$, i.e., for (3.79) it is $\frac{4}{5}$, for (3.80) it 1, and it is 2 for (3.81). Of course, the ratio between $f_{20}^{(1)}$ and $f_{11}^{(2)}$ should not be a surprise; the ratio stems from the entries in the first of the left eigenvectors in (3.61).

**Rescaling of coordinates**

The fact that the coefficient relations are not obeyed, does not stand in the way of an excellent visual agreement of their trajectories near the origin as we saw in Fig. 3.8 and Fig. 3.9. The reason is that the parameters $\hat{g}_1$ and $\hat{g}_2$ in (3.71) can be replaced by 1 and $b = \pm 1$ using a rescaling of coordinates. By setting

$$y_1 \mapsto \alpha x_1, \quad y_2 \mapsto \beta x_2, \quad t \mapsto \gamma t,$$

(3.82)
the vector field (3.71) in normal form becomes

\[
\begin{align*}
\dot{x}_1 &= \frac{\gamma \beta}{\alpha} x_2, \\
\dot{x}_2 &= \frac{\gamma \alpha^2}{\beta} \tilde{g}_1 x_1^2 + \gamma \alpha \tilde{g}_2 x_1 x_2,
\end{align*}
\] (3.83)

Ideally we want to choose \(\alpha, \beta,\) and \(\gamma\) such that the coefficients are all unity; this will not be possible because we do not want the stability to be affected. Consequently, we require \(\gamma > 0\) and \(\alpha \cdot \beta > 0.\)

The linear part should remain to be the same, so we need to set

\[
\frac{\gamma \beta}{\alpha} = 1, \quad \Rightarrow \quad \gamma = \frac{\alpha}{\beta}.
\] (3.84)

This requirement fixes \(\gamma.\)

Next, we equate the coefficient of \(x_1^2 \tilde{g}_2\) to unity,

\[
\frac{\gamma \alpha^2}{\beta} \tilde{g}_1 = 1.
\]

Using (3.84), we get

\[
\alpha \gamma^2 \tilde{g}_1 = 1.
\] (3.85)

This equation fixes \(\alpha.\) Notice that \(\alpha\) has the same sign as \(\tilde{g}_1.\)

We cannot find a value for \(\beta\) to equate the coefficient of \(x_1 x_2 \tilde{g}_2\) to unity,

\[
\gamma \alpha \tilde{g}_2 = \frac{\alpha^2}{\beta} \tilde{g}_2 = 1,
\]

because that would require \(\beta\) to have to same sign of \(\tilde{g}_2\) whereas \(\beta\) needs to have the same sign as \(\alpha\) (which was already linked to the sign of \(\tilde{g}_1\) by (3.85)). Hence, the best we can do is to require

\[
\frac{\alpha^2}{\beta} \tilde{g}_2 = \pm 1.
\]

These computations explain why only the sign of the coefficients is of importance in the vector fields (3.41) and (3.42).
3.2.6 Finding kernel subspaces

Before we discuss how to compute the coefficients in a normal form at third and higher order, we need to discuss one last detail about the the re-ordering into subspaces of \( f_2(y) + \text{ad} J(t_2(y)) \) in Table 3.3. Consider the subspace \( L_3 \). Since it has more coefficients than relations, the subspace \( L_3 \) is a kernel subspace. Because it is the first kernel subspace from the top down, we called this subspace \( K_1 \).

The kernel subspace \( K_1 \) is removed if we set

\[
t_{11}^{(1)} = F_d + \frac{1}{2} f_{02}^{(1)}, \quad t_{02}^{(2)} = F_d - \frac{1}{2} f_{02}^{(1)}.
\]

We find that the parameter \( F_d \) cancels out in the computation of \( \text{ad} J(t_2(y)) \) in (3.73). Thus, \( F_d \) is a free parameter. We can possibly use the free parameter \( F_d \) in the vector

\[
F_d \begin{pmatrix} y_1 y_2 \\ y_2^2 \end{pmatrix}
\]

to simplify our normal form at a later stage.

The 'subspace' \( L_0 \) in Table 3.3 contains the coefficient \( t_{02}^{(1)} \) which directly canceled out in the computation of \( \text{ad} J(t_2(y)) \). As before, we can possibly use the free parameter \( t_{02}^{(1)} \) in the vector

\[
t_{02}^{(1)} \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix}
\]

to simplify our normal form at a later stage. Thus, \( L_0 \) is also a kernel subspace. Since it is the second kernel subspace from the top down in Table 3.3, we called this subspace \( K_2 \).

Let \( \mathcal{K}_2 \) be the span of all vectors from the subspaces \( K_\ell \). We find that

\[
\mathcal{K}_2 = \text{span} \left\{ \begin{pmatrix} y_1 y_2 \\ y_2^2 \end{pmatrix}, \begin{pmatrix} y_2^2 \\ 0 \end{pmatrix} \right\}
\]

equals the kernel of the \( \text{ad} J(\cdot) \)-operator (3.73) for vectors \( t_m(y) \in H_m^2 \) at \( m = 2 \).

The number of free parameters in a kernel subspace is placed in the right-most column of Table 3.3. The check if the dimension of the complementary subspace equals the dimension of the kernel,

\[
\dim \mathcal{G}_2 = \dim \mathcal{K}_2
\]

is some confirmation that no mistakes were made in the computations.
3.2.7 Transformation theory

In the previous paragraphs, we computed three normal form representations up to second order, i.e., (3.62), (3.67) and (3.69). We then presented a procedure to find all possible normal form representations. That procedure yielded explicit expressions for the relations between the coefficients in a normal form and the coefficients in the original vector field up to second order. The same procedure can be used for higher orders.

Ushiki [Usi84] and Gamero et al. [GFR+93] have shown that it is possible to use the free parameters in the kernel to further simplify the complementary subspace at a higher order. However, the vector \( \tilde{f}_m(y) \) in the homological equation (3.50) equals \( f_m(y) \) plus some additional terms of order \( O(\|y\|^{m}) \) formed by transformations which previously simplified the lower orders \( 2, \ldots, m - 1 \). To achieve additional simplification using the free parameters in the kernel, we need to know precisely in which coefficients of \( \tilde{f}_m(y) \) the free parameters enter. For example, we should find out precisely how \( \tilde{f}_3(y) \) relates to \( f_3(y), f_2(y) \) and \( t_2(y) \). Then it is possible to see if the free parameters in the second order kernel \( K_2 \) (which do not affect \( G_2 \)) can be used to simplify the third order complementary subspace \( G_3 \).

Thus, a better understanding is needed of how a transformation to normal form composes the functions \( \tilde{f}_m(y) \) in the homological equation (3.50). The discussion in this section is a stripped down, edited version of §12.2 ff. by Chow and Hale [CH82]. They name Kirchgraber and Stiefel [KS78], Meyer and Schmidt [MS77], and others as the original contributors.

Consider the differential equation with the function \( u : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \)

\[
\dot{y} = u(y, \varepsilon) \tag{3.88}
\]

This function can be obtained, for example, from the vector field (3.43) by the coordinate scaling \( \bar{x} = \varepsilon y \), resulting in

\[
\dot{\bar{x}} = u(\bar{x}, \varepsilon) = \frac{1}{\varepsilon} f(\varepsilon y) = Ay + \sum_{m \geq 2} \varepsilon^{m-1} f_m(y), \quad f_m(y) \in H_m^2 \tag{3.89}
\]

We introduce \( \varepsilon \) as an ordering parameter in the vector field (3.88) to identity and group terms of order \( O(\|y\|^{m}) \). This is done through \( m - 1 \) subsequent differentiations with respect to \( \varepsilon \),

\[
f_m(y) = \frac{\partial^{m-1}}{\partial \varepsilon ^{m-1}} f(y, 0) \tag{3.90}
\]
If we insert a change of coordinates of the form
\[ y = t(z, \varepsilon) = z + \sum_{m \geq 2} \varepsilon^{m-1} t_m(z) \tag{3.91} \]
into the vector field (3.89), we get the following differential equation for \( \dot{z} \)
\[ \dot{z} = v(z, \varepsilon) \equiv \frac{\partial t^{-1}(t(z, \varepsilon), \varepsilon)}{\partial y} u(t(z, \varepsilon), \varepsilon). \tag{3.92} \]
where \( z = t^{-1}(y, \varepsilon) \) is the inverse of (3.91). We want to find the terms of order \( O(\|z\|^m) \) in the expansion of the function \( v(z, \varepsilon) \) through \( m - 1 \) subsequent differentiations with respect to \( \varepsilon \).

However, observe what happens if an arbitrary vector function \( h(y, \varepsilon) \) is evaluated at \( (y, \varepsilon) = (t(z, \varepsilon), \varepsilon) \) and differentiated with respect to \( \varepsilon \);
\[ \frac{\partial h(t(z, \varepsilon), \varepsilon)}{\partial \varepsilon} = \left[ \frac{\partial h(y, \varepsilon)}{\partial \varepsilon} + \frac{\partial h(y, \varepsilon)}{\partial y} \frac{\partial t(z, \varepsilon)}{\partial \varepsilon} \right] \tag{3.93} \]
The part between the square brackets which is not a function of \( (y, \varepsilon) \)
\[ \frac{\partial t(z, \varepsilon)}{\partial \varepsilon} = \sum_{m \geq 2} (m - 1) \varepsilon^{m-2} t_m(z). \]
It is not possible to substitute \( (t(z, \varepsilon), \varepsilon) \) into this expression, because the right-hand side is not a function of \( (y, \varepsilon) \).

As a remedy, assume that the function \( t(z, \varepsilon) \) in (3.91) is the (presumed unique) solution of the equation
\[ \frac{\partial t(z, \varepsilon)}{\partial \varepsilon} = T(t(z, \varepsilon), \varepsilon), \quad t(z, 0) = z. \tag{3.94} \]
for a given function \( T : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \). The function \( T(y, \varepsilon) \) is called the generator of the transformation (3.91). Then, using (3.94), we get
\[ \frac{\partial h(t(z, \varepsilon), \varepsilon)}{\partial \varepsilon} = \left[ \frac{\partial h}{\partial \varepsilon} + \frac{\partial h}{\partial y} T \right] (t(z, \varepsilon), \varepsilon). \tag{3.95} \]
and we can compute the expression within the square brackets completely in terms of \( y \) and \( \varepsilon \).
Repeated differentiations of $h(t(\bar{z}, \varepsilon), \varepsilon)$ with respect to $\varepsilon$ using (3.93) results in

$$h(t(\bar{z}, \varepsilon), \varepsilon) = \sum_{m \geq 0} h_m(\bar{z}) \varepsilon^m / m! = \sum_{m \geq 0} \varepsilon^m / m! \ [D^m h](\bar{z}, 0),$$

(3.96)

where the operator $D$ is defined as

$$Dh = \frac{\partial h}{\partial \varepsilon} + \frac{\partial h}{\partial y} \mathcal{T}.$$  

(3.97)

The formal series (3.96) is the expansion of $h(t(\bar{z}, \varepsilon), \varepsilon)$ with respect to $\varepsilon$.

We now consider the following formal series expansion of the vector field (3.88) with respect to $\varepsilon$,

$$u(y, \varepsilon) = \sum_{m \geq 0} u_m(y) \varepsilon^m / m!,$$  

(3.98)

where $u_m : \mathbb{R}^n \to \mathbb{R}^n$ need not be homogeneous polynomials but can be arbitrary functions. Also, let the formal series in $\varepsilon$ for $v(\bar{z}, \varepsilon)$ and $\mathcal{T}(y, \varepsilon)$ be as

$$v(\bar{z}, \varepsilon) = \sum_{m \geq 0} v_m(\bar{z}) \varepsilon^m / m!, \quad \mathcal{T}(y, \varepsilon) = \sum_{m \geq 0} T_m(y) \varepsilon^m / m!$$

(3.99)

Since $t^{-1}(y, \varepsilon)$ is the inverse of the transformation $y = t(\bar{z}, \varepsilon)$, we have $\bar{z} = t^{-1}(t(\bar{z}, \varepsilon), \varepsilon)$ and using (3.93) gives

$$\frac{\partial t^{-1}(y, \varepsilon)}{\partial \varepsilon} = - \frac{\partial t^{-1}(y, \varepsilon)}{\partial y} \mathcal{T}(y, \varepsilon).$$

(3.100)

With the above results we can differentiate the right-hand side of the vector field (3.92) with respect to $\varepsilon$,

$$\frac{\partial}{\partial \varepsilon} \left[ (\frac{\partial t^{-1}}{\partial y} u)(t(\bar{z}, \varepsilon), \varepsilon) \right] =$$

$$= \left[ \frac{\partial}{\partial \varepsilon} \left( \frac{\partial t^{-1}}{\partial y} u \right) + \frac{\partial}{\partial y} \left( \frac{\partial t^{-1}}{\partial y} u \right) \mathcal{T} \right] (t(\bar{z}, \varepsilon), \varepsilon) =$$

$$= \left[ \frac{\partial t^{-1}}{\partial y} \frac{\partial u}{\partial \varepsilon} - \frac{\partial}{\partial y} \left( \frac{\partial t^{-1}}{\partial y} \mathcal{T} \right) u + \frac{\partial}{\partial y} \left( \frac{\partial t^{-1}}{\partial y} u \right) \mathcal{T} \right] (t(\bar{z}, \varepsilon), \varepsilon).$$

(3.101)
where we applied (3.95) and (3.100), respectively. Simply writing out the last two terms inside the square brackets gives

$$
\frac{\partial}{\partial y} \left( \frac{\partial t^{-1}}{\partial y} u \right) T - \frac{\partial}{\partial y} \left( \frac{\partial t^{-1}}{\partial y} T \right) u = \frac{\partial t^{-1}}{\partial y} \left\{ \frac{\partial u}{\partial y} T - \frac{\partial T}{\partial y} u \right\}.
$$

(3.102)

The expression between the curly brackets is the Lie product of $u$ and $T$,

$$
[u, T] = \frac{\partial u}{\partial y} T - \frac{\partial T}{\partial y} u.
$$

(3.103)

With this definition, (3.101) can be rewritten as

$$
\frac{\partial}{\partial \varepsilon} \left[ \left( \frac{\partial t^{-1}}{\partial y} u \right)(t(z, \varepsilon), \varepsilon) \right] = \left[ \frac{\partial t^{-1}}{\partial y} \left( \frac{\partial}{\partial \varepsilon} + [ , T] \right) u \right] (t(z, \varepsilon), \varepsilon).
$$

(3.104)

Repeated application of (3.104) shows that

$$
\frac{\partial^m u}{\partial \varepsilon^m} = \frac{\partial^m}{\partial \varepsilon^m} \left[ \left( \frac{\partial t^{-1}}{\partial y} u \right)(t(z, \varepsilon), \varepsilon) \right] = \left[ \frac{\partial t^{-1}}{\partial y} \left( \frac{\partial}{\partial \varepsilon} + [ , T] \right)^m u \right] (t(z, \varepsilon), \varepsilon).
$$

(3.105)

At $\varepsilon = 0$, $\partial t^{-1}/\partial y = 1$, and with (3.99) we have

$$
u_m(z) = \frac{\partial^m u}{\partial \varepsilon^m} (z, 0) = \left[ \left( \frac{\partial}{\partial \varepsilon} + [ , T] \right)^m u \right](z, 0).
$$

(3.106)

Now we are ready to make the following statement.

**Theorem 3.10 [CH82]**

Let the sequence $u_i^{(m)}(y)$ for $i, m = 0, 1, 2, \ldots$, be defined by the recursive relation

$$
u_i^{(m)} = u_{i+1}^{(m-1)} + \sum_{j=0}^{i} \binom{i}{j} \left[ u_{i-j}^{(m-1)}, T_j \right], \quad i = 0, 1, 2, \ldots; m = 1, 2, \ldots
$$


$$
u_i^{(0)} = u_i, \quad i = 0, 1, 2, \ldots
$$

where $\binom{i}{j} = \frac{i!}{j!(i-j)!}$ denotes the binomial coefficient, then

$$
u_m = u_0^{(m)}, \quad m = 0, 1, 2, \ldots
$$
The computations can be organized according to the following triangle

\[
\begin{align*}
\text{\(u_0^{(0)}\)} & \leftarrow \text{\(u_0^{(1)}\)} \\
\text{\(u_1^{(0)}\)} & \leftarrow \text{\(u_0^{(2)}\)} \\
\text{\(u_2^{(0)}\)} & \leftarrow \text{\(u_1^{(2)}\)} \leftarrow \text{\(u_0^{(3)}\)} \\
\text{\(u_3^{(0)}\)} & \leftarrow \text{\(u_2^{(2)}\)} \leftarrow \text{\(u_1^{(3)}\)} \leftarrow \text{\(u_0^{(4)}\)} \\
& \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{align*}
\]

(3.107)

The \(i\)th element of the \(m\)th column of the triangle is \(u_i^{(m)}\) which can be computed by knowing the first \(i + 2\) elements of the \((m - 1)\)th column, i.e., by knowing \(u_0^{(m-1)}\) through \(u_{i+1}^{(m-1)}\). The element \(u_0^{(m)}\) on the diagonal of the triangle depends only on the sub-triangle formed by \(u_0^{(m)}\), \(u_0^{(m-1)}\) and \(u_1^{(m-1)}\).

**Proof of Theorem 3.10**

\(\triangleright\) The \(m\)th column of the triangle (3.107) consists of the coefficients in the formal expansion of

\[
\left( \left( \frac{\partial}{\partial \varepsilon} + [-, I] \right)^{m} u \right) (y, 0).
\]

(3.106) then proves the theorem. \(\triangleright\)

The next theorem can be used for the explicit formulas for the series expansion \(y = t(z, \varepsilon)\). Not surprisingly, the theorem looks very similar to the theorem above.

**Theorem 3.11 [CH82]**

\(\triangleright\) Let \(t(z, \varepsilon)\) be a solution of (3.94), \(p(y, \varepsilon)\) a given function of \(y\) and \(\varepsilon\),

\[
p(y, \varepsilon) = \sum_{m \geq 0} p_m(y) \varepsilon^m / m!,
\]

and

\[
g(z, \varepsilon) = p(t(z, \varepsilon), \varepsilon) = \sum_{m \geq 0} g_m(z) \varepsilon^m / m!
\]

Let the sequence \(p_i^{(m)}(z)\) for \(i, m = 0, 1, 2, \ldots\), be defined by the recursive relation

\[
p_i^{(m)} = p_{i+1}^{(m-1)} + \sum_{j=0}^{i} \binom{i}{j} \frac{\partial p_{i-j}^{(m-1)}}{\partial y} T_j, \quad i = 0, 1, 2, \ldots; \ m = 1, 2, \ldots
\]
3.2. Methods for simplification of vector fields

\[ P_i^{(0)} = P_i, \ i = 0, 1, 2, \ldots \]

then

\[ q_m = P_0^{(m)}, \ m = 0, 1, 2, \ldots \]

\[ \triangleright \]

The computations can be organized according to the following triangle

\[
\begin{array}{c}
P_0^{(0)} \\
P_1^{(0)} \leftarrow P_0^{(1)} \\
P_2^{(0)} \leftarrow P_1^{(1)} \leftarrow P_0^{(2)} \\
P_3^{(0)} \leftarrow P_2^{(1)} \leftarrow P_1^{(2)} \leftarrow P_0^{(3)} \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\] (3.108)

**Proof of Theorem 3.11**

\[ \triangleright \] From (3.96) we have

\[ p(t(\bar{z}, \varepsilon), \varepsilon) = \sum_{m \geq 0} \varepsilon^m / m! \left[ D^m p \right](\bar{z}, 0), \]

where

\[ D p = \frac{\partial p}{\partial \varepsilon} + \frac{\partial p}{\partial y} T \]

(see (3.97)). The \( m \)th column of triangle (3.108) satisfies

\[ D^m p(y, \varepsilon) = \sum_{j \geq 0} p_j^{(m)}(y) \varepsilon^m / m! \]

This proves the theorem. \[ \triangleright \]

Explicit formulas for the transformation \( y = t(\bar{z}, \varepsilon) \)

\[ t(\bar{z}, \varepsilon) = \bar{z} + \sum_{m \geq 1} t_m(\bar{z}) \varepsilon^m / m!, \]

are found by applying Theorem 3.11 with \( p(y, \varepsilon) = T(y, \varepsilon) \). This yields an expansion for

\[ T(t(\bar{z}, \varepsilon), \varepsilon) = \sum_{m \geq 0} q_m(y) \varepsilon^m / m!, \]
with \( t_{m+1}(y) = q_m(y) \), \( m = 0, 1, 2, \ldots \), given by the elements on the diagonal of triangle (3.108).

The following lemma shows that the generator \( \mathcal{T}(y, \varepsilon) \) of the transformation (3.91) can be used to transform the vector field (3.88) such that the vector field (3.92) is a normal form.

**Lemma 3.12 [CH82]**

If each \( u_m(y) \) is a homogeneous polynomial in \( y \) of degree \( m + 1 \) and each \( T_m(y) \) is a homogeneous polynomial in \( y \) of degree \( m + 2 \), then \( u_m(z) \), \( t_m(z) \) are homogeneous polynomials in \( z \) of degree \( m + 1 \).

**Proof of Lemma 3.12**

The recursive relations of Theorem 3.10 imply that \( u_i^{(m)}(z) \) is a homogeneous polynomial in \( z \) of degree \( m + i + 1 \). Thus, \( u_m(z) \) satisfies the stated properties. In the same way, the recursive relations of Theorem 3.11 with \( p(y, \varepsilon) = \mathcal{T}(y, \varepsilon) \) imply that \( t_m(y) \) is a homogeneous polynomial in \( y \) of degree \( m + 1 \).

The following example shows how the above theorems are applied.

### 3.2.8 2-D example. Normal form (continued)

Let the expansion of the original vector field be given by

\[
\dot{y} = f(y) = \sum_{m \geq 1} f_m(y), \quad f_m(y) \in \mathbb{H}^2_m, \tag{3.109}
\]

and let the Jacobian matrix of the vector field be the double zero Jordan normal form (3.56). Then, let the expansion of the resulting vector field and the generator of that transformation be given by

\[
\dot{z} = g(z) = \sum_{m \geq 1} g_m(z), \quad g_m(z) \in \mathbb{H}^2_m, \tag{3.110}
\]

and

\[
\mathcal{T}(y) = \sum_{m \geq 2} T_m(y), \quad T_m(y) \in \mathbb{H}^2_m, \tag{3.111}
\]

The expansions (3.98) and (3.99) used in Theorem 3.10 have slightly different definitions. The following relations hold,

\[
u_m(y) = f_{m+1}(y)m!, \quad \nu_m(z) = g_{m+1}(z)m!, \quad T_m(y) = t_{m+2}(y)m!,
\]
also see Lemma 3.12. Because there will no need to compute the actual transformation, we favour the notation $t_m$ over $T_m$. Then, the notation is in correspondence with the previous paragraphs.

Application of the triangle (3.107) results in

$$g_1(\bar{z}) = u_0 = u_0^{(0)} = u_0 = f_1(\bar{z}) = J\bar{z},$$

and

$$g_2(\bar{z}) = v_1 = u_0^{(1)} = u_1^{(0)} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} [u_0^{(0)}, T_0]$$

$$= u_1 + [u_0, T_0]$$

$$= \left\{ f_2 + \text{ad } J(t_2) \right\}(\bar{z}).$$

Thus, the computations at second order using the above method runs similar to the Section 3.2.5. Using the re-ordering into subspaces of $f_2(y) + \text{ad } J(t_2(y))$ in Table 3.3, we found the representation (3.78) of complementary subspace $\mathcal{G}_2$. As a result,

$$g_{20}^{(2)} = f_{20}^{(2)}, \quad g_{11}^{(2)} = f_{11}^{(2)} + 2f_{20}^{(1)}. \quad (3.112)$$

are the only possibly nonzero coefficients in $g_2(\bar{z})$. We also found the expression (3.86) for the kernel $\mathcal{K}_2$.

Using the triangle (3.107) to compute a normal form at third order results in

$$2g_3(\bar{z}) = v_2 = u_0^{(2)} = u_1^{(1)} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} [u_0^{(1)}, T_0]$$

$$= u_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} [u_1^{(0)}, T_0] + \begin{pmatrix} 1 \\ 1 \end{pmatrix} [u_0^{(0)}, T_1] + [u_0^{(1)}, T_0]$$

$$= u_2 + [u_1, T_0] + [u_0, T_1] + [v_1, T_0]$$

$$= \left\{ 2f_3 + [f_2, t_2] + 2 \text{ad } J(t_3) + [g_2, t_2] \right\}(\bar{z}). \quad (3.113)$$

Notice that in (3.113) within the curly brackets $g$-coordinates are used. The vector $\tilde{f}_3(y)$ in the homological equation (3.50) for $m = 3$,

$$\tilde{f}_3(y) + \text{ad } J(t_3(y)) = 0, \quad (3.114)$$

is in fact a combination of three contributions:

$$\tilde{f}_3(y) = f_3(y) + \frac{1}{2}[g_2(y), t_2(y)] + \frac{1}{2}[f_2(y), t_2(y)]. \quad (3.115)$$
\begin{array}{|c|c|c|}
\hline
y_1^3 & \tilde{f}^{(1)}_{30} + \tilde{t}^{(2)}_{30} & \tilde{f}^{(2)}_{30} \\
\hline
y_1^2 y_2 & \tilde{f}^{(1)}_{21} + \tilde{t}^{(2)}_{21} - 3\tilde{t}^{(1)}_{30} & \tilde{f}^{(2)}_{21} - 3\tilde{t}^{(2)}_{30} \\
\hline
y_1 y_2^2 & \tilde{f}^{(1)}_{12} + \tilde{t}^{(2)}_{12} - 2\tilde{t}^{(1)}_{21} & \tilde{f}^{(2)}_{12} - 2\tilde{t}^{(2)}_{21} \\
\hline
y_2^3 & \tilde{f}^{(1)}_{03} + \tilde{t}^{(2)}_{03} - \tilde{t}^{(1)}_{12} & \tilde{f}^{(2)}_{03} - \tilde{t}^{(2)}_{12} \\
\hline
\end{array}

a) Mapping.

\begin{array}{|c|c|c|}
\hline
C_1 & (\tilde{f}^{(1)}_{30} + \tilde{t}^{(2)}_{30})y_1^3 & (\tilde{f}^{(2)}_{21} - 3\tilde{t}^{(2)}_{30})y_1^2 y_2 \\
\hline
C_2 & (\tilde{f}^{(2)}_{30})y_1^2 & 1 \\
\hline
I_1 & (\tilde{f}^{(1)}_{21} + \tilde{t}^{(2)}_{21} - 3\tilde{t}^{(1)}_{30})y_1^2 y_2 & (\tilde{f}^{(2)}_{12} - 2\tilde{t}^{(2)}_{21})y_1 y_2^2 \\
\hline
I_2 & (\tilde{f}^{(1)}_{12} + \tilde{t}^{(2)}_{12} - 2\tilde{t}^{(1)}_{21})y_1 y_2 & (\tilde{f}^{(2)}_{03} - \tilde{t}^{(2)}_{12})y_2^3 \\
\hline
K_1 & (\tilde{f}^{(1)}_{03} + \tilde{t}^{(2)}_{03} - \tilde{t}^{(1)}_{12})y_1^3 & 1 \\
\hline
K_2 & \tilde{t}^{(1)}_{03} & 1 \\
\hline
\end{array}

b) Re-ordering into subspaces.

Table 3.4: $\tilde{f}_3(y) + \text{ad} J(t_3(y))$

As can be seen, the terms in $t_2(y)$ which are part of the kernel $K_2$ become part of $\tilde{f}_3(y)$.

Writting out (3.114) with the aid of (3.73) we get Table 3.4. The underlined terms in Table 3.4b are selected to be part of the complementary subspace $G_3$. Subspaces $C_1$ and $C_2$ show that $G_3$ can be written as

$$G_3 = \tilde{g}_3^{(2)} z_1^3 + g_2^{(2)} z_1^2 z_2,$$

where $g^{(2)}_{30} = \tilde{f}^{(2)}_{30}$ and $g^{(2)}_{21} = \tilde{f}^{(2)}_{21} + 3\tilde{f}^{(1)}_{30}$. The complementary subspace $G_3$ is in the form indicated by Theorem 3.9.

Explicit expressions like (3.113) are necessary if one uses computer algebra to compute a normal form. Ushiki [Usi84] showed the computation of the Lie product

$$[G_2, K_2],$$
suffices to identify which of the terms in complementary subspace $G_3$ are reached by the free parameters in the kernel $K_2$, and thus can be removed. Hence, with $K_2$ given in (3.86), we compute
\[
\begin{pmatrix}
\begin{pmatrix}
0 \\
g_{20}^{(2)} y_1^2 + g_{11}^{(2)} y_1 y_2
\end{pmatrix},
\begin{pmatrix}
y_1 y_2 \\
y_2^3
\end{pmatrix}
\end{pmatrix} = \begin{pmatrix}
-g_{20}^{(2)} y_1^3 - g_{11}^{(2)} y_1^2 y_2 \\
0
\end{pmatrix},
\]
and
\[
\begin{pmatrix}
\begin{pmatrix}
0 \\
g_{20}^{(2)} y_1^2 + g_{11}^{(2)} y_1 y_2
\end{pmatrix},
\begin{pmatrix}
y_2^2 \\
0
\end{pmatrix}
\end{pmatrix} = \begin{pmatrix}
-2g_{20}^{(2)} y_1^2 y_2 - 2g_{11}^{(2)} y_1 y_2^2 \\
2g_{20}^{(2)} y_1 y_2^2 + g_{11}^{(2)} y_2^3
\end{pmatrix}.
\]
We observe that the kernel vector $y_2^3 \partial_1$ reaches only the image, but that the kernel vector $y_1 y_2 \partial_1 + y_2^2 \partial_2$ reaches $y_3^3 \partial_1$, which is part of the complementary subspace, see $C_1$ in Table 3.4. Thus, instead of using $t_{30}^{(2)}$ to remove $f_{30}^{(1)}$, use the kernel vector $y_1 y_2 \partial_1 + y_2^2 \partial_2$. This way, $t_{30}^{(2)}$ is left to remove $f_{21}^{(2)}$, and therewith, subspace $C_1$ is removed altogether! This does require $g_{20}^{(2)}$ to be nonzero. The vector $y_2^3 \partial_1$ could not be used to simplify $G_3$, but can possibly be of use to simplify $G_4$.

In conclusion, if $g_{20}^{(2)} \neq 0$,
\[
\begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= g_{20}^{(2)} z_2^2 + g_{11}^{(2)} z_1 z_2 + g_{30}^{(2)} z_1^3.
\end{align*}
\]
represents a normal form for (3.56) up to third degree.

Simplified normal forms like (3.117) are called hypernormal forms. Using computer algebra, Gamero et al. [GFP91] were able to obtain a hypernormal form up to 8th degree with an explicit expression for the coefficients $g_{k0}^{(2)}$ and $g_{k-1}^{(2)}$.

### 3.3 Versal deformation of vector fields

Consider the parameterized vector field
\[
\dot{x} = f(x; \mu), \quad x \in U \subset \mathbb{R}^n, \quad \mu \in \mathcal{V} \subset \mathbb{R}^p,
\]
where $f$ is a $C^r$, $r \geq 1$, parameter-dependent vector field defined on the cross-product of the region $U$ around $x_0$ in the phase-space $\mathbb{R}^n$ and the region $\mathcal{V}$
around \( \mu_0 \) in the parameter-space \( \mathbb{R}^p \). Suppose that (3.118) has a critical point at \( (x, \mu) = (x_0, \mu_0) \), i.e.,
\[
    f(x_0; \mu_0) = 0. \tag{3.119}
\]
The previous sections classified this critical point using the vector field
\[
    \dot{\xi} = f(\xi; \mu_0), \quad \xi = x - x_0, \tag{3.120}
\]
in two steps; 1) transformation to Jordan form of the Jacobian matrix, and 2) transformation of the nonlinear part to a normal form. We will now look at vector fields in the neighborhood of the vector field (3.120), that is, for \( \|\mu - \mu_0\| \) sufficiently small. We want to find out how the topological classification of the critical point is likely to change under the influence of perturbations acting on the vector field.

As before, we will analyze the vector field (3.118) in two steps; first, we investigate which parameters change the topological classification of the linearized vector field in \( (x_0, \mu_0) \), given by
\[
    \dot{\xi} = D_xf(x_0; \mu_0)\xi, \quad \xi \in \mathbb{R}^n. \tag{3.121}
\]
Second, based on this knowledge, we see which parameters change the topological classification of the nonlinear vector field. Our discussion closely follows Wiggins [Wig90], which in turn follows Arnol'd [Arn88].

### 3.3.1 Versal deformation of families of matrices

Let \( \mathcal{M} \) be the space of all \( n \times n \) matrices with (for the moment) complex entries, and let \( A_0 \) be such a matrix. Consider the following parameter-dependent vector field,
\[
    \dot{x} = A(\mu)x, \quad A(\mu_0) = A_0, \tag{3.122}
\]
where \( A : \mathbb{C}^p \mapsto \mathcal{M}, \ p \geq 1 \).

If the matrix \( A_0 \) has multiple eigenvalues, then it is not a stable process to transform \( A(\mu) \) to a Jordan form around \( \mu = \mu_0 \). A linear coordinate transformation \( x = M(\mu)y, \ M \in \mathcal{M}, \ M \) being invertible, leads to
\[
    \dot{y} = J(\mu)y \equiv M^{-1}(\mu)A(\mu)M(\mu)y. \tag{3.123}
\]
For fixed values of \( \mu \), the matrices \( A(\mu) \) and \( J(\mu) \) are called similar and \( M(\mu) \) is the conjugating matrix. The matrix \( J(\mu) \) has the same eigenvalues and dimensions of Jordan blocks as \( A(\mu) \).
However, consider the following example for continuous $\mu \in \mathbb{C}$,

$$A(\mu) = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix},$$  \hspace{1cm} (3.124)

where $\lambda \in \mathbb{C}$ is given. The Jordan canonical form of $A(\mu)$ for nonzero $\mu$ is

$$J(\mu) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \hspace{1cm} \mu \neq 0.$$  

This matrix has two repeated eigenvalues $\lambda$, i.e., one Jordan blocks of size 2. A conjugating matrix is given by

$$M(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & 1/\mu \end{pmatrix}, \hspace{1cm} \mu \neq 0.$$  \hspace{1cm} (3.125)

For $\mu = 0$, $A(0) = A_0$ equals its Jordan canonical form, which is

$$J(0) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

and which has any invertible matrix as its conjugating matrix. This matrix has two eigenvalues $\lambda$, i.e., two Jordan blocks of size 1. Clearly, no continuous mapping $M(\mu)$ can be found about $\mu = 0$.

With the above example in mind, what are the answers to the following questions:

1. What is the simplest form to which a family of matrices (depending analytically on the parameters) can be reduced by a change of parameters (which depends analytically on the parameters)?

2. Especially, what is the minimum number of parameters?

The answer first requires some definitions.

**Definition 3.13**

A deformation of a matrix $A_0 \in \mathcal{M}$ is an analytic mapping,

$$A : \Lambda \rightarrow \mathcal{M},$$

$$\mu \rightarrow A(\mu),$$

where $\Lambda \in \mathbb{C}^\ell$ is some parameter space around $\mu_0$ and

$$A(\mu_0) = A_0.$$
Another name for a deformation is a family, the variables $\mu_i$, $i = 1, \ldots, \ell$, are called the parameters, and $\Lambda$ is called the base of the family. Two deformations $A_1(\mu)$ and $A_2(\mu)$ are called equivalent if there exists a deformation $M(\mu)$ of the identity matrix $I$ with the same base such that

$$A_2(\mu) = M^{-1}(\mu)A_1(\mu)M(\mu), \quad M(\mu_0) = I.$$ 

If the above relation holds for fixed $\mu$, we would say that $A_1(\mu)$ and $A_2(\mu)$ are similar. The definition, however, requires $M(\mu)$ to be a deformation, and thus that is an analytic mapping, so it must be continuous in $\mu$.

The following definition will be useful for reparametrizing families of matrices in order to reduce the number of parameters.

**Definition 3.14**
Let $\Sigma \subset \mathbb{C}^n$, $\Lambda \subset \mathbb{C}^\ell$ be open sets. Consider the analytic parameter mapping

$$\phi: \Sigma \to \Lambda, \quad \eta \to \phi(\eta),$$

with $\phi(\eta_0) = \mu_0$. The family induced from $A$ by the mapping $\phi$ is called $(\phi^*A)(\eta)$ and is defined by

$$(\phi^*A)(\eta) \equiv A(\phi(\eta)), \quad \eta \in \Sigma.$$ 

**Definition 3.15**
A deformation $A(\mu)$ of a matrix $A_0$ is said to be versal if any deformation $B(\eta)$ of $A_0$ is equivalent to a family induced from $A$ by some appropriate mapping $\phi(\eta)$. In other words,

$$B(\eta) = M^{-1}(\eta)A(\phi(\eta))M(\eta)$$

for some change of parameters $\phi: \Sigma \to \Lambda$, with $M(\eta_0) = I$ and $\phi(\eta_0) = \mu_0$. A versal deformation is said to be universal if the inducing mapping $\phi$ is determined uniquely by the deformation $B$. A versal deformation is said to be miniversal if the dimension of the parameter space is the smallest possible for a versal deformation.

**Example versal deformations**
Let us put these definitions to work for the matrix

$$A_0 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \quad (3.126)$$
3.3. Versal deformation of vector fields

An obvious versal deformation of $A_0$ at $\mu = \mu_0 \equiv 0$ is given by

$$B(\eta) \equiv \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix},$$

(3.127)

where $\eta \equiv (\eta_1, \eta_2, \eta_3, \eta_4) \in \mathbb{C}^4$. The miniversal deformation is given by

$$A(\mu) \equiv \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mu_1 & \mu_2 \end{pmatrix},$$

(3.128)

where $\mu \equiv (\mu_1, \mu_2) \in \mathbb{C}^2$. Miniversality is partly proven by showing that $B(\eta)$ is equivalent to a deformation induced from $A(\mu)$ at $\mu = \mu_0$. Let

$$M(\eta) = \begin{pmatrix} 1 & 0 \\ \eta_1 & 1 + \eta_2 \end{pmatrix}, \quad M^{-1}(\eta) = \frac{1}{1+\eta_2} \begin{pmatrix} 1 + \eta_2 & 0 \\ -\eta_1 & 1 \end{pmatrix},$$

then $(\phi^* A)(\eta)$ is equal to $A(\mu)$, if

$$A(\mu) = M(\eta)B(\eta)M^{-1}(\eta)$$

$$= \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \eta_3(1+\eta_2) - \eta_1 \eta_4 & \eta_1 + \eta_4 \end{pmatrix}$$

holds. This condition is satisfied by the following inducing mapping

$$\phi(\eta) = (\phi_1(\eta), \phi_2(\eta)) = (\eta_3(1+\eta_2) - \eta_1 \eta_4, \eta_1 + \eta_4) \equiv (\mu_1, \mu_2) \equiv \mu.$$

Finally, we note that the matrix $A(\mu)$ in (3.124) is a deformation, but is not versal. Also, the conjugating matrix $M(\mu)$ for nonzero $\mu$ in (3.125) is not a deformation of the identity matrix.

Miniversal deformation of matrices

The objective is to construct a miniversal deformation $A(\mu)$ of $A_0$ at $\mu_0$. To this end, the deformation should not include any matrix similar to $A_0$ other than $A_0$ itself. The matrices similar to a particular matrix form a special manifold in $\mathcal{M}$.

Consider the Lie group $G = GL(n, \mathbb{C})$ of all nonsingular $n \times n$ matrices with complex entries. $GL(n, \mathbb{C})$ is a submanifold of $\mathcal{M}$.

The group $G$ acts on $\mathcal{M}$ according to the formula

$$\text{Ad}_G : \mathcal{M} \rightarrow \mathcal{M},$$

$$M \rightarrow G^{-1} MG \equiv \text{Ad}_G M, \quad G \in G.$$  

(3.129)
Figure 3.10: The mapping $A(\mu)$ is transversal at $\mu_0$ to the orbit $N$ of $A_0$.

The orbit of an arbitrary fixed matrix $A_0 \in \mathcal{M}$ under the action of $G$ is the set of all 'points' $M \in \mathcal{M}$ such that $M = G^{-1} A_0 G$ for all $G \in G$. Thus, from (3.129), the orbit of $A_0$ under $G$ consists of all matrices similar to $A_0$. This orbit forms a smooth submanifold of $\mathcal{M}$, denoted by $N$. Note that the orbit of a singular matrix contains only singular matrices whilst the orbit of a nonsingular matrix consists of only nonsingular matrices.

To make sure that the deformation $A(\mu)$ does not include any matrices equivalent to $A_0$, let $A(\mu)$ be a mapping from some manifold $\Lambda \subset \mathbb{C}^\ell$ to some manifold transversal to the orbit $N$ of $A_0$ at $\mu = \mu_0 \in \Lambda$, see Fig. 3.10. The mapping $A(\mu)$ is called transversal to the orbit $N$ at $\mu_0$ if the tangent space to $\mathcal{M}$ at $A(\mu_0)$ is the sum

$$T_{A(\mu_0)}\mathcal{M} = T_{A(\mu_0)}N + DA(\mu_0) \cdot T_{\mu_0} \Lambda,$$

(3.130)

where $DA(\mu_0)$ denotes the derivative of $A$ at $\mu_0$. The following result holds.

**Proposition 3.16**

- The deformation $A(\mu)$ is versal. If the dimension of the parameter space $\Lambda$ is equal to the co-dimension of the orbit of $A_0$, then the deformation is miniversal.

Suppose that $B(\eta)$, $\eta \in \Sigma \subset \mathbb{C}^m$, $m \geq \ell$, is some arbitrary versal deformation of $A_0$, i.e., for some $\eta_0 \in \Sigma$, $B(\eta_0) = A_0$. To prove Proposition 3.16, we should be able to find an deformation $C(\eta)$ of the identity matrix $I$ and coordinate transformation $\phi(\eta)$ such that

$$B(\eta) = C^{-1}(\eta)A(\phi(\eta))C(\eta).$$

(3.131)
The parameters $\mu$ in the deformation $A(\mu)$ of $A_0$ are thought of as perturbation parameters about $\mu = \mu_0$. The same observation holds for the parameters $\eta$ in the deformation $B(\eta)$, i.e., $\|\eta - \eta_0\|$ is sufficiently small. This means that the deformation $C(\eta)$ maps to matrices close to the identity matrix $I$. These matrices are all nonsingular (and thus invertible).

The identity matrix $I$ is part of the centralizer of a matrix $A_0$, which is the set of all matrices commuting with $A_0$. Notation:

\[ Z_0 = \{ M \in \mathcal{M} \mid [A_0, M] = 0 \}, \quad [M_1, M_2] \equiv M_1 M_2 - M_2 M_1. \]

The set of matrices near the identity matrix $I$ has dimension $n^2$. Within this set, there exists a subspace $\{I\} + Z_0$.

Let $P$ be a smooth submanifold of the set of nonsingular matrices near the identity matrix $I$, intersecting the subspace $\{I\} + Z_0$ transversely at $I$ and having dimension equal to the co-dimension of the centralizer; see Fig. 3.11. Based on (3.131), consider the mapping

\[ \Phi : P \times \Lambda \to \mathbb{C}^n, \quad \Phi : (P, \mu) \to P^{-1}A(\mu)P \equiv \Phi(P, \mu). \quad (3.132) \]

**Lemma 3.17**

\( \Downarrow \) The mapping $\Phi$ is a local diffeomorphism in the neighborhood of $(I, \mu_0)$. \( \Uparrow \)

See Fig. 3.12.

The requirements for $\Phi$ being a local diffeomorphism are that $\dim(P \times \Lambda) = \dim(\mathcal{M}) = n^2$, and that the derivative of $\Phi$ at $(I, \mu_0)$, denoted $D\Phi(I, \mu_0)$, is an isomorphism between linear spaces of dimension $n^2$.

Consider the way $D\Phi(I, \mu_0)$ acts on an element of $T_{(I, \mu_0)}(P \times \Lambda)$. Let
Figure 3.12: The manifolds $P$ and $\Lambda$ and the local diffeomorphism $\Phi$. $N$ is the orbit of the matrix $A_0$.

$$(\hat{P}, \hat{\mu}) \in T_{(l, \mu_0)}(P \times \Lambda);$$ then we have

$$D\Phi(l, \mu_0)(\hat{P}, \hat{\mu}) = (D_P \Phi(l, \mu_0), \mu_0) (\hat{P}, \hat{\mu}) = [(A_0, \hat{P}), DA(\mu_0) \hat{\mu}].$$

The equivalence of the second partial derivative is easy to verify using its definition (3.132);

$$D_{\mu} \Phi(l, \mu_0)(\hat{P}, \hat{\mu}) = l^{-1} DA(\mu_0) \mid \hat{\mu} = DA(\mu_0) \hat{\mu}. $$

In the first partial derivative, the parameter dependence is of no concern. We should evaluate the derivative in the direction $\hat{P}$.

$$D_P \Phi(l, \mu_0) \hat{P} \equiv \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon \hat{P})^{-1} A(\mu_0)(1 + \varepsilon \hat{P}) - l^{-1} A(\mu_0) l}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{(1 - \varepsilon \hat{P}) A_0 (1 + \varepsilon \hat{P}) + O(\varepsilon^2) - A_0}{\varepsilon} = A_0 \hat{P} - \hat{P} A_0 \equiv [A_0, \hat{P}].$$

This proves (3.133).

It is our assumption for miniversality that

$$\dim \Lambda = n^2 - \dim N.$$  

Also, by its construction

$$\dim P = n^2 - \dim Z_0.$$
From the definition of the orbit of $A_0$ in (3.129), the definition of the mapping $\Phi$ in (3.132), and the result in (3.134), we see that

$$\dim Z_0 = n^2 - \dim N.$$  

Thus, $\dim(P \times \Lambda) = n^2$. For a demonstration that (3.133) is an isomorphism see Wiggins [Wig90], page 310 f.f.

We now know that for every $\eta \in \Sigma$, we are able to find some $P \in P$ and $\mu \in \Lambda$, such that

$$B(\eta) = \Phi(P, \mu).$$

Then, the deformation $C(\eta)$ and the coordinate transform $\phi(\eta)$ are found using the natural projections $\pi_1$ and $\pi_2$ onto $P$ and $\Lambda$ of $P \times \Lambda$, respectively, and the fact that $\Phi$ is a local diffeomorphism (and thus invertible),

$$P = C(\eta) \equiv \pi_1 \Phi^{-1}(B(\eta)),
\mu = \phi(\eta) \equiv \pi_2 \Phi^{-1}(B(\eta)).$$

(3.135)

The manifold $P$ transversal to the orbit of $A_0$ at $\mu_0$ is found as follows. Lemma 3.17 shows that a miniversal deformation of $A_0$ is a family of matrices

$$A(\mu) = A_0 + B,$$

where the variables $\mu$ are the entries of $B$, and $B$ is in the orthogonal complement of the orbit of $A_0$. The following lemma informs us how to compute the matrix $B$.

**Lemma 3.18**

$\blacktriangleleft$ A matrix $B$ is orthogonal to the orbit of $A_0$ if and only if

$$[A_0, B^\ast] = 0,$$

(3.136)

where $B^\ast$ denotes the transpose and complex conjugate of $B$. $\triangleright$

The result is that the form of $B$ can be ‘read off’ $A_0$ if it is in Jordan canonical form.

**Proof of Lemma 3.18**

$\blacktriangleleft$ Matrices tangent to the orbit of $A_0$ are matrices representable in the form

$$[M, A_0], \quad M \in \mathcal{M}.$$ 

Orthogonality of $B$ to the orbit of $A_0$ means that, for any $X \in \mathcal{M},$

$$([X, A_0], B) = 0,$$

(3.137)
where $\langle \cdot, \cdot \rangle$ denotes an inner product on $\mathcal{M}$, say

$$\langle A_1, A_2 \rangle \equiv \text{tr}(A_1 A_2^*) .$$

This definition makes the zero on the right-hand side of (3.137) a scalar.

