Stellingen behorende bij het proefschrift 'Low-frequency wave generation due to breaking wind waves'

1. Het is mogelijk het energiespectrum van laagfrequentewatergolven te bepalen uit het energiespectrum van windgolven, die beschreven worden met het randomfasemodel, zonder gebruik te maken van de relative fasen van de spectrale windgolfcomponenten. 
_Dit proefschrift, hoofdstukken III en V._

2. De aansturing van een golfschot voor de opwekking van golven, correct tot op tweede orde in de golfsteinheid, bepaald met behulp van de methode van de meerschalen ontwikkeling (multiple scales analysis), geeft goede resultaten. Bovendien is het aanstuurssignaal veel sneller en eenvoudiger te berekenen dan met de conventionele spectrale methode. 
_Dit proefschrift, hoofdstuk IV._

3. Complex Harmonic Principal Component Analysis is een analysemethode, die in golfonderzoek gebruikt kan worden om de invloed van ruis op de interpretatie van meetgegevens sterk te verminderen. Hierin voldoet de methode beter dan de gebruikelijke spectrale methoden.

4. De faseverschuiving tussen golfgroepgebonden laagfrequente golven en de omhullende van de windgolven, die op een horizontale bodem 180 graden is, neemt op een hellende bodem in ondiep water sterk af, soms tot 90 graden. 
_List, 1992, J. Geophysical Research, 97, p. 5623; dit proefschrift, hoofdstukken III en V._

5. Voor het beschrijven en analyseren van sterk niet-lineaire watergolven dient naar andere basisfuncties gezocht te worden dan de gebruikelijke sinussen en cosinussen. Empirische orthogonale functies kunnen hierbij van groot belang zijn.
6. Het combineren van verschillende remote-sensingtechnieken zal de oceanografie met daarin inbegrepen het golfonderzoek een enorme impuls geven.

7. Personen die een bepaalde verantwoordelijkheid niet kunnen dragen moet die verantwoordelijkheid zoveel mogelijk ontnomen worden. Toepassing van deze stelling op snelheidsovertreders geeft dat zij hun rijbevoegdheid verliezen.

8. Gezien de problemen met de verwerking van radioactief afval is kernenergie de duurste energiebron die wij gebruiken.

9. Het nieuwe huismerk van de Universiteit Utrecht getuigt van weinig inzicht in de financiële en milieu-aspecten van een driekleurendruk op witter dan wit papier.

10. Mensen die duurzaam geproduceerde producten gebruiken, waaronder duurzaam geproduceerd voedsel, zijn twee keer duurder uit dan mensen die dit niet doen: nu omdat deze producten duurder zijn, straks omdat ze moeten meebetalen aan het opruimen van de rotzooi van een ander.
LOW FREQUENCY WAVE GENERATION DUE TO BREAKING WIND WAVES

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Technische Universiteit Delft, op gezag van de Rector Magnificus, prof. drs. P.A. Schenck, in het openbaar te verdedigen ten overstaan van een Commissie aangewezen door het College van Dekanen op dinsdag 15 december 1992 te 10.00 uur

door

Petrus Johannis van Leeuwen
geboren 13 juli 1963 te Lisse

1992
Dit proefschrift is goedgekeurd door de promotor
prof. dr. ir. J.A. Battjes
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SAMENVATTING

Wanneer windgolven naar een kust lopen en breken blijkt dat er zich laagfrequente golven ontwikkelen met een frequentie die een orde van grootte lager is dan die van de invallende golven. We kunnen laagfrequente bewegingen loodrecht op de kust, de zogenaamde surfbeat, onderscheiden van bewegingen met zowel snelheidscomponenten loodrecht op de kust als parallel aan de kust, de edge waves. We concentreren ons in dit proefschrift op de surfbeat. Het 'beat' gedeelte staat op het feit dat de laagfrequente golven sterk gecorreleerd zijn met de golfgroepen, die door de windgolven gevormd worden. Verder blijkt dat de laagfrequente golven zich sterk ontwikkelen in de zone waar de windgolven breken, de zogenaamde 'surf zone'.