Using the definition, (3.137) becomes

$$0 = \text{tr}([X,A_0]^* B^*) = \text{tr}(X A_0 B^* - A_0 X B^*) .$$

Since $\text{tr}(A_1 A_2) = \text{tr}(A_2 A_1)$ and $\text{tr}(A_1 + A_2) = \text{tr} A_1 + \text{tr} A_2$, the latter equation leads to

$$\text{tr}(X A_0 B^* - A_0 X B^*) = \text{tr}(X A_0 B^*) - \text{tr}(A_0 X B^*)$$
$$= \text{tr}(A_0 B^* X) - \text{tr}(B^* A_0 X)$$
$$= \text{tr}([A_0, B^*] X)$$
$$= \langle [A_0, B^*], X \rangle .$$

Since $X$ was arbitrary, the latter result implies (3.136) in Lemma 3.18.

Using the above concepts, Arnol’d [Arn88] proved the following two theorems which provide the minimal dimension $\ell$ for a given matrix $A_0$, and explain how to construct the manifolds $P$ and $\Lambda$.

Denote by $\lambda_1, \ldots, \lambda_s$, the eigenvalues of the $s$ Jordan blocks of the matrix $A_0$ and let $n_1(\lambda_1) \geq n_2(\lambda_2) \geq \ldots \geq n_s(\lambda_s)$ be the dimensions of these Jordan blocks belonging to the $\lambda_i$, beginning with the largest ones.

**Theorem 3.19**

$\blacktriangleleft$ The smallest number of parameters of a versal deformation of the matrix $A_0$ is equal to

$$d = n_1(\lambda_1) + 3n_2(\lambda_2) + 5n_3(\lambda_3) + \ldots + (2s - 1)n_s(\lambda_s) .$$

(3.138)

$\blacktriangleright$

The miniversal deformations can be chosen in different ways. In particular, the three (linear part) normal forms described in the next theorem are versal deformations of a matrix reduced to the upper triangular Jordan normal form.

**Theorem 3.20**

$\blacktriangleleft$ Every matrix $A_0$ has a miniversal deformation; the number of its parameters is equal to the co-dimension of the orbit $N$ of $A_0$ or, equivalently, to the dimension of the centralizer of $A_0$. If $A_0$ is in Jordan canonical form, then for a miniversal deformation we may take a $d$-parameter ($d$ from (3.138)) normal form $A_0 + B$. 

$\blacktriangleright$
where the blocks are of the form depicted in Fig. 3.13 (described below). In other words, any complex matrix close to a given matrix can be reduced to the $d$-parameter normal form $A_0 + B$ (where $A_0$ is the Jordan canonical form of the given matrix), so that the reducing mapping and the parameters of the normal form depend on the elements of the original matrix.

The heavy-black outlined squares in Fig. 3.13 represent the boundaries of the Jordan blocks. The gray areas depict the location of the parameters. Their number, given by (3.138), is easily checked in case of the first two normal forms in Fig. 3.13. In the third normal form in Fig. 3.13, all entries are equal on the (off) diagonal line(s) in the lower triangles. The advantage of the latter representation is the orthogonality of the versal deformation to the corresponding orbit of $A_0$.

Note that $A(\mu)$ in (3.128) corresponds with the first normal form Fig. 3.13.

The three normal forms in Fig. 3.13 are representations for matrices in the sense of Lemma 3.18. Why these normal forms are as they are has to do with the theory of matrices, see [Arn88]. We will not address this subject here, but continue our example from page 104.

Example miniversal deformation (continued)

We continue our proof that $A(\mu)$ in (3.128) is indeed a miniversal deformation of $A_0$ in (3.126), i.e., of

$$A_0 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$
First find all matrices, \( M \in \mathcal{M} \), which commute with \( A_0 \),

\[
[M, A_0] = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = \begin{pmatrix} -m_3 & m_1 - m_4 \\ 0 & m_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

Thus, \( m_3 = 0 \), \( m_1 = m_4 \) and \( m_2 \) are arbitrary. Therefore, take

\[
B^* = \begin{pmatrix} m_1 & m_2 \\ 0 & m_1 \end{pmatrix}.
\]

From the arguments in Lemma 3.18, the matrix orthogonal to the orbit \( A_0 \) is given by

\[
B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{m_1} & 0 \\ \frac{1}{m_2} & \frac{1}{m_1} \end{pmatrix}.
\]

Therefore,

\[
A(\mu) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \mu_1 & 0 \\ \mu_2 & \mu_1 \end{pmatrix},
\]

is a miniversal deformation of matrix \( A_0 \). This matrix is the third normal form representation in Fig. 3.13.

The first normal form in Fig. 3.13 leads to another matrix representation,

\[
\tilde{B} = \begin{pmatrix} 0 & 0 \\ \frac{m_2}{m_1} & \frac{1}{m_1} \end{pmatrix}.
\]

To check whether or not \( \tilde{B} \) is in any way transverse to \( A_0 \) it is sufficient to show that

\[
\langle \tilde{B}, B \rangle \equiv \text{tr}(\tilde{B} B^*) \neq 0.
\]

Doing the computation with the above matrices,

\[
\tilde{B} B^* = \begin{pmatrix} 0 & 0 \\ \frac{m_2}{m_1} & \frac{1}{m_1} \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ 0 & m_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{m_2}{m_1}m_1 + \frac{1}{m_1}m_1 \end{pmatrix},
\]

so that

\[
\langle \tilde{B}, B \rangle \equiv \text{tr}(\tilde{B} B^*) = |m_2|^2 + |m_1|^2 \neq 0,
\]

if both matrices are different from the all zero matrix.

Since \( m_1 \) and \( m_2 \) are arbitrary, the deformation of \( A_0 \) in (3.128),

\[
A(\mu) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mu_1 & \mu_2 \end{pmatrix},
\]

is indeed miniversal.
3.3. Versal deformation of vector fields

Decomplexification

The above discussion concerned matrices with complex entries simply because the Jordan canonical form has its eigenvalues directly on the diagonal. However, the matrices of interest in this thesis are real-valued. Fortunately, the results for versal deformation of matrices with complex entries easily ‘extend’ to the versal deformations of matrices with real entries, see [Arn88].

Note that complex eigenvalues for the Jacobian matrix of real-valued vector fields always come as conjugated pairs. Hence, such a Jordan block can be written as

\[
\begin{pmatrix}
A_r & -A_i \\
A_i & A_r
\end{pmatrix},
\]

where both $A_r$ and $A_i$ are $n \times n$ matrices. The complex representation of this matrix reads

\[ A = A_r + iA_i. \]

Again, consider the miniversal deformation

\[ A(\mu) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mu_1 & \mu_2 \end{pmatrix}, \quad (3.140) \]

Let $\lambda = \lambda_r + i\lambda_i$, $\mu_k = \mu_r^{(k)} + i\mu_i^{(k)}$, $k \in \{1, 2\}$, where $\lambda_r$, $\lambda_i$, $\mu_r^{(k)}$, and $\mu_i^{(k)}$ are real. The decomplexification of (3.140) is given by

\[ \begin{pmatrix} \lambda_r & 1 & -\lambda_i & 0 \\ 0 & \lambda_r & 0 & -\lambda_i \\ \lambda_i & 0 & \lambda_r & 1 \\ 0 & \lambda_i & 0 & -\lambda_r \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mu_r^{(1)} & \mu_r^{(2)} & -\mu_i^{(1)} & -\mu_i^{(2)} \\ \mu_i^{(1)} & \mu_i^{(2)} & \mu_r^{(1)} & \mu_r^{(2)} \end{pmatrix}. \quad (3.141) \]

Naturally, if $\lambda$, $\mu_1$, and $\mu_2$ in (3.140) each are part of $\mathbb{R}$ only, then the deformation (3.140) can be used directly as the deformation of a real-valued vector field.

3.3.2 Versal deformation of nonlinear vector fields

Finding a versal deformation (or versal unfolding) of a vector field, say $f$, runs similar to finding versal deformation of matrices: In the set of all vector fields a submanifold should be found transversal to the manifold consisting of vector fields $C^0$ equivalent with $f$.

There is one major theoretical pitfall: The space of all vector fields is infinite dimensional. Somehow we need to circumvent this problem.
Chapter 3. Bifurcation in General

$k$-Jets, $k$-determinacy and co-dimension

Let $\mathcal{J}^k(f, x)$ denote the natural projection of a map $f(x) \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, $r \geq k$, onto its $k$-jet.

**Definition 3.21**

Two $C^r(\mathbb{R}^n, \mathbb{R}^m)$ maps $f, g$ are said to be $k$-jet equivalent (with $0 \leq k < r$) in $x = 0$ if $f - g = O(\|x\|_n^{k+1})$, i.e., $\exists c > 0$ and $\exists \delta > 0$ such that $\forall \|x\|_n < \delta$, $\|(f - g)(x)\|_m < c\|x\|_n^{k+1}$, where $\|\cdot\|_\ell$ denotes the Euclidean norm on $\mathbb{R}^\ell$.

Notation: $\mathcal{J}^k f = \mathcal{J}^k g$.

Hence, a Taylor series expansion is $k$-jet equivalent with a Taylor polynomial of degree $k$.

**Definition 3.22**

A map $f(x) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ is $k$-determinate at $x = 0$ if whenever a smooth map $g(x) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ has $\mathcal{J}^k g = \mathcal{J}^k f$, then the flows of $f(x)$ and $g(x)$ are topological equivalent.

The objective is to truncate the series of a normal form such that the higher order terms left out do not change the topological classification of the $k$-jet.

**Definition 3.23**

The $k$-jet of $f$ at $x_0$ is the equivalence class of maps $k$-jet equivalent with $f^{(k)}$, where $f^{(k)}$ is the Taylor polynomial of $f$ of degree $k$ at $x_0$.

Denote the $k$-jet of $f$ at $x_0$ by the following $(k + 2)$-tuple,

$$J^k_{x_0}(f) \equiv \left( x, f(x_0), Df(x_0)x, \ldots, \sum_{i=k} D_i f(x_0)(x - x_0)^i \right),$$

(for an explanation of the notations, see Theorem B.4). The set of all $k$-jets of $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ at all points $x_0 \in \mathbb{R}^n$ is called the space of $k$-jets. Notation: $J^k(\mathbb{R}^n, \mathbb{R}^m)$.

One can think of the spaces $J^k(\mathbb{R}^n, \mathbb{R}^m)$ as finite-dimensional approximations of the infinite-dimensional space $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$. The spaces $J^k(\mathbb{R}^n, \mathbb{R}^m)$ can be identified with $\mathbb{R}^p$ for an appropriate $p$. These spaces have a nice linear vector space structure.

For example: $J^0(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^n \times \mathbb{R}^m$. Also, $J^1(\mathbb{R}^2, \mathbb{R}^2) = \mathbb{R}^8$.

**Definition 3.24 Co-dimension of a critical point**

Consider $J^k(\mathbb{R}^n, \mathbb{R}^n)$ and the subset of $J^k(\mathbb{R}^n, \mathbb{R}^n)$ consisting of elements from
3.3. Versal deformation of vector fields

$C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ which have a critical point at the origin ($k$ large enough). We denote this subset by $F$ and note that $F$ has co-dimension $n$ in $J^k(\mathbb{R}^n, \mathbb{R}^n)$. Consider the $k$-jet of an element of $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ that has a nonhyperbolic fixed point. Then this $k$-jet lies in a subset of $F$ defined by the conditions on its derivatives. Suppose that this subset of $F$ has co-dimension $b$ in $J^k(\mathbb{R}^n, \mathbb{R}^n)$. Then we define the co-dimension of the critical point to be $b - n$.

For example, the critical point in the origin of a map $f \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ that has a linear part similar to the Jordan normal form (3.56) has (at least) co-dimension 2.

**Definition 3.25**
For a map $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, the map

$$ \hat{f} : \mathbb{R}^n \rightarrow J^k(\mathbb{R}^n, \mathbb{R}^m), $$

$$ \hat{x} \rightarrow J^k_{\hat{x}}(f) \equiv \hat{f}, $$

is called the $k$-jet extension of $f$.

**Definition 3.26**
Let $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and $M \subset \mathbb{R}^m$ be a submanifold. The map is said to be transversal to $M$ at a point $\hat{x} \in \mathbb{R}^n$ if either $f(\hat{x}) \not\in M$ or the tangent space to $M$ at $f(\hat{x})$ and the image of the tangent space to $\mathbb{R}^n$ at $\hat{x}$ under $Df(\hat{x})$ are transversal, i.e.,

$$ T_{f(\hat{x})} \mathbb{R}^m = T_{f(\hat{x})} M + Df(\hat{x}) \cdot T_{\hat{x}} \mathbb{R}^n. $$

The map is said to be transversal to $M$ if it is transversal to $M$ at any point $\hat{x} \in \mathbb{R}^n$.

**Theorem 3.27 (Thom)**

Let $C$ be a submanifold of $J^k(\mathbb{R}^n, \mathbb{R}^m)$. The set of maps $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ whose $k$-jet extensions are transversal to $C$ is an everywhere dense countable intersection of open sets in $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

For a proof, see Arnol'd [Arn88].

A set $\mathcal{M}$ is an 'everywhere dense intersection in $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$' if every open set in $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ has a element from $\mathcal{M}$. A good analogy is the combination of the sets $Q$ and $\mathbb{P}$; $Q$ is countable, an intersection of $\mathbb{R}$, and everywhere dense in $\mathbb{R}$.

The theorem has the following two consequences:

- almost every polynomial function of degree $k$ is the $k$th Taylor polynomial of vector functions having versal deformations, and
• any vector function that has such a kth degree Taylor polynomial has a versal deformation,

according to Shirer and Wells [SW83], page 57.

The Taylor series expansions of vector fields play the role of the set of maps \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \). The k-jet extensions are the truncated series at order k. What we want to show is that not only does the dynamical behavior of the k-jet represent the dynamical behavior of the set of maps \( f \), it also represents the dynamical behavior of maps in \( C^\infty(\mathbb{R}^n, \mathbb{R}^m) \). To this end, it must be shown that the k-jet extensions of a set of maps \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) are transversal to \( C \), which is some submanifold of \( J^k(\mathbb{R}^n, \mathbb{R}^m) \).

The above theorem nicely takes care of the aforementioned pitfall. However, it does not help us much seen from a practical point of view: We still do not know how to construct a (mini)versal deformation of a vector field.

In practice, the procedure is as follows: Find the co-dimension of the critical point and use exactly that many parameters in the deformation. Then use the Thom transversality theorem to see whether the obtained deformation is versal. The following example, however, shows that finding the right versal deformation of a vector field is more an art than a science.

### 3.3.3 2-D example. Topological normal form

We found that two-dimensional vector fields with a linear part similar to the Jordan normal form (3.56) have co-dimension 2 if all (leading) coefficients in the normal form are nonzero. It was previously shown that (3.139),

\[
\begin{pmatrix}
0 & 1 \\
\mu_1 & \mu_2
\end{pmatrix},
\]

is a miniversal deformation of (3.56). The resulting 2-parameter unfolding of the normal form (3.71) is given by

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \mu_1 x_1 + \mu_2 x_2 + x_1^2 + bx_1 x_2.
\end{align*}
\]

where \( b = \pm 1 \). We will explain the coefficients 1 and \( b \) shortly hereafter.

This vector field has two critical points

\((0,0)^t, \quad \text{and} \quad (-\mu_1,0)^t.\)

if \( \mu_1 \neq 0 \). Note that the origin remains a critical point regardless whether the parameters \( \mu_1 \) and \( \mu_2 \) are varied.
Substitution of the coordinate transformation
\begin{align*}
  x_1 &\to x_1 + \mu_0, \\
  x_2 &\to x_2,
\end{align*}
in (3.142) results in
\begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= (\mu_0 \mu_1 + \mu_0^2) + (\mu_1 + 2\mu_0)x_1 + (\mu_2 + b\mu_0)x_2 + x_1^2 + bx_1x_2
\end{align*}
By setting $\mu_0 = -\frac{1}{2}\mu_1$, we remove the coefficient of $x_1\partial_2$. Effectively we positioned the origin of the resulting vector field in between the two critical points of the original vector field. By a reparametrization
\[ \mu_0 \mu_1 + \mu_0^2 = -\frac{1}{4}\mu_1^2 \mapsto \eta_1, \quad \mu_2 + b\mu_0 = \mu_2 - \frac{1}{2}b\mu_1 \mapsto \eta_2, \]
we get
\begin{align*}
  \dot{x}_1 &= x_2 \\
  \dot{x}_2 &= \eta_1 + \eta_2 x_2 + x_1^2 + bx_1x_2. \tag{3.143}
\end{align*}
The critical points lie on intersection of the lines $x_2 = 0$ and $\eta_1 + x_1^2 = 0$. If we allow positive values for $\eta_1$, the critical points can annihilate themselves. This behavior is an extension of the dynamical behavior of the vector field (3.142) and we will work with the vector field (3.143) instead.

### 3.3.4 $k$-Determinacy and transversality
In our presentation of the topological normal forms (3.142) and (3.143) for the running two-dimensional example we neglected to discuss an important detail: These normal forms are 2-determinate. Effectively this statement means that third and higher order terms do not alter the dynamical behavior. Such a statement does need a proof.

In this section, we discuss two theorems by Mather on $k$-determinacy. The discussion follows Shirer and Wells [SW83].

**Definition 3.28**
A smooth $n \times q$ matrix function $N(x)$ satisfies the **transversality condition** with respect to $f(x)$ near the origin $x = 0$ if and only if every smooth $n$-vector function $Y(x)$ may be written near the origin as
\[ Y(x) = Df(x)g(x) + H(x)f(x) + N(x)\mu, \tag{3.144} \]
where $g(x)$ is a suitable smooth $n$-vector function, $H(x)$ is a suitable smooth $n \times n$ matrix function and $\mu$ is a suitable constant $q$-vector.
It is further noted that the above quantities \( g(x), H(x) \) and \( \mu \) all depend on \( X(x) \).

**Theorem 3.29 (Mather)**

- The deformation \( v(x; \mu) \) of \( u(x) \) is versal if and only if the matrix

\[
N(x) = \left[ \frac{\partial v_i}{\partial \mu_j}(x; 0) \right]
\]

satisfies the transversality condition with respect to \( u(x) \). ▽

For a proof, see Mather [Mat68].

Let \( v^{[1]}(x; \mu) \) be the first degree Taylor expansion of \( v(x; \mu) \) with respect to \( \mu \),

\[
v^{[1]}(x; \mu) \equiv v(x; 0) + N(x) \cdot \mu.
\]

As a direct result of Theorem 3.29, \( v(x; \mu) \) is versal if and only if \( v^{[1]}(x; \mu) \) is versal.

Also, if a matrix \( Q(x) \) satisfies the transversality condition with respect to \( u(x) \), then

\[
v(x; \mu) = u(x) + Q(x) \cdot \mu
\]

is a versal deformation of \( u(x) \). \( Q(x) \) is called the *unfolding matrix* of \( u(x) \).

Let \( \mathcal{X}^n = \{ f(x) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \mid f(0) = 0 \} \). Let \( \mathcal{P}^n(k) \equiv \{ H^n \cup \cdots \cup H^0_k \} \), i.e., the subset of \( \mathcal{X}^n \) whose components are polynomials of degree smaller or equal to \( k \). Let \( \mathcal{I}^k(f) \subset \mathcal{P}^n_k \) consist of the \( k \)-jet extensions of those vectors \( p(x) \) which may be written as

\[
p(x) = Df(x)g(x) + H(x)f(x)
\]

near the origin, for some \( g(x) \in \mathcal{P}^n(k) \) and some smooth \( n \times n \) matrix \( H(x) \) whose entries are all polynomials of degree less or equal to \( k - 1 \). Note that \( \mathcal{I}^k(f) \) is a finite dimensional space.

**Theorem 3.30 (Mather)**

- Suppose that for some \( k \) we have

\[
H^n_k \subset \mathcal{I}^k(f),
\]

(3.147)
then $f(x)$ has a versal deformation. If (3.147) holds, then let $N_1(x), \ldots, N_q(x)$ be members of $P^n(k)$ which together with $X^k(f)$ span all of $P^n(k)$. Then

$$u(x; \mu) = f(x) + \sum_{i=1}^{q} N_i(x)\mu_i$$

(3.148)

is a versal deformation of $f(x)$. Furthermore, if $f(x)$ has a versal deformation, then (3.147) holds for some $k$. >

For a proof, see Mather [Mat68].

It is not always possible to use Mather’s theorem (3.30). Poston and Stewart [PS76] present examples (from catastrophe theory) for which $k$-determinacy holds, say for $k = 4$, while Mather’s condition (3.147) can only be proven for $k \geq 5$.

### 3.3.5 2-D example. 2-determinacy

Bogdanov [Bog75] [Bog81] was the first to show that the vector field (3.143), i.e.,

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \mu_1 + \mu_2 x_2 + x_1^2 + bx_1 x_2,
\end{align*}$$

(3.149)

(in a somewhat different form) is 2-determinate. Consequently, third and higher order terms do not alter the dynamical behavior. Here, we give a proof using Mather’s Theorem 3.30.

The first thing that needs to be checked is that the unperturbed right-hand side of the vector field (3.149),

$$u(x) = \begin{pmatrix} x_2 \\
x_1^2 + bx_1 x_2 \end{pmatrix},$$

(3.150)

has a versal deformation. Therefore, we compute the Jacobian,

$$Du(x) = \begin{pmatrix} 0 & 1 \\
2x_1 + bx_2 & bx_1 \end{pmatrix},$$

(3.151)
and see if \( H_2^2 \subset \mathcal{I}^2(u) \). The following list shows that this is indeed the case.

\[
\begin{pmatrix}
  x_1^2 \\
  0
\end{pmatrix}
= J_2 \{ D_u(x) \begin{pmatrix}
  0 \\
  x_1^2
\end{pmatrix} \},
\]

\[
\begin{pmatrix}
  x_1 x_2 \\
  0
\end{pmatrix}
= \begin{pmatrix}
  x_1 & 0 \\
  0 & 0
\end{pmatrix} u(x),
\]

\[
\begin{pmatrix}
  x_2 \\
  0
\end{pmatrix}
= \begin{pmatrix}
  x_2 & 0 \\
  0 & 0
\end{pmatrix} u(x),
\]

\[
\begin{pmatrix}
  0 \\
  x_1^2
\end{pmatrix}
= D_u(x) \begin{pmatrix}
  x_1 & 0 \\
  0 & 0
\end{pmatrix} + \begin{pmatrix}
  0 & 0 \\
  0 & -1
\end{pmatrix} u(x),
\]

\[
\begin{pmatrix}
  0 \\
  x_1 x_2
\end{pmatrix}
= D_u(x) \begin{pmatrix}
  -\frac{b}{2} x_1 & 0 \\
  0 & 0
\end{pmatrix} + \begin{pmatrix}
  0 & 0 \\
  0 & \frac{2}{b}
\end{pmatrix} u(x),
\]

\[
\begin{pmatrix}
  0 \\
  x_2
\end{pmatrix}
= \begin{pmatrix}
  0 & 0 \\
  x_2 & 0
\end{pmatrix} u(x).
\]

Thus, the vector field (3.149) with \( \mu_1 = \mu_2 \) has a versal deformation.

Second, to see if the vector field (3.149) is a versal deformation, we should check whether the vectors

\[
N_1(x) = \begin{pmatrix}
  0 \\
  1
\end{pmatrix}, \quad N_2(x) = \begin{pmatrix}
  0 \\
  x_2
\end{pmatrix}
\]

together with \( \mathcal{I}^2(u) \) spans \( P^2(2) \). Having shown that \( H_2^2 \subset \mathcal{I}^2(u) \), all that needs verification is whether

\[
P_2^2 = \text{span} \{ N_1(x), N_2(x), \mathcal{I}^2(u) \}. \tag{3.152}
\]

The following list shows that remaining vectors indeed lie in \( \mathcal{I}^2(u) \).

\[
\begin{pmatrix}
  1 \\
  0
\end{pmatrix}
= D_u(x) \begin{pmatrix}
  -\frac{b}{2} x_1 \\
  1
\end{pmatrix} + \begin{pmatrix}
  0 & 0 \\
  \frac{b}{2} & 0
\end{pmatrix} u(x),
\]

\[
\begin{pmatrix}
  x_1 \\
  0
\end{pmatrix}
= D_u(x) \begin{pmatrix}
  -b x_1 \\
  x_1
\end{pmatrix} + \begin{pmatrix}
  0 & 0 \\
  0 & b
\end{pmatrix} u(x),
\]

\[
\begin{pmatrix}
  x_2 \\
  0
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 \\
  0 & 0
\end{pmatrix} u(x),
\]

\[
\begin{pmatrix}
  0 \\
  x_1
\end{pmatrix}
= D_u(x) \begin{pmatrix}
  \frac{b}{2} \\
  0
\end{pmatrix} + \begin{pmatrix}
  0 & 0 \\
  -\frac{b}{2} & 0
\end{pmatrix} u(x).
\]

Thus, the vector field (3.149) is a versal deformation.

Finally, we note that a quick survey shows that no combination of \( g(x) \) and \( H(x) \) in (3.146) is capable of producing the vectors \( N_1(x) \) and \( N_2(x) \).
3.4 Analysis of the dynamics

With the proof of 2-determinacy of the topological normal form for our running two-dimensional example we have established our objective set in Section 3.1.5. The next chapter applies the above techniques to vector fields describing fluid flow. We cannot, however, leave this chapter without presenting one of the bifurcation diagrams of the topological normal form.

To study the local dynamics of the parameterized family of vector fields (3.149) we need to

1. find the critical points and study the nature of their stability;
2. study the bifurcations associated with the critical points;
3. based on a consideration of the local dynamics, infer if global bifurcations must be present.

We will not go into all the details involved with each individual step of the analysis. Instead, we present only the bifurcation diagram of the vector field (3.149) for the case $b = +1$ depicted in Fig. 3.14. The labels Poincaré-Andronov-Hopf and Saddle-node refer to the type of elementary bifurcations that take place as the location $(\mu_1, \mu_2)$ crosses the bifurcation lines $\mu_1 = -\mu_2^2$ and $\mu_1 = 0$, respectively. For a thorough discussion of the bifurcation diagram of the vector field (3.149), see Guckenheimer and Holmes [GH83] or Wiggins [Wig90]. The case $b = -1$ is discussed by Kuznetsov [Kuz95].

3.5 Discussion

This chapter reviewed several techniques to reduce the complexity of the representation of a vector field. The objective was to find a so-called topological normal form: a simple polynomial vector field of degree $k$ with a certain degeneracy that is topological equivalent to all vector field with the same degeneracy in their $k$th degree series expansion. The local dynamics generated by a topological normal form can be analyzed because it has a finite number of coefficients and a finite number of parameters.

To find the topological normal form we had to: transform the linear part to a Jordan normal form, transform the nonlinear part to normal form, truncate at $k$th degree, find a miniversal deformation, and check whether the resulting vector field is $k$-determinate. We introduced a procedure to find all alternative complementary subspaces which helped to explain the existence of a wide variety of normal forms. With that procedure we found explicit expressions for the
relations between the coefficients of the expansions of a vector field and the coefficients of another $C^k$, $k \geq 2$, equivalent vector field. These coefficient relations will be of importance in the next chapter where we compute the topological normal form of vector fields that describe the flow of fluid near a boundary surface.
Chapter 4

Bifurcation in Fluid Flow

Honni soit qui mal y pense.

Introduction

The previous chapter showed that a Taylor-polynomial approximation of a vector field and bifurcation of a nonhyperbolic critical point enables us to study flow patterns with possibly more than one critical point. To analyze the corresponding local flow pattern it is necessary to seek a so-called topological normal form; given a set of vector fields which have a certain degeneracy in common, we should construct a simple polynomial vector field that is topological equivalent to all of the vector fields in that set. To this end, we need to transform the linear part to Jordan form, transform the nonlinear part to a normal form, truncate, rescale, and find a miniversal deformation.

This chapter seeks to construct topological normal forms for vector fields describing fluid flow in a small neighborhood of a critical point on a boundary surface in three-dimensional space. These vector fields were introduced in Chapter 2. However, application of bifurcation theory, such as explained in the previous chapter, will lead to topological normal forms which do no longer apply to fluid flow. Moreover, they will have an unexpected high co-dimension. To overcome these difficulties, a new approach to bifurcation analyses will be set up in this chapter taking into account the special character of fluid-flow vector fields. To demonstrate our approach, we will once again use two-dimensional vector fields as examples.
4.1 2D local flow patterns

Using the techniques discussed in the previous chapter, this section computes a topological normal form for vector fields describing two-dimensional fluid flow near a boundary critical point. We will consider nonhyperbolic critical points, specifically, those in which the Jacobian matrix of the vector field has two repeated zero eigenvalues. First, however, we discuss the possible hyperbolic critical points.

Chapter 2 showed that the set $\mathcal{V}^3_\nu$ contains the vector fields describing viscous, incompressible fluid flow about a point on the boundary surface in three-dimensional space. The two-dimensional equivalent of this set is

$$\mathcal{V}^2_\nu = \{ \mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2 \mid \nabla \cdot (x_2 \mathbf{u}(x)) = 0 \}.$$

Here $x_2 = 0$ represents the boundary ‘surface’.

Let the vector field $\mathbf{u}(x) \in \mathcal{V}^2_\nu$ have the following Taylor series expansion

$$\dot{x} = \mathbf{u}(x) = A \mathbf{x} + \sum_{m \geq 2} \mathbf{u}_m(x), \quad (4.1)$$

with each $\mathbf{u}_m(x) \in \mathbb{R}^2_m$ denoted as

$$\mathbf{u}_m(x) = \sum_{i+j=m} \begin{pmatrix} u_{i,j}^{(1)} \\ u_{i,j}^{(2)} \end{pmatrix} x_1^i x_2^j. \quad (4.2)$$

The following relations hold between the coefficients in the Taylor series expansion

$$(i+1)u_{i+1,j}^{(1)} + (j+2)u_{i,j+1}^{(2)} = 0, \quad i + j = m - 1 \geq 0$$

$$u_{m,0}^{(2)} = 0, \quad m \geq 0 \quad (4.3)$$

where $i$ and $j$ are nonnegative integers.

The general form of the linear part of (4.1) is given by

$$A = \begin{pmatrix} u_{10}^{(1)} & u_{01}^{(1)} \\ 0 & -\frac{1}{2} u_{20}^{(1)} \end{pmatrix}. \quad (4.4)$$

For nonzero $u_{10}^{(1)}$, the origin is a hyperbolic critical point. Being hyperbolic, the local flow pattern near the origin is structurally stable and the topological properties only depend on the sign of $u_{10}^{(1)}$, see Fig. 2.3.
4.1.1 Jordan normal form

Assume that \( u_{10}^{(1)} = 0 \). In that case, the origin is a nonhyperbolic critical point. A linear coordinate transformation, \( \tilde{x} = M y \), in the vector field (4.1) results in

\[
\dot{y} = \psi(y) \equiv M^{-1} u(M y).
\]

(4.5)

The linear part of this vector field is brought into Jordan canonical form by the transformation matrix

\[
M = \begin{pmatrix}
u_{10}^{(1)} & 0 \\ 0 & 1
\end{pmatrix},
\]

in which case the Taylor series expansion reads

\[
\psi(y) = J y + \sum_{m \geq 2} \psi_m(y),
\]

where

\[
J \equiv M^{-1} A M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
\]

(4.6)

and

\[
\psi_m(y) \equiv M^{-1} u_m(M y),
\]

Since the matrix \( M \) only scales the \( x_1 \) coordinate, it is easy to see that again \( \psi(y) \in \mathbb{V}_u^2 \) holds. Thus, if we write the homogeneous polynomials in the series expansion of \( \psi(y) \) as

\[
\psi_m(y) = \sum_{i+j-m} \begin{pmatrix} \psi_{i,j}^{(1)} \\ \psi_{i,j}^{(2)} \end{pmatrix} y_i^1 y_j^2,
\]

(4.7)

the following coefficient relations hold

\[
(i + 1)\psi_{i+1,j}^{(1)} + (j + 2)\psi_{i,j+1}^{(2)} = 0, \quad i + j = m - 1 \geq 0
\]

\[
\psi_{m0}^{(2)} = 0 \quad m \geq 0,
\]

(4.8)

where \( i \) and \( j \) are nonnegative integers.

4.1.2 Normal form and co-dimension

The Jordan form \( J \) in (4.6) equals the one in the running example in the previous chapter. In that chapter, we found that the polynomial vector field (3.143) is a
topological normal form for general vector fields with a Jacobian matrix similar to $J$ under certain nondegeneracy conditions. Using transformation theory we found direct expressions of the normal-form coefficients in terms of the coefficients from the original vector field. To see if the vector field (3.143) is a useful representation of the topological features of vector fields in $\mathbb{V}_\nu^2$, we substitute the relations between the original coefficients in the expressions of the normal-form coefficients.

After substitution of the near-identity transformation

$$y = z + \sum_{m \geq 2} t_m(z),$$

with each $t_m(z) \in \mathcal{H}^n_m$, into the general vector field

$$\dot{y} = f(y) = Jy + \sum_{m \geq 2} f_m(y),$$

with each $f_m(y) \in \mathcal{H}^n_m$, we found that the second-order complementary subspace of $J$ is given by (3.78), i.e.

$$\mathcal{G}_2 = [g^{(2)}_{z_1} z_1^2 + g^{(2)}_{z_1 z_2} z_1 z_2] \mathcal{G}_2.$$  \hspace{1cm} (4.9)

The normal-form coefficients are related to the coefficients from the original vector field as

$$g^{(2)}_{z_1 z_1} = f^{(2)}_{z_1}, \quad g^{(2)}_{z_1 z_2} = f^{(2)}_{z_1 z_2} + 2 f^{(1)}_{z_2}.$$  

Let the vector field $f(y)$ equal the vector field $v(y)$ in (4.5). Using the coefficient relations (4.8), $f^{(2)}_{z_1} = v^{(2)}_{z_1} = 0$, $f^{(1)}_{z_1} = v^{(1)}_{z_1}$, and $f^{(2)}_{z_2} = v^{(2)}_{z_2}$, and $f^{(1)}_{z_1} = v^{(1)}_{z_1}$, and $f^{(1)}_{z_2} = v^{(1)}_{z_2}$. We find that $g^{(2)}_{z_1 z_1} = 0$ and $g^{(2)}_{z_1 z_2} = v^{(1)}_{z_1 z_2} \neq 0$ (in general), and that the second-order complementary subspace is given by

$$\mathcal{G}_2^a = v^{(1)}_{z_1 z_2} z_1 z_2 \mathcal{G}_2.$$  \hspace{1cm} (4.10)

The index $a$ is short for application.

We find that the nondegeneracy condition $g^{(2)}_{z_1 z_1} \neq 0$ does not hold for vector fields in $\mathbb{V}_\nu^2$. Hence, the polynomial vector field (3.143) cannot be used as a topological normal form.

We have found that a representation of the 2-jet of the vector fields in $\mathbb{V}_\nu^2$—with a Jacobian matrix in the critical point in the origin that has a double-zero eigenvalue—is given by

$$\dot{z}_1 = z_2,$$

$$\dot{z}_2 = v^{(1)}_{z_1 z_2} z_1 z_2.$$  \hspace{1cm} (4.11)
This polynomial vector vector cannot represent the topological features of the critical point because the line \( z_2 = 0 \) is a line of critical points. Our assumption was that the origin is the only critical point in the vector field (4.5). The abnormality is caused by premature truncation of our normal form. To see if a third-order polynomial vector field can act as a topological normal form, we need to compute \( \mathcal{G}_3 \).

In Chapter 3, based on Table 3.4b, we found the following representation of the third-order complementary subspace;

\[
\mathcal{G}_3 = [g_{30}^{(2)} z_1^3 + g_{21}^{(2)} z_1^2 z_2] \partial_2,
\]

(4.12)

where \( g_{30}^{(2)} = \bar{f}_{30}^{(2)} \) and \( g_{21}^{(2)} = \bar{f}_{21}^{(2)} + 3 \bar{f}_{30}^{(1)} \), see (3.116). It was said that \( g_{21}^{(2)} \) could be removed using the free parameters in the kernel \( \mathcal{K}_2 \). However, this result does not apply here because \( g_{30}^{(2)} = 0 \).

Note the tilde above the coefficients. It indicates that the coefficients also contain Lie-products of the lower-order coefficients \( t_{ij}^{(s)} \) of the (generator of the) transformation and lower-order coefficients \( \bar{f}_{ij}^{(s)} \) of the vector field. Using (3.115) (found with transformation theory as explained in Section 3.2.7)

\[
\bar{f}_3(y) = f_3(y) + \frac{1}{2} [g_2(y), t_2(y)] + \frac{1}{2} [f_2(y), t_2(y)].
\]

(4.13)

we find

\[
\begin{align*}
\bar{f}_{30}^{(1)} &= f_{30}^{(1)} + \frac{1}{2} f_{11}^{(1)} t_{20}, \\
\bar{f}_{30}^{(2)} &= f_{30}^{(2)} + \frac{1}{2} g_{11}^{(1)} t_{20} - t_{20} f_{20}, \\
\bar{f}_{21}^{(2)} &= f_{21}^{(2)} + \frac{1}{2} g_{11}^{(1)} t_{20} - \frac{1}{2} f_{20} t_{11}^{(1)} - \frac{1}{2} f_{20} t_{11}^{(2)} - \frac{3}{2} f_{11}^{(1)} t_{20}
\end{align*}
\]

(only the relevant coefficients are given here). The coefficients \( t_{20}^{(1)}, t_{20}^{(2)} \) and \( t_{11}^{(2)} \) are found using Table 3.3,

\[
t_{20}^{(1)} = \frac{1}{2} f_{11}^{(1)} + \frac{1}{2} f_{02}^{(2)}, \quad t_{20}^{(2)} = -f_{20}^{(1)}, \quad t_{11}^{(2)} = f_{02}.
\]

We can now substitute the coefficients \( v_{ij}^{(s)} \) into the coefficients \( \bar{f}_{ij}^{(s)} \) to find

\[
\begin{align*}
\bar{f}_{30}^{(1)} &= v_{30}^{(1)} - \frac{1}{2} v_{11}^{(1)} v_{20}^{(1)}, \\
\bar{f}_{30}^{(2)} &= -\frac{1}{2} (v_{20}^{(1)})^2, \\
\bar{f}_{21}^{(2)} &= -v_{11}^{(1)} + \frac{3}{2} v_{20}^{(1)} v_{11}^{(1)}
\end{align*}
\]

and thus that \( g_{30}^{(2)} = -\frac{1}{2} (v_{20}^{(1)})^2, \ g_{21}^{(2)} = 2v_{30}^{(1)} \). We find that the third-order complementary subspace is given by

\[
\mathcal{G}_3 = [-\frac{1}{2} (v_{20}^{(1)})^2 z_1^3 + 2v_{30}^{(1)} z_1^2 z_2] \partial_2.
\]

(4.14)
Note that the first normal-form coefficient is negative whereas the second is nonzero (in general).

We discussed transformation theory in Section 3.2.7 with the intent to simplify the higher-order complementary subspaces using kernel terms. In (4.13), for example, kernel terms are part of the two Lie products. However, we find that the normal-form coefficient $g_{20}^{(2)}$ in $G_3^2$ is completely governed by these two Lie products. As a result, there exists an unexpected complicated relation between third and second order normal-form coefficients.

Combining the above results, we find that a representation of the 3-jet is given by

\[ \begin{align*}
\dot{z}_1 &= z_2, \\
\dot{z}_2 &= v_{20}^{(1)} z_1 z_2 - \frac{1}{2} (v_{20}^{(1)})^2 z_1^2 + 2v_{30}^{(1)} z_1^2 z_2. 
\end{align*} \]  

(4.15)

It is easy to see that the origin is the only critical point of the vector field (4.15).

Prior to transformation to normal form, the line $y_2 = 0$ is an invariant manifold in the vector field (4.5). The vector field (4.15) contains (an approximation of) this invariant manifold. However, an expression for the invariant manifold is hard to find because it no longer coincides with a coordinate line. As a result, it is difficult to determine the –for us important– stability properties of the skin-friction lines.

Another difficulty arises when we want to add bifurcation parameters to the vector field (4.15). Because the normal-form coefficient $g_{20}^{(2)}$ from $G_2$ is zero in $G_3^2$, the boundary critical point has co-dimension 3 (remember that the co-dimension of the matrix $J$ already equals 2). The analysis by Bakker [Bak88], however, used two perturbation parameters. Even then, the resulting bifurcation diagram suggested that the vector field needed only one bifurcation parameter, viz. $\mu_c$ in Fig. 2.6.

We conclude that we need to find more practical simplified representations for vector fields in $V_\nu$. In the case of the above example our task is the following. The normal form coefficients in (4.15) are formed by two coefficients from the original vector field, viz. $v_{20}^{(1)}$ and $v_{30}^{(1)}$. We need to develop a method that produces a simplified vector field that contains these coefficients, but has a form that allows for efficient determination of, for example, the dynamical behavior on the boundary surface.

The next section discusses versal deformations of the linear vector field containing the matrix $A$ in (4.4). The discussion will give an idea of the conditions needed to obtain ‘practical’ simplified representations for vector fields in $V_\nu$. 
4.1.3 Hyperbolicity and versal deformations

In Chapter 2, it was argued that for nonzero $u_{10}^{(1)}$ the local flow pattern of the vector field (4.1) can be represented by its linearized vector field,

$$\dot{\xi} = A\xi,$$

where $A$ is given by (4.4). In that case, $A$ has two distinct, nonzero eigenvalues:

$$\lambda_1 = u_{10}^{(1)}, \quad \text{and} \quad \lambda_2 = -\frac{1}{2}u_{10}^{(1)}.$$  \hspace{1cm} (4.17)

Because the eigenvalues both have a nonzero real part, the origin is, by definition, a hyperbolic critical point of the vector field (4.1).

From Hartman-Grobman’s theorem discussed in Chapter 3, we know that the local flow patterns of (4.1) and (4.16) are $C^0$ conjugate. We also found that hyperbolic critical points are structurally stable in the set of planar vector fields (at least differentiable once); if we subject the vector field (4.1) to small perturbations we will find that the local flow patterns near the origin of the resulting vector field are all topological equivalent.

We want to know the practical implications of this topological equivalence with respect to the classification of local flow patterns. To make the discussion specific, consider the following example of a miniversal deformation.

Let the Jacobian matrix $A$ of the linearized vector field (4.16) equal the Jacobian matrix $A$ in (4.4) corresponding to hyperbolic critical points in vector fields from $\mathcal{V}_p^2$. The transformation matrix

$$M = \begin{pmatrix} 1 & -u_{01}^{(1)} \\ 0 & u_{10}^{(1)} \end{pmatrix}$$

brings $A$ in the following Jordan normal form;

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$  \hspace{1cm} (4.18)

To find a miniversal deformation, we need to the centralizer of this matrix. Computation of the centralizer runs as follows. Find all matrices which commute with $J$,

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = (\lambda_1 - \lambda_2) \begin{pmatrix} 0 & c \\ -b & 0 \end{pmatrix},$$
i.e., \( b = c = 0, a \) and \( d \) arbitrary. Relabelling by letting \( a = \mu_1 \) and \( d = \mu_2 \), we obtain the following miniversal deformation of \( J \),

\[
\begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix} + \begin{pmatrix}
\mu_1 & 0 \\
0 & \mu_2
\end{pmatrix}.
\tag{4.19}
\]

We should use the transpose and complex conjugate of the second matrix according to Lemma 3.18 but since this matrix is diagonal and part of \( \mathbb{R}^{2\times 2} \) the same matrix results. Because the deformation has two parameters, \( \mu_1 \) and \( \mu_2 \), the co-dimension\(^1 \) \( J \) is 2.

In general, the above computation would be nothing more than a formality. Because, when we relabel by letting \( \tilde{\lambda}_1 = \lambda_1 + \mu_1 \) and \( \tilde{\lambda}_2 = \lambda_2 + \mu_2 \), we find the matrix

\[
\tilde{J} = \begin{pmatrix}
\tilde{\lambda}_1 & 0 \\
0 & \tilde{\lambda}_2
\end{pmatrix}.
\]

This matrix also has two distinct eigenvalues (if \( \mu_1 \) and \( \mu_2 \) are both small enough). The sign of \( \tilde{\lambda}_1 \) and \( \tilde{\lambda}_2 \) equals the sign of \( \lambda_1 \) and \( \lambda_2 \), respectively, and therefore, there exists a homeomorphism which maps the local flow pattern induced by (4.18) onto the local flow pattern induced by (4.19). Hence, \( J \) and \( \tilde{J} \) generate linear vector fields which are topological equivalent.

However, this topological equivalence has no meaning with respect to the classification of local flow patterns. The matrix \( J \) cannot be the Jacobian matrix of any vector field in \( \mathcal{V}^2 \) since the coefficient relation

\[
\tilde{\lambda}_1 + 2\tilde{\lambda}_2 = \lambda_1 + \mu_1 + 2(\lambda_2 + \mu_2) = \mu_1 + 2\mu_2 \neq 0,
\tag{4.20}
\]

for general \( \mu_1 \) and \( \mu_2 \). This result means that conservation of mass no longer occurs and that is not acceptable.

There is, of course, a way to make sure that the miniversal deformation remains within \( \mathcal{V}^2 \): simply require that coefficient relation

\[
\mu_1 + 2\mu_2 = 0,
\tag{4.21}
\]

holds. Any versal deformation of the vector field (4.1) has perturbation parameters satisfying precisely this coefficient relation! But this situation holds prior to the computation of the normal form through a change of coordinates. In the new coordinate system we cannot assume that (4.21) holds.

\(^1\)The Jacobian matrix in a hyperbolic critical point can have a nonzero co-dimension.
For the sake of argument, assume that the relation (4.21) does hold. Since \( \mu_2 \) is no longer independent of \( \mu_1 \), the co-dimension drops to 1. This time, the result makes more sense: Relabel by letting \( \lambda_1 = \lambda \) and \( \mu_1 = \mu \), and we obtain
\[
\begin{pmatrix}
\lambda & 0 \\
0 & -\frac{1}{2}\lambda
\end{pmatrix} + \begin{pmatrix}
\mu & 0 \\
0 & -\frac{1}{2}\mu
\end{pmatrix}.
\]
For small enough \( \mu \), this deformation cannot lead to a change in the topological classification of the local flow pattern, as expected.

Discussion

Consider the way in which bifurcation parameters were introduced in Chapter 3. It was only after transformation of the linear part to Jordan normal form, transformation of the nonlinear part to a normal form, and truncation and rescaling of that normal form, that we computed the co-dimension, and plugged in precisely that number of parameters into the vector field. In Section 3.3.3, we used a miniversal deformation of the Jacobian matrix in Jordan normal form, only to find that the origin in the topological normal form remained a critical point regardless of the values of the parameters. So we fiddled about and found another topological normal form that did truly bifurcate the origin; once perturbed, the critical point in the origin splits into two critical points which branched off from the origin. That situation still did not satisfy us and we extended the domain of one of the two perturbation parameters such that the two critical points can annihilate one another.

We want to study the bifurcation behavior in local flow patterns of vector fields in the sets \( \mathcal{V}_\nu^2 \) and \( \mathcal{V}_\nu^3 \). We also want to see a bifurcation in the origin. But consider how difficult to control the analysis becomes if we were to ‘fiddle about’ as just described: We would never know if we used a vector field from \( \mathcal{V}_\nu^2 \) or some vector field outside \( \mathcal{V}_\nu^2 \).

The computations above showed us that a vector field in \( \mathcal{V}_\nu^2 \) can be topological equivalent to a planar vector field which does not belong to \( \mathcal{V}_\nu^2 \). We also met a hyperbolic critical point of a vector field in \( \mathcal{V}_\nu^2 \) of co-dimension 2 in the set of all planar vector fields. The same hyperbolic critical point had co-dimension 1 within the more restricted set of vector fields \( \mathcal{V}_\nu^2 \).

Something similar happened when we computed a normal form for a non-hyperbolic critical point with the double-zero eigenvalue matrix (4.6). Without realizing it, we were actually looking for a topological-equivalent vector field in the set of all planar vector fields. And when we did, the co-dimension increased by 2 because two crucial coefficients in the normal form vanished.
We are, however, not at all interested in the embedding or stability of vector fields in $V^2_v$ with respect to the set of all planar vector fields. We *a priori* know that they form a lower dimensional subset (manifold?) in the set of all planar vector fields: The coefficients in the expansion of a vector field in $V^2_v$ satisfy certain relations not satisfied by the coefficients in the expansion of a general planar vector field.

The double-zero eigenvalue matrix (4.6) has co-dimension 2 in the set of all planar vector fields. We suspect that it has co-dimension 1 based the figures presented by Bakker [Bak88], discussed in Chapter 2. If we look for a normal form or miniversal deformation, we should only take topological equivalent vector fields from $V^2_v$ into account. That way, we make sure that the critical point remains to be co-dimension 1.

Fig. 4.1 presents an outline of the general approach towards the computation of a topological normal form and our approach in the case of fluid flow which will be developed in the next sections. In step 4 of the general approach, we need to superposition a miniversal deformation. In the previous steps, the computations are performed as if there are no perturbation parameters present. In practice, the vector fields have a general versal deformation in each step; each coefficient in the Taylor expansion of the vector field has an accompanying perturbation parameter. The computations of the Jordan normal form for the Jacobian matrix, for example, will change the arrangement of the perturbation parameters somewhat. However, after reparameterization, once again perturbation parameters appear at the same positions as before. For that reason, taking the perturbation parameters into account in the computations on general vector fields needlessly complicates things.

In the case of vector fields describing fluid flow, a general versal deformation has perturbation parameters with specific physical properties and relations exist between them. This time, reparameterization after the computation of the Jordan normal form, for example, is no longer straightforward as it was in the general approach.

In the remainder of this chapter, we discuss the steps B, D, E, F and G.

### 4.2 Parameter dependent vector field expansions

In this section, we discuss a general versal deformation of vector fields that relates to changes in global physical parameters. In Fig. 4.1, it is step B.

Let $(x_1 \ldots x_n)$ be some local, $n$-dimensional coordinate system. Suppose that we have a vector field which depends on parameters

$$ \dot{x} = u(x; \mu), \quad x \in U \subset \mathbb{R}^n, \quad \mu \in V \subset \mathbb{R}^p, $$

(4.22)
Figure 4.1: Computation of a topological normal form. The general approach (left) and our approach for fluid flow (right). The steps in the two dashed boxes apply to boundary critical points.
Here, the phase-space $U$ is a region in $\mathbb{R}^n$ surrounding the origin $x = 0$, and the parameter-space $V$ is a region in $\mathbb{R}^p$ surrounding the origin $\mu = 0$. The perturbation-parameter vector is defined as

$$\mu \equiv (\mu_1, \mu_2, \ldots, \mu_p)^t.$$

The dimension $p$ of the parameter space is arbitrary (for now).

The Taylor polynomial of index $k$ about the origin $x = 0$ of the vector field (4.22) reads

$$u^{[k]}(x; \mu) = \sum_{m=0}^{k} u_m(x; \mu),$$

(4.23)

where each $u_m(x; \mu) \in H^n$. Denote each of the $n$ component functions of $u_m(x; \mu)$ as follows

$$u^{(\ell)}_m(x; \mu) = \sum_{i_1 + \ldots + i_n = m} u^{(\ell)}_{i_1 \ldots i_n}(\mu) x_1^{i_1} \ldots x_n^{i_n},$$

(4.24)

where $i_1, \ldots, i_n$ are $n$ nonnegative integers, and $\ell \in \{1, \ldots, n\}$. Taylor expand each of the coefficients in the (4.24), this time about the origin $\mu = 0$ in the $p$-dimensional parameter space $V$,

$$u^{(\ell)}_{i_1 \ldots i_n}(\mu) = u^{(\ell)}_{i_1 \ldots i_n}(0) + \sum_{m=1}^{p} \frac{\partial u^{(\ell)}_{i_1 \ldots i_n}(\mu)}{\partial \mu_m} \bigg|_{\mu=0} \mu_m + O(\|\mu\|^2).$$

(4.25)

We reparameterize the variables in (4.25) as follows

$$u^{(\ell)}_{i_1 \ldots i_n} \equiv u^{(\ell)}_{i_1 \ldots i_n}(0),$$

$$\eta^{(\ell)}_{i_1 \ldots i_n} \equiv \sum_{m=1}^{p} \frac{\partial u^{(\ell)}_{i_1 \ldots i_n}(\mu)}{\partial \mu_m} \bigg|_{\mu=0} \mu_m + O(\|\mu\|^2),$$

(4.26)

We define a new perturbation-parameter vector

$$\eta \equiv (\eta^{(1)}_0, \ldots, \eta^{(n)}_0, \ldots, \eta^{(1)}_1, \ldots, \eta^{(n)}_1, \ldots, \eta^{(1)}_{n-1}, \ldots, \eta^{(n)}_{n-1}, \ldots, \eta^{(1)}_k, \ldots, \eta^{(n)}_k)^t.$$
and its parameter space

$$\mathcal{W} \equiv \{ \eta \in \mathbb{R}^m \mid \| \eta \| \ll 1 \}. \quad (4.27)$$

The finite dimension of $\mathcal{W}$, $m$, does not depend on the dimension of the parameter space $p$, but on the dimension of the phase space $n$ and the index of the Taylor polynomial $k$.

The result is that all coefficients in the Taylor polynomial get their own perturbation parameter:

$$u^{(\ell)}_m(x; \eta) = \sum_{i_1 + \ldots + i_n = m} (u^{(\ell)}_{i_1 \ldots i_n} + \eta^{(\ell)}_{i_1 \ldots i_n}) x_1^{i_1} \ldots x_n^{i_n}. \quad (4.28)$$

Notice that this time $\eta \in \mathcal{W}$.

If the parameters $\eta^{(\ell)}_{i_1 \ldots i_n}$ are to act truly as a perturbation on the coefficients $u^{(\ell)}_{i_1 \ldots i_n}$, their order has to be much smaller. Set all $\eta^{(\ell)}_{i_1 \ldots i_n} = \mathcal{O}(\delta)$, with $\delta \ll 1$. We rewrite (4.28) as

$$u^{(\ell)}_m(x; \mu) = \sum_{i_1 + \ldots + i_n = m} (u^{(\ell)}_{i_1 \ldots i_n} + \delta \eta^{(\ell)}_{i_1 \ldots i_n}) x_1^{i_1} \ldots x_n^{i_n}. \quad (4.29)$$

This way, when we compute, for example, the coefficient relations, we can distinguish coefficients from perturbation parameters.

**Vector fields describing fluid flow near a boundary surface**

We suppose that the vector field (4.22) is part of the following set

$$\mathcal{V}_\nu^n = \{ v : U \times \mathcal{W} \to \mathbb{R}^n \mid \nabla \cdot (x_n v(x; \mu)) = 0 \}, \quad (4.30)$$

where $U$ is a region surrounding the origin in $\mathbb{R}^n$, and $\mathcal{W}$ is as defined in (4.27). This set is a generalization of (2.87). The dimensions of interest are $n = 2$ and $n = 3$.

We denote the Taylor polynomial of index $k$ as

$$\dot{x} = u^{[k]}(x; \mu) = \sum_{m=0}^{k} u_m(x) + \delta \sum_{m=0}^{k} \mu_m(x), \quad (4.31)$$
where \( u_m(x) \) and \( \mu_m(x) \) are part of \( H^m_r \),

\[
\begin{align*}
u_m(x) &= \sum_{i_1+\ldots+i_n=m} \begin{pmatrix} u_{i_1\ldots i_n}^{(1)} \\ \vdots \\ u_{i_1\ldots i_n}^{(n)} \end{pmatrix} x_1^{i_1} \ldots x_n^{i_n}, \\
\mu_m(x) &= \sum_{i_1+\ldots+i_n=m} \begin{pmatrix} \mu_{i_1\ldots i_n}^{(1)} \\ \vdots \\ \mu_{i_1\ldots i_n}^{(n)} \end{pmatrix} x_1^{i_1} \ldots x_n^{i_n}.
\end{align*}
\] (4.32)

Coefficient relations

The same analysis which for \( u_m(x) \in \mathbb{V}_0^3 \) lead to (2.62), for the functions \( u_m(x) \in \mathbb{V}_0^n \) in 4.31 leads to

\[
(i_1 + 1)u_{i_1+1\ldots i_n}^{(1)} + \ldots + (i_{n-1} + 1)u_{i_1\ldots i_n-2\ i_{n-1}+1\ i_n}^{(n-1)} + (i_n + 2)u_{i_1\ldots i_{n-1}\ i_{n}+1}^{(n)} = 0,
\] (4.33)

with \( i_1 + \ldots + i_n = m - 1 \geq 0 \), and

\[
u_{i_1\ldots i_{n-1}\ 0}^{(n)} = 0,
\] (4.34)

with \( i_1 + \ldots + i_{n-1} = m \geq 0 \). In both (4.33) and (4.34), \( i_1, \ldots, i_n \) are nonnegative integers.

Similar coefficient relations hold for the coefficients of the vectors \( \mu_m(x) \).

### 4.3 Jordan normal form

In this section, we will formulate the conditions on conjugating matrices such that after transformation to Jordan normal form, the resulting vector field is still an element of \( \mathbb{V}_0^n \) (step D in Fig. 4.1).

Consider a vector field \( \dot{x} = u(x; \mu) \), with a critical point in the origin. Using the previously introduced notations, this vector field is written as

\[
\dot{x} = u(x; \mu) \quad u^{[k]}(x; \mu) = A x + \sum_{m=2}^{k} u_m(x) + \delta \sum_{m=0}^{k} \mu_m(x),
\] (4.35)
where each \( u_m(x), \mu_m(x) \in \mathbb{H}_m^\alpha \). Notice that we rewrote the first-order vector function \( u_1(x) \) as a matrix-vector multiplication.

We start by discussing some of the properties satisfied by the Jacobian matrix \( A \). These properties will be used to reduce the complexity of the vector field representation through a transformation to Jordan normal form. The transformation is set up such that the resulting vector field again is an element of \( \mathbb{V}^\alpha \).

### 4.3.1 Eigenvalues

**Lemma 4.1**  
\( \triangleright \) The eigenvalues \( \lambda_i, i \in \{1, \ldots, n\} \), of the matrix \((A_{i,j}) = A\) in (4.35) satisfy the relation

\[
\lambda_1 + \cdots + \lambda_{n-1} + 2\lambda_n = 0,
\]

where \( \lambda_n = A_{n,n} \) (no summation over \( n \)). Furthermore, \( A_{n,n} = -\frac{1}{2} \text{tr} A_n \), where \( A_n \) is the \( n-1 \times n-1 \) submatrix of \( A \) found by deleting its \( n \text{th} \) column and \( n \text{th} \) row. \( \triangleright \)

Note that (4.36) leads to a resonance of order \( n + 2 \).

**Proof of Lemma 4.1**  
\( \triangleright \) Using the coefficient relations (4.33) and (4.34), the coefficients \( A_{i,j} \) of the Jacobian matrix \( A \) are related as follows,

\[
A_{11} + \cdots + A_{n-1, n-1} + 2A_{n,n} = u^{(1)}_{10} + \cdots + u^{(n-1)}_{01} + 2u^{(n)}_{00} = 0,
\]

and

\[
A_{n,k} = u^{(n)}_{i_1 \cdots i_k \cdots i_{n-1} 0} = 0,
\]

where \( k \in \{1, \ldots, n-1\}, i_1 + \cdots + i_{n-1} = 1, \) and \( i_k = 1 \). These relations enable us to write \( A \) as

\[
A = (A_{i,j}) = \begin{pmatrix}
& A_{1n} & \\
& A_n & \vdots \\
& 0 & \cdots & 0
\end{pmatrix},
\]

where \( A_{n,n} = -\frac{1}{2} \text{tr} A_n \) via (4.37).
Since $A_{n,n}$ is the right-most entry at the otherwise all zero $n$th row in $A$, it must be an eigenvalue. It is this eigenvalue which is called $\lambda_n$, and which is always a real number since $A \in \mathbb{R}^{n^2}$. The first $n - 1$ coefficients in the left-hand side of the coefficient relation (4.37) equal the trace of the submatrix $A_n$, $\text{tr} A_n$. Also, the trace of a matrix equals the sum of its eigenvalues. Since the remaining $\lambda_i, i \in \{1, \ldots, n - 1\}$, stem from $A_n$, the lemma is proven. \(\triangleright\)

### 4.3.2 Linear coordinate transformations

Let $M$ be an (invertible) $n \times n$ matrix with coefficients $m_{i,j}, 1 \leq i, j \leq n$. Substitution of the coordinate transformation,

$$x = My,$$  \hspace{1cm} (4.40)

into the vector field (4.22) leads to

$$\dot{y} = \psi(y; \mu) \equiv M^{-1}y(My; \mu).$$  \hspace{1cm} (4.41)

Using (4.35), and after reparameterization, the Taylor polynomial of degree $k$ can be written as

$$\psi^{[k]}(y; \mu) = M^{-1} A_M y + \sum_{m=2}^{k} \mu_m(y) + \delta \sum_{m=0}^{k} \mu_m(y),$$  \hspace{1cm} (4.42)

where $\psi_m(y) \equiv M^{-1} \psi_m(My)$ and $\mu_m(y) \equiv M^{-1} \mu_m(My)$. It is not difficult to see that again $\psi_m(y), \mu_m(y) \in H^n$. The transformation (4.40) can be used to simplify the Jacobian matrix of $\psi(y; \mu)$ to a Jordan normal form.

Since $M$ is invertible, (4.40) is analytic. Then, by definition, the local flow pattern of the vector field (4.41) is $C^k$ equivalent to the local flow pattern of the vector field (4.31), $k$ depending on the differentiability of $u(x; \mu)$.

The following properties hold.

**Lemma 4.2 Volume preservation**

Let $V^n \equiv \{u : U \times V \rightarrow \mathbb{R}^n | \nabla \cdot u = 0\}$. If $u(x; \mu) \in V^n$, $\psi(y; \mu)$ defined by (4.41) again is an element of $V^n$. \(\triangleright\)

**Lemma 4.3 Fluid-flow preservation**

If $u(x; \mu) \in V^n$, and the linear transformation (4.40) is such that $x_n \mapsto Cy_n$, with $C \neq 0$, then $\psi(y; \mu)$ defined by (4.41) again is an element of $V^n$. \(\triangleright\)
The proofs for these lemmas are straightforward and given below.

**Proof of Lemma 4.2**
- Using summation convention, the continuity equation is written as

\[ \nabla_z \cdot u(x) = 0 \iff \frac{\partial}{\partial x_i} u_i(x) = 0, \]  

(4.43)

with omission of the - for this proof unimportant - perturbation parameters \( \mu \). Inserting the transformation \( x = My \) (or \( x_i = m_{ij} y_j \)) into (4.43) we get

\[ \frac{\partial y_k}{\partial x_i} \frac{\partial}{\partial y_k} u_i(My) = m^{-1}_{kij} \frac{\partial}{\partial y_k} u_i(My) = 0, \]

where \( y_i = m^{-1}_{ij} x_j \) denotes the \( i \)th entry in the vector \( y = M^{-1} x \), and \( m^{-1}_{ij} \) denote the coefficients in the matrix \( M^{-1} \). The coefficients \( m^{-1}_{kij} \) can be put under the partial derivative operator, so that

\[ \frac{\partial}{\partial y_k}(m^{-1}_{kij} u_i(My)) = \nabla_y \cdot M^{-1} u(My) = 0, \]

which with (4.42) equals \( \nabla_y \cdot u(y) = 0 \).

**Proof of Lemma 4.3**
- The no-slip boundary condition (2.5) prescribes \( u(x_n = 0) = 0 \). The surface \( x_n = 0 \) is mapped onto \( y_n = 0 \) if the matrix \( M \) results in \( x_n \mapsto C y_n, \ C \neq 0 \). This fact combined with Lemma 4.2 makes the desired result trivial.

Note that both proofs do not depend on the fact whether a vector field is a series.

### 4.3.3 Decomplexification

Jordan canonical form theorem states that a matrix \( M \) whose columns \( m_i, \ 0 \leq i \leq n \), are (generalized) eigenvectors of a matrix \( A \) transforms that matrix into an Jordan canonical form \( J \) via

\[ J = M^{-1} A M, \]

(4.44)

The Jordan canonical form \( J \) has the same eigenvalues as \( A \) but they are located on the diagonal. The other entries in the Jordan form are zero except possibly
the entries 1 directly above\(^2\) repeated eigenvalues. The theorem is a well-known result from linear algebra, e.g. see Strang [Str80].

The Jordan canonical form requires \(J\) to be part of \(C^2\). However, for interpretation of the flow patterns, it is preferable that it remains within \(\mathbb{R}^2\). Since \(A\) is an element of \(\mathbb{R}^2\) the complex eigenvalues appear as complex conjugated pairs. Let \((\lambda, \lambda^c)\) be such a pair. Instead of writing
\[
\left( \begin{array}{cc}
\lambda & 0 \\
0 & \lambda^c
\end{array} \right),
\]
we use the equivalent representation
\[
\left( \begin{array}{cc}
\text{Re} \lambda & -\text{Im} \lambda \\
\text{Im} \lambda & \text{Re} \lambda
\end{array} \right),
\]
where \(\text{Re} \lambda = \frac{1}{2}(\lambda + \lambda^c)\) and \(\text{Im} \lambda = \frac{1}{2}(\lambda - \lambda^c)\). In that case, \(M\) also remains within \(\mathbb{R}^2\), and therewith all higher order vectors \(y_m\) and perturbations vectors \(\mu_m(y)\) in the vector field (4.42) remain to be maps from \(\mathbb{R}^n\) to \(\mathbb{R}^n\).

### 4.3.4 Eigenvector basis

Let \(a_i, i \in \{1, \ldots, n\}\) be (generalized) eigenvectors of the matrix \(A\). The eigenvalues of Jacobian matrices of vector fields in \(V_\nu\) satisfy the relation (4.36), see Lemma 4.1. Naturally, the eigenvalues of the Jacobian matrix of \(\psi(y; \mu)\), \(J\), satisfy the same relation for all conjugating matrices \(M\). However, for the interpretation of the flow pattern, it is preferable that \(J\) has the form given in (4.39). Specifically, \(y_n = 0\) should be an invariant manifold. Therefore, a special base \((m_1, \ldots, m_n)\) for use in the transformation matrix \(M\) must be found within the (generalized) eigenvectors \(a_i, i \in \{1, \ldots, n\}\) of the matrix \(A\). The requirement to be satisfied by this base is specified by Lemma 4.3: The transformation matrix \(M\) should map \(x_n = 0\) directly to \(y_n = 0\). The following lemma claims that such base can always be found even in the case of repeated eigenvalues.

**Lemma 4.4**

\(<\) The Jacobian matrix \(A\) of every vector field in \(V_\nu\) has an eigenvector basis \((m_1, \ldots, m_n)\) such that \(m_n\) is the only eigenvector transversal to the surface \(x_n = 0\). \(>\)

In other words, \(m_n\) is the only eigenvector with a nonzero \(n\)th entry. Fig. 4.2

\(^2\)Or below, whichever one prefers.
shows the geometrical picture in the case $n = 3$. The new coordinate system $(y_1 y_2 y_3)$ is aligned along the direction of the vectors $m_i$, $i \in \{1, 2, 3\}$.

**Proof of Lemma 4.4**

Denote the $n$ eigenvalues of $A$ by $\lambda_1, \ldots, \lambda_n$, where $\lambda_i$, $i \in \{1, \ldots, n-1\}$, belong to $A_n$, and $\lambda_n = A_{n,n} = -\frac{1}{2} \text{tr} A_n$.

Let $\lambda_n$ be different from the eigenvalues $\lambda_i$, $i \in \{1, \ldots, n-1\}$, from $A_n$. Denote by $m_i = (m_i^{(1)}, \ldots, m_i^{(n-1)})^t$ a (generalized) eigenvector of $A_n$ corresponding to an eigenvalue $\lambda_i$. The vector $\hat{m}_i = (m_i^{(1)}, \ldots, m_i^{(n-1)}, C_i)^t$ then is a (generalized) eigenvector of $A$ if and only if $C_i = 0$. Since there always exist a full basis of (generalized) eigenvectors for $A$, the eigenvector corresponding to $\lambda_n$ must have a nonzero $n$th entry, which proves Lemma 4.4 for this case.

Let $\lambda_n$ be equal to at least one of the eigenvalues $\lambda_i$, $i \in \{1, \ldots, n-1\}$, from $A_n$. The only difficult case—the one we will assume from hereon—is that $\lambda_n$ belongs to a $k \times k$ Jordan block, $J_k$, $k > 1$. Naturally, more than one Jordan block with eigenvalue $\lambda_n$ may exist. Here, without loss of generality, one single Jordan block is assumed. The submatrix $A_n$ has one or more eigenvalues equal to $\lambda_n$. Being a repeated eigenvalue, only one eigenvector in $A$ belonging to $\lambda_n$.
in \( J_k \) can be found. The Jordan form of \( A \) looks like

\[
J = \begin{pmatrix} J_{n-k} & 0 \\ 0 & J_k \end{pmatrix}, \quad J_k = \begin{pmatrix} \lambda_n & 1 & 0 & \cdots & 0 \\ 0 & \lambda_n & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \lambda_n & 1 \\ 0 & \cdots & 0 & 0 & \lambda_n \end{pmatrix}
\]

where \( J_k \) is a \( k \times k \) matrix and \( J_{n-k} \) is a \( (n-k) \times (n-k) \) matrix which contains the remaining Jordan blocks.

Let \( \overline{m} \) be the only eigenvector of \( A \) belonging to \( \lambda_n \), i.e.

\[
(A - \lambda_n I) \overline{m} = 0. \tag{4.45}
\]

The generalized eigenvectors are found using the recursive formula

\[
(A - \lambda_n I) \overline{m}_{i+1} = \overline{m}_i, \quad \overline{m}_1 = \overline{m}, \tag{4.46}
\]

for \( i = 1, \ldots, k - 1 \). Since the \( n \)th row of \( (A - \lambda_n I) \) contains only zero entries, it is arbitrary whether the \( n \)th entry of \( \overline{m} \) is zero or nonzero. However, for the same reason, if, in the process of constructing the generalized eigenvectors of \( \overline{m} \), one uses a nonzero \( n \)th entry in \( \overline{m}_i \) in (4.46) no other generalized eigenvectors can be found. Therefore, \( \overline{m} \) cannot have a nonzero \( n \)th entry whilst only the last generalized eigenvector can have a nonzero entry.

Let \( \overline{m}_k = (m_k^{(1)}, \ldots, m_k^{(n-1)}, C)^t, \ C \neq 0 \). Substitution of \( \overline{m}_k \) into (4.46) leads to

\[
(A_n - \lambda_n I) \overline{m}_{i+1} = \overline{m}_i - C(A_{1n}, \ldots, A_{n-1,n})^t, \quad \overline{m}_1 = \overline{m}, \tag{4.47}
\]

where the tilde above a vector denotes the fact that that vector has dimension \( n - 1 \) and is found by omission of the \( n \)th entry in the vector on which it operates. Since in all but the last computation of the eigenvectors, \( C \) equals zero, the vectors \( \overline{m}_i, \ i \in \{2, \ldots, n - 1\} \), are generalized eigenvectors of \( A_n \). Jordan form theorem guarantees that these vectors can always be found. \( \blacktriangleright \)

In constructing generalized eigenvectors for some eigenvalue \( \lambda \), one should always check whether there exist vectors in the column space of \( A - \lambda I \) which are part of the span of eigenvectors. These vectors should always become part of the conjugating matrix \( M \). Suppose that the \( i \)th column of \( A - \lambda I \) is an eigenvector. Then the vector \( \epsilon_i \), which has a 1 as the \( i \)th entry and zeros elsewhere, is a corresponding generalized eigenvector and should become part of \( M \).
4.3.5 2-D example. Jordan normal form

Consider the Jacobian matrix of a vector field in $\mathbb{V}_r^2$. The general form of this matrix is given by

$$
A = \begin{pmatrix}
  u_{10}^{(1)} & u_{01}^{(1)} \\
  0 & -\frac{1}{2} u_{10}^{(1)}
\end{pmatrix}.
$$

Depending on its coefficients, $A$ is similar to one of the following three Jordan forms:

$$
J_\lambda = \begin{pmatrix}
  \lambda & 0 \\
  0 & -\frac{1}{2} \lambda
\end{pmatrix}, \quad J_{R0} = \begin{pmatrix}
  0 & 1 \\
  0 & 0
\end{pmatrix}, \quad J_{A0} = \begin{pmatrix}
  0 & 0 \\
  0 & 0
\end{pmatrix}
$$

(\text{R0 from repeated zero, \textit{A0 from all zero}). A conjugating matrix for the first Jordan form reads}

$$
M = \begin{pmatrix}
  1 & -u_{01}^{(1)} \\
  0 & \frac{3}{2} u_{10}^{(1)}
\end{pmatrix}.
$$

Naturally, we have to assume $\lambda = u_{10}^{(1)} \neq 0$. In [Bak88] it is mentioned that the angle between the separatrix and the boundary surface equals

$$
\tan \theta = -\frac{3}{2} \frac{u_{10}^{(1)}}{u_{01}^{(1)}},
$$

(4.49)

see Fig. 2.3. More precise, it is the angle of the separatrix (corresponding to the second eigenvalue $-\frac{1}{2} u_{10}^{(1)}$) and the tangent plane to the boundary surface. A physical interpretation of (4.49) is given by

$$
\tan \theta = -3 \frac{\tau_{x1}^{(1)}(0,0)}{p_{x1}(0,0)} = 3 \frac{p_{x2}(0,0)}{p_{x1}(0,0)},
$$

using the Navier-Stokes equations (2.2). This result dates back to Legendre [Leg55] and Oswatitsch [Osw58] and shows that separation may be expected in flows with adverse pressure gradients and decreasing shear stress.

A conjugating matrix for $J_{R0}$ is given by

$$
M_{R0} = \begin{pmatrix}
  u_{01}^{(1)} & 0 \\
  0 & 1
\end{pmatrix}.
$$

This time, $u_{10}^{(1)} = 0$ and $u_{01}^{(1)} \neq 0$. Note that the first column of $M_{R0}$ equals the second column in $A - \lambda I \Leftrightarrow A$ which happens to be the only eigenvector for the repeated eigenvalue $\lambda = 0$. The second column of $M_{R0}$ equals the generalized eigenvector. $J_{R0}$ is used throughout this thesis as an example of a bifurcation analysis.
4.4 Normal form

In this section, we will formulate the conditions on near-identity transformations such that after transformation to normal form, the resulting vector field is still an element of $\mathcal{V}_\nu^n$ (step E in Fig. 4.1).

The linear coordinate transformation $\mathcal{z} = M \mathcal{y}$ made it possible to map the local flow pattern of a general vector field $\mathcal{u}(\mathcal{z}; \mu) \in \mathcal{V}_\nu^n$, with $\mathcal{u}(0; \mathcal{0}) = \mathcal{0}$, to the local flow pattern of another, simplified vector field (4.42),

$$\dot{\mathcal{y}} = \mathcal{u}(\mathcal{y}; \mu),$$

$$\mathcal{u}^{[k]}(\mathcal{y}; \mu) = J \mathcal{y} + \sum_{m=2}^{k} \mathcal{u}_m(\mathcal{y}) + \delta \sum_{m=0}^{k} \mu_m(\mathcal{y}),$$

(4.50)

where the matrix $J$ equals a Jordan form. The conjugating transformation matrix $M$ was constructed in such a way that $\mathcal{u}(y; \mu) \in \mathcal{V}_\nu^n$ again holds.

This section searches for $\mathcal{C}^k$, $k \geq 2$, equivalent vector fields in $\mathcal{V}_\nu^n$ with a minimal number of independent coefficients in the nonlinear part. The resulting vector field is a normal form within $\mathcal{V}_\nu^n$.

4.4.1 Nonlinear transformations

The transformation technique described in Section 3.2.7 produces explicit expressions of the coefficients in a vector field after transformation in terms of the coefficients in the original vector field. This enables us to check whether that vector field again is an element of $\mathcal{V}_\nu^n$.

The vector fields considered in Section 3.2.7 were of the form

$$\dot{\mathcal{y}} = \mathcal{u}(\mathcal{y}, \varepsilon) = J \mathcal{y} + \sum_{m \geq 2} \varepsilon^{m-1} \mathcal{u}_m(\mathcal{y}),$$

(4.51)

where $\mathcal{u} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and $\mathcal{u}_m \in \mathcal{H}_m^n$. The parameter $\varepsilon$ entered the vector field by substitution of the coordinate scaling,

$$\mathcal{x} = \varepsilon \mathcal{y},$$

(4.52)

into the vector field

$$\dot{\mathcal{x}} = \mathcal{u}(\mathcal{x}) = J \mathcal{x} + \sum_{m \geq 2} \mathcal{u}_m(\mathcal{x}).$$

(4.53)

The powers in $\varepsilon$ were used to collect monomials of equal index through differentiations with respect to that parameter and then setting $\varepsilon = 0$. 

If we substitute the coordinate scaling \( y \mapsto \varepsilon y \) into the vector field (4.50) we get
\[
\dot{y} = v(y; \mu, \varepsilon) = J y + \sum_{m \geq 2} \varepsilon^{m-1} u_m(y) + \delta \sum_{m \geq 0} \varepsilon^{m-1} \mu_m(y),
\]
(4.54)
where \( y : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^n \), and \( u_m, \mu_m \in H_m^\kappa \). This time it is impossible to differentiate with respect to \( \varepsilon \) and insert \( \varepsilon = 0 \) due to the term \( \delta \varepsilon^{-1} \mu_0(y) \). The coordinate scaling with a factor \( \varepsilon \) was nothing more than a trick and it should not have such a dramatic effect as this. We can, however, easily counteract this effect.

Remember the role of the parameter \( \delta \). It was introduced to collect the terms corresponding to the perturbation parameters \( \mu \). These perturbation parameters are small, so we said \( \delta \ll 1 \). Let us make this parameter even smaller and set \( \delta \ll \varepsilon^\kappa, \kappa \geq 1, \kappa \in \mathbb{N} \). The interpretation of this scaling is as follows: The sphere of interest is of size \( \varepsilon \). Then, the length scale of perturbations acting in this sphere must at least be of order in \( \varepsilon \) or smaller, because otherwise their influence cannot be considered to be local.

Therefore, let \( \delta = \varepsilon^\kappa \delta, \delta \ll 1 \). With this rescaling in place, all formula manipulations explained in Section 3.2.7 once again run without any problem, with \( \delta \) being the discriminating parameter between the unperturbed vector field and its perturbation. Although this substitution will not always be explicitly mentioned, it is always intended.

For a given function \( T : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \), let \( t(z, \varepsilon), z \in \mathbb{R}^n \), and \( \varepsilon \in \mathbb{R} \), be the solution of
\[
\frac{\partial t(z, \varepsilon)}{\partial \varepsilon} = T(t(z, \varepsilon), \varepsilon), \quad t(z, 0) = z
\]
(4.55)

Let \( z = t^{-1}(y, \varepsilon) \) be the inverse of the transformation of
\[
y = t(z, \varepsilon).
\]
(4.56)

Substitution of the (4.56) into vector field (4.54) results in the following \( C^k \)-equivalent vector field
\[
\dot{z} = w(z; \mu, \varepsilon) \equiv \frac{\partial t^{-1}(t(z, \varepsilon), \varepsilon)}{\partial y} v(t(z, \varepsilon); \mu, \varepsilon).
\]
(4.57)
If \( v(y; \mu, \varepsilon) \) is \( C^r, r \geq 2 \), then \( 2 \leq k \leq r \).
PROPOSITION 4.5

If both the vector field \( v(y; \mu, \varepsilon) \in \mathbb{V}_v^n \) and the generator \( T(y, \varepsilon) \in \mathbb{V}_v^n \) of the transformation \( y = t(x, \varepsilon) \), then the vector field \( w(z; \mu, \varepsilon) \) defined in (4.57) again is an element of \( \mathbb{V}_v^n \).

To prove Proposition 4.5, we should first look into the main ingredient of simplification through \( C^k \), \( k \geq 2 \), equivalence, which is the Lie product:

\[
[X, Y](x) = \frac{\partial X(x)}{\partial x} Y(x) - \frac{\partial Y(x)}{\partial x} X(x),
\]

where both \( X, Y : \mathbb{R}^p \to \mathbb{R}^n \). See, for example, (3.102) in Section 3.2.7. The Lie product forms part of a broad theory called Lie algebra.

Lie algebra

The following definition gives precise conditions which need to be satisfied in order to call a set of vector fields a Lie algebra.

DEFINITION 4.6 Lie algebra

A Lie algebra \( g = (F, [\cdot, \cdot]) \) is a vector space over a field \( F \) on which a product \( [\cdot, \cdot] \), called the Lie bracket, is defined, with the properties

1. \( X, Y \in g \) imply \( [X, Y] \in g \),
2. \( [X, \alpha Y + \beta Z] = \alpha [X, Y] + \beta [X, Z] \) for \( \alpha, \beta \in F \) and \( X, Y, Z \in g \),
3. \( [X, Y] = -[Y, X] \), and
4. \( [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \).

Properties (1) and (2) make the Lie bracket a bilinear operation. Property (3) is called skew symmetry from which among other things follows that \( [X, X] = 0 \) for all \( X \in g \). Property (4) is known as the Jacobi identity. If \( F \) is the field of real numbers \( \mathbb{R} \) we say \( g \) is a real Lie algebra; and if \( F = \mathbb{C} \) we say \( g \) is complex, see [SW86]. Only the first case is treated here and the additive ‘real’ is omitted.

Consider the sets of vector fields introduced so far:

\[
\mathbb{V}^n = \left\{ u : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \mid \nabla \cdot u(x; \mu) = 0 \right\},
\]

\[
\mathbb{V}_v^n = \left\{ u : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n \mid \nabla \cdot (x_n u(x; \mu)) = 0 \right\}.
\]

It is not difficult to show that the Lie product \( [\cdot, \cdot] \) in (4.58) is indeed a Lie bracket. Given the important role of the Lie product in transformation theory,
and therewith in the transformation to a normal form, we want to know whether $(\mathbb{V}_n, [\ , \ ]_v)$ and $(\mathbb{V}_n, [\ , \ ]_{\mathbb{V}_n})$ both are a Lie algebra. Property 1 in Definition 4.6 is the only difficult part to prove. It will also be the most useful property for our application. This requires use to prove the following to lemmas.

**Lemma 4.7** Volume preservation

- If $f(x), g(x) \in \mathbb{V}_n$, their Lie product also lies in $\mathbb{V}_n$. △

**Lemma 4.8** Fluid-flow preservation

- If $f(x), g(x) \in \mathbb{V}_n$, their Lie product also lies in $\mathbb{V}_n$. △

Since the proof of Lemma 4.7 runs similarly to that of Lemma 4.8 only a proof of the latter is given.

**Proof of Lemma 4.8**

- We have to check whether

\[ \nabla \cdot \left( x_n [f(x), g(x)] \right) = 0 \quad (4.61) \]

given that

\[ \nabla \cdot (x_n f(x)) = \nabla \cdot (x_n g(x)) = 0. \quad (4.62) \]

Write out the left-hand side of (4.61),

\[ \nabla \cdot \left( x_n \frac{\partial f(x)}{\partial x} g(x) - x_n \frac{\partial g(x)}{\partial x} f(x) \right). \]

Using summation convention the latter expression can be written as

\[ \partial_j \{ x_n g_i (\partial_i f_j) - x_n f_i (\partial_i g_j) \}. \quad (4.63) \]

Similarly, (4.62) becomes

\[ \partial_i (x_n f_i) = \partial_i (x_n g_i) = 0. \quad (4.64) \]

Since $\partial_i x_n = \delta_{i,n}$ and $x_n \partial_i f_j = \partial_i (x_n f_j) - f_j \delta_{i,n}$, expression (4.63) can be rewritten as

\[ \partial_j \{ x_n g_i (\partial_i f_j) - x_n f_i (\partial_i g_j) \} = \]

\[ \partial_j \{ \partial_i (x_n f_j) g_i - f_j g_n - \partial_i (x_n g_j) f_i + g_j f_n \} = \]

\[ \partial_i (x_n f_j) \partial_j g_i - g_n \partial_j f_j - f_j \partial_j g_n - \partial_i (x_n g_j) \partial_j f_i + f_n \partial_j g_j + g_j \partial_j f_n = \]

\[ f_j \partial_j g_n + x_n \partial_i f_j \partial_j g_i + x_n \partial_i g_i \partial_j f_j - f_j \partial_j g_n + \]

\[ - g_j \partial_j f_n - x_n \partial_i g_j \partial_j f_i - x_n \partial_i f_j \partial_j g_j + g_j \partial_j f_n \]
where the relations in (4.64) have already been inserted. The last result indeed equals zero. ▶

Chapter 5 also looks at local flow patterns with a plane of symmetry. We therefore introduce the following set of vector fields:

$$\mathcal{V}_{S}^{n,i} = \left\{ u: \mathbb{R}^{n} \times \mathbb{R}^{p} \rightarrow \mathbb{R}^{n} \middle| \begin{array}{l}
u^{(j)}(x,\mu) \text{ even in } x_{i}, j \neq i, \text{ and} \\
u^{(i)}(x,\mu) \text{ uneven in } x_{i},
\end{array} \quad i,j \in \{1,\ldots,n\} \right\}. \quad (4.65)$$

**Lemma 4.9** Mirror symmetry preservation
< If \( f, g \in \mathcal{V}_{S}^{n,i} \), their Lie product also lies in \( \mathcal{V}_{S}^{n,i} \). ▶

**Proof of Lemma 4.9**
▶ Let \( f(\tilde{x}), g(\tilde{x}) \in \mathcal{V}_{S}^{3,1} \) (the proof runs similar for other dimensions \( n \) and coordinates axes \( x_{i} \)). In that case, \( f^{(1)}(\tilde{x}) \) and \( g^{(1)}(\tilde{x}) \) only have terms with uneven powers in \( x_{1} \) and \( f^{(2)}(\tilde{x}), f^{(3)}(\tilde{x}), g^{(2)}(\tilde{x}), g^{(3)}(\tilde{x}) \) only have terms with even powers in \( x_{1} \). Since \( [f, g] = \bar{D}f \bar{g} - \bar{D}g \bar{f} \), writing out the first part in full (the second part runs similar) gives,

$$\begin{pmatrix}
\partial_{1} f^{(1)} g^{(1)} + \partial_{2} f^{(1)} g^{(2)} + \partial_{3} f^{(1)} g^{(3)} \\
\partial_{1} f^{(2)} g^{(1)} + \partial_{2} f^{(2)} g^{(2)} + \partial_{3} f^{(2)} g^{(3)} - \ldots
\end{pmatrix}$$

It is now easy to see that the first row again has only terms with uneven powers in \( x_{1} \) and the second and third row both have only terms with even powers in \( x_{1} \). Thus, Lemma 4.9 is proven. ▶

We are now ready to prove Proposition 4.5. Let the series expansion in \( \varepsilon \) of the generator \( \mathcal{T}(y, \varepsilon) \) be as

$$\mathcal{T}(y, \varepsilon) = \sum_{m=2}^{k} \varepsilon^{m-1} t_{m}(y), \quad (4.66)$$

where \( t_{m}(y) \in H_{m}^{n} \).

**Proof of Proposition 4.5**
▶ If \( \mathcal{T}(y, \varepsilon) \in \mathcal{V}_{\mu}^{n} \), every one of its homogeneous polynomial vectors \( t_{m}(y) \in \mathcal{V}_{\mu}^{n} \). The transformation technique discussed in Section 3.2.7 solely consists of Lie products organized in the triangle (3.107). The original vector field is an element in \( \mathcal{V}_{\mu}^{n} \). Initially, the triangle (3.107) involves Lie products from elements in \( \mathcal{V}_{\mu}^{n} \).
4.4. Normal form

From Lemma 4.8 we know that $\mathcal{V}_n^\circ$ is a Lie algebra which means that the Lie product of two vector fields in $\mathcal{V}_n^\circ$ is again a vector field in $\mathcal{V}_n^\circ$. As a result, throughout the triangle (3.107) Lie products are computed from elements in $\mathcal{V}_n^\circ$. Thus, Lemma 4.8 implies Proposition 4.5.

**Remark 4.10**

Similar propositions as Proposition 4.5 hold for vector fields in $\mathcal{V}_n^u$ and $\mathcal{V}_S^m$.

Limiting the generator to $\mathcal{V}_n^\circ$ in the transformation to a normal form brings along with it that a few terms in the nonlinear part persist which otherwise would be removed because they lie in the image of the ad $J$ operator. For this reason, simplified representations obtained with Proposition 4.5 will sometimes be called *physical normal forms*.

Vector fields in $\mathcal{V}_n^u$ are *volume-preserving* flows. Volume-preserving normal forms (and deformations) have been studied extensively. Van der Meer [vdM85] provides a historical background of normal forms in the volume-preserving case. Also, MacKay [Mac94] presents a nice list of contributions on volume-preserving flows in general.

Broer [Bro79] [Bro81a] [Bro81b] presented a bifurcation analysis of volume preserving vector fields in three dimensions with rotational symmetry. Initially, these vector fields have only rotational symmetry in the linear part. Coordinate transformations are then used to achieve rotational symmetry in the nonlinear part. Volume-preservation is then imposed. Limiting the generator of a transformation to volume-preserving vector fields (i.e., $\mathcal{V}_3^u$) as Remark 4.10 suggests, results in identical normal forms.

The representation of the perturbation terms are also effected by the transformation. However, these terms remain to be part of $\mathcal{V}_n^u$. Therefore, reparameterization of the perturbation parameters is all that is needed.

4.4.2 2-D example. Normal form

The requirement that $\mathcal{T}(y, z)$ is an element of $\mathcal{V}_n^u$ is equivalent with the requirement that every $t_m(y)$ in (4.66) is an element of $\mathcal{H}_m^2 \cap \mathcal{V}_k^2$. Thus, the following coefficient relations also hold

$$(i + 1)t_{i+1,j}^{(1)} + (j + 2)t_{i,j+1}^{(2)} = 0, \quad i + j = m - 1 \geq 0 \quad t_{m,0}^{(2)} = 0, \quad m \geq 0$$

see (4.3).
Table 4.1: $u_2(y) + \text{ad} J(t_2(y))$, with $u_2(y)$, $t_2(y) \in H_2^2 \cap \mathbb{V}_\nu^2$.

Computation of the adjoint operation of $J \equiv J_{R0}$, see (4.48), acting on general $t_m(y) \in H_m^2$ leads to

$$\text{ad} J(t_m(y)) = \sum_{i+j=m} \left( \begin{array}{c}
t_{1}^{(i)} y_{i}^{j} - i t_{1}^{(1)} y_{i}^{i-1} y_{j}^{j+1} \\
t_{1}^{(2)} y_{i}^{i-1} y_{j}^{j+1}
\end{array} \right).$$ (4.67)

The objective is to simplify$^3$

$$u_m(y) + \text{ad} J(t_m(y)) = \sum_{i+j=m} \left( \begin{array}{c}
u_{1}^{(i)} y_{i}^{j} + t_{1}^{(2)} y_{i}^{i-1} y_{j}^{j+1} \\
u_{1}^{(2)} y_{i}^{i-1} y_{j}^{j+1}
\end{array} \right),$$ (4.68)

for each $m \geq 2$. Table 4.1a shows the coefficients of this mapping for each monomial in $t_2(y)$. The coefficient relations due to $\mathbb{V}_\nu^2$ are already substituted. Notice that since $t_2(y) \in H_2^2 \cap \mathbb{V}_\nu^2$ also $u_2(y) + \text{ad} J(t_2(y)) \in H_2^2 \cap \mathbb{V}_\nu^2$ holds.

The re-ordering into subspaces in Table 4.1b$^4$ is based on repeated coefficients $t_{i}^{(s)}$ and $u_{i}^{(s)}$. From this table, the complementary subspace and the kernel are

$^3$As in Chapter 3, we prefer the notation $u_m(y)$ rather than $v_m(y)$. Also, normal-form coefficients are denoted with an $n$ rather than $a$.

$^4$The omission of the monomials $y_i y_j$ in Table 4.1b may be somewhat enigmatic. With the monomials, as in Table 3.3, the interpretation of the contribution of $C_1$, $G_2$ is more easily
<table>
<thead>
<tr>
<th>$y_1^3$</th>
<th>$y_1^2 y_2$</th>
<th>$y_1 y_2^2$</th>
<th>$y_2^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial_1$</td>
<td>$u_{30}^{(1)}$</td>
<td>$u_{21}^{(1)} - \frac{9}{2} t_{30}^{(1)}$</td>
<td>$u_{12}^{(1)} - \frac{8}{3} t_{21}^{(1)}$</td>
</tr>
<tr>
<td>$\partial_2$</td>
<td>0</td>
<td>$-\frac{3}{2} u_{30}^{(1)}$</td>
<td>$-\frac{2}{3} u_{21}^{(1)} + 3 t_{30}^{(1)}$</td>
</tr>
</tbody>
</table>

a) Mapping.

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$u_{30}^{(1)}$</th>
<th>$-\frac{3}{2} u_{30}^{(1)}$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_1$</td>
<td>$u_{21}^{(1)} - \frac{9}{2} t_{30}^{(1)}$</td>
<td>$-\frac{2}{3} u_{21}^{(1)} + 3 t_{30}^{(1)}$</td>
<td>-</td>
</tr>
<tr>
<td>$I_2$</td>
<td>$u_{12}^{(1)} - \frac{8}{3} t_{21}^{(1)}$</td>
<td>$-\frac{1}{4} u_{12}^{(1)} + \frac{2}{3} t_{21}^{(1)}$</td>
<td>-</td>
</tr>
<tr>
<td>$I_3$</td>
<td>$u_{03}^{(1)} - \frac{5}{4} t_{12}^{(1)}$</td>
<td>$t_{03}^{(1)}$</td>
<td>1</td>
</tr>
</tbody>
</table>

b) Re-ordering into subspaces.

Table 4.2: $u_3(y) + \text{ad } J(t_3(y))$, with $u_3(y)$, $t_3(y) \in H^2_3 \cap \mathbb{V}_\nu^2$.

found as

$$G_2^\nu = n_{20}^{(1)} (z_1^2 \partial_1 - z_1 z_2 \partial_2), \quad \mathcal{K}_2^\nu = t_{02}^{(1)} y_2^2 \partial_1, \quad (4.69)$$

where $n_{20}^{(1)} = u_{20}^{(1)}$ which is in general nonzero. The index $\nu$ in $G_2^\nu$ and $\mathcal{K}_2^\nu$ indicates that this representation of the complementary subspace and the kernel are found using a restricted generator and that the result only applies to vector fields describing fluid flow.

Computation of the third degree terms runs similarly. As before the complementary subspace needs to be computed using $\text{ad } J(t_3(y))$. The result is listed in Table 4.2. As before, the complementary subspace and the kernel are found by a re-ordering into subspaces based on repeated coefficients $t_{i,j}^{(s)}$ and $u_{i,j}^{(s)}$, and the result is

$$G_3^\nu = n_{30}^{(1)} (z_1^2 \partial_1 - \frac{3}{2} z_1 z_2 \partial_2), \quad \mathcal{K}_3^\nu = t_{03}^{(1)} y_2^3 \partial_1, \quad (4.70)$$

made. However, this addition spreads the tables of three-dimensional vector fields over multiple pages. To guide the interpretation, the computation of the mapping and its re-ordering into subspaces are given side by side, or at least close by.
where \( n_{30}^{(1)} = \tilde{u}_{30}^{(1)} \). As in Chapter 3, the tilde in \( \tilde{u}_{30}^{(1)} \) indicates that this coefficient equals the sum of \( u_{30}^{(1)} \) plus products of coefficients formed by additional Lie-products.