Het doel van het onderzoek was het ontwikkelen van een model dat het ontstaan en de ontwikkeling van deze laagfrequente golven beschrijft.

De redenen voor dit onderzoek zijn meervoudig. Ten eerste, windgolven zijn zo steil dat ze breken op het strand, zoals iedereen uit eigen ervaring weet. De laagfrequente golven zijn niet steil genoeg om te breken en zullen reflecteren aan de kustlijn en terugkeren naar open zee. Een gevolg hiervan is dat de energie die in de windgolven vertegenwoordigd is snel afneemt naar de kust toe, terwijl die van de laagfrequente golven juist toeneemt naar de kust door het ontstaan van staande golven. Hierdoor is de waterbeweging in de buurt van de kustlijn sterk gedomineerd door de laagfrequente golven.

Ten tweede blijkt uit waarnemingen dat de laagfrequente golven van groot belang zijn voor sedimenttransport en zandbankvorming. Dit laatste is aan te voelen door te beseffen dat de golflengte van de laagfrequente golven van dezelfde orde van grootte is als de afstand van zandbanken uit de kust, en ook tussen zandbanken onderling.

Een derde reden is het feit dat de frequenties van de laagfrequente golven vergelijkbaar zijn met de eigen frequenties van kleine havens, zodat resonantie-effecten kunnen optreden. De
scheepvaart kan hier grote last van ondervinden.

Als laatste maar zeker niet minste reden geldt die in het kader van het zuiver wetenschappelijk onderzoek: Men heeft laagfrequente golven gemeten; waar komen ze vandaan?

Het onderzoek heeft zich voornamelijk gericht op de laagfrequente golven die loodrecht op de kust bewegen, deels door de vereenvoudiging van het systeem van twee naar een dimensie (de vergelijkingen zijn dieptegeïntegreerd omdat de vertikale bewegingen verwaarloosbaar zijn) en deels omdat er aanwijzingen zijn dat deze categorie het grootste deel van de laagfrequente energie loodrecht op de kust vertegenwoordigt.

Er is een model ontwikkeld voor de aansluiting van deze laagfrequente golven door brekende onregelmatige golven, en de bijbehorende respons. Verificatie van het model is gebeurd door vergelijking met natuurmetingen gedaan in Duck, North Carolina, U.S.A. en met metingen in een golfgoot. De laboratorium experimenten werden uitgevoerd met en zonder zandbanken voor het onderzoek van resonanties.

Het proefschrift is als volgt opgebouwd. In hoofdstuk 1 staat een inleiding in het engels. Hoofdstuk 2 bevat een kort historisch overzicht van waarnemingen over en modellen van het verschijnsel surfbeat. Ook wordt hier nader ingegaan op de betekenis van surfbeat.

In het volgende hoofdstuk wordt het ontwikkelde model voor het ontstaan en de opsluiting van de laagfrequente golven aan een kust beschreven. Het model gaat uit van de gelineariseerde ondiepwater vergelijkingen. Deze worden omgeschreven naar een vergelijking die de aandrijving en evolutie van laagfrequente golven beschrijft. De aandrijvende kracht van de laagfrequente golven bevat de afgeleide naar de ruimtevariabele van de golfspanning, die veel weg heeft van de Reynoldse spanningen in turbulentie theorieën. Golfspanning geeft de horizontale impulsoverdracht aan door een vertikaal vlak, gemiddeld over een windgolfperiode. De golfspanningsgradiënten treden op bij gradiënten in de windgolfhoogten, zodat windgolfgroepen en brekende windgolven laagfrequente golven zullen opwekken.

Het bijzondere van het hier ontwikkelde model is dat het het energiespectrum van de laagfrequente golven direct uit het energiespectrum van de windgolven wordt berekend. Hierbij wordt de fase-informatie van de Fourier componenten van de windgolven niet gebruikt: het idee is dat een specifieke realisatie van het windgolfveld met een zeker energiespectrum geen invloed heeft op het energiespectrum van de laagfrequente golven. Dit in tegenstelling met de al bestaande modellen, die allen (voor zover aan de auteur bekend) starten met een tijdreeks van de windgolven.
Het theoretische model is geïmplementeerd op een PC en uitvoerig getest. Voor eenvoudige bodemtopografieën zijn analytische oplossingen gevonden die door het numerieke model goed weergegeven worden.