We remark that it is not possible to further simply the normal form entries from \( G_5^\nu \) using the kernel \( \kappa_2^\nu \). The relevant Lie product

\[
\begin{bmatrix} 2 \nu_2, \kappa_2^\nu \end{bmatrix} = \begin{pmatrix} \begin{pmatrix} n_{20}^{(1)} y_1^2 \\ -n_{20}^{(1)} y_1 y_2 \end{pmatrix} \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} \ell_0^{(1)} y_2 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 4n_{20}^{(1)} \ell_0^{(1)} y_1 y_2 \\ -n_{20}^{(1)} \ell_0^{(1)} y_2^2 \end{pmatrix} \end{pmatrix}
\]

lies completely in the image of \( \text{ad} J(\cdot) \), see subspace \( I_2 \) of Table 4.2b.

In conclusion, up to third order, the vector field

\[
\begin{align*}
\dot{z}_1 &= z_2 + n_{20}^{(1)} z_1^2 + n_{30}^{(1)} z_1^3, \\
\dot{z}_2 &= -n_{20}^{(1)} z_1 z_2 - \frac{3}{2} n_{30}^{(1)} z_1^2 z_2.
\end{align*}
\tag{4.71}
\]

is topological equivalent with any (unperturbed) vector field in \( V^2_\nu \) with a Jacobian matrix similar to \( J_{R0} \).

\( G_k^\nu \) of double zero eigenvalue

The normal form representation of (4.71) differs a lot in appearance from the general normal form derived in (3.117). The latter has all normal-form coefficients on the second row, i.e.

\[
G_k = n_{k_0}^{(2)} z_1^k \partial_2 + n_{k_1}^{(2)} z_1^{k-1} z_2 \partial_2.
\tag{4.72}
\]

The normal form (4.71), on the other hand, uses both rows. Section 4.1 computed a normal form for vector fields in \( V^2_\nu \) based on (4.72), and found that

\[
G_k^a = v_{20}^{(1)} z_1 z_2 \partial_2, \quad G_3^a = \left(-\frac{1}{2} (v_{20}^{(1)})^2 z_1^3 + 2v_{30}^{(1)} z_1 z_2 \right) \partial_2.
\]

The index \( a \) in \( G_k^a \) denotes that we substituted the coefficient relations from our application into a general representation of the complementary subspace.

Note that the number of degrees of freedom in the normal-form coefficients is 2, precisely the same as in the normal form (4.71). The amount of work to find the normal form (4.71) is, however, considerable less.

The complementary subspace \( G_k^\nu \) in (4.72) has the elegance of being able to use a single formula for all \( k \geq 2 \). Within \( V^2_\nu \), it is also possible to find the complementary subspace \( G_k^\nu \) in one single formula through straightforward index manipulation.
4.4. Normal Form

**Proposition 4.11**

- The complementary subspace at order \(k\) of the adjoint operation of \(J_{R0}\) within \(\mathbb{V}_\nu^2\) equals

\[
G_k^c = n_{k_0}^{(1)}(z_1^k \partial_1 - \frac{k}{2} z_1^{k-1} z_2 \partial_2).
\]

\[\text{Eq. (4.73)}\]

**Proof of Proposition 4.11**

- The adjoint operation of \(J_{R0}\) acting on \(H_\nu^2\) is given in (4.67) and the expression which needs simplification by (4.68), repeated here for convenience, is

\[
u_k(y) + \text{ad } J(t_k(y)) = \sum_{i+j=k} \begin{pmatrix}
  u_{i,j}^{(1)} y_i y_j + t_{i,j}^{(2)} y_i y_j - i t_{i,j}^{(1)} y_1^{i-1} y_2^{j+1} \\
  u_{i,j}^{(2)} y_i y_j - i t_{i,j}^{(2)} y_1^{i-1} y_2^{j+1}
\end{pmatrix}.
\]

\[\text{Eq. (4.74)}\]

In addition, the following coefficient relations holds for \(u_k(y) \in \mathbb{V}_\nu^2\),

\[
(i + 1) u_{i+1,j}^{(1)} + (j + 2) u_{i,j+1}^{(2)} = 0, \quad i + j = k - 1
\]

\[
u_k^{(2)} = 0,
\]

and for \(t_k(y) \in \mathbb{V}_\nu^2\),

\[
(i + 1) t_{i+1,j}^{(1)} + (j + 2) t_{i,j+1}^{(2)} = 0, \quad i + j = k - 1
\]

\[t_k^{(2)} = 0,
\]

where \(i\) and \(j\) are nonnegative integers, see (4.3). Thus, the three sums in the top row of (4.74) can be rewritten as

\[
\sum_{i+j=k} u_{i,j}^{(1)} y_i y_j = u_{k,0}^{(1)} y_1^k + \sum_{i+j=k-1} u_{i,j+1}^{(1)} y_i y_j^{j+1},
\]

\[
\sum_{i+j=k} t_{i,j}^{(2)} y_i y_j = \sum_{i+j=k-1} t_{i,j+1}^{(2)} y_i y_j^{j+1} = - \sum_{i+j=k-1} \frac{i+1}{j+2} t_{i,j+1}^{(1)} y_1 y_2^{j+1},
\]

\[
\sum_{i+j=k} i t_{i,j}^{(1)} y_1^{i-1} y_2^{j+1} = \sum_{i+j=k-1} (i + 1) t_{i+1,j}^{(1)} y_1 y_2^{j+1}.
\]

Note that the three right-most sums are written such that all monomials are of equal index. Then it is easy to see that

\[
t_{i+1,j}^{(1)} = \frac{j+2}{(j+3)(i+1)} u_{i,j+1}^{(1)}, \quad i + j = k - 1,
\]
and that \( u_{k0}^{(1)} y_{1}^{k} \partial_{1} \) moves into \( \mathbb{G}'_{k} \).

Because of the above coefficient relations, we also find that

\[
t^{(2)}_{i,j+1} = -\frac{i+1}{j+2} t^{(1)}_{i+1,j} = -\frac{1}{j+3} u^{(1)}_{i,j+1}, \quad i + j = k - 1.
\]

If we re-write the sums in the second row similarly as above, it is easy to verify that this solution removes that row with exception of the term \( u_{k-11}^{(2)} y_{1}^{k-1} y_{2} \partial_{2} \) which also moves into \( \mathbb{G}'_{k} \).

Combining results, the complementary subspace is given by

\[
u_{k0}^{(1)} y_{1}^{k} \partial_{1} + u_{k-11}^{(2)} y_{1}^{k-2} y_{2} \partial_{2}.
\]

Since \( u_{k-11}^{(2)} = -\frac{k}{2} u_{k0}^{(1)} \), Proposition 4.11 is proven. ▶

This result will be used in Section 4.7.

## 4.5 Miniversal deformations

In this section, we will present a scheme to reduce a general versal deformation of a vector field to a miniversal deformation. This way, we can find (mini)versal deformations for vector fields in \( \mathbb{V}_{p}^{n} \) (step E in Fig. 4.1).

### 4.5.1 Perturbation-parameter reduction scheme

After reduction to Jordan form and normal form, we have found the following vector field

\[
\dot{z} = w(z; \mu)
\]

\[
w^{[k]}(z; \mu) = J \dot{z} + \sum_{m \geq 2} \varepsilon^{m-1} w_{m}(z) + \delta \sum_{m \geq 0} \varepsilon^{m-1} \mu_{m}(z),
\]

(4.75)

where \( w \in \mathbb{V}_{p}^{n} \), and each \( w_{m}, \mu_{m} \in \mathbb{H}_{m}^{n} \). The entries in the perturbation-parameter vector \( \mu \) are the coefficients of the homogeneous-polynomial vector function \( \mu_{m}(z) \), denoted as \( \mu_{i_{1}, \ldots, i_{n}}^{(k)} \) with \( \sum_{k=1}^{n} i_{k} = m, \ell \in \{1, \ldots, n\} \). In the case of a two-dimensional vector field, for example, the vector is defined as

\[
\mu \equiv (\mu_{00}^{(1)}, \mu_{00}^{(2)}, \mu_{01}^{(1)}, \mu_{01}^{(2)}, \mu_{10}^{(1)}, \mu_{10}^{(2)}, \mu_{02}, \ldots).
\]

The objective is to reduce the number of parameters in this vector.
Let the near-identity transformation \( z = t(x; \mu, \epsilon, \delta), \quad x \in \mathbb{R}^n \), have the following truncated expansion in \( \delta \)

\[
    z = t(x; \mu, \epsilon, \delta) = x + \delta G(x; \mu, \epsilon).
\]  

(4.76)

Transformation (4.76) is a so-called infinitesimal transformation since \( \delta \ll 1 \). A general expansion of \( G(x; \mu, \epsilon) \) in \( \epsilon \) reads

\[
    G(x; \mu, \epsilon) = \sum_{m \geq 0} \epsilon^{m-1} G_m(x; \mu).
\]

(4.77)

with each \( G_m \in H_m^* \). The function \( G(x; \mu, \epsilon) \) will be called the generating function.

Let \( u(x; \mu) \) (once again) denote the vector field which results after substitution of the near-identity transformation (4.76) into the vector field (4.75). This vector field is found as

\[
    \dot{x} = u(x; \mu)
    = \left(1 + \delta D G(x; \mu, \epsilon)\right)^{-1} \left\{ J x + \delta J G(x; \mu, \epsilon) + \sum_{m \geq 2} \epsilon^{m-1} w_m \left( x + \delta G(x; \mu, \epsilon) \right) \right.
    \]

\[
    + \delta \sum_{m \geq 0} \epsilon^{m-1} \mu_m \left( x + \delta G(x; \mu, \epsilon) \right) \right\}.
\]

(4.78)

Using the equivalence

\[
    \left(1 + \delta D G(x; \mu, \epsilon)\right)^{-1} = 1 - \delta D G(x; \mu, \epsilon) \left(1 + \delta D G(x; \mu, \epsilon)\right)^{-1}
\]

it is possible to rewrite (4.78) as

\[
    \dot{x} = u(x; \eta) = J x + \sum_{m \geq 2} \epsilon^{m-1} w_m (x) + \delta \sum_{m \geq 0} \epsilon^{m-1} \eta_m (x) + \mathcal{O}(\delta^2),
\]

(4.79)

where

\[
    \sum_{m \geq 0} \epsilon^{m-1} \eta_m (x) \equiv \sum_{m \geq 0} \epsilon^{m-1} \mu_m (x) + \left( J G(x; \mu, \epsilon) - D G(x; \mu, \epsilon) J x \right)
    + \sum_{m \geq 2} \epsilon^{m-1} \left( D w_m (x) G(x; \mu, \epsilon) - D G(x; \mu, \epsilon) w_m (x) \right).
\]

(4.80)

The right-hand side of (4.79) shows that the factor \( \delta \) nicely separates the normal form and the perturbation terms. In fact, transformation (4.76) does not effect the representation of the normal form in any way.
Note that the vector field (4.79) depends on the perturbation-parameter vector $\eta$. As before, the coefficients of $\eta_m(\bar{x})$, $m \geq 0$, span this vector. For example, in the two-dimensional case, we have

$$\eta = (\eta_{00}^{(1)}, \eta_{00}^{(2)}, \eta_{01}^{(1)}, \eta_{01}^{(2)}, \eta_{01}^{(2)}, \ldots)$$

We get explicit expressions for the new perturbation parameters $\eta_i^{(\varepsilon)}$ in terms of the old perturbation parameters $\mu_i^{(\varepsilon)}$ by inserting (4.77) into (4.80). By rewriting the result using the Lie-brackets we get

$$\sum_{m \geq 0} \varepsilon^{m-1} \eta_m(\bar{x}) = \sum_{m \geq 0} \varepsilon^{m-1} \mu_m(\bar{x}) + \sum_{m \geq 0} \varepsilon^{m-1} \left[ J \bar{x}, G_m(\bar{x}; \mu) \right]$$

$$+ \sum_{m \geq 2} \varepsilon^{m-1} \left[ w_m(\bar{x}), \sum_{k \geq 0} \varepsilon^k G_k(\bar{x}; \mu) \right]. \quad (4.81)$$

The above expression shows that by sorting terms of equal power in $\varepsilon$, we can use the adjoint operation $\text{ad} J G_m(\bar{x}; \mu) = [J \bar{x}, G_m(\bar{x}; \mu)]$ to simplify the perturbation vector $\eta$. These computations can be done degree by degree for increasing $m \geq 0$. The perturbation terms which cannot be removed are located in the complementary subspace of the $\text{ad} J$ operator. It is easy to see that (4.81) organizes itself in a Lie-triangle.

Note that (4.81) contains a Lie products of the normal form and the generating function. The term $[w_p(\bar{x}), G_q(\bar{x}; \mu)]$ is multiplied by the factor $\varepsilon^{p+q-2}$ and the term $\mu_m(\bar{x})$ by $\varepsilon^{m-1}$. For matching powers it is therefore possible to further reduce the perturbations vector $\eta$. For example, the kernel of $\text{ad} J G_0(\bar{x}; \mu)$ is used via $[w_2(\bar{x}), G_0(\bar{x}; \mu)]$ to simplify $\eta_2(\bar{x})$. Note that the kernel $\mathcal{K}_0$ should be used since the image is already used in the simplification of $\eta_0(\bar{x})$.

### 4.5.2 Deformation of matrices

Section 3.3.1 explained how to construct a deformation for families of matrices. It showed that the co-dimension of a critical point equals the dimension of the centralizer for the Jacobian matrix of its vector field.

The adjoint of a matrix $A$ operating on a matrix-vector multiplication $B \bar{x}$ is closely related to the bracket operation for the two matrices, since

$$\text{ad} A(B \bar{x}) \equiv [A \bar{x}, B \bar{x}] = A B \bar{x} - B A \bar{x} \equiv [A, B] \bar{x}.$$

The centralizer for the matrix $A$ is the set of all matrices $B$ for which the bracket operation with the matrix $A$ results in the all zero matrix. We already found that
the re-ordering of subspaces in the adjoint operator of the matrix $A$ produces an expression of the kernel $\mathcal{K}_1$. This kernel is equivalent to the centralizer.

The deformation of a matrix is found as the manifold transversal to the centralizer of that matrix. The dimension of that manifold equals the dimension of the centralizer, see Section 3.3.1. However, the dimension of the kernel also equals the dimension of the complementary subspace. This can be seen as follows.

The number of linear equations as a result of the homological equation (3.50), which in this case becomes

$$\eta_1(x) = \mu_1(x) + \text{ad} A(G_1(x; \mu)) = 0,$$

is equal to the number of parameters in the homogeneneous polynomial $\mu_1(x)$. Since the number of coefficients in the homogeneous polynomial $G_1(x; \mu)$ equals that number of parameters, there are as many coefficients as there are linear equations. The number of linearly independent equations versus the number of coefficients in a subspace determines whether that subspace is part of the complementary subspace, the image, or the kernel. The image consists of subspaces in which the number of equations equals the number of coefficients. Then, the coefficients lacking in the complementary subspace are precisely those which are redundant in the kernel. Therefore, the dimension of the complementary subspace equals the dimension of the kernel. This result can also be used to check the correctness of the re-ordering into subspaces.

A proof that the complementary subspace $\mathcal{G}_1$ is a manifold transversal to the kernel $\mathcal{K}_1$ is somewhat beyond the scope of this thesis. This result, however, holds for every case to be discussed here.

### 4.5.3 2-D example. Miniversal deformation of a matrix

Maybe the best supporting evidence that the above bifurcation scheme works is to present an example. The three normal forms of miniversal deformations according to Fig. 3.13 for the repeated zero case

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(4.82)

are

$$J(\mu) = \begin{pmatrix} 0 & 1 \\ \mu_1 & \mu_2 \end{pmatrix}, \quad J(\hat{\mu}) = \begin{pmatrix} \hat{\mu}_1 & 1 \\ \hat{\mu}_2 & 0 \end{pmatrix}, \quad \text{and} \quad J(\bar{\mu}) = \begin{pmatrix} \bar{\mu}_1 & 1 \\ \bar{\mu}_2 & \bar{\mu}_1 \end{pmatrix},$$

(4.83)

respectively. We should be able to construct every one of these matrices using the complementary subspace of $\text{ad} J$ over $\mathbb{H}^2_1$. Table 4.3 shows the computation
Table 4.3: \( \mu_1(y) + \mathrm{ad} \mathbf{J}(y) \), with \( \mu_1(y), t_1(y) \in \mathbb{H}_1^2 \).

of the subspaces of the adjoint operation of the matrix \( \mathbf{J} \) acting on a vector\(^5\) \( t_1(x) \in \mathbb{H}_1^2 \). Note that the dimension of both the complementary subspace and the kernel is 2.

The only parameter that controls the form of the deformation is \( t_{10}^{(2)} \). The following three selections achieve the desired normal forms of miniversal deformations in (4.83),

\[
\begin{align*}
 t_{10}^{(2)} &= -\mu_{10}^{(1)} & \longrightarrow & \left\{ \begin{array}{l}
 \mu_1 = \mu_{10}^{(2)}, \\
 \mu_2 = \mu_{10}^{(1)} + \mu_{01}^{(2)}, 
\end{array} \right. \\
 t_{10}^{(2)} &= \mu_{01}^{(2)} & \longrightarrow & \left\{ \begin{array}{l}
 \mu_1 = \mu_{01}^{(1)} + \mu_{01}^{(2)}, \\
 \mu_2 = \mu_{10}^{(2)}, 
\end{array} \right. \\
 t_{10}^{(2)} &= \frac{1}{2} \mu_{01}^{(2)} - \frac{1}{2} \mu_{10}^{(1)} & \longrightarrow & \left\{ \begin{array}{l}
 \mu_1 = \frac{1}{2} \mu_{10}^{(1)} + \frac{1}{2} \mu_{01}^{(2)}, \\
 \mu_2 = \mu_{10}^{(2)}. 
\end{array} \right.
\end{align*}
\]

In everyone of the three cases, \( t_{01}^{(2)} - t_{10}^{(1)} = -\mu_{01}^{(1)} \).

\(^5\)We prefer to use \( t_k(x) \) instead of \( G_k(x; \mu) \) in correspondence with previous computations done to find normal forms. The presence of the perturbation-parameter vector \( \mu_k(x) \) in the tables indicates the nature of their application.
4.5. Miniversal deformations

\[
\begin{array}{c|c|c}
\partial_1 & \mu_{00}^{(1)} + \iota_{00}^{(2)} \\
\partial_2 & \mu_{00}^{(2)} \\
\end{array}
\]

a) Mapping.

\[
\begin{array}{c|c|c}
\partial_1 & \partial_2 \\
C_1 & \mu_{00}^{(2)} & 1 \\
I_1 & \mu_{00}^{(1)} + \iota_{00}^{(2)} \\
K_1 & \iota_{00}^{(1)} & 1 \\
\end{array}
\]

b) Re-ordering into subspaces.

Table 4.4: \( \mu_0(y) + \text{ad} J(t_0(y)) \), with \( \mu_0(y), t_0(y) \in H_0^2 \).

4.5.4 2-D example. Miniversal deformation of a vector field

To compute the deformation of a vector field about a particular critical point, one needs to know the co-dimension in that critical point. The co-dimension equals the sum of the dimension of the deformation of Jacobian matrix of that vector field and the number of zero coefficients in the normal form corresponding to that Jacobian matrix, see Definition 3.24. Using this definition, Section 3.3 derived a miniversal deformation of the double zero eigenvalue. It did so by stating that the co-dimension is 2 and thus that the two parameters in \( J(\mu) \) suffice. However, using this deformation, the origin remains a critical point for all values of parameters \( \mu_1 \) and \( \mu_2 \). After substitution of a translation and a reparameterization, we found (3.149). This deformation uses a zero order term.

Our bifurcation scheme is especially suited to find deformations of vector fields such as (3.149). First we compute \( \text{ad} J(\cdot) \) over \( H_0^2 \), see Table 4.4. The complementary subspace and kernel are given by

\[
G_0 = \eta_{00}^{(2)} \partial_2, \quad \mathcal{K}_0 = \iota_{00}^{(1)} \partial_1,
\]

where \( \eta_{00}^{(2)} = \mu_{00}^{(2)} \).

Second, we compute \( \text{ad} J(\cdot) \) over \( H_1^2 \). We did this already, see Table 4.3. Let
us opt for the first form,

$$G_1 = \eta_{10}^{(2)} y_1 \partial_2 + \eta_{01}^{(2)} y_2 \partial_2. \quad (4.86)$$

The parameters are defined as $\eta_{10}^{(2)} \equiv \tilde{\mu}_{10}^{(2)}$ and $\eta_{01}^{(2)} \equiv \tilde{\mu}_{01}^{(1)} + \tilde{\mu}_{01}^{(2)}$.

Note the tilde above the old perturbation parameters. This notation indicates that, for example, $\tilde{\mu}_{10}^{(2)}$ equals the sum of $\mu_{10}^{(2)}$ plus additional coefficients formed by other Lie-products. As mentioned in Section 4.5.1, parameters from $\mathcal{K}_0$ can be used to reduce the number of first-order perturbation parameters in $G_1$. To this end, the Lie product of $G_2$ with $\mathcal{K}_0$ needs to be computed. A suitable normal-form representation of the nonlinear part with respect to (4.86) is given by (3.70), repeated here,

$$G_k = n_{k0}^{(2)} y_1^k \partial_2 + n_{k1}^{(2)} y_1^{k-1} y_2 \partial_2, \quad k \geq 2. \quad (4.87)$$

Thus, in this specific case

$$[G_2, \mathcal{K}_0] = t_{00}^{(1)} (2n_{20}^{(2)} y_1 + n_{11}^{(2)} y_2) \partial_2. \quad (4.88)$$

Comparing with (4.86), we see that the parameter $t_{00}^{(1)}$ reaches both $\eta_{10}^{(2)}$ and $\eta_{01}^{(2)}$ in $G_1$. We denote this result as follows

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$\eta_{10}^{(2)}$</th>
<th>$\mu_{10}^{(2)}$</th>
<th>$2t_{00}^{(1)}n_{20}^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2$</td>
<td>$\eta_{01}^{(2)}$</td>
<td>$\mu_{01}^{(1)}$</td>
<td>$\mu_{01}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td>$t_{00}^{(1)}n_{11}^{(2)}$</td>
<td>$t_{00}^{(1)}n_{11}^{(2)}$</td>
<td>$t_{00}^{(1)}n_{11}^{(2)}$</td>
</tr>
</tbody>
</table>

The first column displays the name of the subspaces in Table 4.3b which become part of the complementary subspace $G_1$. The second column displays the parameters in those subspaces (and with it their location and multiplying monomials). The third and last column displays the result of the computation of the Lie product of the kernel times the relevant complementary subspace.

From the table, it can be concluded that, if the normal-form coefficients $n_{20}^{(2)} \neq 0$ or $n_{11}^{(2)} \neq 0$, $t_{00}^{(1)}$ can be set such as to remove either $\eta_{10}^{(2)}$ or $\eta_{01}^{(2)}$ in (4.86). We find that the resulting deformation up to second order reads

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \mu_{00}^{(2)} + \mu_{10}^{(2)} x_1 + n_{20}^{(2)} x_1^2 + n_{11}^{(2)} x_1 x_2. \quad (4.89)$$

Note that this vector field has two bifurcation parameters\textsuperscript{6}, corresponding to the co-dimension of the matrix $J$ in (4.82).

\textsuperscript{6}We prefer the notation $\mu$ versus $\eta$ in the final presentation of a vector field.
4.5.5 2-D example. Co-dimension 3 degeneracy

In the previous example, the higher-order perturbation-parameter vectors \( \eta_{m_0} (x) \), \( m \geq 2 \), can be simplified such that they lie in the complementary subspace \( G_m \) in (4.87). These perturbation parameters can be removed by a reparametrization. As such there is no reason to perform the actual computations.

This procedure fails if one or more normal-form coefficients vanish. The double zero eigenvalue has a co-dimension 3 degeneracy if, for example, the normal-form coefficient \( n_{20}^{(2)} \equiv 0 \). For this case, Dumortier et al. [DR90] [DRSZ91] studied the following polynomial vector field

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \mu_{00}^{(2)} + \mu_{10}^{(2)} x_1 + \mu_{01}^{(2)} x_2 + n_{11}^{(2)} x_1 x_2 + n_{30}^{(2)} x_1^3,
\end{align*}
\]  

(4.90)

with \( n_{30}^{(2)} = \pm 1 \) and \( n_{11}^{(2)} > 0 \). Because we already computed the two-parameter deformation (4.89), it will be easy to find the three-parameter deformation (4.90).

Because the normal-form coefficient \( n_{11}^{(2)} \neq 0 \), \( t_{00}^{(2)} \) from \( K_0 \) can be used to remove the parameter \( \mu_{01}^{(2)} \). Note, however, that the vector field (4.90) still contains this parameter. There is a good reason not to remove the parameter: Since \( n_{20}^{(2)} \) vanishes, \( \mu_{20}^{(2)} \) becomes important. This parameter cannot be removed with the Lie product \( [G_2, K_1] \). It is, however, possible to utilize \( K_0 \) to remove \( \mu_{20}^{(2)} \) instead of \( \mu_{01}^{(2)} \). For, provided that \( n_{30}^{(3)} \neq 0 \), the Lie product \( [G_3, K_0] \) reaches \( \mu_{20}^{(2)} \).

4.5.6 Miniversal deformations within a set of vector fields

The derivation of the bifurcation parameters through the systematic reduction of perturbation parameters via (4.81) is completely governed by Lie products. Thus, through application of the lemma’s used in the derivation of a normal form, the following property is easily proven.

**Proposition 4.12**

\(<\) The vector fields in \( V^n \), \( V^n_\sigma \) and \( V^n_S \) all have miniversal deformations within their set of vector fields, found using a generating function part of \( V^n \), \( V^n_\sigma \) and \( V^n_S \), respectively. \(>\)

We will illustrate the procedure with an example.
a) Mapping.

\[
\begin{array}{cc}
\partial_1 & \mu^{(1)}_{00} \\
\partial_2 & 0 \\
\end{array}
\]

b) Re-ordering into subspaces.

\[
\begin{array}{ccc}
C_1 & \mu^{(1)}_{00} & 1 \\
K_2 & t^{(1)}_{00} & 1 \\
\end{array}
\]

Table 4.5: \( \mu'_{0}(y) + \text{ad} \, J(t_{0}(y)) \), with \( \mu'_{0}(y), t_{0}(y; \mu) \in H_{0}^{2} \cap V_{\nu}' \).

### 4.5.7 2-D example. Topological normal form

In this example, we compute the unfolding of the normal form (4.71) within \( V_{\nu}' \). Hence, the objective is to simplify

\[
\begin{align*}
\dot{z}_1 &= z_2 + n^{(1)}_{20} z_1^2 + n^{(1)}_{30} z_1^3 + \mathcal{O}(\|z\|^2) + \delta \sum_{m \geq 0} \sum_{i+j=m} \mu^{(1)}_{i,j} z_i z_j, \\
\dot{z}_2 &= -n^{(1)}_{20} z_1 z_2 - \frac{3}{2} n^{(1)}_{30} z_1^2 z_2 + \mathcal{O}(\|z\|^2) + \delta \sum_{m \geq 0} \sum_{i+j=m} \mu^{(2)}_{i,j} z_i z_j,
\end{align*}
\]

with the near-identity transformation \( z = t(x; \mu, \varepsilon, \delta) \) in (4.76). Compute the adjoint of the matrix \( J \) over \( H_{0}^{2} \cap \mathcal{V}_{\nu}' \) and \( H_{1}^{2} \cap \mathcal{V}_{\nu}' \), see Table 4.5 and Table 4.6.

The complementary subspace and the kernel are

\[
G^{\nu}_0 = \eta^{(1)}_{00} \partial_1, \quad K^{\nu}_0 = t^{(1)}_{00} \partial_1,
\]

and

\[
G^{\nu}_1 = \eta^{(1)}_{10} (x_1 \partial_1 - \frac{1}{2} x_2 \partial_2), \quad K^{\nu}_1 = t^{(1)}_{01} x_2 \partial_1,
\]

where \( \eta^{(1)}_{00} \equiv \mu^{(1)}_{00} \), and \( \eta^{(1)}_{10} \equiv \mu^{(1)}_{10} \).

According to (4.81), parameters in \( K^{\nu}_0 \) can be used to simplify terms in \( G^{\nu}_1 \). Therefore, compute the Lie product of \( G^{\nu}_2 \) and \( K^{\nu}_0 \), i.e.,

\[
[n^{(1)}_{20} (x_1^2 \partial_1 - x_1 x_2 \partial_2), t^{(1)}_{00} \partial_1] = n^{(1)}_{20} t^{(1)}_{00} (2x_1 \partial_1 - x_2 \partial_2).
\]
4.6. Projection of the series onto a polynomial

\[
\begin{array}{|c|c|c|}
\hline
y_1 & y_2 \\
\hline
\partial_1 & \mu_{10}^{(1)} & \mu_{01}^{(1)} - \frac{3}{2} t_{10}^{(1)} \\
\partial_2 & 0 & -\frac{1}{2} \mu_{10}^{(1)} \\
\hline
\end{array}
\]

a) Mapping.

\[
\begin{array}{|c|c|c|c|}
\hline
\partial_1 & \partial_2 \\
\hline
C_1 & \mu_{10}^{(1)} & -\frac{1}{2} \mu_{10}^{(1)} & 1 \\
I_1 & \mu_{01}^{(1)} - \frac{3}{2} t_{10}^{(1)} & & \\
K_1 & t_{01}^{(1)} & 1 & \\
\hline
\end{array}
\]

b) Re-ordering into subspaces.

Table 4.6: \( \mu_1(y) + \text{ad} J(t_1(y)) \), with \( \mu_1(y), t_1(y; \mu) \in H_1^2 \cap V_\nu^2 \).

The resulting term enters only the subspace \( C_1 \) in Table 4.6. Provided that the normal-form coefficient \( n_{20}^{(1)} \neq 0, t_{00}^{(1)} \) can be used to remove the perturbation parameter \( \eta_{10}^{(1)} \).

In conclusion, the simplest possible representative in \( V_\nu^2 \) which has two repeated zero-eigenvalues up to third order reads

\[
\begin{align*}
\dot{x}_1 &= u_1(x; \mu) = \mu_{00} + x_2 + n_{20} x_1^2 + n_{30} x_1^3 + \ldots, \\
\dot{x}_2 &= u_2(x; \mu) = -n_{20} x_1 x_2 - \frac{3}{2} n_{30} x_1^2 x_2 + \ldots,
\end{align*}
\]

(with omission of the superscripts \( ^{(1)} \) and using \( \mu \) instead of \( \eta \)). The two bifurcation diagrams presented in [Bak88] depend on one parameter, viz. \( \mu_c \) in Fig. 2.6. The above discussion explains why the unfolding of the third-order saddle – with \( n_{20} \neq 0 \) – is co-dimension 1. Fig. 2.6 clearly shows that the tangent space in the direction of the parameter \( \mu_{00} \) at \( \mu = (\mu_{00}, \mu_{10}) = (0, 0) \) is transversal to the line \( \mu_c = 0 \). On the other hand, the tangent space in the direction of the parameter \( \mu_{10} \) at \( \mu = (0, 0) \) is tangent to the line \( \mu_c = 0 \).

4.6 Projection of the series onto a polynomial

The previous sections showed how to simplify the representation of a vector field. We did this by transformation to Jordan form, transformation to a normal
form, and reduction to a miniversal deformation. Despite our efforts, we still end up with a series representation which, of course, has an infinite number of coefficients. In order to systematically analyze the dynamical behavior it is a necessity to project the series to a polynomial (or k-jet). This is step G in Fig. 4.1.

For our application, for example in the case of vector fields in $V^2_\nu$ with a double zero eigenvalue, we would like to know whether it suffices to analyze the dynamical behavior of a polynomial vector field, such as

$$
\dot{x}_1 = u_1(x; \mu) = \mu_{00} + x_2 + n_{20}x_1^2 \\
\dot{x}_2 = u_2(x; \mu) = -n_{20}x_1x_2,
$$

instead of the series given in (4.92). To be specific, we want to know whether it represents all possible local flow patterns described by that series.

We are only interested in the topological properties of the series’ flow pattern in a small neighborhood near the point of expansion of that series. This point of view allowed us to simplify the series representation through several invertible coordinate transformations. Up till this point we only used diffeomorphisms to reduce the number of coefficients in the nonlinear part. Homeomorphisms, on the other hand, are a much broader class of transformations. They are needed to show, for example, why the phase portraits near two critical points classified as saddles with different values for $p$ and $q$ in the $p$-$q$ chart are topological equivalent, see Fig. 1.6.

From theorem by Thom, discussed in Chapter 3, we know that a series can be mapped onto its truncated series with the help of some appropriate homeomorphism, i.e., onto a k-jet, for high enough $k$. After mapping to a truncated series one should verify that the neglected terms really do not influence the dynamical behavior, for example using the theorems by Mather, also discussed in Chapter 3.

However, an important question that needs answering is: At which order can the series be truncated without loss of topological equivalence? The theorems by Thom and Mather can only tell whether a proposed truncation is correct or not. These theorems do not tell us at which order $k$ to truncate.

Moreover, $k$-jets are not always the most efficient polynomial vector-fields. Once again, consider the co-dimension 3 degeneracy from Section 4.5.5. Dumortier et al. [DR90] [DRSZ91] study the following topological normal form

$$
\dot{x} = y \\
\dot{y} = \alpha x^3 + \mu_2 x + \mu_1 + y(\beta x + \mu_3), \quad \alpha = \pm 1, \beta > 0.
$$

(4.93)

This polynomial vector field is the same as (4.90) but rewritten to demonstrate
that it is a Liénard equation, i.e.

\[ \dot{x} = y, \quad \dot{y} = -g(x) + y f(x). \]

The complementary subspace corresponding to the double zero eigenvalue is
given by \( G_k \) in (4.72), repeated here for convenience,

\[ G_k = n_{k_0}^{(2)} y_1 y_2 + n_{k-1}^{(2)} y_1^{k-1} y_2, \quad k \geq 2. \]

A co-dimension 3 singularity occurs if \( n_{20}^{(2)} = 0 \) and \( n_{30}^{(2)} = \alpha \neq 0 \). The 3-jet contains, among others, the third-order term \( n_{12}^{(2)} x^2 y_2 \). Yet this term is not present in the topological normal form (4.93).

As we showed in Section 3.2.8, it is possible to use a diffeomorphism to remove the third-order term \( n_{12}^{(2)} x^2 y_2 \) for the double zero case but only if \( n_{20}^{(2)} \neq 0 \). Clearly, this situation does not apply here. Still, the third-order terms can be neglected, see [DR90].

In three-dimensional space, which has our primary interest, the situation is even more of an issue. If, for example, it is a necessity to include third-order terms, the 3-jet typically introduces not one but several topological-unimportant normal-form terms\(^7\).

In our computations on topological normal forms for fluid-flow vector fields, we observed that, much to our own surprise, it is possible to identify the topological-unimportant normal-form terms \textit{a priori}. Below, we will state our findings.

### 4.6.1 Congruence: more of the same

Chapter 3 discussed the process of finding topological normal form for general two-dimensional vector fields in the case of a double zero eigenvalue. In the case that neither \( n_{20}^{(2)} \) nor \( n_{11}^{(2)} \) vanishes, the resulting polynomial vector field read

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \mu_1 + \mu_2 x_1 + \alpha x_1^2 + \beta x_1 x_2, \quad \alpha = \pm 1, \beta > 0.
\end{align*}
\]

see (3.149). Note that this vector field equals the 2-jet.

Consider the terms which were not taken into account from the complementary subspace (3.70). These are the monomials \( x_1^3, x_1^4, \) etc., and, on the other

\(^7\)Bruno [Bru89] has developed a very elegant method to remove topological unimportant terms using a so-called \textit{open Newton polygon}. However, his idea of a normal form is very different than ours. We have not found a way to reformulate his results such that they can be applied to fluid-flow vector fields.
hand, the monomials $x^2_1 x_2$, $x^3_2 x_2$, etc. In a way, $x^2_1$ incorporates the topological properties of the first type of monomials and $x_1 x_2$ takes care of the second type. In both cases, $x_1$ is the common monomial. The higher-order monomials are formed as $x^{k-2}_1 x^2_1$ and $x^{k-2}_1 x_1 x_2$, for $k \geq 3$.

We suspect that if, at a certain order, the complementary subspace produces only terms have increased index of some common monomial, that these terms need not be included in the topological normal form. This observation forms the indication for our truncations. We will now give a precise statement of our findings and give some evidence to back up our claim.

Consider a map $u(x) \in H^n_p$. Let $\prod_{i=1}^n x_i^{c_i}$, $c_i$ non-negative, $c = \sum_{i=1}^n c_i \geq 1$ be the polynomial common to all the terms in the vector components $u^{(\ell)}(x)$ of $u(x)$. Then we can write the $n$ vector components as

$$u^{(\ell)}(x) = u^{(\ell)} \prod_{i=1}^n x_i^{c_i}, \quad \ell = 1, \ldots, n,$$

(4.94)

with $\sum_{i=1}^n u_i^{(\ell)} = p$. Note that $u_i^{(\ell)} \geq c_i$ (because of the common polynomial).

**Definition 4.13**

A map $p(x) \in H^n_q$, $q > p$, is said to be congruent to the map $u(x)$ if $n$ non-negative integers $a_i$ can be found such that

$$p^{(\ell)}(x) = p^{(\ell)} \prod_{i=1}^n x_i^{u_i^{(\ell)} + a_i}, \quad \ell = 1, \ldots, n,$$

(4.95)

with $\sum_{i=1}^n a_i \geq 1$, $a_i = 0$ if $c_i = 0$. Furthermore, if $p^{(\ell)} = 0$, $u^{(\ell)} = 0$, and if $u^{(\ell)} \neq 0$, $p^{(\ell)} \neq 0$.

Note the independence with respect to the index $\ell$ of $a_i$ in (4.95).

For example, consider the complementary subspace $G_k$ of the double zero eigenvalue, see (3.70). Here, $y^k_1 \partial_2$, $k \geq 3$, is congruent to $y^2_1 \partial_2$ while $y^{k-1}_1 y_2 \partial_2$, $k \geq 3$, is congruent to $y_1 y_2 \partial_2$.

As another example, consider the case of two-dimensional fluid-flow vector fields with a double zero eigenvalue. The complementary subspace $G'_k$ within $\mathbb{V}^2_\ell$ consists of vectors

$$\left( x^k_1 - \frac{k}{2} x^{k-1}_1 x_2 \right)$$

see (4.73) Here, again, the common monomial is $x_1$. 
The complementary subspace $G_k^\nu$ within $V^2_\nu$ consists of vectors

$$\mathbf{u}(x) = \left( \begin{array}{c} x_1^2 \\ -x_1x_2 \end{array} \right), \text{ and } p_k(x) = \left( \begin{array}{c} x_1^k \\ -\frac{k}{2} x_1^{k-1}x_2 \end{array} \right), \text{ for } k \geq 3. \tag{4.96}$$

Here, $p_k(x), k \geq 3$, is congruent to $\mathbf{u}(x)$. The common polynomial in $u(x)$ is $x_1^1$, thus $p_1 = 1$ and $p_2 = 0$ in (4.95). Clearly, with $a_1 = k - 2, a_2 = 0$ in (4.95), the definition holds.

Our observation is the following:-

**Conjecture 4.14**

Congruent terms in the complementary subspace do not influence the topological classification of unfolded critical points.

We make no claim on about the influence of congruent terms on global bifurcations,

The above statement is quite bold and may not be true after all. However, the following provides some heuristic evidence of Conjecture 4.14.

Consider the $k$-jet

$$\dot{x} = u_k(x; \mu). \tag{4.97}$$

The analysis of the flow pattern of (4.97) starts with the computation of the location of its critical points. Since (4.97) represents an unfolding, we are only interested in critical points which return to the origin for $\mu \to 0$.

Let the parameter-space $V$ be an open subset such that $0 \in \partial V$ and for all $\mu \in V$, (4.97) has only hyperbolic critical points. Let $x_0(\mu)$ be such a critical point. Hence, the Jacobian of $u_k(x; \mu)$ at $x_0(\mu)$,

$$A_0(\mu) \equiv D_xu_k(x_0(\mu); \mu),$$

has only eigenvalues with nonzero real part.

Consider the effect of the introduction of a congruent term $p(x)$ in (4.97),

$$\dot{x} = v(x; \mu) = u_k(x; \mu) + p(x). \tag{4.98}$$

Since $p(x)$ is congruent, there exists a term in the vector $u_k(x; \mu)$, say $u(x)$, such that the requirements of Definition 4.13 hold. Thus, a polynomial exists, $\sum_{i=1}^n x_i^c_i$, with $c_i$ non-negative, $c = \sum_{i=1}^n p_i \geq 1$, which is common to all component functions in $u(x)$. 
In general, \( x_0(\mu) \) is not a critical point of \( u(x; \mu) \) but it should be a good first approximation in an iterative procedure. A better approximation of the critical point
\[
x_{i+1}(\mu) = x_i(\mu) + \delta x_i(\mu), \quad i \geq 0.
\]
can be found using the first-order Taylor polynomial
\[
u(x_i(\mu) + \delta x_i(\mu); \mu) = u(x_i(\mu); \mu) + D\nu(x_i(\mu); \mu) \delta x_i(\mu) + \ldots = 0.
\]
\[
\Rightarrow \quad \delta x_i(\mu) = -\left(D\nu(x_i(\mu); \mu)\right)^{-1} u(x_i(\mu); \mu). \quad (4.99)
\]
For \( i = 0 \), this recursive formula written out results in
\[
D\nu(x_0(\mu); \mu) = A_0(\mu) + Dp(x_0(\mu)).
\]
By the assumptions made, the first matrix on the right-hand side is hyperbolic.

A direct result of the definition is that each column in the Jacobian of \( p(x) \) is congruent to the column in the Jacobian of \( u(x) \) with the same combination of \( a_i \)'s. The component function of the sum \( u(x) + p(x) \) may be written as
\[
(u^\ell + p^\ell x_i^{a_i}) \prod_{k=1}^{n} x_k^{u_k^\ell}.
\] \quad (4.100)

By the definition, \( a_i = 0 \) if \( c_i = 0 \), meaning that no 'new' entries in the Jacobian of \( u(x; \mu) \) appear due to the introduction of \( p(x) \). For example, the partial derivative \( \partial_j \), with \( a_j > 0 \) (and thus \( u_j^\ell > c_j > 0 \)), may be written as
\[
(u^\ell + p^\ell (1 + a_j/u_j^\ell) x_i^{a_i}) u_j^\ell x_j^{u_j^\ell - 1} \prod_{k=1, k \neq j}^{n} x_k^{u_k^\ell}.
\]
For small enough \( x = x_i(\mu) \), the second term within the brackets is much smaller than the first. For this reason, the hyperbolicity of the Jacobian matrix cannot be influenced (much).

### 4.6.2 2D-example. Co-dimension 3 degeneracy (continued)

To conclude the discussion on congruence, once again consider two-dimensional vector fields with a double zero eigenvalue. In the case of a co-dimension 3 degeneracy, the topological normal form is given by (4.90). Conjecture 4.14 indeed predicts that the term \( n_{21}^{(2)} x_1^2 x_2 \partial_2 \) need not be included into the topological normal form because it is congruent to \( n_{11}^{(2)} x_1 x_2 \partial_2 \). For a proof on the removal of \( n_{12}^{(2)} x_1^2 x_2 \partial_2 \), see Dumortier and Rousseau [DR90].
4.7 2-D local flow patterns

To conclude the discussion in this chapter, let us analyse the dynamical behavior of our two-dimensional example of the double zero eigenvalue in $V_y^2$.

Proposition 4.11 showed that the complementary subspace of the adjoint operation of $J_{R0}$ on $H_k^2 \cap V_y^2$ is given by

$$G_y^\nu = n_{k0}(x_1^k \partial_1 - \frac{k}{2} x_1^{k-1} x_2 \partial_2)$$

(with omission of the superscript $(1)$). As stated before, the complementary subspaces $G_y^\nu$, $k \geq 3$ are all congruent to $G_y$. Therefore, the vector field of interest is given by

$$\begin{align*}
\dot{x}_1 &= u_1(x; \mu) = \mu_{00} + x_2 + n_{20} x_1^2, \\
\dot{x}_2 &= u_2(x; \mu) = -n_{20} x_1 x_2. 
\end{align*}$$

(4.101)

The number of cases to be studied can be reduced by the rescaling of coordinates $x_i \to f_i x_i$, $i \in \{1, 2\}$, and $t \to f_t t$, where $f_1$, $f_2$ and $f_t$ are nonzero. In order to distinguish flow separation from flow attachment, the scale factors of $x_2$ and the time parameter $t$ should be positive. We find that $n_{20} = \pm 1$ are the only important cases.

The critical points of (4.101) are

$$\begin{align*}
x_S(\mu) &= \left( \pm \sqrt{-\mu_{00}/n_{20}}, 0 \right), \\
x_C(\mu) &= (0, -\mu_{00}).
\end{align*}$$

(4.102)

Note that – unlike in the case of the vector field (2.92) and its critical points (2.94) and (2.95) – no critical points are found which do not return to the origin for $\mu \to 0$.

For $\mu_{00}/n_{20} > 0$, $x_S(\mu)$ does not describe any critical points. For $\mu_{00} = 0$, the degeneracy returns, thus, let $\mu_{00}/n_{20} < 0$. Transformation to $x_S(\mu)$ via $x = x_S(\mu) + \xi$, produces the following linearization,

$$\dot{\xi} = \begin{pmatrix} \pm 2 \sqrt{-\mu_{00}/n_{20}} & 0 \\ 0 & \mp \sqrt{-\mu_{00}/n_{20}} \end{pmatrix} \xi.$$  

(4.103)

The trace of the linearized matrix equals $p = \pm \sqrt{-\mu_{00}/n_{20}}$, and its determinant is given by $q = 2\mu_{00}/n_{20} < 0$, notations conform (3.25). Since $q$ is negative, the flow pattern near $x_S(\mu)$ is a saddle.

---

8If we require $f_2$ and $f_t$ to have the same sign, we also have to study the dynamical behavior of the 'flow' under the surface boundary.
On the other hand, transformation to \( x_C(\mu) \) via \( x = x_C(\mu) + \xi \), leads to

\[
\dot{\xi} = \begin{pmatrix}
0 & 1 \\
\mu_{00} n_{20} & 0
\end{pmatrix} \xi.
\]  \hspace{1cm} (4.104)

Only the case \( \mu_{00} < 0 \) (above the boundary surface) is of interest. The trace of the linearized matrix equals \( p = 0 \) and its determinant is given by \( q = -\mu_{00} n_{20} \). For \( n_{20} > 0 \), the flow flow pattern near \( x_C(\mu) \) is a center, whilst for \( n_{20} < 0 \), it is a saddle. Naturally, the center-case requires further investigation, which, however, we will omit here.

Calculation of the eigenvalues for the two linearizations, (4.103) and (4.104), shows an agreement with the findings of Remark 4.10. In the case (4.103), the two eigenvalues are \( \lambda_1 = \pm 2\sqrt{-\mu_{00} n_{20}} \) and \( \lambda_2 = \mp \sqrt{-\mu_{00} n_{20}} \). These two eigenvalues indeed satisfy the relation \( \lambda_1 + 2\lambda_2 = 0 \) as any representative in \( V^2 \) is supposed to. In the other case, (4.104), the fact that always \( p = 0 \) is enough proof that \( \lambda_1 + \lambda_2 = 0 \), which corresponds to \( V^2 \), as expected.

### 4.8 Discussion

This chapter set up a specially tailored bifurcation analysis for vector fields in the set \( V^n \). The vector fields in this set describe the fluid flow near a point on the boundary surface in two-dimensional or three-dimensional space, \( n \) being 2 or 3, respectively. The analysis was set up such that every step towards the topological normal form resulted in a vector field from \( V^n \).

As a result, their unfolded critical points have exactly the expected topological properties given their position with respect to the no-slip boundary surface or to the plane of symmetry. For example, every unfolded three-dimensional boundary critical point satisfies the eigenvalue relation

\[
\lambda_3 = -\frac{1}{2}(\lambda_1 + \lambda_2).
\]  \hspace{1cm} (4.105)

This means that the local flow pattern near such a critical point falls within the topology sketched in Fig. 1.5.

The techniques developed in this chapter make it possible to keep track of relations between the coefficients in the Taylor expansion of the vector field in the original Cartesian coordinate system with the curved boundary surface all the way to the topological normal form. The next chapter will use these techniques to compute topological normal forms for vector fields from the set \( V^3 \) of co-dimension 1 and co-dimension 2 in that set.

One important issue still open will be addressed in the next chapter; the validity of the ‘exclusion’ of the vorticity transport equation from the set of vector
fields $\mathcal{V}_\nu^n$. For, the set $\mathcal{V}_\nu^n$ represents vector fields which obey the continuity equation and the no-slip condition. The vector fields which in addition obey the vorticity transport equation form a subset in this set. The question to be answered is why we did not look for a topological normal form in that subset.
Chapter 5

Topological Normal Forms for Fluid-flow Vector Fields

Jonathan is zo'n jongen die almaar verhaaltjes in zijn hoofd krijgt. Daar kan hij niks aan doen. Dat gaat vanzelf. Hij kan ze niet tegenhouden. Hij kan ze ook niet zelf bedenken. Ze groeien aan een verhalenboompje. En als er een verhaaltje rijp is valt het eraf. Zo gaat dat ongeveer. —Guus Kuijer, De tranen knallen uit mijn kop

Introduction

In the previous chapter, we set up a new approach to find topological normal forms for vector fields describing fluid flow. A topological normal form is needed to study the dynamical behavior of local flow patterns about a nonhyperbolic critical point. This in turn enables a classification of the possible fundamental local flow patterns with possibly more than one critical point.

To make the classification systematic, it is desirable to first study the local flow patterns with a low degree of degeneracy. The idea is that the local flow patterns found this way are the most likely to occur. In a nonhyperbolic critical point, one or more of the eigenvalues in the Jacobian matrix has a zero real part. Appendix A shows that this situation can occur in various ways. In the case of a nonhyperbolic critical point, the local flow pattern about a critical point is said to be degenerate. Further degeneracies occur in the case of vanishing normal-form coefficients.

This chapter constructs topological normal forms within $V^3$ for four types of degeneracies. We confine the discussion to nonhyperbolic critical points of
co-dimension 1 and of co-dimension 2.

At the end of Chapter 2, we raised the question if we should concentrate the subset in \( V^3_\nu \) of those vector fields which in addition satisfy the vorticity transport equation. We have postponed answering this question because our argument requires example topological normal forms. The argument will be given in the concluding section of this chapter.

## 5.1 Main results

The propositions below list a number of topological normal forms. For each topological normal form it is indicated in what way removing the so-called congruent terms and rescaling of coordinates leads to a further reduction of the number of cases to be studied.

**Proposition 5.1**

\(<\) Within \( V^3_\nu \), a topological normal form for perturbations of vector fields with a critical point having one zero eigenvalue –with a corresponding eigenvector tangent to the boundary surface– is given by

\[
\begin{align*}
\dot{x}_1 &= \mu_{000}^{(1)} + n_{200}^{(1)} x_1^2 \\
\dot{x}_2 &= \lambda x_2 + n_{110}^{(3)} x_1 x_2 \\
\dot{x}_3 &= -\frac{1}{2} \lambda x_3 - \frac{1}{5} (n_{200}^{(1)} + n_{110}^{(2)}) x_1 x_3
\end{align*}
\]

To find the fundamental local flow patterns, it is sufficient to analyze the case \( n_{200}^{(1)} = 1 \) and \( n_{110}^{(2)} = 0 \). \( \triangleright \)

**Proposition 5.2**

\(<\) Define the vectors

\[
\begin{align*}
r_{\vartheta} &= \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}, \\
r_{r} &= \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}.
\end{align*}
\]

(5.1)

Within \( V^3_\nu \), a topological normal form for perturbations of vector fields that have rotational symmetry with respect to an axis transversal to the boundary surface and with a critical point having a Jacobian matrix with three eigenvalues with a zero real part is given by

\[
\begin{align*}
\dot{x} &= \omega r_{\vartheta} + \mu (r_{\vartheta} - x_3 \vartheta_3) + g_{10} x_3 (r_{\vartheta} - \frac{1}{2} x_3 \vartheta_3) + f_1 x_3 r_{\vartheta} \\
&\quad + g_{20} x_3^2 (r_{\vartheta} - \frac{1}{2} x_3 \vartheta_3) + g_{01} r^2 (r_{\vartheta} - 2 x_3 \vartheta_3) + f_{01} r^3 \vartheta_\vartheta.
\end{align*}
\]

(5.2)
where \( r^2 = x_1^2 + x_2^2 \). The fundamental local flow patterns of this vector field are most easily found by an analysis of

\[
\dot{r} = r(\mu + g_{10} z + g_{01} r^2), \\
\dot{z} = -z(\mu + \frac{2}{3}g_{10} z + 2g_{01} r^2).
\]

It suffices to study the cases \( g_{10} = \pm 1, g_{01} = \pm 1 \).  

**Proposition 5.3**

\( \triangleleft \) Within \( \mathcal{V}_\mu^3 \), a topological normal form for perturbations of vector fields with a critical point having three zero eigenvalues and a one-dimensional eigenvector space is given by

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \mu_1 + \mu_2 x_1 + x_3 + n^{(2)}_{200} x_1^2 + n^{(2)}_{110} x_1 x_2 + n^{(2)}_{300} x_1 x_2 + n^{(2)}_{210} x_1^2 x_2 \\
\dot{x}_3 &= -\frac{1}{3} n^{(2)}_{110} x_1 x_3 - \frac{1}{2} n^{(2)}_{210} x_1^2 x_3
\end{align*}
\]

It is sufficient to analyze the cases \( n^{(2)}_{200} = \pm 1, n^{(2)}_{110} = 1, \) and \( n^{(2)}_{300} = n^{(2)}_{210} = 0 \).  

**Proposition 5.4**

\( \triangleleft \) Within \( \mathcal{V}_\mu^3 \), a topological normal form for perturbations of vector fields with a critical point having three zero eigenvalues and a two-dimensional eigenvector space is given by

\[
\begin{align*}
\dot{x}_1 &= n^{(1)}_{110} x_1 x_2 + n^{(1)}_{120} x_1 x_2^2 \\
\dot{x}_2 &= \mu_1 + \mu_2 x_2 + x_3 + n^{(2)}_{200} x_2^2 + n^{(2)}_{020} x_2^2 + n^{(2)}_{210} x_2^2 x_2 + n^{(2)}_{030} x_2^2 \\
\dot{x}_3 &= -\frac{1}{2} \mu_2 x_3 - \frac{1}{2} (n^{(1)}_{110} + 2n^{(2)}_{020}) x_2 x_3 - \frac{1}{2} n^{(2)}_{210} x_1 x_3 - \frac{1}{2} (n^{(1)}_{120} + 3n^{(2)}_{030}) x_2^2 x_3
\end{align*}
\]

It is sufficient to analyze the cases \( n^{(2)}_{200} = \pm 1, n^{(2)}_{020} = \pm 1, \) and \( n^{(2)}_{210} = \pm 1, \) with \( n^{(1)}_{120} = n^{(2)}_{030} = 0 \).  

**Proposition 5.5**

\( \triangleleft \) The co-dimension within \( \mathcal{V}_\mu^3 \) of the Jordan normal forms \( J_{3A}, J_{3B} \) and \( J_{3D} \) listed in Table A.2 is 3, 4 and 6, respectively.  

The next five sections present the proofs of the above propositions.
5.2 Single zero eigenvalue

De Winkel [Win89] was the first to present bifurcation diagrams related to three-dimensional local flow patterns about a nonhyperbolic critical point with a single zero eigenvalue. Appendix A shows that there are three possible Jordan normal forms with one vanishing eigenvalue. Using the definition of the eigenvalues as given by (1.25), either $\lambda_1 = 0$, $\lambda_2 = 0$, or $\lambda_3 = 0$. In the first two cases, the corresponding (normalized) eigenvector has a direction tangential to the boundary surface. In the third case, the corresponding (normalized) eigenvector has a direction transversal to the boundary surface.

In what follows, we assume a critical point in a vector field from $\mathcal{V}^3_\nu$ such that the Jacobian matrix has a single zero eigenvalue and the corresponding (normalized) eigenvector has a direction tangential to the boundary surface. Using Lemma 4.3 and Table A.2 it is easy to show that there exists a vector field in $\mathcal{V}^3_\nu$ that has a Jacobian matrix identical to the Jordan normal form

$$J_{1B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\frac{1}{2} \lambda \end{pmatrix}.$$  \hspace{1cm} (5.3)

and which is $C^r$-equivalent to our vector field. Hence, without loss of generality, we can study the vector fields

$$\dot{x} = u(x; \mu), \quad u(x; \mu) \in \mathcal{V}^3_\nu, \quad \frac{\partial u(0; 0)}{\partial x} = J_{1B}.$$  

This section will calculate a topological normal form for such vector fields using two different approaches. First, we will follow De Winkel [Win89] and use center manifold theory. Second, we will compute a topological normal form with $\mathcal{V}^3_\nu$ using the machinery developed in the previous chapter. Afterwards, we will discuss the resulting vector fields.

Center manifold approach

De Winkel [Win89] computed an approximation of the center manifolds for the vector field,

$$\begin{align*}
\dot{x}_1 &= f_1(x), \\
\dot{x}_2 &= \lambda x_2 + f_2(x), \\
\dot{x}_3 &= -\frac{1}{2} \lambda x_3 + f_3(x),
\end{align*}$$  \hspace{1cm} (5.4)

with each $f_i(x) = O(\|x\|^2)$, see also Theorem 3.8. Notice that perturbation parameters are not (yet) included. Computation of the center manifold runs as follows.
5.2. Single zero eigenvalue

Figure 5.1: Arrangement of invariant manifolds for the vector field (5.4) in the
case $\lambda < 0$. In the case $\lambda > 0$, $W^u$ and $W^s$ are reversed.

Center manifolds are tangent to the center subspace $E^c = \text{span}\{(1,0,0)^t\}$. Thus, we can represent the center manifolds locally as a curve

$$W^c = \{ \bar{x} = (x_1, x_2, x_3)^t \mid x_2 = h_2(x_1), \ x_3 = h_3(x_1) \}$$

where $h_2$ and $h_3$ are defined in some neighborhood of the origin, and $h_i(0) = Dh_i(0) = 0, \ i = \{2,3\}$. Consider the projection of the vector field on $W^c$ onto $E^c$,

$$\dot{x}_1 = f_1(x_1, h_2(x_1), h_3(x_1)).$$

(5.5)

The solutions of (5.5) will provide a good approximation of the flow of $\dot{x}_1 = f_1(\bar{x})$ restricted to $W^c$.

Let $h_n^{(j)}$, $n \geq 2$, be the coefficients in the Taylor series expansion of the functions $h_j(x_1)$, $j \in \{2,3\}$ about $x_1 = 0$. Substitution of these expansions in (5.4) and equating terms with equal monomial up to $O(|x_1|^2)$ results in $h_2^{(2)} = -f_2^{(2)}/\lambda$ and $h_2^{(3)} = 0$. Here, $f_s^{(i)}$ denote the coefficients in the Taylor series expansion of the functions $f_s(\bar{x})$, $s \in \{1,2,3\}$ about $\bar{x} = 0$. Actually, the computations indicate that $h_3(x_1) \equiv 0$. The result is that $W_c$ consists of skin-friction lines.

A normal form of the vector field on the center manifolds is given by

$$\dot{x}_1 = n x_1^2, \quad \bar{x} \in W^c,$$

(5.6)

where $n = f_2^{(1)}$. In fact, if $f_2^{(1)} \neq 0$, one needs only study the case $n = 1$: The coordinate scaling $x_1 \to x_1/f_2^{(1)}$ applied to (5.6) results in $n \to 1$. If $f_2^{(1)} < 0$, this coordinate scaling effectively changes the stability character, for example, it changes the direction of the arrows on $W^c$. Fig. 5.2 sketches the local flow pattern of the vector field (5.4) in the case $f_2^{(1)} > 0$. In that figure, the invariant manifolds are made to coincide onto their corresponding invariant subspace. As a result, the local flow patterns have mirror symmetry. In general, all manifolds can be curved.
Figure 5.2: Local flow pattern of the vector field (5.4) in the case \( f^{(1)}_{200} > 0 \). (Note: only one half is shown.)

Figure 5.3: Bifurcation of vector field (5.4) with \( \lambda < 0 \) and \( f^{(2)}_{200} > 0 \). (Note: only one half is shown.)

The following one-dimensional vector field is a miniversal deformation of the vector field (5.6);

\[
\dot{x}_1 = \mu + x_1^2, \quad x_1 \in W^c,
\]

where \( \mu \) is a small parameter. The vector field describes the dynamical behavior on the center manifold.

The vector field (5.7) has two critical points at \( x_1 = \pm \sqrt{-\mu} \) for \( \mu < 0 \) and none for \( \mu > 0 \). Obviously, the degeneracy re-appears at \( \mu = 0 \). If we superimpose the unstable and stable manifolds, the dynamical behavior of the vector field (5.4) is as sketched in Fig. 5.3.

The dynamical behavior of the vector field (5.7) is the well-known saddle-node bifurcation. In the terminology of Strogatz [Str94], the remnant of the annihilation of a saddle and a node produces a ghost (or bottleneck) region. It
is called a ghost region because streamlines in the neighborhood of the origin are still influenced by a critical point that is no longer present. In the boundary surface, skin-friction lines are sucked into the region surrounding the center-manifold. Above the boundary surface streamlines are bent sharply upwards when they leave that region. Physically, the resulting local flow pattern for \( \mu > 0 \) resembles free-shear layer separation as shown in Fig. 1.11. We find that saddle-node bifurcation provides a possible explanation as to what generates this type of flow pattern.

In general, the three-dimensional local flow pattern near the unfolded critical points on the center manifold needs to be constructed by adding the stable and unstable manifolds. However, it is necessary for a classification within the topology discussed in Chapter 1 to have an expression of the linearized vector field about the unfolded critical points on the center manifolds.

To analyze the three-dimensional local flow pattern of the vector field (5.4), De Winkel [Win89] attempted the following. The two-dimensional vector field

\[
\begin{align*}
\dot{x}_2 &= \lambda x_2 \\
\dot{x}_3 &= -\frac{1}{2} \lambda x_3
\end{align*}
\]  

(5.8)

describes the flow on the stable and unstable subspace, and therewith on the stable and unstable manifold. The one-dimensional vector field (5.7), that resulted after projection of the center manifold in the vector field (5.4) on to the center subspace, describes the flow on the center manifold.

The combination of the vector fields (5.8) and (5.7) reads

\[
\begin{align*}
\dot{x}_1 &= \mu + x_1^2 \\
\dot{x}_2 &= \lambda x_2 \\
\dot{x}_3 &= -\frac{1}{2} \lambda x_3.
\end{align*}
\]  

(5.9)

If \( \mu < 0 \), the vector field (5.9) has two unfolded critical points: \( (\pm \sqrt{-\mu}, 0, 0)^t \).

The linearized vector fields around them read

\[
\dot{\xi} = A \xi,
\]

\[
A \equiv \left( \begin{array}{ccc}
\pm 2\sqrt{-\mu} & 0 & 0 \\
0 & \lambda & 0 \\
0 & 0 & -\frac{1}{2} \lambda
\end{array} \right).
\]

Clearly the coefficient relation \( A_{11} + A_{22} + 2A_{33} = 0 \) does not hold. Thus, the unfolded critical points fall outside the known topology discussed in Chapter 1.

**Topological normal form within \( \mathbb{V}_3^\nu \)**

Notice that the vector field (5.9) does not belong to \( \mathbb{V}_3^\nu \). It is for this reason that the unfolded critical points fall outside the known topology. In this section, we
shall use the machinery developed in the previous chapter to find a topological normal form within $\mathcal{V}_r^3$ for the vector field (5.4).

The Jacobian matrix of the vector field (5.4) is already a Jordan normal form. Thus, the first step is to transform the nonlinear part of the vector field (5.4) to a normal form within $\mathcal{V}_r^3$. To this end, we need to substitute a near-identity transformation in the vector field. As explained in Chapter 4, we only need to look at the complementary subspace $\mathcal{G}_m^r$ and the kernel $\mathcal{K}_m^r$ of the adjoint operator of the Jacobian matrix on the homogeneous polynomials $t_m(x) \in \mathcal{V}_r^3 \cap H_m^3$ from the generator of the transformation. We shall combine this step with the computation of the versal deformation.

In the case of $J_{1B}$, the complementary subspace $\mathcal{G}_m$ and the kernel $\mathcal{K}_m$ are easy to identify due to the special form of its image;

$$\text{ad} \ J_{1B}(t_m(x)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -\frac{1}{2} \lambda \end{bmatrix} x, \ t_m(x) = \sum_{i+j+k=m} \lambda \begin{pmatrix} (-j + \frac{1}{2} k) s_i^{(1)} \ j \ k \\ (1 - j + \frac{1}{2} k) s_i^{(2)} \ j \ k \\ (-\frac{1}{2} - j + \frac{1}{2} k) s_i^{(3)} \ j \ k \end{pmatrix} x_i^1 x_i^2 x_i^3. \quad (5.10)$$

Here, we have denoted $t_m(x) \in H_m^3$ as before as

$$t_m(x) = \sum_{i+j+k=m} \begin{pmatrix} s_i^{(1)} \ j \ k \\ s_i^{(2)} \ j \ k \\ s_i^{(3)} \ j \ k \end{pmatrix} x_i^1 x_i^2 x_i^3. \quad (5.11)$$

As can be seen in (5.10), no mixing of the coefficients of $t_m(x)$ takes place in (5.10) across the rows nor in the rows. As a result, $\mathcal{G}_m = \mathcal{K}_m$. We find that the complementary subspaces are $\mathcal{G}_m^r = n_{000}^{(1)} \mathcal{G}_1$, and for $m \geq 1$,

$$\mathcal{G}_m^r = \sum_{i+j+k=m} \begin{pmatrix} n_i^{(1)} \ j \ k & n_i^{(2)} \ j \ k \\ -n_i^{(1)} \ j \ k & -n_i^{(2)} \ j \ k \\ n_i^{(3)} \ j \ k & n_i^{(3)} \ j \ k \end{pmatrix} x_i^1 x_i^2 x_i^3, \quad (5.12)$$

where we substituted the coefficient relations (2.62).

We assume that $n_{200}^{(1)} \equiv f_{200}^{(1)} \neq 0$ and $n_{110}^{(2)} \equiv f_{110}^{(2)} \neq 0$. Then, the complementary subspace $\mathcal{G}_3^r$ only introduces congruent terms. On the other hand, $\mathcal{G}_3^r$ does produce non-congruent terms but it is not difficult to show that these
5.2. Single zero eigenvalue

terms also do not influence the topological classification of the unfolded critical points. We therefore truncate our normal form at second order.

From Section 4.5 we know that bifurcation parameters are part of the complementary subspace $G'_0$ and $G'_1$. This means that

\[
\begin{align*}
\dot{x}_1 &= \mu^{(1)}_{000} + \mu^{(1)}_{100} x_1 + n^{(1)}_{200} x_1^2 \\
\dot{x}_2 &= (\mu^{(2)}_{010} + \lambda) x_2 + n^{(2)}_{110} x_1 x_2 \\
\dot{x}_3 &= -\frac{1}{2}(\mu^{(1)}_{100} + \mu^{(2)}_{010} + \lambda) x_3 - (n^{(1)}_{200} + \frac{1}{2} n^{(2)}_{110}) x_1 x_3
\end{align*}
\tag{5.13}
\]

is an initial representation of a versal deformation of the 2-jet of $J_{1B}$. We still need to incorporate the interaction of the normal form and the free-parameters in the kernel to get a reduction of this vector field onto a miniversal deformation.

In (5.13), the vector

\[
\begin{pmatrix}
\mu^{(1)}_{100} x_1 \\
\mu^{(2)}_{010} x_2 \\
-\frac{1}{2}(\mu^{(1)}_{100} + \mu^{(2)}_{010}) x_3
\end{pmatrix} = \mu^{(1)}_{100} \begin{pmatrix} x_1 \\ 0 \\ -\frac{1}{2} x_3 \end{pmatrix} + \mu^{(2)}_{010} \begin{pmatrix} 0 \\ x_2 \\ -\frac{1}{2} x_3 \end{pmatrix}.
\tag{5.14}
\]

could not be removed because it is an element of $G'_0$. The vector multiplied by $\mu^{(1)}_{100}$ can be removed because it is contained in the image of the Lie product of the kernel $K'_0 = \text{span}\{t^{(1)}_{000} \partial_1\}$ and the complementary subspace $G'_2$. The perturbation parameter $\mu^{(2)}_{010}$ multiplies the same vector as the eigenvalue $\lambda$ does. Since $\mu^{(2)}_{010}$ is small, it cannot change the sign of $\mu^{(2)}_{010} + \lambda$. Therefore, the second vector can be ‘removed’ by a reparametrization of the eigenvalue; $\lambda + \mu^{(2)}_{010} \rightarrow \lambda$.

Combining results, the vector field (5.4) has co-dimension 1, and a miniversal deformation is given by

\[
\begin{align*}
\dot{x}_1 &= \mu^{(1)}_{000} + n^{(1)}_{200} x_1^2 \\
\dot{x}_2 &= \lambda x_2 + n^{(2)}_{110} x_1 x_2 \\
\dot{x}_3 &= -\frac{1}{2} \lambda x_3 - (n^{(1)}_{200} + \frac{1}{2} n^{(2)}_{110}) x_1 x_3
\end{align*}
\tag{5.15}
\]

The entry $n^{(2)}_{110}(x_1 x_2 \partial_2 - \frac{1}{2} x_1 x_3 \partial_3)$ in (5.15) cannot influence the classification of the unfolded critical points. We demonstrate this using the following argument.

The vector field (5.15) has two unfolded critical points;

\[
(\pm \sqrt{-\mu^{(1)}_{000} / n^{(1)}_{200}}, 0, 0)^t.
\]
The linearized vector field about the unfolded critical points can be written as

$$\dot{\xi} = A \xi, \quad A = J_{1B} \pm \sqrt{-\mu_{000}/n_{200}^{(1)}} \begin{pmatrix} 2n_{200}^{(1)} & 0 & 0 \\ 0 & n_{110}^{(2)} & 0 \\ 0 & 0 & -(n_{200}^{(1)} + \frac{1}{2}n_{110}^{(2)}) \end{pmatrix}.$$  

Notice that the matrix $A$ obeys the coefficient relation $A_{11} + A_{22} + 2A_{33} = 0$. The eigenvalues of the matrix $A$ can be read off of the diagonal,

$$\lambda_1 = \pm 2n_{200}^{(1)} \sqrt{-\mu_{000}/n_{200}^{(1)}}, \quad \lambda_2 = \lambda \pm n_{110}^{(2)} \sqrt{-\mu_{000}/n_{200}^{(1)}},$$

$$\lambda_3 = -\frac{1}{2} \lambda \mp \frac{1}{2} (n_{200}^{(1)} + n_{110}^{(2)}) \sqrt{-\mu_{000}/n_{200}^{(1)}}.$$  

Regardless of the value of the (fixed) coefficient $n_{110}^{(2)}$, small $\mu_{000}$ cannot change the sign of the eigenvalues $\lambda_2$ and $\lambda_3$; their sign equals the sign of $\lambda$.

In conclusion, the vector field

$$\begin{align*}
\dot{x}_1 &= \mu_{000}^{(1)} + n_{200}^{(1)} x_1^2, \\
\dot{x}_2 &= \lambda x_2, \\
\dot{x}_3 &= -\frac{1}{2} \lambda x_3 n_{200}^{(1)} x_1 x_3
\end{align*} \quad (5.16)$$

models the dynamical behavior of versal deformations of vector fields in $\mathcal{V}_\nu^3$ with a Jacobian matrix similar to $J_{1B}$.

Finally, note that the precise value of $n_{200}^{(1)}$ is not of importance:- Substitution of a rescaling of coordinates, $x_i \rightarrow f_i x_i$ and $t \rightarrow f_i t$, with $f_1 = 1/n_{200}^{(1)}$, $f_2 = f_3 = f_t = 1$, shows that a nonzero $n_{200}^{(1)}$ can be replaced by 1. Herewith, Proposition 5.1 is proven.

**Discussion**

One cannot help but notice that finding a normal form within $\mathcal{V}_\nu^3$ requires more effort than finding its one-dimensional co-dimension 1 center manifold. Furthermore, we could easily recognize the one-parameter miniversal deformation whereas we were forced to perform computations to find an appropriate one-parameter deformation within $\mathcal{V}_\nu^3$. Moreover, some further work was needed to show that certain terms were irrelevant for the topological classification of the local flow patterns.

The computational overhead involved in finding a topological normal form within $\mathcal{V}_\nu^3$ results from the fact that we are verifying, to some extent, the proofs
of center manifold theory and general-purpose miniversal deformations. Our reason to discuss the case of one vanishing eigenvalue is that it demonstrates that the machinery developed to find topological normal forms within $\mathcal{V}_\nu^3$ works:-

The topological normal form found for this case correctly has a one-parameter deformation. Also, it describes the topologically correct local flow patterns.

The true strength of using topological normal forms within $\mathcal{V}_\nu^3$ becomes apparent when dealing with vector fields which have nilpotent Jacobian matrices. Such vector fields are the topic of the next sections.

5.3 Rotational invariance

We assume a general vector field in $\mathcal{V}_\nu^3$ that has a critical point in the origin, and has a Jacobian matrix in that point with a pair of complex conjugated eigenvalues with zero real part and a zero eigenvalue. According to Lemma 4.3 and Lemma 4.4, a linear coordinate transformation exists which maps this vector field to another, $C^r$-equivalent, $r$ arbitrary, vector field in $\mathcal{V}_\nu^3$ that has the following Jacobian matrix,

$$J_{4B} = \begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (5.17)$$

where $\omega > 0$ (the case $\omega < 0$ is identical apart from a reversal in the time direction), also see Appendix A. The matrix $J_{4B}$ is a Jordan normal form, and its eigenvalues are $i\omega$, $-i\omega$, and 0. Without loss of generality, we can study the vector fields

$$\dot{x} = u(x; \mu), \quad u(x; \mu) \in \mathcal{V}_\nu^3, \quad \frac{\partial u(0; 0)}{\partial x} = J_{4B}.$$

This section will calculate a topological normal form for such vector fields.

Normal form

The first step towards Proposition 5.2 is the transformation of the nonlinear part of the vector field $u(x; \mu)$ to a normal form within $\mathcal{V}_\nu^3$. To this end, we need to substitute a near-identity transformation in the vector field. As explained in Chapter 4, we only need to look at the complementary subspace $\mathcal{G}_m^\nu$ and the kernel $\mathcal{K}_m^\nu$ of the adjoint operator of the Jacobian matrix on the homogeneous

1All eigenvalues of a nilpotent matrix have a zero real part.
polynomials $t_m(x) \in \mathbb{V}_n^3 \cap H_m^3$ from the generator of the transformation. In the case of the matrix $J_4B$, we obtain

$$\text{ad} J_4B(t_m(x)) = \sum_{i+j+k=m} \omega \left( \begin{array}{c} -t_{i,j}^{(2)} x_1^i x_2^j x_3^k + t_{i,j}^{(1)} (i x_1^{i-1} x_2^{j+1} x_3^k - j x_1^{i+1} x_2^j x_3^{k-1}) \\ t_{i,j}^{(1)} x_1^i x_2^j x_3^k + t_{i,j}^{(2)} (i x_1^{i-1} x_2^{j+1} x_3^k - j x_1^{i+1} x_2^j x_3^{k-1}) \\ t_{i,j}^{(3)} (i x_1^{i-1} x_2^{j+1} x_3^k - j x_1^{i+1} x_2^j x_3^{k-1}) \end{array} \right),$$

(5.18)

for general homogeneous polynomials $t_m(x) \in H_m^3$, denoted as in (5.11).

The rotational invariance of the matrix $J_4B$ suggests to find a normal form with the same symmetry. Using general $t_m(x) \in H_m^3$, Takens [Tak74] indeed showed that both the complementary subspace $G_m$ and the kernel $K_m$ of the adjoint operation $\text{ad} J_4B(\cdot)$ consist of terms of the form

$$f(r^2, x_3) r \partial_\theta, \quad g(r^2, x_3) r \partial_r, \quad \text{and} \quad h(r^2, x_3) \partial_\theta,$$

(5.19)

with $f(0,0) = g(0,0) = h(0,0) = 0$, and $(\partial_3 h)(0,0) = 0$, and where $r^2 = x_1^2 + x_2^2$.

The vectors $r \partial_r$ and $r \partial_\theta$ are defined in (5.1).

This means that the normal-form entries of degree $m$ can be written as

$$\sum_{i+1+2j=m} \left( \begin{array}{c} (f_{i,j} x_2 + g_{i,j} x_1) x_3^j (x_1^2 + x_2^2)^i \\ (-f_{i,j} x_1 + g_{i,j} x_2) x_3^j (x_1^2 + x_2^2)^i \\ h_{i+1,j} x_3^{i+1} (x_1^2 + x_2^2)^j \end{array} \right).$$

(5.20)

The indices $i$ and $j$ correspond to the powers of $x_3$ and $x_1^2 + x_2^2$, respectively. The vector (5.20) is part of $\mathbb{V}_n^3$ if

$$\partial_1 ((f_{i,j} x_2 + g_{i,j} x_1) x_3^{i+1} (x_1^2 + x_2^2)^j) + \partial_2 ((-f_{i,j} x_1 + g_{i,j} x_2) x_3^{i+1} (x_1^2 + x_2^2)^j) + \partial_3 (h_{i+1,j} x_3^{i+2} (x_1^2 + x_2^2)^j) = 0,$$

which is equivalent to the requirement

$$h_{i+1,j} = -\frac{2j+2}{i+2} g_{i,j},$$

(5.21)

As can be seen, no restrictions are placed on the coefficients $f_{i,j}$. 

5.3. Rotational Invariance

Define the following vectors found by rewriting the vector (5.20) after substitution of (5.21),

\[
 f_{i,j}(x) \equiv \sum_{i+1+2j=m} f_{i,j} \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} x_2^i(x_1^2 + x_2^2)^j, \\
 g_{i,j}(x) \equiv \sum_{i+1+2j=m} g_{i,j} \begin{pmatrix} x_1 \\ x_2 \\ -\frac{2j+2}{i+2}x_3 \end{pmatrix} x_3^i(x_1^2 + x_2^2)^j. 
\]

The sum of these vectors spans the complementary subspace \(G'_m\) and the kernel \(K'_m\) of the \(\text{ad} \, J_{4B}(\cdot)\) operator. The latter property is easily proven by writing out the expressions

\[
 \text{ad} \, J_{4B}(f_{i,j}(x)), \quad \text{ad} \, J_{4B}(g_{i,j}(x)), 
\]

and verifying that they are indeed equal to zero. This result enables us to set up a proof similar to Takens' [Tak74] to show that (5.23) also describes the complementary subspace \(G'_m\).

In Chapter 3, we showed that a re-ordering into independent subspaces of the adjoint operation of a matrix can be used to find all possible representations of the complementary subspace. The above representation enables us to make an intelligent choice in the selection of a complementary subspace.

Table 5.1b shows the independent subspaces in the mapping of the adjoint operation at \(m = 2\). To avoid confusion, lines are added in subspaces when they contain too many entries. In that case, the entries are ordered such that the 'new' coefficients relations can easily be verified. For example, the entries above the line in the image subspace \(I_1\) compose the vector

\[
 \begin{pmatrix} v^{(1)}_{200}x_1^2 \\ v^{(2)}_{110}x_1x_2 \\ v^{(3)}_{101}x_1x_3 \end{pmatrix}
\]

where \(v^{(1)}_{200} = u^{(1)}_{200} - t^{(2)}_{200} - t^{(1)}_{110}, v^{(2)}_{110} = u^{(2)}_{110} + t^{(1)}_{110} - 2t^{(2)}_{200} - 2t^{(2)}_{020}\) and \(v^{(3)}_{101} = -v^{(1)}_{200} - \frac{1}{2}v^{(2)}_{110} + \frac{1}{2}t^{(1)}_{110} + t^{(2)}_{020}\). It is easy to see that the coefficient relation

\[
2v^{(1)}_{200} + v^{(2)}_{110} + 2v^{(3)}_{101} = 0
\]

holds, see (2.62).
Table 5.1: $u_2(x) + \text{ad} J_A(t_2(x))$, with $u_2(x), t_2(x) \in \mathbb{V}_6^3 \cap H_2^3$
The subspaces $C_1$ and $C_2$ in Table 5.1b. contribute to the complementary subspace and the kernel as

$$G_2' = f_{10} x_3 r \partial_\theta + g_{10} (x_3 x_3 - \frac{2}{3} x_3^2 \partial_3),$$

$$(5.24)$$

$$K_2' = \tilde{f}_{10} x_3 r \partial_\theta + \tilde{g}_{10} (x_3 x_3 - \frac{2}{3} x_3^2 \partial_3),$$

$$(5.25)$$

where

$$f_{10} = \frac{1}{2} u_{101}^{(1)} - \frac{1}{2} u_{101}^{(2)}, \quad g_{10} = \frac{1}{2} u_{101}^{(1)} + \frac{1}{2} u_{011}^{(2)},$$

and where $\tilde{f}_{10}$ and $\tilde{g}_{10}$ are free parameters.

From (5.24) we see that $x_3$ is a common factor in $G_2'$. If the vector field is truncated at second order, all terms in the nonlinear part of the normal form within $\mathcal{V}'_\nu$ will vanish on the boundary surface $x_3 = 0$. This situation did not occur in the original vector field. It is an abnormality caused by premature truncation of the normal form.

From Table 5.3, we find that a third-order complementary subspace and the kernel are given by

$$G_3' = f_{01} r^3 \partial_\theta + g_{01} (r^3 \partial_r - 2 r^2 x_3 \partial_3)$$
$$+ f_{20} x_3^2 r \partial_\theta + g_{20} (x_3^2 r \partial_r - \frac{1}{2} x_3^2 \partial_3),$$

$$(5.26)$$

$$K_3' = \tilde{f}_{01} r^3 \partial_\theta + \tilde{g}_{01} (r^3 \partial_r - 2 r^2 x_3 \partial_3)$$
$$+ \tilde{f}_{20} x_3^2 r \partial_\theta + \tilde{g}_{20} (x_3^2 r \partial_r - \frac{1}{2} x_3^2 \partial_3).$$

$$(5.27)$$

We omit the expressions for the coefficients in $G_3'$.

The complementary subspace $G_3'$ can be somewhat simplified using the free parameters in the second-order kernel $K_3'$. To see which terms can be removed we compute the following Lie product

$$[K_3', G_2'] = \frac{2}{3} (-g_{10} \tilde{f}_{10} + f_{10} \tilde{g}_{10}) x_3^2 r \partial_\theta$$

Thus, if not both $f_{10}$ and $g_{10}$ are zero, a transformation exists which removes $f_{20} x_3^2 r \partial_\theta$ from $G_3'$.

The above computations of $G_2'$ in (5.24) and of $G_3'$ in (5.26) prove the nonlinear part of the topological normal form (5.2) in Proposition 5.2. Finally, note that $g_{20} x_3^2 (r \partial_r - \frac{1}{2} x_3 \partial_3)$ in $G_3'$ is congruent to $g_{10} x_3 (r \partial_r - \frac{2}{3} x_3 \partial_3)$ in $G_2'$, and is therefore not expected to be much of influence in a topological classification of fundamental local flow patterns.
<table>
<thead>
<tr>
<th>$x_1^3$</th>
<th>$\partial_1$</th>
<th>$\partial_2$</th>
<th>$\partial_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^3$</td>
<td>$u_{300}^{(1)} - t_{300}^{(2)} - t_{210}^{(1)}$</td>
<td>$u_{210}^{(2)} + t_{210}^{(1)}$</td>
<td>$u_{300}^{(2)} + t_{300}^{(1)} - t_{210}^{(2)}$</td>
</tr>
<tr>
<td>$x_1^3 x_2$</td>
<td>$u_{210}^{(1)} - t_{210}^{(2)} + 3t_{210}^{(1)} - 2t_{120}^{(1)}$</td>
<td>$u_{210}^{(2)} + t_{210}^{(1)} + 3t_{300}^{(2)} - 2t_{120}^{(2)}$</td>
<td>$u_{210}^{(1)} - t_{210}^{(2)} + 3t_{300}^{(2)} - 2t_{120}^{(2)}$</td>
</tr>
<tr>
<td>$x_1^2 x_3$</td>
<td>$u_{201}^{(1)} - t_{201}^{(2)} - t_{111}^{(1)}$</td>
<td>$u_{201}^{(2)} + t_{201}^{(1)} - t_{111}^{(2)}$</td>
<td>$u_{201}^{(1)} - t_{201}^{(2)} - t_{111}^{(1)}$</td>
</tr>
<tr>
<td>$x_1^2 x_3$</td>
<td>$u_{120}^{(1)} + t_{120}^{(1)} + 2t_{210}^{(2)} - 3t_{030}^{(1)}$</td>
<td>$u_{120}^{(2)} + t_{120}^{(1)} + 2t_{210}^{(2)} - 3t_{030}^{(1)}$</td>
<td>$u_{120}^{(1)} + t_{120}^{(1)} + 2t_{210}^{(2)} - 3t_{030}^{(1)}$</td>
</tr>
<tr>
<td>$x_1^2 x_3$</td>
<td>$u_{111}^{(1)} - t_{111}^{(2)} + 2t_{201}^{(1)} - 2t_{021}^{(1)}$</td>
<td>$u_{111}^{(2)} + t_{111}^{(1)} + 2t_{201}^{(2)} - 2t_{021}^{(2)}$</td>
<td>$u_{111}^{(1)} - t_{111}^{(2)} + 2t_{201}^{(1)} - 2t_{021}^{(1)}$</td>
</tr>
<tr>
<td>$x_1^2 x_3$</td>
<td>$u_{102}^{(1)} - t_{102}^{(2)} - t_{012}^{(1)}$</td>
<td>$u_{102}^{(2)} + t_{102}^{(1)} - t_{012}^{(2)}$</td>
<td>$u_{102}^{(1)} - t_{102}^{(2)} - t_{012}^{(1)}$</td>
</tr>
<tr>
<td>$x_2^3$</td>
<td>$u_{030}^{(1)} - t_{030}^{(2)} + t_{120}^{(1)}$</td>
<td>$u_{030}^{(2)} + t_{030}^{(1)} + t_{120}^{(2)}$</td>
<td>$u_{030}^{(1)} - t_{030}^{(2)} + t_{120}^{(1)}$</td>
</tr>
<tr>
<td>$x_2^3 x_3$</td>
<td>$u_{021}^{(1)} - t_{021}^{(2)} + t_{111}^{(1)}$</td>
<td>$u_{021}^{(2)} + t_{021}^{(1)} + t_{111}^{(2)}$</td>
<td>$u_{021}^{(1)} - t_{021}^{(2)} + t_{111}^{(1)}$</td>
</tr>
<tr>
<td>$x_2^3 x_3$</td>
<td>$u_{012}^{(1)} - t_{012}^{(2)} + t_{102}^{(1)}$</td>
<td>$u_{012}^{(2)} + t_{012}^{(1)} + t_{102}^{(2)}$</td>
<td>$u_{012}^{(1)} - t_{012}^{(2)} + t_{102}^{(1)}$</td>
</tr>
<tr>
<td>$x_3^3$</td>
<td>$u_{003}^{(1)} - t_{003}^{(2)}$</td>
<td>$u_{003}^{(2)} + t_{003}^{(1)}$</td>
<td>$u_{003}^{(1)} - t_{003}^{(2)}$</td>
</tr>
</tbody>
</table>

Table 5.2: $u_3(x) + \text{ad}_{J_4B}(t_3(x))$, with $u_3(x), t_3(x) \in \mathbb{R}^3 \cap H^3$, mapping.
<table>
<thead>
<tr>
<th>( \partial_1 )</th>
<th>( \partial_2 )</th>
<th>( \partial_3 )</th>
<th>( C_1, K_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{300}^{(1)} - t_{300}^{(2)} - t_{210}^{(1)} )</td>
<td>( t_{210}^{(2)} + t_{210}^{(1)} )</td>
<td>( -\frac{3}{2} t_{300}^{(3)} - \frac{1}{2} t_{210}^{(2)} )</td>
<td>1</td>
</tr>
<tr>
<td>( 2t_{210}^{(1)} - 3t_{030}^{(1)} )</td>
<td>( t_{030}^{(1)} + t_{030}^{(1)} )</td>
<td>( -\frac{1}{2} t_{120}^{(1)} + \frac{3}{2} t_{030}^{(2)} )</td>
<td></td>
</tr>
</tbody>
</table>

| \( C_2, K_2 \) |
| --- | --- | --- | --- |
| \( t_{210}^{(1)} - t_{210}^{(2)} \) | \( t_{120}^{(2)} + t_{120}^{(1)} \) | \( -t_{210}^{(1)} + \frac{1}{2} t_{120}^{(2)} \) | |
| \( +3t_{300}^{(1)} - 2t_{120}^{(1)} \) | \( +2t_{210}^{(2)} - 3t_{030}^{(2)} \) | \( -3t_{300}^{(1)} - \frac{1}{2} t_{210}^{(2)} \) | |
| \( -t_{030}^{(1)} + t_{030}^{(1)} \) | \( t_{030}^{(1)} + t_{030}^{(1)} \) | \( +t_{120}^{(1)} + 3t_{030}^{(2)} \) | |

| \( C_3, K_3 \) |
| --- | --- | --- | --- |
| \( t_{102}^{(1)} - t_{102}^{(2)} - t_{012}^{(1)} \) | \( t_{012}^{(2)} + t_{012}^{(1)} + t_{102}^{(2)} \) | \( -\frac{1}{4} t_{102}^{(1)} - \frac{1}{4} t_{012}^{(2)} \) | 1 |

| \( C_4, K_4 \) |
| --- | --- | --- | --- |
| \( t_{012}^{(1)} - t_{012}^{(2)} + t_{012}^{(1)} \) | \( t_{102}^{(2)} + t_{102}^{(1)} - t_{012}^{(2)} \) | | 1 |

| \( I_1 \) |
| --- | --- | --- | --- |
| \( t_{201}^{(1)} - t_{201}^{(2)} - t_{111}^{(1)} \) | \( t_{111}^{(2)} + t_{111}^{(1)} \) | \( -\frac{2}{3} t_{201}^{(1)} - \frac{1}{3} t_{111}^{(2)} \) | |
| \( +2t_{201}^{(2)} - 2t_{021}^{(2)} \) | \( +2t_{201}^{(2)} - 2t_{021}^{(2)} \) | \( +\frac{2}{3} t_{111}^{(1)} + \frac{4}{3} t_{021}^{(1)} \) | |
| \( t_{111}^{(1)} - t_{111}^{(2)} \) | \( t_{021}^{(2)} + t_{021}^{(1)} + t_{111}^{(2)} \) | \( -\frac{1}{3} t_{111}^{(1)} - \frac{2}{3} t_{021}^{(2)} \) | |
| \( +2t_{201}^{(1)} - 2t_{021}^{(2)} \) | \( +2t_{201}^{(1)} - 2t_{021}^{(2)} \) | \( -\frac{2}{3} t_{201}^{(1)} - \frac{1}{3} t_{111}^{(2)} \) | |

| \( I_2 \) |
| --- | --- | --- | --- |
| \( t_{003}^{(1)} - t_{003}^{(2)} \) | | | |
Table 5.4: $\mu_0(x) + \text{ad} J_{4B}(t_0(x))$, with $\mu_0(x), t_0(x) \in \mathcal{V}_\nu^3 \cap H_0^3$

**Versal deformation**

The second step towards Proposition 5.2 is the computation of the versal deformation within $\mathcal{V}_\nu^3$. Table 5.4 and Table 5.5 present the adjoint operation of $J_{4B}$ on $\mathcal{V}_\nu^3 \cap H_0^3$ and $\mathcal{V}_\nu^3 \cap H_1^3$, respectively. Table 5.4 shows that there is no complementary subspace nor kernel at zero order.

The representative terms in the complementary subspaces $C_1$ and $C_2$ in Table 5.5 are

$$\mathcal{G}^\nu_1 = \mu_1 (r \partial_r - x_3 \partial_3) + \mu_2 r \partial_\theta, \quad (5.28)$$

where $\mu_1 = \frac{1}{2}(\mu_{100}^{(1)} + \mu_{010}^{(2)})$, and $\mu_2 = \frac{1}{2}(\mu_{010}^{(1)} - \mu_{100}^{(2)})$. The term $\mu_2 r \partial_\theta$ equals the 1-jet of the vector field and can be omitted if we reparametrize $\omega$.

The above computation of $\mathcal{G}^\nu_1$ in (5.28) prove the deformation part of the topological normal form (5.2) in Proposition 5.2.

**Scaling**

All terms in the direction $r \partial_\theta$ other than $\omega r \partial_\theta$ do not change the topological classification. Substituting the coordinate transformation $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, and $x_3 = z$ into the topological normal form (5.2) results in

$$\dot{r} = r(\mu + g_{10} z + g_{01} r^2),$$
$$\dot{z} = -z(\mu + \frac{2}{3} g_{10} z + 2 g_{01} r^2). \quad (5.29)$$
5.3. Rotational invariance

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial_1$</td>
<td>$\mu_{100}^{(1)} - t_{100}^{(2)} - t_{010}^{(1)}$</td>
<td>$\mu_{010}^{(1)} - t_{010}^{(2)} + t_{100}^{(1)}$</td>
<td>$(\mu_{001} - t_{001})^{(1)}$</td>
</tr>
<tr>
<td>$\partial_2$</td>
<td>$\mu_{100}^{(2)} + t_{100}^{(1)} - t_{010}^{(2)}$</td>
<td>$\mu_{010}^{(2)} + t_{010}^{(1)} + t_{100}^{(2)}$</td>
<td>$(\mu_{001} + t_{001})^{(1)}$</td>
</tr>
<tr>
<td>$\partial_3$</td>
<td>0</td>
<td>0</td>
<td>$-\frac{1}{2} \mu_{100}^{(1)} - \frac{1}{2} \mu_{010}^{(2)}$</td>
</tr>
</tbody>
</table>

a) Mapping.

<table>
<thead>
<tr>
<th></th>
<th>$\partial_1$</th>
<th>$\partial_2$</th>
<th>$\partial_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1, K_1$</td>
<td>$\mu_{100}^{(1)} - t_{100}^{(2)} - t_{010}^{(1)}$</td>
<td>$\mu_{010}^{(2)} + t_{010}^{(1)} + t_{100}^{(2)}$</td>
<td>$-\frac{1}{2} \mu_{100}^{(1)} - \frac{1}{2} \mu_{010}^{(2)}$</td>
</tr>
<tr>
<td>$C_2, K_2$</td>
<td>$\mu_{010}^{(1)} - t_{010}^{(2)} + t_{100}^{(1)}$</td>
<td>$\mu_{100}^{(2)} + t_{100}^{(1)} - t_{010}^{(2)}$</td>
<td></td>
</tr>
<tr>
<td>$I_1$</td>
<td>$-t_{001}^{(2)} + u_{001}^{(1)}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I_2$</td>
<td>$t_{001}^{(1)} + u_{001}^{(2)}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Re-ordering into subspaces.

Table 5.5: $\mu_1(\mathbf{x}) + \text{ad} J_{4B}(t_1(\mathbf{x}))$, with $\mu_1(\mathbf{x}), t_1(\mathbf{x}) \in V_\nu \cap H_1^3$

Substitution of the rescaling of coordinates $r \rightarrow f_rr$, $z \rightarrow f_zz$, and $t \rightarrow f_tt$ produces the following list of equations

$$
\begin{align*}
&f_rf_zg_{10} = 1, \\
&f_tf_z^2g_{01} = 1.
\end{align*}
$$

(5.30)

Both $f_r$ and $f_z$ should be kept positive, so all that can be done is to replace $g_{10}$ and $g_{01}$ by $\pm 1$.

Critical points

The unfolded critical points of the vector field (5.29) are

$$
O \equiv (0, 0), \quad A \equiv (0, -\frac{3}{2} \frac{\mu}{g_{10}}), \quad B \equiv (\sqrt{-\frac{\mu}{g_{01}}}, 0),
$$

$$
C \equiv (\frac{1}{2} \sqrt{-\frac{\mu}{g_{01}}}, -\frac{3}{4} \frac{\mu}{g_{10}}).
$$

(5.31)

If $\mu g_{10} < 0$ and $\mu g_{01} < 0$, $A$ lies precisely in between $A$ and $B$.

The eigenvalues $(\lambda_1, \lambda_2)$ of the linearized vector field about the points $O$, $A$, $B$, and $C$ are $(\mu, -\mu), (-\frac{3}{2} \mu, \mu), (-2\mu, \mu)$, and $(-\frac{3}{4} \mu, \mu, \frac{3}{4} \sqrt{2} \mu, \sqrt{2} \mu)$. Hence,
(a) Case $g_{10} = 1, g_{01} = -1$. In case $g_{10} = -1, g_{01} = 1$, the sign of $\mu$ and the time direction needs to be reversed.

(b) Case $g_{10} = g_{01} = -1$. In case $g_{10} = g_{01} = 1$, the sign of $\mu$ and the time direction needs to be reversed.

Figure 5.4: Bifurcation diagrams of the local flow pattern near the origin of the vector field (5.29).

the first three unfolded critical points are saddles whereas the fourth is a spiral. Fig. 5.4 sketches the bifurcation diagrams of the local flow pattern near the origin of the vector field (5.29). A detailed treatment of the bifurcation behavior can be found in De Winkel’s thesis [Win96].

### 5.4 Triple zero eigenvalue

We assume a general vector field in $\mathbb{V}_d^3$, which has a critical point in the origin, and has a Jacobian matrix in that point with three zero eigenvalues and a one-dimensional eigenvector space. According to Lemma 4.3 and Lemma 4.4, a
linear coordinate transformation exists which maps this vector field to another, C*-equivalent, \( r \) arbitrary, vector field in \( \mathcal{V}_\nu^3 \) which has the following Jacobian matrix,

\[
J_{3C} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
\]

(5.32)

also see Appendix A. The matrix \( J_{3C} \) is a Jordan normal form. Without loss of generality, we can study the vector fields

\[
\dot{x} = u(x; \mu), \quad u(x; \mu) \in \mathcal{V}_\nu^3, \quad \frac{\partial u(0; 0)}{\partial x} = J_{3C}.
\]

In this section, we will construct a topological normal form within \( \mathcal{V}_\nu^3 \) for these vector fields.

**Normal form**

The first step towards Proposition 5.3 is the transformation of the nonlinear part of the vector field \( u(x; \mu) \) to a normal form within \( \mathcal{V}_\nu^3 \). To this end, we need to substitute a near-identity transformation in the vector field. As explained in Chapter 4, we only need to look at the complementary subspace \( g_m^\nu \) and the kernel \( \mathcal{K}_m^\nu \) of the adjoint operator of the Jacobian matrix on the homogeneous polynomials \( t_m(x) \in \mathcal{V}_\nu^3 \cap H^3_m \) from the generator of the transformation. In the case of the matrix \( J_{3C} \), we obtain

\[
\text{ad} J_{3C}(t_m(x)) = \sum_{i+j+k=m} \left( t_{ijk} (x_1 x_2 x_3) - t^{(1)}_{ijk} (x_1 x_2 x_3) + j x_1 x_2 x_3 + j x_1 x_2 x_3 - t^{(3)}_{ijk} (x_1 x_2 x_3) + j x_1 x_2 x_3 + j x_1 x_2 x_3) \right).
\]

(5.33)

Table 5.6a contains the result of the adjoint of \( J_{3C} \) operating on \( t_2(x) \in \mathcal{V}_\nu^3 \cap H^3_2 \). Table 5.6b shows the re-ordering of Table 5.6a into independent subspaces. Unlike the case of rotational invariance discussed in the previous section, there is no preferred complementary-subspace representation. To maximize the occurrence of congruent terms, it is desirable to make a consistent selection. To this end, we remove all entries that part in a coefficient relation, and if that is not possible, we remove the first row. For example, in the subspace \( C_3 \) in
### Table 5.6: $u(x) + \text{ad} J_{JC}(t_{2}(x))$, with $u(x), t_{2}(x) \in V_{c}^{3} \cap H_{2}^{3}$. 

<table>
<thead>
<tr>
<th>$\partial_1$</th>
<th>$\partial_2$</th>
<th>$\partial_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{1}^{2}$</td>
<td>$u_{200}^{(1)} + t_{200}^{(2)}$</td>
<td>$u_{200}^{(2)}$</td>
</tr>
<tr>
<td>$x_{1}x_{2}$</td>
<td>$u_{110}^{(1)} + t_{110}^{(2)} - 2t_{200}^{(1)}$</td>
<td>$u_{110}^{(2)} - 2t_{200}^{(2)}$</td>
</tr>
<tr>
<td>$x_{1}x_{3}$</td>
<td>$u_{101}^{(1)} + t_{101}^{(2)} - t_{110}^{(1)}$</td>
<td>$u_{101}^{(2)} - t_{200}^{(2)} - \frac{3}{2}t_{110}^{(2)}$</td>
</tr>
<tr>
<td>$x_{2}^{2}$</td>
<td>$u_{020}^{(1)} + t_{020}^{(2)} - t_{110}^{(1)}$</td>
<td>$u_{020}^{(2)} - t_{110}^{(2)}$</td>
</tr>
<tr>
<td>$x_{2}x_{3}$</td>
<td>$u_{011}^{(1)} + t_{011}^{(2)}$</td>
<td>$u_{011}^{(2)} - \frac{1}{2}t_{110}^{(1)}$</td>
</tr>
<tr>
<td>$-t_{101}^{(1)} - 2t_{020}^{(1)}$</td>
<td>$-t_{101}^{(2)} - 3t_{020}^{(2)}$</td>
<td>$-\frac{1}{2}u_{110}^{(1)} + u_{020}^{(2)}$</td>
</tr>
<tr>
<td>$x_{3}^{2}$</td>
<td>$u_{002}^{(1)} + t_{002}^{(2)} - t_{011}^{(1)}$</td>
<td>$u_{002}^{(2)} - \frac{1}{3}t_{101}^{(1)} - \frac{4}{3}t_{001}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### a) Mapping.

<table>
<thead>
<tr>
<th>$\partial_1$</th>
<th>$\partial_2$</th>
<th>$\partial_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{1}$</td>
<td>$u_{200}^{(2)}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$C_{2}$</td>
<td>$u_{200}^{(1)} + t_{200}^{(2)}$</td>
<td>$u_{110}^{(2)} - 2t_{200}^{(2)}$</td>
</tr>
<tr>
<td>$C_{3}$</td>
<td>$u_{110}^{(1)} + t_{110}^{(2)} - 2t_{200}^{(1)}$</td>
<td>$u_{020}^{(2)} - t_{110}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$I_{1}$</td>
<td>$u_{101}^{(1)} + t_{101}^{(2)} - t_{110}^{(1)}$</td>
<td>$u_{011}^{(2)} - \frac{1}{2}t_{110}^{(1)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_{1}$</td>
<td>$u_{011}^{(1)} + t_{011}^{(2)} - t_{110}^{(1)}$</td>
<td>$u_{002}^{(2)} - \frac{1}{3}t_{101}^{(1)} - \frac{4}{3}t_{011}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_{2}$</td>
<td>$u_{002}^{(1)} + t_{002}^{(2)} - t_{011}^{(1)}$</td>
<td>$1$</td>
</tr>
<tr>
<td>$K_{3}$</td>
<td>$t_{002}^{(1)}$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
Table 5.6b, the entries above the line compose the vector
\[
\begin{pmatrix}
  v_{110}^{(1)} x_1 x_2 \\
  v_{020}^{(2)} x_2 \\
  v_{011}^{(3)} x_2 x_3
\end{pmatrix}
\]
where \( v_{110}^{(1)} = u_{110}^{(1)} + t_{110}^{(1)} - 2u_{200}^{(1)} \), \( v_{020}^{(2)} = u_{020}^{(2)} + t_{110}^{(2)} \), and \( v_{011}^{(3)} = -\frac{1}{2} u_{110}^{(1)} - u_{020}^{(2)} + t_{200}^{(1)} + \frac{1}{2} t_{110}^{(2)} \). It is easy to see that the coefficient relation
\[
v_{110}^{(1)} + 2v_{020}^{(2)} + 2v_{011}^{(3)} = 0
\]
holds, see (2.62). We remove these three coefficients, leaving the underlined term in subspace \( C_3 \).

Following these guidelines, our representation of the complementary subspace is given by
\[
G''_3 = n_{200}^{(2)} x_1^2 \partial_2 + n_{110}^{(2)} (x_1 x_2 \partial_2 - \frac{1}{2} x_1 x_2 \partial_3) + n_{101}^{(2)} x_1 x_3 \partial_2,
\]
where
\[
\begin{align*}
  n_{200}^{(2)} &= u_{200}^{(2)}, \\
  n_{110}^{(2)} &= u_{110}^{(2)} + 2u_{200}^{(2)}, \\
  n_{101}^{(2)} &= u_{101}^{(2)} - 2u_{020}^{(2)} - \frac{1}{2} u_{110}^{(1)}.
\end{align*}
\]
Note that normal-form coefficients are combinations of all the coefficients from the subspaces which they represent. Also from Table 5.6b, we see that the second-order kernel is given by
\[
K''_2 = t_{002}^{(1)} x_3 \partial_1 + F_1 (-4x_1 x_3 \partial_1 + \frac{5}{2} x_2^2 \partial_1 + x_2 x_3 \partial_2 + x_3^2 \partial_3) \\
+ F_2 (x_2 x_3 \partial_1 + x_2^2 \partial_2)
\]

To see if inclusion of normal-form terms of higher order will affect the topological classification of the fundamental local flow patterns, we compute the third order normal form terms. Thus, compute the ad \( J_{3C}(\cdot) \)-operation over \( H^3_{\mathcal{S}} \cap V', \) see Table 5.7 and Table 5.8.

From these tables, the third-order complementary subspace is found as
\[
G''_3 = n_{300}^{(2)} x_1^3 \partial_2 + n_{210}^{(2)} (x_1^2 x_2 \partial_2 - \frac{1}{2} x_1^2 x_3 \partial_3) + n_{201}^{(2)} x_1^2 x_3 \partial_2 \\
+ n_{111}^{(2)} (x_1 x_2 x_3 \partial_2 - \frac{1}{3} x_2 x_3^2 \partial_3),
\]
<table>
<thead>
<tr>
<th>$x_1^3$</th>
<th>$\partial_1$</th>
<th>$\partial_2$</th>
<th>$\partial_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1^3 x_2$</td>
<td>$u_{300}^{(1)} + t_{210}^{(2)}$</td>
<td>$u_{300}^{(2)}$</td>
<td>$u_{210}^{(2)} - 3t_{300}^{(2)} - 3t_{210}^{(2)}$</td>
</tr>
<tr>
<td>$x_1^2 x_3$</td>
<td>$u_{201}^{(1)} + t_{210}^{(2)} - t_{210}^{(1)}$</td>
<td>$u_{201}^{(2)} - \frac{3}{2}t_{300}^{(1)} - \frac{3}{2}t_{210}^{(2)}$</td>
<td>$-\frac{3}{2}u_{300}^{(1)} - \frac{1}{3}u_{210}^{(2)}$</td>
</tr>
<tr>
<td>$x_1 x_2^2$</td>
<td>$u_{120}^{(1)} + t_{120}^{(2)} - 2t_{210}^{(1)}$</td>
<td>$u_{120}^{(2)} - 2t_{210}^{(2)}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$x_1 x_2 x_3$</td>
<td>$u_{111}^{(1)} + t_{111}^{(2)} - 2t_{201}^{(1)} - 2t_{120}^{(1)}$</td>
<td>$u_{111}^{(2)} - t_{210}^{(1)} - 3t_{120}^{(2)} - 2t_{201}^{(2)}$</td>
<td>$-u_{210}^{(1)} - u_{120}^{(2)} + 3t_{300}^{(1)} + t_{210}^{(2)}$</td>
</tr>
<tr>
<td>$x_1 x_3^2$</td>
<td>$u_{102}^{(1)} + t_{102}^{(2)} - t_{111}^{(1)}$</td>
<td>$u_{102}^{(2)} - \frac{2}{3}t_{201}^{(1)} - \frac{4}{3}t_{111}^{(2)}$</td>
<td>$-\frac{2}{3}u_{201}^{(1)} - \frac{1}{3}u_{111}^{(2)} + t_{210}^{(1)} + t_{120}^{(2)}$</td>
</tr>
<tr>
<td>$x_2^3$</td>
<td>$u_{030}^{(1)} + t_{030}^{(2)} - t_{120}^{(1)}$</td>
<td>$u_{030}^{(2)} - t_{120}^{(2)}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$x_2^2 x_3$</td>
<td>$u_{021}^{(1)} + t_{021}^{(2)} - t_{111}^{(1)} - 3t_{030}^{(1)}$</td>
<td>$u_{021}^{(2)} - \frac{1}{2}t_{120}^{(1)} - \frac{1}{2}t_{030}^{(1)} - \frac{1}{2}t_{111}^{(2)}$</td>
<td>$-\frac{1}{3}u_{120}^{(1)} - \frac{1}{3}u_{030}^{(2)} + t_{210}^{(1)} + t_{120}^{(2)}$</td>
</tr>
<tr>
<td>$x_2 x_3^2$</td>
<td>$u_{012}^{(1)} + t_{012}^{(2)} - t_{102}^{(1)} - 2t_{021}^{(1)}$</td>
<td>$u_{012}^{(2)} - \frac{1}{2}t_{111}^{(1)} - \frac{3}{2}t_{021}^{(1)} - \frac{3}{2}t_{102}^{(2)}$</td>
<td>$-\frac{1}{3}u_{111}^{(1)} - \frac{1}{3}u_{021}^{(2)} + \frac{2}{3}t_{201}^{(1)} + \frac{1}{3}t_{111}^{(2)} + t_{120}^{(1)} + 3t_{030}^{(2)}$</td>
</tr>
<tr>
<td>$x_3^3$</td>
<td>$u_{003}^{(1)} + t_{003}^{(2)} - t_{012}^{(1)}$</td>
<td>$u_{003}^{(2)} - \frac{1}{4}t_{102}^{(1)} - \frac{5}{4}t_{012}^{(2)}$</td>
<td>$-\frac{1}{4}u_{102}^{(1)} - \frac{1}{4}u_{012}^{(2)} + \frac{1}{3}t_{111}^{(1)} + \frac{2}{3}t_{021}^{(2)}$</td>
</tr>
</tbody>
</table>

Table 5.7: Mapping of $u_3(x) + \text{adj}_3(C(t_3(x)))$, with $u_3(x), t_3(x) \in V_3 \cap H_3^3$. 
<table>
<thead>
<tr>
<th>( \partial_1 )</th>
<th>( \partial_2 )</th>
<th>( \partial_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>( u^{(2)}_{300} )</td>
<td>0</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( u^{(1)}<em>{300} + t^{(2)}</em>{300} )</td>
<td>( u^{(2)}<em>{210} - 3t^{(2)}</em>{300} )</td>
</tr>
<tr>
<td>( C_3 )</td>
<td>( u^{(1)}<em>{210} + t^{(2)}</em>{210} - 3t^{(1)}_{300} )</td>
<td>( u^{(2)}<em>{120} - 2t^{(2)}</em>{210} )</td>
</tr>
<tr>
<td>( C_4 )</td>
<td>( u^{(1)}<em>{201} + t^{(2)}</em>{201} - t^{(1)}_{210} )</td>
<td>( u^{(2)}<em>{111} - t^{(1)}</em>{210} )</td>
</tr>
<tr>
<td>( I_1 )</td>
<td>( u^{(1)}<em>{111} + t^{(2)}</em>{111} - 2t^{(1)}<em>{201} - 2t^{(1)}</em>{120} )</td>
<td>( u^{(2)}<em>{111} - \frac{1}{2}t^{(1)}</em>{120} )</td>
</tr>
<tr>
<td>( K_1 )</td>
<td>( u^{(1)}<em>{102} + t^{(2)}</em>{102} - t^{(1)}_{102} )</td>
<td>( u^{(2)}<em>{102} - \frac{1}{3}t^{(1)}</em>{111} - \frac{5}{4}t^{(2)}_{012} )</td>
</tr>
<tr>
<td>( K_2 )</td>
<td>( u^{(1)}<em>{012} + t^{(2)}</em>{012} - t^{(1)}_{021} )</td>
<td>( u^{(2)}<em>{012} - \frac{1}{3}t^{(1)}</em>{111} - \frac{5}{4}t^{(2)}_{012} )</td>
</tr>
<tr>
<td>( K_3 )</td>
<td>( u^{(1)}<em>{003} + t^{(2)}</em>{003} - t^{(1)}_{012} )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( K_4 )</td>
<td>( t^{(1)}_{003} )</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 5.8: Re- ordering into subspaces of \( u_3(x) + \text{ad} J_3C(t_3(x)) \), with \( u_3(x), t_3(x) \in V^3_9 \cap H^3_9 \).
where
\[
\begin{align*}
  n^{(2)}_{300} &= \tilde{u}^{(2)}_{300}, \\
  n^{(2)}_{210} &= \tilde{u}^{(2)}_{210} + 3\tilde{u}^{(1)}_{300}, \\
  n^{(2)}_{201} &= \tilde{u}^{(2)}_{201} - \frac{1}{2}\tilde{u}^{(1)}_{210} - \tilde{u}^{(2)}_{120}, \\
  n^{(2)}_{111} &= \tilde{u}^{(2)}_{111} + 2\tilde{u}^{(1)}_{201} - \frac{3}{2}\tilde{u}^{(1)}_{120} - \frac{2}{3}\tilde{u}^{(2)}_{030}.
\end{align*}
\] (5.38)

The tildes in (5.38) indicate that the coefficients contain additional coefficients formed by Lie products in the triangle (3.107). These Lie products will allow us to simplify \( G'_3 \).

**Hypernormal form**

Only one term in \( G'_3 \) is not congruent to a term in \( G'_2 \), viz.,

\[
n^{(2)}_{111}(x_1x_2x_3\partial_2 - \frac{1}{3}x_2x_3^2\partial_3).
\]

However, this term can be removed by a free parameter from the kernel \( K_2 \). These parameters are part of the Lie products in the triangle (3.107). The relevant relation is given by (3.115). It is not necessary to compute this relation in full but it suffices to investigate the Lie product of \( G'_2 \) with \( K'_2 \). Table 5.9 shows the relevant portion of the resulting expression for each of the subspaces \( C_i, i \in \{1, 2, 3, 4\} \), in Table 5.8. We see that if \( n^{(2)}_{200} \neq 0, F_1 \) can be used to remove the remainder of \( C_3 \), and under the same condition, \( F_2 \) can be used to remove the remainder of \( C_4 \).

The subspaces \( C_1 \) and \( C_2 \) in Table 5.8 are not reached by the Lie product of \( G'_2 \) and \( K'_2 \). Their representatives have congruent terms in \( G'_2 \).

We conclude that third-order terms have no influence on the dynamical behavior.

The coefficient \( n^{(2)}_{101} \) can be removed by extending the transformation to a nilpotent Jordan normal form to

\[
\mathcal{X} = M\mathcal{X}, \quad \mathcal{X} = e^{B\gamma}\mathcal{X},
\] (5.39)

where \( M \) is the conjugating matrix as before and \( B \) belongs to the centralizer of \( J_3C \), i.e., \( J_3C B = B J_3C \). The second transformation in (5.39) does not alter the Jordan normal form. Denote the centralizer of a matrix \( A \) as

\[
\mathcal{Z}_A = \text{span}\{1, A, M_1, \ldots, M_p\}.
\]

It can be shown, see Gamero [Gam90], that the matrices \( 1, A \) and the parameter \( \gamma \) do not contribute to the simplification of a normal form. Therefore, take \( \gamma = 1 \).
Table 5.9: The Lie product of $G_2^\nu$ and $K_2^\nu$ (fragment). The labels $C_i$, $i \in \{1, 2, 3, 4\}$, refer to the subspaces in Table 5.8.

\[
\begin{array}{|c|c|c|c|}
\hline
C_1 & n_{300}^{(2)} & u_{300}^{(2)} & 0 \\
\hline
C_2 & n_{210}^{(2)} & u_{210}^{(2)} & 0 \\
\hline
& u_{300}^{(1)} & 0 & \\
\hline
C_3 & n_{201}^{(2)} & u_{201}^{(2)} & -9F_1n_{200}^{(2)} \\
\hline
& u_{210}^{(1)} & -5F_1n_{200}^{(2)} & \\
\hline
& u_{120}^{(2)} & 5F_1n_{200}^{(2)} & \\
\hline
C_4 & n_{111}^{(2)} & u_{111}^{(2)} & 2F_2n_{200}^{(2)} - \frac{7}{2}F_1n_{110}^{(2)} \\
\hline
& u_{201}^{(1)} & -2F_2n_{200}^{(2)} - 2F_1n_{110}^{(2)} & \\
\hline
& u_{120}^{(1)} & -5F_1n_{110}^{(2)} & \\
\hline
& u_{030}^{(2)} & -\frac{5}{2}F_1n_{110}^{(2)} & \\
\hline
\end{array}
\]

and

\[
B = \sum_{i=1}^{p} \alpha_i M_i.
\]

In the case of $J_3^C$ we have

\[
\text{span}\{M_1, \ldots, M_p\} = \text{span}\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\},
\]

which makes the proposed change of coordinates

\[
X = e^B \bar{x} = \begin{pmatrix} 1 & 0 & \alpha_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \bar{x}.
\]

This transformation was first proposed by Gamero et al. [GFR+93] in the general case. Note that the change of coordinates (5.41) the conditions stated in Lemma 4.3 to insure that the resulting vector field is again an element of $\mathbb{V}^3_\nu$.

Let $\bar{u}(\bar{x}, \mu)$ denote the vector field after application of (5.39). The normal form coefficient $n_{101}^{(2)}$ then reads

\[
n_{101}^{(2)} = \bar{u}_{101}^{(2)} - 2\bar{u}_{020}^{(2)} - \frac{1}{2} \bar{u}_{110}^{(2)},
\]
and similar results hold for the other coefficients in $G_2'$. By computing $\tilde{u}_2(x) = e^{-B} u_2( e^B x )$ we get

$$
\begin{align*}
\hat{u}_{101}^{(2)} &= u_{101}^{(2)} + 2\alpha_1 u_{200}^{(2)}, \\
\hat{u}_{020}^{(2)} &= u_{020}^{(2)}, \\
\hat{u}_{110}^{(2)} &= u_{110}^{(2)}.
\end{align*}
$$

(5.43)

The parameter $\alpha_1$ enters other coefficients than just $\hat{u}_{101}^{(2)}$. However, all these coefficients are part in the image of ad $J_{3C}(\cdot)$-operator, see Table 5.6b. Therefore, under the condition $u_{200}^{(2)} \neq 0$, which generally holds, the parameter $\alpha_1$ can be used to remove $n_{101}^{(2)}$.

**Versal deformation**

The second step towards Proposition 5.3 involves the computation of the bifurcation terms. A transformation $x \mapsto t(x, \mu)$ is constructed so as to remove as much of the perturbation terms $\mu_{i,j,k}^{(s)}$ as possible. The remaining parameters are part of the complementary subspace of the ad $J_{3C}(\cdot)$-operator, see (5.33). In the formal series of $t(x, \mu)$ in $\varepsilon$ let the homogeneous polynomial $t_n(x)$ be the terms of order $\varepsilon^{a+n-1}$, $n \geq 0$. As usual, the coefficients of $t_m(x)$ are denoted by $t_{i,j,k}^{(s)}$, $s \in \{1, 2, 3\}$ and $i, j, k \geq 0$ with $i + j + k = m$.

The result of the adjoint of $J_{3C}$ acting on $t_n(x) \in V_{\nu}^3$ for both $n = 0$ and $n = 1$ is given by Table 5.10a and Table 5.11a, respectively. From the re-ordering into subspaces given by Table 5.10b and Table 5.11b we conclude that the complementary subspace and the kernel are given by

$$
G_0' = \dot{\mu}_{000}^{(2)} \Omega_2, \quad K_0' = \dot{t}_{000}^{(1)} \Omega_1,
$$

(5.44)
Table 5.10: $\mu_0(x) + \text{ad} J_3 C(t_0(x))$, with $\mu_0(x), t_0(x) \in V_\nu^3 \cap H_0^3$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial_1$</td>
<td>$\mu_1^{(1)} + t_{100}^{(2)}$</td>
<td>$t_{100}^{(1)} + t_{010}^{(2)} - t_{100}^{(1)}$</td>
</tr>
<tr>
<td>$\partial_2$</td>
<td>$\mu_1^{(2)}$</td>
<td>$\mu_{010} - t_{100}^{(2)}$</td>
</tr>
<tr>
<td>$\partial_3$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5.11: $\mu_1(x) + \text{ad} J_3 C(t_1(x))$, with $\mu_1(x), t_1(x) \in V_\nu^3 \cap H_1^3$
only the computation of the Lie product of $\mathcal{K}_0$ and $n_2(x)$ is needed,

$$\left[\mathcal{K}_0, n_2(x)\right] = \begin{pmatrix} t_{000}^{(1)} & 0 \\ 0 & n_{200}^{(2)} x_1 + n_{110}^{(2)} x_1 x_2 \\ 0 & -\frac{1}{2} n_{110}^{2} x_1 x_3 \end{pmatrix} \left[ 0 \right].$$

(5.46)

The resulting vector reaches the following positions:

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$\mu_{100}^{(2)}$</th>
<th>$-2t_{000}^{(1)} n_{200}^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2$</td>
<td>$\mu_{010}^{(2)}$</td>
<td>$-t_{000}^{(1)} n_{110}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td>$\mu_{100}^{(1)}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Thus, it is possible to remove either $C_1$ or $C_2$. As can be seen in Proposition 5.3, we choose to remove $C_2$.

**Scaling**

The third and final step is to use a scaling of the coordinates and the time parameter to reduce the number of cases to be studied. Substitution of $x_i \rightarrow f_i x_i$ and $t \rightarrow f_t t$ into the vector field in Proposition 5.3 produces to following list of requirements:

$$A; \quad f_t f_2 = f_1, \quad C; \quad f_t f_3 = f_2,$$

$$B; \quad f_t f_1 n_{200}^{(1)} = f_2, \quad D; \quad f_t f_1 n_{110}^{(2)} = 1.$$

For the correct interpretation of the flow patterns, it is necessary to require that the sign of $f_3$ and $f_t$ remains positive. As a result, not all of the requirements can be satisfied at the same time. Requirement $D$ fixes the sign of $f_t f_1$ and the value of $f_1$. Then, requirement $B$ can only be satisfied in absolute value. Hence, nonzero $n_{200}^{(2)}$ can be replaced with $\pm 1$ while nonzero $n_{110}^{(2)}$ can be replaced with 1. In conclusion,

$$\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \mu_1 + \mu_2 x_1 + x_3 \pm x_1^2 + x_1 x_2, \\
\dot{x}_3 &= -\frac{1}{2} x_1 x_3
\end{align*}$$

(5.47)

is the vector field which models the dynamical behavior of vector field with a Jacobian matrix similar to $J_{3C}$. 
Using the above tables, it is easy to verify that the vector field
\[ \dot{x}_1 = \mu_2 x_1 + x_2 + x_1^2, \]
\[ \dot{x}_2 = \mu_1 + x_3 \pm x_1^2, \]
\[ \dot{x}_3 = -\frac{1}{2} \mu_2 x_3 - x_1 x_3, \] (5.48)
is also a topological normal form. This representation very much resembles the vector field (1.27) analyzed by Bakker [Bak90]. In fact, critical point analysis of the vector field (5.48) produces the same topology of local flow patterns as sketched in the bifurcation diagrams Fig. 1.12b and Fig. 1.13b.

Bibliographical notes; a comparison

The general-purpose second-order complementary subspace of the adjoint operation of \( J_{3C} \) is given by
\[ \mathcal{G}_2 = \left[ n_{200}^{(3)} y_1^2 + n_{110}^{(3)} y_1 y_2 + n_{101}^{(3)} y_1 y_3 + n_{020}^{(3)} y_2^2 \right] \partial_3, \]
where \( n_{200}^{(3)} = f_{200}^{(3)}, n_{110}^{(3)} = 2 f_{200}^{(2)} + f_{110}^{(3)}, n_{101}^{(3)} = 2 f_{200}^{(1)} + f_{110}^{(2)} + f_{101}^{(3)}, \) and \( n_{020}^{(3)} = 2 f_{200}^{(1)} + f_{110}^{(2)} + f_{020}^{(3)} \), see, for example, Ushiki [Usi84], Gamero et al. [GFR+93], based on methods developed by Cushman and Sanders [CS86] and Elphick et al. [ETB+87], obtained similar results.

For a vector field \( \mathcal{U}(\dot{x}) \in \mathcal{V}_\nu \) we have \( u_{200}^{(3)} = u_{110}^{(3)} = u_{020}^{(3)} = 0, \) and \( u_{101}^{(3)} = -u_{200}^{(1)} - \frac{1}{2} u_{110}^{(2)} \), and thus
\[ \mathcal{G}_2^a = \left[ n_1 y_1 y_2 + n_2 \left( \frac{1}{2} y_1 y_3 + y_2^2 \right) \right] \partial_3, \]
where \( n_1 = 2 u_{200}^{(2)} \) and \( n_2 = 2 u_{200}^{(1)} + u_{110}^{(2)} \). As can be seen, only two degrees of freedom remain.

The 2-jet reads
\[ \dot{y}_1 = y_2, \]
\[ \dot{y}_2 = y_3, \]
\[ \dot{y}_3 = n_1 y_1 y_2 + n_2 \left( \frac{1}{2} y_1 y_3 + y_2^2 \right). \] (5.49)

Before transformation to normal form, the boundary surface was a coordinate plane and an invariant manifold of the vector field. It is difficult to find an invariant manifold in the vector field (5.49). Further note that the \( y_1 \)-axis -- on which \( y_2 = y_3 = 0 \) -- is a line of critical points. Because of this abnormality,
it will be necessary to compute a normal-form up to third order to get a good enough approximation of the 'flow' on the boundary surface.

As was argued in Chapter 4, we see that general-purpose normal forms is not a practical representation for vector fields in $V^3_\nu$. Moreover, we see that the representation in (5.48) is more efficient; no third-order terms are necessary.

### 5.5 Mirror symmetry

We assume a general vector field $\mathbf{u}(\mathbf{x}, \nu)$ in $V^3_\nu$ which has a critical point in the origin, and has a local flow pattern with a plane of symmetry $x_1 = 0$, denoted as $V^{3,1}_S$. The vector field is written as

$$\dot{\mathbf{x}} = A \mathbf{x} + \sum_{m \geq 2} u_m(\mathbf{x}) + \delta \sum_{m \geq 0} \mu_m(\mathbf{x}). \quad (5.50)$$

We shall first briefly look at the hyperbolic mirror-symmetric critical points. From thereon we concentrate on finding a topological normal form for the non-hyperbolic mirror-symmetric critical point with three zero eigenvalues with a two-dimensional eigenvector space.

### Hyperbolic critical points

As a result of the assumption of symmetry in the coordinate plane $x_1 = 0$, the component functions $u_m^{(1)}(\mathbf{x})$ in (5.50) are uneven in $x_1$ whereas the component functions $u_m^{(2)}(\mathbf{x})$ and $u_m^{(3)}(\mathbf{x})$ are even in $x_1$. Therefore, the most general form of a Jacobian matrix in a hyperbolic mirror-symmetric critical point is given by

$$A = \begin{pmatrix} u_{100}^{(1)} & 0 & 0 \\ 0 & u_{010}^{(2)} & u_{001}^{(2)} \\ 0 & 0 & -\frac{1}{2}(u_{100}^{(1)} + u_{010}^{(2)}) \end{pmatrix} \quad (5.51)$$

The two matrix invariants are $p = u_{100}^{(1)} + u_{010}^{(2)}$ and $q = u_{100}^{(1)} u_{010}^{(2)}$, see (1.24). The latter invariant is nonzero through the assumption of hyperbolicity. The discriminant of the quadratic equation for the eigenvalues restricted to the boundary surface equals

$$D = p^2 - 4q = (u_{100}^{(1)} - u_{010}^{(2)})^2,$$

see (1.25). This means that $D \geq 0$ in a hyperbolic mirror-symmetric critical point. Hence, from the topology of boundary critical points depicted in Fig. 1.5, as expected, the spiral cannot appear in the intersection of the boundary surface $x_3 = 0$ and the plane of symmetry $x_1 = 0$. 
Nonhyperbolic critical points

All eigenvalues of the Jacobian matrix $A$ in (5.51) are located directly on the main diagonal. Hence, the possible degenerate cases are easy to find:

- Either $u_{100}^{(1)} = 0$ or $u_{010}^{(2)} = 0$; one zero eigenvalue in the direction tangential to the boundary surface,

- $u_{100}^{(1)} + u_{010}^{(2)} = 0$; one zero eigenvalue in the direction transversal to the boundary surface, and

- $u_{100}^{(1)} = 0$ and $u_{010}^{(2)} = 0$; three zero eigenvalues.

The first two cases have been treated in a more general setting, see Proposition 5.1. Therefore, we concentrate on the third case.

We assume that the coefficient $u_{001}^{(2)} \neq 0$. The actual value of $u_{001}^{(2)}$ is not important. Because, by using a linear coordinate transformation, $A$ can be mapped to the matrix

$$J_{3B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (5.52)

for example, using the conjugating matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_{001}^{(2)} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ \hspace{1cm} (5.53)

The transformation matrix $M$ satisfies the necessary conditions of Lemma 4.3 to insure that the resulting vector field is once again part of $V_{\nu}^3$. Furthermore, since $M$ only scales the coordinate $x_2$, the resulting vector field also belongs to $V_{S}^{3,1}$. Thus, without loss of generality, we can study the vector fields

$$\dot{x} = u(x; \mu), \quad u(x; \mu) \in V_{\nu}^3, \quad \frac{\partial u(0; 0)}{\partial x} = J_{3B}.$$

In what follows, we will show that the vector field (5.63) is a topological normal form for these vector fields.

Normal form

The first step towards the topological normal form (5.63) is the transformation of the nonlinear part of the vector field $u(x; \mu)$ to a normal form within $V_{\nu}^3 \cap V_{S}^{3,1}$. 
To this end, we need to substitute a near-identity transformation in the vector field. As explained in Chapter 4, we only need to look at the complementary subspace $\mathcal{G}^\nu_m$ and the kernel $\mathcal{K}^\nu_m$ of the adjoint operator of the Jacobian matrix on the homogeneous polynomials $t_m(\bar{x}) \in \mathbb{V}_\nu^3 \cap \mathbb{V}_S^{3,1} \cap H_m^3$ from the generator of the transformation. In the case of the matrix $J_{3B}$, we obtain

$$
ad J_{3B}(t_m(\bar{x})) = \sum_{i+j+k=n} \begin{pmatrix} -j t^{(1)}_{i,j,k} x_1^i x_2^{j-1} x_3^{k+1} \\ t^{(3)}_{i,j,k} x_1^i x_2^{j-1} x_3^{k+1} - j t^{(2)}_{i,j,k} x_1^i x_2^{j-1} x_3^{k+1} \\ -j t^{(3)}_{i,j,k} x_1^i x_2^{j-1} x_3^{k+1} \end{pmatrix}.$$  \hspace{1cm} (5.54)

for $t_m(\bar{x}) \in H_m^3$.

Table 5.12a shows the coefficients which remain in the calculation of

$$u_2(\bar{x}) + \text{ad } J_{3B}(t_2(\bar{x})),
$$

where $u_2(y), t_2(y) \in \mathbb{V}_\nu^3 \cap \mathbb{V}_S^{3,1} \cap H_2^3$. Table 5.12b shows the re-ordering into independent subspaces. Note that there are no coefficients $t_{i,j,k}$ in the subspaces $C_1$ and $C_2$. Hence, the complementary subspace is given by

$$\mathcal{G}^\nu_2 = n^{(1)}_{110} x_1 x_2 \partial_1 + n^{(2)}_{020} x_2^2 \partial_2 + n^{(2)}_{200} x_1^2 \partial_2 - \left( \frac{1}{2} n^{(1)}_{110} + n^{(2)}_{020} \right) x_2 x_3 \partial_3,
$$

(5.55)

where $n^{(s)}_{i,j,k} = u^{(s)}_{i,j,k}$.

To compute the kernel $\mathcal{K}^\nu_2$ set $t^{(1)}_{010} = 4F_1 + u^{(2)}_{002}$ and $t^{(2)}_{011} = -F_1 + \frac{1}{2} u^{(2)}_{002}$.

$$\mathcal{K}^\nu_2 = t^{(2)}_{200} x_1^2 \partial_2 + t^{(2)}_{002} x_3^2 \partial_2 + F_1(4x_1 x_3 \partial_1 + x_1^2 \partial_2 - x_3^2 \partial_3),
$$

(5.56)

The free parameters $t^{(2)}_{200}, t^{(2)}_{002}$ and $F_1$ can be used to possibly simplify the third order terms of the normal form.

As in the case of rotational symmetry, third order terms are needed in the normal form within $\mathbb{V}_\nu^3 \cap \mathbb{V}_S^{3,1}$ of $J_{3B}$. It is easy to see why:- The unperturbed vector field truncated after second order reads

$$\dot{x}_1 = n^{(1)}_{110} x_1 x_2, \quad \dot{x}_2 = x_3 + n^{(2)}_{020} x_2^2 + n^{(2)}_{200} x_1^2, \quad \dot{x}_3 = -\left( \frac{1}{2} n^{(1)}_{110} + n^{(2)}_{020} \right) x_2 x_3.
$$

(5.57)

Every point on the parabola

$$x_3 + n^{(2)}_{200} x_1^2 = 0,$$
### 5.5. Mirror Symmetry

<table>
<thead>
<tr>
<th>( x_1^2 )</th>
<th>( \partial_1 )</th>
<th>( \partial_2 )</th>
<th>( \partial_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( u^{(2)}_{110} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_1 x_2 )</td>
<td>( u^{(1)}_{110} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_1 x_3 )</td>
<td>( u^{(1)}<em>{101} - \ell^{(1)}</em>{110} )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_2^2 )</td>
<td>0</td>
<td>( u^{(2)}_{020} )</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 x_3 )</td>
<td>0</td>
<td>( u^{(2)}<em>{011} - \frac{1}{3} \ell^{(1)}</em>{110} - 3 \ell^{(2)}_{020} )</td>
<td>(-\frac{1}{2} u^{(1)}<em>{110} - u^{(2)}</em>{020} )</td>
</tr>
<tr>
<td>( x_3^2 )</td>
<td>0</td>
<td>( u^{(2)}<em>{002} - \frac{1}{3} \ell^{(1)}</em>{101} - \frac{4}{3} \ell^{(2)}_{011} )</td>
<td>(-\frac{1}{3} u^{(1)}<em>{101} - \frac{1}{3} u^{(2)}</em>{011} ) (+ \frac{1}{2} \ell^{(1)}<em>{110} + \ell^{(2)}</em>{020} )</td>
</tr>
</tbody>
</table>

#### a) Mapping.

<table>
<thead>
<tr>
<th>( \partial_1 )</th>
<th>( \partial_2 )</th>
<th>( \partial_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_1 )</td>
<td>( u^{(1)}_{110} )</td>
<td>( u^{(2)}_{020} )</td>
</tr>
<tr>
<td>( C_2 )</td>
<td>( u^{(2)}_{120} )</td>
<td>1</td>
</tr>
<tr>
<td>( I_1 )</td>
<td>( u^{(1)}<em>{101} - \ell^{(1)}</em>{110} )</td>
<td>( u^{(2)}<em>{011} - \frac{1}{2} \ell^{(1)}</em>{110} - 3 \ell^{(2)}_{020} )</td>
</tr>
<tr>
<td>( K_1 )</td>
<td>( u^{(2)}<em>{002} - \frac{1}{3} \ell^{(1)}</em>{101} - \frac{4}{3} \ell^{(2)}_{011} )</td>
<td>1</td>
</tr>
<tr>
<td>( K_2 )</td>
<td>( t^{(2)}<em>{100}, t^{(2)}</em>{102} )</td>
<td>2</td>
</tr>
</tbody>
</table>

#### b) Re-ordering into subspaces.

Table 5.12: \( \mu_2(z) + \text{ad} J_{3B}(t_2(z)) \), with \( \mu_2(z), t_2(z) \in \mathbb{V}^3_S \cap \mathbb{V}^3_S \cap \mathbb{H}_s^3 \)
in the plane \( x_2 = 0 \) is a critical point for the vector field (5.57). The parabola of critical points appears after computation and truncation of the normal form. In general, this parabola is not present in the untruncated original vector field. Therefore, the parabola of critical points is an abnormality caused by premature truncation.

To show this, we compute the third-order normal form terms. The reordering into subspaces of \( u_3(x) + \text{ad} J_{3B}(t_3(x)) \), in Table 5.13b, shows that the complementary subspace is given by

\[
G'_3 = n^{(1)}_{300} x_1^3 \partial_1 + n^{(2)}_{210} x_1^2 x_2 \partial_2 - (\frac{3}{2} n^{(1)}_{300} + \frac{1}{2} n^{(2)}_{210}) x_1^2 x_3 \partial_3
+ n^{(1)}_{110} x_1 x_2^2 \partial_1 + n^{(2)}_{030} x_2^3 \partial_2 - (\frac{1}{2} n^{(1)}_{110} + \frac{3}{2} n^{(2)}_{030}) x_2^2 x_3 \partial_3.
\]

(5.58)

where \( n^{(s)}_{i,j,k} = \bar{u}^{(s)}_{i,j,k} \). The reason for using the tilde is explained in Chapter 3.

The two terms in \( C_2 \) of \( G'_3 \) are congruent to terms in \( C_1 \) of \( G'_2 \). There are no congruent terms in \( C_1 \) of \( G'_2 \). Fortunately, one can be removed using the kernel \( \mathcal{K}'_2 \). To this end, we compute the Lie product of \( G'_2 \) and \( \mathcal{K}'_2 \). Table 5.14 shows the relevant information. We observe that either \( n^{(1)}_{300} \) or \( n^{(2)}_{210} \) from \( C_1 \) in \( G'_3 \) can be removed, if \( n^{(1)}_{110} \neq 0 \) or \( n^{(1)}_{110} \neq n^{(2)}_{030} \), respectively. The reader can easily check that either one removes the parabola of critical points. We choose to remove \( n^{(1)}_{300} \) for Proposition 5.4.

Finally, we mention that the complementary subspace \( G'_4 \) produces only terms congruent to terms in \( G'_2 \) and \( G'_3 \). Therefore, we do not expect fourth (and higher) order terms to influence the topological classification of fundamental local flow patterns.

**Versal deformation**

The versal deformation within \( \mathbb{V}_1^3 \cap \mathbb{V}_2^{3,1} \) is found by computing the adjoint of \( J_{3B} \) acting on \( H_3^3 \cap \mathbb{V}_2^3 \cap \mathbb{V}_2^{3,1} \) and \( H_1^3 \cap \mathbb{V}_2^3 \cap \mathbb{V}_2^{3,1} \). Table 5.15 and Table 5.16 show the resulting reordering into subspaces, from which we conclude that

\[
G'_0 = \mu^{(2)}_{000} \partial_2, \quad \mathcal{K}'_0 = t^{(2)}_{000} \partial_2.
\]

(5.59)

\[
G'_1 = \mu^{(1)}_{100} x_1 \partial_1 + \mu^{(2)}_{010} x_2 \partial_2 - \frac{1}{2} (\mu^{(1)}_{100} + \mu^{(2)}_{010}) x_3 \partial_3,
\]

(5.60)

and

\[
\mathcal{K}'_1 = F_0 (3x_1 \partial_1 - x_2 \partial_2 - x_3 \partial_3) + t^{(2)}_{001} x_3 \partial_2.
\]

(5.61)
5.5. Mirror symmetry

\[
\begin{array}{|c|c|c|c|}
\hline
\partial_1 & \partial_2 & \partial_3 \\
\hline
x_1^3 & u_{300}^{(1)} & 0 & 0 \\
\hline
x_1^2 x_2 & 0 & u_{210}^{(2)} & 0 \\
\hline
x_1 x_3^2 & 0 & u_{201}^{(2)} - \frac{3}{2} t_{300}^{(1)} - \frac{3}{2} t_{210}^{(2)} - \frac{3}{2} u_{300}^{(1)} - \frac{1}{2} u_{210}^{(2)} & 0 \\
\hline
x_1 x_2 & 0 & u_{120}^{(1)} & 0 \\
\hline
x_1 x_2 x_3 & 0 & u_{111}^{(1)} - 2 t_{120}^{(1)} & 0 \\
\hline
x_1 x_3^2 & 0 & u_{102}^{(1)} - t_{111}^{(1)} & 0 \\
\hline
x_2 & 0 & u_{030}^{(2)} & 0 \\
\hline
x_2^2 x_3 & 0 & u_{021}^{(2)} - \frac{1}{2} t_{120}^{(1)} - \frac{1}{2} t_{030}^{(1)} - \frac{1}{2} u_{120}^{(1)} - \frac{1}{2} u_{030}^{(2)} & 0 \\
\hline
x_2 x_3^2 & 0 & u_{012}^{(2)} - \frac{1}{2} t_{111}^{(1)} - \frac{1}{2} t_{021}^{(2)} - \frac{1}{2} t_{111}^{(1)} - \frac{1}{2} u_{021}^{(2)} & 0 \\
\hline
x_3^3 & 0 & u_{003}^{(2)} - \frac{1}{4} t_{102}^{(1)} - \frac{5}{4} t_{012}^{(2)} - \frac{1}{4} t_{012}^{(1)} - \frac{1}{4} u_{012}^{(2)} & 0 \\
\hline
\end{array}
\]

\[a) \text{ Mapping.}\]

\[
\begin{array}{|c|c|c|c|}
\hline
\partial_1 & \partial_2 & \partial_3 \\
\hline
C_1 & u_{300}^{(1)} & u_{210}^{(2)} & -\frac{3}{2} u_{300}^{(1)} - \frac{1}{2} u_{210}^{(2)} \\
\hline
C_2 & u_{120}^{(1)} & u_{030}^{(2)} & -\frac{1}{2} u_{120}^{(1)} - \frac{3}{2} u_{030}^{(2)} \\
\hline
K_1 & u_{201}^{(2)} - \frac{3}{2} t_{300}^{(1)} - \frac{3}{2} t_{210}^{(2)} & 1 \\
\hline
K_2 & u_{003}^{(2)} - \frac{1}{4} t_{102}^{(1)} - \frac{5}{4} t_{012}^{(2)} & 1 \\
\hline
K_3 & t_{201}^{(2)}, t_{003}^{(2)} & 2 \\
\hline
\end{array}
\]

\[b) \text{ Re-ordering into subspaces (fragment).}\]

Table 5.13: \( u_3(x) + \text{ad} J_{3B} (t_3(x)) \), with \( u_3(x), t_3(x) \in V_3^3 \cap V_{3,1}^3 \cap H_3^3 \)
Table 5.14: The Lie product of $\mathcal{G}_2^\nu$ and $\mathcal{K}_2^\nu$ (fragment). The labels $C_i$, $i \in \{1, 2\}$, refer to the subspaces in Table 5.13b.

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$u_{300}^{(1)}$</th>
<th>$n_{110}^{(1)} n_{200}^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_{210}^{(2)}$</td>
<td>$2 \left( n_{020}^{(2)} - n_{110}^{(1)} \right) t_{200}^{(2)}$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$u_{120}^{(1)}$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$u_{030}^{(2)}$</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\partial_1$</th>
<th>$\partial_2$</th>
<th>$\partial_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>$\mu_{000}^{(2)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

a) Mapping.

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$\mu_{000}^{(2)}$</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>$t_{000}^{(2)}$</td>
<td>1</td>
</tr>
</tbody>
</table>

b) Re-ordering into subspaces.

Table 5.15: $\mu_0(x) + \text{ad } J_{3B}(t_0(x))$, with $\mu_0(x), t_0(x) \in \mathbb{V}_v^3 \cap \mathbb{V}_S^{3,1} \cap H_0^3$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$\mu_{100}^{(1)}$</th>
<th>$0$</th>
<th>$0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>$0$</td>
<td>$\mu_{010}^{(2)}$</td>
<td>$\mu_{001}^{(2)} - \frac{1}{2} t_{100}^{(1)} - \frac{3}{2} t_{010}^{(2)}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-\frac{1}{2}(\mu_{100}^{(1)} + \mu_{010}^{(2)})$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\partial_1$</th>
<th>$\partial_2$</th>
<th>$\partial_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\mu_{100}^{(1)}$</td>
<td>$\mu_{010}^{(2)}$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$0$</td>
<td>$\mu_{010}^{(2)}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

a) Mapping.

<table>
<thead>
<tr>
<th>$C_1$</th>
<th>$\mu_{100}^{(1)}$</th>
<th>$\mu_{010}^{(2)}$</th>
<th>$-\frac{1}{2}(\mu_{100}^{(1)} + \mu_{010}^{(2)})$</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_1$</td>
<td>$\mu_{001}^{(2)} - \frac{1}{2} t_{100}^{(1)} - \frac{3}{2} t_{010}^{(2)}$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K_2$</td>
<td>$t_{001}^{(2)}$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

b) Re-ordering into subspaces.

Table 5.16: $\mu_1(x) + \text{ad } J_{3B}(t_1(x))$, with $\mu_1(x), t_1(x) \in \mathbb{V}_v^3 \cap \mathbb{V}_S^{3,1} \cap H_1^3$
The first term in $\mathcal{K}_1$ is found by setting $t_{100}^{(1)} = 3F_0 + \frac{1}{2} \mu_{001}^{(2)}$ and $t_{010}^{(2)} = -F_0 + \frac{1}{2} \mu_{001}^{(2)}$ in the subspace $K_1$ in Table 5.16b.

To see if we can remove terms from $G_i^\nu$ in (5.60), we compute the Lie product of $G_2^\nu$ and $K_0^\nu$,

$$[G_2^\nu, K_0^\nu] = \left[ \begin{pmatrix} n_{110}^{(1)} x_1 x_2 \\ n_{200} x_1^2 + n_{020} x_2^2 \\ -\frac{1}{2} (n_{010}^{(1)} + 2n_{020}) x_2 x_3 \end{pmatrix}, \begin{pmatrix} 0 \\ t_{000}^{(2)} \\ 0 \end{pmatrix} \right] =$$

$$t_{000}^{(2)} \begin{pmatrix} n_{110}^{(1)} x_1 \\ 2n_{020} x_2 \\ -\frac{1}{2} (n_{010}^{(1)} + 2n_{020}) x_3 \end{pmatrix}.$$

(5.62)

We observe that it is possible to remove either $\mu_{100}^{(1)}$ or $\mu_{010}^{(2)}$ if $n_{110}^{(1)} \neq 0$ or $n_{020}^{(2)} \neq 0$, respectively. We choose to remove $\mu_{100}^{(1)}$ resulting in the following topological normal form

$$\dot{x}_1 = n_{110}^{(1)} x_1 x_2,$n_{010}^{(1)} x_1 x_2,$n_{020}^{(1)} x_1 x_2,$n_{210}^{(1)} x_1 x_2.$

$$\dot{x}_2 = \mu_1 + \mu_2 x_2 + x_3 + n_{200} x_1^2 + n_{020} x_2^2 + n_{210} x_1 x_2,$n_{210} x_1 x_2$.

$$\dot{x}_3 = -\frac{1}{2} \mu_2 x_3 - \frac{1}{2} (n_{110}^{(1)} + 2n_{020}) x_2 x_3 - \frac{1}{2} n_{210} x_1^2 x_3.$$

(5.63)

(without the congruent terms).

Scaling

The last step towards the topological normal form (5.63) is to use a scaling of the coordinates and the time parameter to reduce the number of cases to be studied. Substitution of $x_i \rightarrow f_i x_i$ and $t \rightarrow f_t t$ into the vector field (5.63) produces to following list of requirements:

$$A; \quad f_1 f_2 n_{110}^{(1)} = 1,$n_{010}^{(1)} n_{210}^{(2)}$.

$$B; \quad f_1 f_2 n_{200}^{(2)} = f_2,$n_{110}^{(1)}$.

$$C; \quad f_1 f_2 n_{110}^{(2)} = 1.$$

Not all of these requirements can be met. For example, the first and last equation are incompatible.

For the correct interpretation of the flow patterns, it is necessary to keep $f_1 > 0$ and $f_3 > 0$. Hence, equation $D$ fixes the sign of $f_2$. With $f_2 = |n_{200}^{(2)}/n_{210}^{(2)}|$,\n
\[ f_t = 1 / | f_2 n_{020}^{(2)} |, \text{ and } f_t f_1^2 = 1 / | n_{210}^{(2)} |, \] all coefficients in the second row of (5.63) can be replaced by ±1.

### 5.6 Higher co-dimensions

The topological normal form found for \( J_{3C} \) and for \( J_{3B} \) with mirror symmetry are both co-dimension 2. However, it will be more laborious to find all possible fundamental local flow patterns in the case of \( J_{3B} \) than the case of \( J_{3C} \). For, the former has a topological normal normal form with more coefficients equal to ±1 and has more unfolded critical points.

The critical points listed in Proposition 5.5 have an even higher co-dimension. We know the co-dimension of a vector field which has Jacobian matrix \( J_{3A} \) by computing the dimension of the complementary subspace of \( \text{ad} \ J_{3A}(\cdot) \)-operator (similarly for \( J_{3B} \)). Table 5.17 and Table 5.18 provide the proof needed for Proposition 5.5. Calculation of the co-dimension of \( J_{4B} \) is trivial; one only has to count the number of independent coefficients in the general form of the Jacobian matrix, see, for example, (1.19).

### 5.7 Discussion

#### 5.7.1 About the vorticity transport equation

This chapter successfully computed topological normal forms within \( \mathbb{V}_\nu^3 \) for four types of degeneracies. These topological normal forms represent vector fields which satisfy the continuity equation and the no-slip condition at a possibly curved boundary. The question we now like to answer is whether the topological normal forms computed in this chapter also represent vector fields which in addition satisfy the vorticity transport equation.

Topological normal form within \( \mathbb{V}_\nu^3 \) contain reparametrized normal-from coefficients. They result after three transformations:- In reverse order, first, a transformation to a normal form, second, a transformation to make the Jacobian matrix a Jordan normal form, and third, a transformation to make the possibly curved boundary surface coincide with a coordinate plane. Transformation theory as discussed in Chapter 3 enables us to express the coefficients in these polynomial vector fields in terms of the coefficients from the Taylor series expansion in the original Cartesian coordinate system. Appendix C, for example, computes the coefficient relations due to the vorticity transport equation after the transformation to make the possibly curved boundary surface coincide.
### Table 5.17: $\mu_1(\bar{x}) + \text{ad} J_{3A}(t_1(\bar{x}))$, with $\mu_1(\bar{x}), t_1(\bar{x}) \in \mathbb{V}_3 \cap H_1^3$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial_1$</td>
<td>$\mu^{(1)}<em>{100} + t^{(2)}</em>{100}$</td>
<td>$\mu^{(1)}<em>{010} + t^{(2)}</em>{010} - t^{(1)}_{100}$</td>
</tr>
<tr>
<td>$\partial_2$</td>
<td>$\mu^{(2)}_{100}$</td>
<td>$\mu^{(2)}<em>{010} - t^{(2)}</em>{100}$</td>
</tr>
<tr>
<td>$\partial_3$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

#### a) Mapping.

<table>
<thead>
<tr>
<th>$\partial_1$</th>
<th>$\partial_2$</th>
<th>$\partial_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$\mu^{(1)}<em>{100} + t^{(2)}</em>{100}$</td>
<td>$\mu^{(2)}<em>{010} - t^{(2)}</em>{100}$</td>
</tr>
<tr>
<td>$C_2$</td>
<td></td>
<td>$\mu^{(2)}_{100}$</td>
</tr>
<tr>
<td>$C_3$</td>
<td></td>
<td>$\mu^{(2)}_{001}$</td>
</tr>
<tr>
<td>$I_1$</td>
<td>$\mu^{(1)}<em>{001} - t^{(2)}</em>{001}$</td>
<td></td>
</tr>
<tr>
<td>$K_1$</td>
<td>$\mu^{(1)}<em>{010} + t^{(1)}</em>{010} - t^{(1)}_{100}$</td>
<td></td>
</tr>
<tr>
<td>$K_2$</td>
<td>$t^{(1)}<em>{010}; t^{(1)}</em>{001}$</td>
<td></td>
</tr>
</tbody>
</table>

#### b) Re-ordering into subspaces.
### Table 5.18: $\mu_1(x) + \text{ad } J_{3B}(t_1(x))$, with $\mu_1(x), t_1(x) \in \mathbb{V}_r^3 \cap H_1^3$

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\partial_1$</td>
<td>$\mu^{(1)}_{100}$</td>
<td>$\mu^{(1)}_{010}$</td>
</tr>
<tr>
<td>$\partial_2$</td>
<td>$\mu^{(2)}_{100}$</td>
<td>$\mu^{(2)}_{010}$</td>
</tr>
<tr>
<td>$\partial_3$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

a) **Mapping.**

| $C_1$ | \( \mu^{(1)}_{100} \) | \( \mu^{(2)}_{010} \) | $-\frac{1}{2} \mu^{(1)}_{100} - \frac{1}{2} \mu^{(2)}_{010}$ | 2 |
| $C_2$ | \( \mu^{(1)}_{010} \) | 1 |
| $C_3$ | \( \mu^{(2)}_{100} \) | 1 |
| $I_1$ | \( \mu^{(1)}_{001} - t^{(1)}_{010} \) | |
| $K_1$ | \( \mu^{(2)}_{001} - \frac{1}{2} t^{(1)}_{100} - \frac{3}{2} t^{(2)}_{010} \) | 1 |
| $K_2$ | $\iota^{(1)}_{001}, \iota^{(2)}_{100}, \iota^{(2)}_{001}$ | 3 |

b) **Re-ordering into subspaces.**
with a coordinate plane. To avoid too lengthy expressions, however, we used reparametrization of coefficients whenever possible.

Using $\mathbb{V}_0^2$ as an example, Section 4.1 showed that substitution of the relations between the coefficients from the original Taylor series expansion resulted in a zero normal-form coefficient, compare (4.9) with (4.10). We also found a direct dependence between higher-order and lower-order normal-form coefficients, see (4.14). These observations made us look for (topological) normal forms within $\mathbb{V}_0^3$ instead.

The coefficients of the four topological normal forms in Proposition 5.1, Proposition 5.2, Proposition 5.3 and Proposition 5.4 depend in complicated ways on the coefficients from the original Taylor series expansion. In our computations we have not found, however, that one of these topological-normal-form coefficients vanishes as a direct result of the relations between the coefficients from the original Taylor series expansion due to the vorticity transport equation, not even in the (degenerate) case of a flat boundary surface. It is for this reason that we conclude that the bifurcation behavior of the topological normal forms within $\mathbb{V}_0^3$ computed in this chapter also represents vector field satisfying the vorticity transport equation.

5.7.2 About viscosity

In this thesis, we concentrated on bifurcation in viscous fluid flow. It is also possible to use our approach to find a topology of fundamental inviscid fluid flow patterns. In that case, for points on a boundary surface, the so-called nonpermeance condition holds;

$$\mathbf{n}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) = 0, \quad \text{for } \mathbf{x} \in \partial B.$$  \hspace{1cm} (5.64)

i.e., no fluid passes through the boundary surface $\partial B$. Here, $\mathbf{n}(\mathbf{x})$ is the normal to the boundary surface. Just as before, the curved boundary surface should be transformed to the tangent plane in the point about which the Taylor expansion of the vector field takes place.

Consider the following set of vector fields

$$\mathbb{V}_0^n = \left\{ \mathbf{u} : U \times \mathcal{W} \rightarrow \mathbb{R}^n \left| \begin{array}{l} \nabla \cdot \mathbf{u}(\mathbf{x}; \mu) = 0, \text{ and } \\ u^{(n)}(\mathbf{x}; \mu) = 0, \text{ on } x_n = 0 \end{array} \right. \right\}. \hspace{1cm} (5.65)$$

The subscript 0 in $\mathbb{V}_0^n$ indicates that the vector fields in this set describe the inviscid fluid flow near a boundary surface, i.e., $\nu \equiv \mu/\rho = 0$. The coordinate plane $x_n = 0$ is an invariant manifold.
The (unfolded) boundary critical points of a topological normal form in $\mathcal{V}_0^3$ satisfy the eigenvalue relation
\[ \lambda_3 = - (\lambda_1 + \lambda_2). \]

Fig. 1.3 and Fig. 1.5 which sketch the topology of local flow patterns about a hyperbolic critical point in viscous fluid above and on the boundary surface, respectively. The critical points also have an invariant manifold at $x_3 = 0$, but satisfy the eigenvalue relation
\[ \lambda_3 = - \frac{1}{2} (\lambda_1 + \lambda_2). \]

The difference is only a scaling in the direction transversal to the invariant manifold $x_3 = 0$. The result is that Fig. 1.3 and Fig. 1.5 also depict the possible local flow patterns about a hyperbolic critical point in inviscid fluid.

**Final remark**

The relative unimportance of both the vorticity transport equation and viscosity might be strange at first sight. Remember, however, that we only considered local flow patterns. Hyperbolic unfolded critical points found in a bifurcation analysis still remain infinitesimally close to the location of the original nonhyperbolic critical point. The radius of the region of validity is probably much smaller than the height of the boundary layer directly above the boundary surface.

Our steady-flow-only analysis of bifurcation of nonhyperbolic critical points enables us to determine the kind of changes in flow topology that are possible. Of course, actual changes in a flow topology can only happen in an unsteady setting. The vorticity transport equation and viscosity may then very well play an important role.
Chapter 6

Degenerate Local Flow Patterns with Mirror Symmetry

At their first appearance innovators are always derived as fools and madman. —Aldous Huxley

Introduction

In Chapter 1, we discussed the classification presented by Dallmann [Dal83] of mirror-symmetric local flow patterns about a point on a boundary surface. We argued that the classification was incorrect being based on critical points found in a truncated Taylor series approximation of the three-dimensional vector field. As explained in Chapter 4, classification of local flow patterns requires a topological normal form.

This chapter focusses on the possible flow patterns described by the degeneracy of the topological normal form given in Proposition 5.4. To find them, we need to remove the perturbation parameters from the Proposition 5.4. The resulting vector field reads

\[
\begin{align*}
\dot{x}_1 &= n_{110}^{(1)} x_1 x_2 \\
\dot{x}_2 &= x_3 + n_{200}^{(2)} x_1^2 + n_{020}^{(2)} x_2^2 + n_{210}^{(2)} x_1^2 x_2 \\
\dot{x}_3 &= -\frac{1}{2} (n_{110}^{(1)} + 2n_{020}^{(2)}) x_2 x_3 - \frac{1}{2} n_{210}^{(2)} x_1^2 x_3
\end{align*}
\]  

(6.1)
According to Proposition 5.4, we need to analyse the eight cases $n_{200}^{(2)} = \pm 1$, $n_{020}^{(2)} = \pm 1$, and $n_{210}^{(2)} = \pm 1$. In addition, different values for the coefficient $n_{110}^{(1)}$ can also lead to changes in flow topology. This chapter analyses the local flow patterns of (6.1) in three invariant manifolds: the boundary surface $x_3 = 0$, the plane of symmetry $x_1 = 0$, and a certain curved surface $x_3 = x_3(x_1, x_2)$. Combining the resulting two-dimensional flow patterns in these three manifolds facilitates us to construct the three-dimensional flow patterns.

6.1 Skin-friction patterns

Flow patterns on the boundary surface $x_3 = 0$ in (6.1) represent skin-friction patterns, and are governed by the vector field

$$
\dot{x}_1 = n_{110}^{(1)} x_1 x_2 \\
\dot{x}_2 = n_{200}^{(2)} x_1^2 + n_{020}^{(2)} x_2^2 + n_{210}^{(2)} x_1 x_2
$$

(6.2)

see (6.1). Here $n_{200}^{(2)} = \pm 1$, $n_{020}^{(2)} = \pm 1$, and $n_{210}^{(2)} = \pm 1$, and $n_{110}^{(1)}$ is a free coefficient. Note that the vector field (6.2) is symmetric with respect to the 'line' $x_1 = 0$; the substitution

$$
x_1 \rightarrow -x_1,
$$

does not lead to a new vector field representation.

The term $n_{210}^{(2)} x_1^2 x_2^2$ does not contribute to the qualitative character of the local flow patterns of (6.2). To prove this we introduce a blow-up of coordinates,

$$
x_1 = r \cos \varphi, \\
x_2 = r \sin \varphi,
$$

(6.3)

$r \geq 0, \varphi \in [0, 2\pi)$, so that

$$
\dot{r} = \dot{x}_1(r, \varphi) \cos \varphi + \dot{x}_2(r, \varphi) \sin \varphi, \\
r \dot{\varphi} = -\dot{x}_1(r, \varphi) \sin \varphi + \dot{x}_2(r, \varphi) \cos \varphi.
$$

(6.4)

A blow-up 'smears out' the original critical point over the line $r = 0$ in the $r, \varphi$-plane (with $\varphi \in [0, 2\pi]$) so that its qualitative character can be analysed.

Substitution of the change of coordinates (6.3) into the vector field (6.2) results in

$$
\dot{r} = r \left\{ (n_{110}^{(1)} + n_{200}^{(2)}) \cos^2 \varphi \sin \varphi + n_{020}^{(2)} \sin^3 \varphi + r n_{210}^{(2)} \cos^2 \varphi \sin^2 \varphi \right\}, \\
\dot{\varphi} = (n_{020}^{(2)} - n_{110}^{(1)}) \sin^2 \varphi \cos \varphi + n_{200}^{(2)} \cos^3 \varphi + r n_{210}^{(2)} \cos^3 \varphi \sin \varphi.
$$

(6.5)
Through reparameterization, we removed the common factors \( r \).

On the line \( r = 0 \) in the \( r, \varphi \)-plane, (6.5) has blow-up critical points at \( \varphi = \varphi_0 \) satisfying

\[
\cos \varphi_0 = 0, \quad \text{or} \quad \tan^2 \varphi_0 = \frac{n_{200}^{(2)}}{n_{110}^{(1)} - n_{020}^{(2)}}.
\] (6.6)

By showing that the blow-up critical points are not effected by the parameter \( n_{210}^{(2)} \), we have proven our case.

The linearized vector field of (6.5) about \( r = 0, \varphi = \varphi_0 \) equals

\[
\begin{align*}
\dot{r}_\ell &= (A_{11} r_\ell + A_{12} \varphi_\ell) \sin \varphi_0, \\
\dot{\varphi}_\ell &= (A_{21} r_\ell + A_{22} \varphi_\ell) \sin \varphi_0,
\end{align*}
\] (6.7)

where \( r_\ell = r \) and \( \varphi_\ell = \varphi - \varphi_0 \) are local coordinates, and where

\[
\begin{align*}
A_{11} &= (n_{110}^{(1)} + n_{200}^{(2)}) \cos^2 \varphi_0 + n_{020}^{(2)} \sin^2 \varphi_0, \\
A_{12} &= 0, \\
A_{21} &= n_{210}^{(2)} \cos^3 \varphi_0, \\
A_{22} &= (2n_{020}^{(2)} - 2n_{110}^{(1)} - 3n_{200}^{(2)}) \cos^2 \varphi_0 + (n_{110}^{(1)} - n_{020}^{(2)}) \sin^2 \varphi_0.
\end{align*}
\] (6.8)

The invariants of the Jacobian matrix of the linearized vector field,

\[ A_0 = \begin{pmatrix}
A_{11} \sin \varphi_0 & A_{12} \sin \varphi_0 \\
A_{21} \sin \varphi_0 & A_{22} \sin \varphi_0
\end{pmatrix},
\]

are the trace \( p \) and the determinant \( q \), see (1.24). The coefficient \( n_{210}^{(2)} \) enters neither the expression for the trace \( p \) nor for the determinant \( q \). This argument concludes our proof.

We will omit the term \( n_{210}^{(2)} x_1^2 x_2 \) in the remainder of this section. The omission produces an additional symmetry:

\[
x_2 \to -x_2, \quad n_{110}^{(1)} \to -n_{110}^{(1)}, \quad n_{200}^{(2)} \to -n_{200}^{(2)}, \quad n_{020}^{(2)} \to -n_{020}^{(2)}.
\] (6.9)

This symmetry will be used to relate the skin-friction patterns obtained for \( n_{200}^{(2)} > 0 \) with the skin-friction patterns for \( n_{200}^{(2)} < 0 \).

Consider the blow-up critical points at \( \varphi_0 = \frac{1}{2} \pi \) and \( \varphi_0 = \frac{3}{2} \pi \), i.e. where \( \cos \varphi_0 = 0 \). Because \( A_{12} = 0 \), the matrix \( A_0 \) is upper-triangular and the eigenvalues equal the elements on the diagonal. They are

\[
\lambda_r = A_{11} \sin \varphi_0, \quad \lambda_\varphi = A_{22} \sin \varphi_0.
\] (6.10)
or, for the specific values for the $\varphi_0$ on hand,

$$\lambda_r = (\pm) n_{020}^{(2)}, \quad \lambda_\varphi = (\pm) \left( n_{110}^{(1)} - n_{020}^{(2)} \right),$$

where the plus-sign within brackets occurs at $\varphi_0 = \frac{1}{2} \pi$, and the minus-sign occurs at $\varphi_0 = \frac{3}{2} \pi$.

Based on the above results, the next list presents various regions and the topological classification of the blow-up critical points.

I. If $n_{020}^{(2)} > 0$ and $n_{020}^{(2)} > n_{110}^{(1)}$, the critical points at $(0, \frac{1}{2} \pi)$ and $(0, \frac{3}{2} \pi)$ are saddles, and $\lambda_r$ is positive and negative, respectively.

II. If $n_{020}^{(2)} < 0$, and $n_{020}^{(2)} > n_{110}^{(1)}$, the critical point at $(0, \frac{1}{2} \pi)$ is a stable node, and the critical point at $(0, \frac{3}{2} \pi)$ is an unstable node.

III. If $n_{020}^{(2)} < 0$ and $n_{020}^{(2)} < n_{110}^{(1)}$, the critical points at $(0, \frac{1}{2} \pi)$ and $(0, \frac{3}{2} \pi)$ are saddles, and $\lambda_r$ is negative and positive, respectively.

IV. If $n_{020}^{(2)} > 0$, and $n_{020}^{(2)} < n_{110}^{(1)}$, the critical point at $(0, \frac{1}{2} \pi)$ is an unstable node, and the critical point at $(0, \frac{3}{2} \pi)$ is a stable node.

The blow-up critical points are sketched in Fig. 6.1 (this figure also includes upcoming results).

Consider the blow-up critical points at $\tan^2 \varphi_0 = n_{200}^{(2)}/(n_{110}^{(1)} - n_{020}^{(2)})$. We find four critical points at: $\varphi_0 = \theta$, $\varphi_0 = \pi - \theta$, $\varphi_0 = \pi + \theta$, and $\varphi_0 = 2\pi - \theta$, under the condition that

$$n_{200}^{(2)} < 0 \text{ and } n_{020}^{(2)} > n_{110}^{(1)}, \quad \text{or} \quad n_{200}^{(2)} > 0 \text{ and } n_{020}^{(2)} < n_{110}^{(1)},$$

where $\theta = \arctan \sqrt{n_{200}^{(2)}/(n_{110}^{(1)} - n_{020}^{(2)})}$. From (6.8) and (6.10), we get

$$\lambda_r = \frac{-n_{110}^{(1)}}{n_{110}^{(1)} - n_{020}^{(2)}} (n_{020}^{(2)} - n_{110}^{(1)} - n_{200}^{(2)}) \cos^2 \varphi_0 \sin \varphi_0, \quad (6.11)$$

$$\lambda_\varphi = 2(n_{020}^{(2)} - n_{110}^{(1)} - n_{200}^{(2)}) \cos^2 \varphi_0 \sin \varphi_0.$$

It is not difficult to determine the sign of the eigenvalues given the values of the coefficients $n_{110}^{(1)}$, $n_{200}^{(2)}$, and $n_{020}^{(2)}$. Through the assumptions made, in the region $n_{020}^{(2)} > n_{110}^{(1)}$, $n_{200}^{(2)} < 0$ holds, and therefore $n_{020}^{(2)} > n_{110}^{(1)} + n_{200}^{(2)}$. This factor is present in the above expression for the eigenvalues. Evaluation of $p = \lambda_r + \lambda_\varphi$
Figure 6.1: Blow-up of skin-friction patterns for $n_{200}^{(2)} > 0$. Use substitution (6.9) to obtain the blow-up for $n_{200}^{(2)} < 0$. 
shows that its sign does not change in the region. Finally, the common factor 
\( \sin \varphi_0 \) changes the sign of both eigenvalues, and thus changes the stability of the 
phase portrait.

Using the same labels as in Fig. 6.1, we have

III\(_+\). four additional saddles, with \( \lambda_r \) is positive, positive, negative, and negative, at \( \varphi_0 = \theta, \varphi_0 = \pi - \theta, \varphi_0 = \varphi = \pi + \theta, \) and \( \varphi_0 = 2\pi - \theta, \) respectively,

III\(_-\). four additional nodes, with \( \lambda_r \) is negative, negative, positive, and positive, at \( \varphi_0 = \theta, \varphi_0 = \pi - \theta, \varphi_0 = \varphi = \pi + \theta, \) and \( \varphi_0 = 2\pi - \theta, \) respectively, or

IV. four additional saddles, with \( \lambda_r \) as in III\(_+\).

Fig. 6.1 presents the blow-up of the skin-friction patterns for \( n_{200}^{(2)} > 0. \) The blow-up of the skin-friction patterns for \( n_{200}^{(2)} < 0 \) are found through substitution of (6.9). The 'difference' between the topologically equivalent cases I\(_+\) and I\(_-\) is based on the sign of the coefficient \( n_{110}^{(1)} \). In the former case, the skin-friction lines converge to the symmetry plane \( x_1 = 0, \) whereas they diverge in the latter case.

Fig. 6.2 sketches the skin-friction patterns in the \( x_1-x_2 \) plane. Note the dashed lines in these patterns; they represent the two isoclines \( \dot{x}_2 = 0, \) found at

\[
x_2^2 = -n_{200}^{(2)}/n_{020}^{(2)} x_1^2
\]

under omission of the term \( n_{210}^{(2)} x_1^2 x_2 \partial_2 \) in (6.2).

6.2 Flow patterns in the plane of symmetry

By construction, the coordinate plane \( x_1 = 0 \) is the plane of symmetry for the vector fields described by Proposition 5.4. Consequently, \( x_1 = 0 \) is an invariant manifold in the vector field of its degeneracy (6.1). The flow pattern in this manifold is the subject of discussion in this section.

On \( x_1 = 0, \) the vector field (6.1) reduces to

\[
\begin{align*}
\dot{x}_2 &= x_3 + n_{020}^{(2)} x_2^2 \\
\dot{x}_3 &= -\frac{1}{2}(n_{110}^{(1)} + 2n_{020}^{(2)}) x_2 x_3
\end{align*}
\]

As can be seen, the coefficient \( n_{210}^{(2)} \) is again not of interest.

A blow-up of coordinates via \( x_2 = r \cos \varphi, \ x_3 = r \sin \varphi \) only informs us that \( \varphi = 0 \) and \( \varphi = \pi \) are invariant lines, corresponding with the boundary
Figure 6.2: Skin-friction patterns for $n_{200}^{(2)} > 0$. Use substitution (6.9) to obtain the flow patterns for $n_{200}^{(2)} < 0$. 
surface \( x_3 = 0 \). To find the flow patterns, we make use of a classical result by Andronov et al. [ALGM73], discussed in Appendix B as Theorem B.5. Observe that the vector field (6.13) is of the form (B.3) with \( P(x, y) = n_{020}^{(2)} x^2 \), and \( Q(x, y) = -\frac{1}{2}(n_{110}^{(1)} + 2n_{020}^{(2)})xy \). Application of Theorem B.5 is, in this case, very straightforward since there exists an explicit expression for the function \( \varphi(x) \), viz,

\[
\varphi(x) = -n_{020}^{(2)} x^2.
\]

With this expression, the functions \( p(x) \) and \( q(x) \) defined in (B.4) become

\[
q(x) = q_m x^m = \frac{1}{2}(n_{110}^{(1)} + 2n_{020}^{(2)})n_{020}^{(2)} x^3, \\
p(x) = p_n x^n = \frac{1}{2}(2n_{020}^{(2)} - n_{110}^{(1)}) x,
\]

Thus, \( m = 3 \), and odd. If \( q_3 = \frac{1}{2}(n_{110}^{(1)} + 2n_{020}^{(2)})n_{020}^{(2)} > 0 \), the critical point is a topological saddle. If not, since \( p_1 \neq 0 \) in general, \( n = \ell = 1 \), and \( \lambda = p_1^2 + 4(1 + 1)q_3 = \frac{1}{4}(n_{110}^{(1)} + 6n_{020}^{(2)})^2 \), which is always positive, and the critical point has an elliptic sector (possibly ‘under’ the boundary surface).

The above theorem provides the topological classification of the flow patterns, but we still need to know what part of these flow patterns is in the visible half space \( x_3 > 0 \). Therefore, we compute trajectories of (6.13) approaching the origin. It is not difficult to see that \( x_3 = C x_2^2 \) describes one of these trajectories. The factor \( C \) is found to be either \( C = 0 \) (representing the boundary surface) or \( C = -\frac{1}{2}(n_{110}^{(1)} + 6n_{020}^{(2)}) \). The latter trajectory parabola is ‘visible’ if \( n_{110}^{(1)} + 6n_{020}^{(2)} < 0 \). On the parabola \( x_3 = -n_{020}^{(2)} x_2^2 \), we have \( \dot{x}_2 = 0 \). A higher order degeneracy occurs of this isocline equals the aforementioned parabola. This situation occurs when \( n_{020}^{(2)} = 0 \) or \( n_{110}^{(1)} + 2n_{020}^{(2)} = 0 \). Thus, for \( n_{110}^{(1)} > 0 \), when \( n_{110}^{(1)} + 2n_{020}^{(2)} \) changes sign from negative to positive, the position of the trajectory parabola changes from above to below the isocline, respectively.

With the above results, the flow patterns in the symmetry plane can be constructed, see Fig. 6.3. The region around the origin indicated with a circular arc depicts the region of topological saddles. Note the heavy-black parabola in the flow pattern with \( n_{110}^{(1)} + 6n_{020}^{(2)} < 0 \). It is the aforementioned parabola \( x_3 = C x_2^2 \). The parabola plays a special role in the flow patterns in that it separates the parabolic and elliptical sector.

Strictly speaking, an elliptic sector requires evaluation of properties at infinity. For further details about topological classification of two-dimensional vector fields such as (6.13), we refer to de Jager [dJ90].
Figure 6.3: Flow patterns in the plane of symmetry.
6.3 Third invariant manifold

Computer aided visualizations of the vector field (6.1) suggested that there exists a third invariant manifold given by

\[ x_3 = x_3(x_1, x_2). \]  \hspace{1cm} (6.16)

In this section, we verify this observation mathematically.

Solution curves remain on an invariant manifold at all time, thus in the case of (6.16) we have

\[ x_3(t) = x_3(x_1(t), x_2(t)). \]

Differentiation of this expression produces the requirement

\[ \dot{x}_3 = \frac{\partial x_3(x_1, x_2)}{\partial x_1} \dot{x}_1 + \frac{\partial x_3(x_1, x_2)}{\partial x_2} \dot{x}_2. \]  \hspace{1cm} (6.17)

Assume that \( x_3(x_1, x_2) = a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2 + \mathcal{O}(|x|^3). \) Zero-order and first-order terms are unlikely, and it is not difficult to show that they are indeed zero through direct computation. The coefficient \( a_{11} \) also equals zero since (6.1) is mirror symmetric with respect to the plane \( x_1 = 0. \) Substituting (6.17) into (6.1) proves this and, and shows that the other two coefficients become

\[ a_{20} = \frac{n_{200}^{(2)}}{n_{110}^{(1)} - n_{020}^{(2)}} \frac{1}{4} (n_{110}^{(1)} + 6n_{020}^{(2)}), \quad a_{02} = -\frac{1}{4} (n_{110}^{(1)} + 6n_{020}^{(2)}). \]  \hspace{1cm} (6.18)

The coefficients contain some familiar terms because the third manifold can intersect the plane of symmetry as well as the boundary surface. The third invariant manifold is visible if it is located above the boundary surface \( x_3 = 0. \) This situation occurs if \( a_{20} > 0 \) or \( a_{02} > 0. \)

The manifold intersects the plane of symmetry \( x_1 = 0 \) in the half space of interest \( x_3 \geq 0 \) if

\[ a_{02} = -\frac{1}{4} (n_{110}^{(1)} + 6n_{020}^{(2)}) > 0. \]

It then runs through the heavy black parabolas in Fig. 6.3.

Using the definition of \( \tan^2 \varphi \) from (6.6), the coefficient \( a_{20} \) can be written

\[ a_{20} = -a_{20} \tan^2 \varphi. \]

In order to have real valued solutions for \( \varphi \), the right-hand side of (6.6) has to be positive. This situation corresponds with the flow patterns III\(_-\), III\(_+\), and IV from Fig. 6.3.
If \( a_{02} < 0 \), there is no visible intersection of the third invariant manifold with the plane of symmetry \( x_1 = 0 \), but there is a visible intersection with the plane \( x_2 = 0 \) if the right-hand side of (6.6) is positive. In that case, the intersection with the boundary surface occurs at

\[
x_2^2 = \frac{n_{200}^{(2)}}{n_{110}^{(1)} - n_{020}^{(2)}} x_1^2.
\]

(6.19)

These are the four separatrixes in the aforementioned flow patterns.

The third invariant manifold intersects the two planes \( x_1 = 0 \) and \( x_2 = 0 \) if \( a_{02} > 0 \) and the right-hand side of (6.6) is negative. The latter condition corresponds with the flow patterns I-, I+, and II from Fig. 6.3. Since both \( a_{20} \) and \( a_{02} \) are positive, there is no intersection with the boundary surface other than the origin.

In summary, there is a parabolic cup-like surface resting on the boundary surface \( x_3 = 0 \) below the line \( n_{110}^{(1)} + 6n_{020}^{(2)} = 0 \) in the regions II and I_ and there is a saddle surface intersecting the boundary surface in the regions III_ and IV. In the latter region, the saddle surface is rotated one quarter compared to the former two. Fig. 6.6 sketches the third manifold.

The flow pattern on the third manifold is given by

\[
\begin{align*}
\dot{x}_1 &= n_{110}^{(1)} x_1 x_2, \\
\dot{x}_2 &= \frac{n_{200}^{(2)}}{n_{110}^{(1)} - n_{020}^{(2)}} \left( \frac{5n_{110}^{(1)} + 2n_{020}^{(2)}}{4} \right) x_1^2 - \frac{1}{4} \left( n_{110}^{(1)} + 2n_{020}^{(2)} \right) x_2^2 + O(\|x\|^3).
\end{align*}
\]

(6.20)

Two features in this vector field require special attention; the two isoclines (approximately),

\[
x_2^2 = \frac{1}{4} \frac{n_{200}^{(2)}}{n_{110}^{(1)} - n_{020}^{(2)}} \frac{5n_{110}^{(1)} + 2n_{020}^{(2)}}{n_{110}^{(1)} + 2n_{020}^{(2)}} x_1^2,
\]

and the two invariant lines given in (6.19). The latter two are found using the blow-up technique. As said before, the lines are also the location where the third invariant manifold intersects with the boundary surface \( x_3 = 0 \).

Fig. 6.4 and Fig. 6.5 sketch the flow pattern on the third invariant manifold for \( n_{200}^{(2)} \) is positive and negative, respectively. The grey arcs indicate where the manifold is invisible. The sign of the parameters \( a_{20} \) and \( a_{02} \) determines if the third invariant manifold is located above the boundary surface. Based on that fact, we conclude this section with a sketch of the third manifold in Fig. 6.6.
Figure 6.4: Flow patterns on the third manifold, $n_{200}^{(2)} > 0$. 
Figure 6.5: Flow patterns on the third manifold, $n_{200}^{(2)} < 0$. 
<table>
<thead>
<tr>
<th>$a_{02}$</th>
<th>$&lt; 0$</th>
<th>$&gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{20}$</td>
<td><img src="image1" alt="Diagram" /></td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>$&lt; 0$</td>
<td><img src="image3" alt="Diagram" /></td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>$&gt; 0$</td>
<td><img src="image5" alt="Diagram" /></td>
<td><img src="image6" alt="Diagram" /></td>
</tr>
</tbody>
</table>

Figure 6.6: The third manifold.
6.4 Discussion

This chapter investigated the degenerate local flow patterns with mirror symmetry using the topological normal form found in Chapter 5. We have computed the skin-friction patterns on the boundary surface, and the flow patterns in the plane of symmetry and on a third manifold. These three manifolds intersect, and we have shown the ramifications of such intersections with respect to the flow pattern. Combined, the flow patterns on the three manifolds provide a good qualitative understanding of the degenerate mirror-symmetric flow patterns possible.

Analysis of the degeneracy is only the first step. To complete the topology we still need to investigate the flow patterns described by the unfolded vector field.

The analysis of degenerate flow patterns required the absence of any sort of perturbations. As a result, degenerate flow patterns cannot be verified using experimental investigations or computer aided simulations. However, these techniques can find the corresponding unfolded flow patterns. A possible means of verification of a degenerate flow pattern is that certain observed changes in flow topology due to variation of flow parameters occur within its class of unfolded flow patterns. In that case, there exist critical flow parameters at which the flow behaves unstable. The unfolded flow patterns provide an indirect proof for the degenerate flow pattern.
Chapter 7

Summary, Conclusions and Recommendations

Behold! Here comes, the Dreamer.
What will become of his dreams?
—Seal, Crazy (Extended Version)

7.1 Summary

This thesis provides a method to study bifurcations in vector fields describing fluid flow about a critical point on a boundary surface. In a critical point, every component of the vector equals zero. Detailed knowledge of the critical points present in a vector field yields important information about the properties of the flow. Analysis of bifurcations makes it possible through systematic classification to generate a topology of fundamental local flow patterns containing more than a single critical point. This topology enables interpretation of complicated flow patterns around aerodynamic configurations. In addition, it provides a verbal tool to clarify dynamic changes in flow patterns.

This thesis pays particular attention to the derivation of topological normal forms. Topological normal forms are highly simplified polynomial vector fields topologically equivalent with entire classes of vector fields. Two vector fields are said to be topologically equivalent if there exists a continuous mapping between them (with a continuous inverse) that maps the flow pattern on one vector field onto the flow pattern of the other while preserving the orientation of the trajectories. This definition is the basis for classifying local flow patterns.

233
about a point. The expectation is that the polynomial vector fields found with bifurcation analysis enable us to describe flow patterns observed in experiments or computer aided simulations.

Chapter 1 introduces the problematic nature of flow visualisation and interpretation using topological properties. In addition, it recapitulates a classification of the so-called hyperbolic critical points. The Jacobian matrix of the vector field in such a point only has eigenvalues with a real part not equal to zero. The topological normal form in the case of a hyperbolic critical point consists solely of a linear vector field. The topological classification of the flow pattern about a hyperbolic critical point is stable: analytical perturbations operating on the vector field only lead to topological equivalent flow patterns.

Classification of non-hyperbolic critical points is more difficult. Normal forms for such points include non-linear terms and the topological classification of their flow patterns changes under the influence of perturbations. Chapter 2 derives a set of vector fields that describes fluid flow about a point on a boundary surface. It is thereby assumed the fluid is incompressible and viscous, and that its flow is steady.

Chapter 3 describes mathematical techniques from bifurcation theory to find topological normal forms, illustrated using two-dimensional vector fields. It furthermore introduces a new method to find every possible representation of the complementary subspace for a given (Jacobian) matrix.

However, as Chapter 4 explains in its introduction, these techniques do not generate suitable normal forms for our application. For this reason, that chapter introduces a new approach. The main point is to let the transformations used to find a normal form also operate on the perturbation parameters. Computing a normal form is needed to find a vector field that is more simpler to analyse. This new approach allowed us to maintain the physical meaning while computing a normal form within a specific set of vector fields ($\mathcal{V}_\nu^3$ defined in (2.87)).

Chapter 5 uses this new approach and derives topological normal forms for four classes of vector fields. The end of the chapter addresses the question whether the set of vector fields $\mathcal{V}_\nu^3$ indeed contains the right physics. As it turns out, the Navier-Stokes equations do not alter to the topology of local flow patterns about a point on a boundary surface.

Chapter 6 starts the analysis of mirror-symmetric flow patterns about a point on a boundary surface by studying the degeneracy. It uses the topological normal form computed in Chapter 5.
7.2 Conclusions and Recommendations

We conclude that there exists a systematic way to find the topology of three-dimensional local flow patterns near a boundary surface by studying polynomial vector field representations. A new method of analysis had to be developed to find polynomial vector field with the correct physical meaning.

In the past, authors simply truncated the Taylor expansion representation of the vector field. Such an approach is only valid in the case of one critical hyperbolic critical point. To find flow patterns with more than one critical point, it is necessary to use bifurcation theory.

However, it turned out impossible to utilize results from the literature. For example, computer algebra aided bifurcation analyses typically report lengthy expressions relating normal-form coefficients in terms of the original Taylor coefficients. We found that in the case of our application, crucial normal-form coefficients vanished, resulting in increased co-dimension, and thus leading to unsuitable deformations (unfoldings). The resulting vector fields contained coefficients with no physical meaning whatsoever.

To generate fundamental local flow patterns it was necessary to deduce bifurcation theorems specially tailored for our application. This thesis introduced two new theorems; one to find all possible representations of the complementary subspace and the kernel of the adjoint operation of a matrix in Jordan normal form, and another to find all possible representations of miniversal deformations. These techniques enabled the development of a new approach to find topological normal forms specially tailored to fluid flows.

We focussed on a specific set of vector fields ($\mathcal{V}_\nu^3$ defined in (2.87)). This set represents vector fields satisfying conservation of mass and the no-slip boundary condition. Using the theorems developed in this thesis, we successfully computed four topological normal forms within $\mathcal{V}_\nu^3$.

Initially, we did not include conservation of momentum simply to ease our analysis. This assumption turned out to be fortunate; conservation of momentum does not change our topological normal forms in any way. In other words, our topological normal forms fully represent steady flow of a viscous, incompressible fluid about a point on a boundary surface.

The theorems developed in this thesis can also be used to study local flow patterns of inviscid fluid. In that case one should compute topological normal forms within the set of vector fields $\mathcal{V}_0^3$ defined in (5.65). The only difference will be that viscosity leads to a scaling of the velocity vector in the direction normal to the boundary surface.

Chapter 6 only started the analysis of the topology of mirror-symmetric flow patterns. We recommend completion, because our exploratory computations
indicate that many interesting separated flow patterns will be found, possibly linking free-shear layer separation to *horse-shoe vortex separation*.

We recommend analysis of the topological normal form for vector fields with three zero eigenvalues and a one-dimensional eigenvector space. Kooij and Bakker [KB89] used a somewhat different polynomial vector field in their analysis. They found that this vector field links free-shear layer separation and Werlé-Legendre separation. It would be worthwhile to verify their results using our topological normal form.

Thom’s transversality theorem, see Theorem 3.27, suggests truncation of a normal form. We have shown that truncation leaves too many unnecessary terms in the resulting polynomial vector field. We would like to see a proof that our so-called congruent normal-form terms do not influence the topological classification as suggested by Conjecture 4.14.

Finally, since we only focussed on steady flows, we would like to know which role the time parameter plays in our bifurcation parameters.
Appendix A

Conjugating Matrices

Introduction

In Chapter 4 we discussed conjugating matrices bringing the Jacobian matrix of vector fields in \( \mathbb{V}_\nu^3 \) into Jordan normal form. We argued that this coordinate transformation should be such that the resulting vector field again is an element of \( \mathbb{V}_\nu^3 \). Section 4.3 demonstrated that every Jacobian matrix found in \( \mathbb{V}_\nu^n \), \( n \geq 2 \), has a conjugation matrices having this property. In this appendix, we will compute a conjugating matrix for every Jacobian matrix found in \( \mathbb{V}_\nu^3 \).

A.1 Jordan normal forms

The Jacobian matrix of a vector field \( u(\mathbf{x}) \in \mathbb{V}_\nu^3 \), describing the flow of a fluid near a boundary critical point, is given by

\[
A = \begin{pmatrix}
  u^{(1)}_{100} & u^{(1)}_{010} & u^{(1)}_{001} \\
  u^{(2)}_{100} & u^{(2)}_{010} & u^{(2)}_{001} \\
  0 & 0 & -\frac{1}{2}(u^{(1)}_{100} + u^{(2)}_{010})
\end{pmatrix}.
\]  

(A.1)

The eigenvalues of the matrix \( A \) in (A.1) are

\[
\lambda_1 = \frac{1}{2}p - \frac{1}{2}\sqrt{D}, \quad \lambda_2 = \frac{1}{2}p + \frac{1}{2}\sqrt{D}, \quad \lambda_3 = -\frac{1}{2}p,
\]

where \( p \equiv u^{(1)}_{100} + u^{(2)}_{010} \), \( q \equiv u^{(1)}_{100}u^{(2)}_{010} - u^{(2)}_{100}u^{(1)}_{010} \), and \( D \equiv p^2 - 4q \).

Table A.1 and Table A.2 list all possible Jordan normal forms in a hyperbolic and nonhyperbolic critical point, respectively. The notation used for these Jordan normal forms is conform Bakker [Bak90].
<table>
<thead>
<tr>
<th>All eigenvalues are real and different:</th>
</tr>
</thead>
</table>
| $J_{1A} = \begin{pmatrix}
\frac{1}{2} p - \frac{1}{2} \sqrt{p^2 - 4q} & 0 & 0 \\
0 & \frac{1}{2} p + \frac{1}{2} \sqrt{p^2 - 4q} & 0 \\
0 & 0 & -\frac{1}{2} p
\end{pmatrix}$ |

<table>
<thead>
<tr>
<th>All eigenvalues are real and two are equal:</th>
</tr>
</thead>
</table>
| $J_{2A} = \begin{pmatrix}
-\frac{1}{2} p & 0 & 0 \\
0 & \frac{3}{2} p & 0 \\
0 & 0 & -\frac{1}{2} p
\end{pmatrix}$ or $\begin{pmatrix}
\frac{3}{2} p & 0 & 0 \\
0 & -\frac{1}{2} p & 0 \\
0 & 0 & -\frac{1}{2} p
\end{pmatrix}$ |
| $J_{2C} = \begin{pmatrix}
-\frac{1}{2} p & 0 & 1 \\
0 & \frac{3}{2} p & 0 \\
0 & 0 & -\frac{1}{2} p
\end{pmatrix}$ or $\begin{pmatrix}
\frac{3}{2} p & 0 & 0 \\
0 & -\frac{1}{2} p & 1 \\
0 & 0 & -\frac{1}{2} p
\end{pmatrix}$ |
| $J_{2B} = \begin{pmatrix}
\frac{1}{2} p & 0 & 0 \\
0 & \frac{3}{2} p & 0 \\
0 & 0 & -\frac{1}{2} p
\end{pmatrix}$ |
| $J_{2D} = \begin{pmatrix}
\frac{1}{2} p & 1 & 0 \\
0 & \frac{1}{2} p & 0 \\
0 & 0 & -\frac{1}{2} p
\end{pmatrix}$ |

<table>
<thead>
<tr>
<th>All eigenvalues are real and equal:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varnothing$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>One eigenvalue is real and two are complex conjugated:</th>
</tr>
</thead>
</table>
| $J_{4B} = \begin{pmatrix}
\frac{1}{2} p & \frac{1}{2} \sqrt{-p^2 + 4q} & 0 \\
-\frac{1}{2} \sqrt{-p^2 + 4q} & \frac{1}{2} p & 0 \\
0 & 0 & -\frac{1}{2} p
\end{pmatrix}$ |

Table A.1: Hyperbolic Jordan normal forms in $\mathbb{V}^3_L$
### Table A.2: Nonhyperbolic Jordan Normal Forms in $\mathbb{V}_3^\lambda$

<table>
<thead>
<tr>
<th>Case</th>
<th>Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>All eigenvalues are real and different:</td>
<td>$J_{1B} = \begin{pmatrix} \frac{1}{2}p - \frac{1}{2}</td>
</tr>
<tr>
<td>All eigenvalues are real and equal:</td>
<td>$J_{3A} = \begin{pmatrix} 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$J_{3B} = \begin{pmatrix} 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$ or $\begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$J_{3C} = \begin{pmatrix} 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td>$J_{3D} = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>One eigenvalue is real and two are complex conjugated:</td>
<td>$J_{4B} = \begin{pmatrix} 0 &amp; \frac{1}{2} \sqrt{4q} &amp; 0 \ -\frac{1}{2} \sqrt{4q} &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
</tr>
</tbody>
</table>
Note that in the case of complex eigenvalues, we arranged the Jordan normal form such that it is an element of $\mathbb{V}_p^3$. Also note that certain Jordan normal form have different names despite the fact that they are topological equivalent. For example, $J_{1B}$ and $J_{1C}$ both have one vanishing eigenvalue. The difference between both cases is that the former has a vanishing eigenvalue whose eigenvector is tangent to the surface boundary while the latter has a vanishing eigenvalue whose eigenvector is transversal to the boundary surface. This geometric difference is enough reason to treat the cases $1B$ and $1C$ separately. Of course, in both cases the boundary surface is an invariant manifold meaning that trajectories of the vector field remain to be part of that surface at all times.

### A.2 Conjugating matrices

#### Eigenvectors

When computing a conjugating matrix for a given Jordan normal form, a natural by-product is a list of requirements by the elements of the matrix $A$ in (A.1) to be satisfied such that conjugation holds. It is more efficient, however, to first draw up a list of requirements stating the conditions under which $A$ is similar to the Jordan normal forms in Table A.1 and Table A.2. Table A.3 presents this list.

Conjugating matrices are not unique because their columns consist of (generalized) eigenvector which are not unique. In most cases listed in Table A.3, the eigenvectors corresponding to $\lambda_1$ and $\lambda_2$ can be written as sum of two parallel vectors as

$$a_i = a_i(l_1, l_2) \equiv l_1 \begin{pmatrix} -u_0^{(1)} & u_1^{(1)} - \lambda_i \\ u_1^{(1)} - \lambda_i & 0 \end{pmatrix} + l_2 \begin{pmatrix} u_0^{(2)} - \lambda_i \\ -u_1^{(2)} \\ 0 \end{pmatrix}.$$  \hspace{1cm} (A.2)

Suitable choices for $l_1$ and $l_2$ will give the appropriate expression for the eigenvectors.

The elements of the third row of the matrix $A - \lambda_3 I$ are all zero. Thus, the eigenvector corresponding to the eigenvalue $\lambda_3$ is the cross-product of the first row and the second row of that matrix and is given by

$$a_3 = \begin{pmatrix} -u_0^{(1)}(u_0^{(2)} + \frac{1}{2}p) + u_1^{(2)}u_0^{(1)} \\ u_0^{(1)}u_1^{(2)} - u_0^{(2)}u_1^{(1)} + \frac{1}{2}p \\ \frac{3}{2}p^2 + q \end{pmatrix}.$$  \hspace{1cm} (A.3)
### A.2. Conjugating matrices

\[ D > 0 \land E \neq 0 \land p \neq 0 \land q \neq 0 \]

\[ D > 0 \land E \neq 0 \land q = 0 \land \begin{cases} p < 0 \Rightarrow \lambda_1 = 0 \\ p > 0 \Rightarrow \lambda_2 = 0 \end{cases} \]

\[ D > 0 \land E \neq 0 \land p = 0 \land q \neq 0 \Rightarrow \lambda_3 = 0 \]

\[ D > 0 \land E = 0 \land \begin{cases} p > 0 \Rightarrow \lambda_1 = \lambda_3 \\ p < 0 \Rightarrow \lambda_2 = \lambda_3 \end{cases} \]

2A occurs if in addition \( u_{001}^{(1)} = u_{001}^{(2)} = 0 \)

\[ D = 0 \land p \neq 0 \Rightarrow \lambda_1 = \lambda_2 \]

2B occurs if in addition \( u_{100}^{(1)} = u_{010}^{(2)} \) and \( u_{010}^{(1)} = u_{010}^{(2)} = 0 \)

\[ D = p = q = 0 \]

\begin{align*}
 u_{100}^{(1)}u_{010}^{(2)} &\neq 0 \\
u_{100}^{(1)}u_{010}^{(2)} &\neq 0 \\
u_{001}^{(1)}u_{001}^{(2)} &\neq 0
\end{align*}

\begin{align*}
u_{100}^{(1)}u_{010}^{(2)} &= 0 \\
u_{010}^{(1)}u_{100}^{(2)} &= 0 \\
u_{001}^{(1)}u_{001}^{(2)} &= 0
\end{align*}

\[ u_{100}^{(1)}u_{010}^{(2)} = 0 \]

\begin{align*}
u_{100}^{(1)}u_{010}^{(2)} &= 0 \\
u_{010}^{(1)}u_{100}^{(2)} &= 0 \\
u_{001}^{(1)}u_{001}^{(2)} &= 0
\end{align*}

\[ u_{100}^{(1)}u_{010}^{(2)} = 0 \]

\begin{align*}
u_{100}^{(1)}u_{010}^{(2)} &= 0 \\
u_{010}^{(1)}u_{100}^{(2)} &= 0 \\
u_{001}^{(1)}u_{001}^{(2)} &= 0
\end{align*}

\[ u_{100}^{(1)}u_{010}^{(2)} = 0 \]

\[ u_{i,j,k} = 0, \forall i + j + k = 1, i,j,k \geq 0, s \in \{1,2,3\} \]

\[ D < 0 \land p \neq 0 \]

\[ D < 0 \land p = 0 \]

\[ D \equiv p^2 - 4q, E = \frac{3}{4}p^2 + q, n = v_{001}^{(1)}v_{010}^{(2)} - v_{010}^{(1)}u_{001}^{(2)} \\
p = u_{100}^{(1)} + u_{010}^{(2)}, q = u_{100}^{(1)}u_{010}^{(2)} - u_{010}^{(1)}u_{100}^{(2)} \]

**Table A.3: Classification of Jordan normal forms**
The vectors \( a_i, i \in \{1, 2, 3\} \), can be used to form a (generalized) eigenvector basis for each of the cases listed in Table A.3.

**All eigenvalues are real and different**

For all three cases 1A, 1B and 1C the eigenvectors are \( a_i(1, 1), i \in \{1, 2\} \), and \( a_3 \), see (A.2) and (A.3). These three vectors are the columns of the conjugating matrix \( M \).

**All eigenvalues are real and two are equal**

**Case 2A/2C**

Let \( E = \frac{3}{4}p^2 + q = 0 \). In that case, we can write the eigenvalues as

\[
\lambda_1 = \frac{1}{2}p - |p|, \quad \lambda_2 = \frac{1}{2}p + |p|, \quad \lambda_3 = -\frac{1}{2}p.
\]

If \( p > 0 \), \( \lambda_1 = \lambda_3 \), whereas if \( p < 0 \), \( \lambda_2 = \lambda_3 \). Because \( E = 0 \), the third component in the vector \( a_3 \) is zero.

The case 2A occurs if \( u_{001}^{(1)} = u_{001}^{(2)} = 0 \). We see that all three components of the vector \( a_3 \) are equal to zero. For this case a full set of eigenvectors is found: \( a_3(1, 1), a_2(1, 1), \) and \((0, 0, 1)^t \). These vectors are the columns of the conjugating matrix \( M \).

The case 2C occurs if not both \( u_{001}^{(1)} \) and \( u_{001}^{(2)} \) vanish. The eigenvector \( a_3 \) has \( \mathbf{w} = (u_{001}^{(1)}, u_{001}^{(2)}, -2p)^t \) as its generalized eigenvector. Thus, the conjugating matrices for \( p > 0 \) and \( p < 0 \) are given given

\[
\begin{pmatrix}
\vdots & \vdots & \vdots \\
\mathbf{a}_3 & \mathbf{a}_2(1, 1, \lambda_2) & \mathbf{w} \\
\vdots & \vdots & \vdots 
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\vdots & \vdots & \vdots \\
\mathbf{a}_1(1, 1, \lambda_1) & \mathbf{a}_3 & \mathbf{w} \\
\vdots & \vdots & \vdots 
\end{pmatrix},
\]

respectively.

**Case 2B/2D**

Let \( D = p^2 - 4q = 0 \). We have \( \lambda_1 = \lambda_2 \), and thus \( a_1(l_1, l_2) = a_2(l_1, l_2) \).

The eigenvector \( a_1(l_1, l_2) \) has the vector \( \mathbf{w} = (-l_2, -l_1, 0)^t \) as its generalized eigenvector. The length of their cross-product is

\[
\|a_1(l_1, l_2) \times \mathbf{w}\| = l_1^2 u_{100}^{(1)} + l_1 l_2 (u_{100}^{(1)} - u_{010}^{(2)}) - l_2^2 u_{100}^{(2)}
\]
from which it follows that a suitable \( l_1 \) and \( l_2 \) can always be found except when \( u_{100}^{(1)} = u_{010}^{(2)} \) and \( u_{010}^{(1)} = u_{100}^{(2)} = 0 \). This is case 2B for which a full set of eigenvectors is found as \( (1,0,0)^t \), \((0,1,0)^t\) and \( \alpha_3 \) as in (A.3).

The case 2D occurs otherwise and the columns of the conjugating matrix are \( \alpha_1(l_1, l_2) \), \( \omega \) and \( \alpha_3 \) as in (A.3).

**All eigenvalues are real and equal**

The matrix \( A \) has three eigenvalues equal to zero if \( p = q = 0 \). Inserting these conditions into the ‘general form’ eigenvectors (A.2) and (A.3) only one eigenvector is found which in certain cases may vanish altogether.

Table A.3 lists the extra conditions which force either one, two or three eigenvector(s). We treat the Jordan normal forms 3A and 3B separately even though both have two eigenvectors because their skin-friction patterns differ. Since the zero matrix is only similar to itself Jordan form type 3D is trivial.

Table A.4, Table A.5 and Table A.6 list the conjugating matrices.

**One eigenvalue is real and two are complex conjugated**

If \( D < 0 \), the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) form a pair of complex conjugated eigenvalues. The eigenvectors \( \alpha_i \) in (A.2) are also complex. We, however, do not want Jordan *canonical* forms but the Jordan normal forms \( J_{4A} \) and \( J_{4B} \), see Table A.2.

Let

\[
\tilde{\alpha}_i \equiv \begin{pmatrix}
-u_{010}^{(1)} + u_{010}^{(2)} - \tilde{\lambda}_i \\
u_{100}^{(1)} - \tilde{\lambda}_i - u_{100}^{(2)} \\
0
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
\vdots & \vdots & \vdots \\
\tilde{\alpha}_1 & \tilde{\alpha}_2 & \tilde{\alpha}_3 \\
\vdots & \vdots & \vdots
\end{pmatrix},
\]

where

\[
\tilde{\lambda}_1 = \frac{1}{2} p + \frac{1}{2} \sqrt{-D}, \quad \tilde{\lambda}_2 = \frac{1}{2} p - \frac{1}{2} \sqrt{-D}.
\]

If \( p \neq 0 \), \( M \) is a conjugation matrix for \( A \) and \( J_{4A} \). If, on the other hand, \( p = 0 \), \( M \) is a conjugation matrix for \( A \) and \( J_{4B} \).
Table A.4: Case 3A

<table>
<thead>
<tr>
<th>A</th>
<th>M</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
  u^{(1)}_{100} & u^{(1)}_{010} & 0 \\
-(u^{(1)}_{100})^2/u_{010} & -u^{(1)}_{100} & 0 \\
0 & 0 & 0
\end{pmatrix}
| \[
\begin{pmatrix}
  u^{(1)}_{010} & 0 & 0 \\
-u^{(1)}_{100} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
| \[
\begin{pmatrix}
  0 & u^{(1)}_{010} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
| \[
\begin{pmatrix}
  u^{(1)}_{010} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
| \[
\begin{pmatrix}
  0 & 0 & 0 \\
0 & u^{(2)}_{100} & 0 \\
0 & 0 & 0
\end{pmatrix}
| \[
\begin{pmatrix}
  0 & 1 & 0 \\
0 & u^{(2)}_{100} & 0 \\
0 & 0 & 1
\end{pmatrix}
| \[
\begin{pmatrix}
  u^{(1)}_{100} & u^{(1)}_{010} & u^{(1)}_{001} \\
-(u^{(1)}_{100})^2/u_{010} & -u^{(1)}_{100} & -(u^{(1)}_{001}u^{(1)}_{100})/u_{010} \\
0 & 0 & 0
\end{pmatrix}
| \[
\begin{pmatrix}
  u^{(1)}_{010} & 0 & 0 \\
-u^{(1)}_{100} & 1 & u^{(1)}_{010} \\
0 & 0 & -u^{(1)}_{001}
\end{pmatrix}
|

Table A.5: Case 3B

<table>
<thead>
<tr>
<th>A</th>
<th>M</th>
</tr>
</thead>
</table>
| \[
\begin{pmatrix}
  0 & 0 & u^{(1)}_{001} \\
0 & 0 & u^{(2)}_{001} \\
0 & 0 & 0
\end{pmatrix}
| \[
\begin{pmatrix}
  -u^{(2)}_{001} & u^{(1)}_{001} & 0 \\
-u^{(1)}_{001} & u^{(2)}_{001} & 0 \\
0 & 0 & 1
\end{pmatrix}
|
\[
\begin{align*}
\text{A} & = \\
&= \begin{pmatrix}
u^{(1)}_{100} & u^{(1)}_{010} & u^{(1)}_{001} \\
-(u^{(1)}_{100})^2/u^{(1)}_{010} & -u^{(1)}_{100} & u^{(1)}_{001} \\
0 & 0 & 0
\end{pmatrix} \\
\text{M} & = \\
&= \begin{pmatrix}u^{(1)}_{010} & 0 & 0 \\
-u^{(1)}_{100} & 1 & u^{(1)}_{010}/n \\
0 & 0 & -u^{(1)}_{001}/n
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{A} & = \\
&= \begin{pmatrix}0 & u^{(1)}_{010} & u^{(1)}_{001} \\
0 & 0 & u^{(1)}_{001} \\
0 & 0 & 0
\end{pmatrix} \\
\text{M} & = \\
&= \begin{pmatrix}u^{(1)}_{010} & 0 & 0 \\
0 & 1 & -u^{(1)}_{001}/(u^{(1)}_{010}u^{(2)}_{001}) \\
0 & 0 & 1/u^{(2)}_{001}
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{A} & = \\
&= \begin{pmatrix}0 & 0 & u^{(1)}_{001} \\
u^{(2)}_{100} & 0 & u^{(2)}_{001} \\
0 & 0 & 0
\end{pmatrix} \\
\text{M} & = \\
&= \begin{pmatrix}0 & 1 & -u^{(2)}_{001}/(u^{(1)}_{001}u^{(2)}_{100}) \\
u^{(2)}_{100} & 0 & 0 \\
0 & 0 & 1/u^{(1)}_{001}
\end{pmatrix}
\end{align*}
\]

Table A.6: Case 3C
Appendix B

Some Basic Definitions and Theorems

Introduction
This appendix discusses some basic definitions and theorems.

B.1 Implicit and Inverse Function Theorems

Consider a map

\[(x, y) \mapsto F(x, y),\]

where

\[F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\]

is a smooth map defined in a neighborhood of \((x, y) = (0, 0)\) such that \(F(0, 0) = 0\). Let \(F_x(0, 0)\) denote the \(n \times n\) matrix of first partial derivatives of \(F\) with respect to \(x\) evaluated at \((0, 0)\);

\[F_x(0, 0) = \left( \frac{\partial F_i(x, y)}{\partial x_j} \right)_{(x, y) = (0, 0)}.\]

**Theorem B.1 Implicit Function Theorem**

- If the matrix \(F_x\) is nonsingular, there is a smooth locally defined function \(y = f(x)\),

\[f : \mathbb{R}^n \to \mathbb{R}^m,\]

247
such that

\[ F(\bar{x}, f(\bar{x})) = 0, \]

for all \( \bar{x} \) in some neighborhood of the origin of \( \mathbb{R}^n \). Moreover,

\[ f_{x}^{\varepsilon}(0) = -[F_{x}(0, 0)]^{-1}F_{y}(0, 0). \]

\[ \triangleright \]

The degree of smoothness of the function \( f \) is the same as that of \( F \).

Let \( m = n \) and consider the map

\[ y = g(x), \]

where

\[ g : \mathbb{R}^n \to \mathbb{R}^n \]

is a smooth map defined in a neighborhood of \( \bar{x} = 0 \) such that \( g(0) = 0 \). The following theorem is a consequence of the Implicit Function Theorem.

**Theorem B.2 Inverse Function Theorem**

\( \triangleright \)

If the matrix \( g_{x}(0) \) is nonsingular, there is a smooth locally defined function \( x = f(y) \),

\[ f : \mathbb{R}^n \to \mathbb{R}^n, \]

such that

\[ g(f(y)) = y, \]

for all \( y \) in some neighborhood of the origin of \( \mathbb{R}^n \).

The function \( f \) is called the inverse function for \( g \), and is denoted by \( f = g^{-1} \).

From Kuznetsov [Kuz95], page 482, ff.

### B.2 Taylor expansion

Let \( x \in U_{\varepsilon}(x_0), x_0 \in \mathbb{R}^n \), where \( U_{\varepsilon}(x_0) = \{ x \in \mathbb{R}^n : \| x - x_0 \| < \varepsilon \} \), \( \varepsilon > 0 \) is a \( n \)-dimensional disc surrounding the point \( x_0 \). The nonempty open set \( U_{\varepsilon}(x_0) \) is called the \( \varepsilon \)-sphere around point \( x_0 \). Furthermore, let \( i = (i_1, i_2, \ldots, i_n)^t \) be an \( n \)-vector of nonnegative integers. The norm of \( i \) is \( |i| = i_1 + i_2 + \ldots + i_n \). The product \( x^i \) is \( x^i = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \). Finally, let the \( |i| \) derivative of a function \( f(x) : \mathbb{R}^n \to \mathbb{R} \) be defined as

\[ D_i f(x) = \frac{\partial|x|}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}} f(x). \]
**Theorem B.3** Scalar Taylor's Theorem

Let \( f : U^1_\varepsilon(x_0) \to \mathbb{R} \). For every \( x \in U^1_\varepsilon(x_0) \), a \( \xi \) exists between \( x_0 \) and \( x \) such that

\[
f(x) = \sum_{k=0}^{m} \frac{1}{k!} \frac{d^k f}{dx^k}(x_0)(x-x_0)^k + \frac{1}{(m+1)!} \frac{d^{m+1} f}{dx^{m+1}}(\xi)(x-x_0)^{(m+1)}.
\]

> 

**Theorem B.4** Taylor's Theorem in three-dimensional space

Let \( f : U^3_\varepsilon(x_0) \to \mathbb{R} \) and \( \bar{x} \in U^3_\varepsilon(x_0) \). There exists a point \( \xi \) on the line segment from the points \( x_0 \) to \( \bar{x} \) such that

\[
f(\bar{x}) = \sum_{|\bar{l}| \leq m} \frac{1}{|\bar{l}|!} D_{\bar{l}} f(\bar{x}_0) (\bar{x} - \bar{x}_0)^{\bar{l}} + \frac{1}{(m+1)!} \sum_{|\bar{l}| = m+1} D_{\bar{l}} f(\xi) (\bar{x} - \bar{x}_0)^{\bar{l}}.
\]

In Mac Laurin series expansion is a Taylor series expansion around the origin. Proofs for both theories are based on the Mean-Value-theorem, see the Appendix in [HK91] by Hale and Koçak.

For \( f : U^3_\varepsilon(x_0) \to \mathbb{R}^3 \) three points \( \xi_\ell, \ell \in \{1, 2, 3\} \), are needed on the line segment from \( x_0 \) and \( \bar{x} \) such that

\[
f^{(\ell)}(\bar{x}) = f^{[k]}_\ell(\bar{x}) + R^{[k]}_\ell(\bar{x}; x_0), \tag{B.1}
\]

where the Taylor polynomial of index \( k \), \( f^{[k]}(\bar{x}) \), and the remainder, \( R^{[k]}(\bar{x}; x_0) \), are defined as

\[
f^{[k]}_\ell(\bar{x}) = \sum_{|\bar{l}| \leq k} \frac{1}{|\bar{l}|!} D_{\bar{l}} f^{(\ell)}(x_0) (\bar{x} - x_0)^{\bar{l}},
\]

\[
R^{[k]}_\ell(\bar{x}; x_0) = \frac{1}{(m+1)!} \sum_{|\bar{l}| = k+1} D_{\bar{l}} f^{(\ell)}(\xi_\ell) (\bar{x} - x_0)^{\bar{l}}. \tag{B.2}
\]

Since always an \( M > 0 \) can be found such that

\[
\max_{\ell \in \{1, 2, 3\}} \max_{|\bar{l}| = m+1} \inf_{\bar{x} \in U^3_\varepsilon(x_0)} \left| D_{\bar{l}} f^{(\ell)}(\xi_\ell) \right| \leq M < \infty,
\]

the remainder can be written as \( R^{[k]}_\ell(\bar{x}; x_0) = O(|\bar{x} - x_0|^{k+1}) = O(\varepsilon^{k+1}) \). Consequently, the difference between the Taylor polynomial \( f^{[k]}(\bar{x}) \) and the function \( f(\bar{x}) \) can be made arbitrarily small using the \( \varepsilon \) parameter.
B.3 A Theorem by Andronov et al.

The following theorem by Andronov et al. [ALGM73], page 362 f.f., provides qualitative information about an important class of two-dimensional vector fields.

**Theorem B.5 Andronov**

Let the origin $(0,0)$ be an isolated critical point of the system

\[
\begin{align*}
\dot{x} &= y + P(x,y), \\
\dot{y} &= Q(x,y),
\end{align*}
\]  

(B.3)

and let $y = \varphi(x)$ be the (series) solution of $y + P(x,y) = 0$ about the origin. Furthermore, let the series expansions of the functions

\[
q(x) \equiv Q(x, \varphi(x)), \quad p(x) \equiv \frac{\partial P}{\partial x}(x, \varphi(x)) + \frac{\partial Q}{\partial y}(x, \varphi(x)),
\]

(B.4)

have the respective forms

\[
q(x) = q_m x^m + \sum_{i>m} q_i x^i, \quad p(x) = p_n x^n + \sum_{i>n} p_i x^i,
\]

(B.5)

where $q_m$ and $p_n$ are the coefficients of the first nonvanishing terms in those series. If $p(x) = 0$, then use $p_n = 0$.

I. Let $m$ be odd.

- If $q_m > 0$, the critical point is a topological saddle-point.
- Otherwise, define $\ell$ via $m = 2\ell + 1$, ($\ell \geq 1$) and let $\lambda \equiv p_n^2 + 4(\ell+1)q_m$. The critical point
  1. is a focus or center if $p_n = 0$ or if $p_n \neq 0$ and $n = \ell$ and $\lambda < 0$,
  2. is a topological node if $p_n \neq 0$, $n$ is even, and if $n < \ell$ or $n = \ell$, $\lambda \geq 0$,
  3. has an elliptic sector if $p_n \neq 0$, $n$ is odd, and if $n < \ell$ or $n = \ell$ and $\lambda \geq 0$.

II. Let $m$ be even. Define $\ell$ via $m = 2\ell$, ($\ell \geq 1$). The critical point is a

- saddle-node if $p_n \neq 0$ and $n < \ell$,
- cusp if $p_n = 0$ or if $p_n \neq 0$ and $n \geq \ell$,

The order of the singularity is by definition equal to $m$. Fig. B.1 sketches the phase portraits. 

\[\triangleright\]
Figure B.1: Higher order planar phase portraits.
Appendix C

Third-order coefficient relations

Introduction

Chapter 2 discussed various properties of the vector field \( \mathbf{v}(y) \) which resulted after substitution of a change of coordinates in the velocity vector field \( \mathbf{u}(x) \). The change of coordinates was necessary to make the curved boundary surface \( x_3 = h(x_1, x_2) \) coincide the coordinate-plane \( y_3 = 0 \). It was said that we still needed to find the coefficient relations between the third-order coefficients in the expansion of the vector field \( \mathbf{v}(y) \) based on the required third-order coefficients in the expansion of the vector field \( \mathbf{u}(x) \). This appendix computes them.

C.1 Computing the generator

The vector field \( \mathbf{v}(y) \) resulted after substitution of the transformation (2.47)

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 = y_3 + h(y_1, y_2)
\end{bmatrix} = \begin{bmatrix}
  y_1 \\
  y_2 \\
  h(y_1, y_2)
\end{bmatrix},
\]

in the velocity vector field \( \mathbf{u}(x) \) which describes the fluid flow near a boundary surface \( x_3 = h(x_1, x_2) \). In what follows, we shall compute the relations between the coefficients of the vector field \( \mathbf{v}(y) \) at \( \mathcal{O}(\|y\|^3) \) due to the vorticity transport equation.

253
Theorem 3.10 showed that it is possible to relate the coefficients in the expansions of these two vector fields using the generator of a transformation. For a given generator, the corresponding transformation can be computed with the aid of Theorem 3.11. In this case, however, we are given a coordinate transformation and not its generator.

Fortunately, it is also possible to use Theorem 3.11 for 'reverse engineering', that is, to find the coefficients in the expansion of the generator

\[ T(x, \epsilon) = \sum_{m \geq 2} T_m(x) \epsilon^{(m-2)/(m-2)!}, \quad T_m(x) \in H^3_m, \]  

(C.1)

for a given transformation

\[ x = t(y, \epsilon) = y + \sum_{m \geq 2} t_m(y) \epsilon^{(m-1)/(m-1)!}, \quad t_m(y) \in H^3_m. \]  

(C.2)

Note that we shifted the subscripts compared to the formulation in Theorem 3.11. Up to \( \mathcal{O}(\epsilon^2) \), we get

\[ T_2(x) = t_2(x), \quad T_3(x) = t_3(x) - \frac{\partial t_2}{\partial x} t_2(x), \]

and so on.

The boundary surface displacement function \( h(y_1, y_2) \) is \( \mathcal{O}((\|y_1, y_2\|^4)^2) \), and the expansion of \( h \) is denoted as

\[ h(y_1, y_2) = \sum_{n \geq 2} \sum_{i+j=n} h_{ij} y_1^i y_2^j. \]

If we correct for the factor \( (m-1)! \) in (C.2) at a later stage, the coefficients of \( t_m(y) \) in the expansion of the transformation are

\[ t^{(1)}_{i+j+k} = t^{(2)}_{i+j+k} = 0, \quad \forall i + j + k = m, \quad t^{(3)}_{i+j+k+1} = 0, \quad \forall i + j + k = m - 1, \]

\[ i^{(3)}_{i+j} = h_{ij}, \quad \forall i + j = m, \]

for \( m \geq 2 \). For this specific transformation, it is easy to see that

\[ \frac{\partial t_m}{\partial x} t_n(x) = 0, \quad m, n \geq 2, \]

so that \( T_m(x) = t_m(x), m \geq 2 \).
C.2 Computing the coefficient relations

To apply Theorem 3.10, we expand the vector fields \( u(x) \) and \( v(y) \) as a series in \( \varepsilon \),

\[
  u(x, \varepsilon) = \sum_{m \geq 1} u_m(x) \varepsilon^{(m-1)/(m-1)!}, \quad u_m(x) \in H_\nu^3, \quad (C.3)
\]

\[
  v(y, \varepsilon) = \sum_{m \geq 1} v_m(y) \varepsilon^{(m-1)/(m-1)!}, \quad v_m(y) \in H_\nu^3, \quad (C.4)
\]

Then, using the triangle (3.107), \( v_3(y) \) is found as

\[
  2v_3(y) = \{2u_3 + 2[u_1, t_3] + [v_2, t_2] + [u_2, t_2]\} (y),
\]

also see (3.113). The factors 2 stem for the factor \((m - 1)!\) in the expansions (C.2), (C.3) and (C.4). The expansion (2.7) of \( u(x) \) and (2.55) of \( v(y) \) do not have this factor.

After substitution of the coefficient relations due to the continuity equation, the vorticity transport equation, and the no-slip condition, we found that

\[
  u_1(x) = \begin{pmatrix} u_{001}^{(1)} x_3 \\ u_{001}^{(2)} x_3 \\ 0 \end{pmatrix},
\]

\[
  u_2(x) = \begin{pmatrix} -h_{20} v_{001}^{(1)} x_1 - h_{11} u_{001}^{(1)} x_1 x_2 - v_{002}^{(2)} u_{001}^{(2)} x_2 \\ v_{011}^{(1)} x_1 x_3 + u_{011}^{(2)} x_2 x_3 + u_{002}^{(2)} x_3 \\ -h_{20} v_{001}^{(2)} x_1 - h_{11} u_{001}^{(2)} x_1 x_2 - v_{002}^{(2)} u_{001}^{(2)} x_2 \\ v_{011}^{(2)} x_1 x_3 + u_{011}^{(2)} x_2 x_3 + u_{002}^{(2)} x_3 \\ 2h_{20} u_{001}^{(1)} + h_{11} u_{001}^{(2)} x_1 x_3 \\ (h_{11} u_{001}^{(1)} + 2h_{02} u_{001}^{(2)} x_2 x_3 - \frac{1}{2} (u_{101}^{(1)} + u_{011}^{(2)}) x_3^2 \end{pmatrix}
\]

see (2.45) and (2.46), respectively. The assigned coefficients of \( u_3(x) \) are as follows

\[
  u_{300}^{(1)} = -h_{30} v_{001}^{(1)} - h_{20} v_{001}^{(1)},
\]

\[
  u_{210}^{(1)} = -h_{21} v_{001}^{(1)} - h_{11} u_{011}^{(1)} - h_{20} v_{011}^{(1)},
\]

\[
  u_{120}^{(1)} = -h_{12} u_{001}^{(1)} - h_{02} u_{011}^{(1)} - h_{11} u_{011}^{(1)},
\]

\[
  u_{030}^{(1)} = -h_{03} u_{001}^{(1)} - h_{02} u_{011}^{(1)}.
\]
\[ u_{300}^{(2)} = -h_{30}u_{001}^{(2)} - h_{20}u_{101}^{(2)}, \]
\[ u_{210}^{(2)} = -h_{21}u_{001}^{(2)} - h_{11}u_{101}^{(2)} - h_{20}u_{011}^{(2)}, \]
\[ u_{120}^{(2)} = -h_{12}u_{001}^{(2)} - h_{02}u_{101}^{(2)} - h_{11}u_{011}^{(2)}, \]
\[ u_{030}^{(2)} = -h_{03}u_{001}^{(2)} - h_{02}u_{011}^{(2)}. \]

\[ u_{300}^{(3)} = -h_{20}(2h_{20}u_{001}^{(1)} + h_{11}u_{001}^{(2)}), \]
\[ u_{210}^{(3)} = -h_{11}(2h_{20}u_{001}^{(1)} + h_{11}u_{001}^{(2)}) - h_{20}(h_{11}u_{001}^{(1)} + 2h_{02}u_{001}^{(2)}), \]
\[ u_{201}^{(3)} = 3h_{30}u_{001}^{(1)} + 3h_{20}u_{001}^{(1)} + h_{21}u_{101}^{(2)} + h_{11}u_{011}^{(2)} + h_{20}u_{011}^{(2)}, \]
\[ u_{120}^{(3)} = -h_{03}(2h_{20}u_{001}^{(1)} + h_{11}u_{001}^{(2)}) - h_{11}(h_{11}u_{001}^{(1)} + 2h_{02}u_{001}^{(2)}), \]
\[ u_{111}^{(3)} = 2h_{21}u_{001}^{(1)} + 2h_{11}u_{101}^{(1)} + 2h_{20}u_{011}^{(1)} + 2h_{12}u_{001}^{(2)} + 2h_{02}u_{011}^{(2)} + 2h_{11}u_{011}^{(2)}, \]
\[ u_{102}^{(3)} = -u_{201}^{(1)} - \frac{1}{2}u_{111}^{(1)}, \]
\[ u_{030}^{(3)} = -h_{02}(h_{11}u_{001}^{(1)} + 2h_{02}u_{001}^{(2)}), \]
\[ u_{021}^{(3)} = h_{12}u_{001}^{(1)} + h_{02}u_{101}^{(1)} + h_{11}u_{011}^{(1)} + 3h_{03}u_{001}^{(2)} + 3h_{02}u_{011}^{(2)}, \]
\[ u_{012}^{(3)} = -\frac{1}{2}u_{111}^{(1)} - u_{201}^{(1)}, \]
\[ u_{003}^{(3)} = -\frac{1}{2}u_{012}^{(1)} - \frac{1}{3}u_{012}^{(2)}. \]

see (2.41) and (2.39).

Chapter 2 computed the vectors \( v_1(y) \) and \( v_2(y) \),

\[ v_1(y) = y^3 \begin{pmatrix} u_{001}^{(1)} \\ u_{001}^{(2)} \\ u_{001}^{(3)} \end{pmatrix}, \]

\[ v_2(y) = y^3 \begin{pmatrix} u_{101}^{(1)} & u_{011}^{(1)} & u_{002}^{(1)} \\ u_{101}^{(2)} & u_{011}^{(2)} & u_{002}^{(2)} \\ 0 & 0 & -\frac{1}{2}(u_{101}^{(1)} + u_{011}^{(2)}) \end{pmatrix} y, \]

see (2.66) and (2.67), respectively.

Then, after some calculations, we find that the assigned coefficients in \( v_3(y) \) are as follows;

\[ v_{300}^{(1)} = v_{210}^{(1)} = v_{120}^{(1)} = v_{030}^{(1)} = 0, \]
\[ v_{300}^{(2)} = v_{210}^{(2)} = v_{120}^{(2)} = v_{030}^{(2)} = 0, \]
\[ v_{300}^{(3)} = v_{210}^{(3)} = v_{120}^{(3)} = v_{030}^{(3)} = v_{201}^{(3)} = v_{111}^{(3)} = v_{021}^{(3)} = 0, \]
as expected, and

\[
\begin{align*}
\nu_{201}^{(1)} &= u_{201}^{(1)} + 2h_{20}u_{002}^{(1)}, \\
\nu_{11}^{(1)} &= u_{11}^{(1)} + 2h_{11}u_{002}^{(1)}, \\
\nu_{021}^{(1)} &= u_{021}^{(1)} + 2h_{02}u_{002}^{(1)}, \\
\nu_{102}^{(1)} &= u_{102}^{(1)}, \\
\nu_{012}^{(1)} &= u_{012}^{(1)}, \\
\nu_{003}^{(1)} &= u_{003}^{(1)}. \\
\nu_{201}^{(2)} &= u_{201}^{(2)} + 2h_{20}u_{002}^{(2)}, \\
\nu_{11}^{(2)} &= u_{11}^{(2)} + 2h_{11}u_{002}^{(2)}, \\
\nu_{021}^{(2)} &= u_{021}^{(2)} + 2h_{02}u_{002}^{(2)}, \\
\nu_{102}^{(2)} &= u_{102}^{(2)}, \\
\nu_{012}^{(2)} &= u_{012}^{(2)}, \\
\nu_{003}^{(2)} &= u_{003}^{(2)}. \\
\nu_{102}^{(3)} &= -(u_{201}^{(1)} + 2h_{20}u_{002}^{(1)}) - \frac{1}{2}(\nu_{11}^{(1)} + 2h_{11}u_{002}^{(2)}), \\
\nu_{012}^{(3)} &= -\frac{1}{2}(u_{11}^{(1)} + 2h_{11}u_{002}^{(1)}) - (\nu_{021}^{(2)} + 2h_{02}u_{002}^{(2)}), \\
\nu_{003}^{(3)} &= -\frac{1}{3}u_{102}^{(1)} - \frac{1}{3}u_{012}^{(2)}.
\end{align*}
\]

From these expressions it is easy to see that the coefficient relations (2.59) and (2.60) hold.

The relations between the coefficients in \(u_3(x)\) due to the vorticity transport equation are

\[
\begin{align*}
-\nu_{111}^{(1)} - 2\nu_{201}^{(2)} - 4\nu_{021}^{(2)} - 6\nu_{003}^{(2)} = \\
2h_{11}(2h_{20}u_{001}^{(1)} + h_{11}u_{001}^{(2)}) + 2(h_{20} + 3h_{02})(h_{11}u_{001}^{(1)} + 2h_{11}u_{002}^{(2)}), \\
4\nu_{201}^{(1)} + 2\nu_{021}^{(2)} + 6\nu_{003}^{(1)} + \nu_{111}^{(2)} = \\
-2(3h_{20} + 2h_{02})(2h_{20}u_{001}^{(1)} + h_{11}u_{001}^{(2)}) - 2h_{11}(h_{11}u_{001}^{(1)} + 2h_{11}u_{002}^{(2)}), \\
u_{102}^{(1)} - \nu_{012}^{(2)} = \\
(3h_{20} + h_{12})u_{001}^{(2)} + (3h_{20} + h_{02})u_{101}^{(2)} + h_{11}u_{001}^{(1)} \\
- (h_{21} + 3h_{03})u_{001}^{(1)} - h_{11}u_{101}^{(1)} - (h_{20} + 3h_{02})u_{011}^{(1)}.
\end{align*}
\]

see (2.42), (2.43) and (2.44). Then, with the above expression for the coefficients of \(v_3(y)\), we get

\[
\begin{align*}
-\nu_{111}^{(1)} - 2\nu_{201}^{(2)} - 4\nu_{021}^{(2)} - 6\nu_{003}^{(2)} = \\
2h_{11}(2h_{20}v_{001}^{(1)} + h_{11}v_{001}^{(2)}) + 2(h_{20} + 3h_{02})(h_{11}v_{001}^{(1)} + 2h_{11}v_{002}^{(2)}) \\
- 2h_{11}v_{002}^{(1)} - 4h_{11}v_{002}^{(2)} - 8h_{02}v_{002}^{(2)}.
\end{align*}
\] (C.5)
\[ 4v_{201}^{(1)} + 2v_{021}^{(1)} + 6v_{003}^{(1)} + v_{111}^{(2)} = \\
-2(3h_{20} + h_{02})(2h_{20}v_{001}^{(1)} + h_{11}v_{001}^{(2)}) - 2h_{11}(h_{11}v_{001}^{(1)} + 2h_{02}v_{001}^{(2)}) \\
+ 8h_{20}v_{002}^{(1)} + 4h_{11}v_{002}^{(1)} + 2h_{11}v_{002}^{(2)} \]  
(C.6)

\[ v_{102}^{(2)} - v_{012}^{(1)} = \\
(3h_{30} + h_{12})v_{001}^{(2)} + (3h_{20} + h_{02})v_{101}^{(2)} + h_{11}v_{011}^{(2)} \\
- (h_{21} + 3h_{03})v_{001}^{(1)} - h_{11}v_{101}^{(1)} - (h_{20} + 3h_{02})v_{011}^{(1)} \]  
(C.7)

Note that we removed every reference to the coefficients from \( u_1(x) \) and \( u_2(x) \).

Through similar computations we can find the relations due to the vorticity transport equation between the coefficients of the vectors \( v_m(y), m \geq 4 \).
Bibliography


[DHR+90] Uwe Dallmann, Achim Hilgenstock, Stefan Riedelbauch, Burkhard Schulte-Werning, and Heinrich Vollmers. On the footprints of three-dimensional separated vortex flows around blunt bodies. In Vortex


Samenvatting

Bifurcatie in stromingen van vloeistof of gas in de omgeving van een begrenzend oppervlak

In dit proefschrift geef ik een methode voor de bestudering van bifurcaties in vectorvelden die stromingen van vloeistof of gas beschrijven in de buurt van kritieke punten op een begrenzend oppervlak. Een kritiek punt in een vectorveld is een punt waar de vector gelijk aan nul is. Kennis van de kritieke punten in een vectorveld geeft belangrijke informatie over de eigenschappen van de stroming. Bestudering van bifurcaties maakt het mogelijk om via een systematische classificatie een topologie van fundamentele lokale stromingspatronen met mogelijk meer dan één kritiek punt te kunnen vinden. Die topologie maakt interpretatie mogelijk van gecompliceerde stromingspatronen die op treden rond aerodynamische configuraties. Tevens bestaat de mogelijkheid om daarmee veranderingen in stromingspatronen te verduidelijken.

Ik heb voornamelijk gekeken naar de afleiding van topologische normaalvormen. Dit zijn sterk vereenvoudigde polynomiale vectorvelden welke topologisch equivalent zijn voor een gehele klassen van vectorvelden. Twee vectorvelden zijn topologisch equivalent indien er lokaal een continue afbeelding bestaat met een continue inverse, die het stromingspatroon van het ene vectorveld op het stromingspatroon van het ander vectorveld afbeeldt onder de voorwaarde dat de oriëntatie van de stroomlijnen behouden blijft. Dit begrip maakt een classificatie mogelijk van structuren in stromingspatronen lokaal rond een punt. De verwachting is dat de via bifurcatie analyse gevonden polynomiale vectorvelden stromingspatronen beschrijven die door middel van een experiment of computer simulatie zijn waar te nemen.

Hoofdstuk 1 geeft een inleiding in de probleematiek van stromingsvisualisatie en interpretatie d.m.v. topologische eigenschappen. Tevens geeft het een classificatie van de zogenaamde hyperbolische kritieke punten. De Jacobiaanse matrix
van het vectorveld in zo'n punt heeft slechts eigenwaarden met een reëel deel dat ongelijk nul is. In het geval van een hyperbolisch kritiek punt bestaat een topologische normaalvorm uit een lineair vectorveld. De topologische structuur van het stromingspatroon rond een hyperbolisch kritiek punt is stabiel; analytische verstoringen werkend op het vectorveld leiden alleen tot topologisch equivalentie stromingspatronen.

Anders is het in het geval van een niet-hyperbolisch kritiek punt. De normaalvorm is niet-lineair en het topologische karakter van het stromingspatroon verandert onder de invloed van verstoringen. Hoofdstuk 2 geeft een afleiding van de verzameling van vectorvelden welke stromingen rond een punt op een begrenzend oppervlak beschrijven. De aannamen zijn dat het gas (of de vloeistof) onsamendrukbare en viskeus is en dat de stroming stationair is.

Hoofdstuk 3 behandelt wiskundige technieken uit de bifurcatie-theorie om topologische normaal vormen te vinden aan de hand van een voorbeeld tweedimensionaal vectorveld. Daarbij introduceer ik een nieuwe methode om alle mogelijke representaties van de toegevoegde ruimte (‘complementary subspace’) te vinden behorende bij (Jacobiaanse) matrices.

Echter, hoofdstuk 4 maakt in zijn inleiding duidelijk dat het niet mogelijk is om voor onze toepassing topologische normaalvormen op de gebruikelijke manier te verkrijgen. Daarom beschrijf ik in dat hoofdstuk een nieuwe aanpak. Kernpunt is om transformaties gebruikt om een normaalvorm te berekenen ook te laten werken op de verstoringparameters. Het berekenen van een normaalvorm dient om een eenvoudiger te analyseren vectorveld te vinden. De nieuwe aanpak maakt het mogelijk om de fysische betekenis te behouden tijdens de berekening van de normaalvorm en de bijbehorende ontvouwing binnen een bepaalde set van vectorvelden ($\mathcal{V}_v$ gedefinieerd in (2.87)).

Hoofdstuk 5 maakt gebruik van de nieuw ontwikkelde aanpak en geeft de afleiding van topologische normaalvormen voor vier typen vectorvelden. Aan het eind van het hoofdstuk ga ik in op de vraag of de verzameling van vectorvelden $\mathcal{V}_v$ inderdaad de juiste fysica bevat. Het blijkt namelijk dat de zogeheten Navier-Stokes vergelijkingen niet van belang is voor de mogelijke topologische structuren van stromingen in de buurt een punt op een begrenzend oppervlak.

Hoofdstuk 6 geeft een aanzet tot de analyse van spiegel-symmetrische stromingspatronen rond een punt op een begrenzend oppervlak. Daarbij wordt gebruik gemaakt van de topologische normaalvorm berekend in Hoofdstuk 5.

Het laatste hoofdstuk geeft een samenvatting en gaat in op de betekenis van de gevonden resultaten. Tevens wordt ingegaan op mogelijke aanvullingen en wensen betreffend toekomstig onderzoek ten aanzien van dit onderwerp.

Roland J.P. Boon
Acknowledgements

Mathematics equals Higher Laziness:
Constant hard work in search of the easy way.
—Matthew Pordage, in [PS76].

If it were not for the generous help I have received during the two years that I have spent in gathering knowledge for this thesis, I would never have finished writing in time. I know now that writing is a skill to be mastered, not to be taken lightly. Moreover, if the final product is going to have any lasting value it is because of the patience others had in reading and revising the chaotic sketches of proofs I presented them with. I would like to say thanks to everybody involved.

A special thanks to the following people.
Marco de Winkel for the chapter 'In de IJskast'. Louis Walpot, Erwin Mooij and Theo Lanen for their friendship. Frits Donker Duyvis for building the best Hi-Fi equipment this side of the universe. Henk Broer for his guidance in the rocky world of nonlinear dynamical systems. Andre Vanderbauwhede for suggesting to prove observed Lie-group behavior. Robert MacKay, Estanislao Gamero Gutierrez, and Duo Wang for sending me stuff. Andre Zegeling for his gossip on famous mathematicians. Milan Medved’ for trying to get famous mathematicians drunk enough and make them spill the beans on there latest results. And last but not least, Jane, for our voyage through life.
Curriculum Vitae

The author was born on Sunday, July 2, 1967, in The Hague, The Netherlands. He successfully studied at the Faculty of Technical Mathematics and Informatics, Delft University of Technology, starting from September 1985. Graduation took place September 1991 under the supervision of professor A.J. Hermans, Mathematical Physics.

Professors P.G. Bakker and J.W. Reyn, also from the Delft University of Technology, Faculty of Aerospace Engineering and of Technical Mathematics and Informatics, respectively, enrolled him for their ongoing research project called 'Topology of 3D Separated Flow Structures', and stationed him at the High Speed Aerodynamics Laboratory. The first 18 months were spent developing a program for visualization of the flow topology of three-dimensional vector fields. More than this, the promoters liked to find a firm mathematical foundation for bifurcations in local flow patterns. The help received from Groningen, Gent, Bratislava, and Seville was a critical factor to find the answers they were looking for.

Producing a thesis took longer than the duration of employment. So, with a nearly finished thesis, after 4 years and a 3 month extension, starting January 1996, the author found himself looking for a job. The work done for the visualization program were used as a selling point, and in April 1996, he was employed as a Software Engineer by Turnkiek Technical Systems B.V. at Amersfoort. After one month of in-house training he was sent to TNO Bouw, Building and Construction Research, Rijswijk. There, he developed an application for sonic integrity testing of foundation piles. Both Turnkiek and TNO strongly encouraged completion of the thesis.

The author still works for the same company and client, and dreams of...
 STELLINGEN

behorende bij het proefschrift "Bifurcation in Fluid Flow near a Boundary Surface"

Roland J.P. Boon, 22 september, 1997

1. Normaalvorm theorie, ontwikkeld om aan de hand van eenvoudiger vormen kwalitatieve uitspraken te doen, bereikt in concrete gevallen zonder specifieke modificaties niet altijd het beoogde doel (voorbeeld, dit proefschrift).

2. Er zijn weinig goede boeken over bifurcatie theorie.

3. De neiging bestaat de realiteitswaarde van een visualisatie van computer gegenereerde data te overschatten.


5. Gezien de snelle veroudering van computers en software is het aan te raden zich bij aanschaf te beperken tot de momentane behoefte.

6. Sinds de invoering van de computer wordt er steeds meer uitgevoerd op het werk.

7. Een goed automatiseringstraject behoeft documentatie van bedrijfs specifieke domeinkennis, waar het in de praktijk nog al eens aan ontbreekt.