Het model is getest door vergelijking met laboratoriumexperimenten en veldwaarnemingen. De laboratoriumexperimenten zijn uitgevoerd in een golfgoot van het Laboratorium voor Vloeiistofmechanica van de TU Delft. Golven worden opgewekt met een hydraulisch gedreven golfschot en in de goot is een geïdealiseerd kustmodel aangebracht.

Het doel van de experimenten in de golfgoot is om de situatie zoals die zich voordoet aan een kust zo goed mogelijk na te bootsen, zodat de aansluiting van de laagfrequente golven op dezelfde manier verloopt als aan die kust. Hiertoe moeten de volgende maatregelen genomen worden: In de golfgoot moeten zo realistisch mogelijke zeegolven gegenereerd worden die naar een helling lopen en daar breken, en golven die van het strand terugkeren naar het golfschot moeten geabsorbeerd worden aan het golfschot. Zo realistisch mogelijk betekent in dit geval niet alleen dat de windgolven zelf moeten worden aangemaakt maar ook de groepsgebonden laagfrequente golven, die van tweede orde zijn. De conventionele methode voor de berekening van deze tweede orde golven, die uitgaat van de niet-lineaire interacties van alle eerste orde spectrale componenten, is opzij gezet omdat deze methode zeer rekenintensief is en dus duur.

In het kader van dit promotie-onderzoek is in samenwerking met het Waterloopkundig Laboratorium een nieuwe methode ontwikkeld voor de opwekking van windgolven tot op tweede orde. Deze methode berust op een meerschalen ontwikkeling waarbij alleen de meest energierijke eerste orde spectrale componenten worden meegenomen. De zogenaamde reflectie compensatie, die golven die van het strand terugkeren naar het golfschot absorbeert is ook in deze methode opgenomen. In paragraaf 4.1 wordt dit verder beschreven.

Het nieuwe aansturingssignaal voor het golfschot is uitvoerig getest en deze tests zijn beschreven in paragraaf 4.3. Het bleek dat ruis een grote invloed had op de interpretatie van de tests en de latere model verificatie. Om de ruis te onderdrukken is gebruik gemaakt van Complex Harmonic Principal Component Analysis, een analysemethode die gebruik maakt van het feit dat ruis ongecorrleereerd is met het eigenlijke signaal. Dit is beschreven in paragraaf 4.2.

In hoofdstuk 5 wordt de vergelijking van het model en de waarnemingen gegeven. De vergelijking met de laboratoriummetingen levert dat de correlaties tussen de omhullende van de windgolven en de laagfrequente golven goed door het model voorspeld worden, maar de energiedichtheden van de golven met hogere frequenties worden sterk overschat door het model. Onderzoek leverde dat alle deze auteur bekende lineare modellen hetzelfde probleem
te zien geven. (Een nietlinear model dat nu in ontwikkeling is (Roelvink, prive-gesprek) heeft dit probleem niet.) Ook bleek dat de veldmetingen wel goed door het model voorspeld kunnen worden. Deze tegenstrijdigheid is tot op heden nog niet geheel opgelost; wel blijkt dat de waterdiepte aan de zeewaartse rand van het model van grote invloed is: Hoe groter deze diepte, hoe slechter de model resultaten. In paragraaf 5.3 wordt uitgebreid op dit probleem ingegaan en wordt een nietlineaire versie van het model voorgesteld.

In hoofdstuk 6 tenslotte volgen een korte samenvatting en de conclusies die uit dit onderzoek getrokken kunnen worden.

Het proefschrift bevat een appendix die handelt over de afleiding van de laagfrequentegolfvergelijking die de basis is voor het ontwikkelde model.
Abstract

This thesis deals with low-frequency waves, which propagate normal to the shore and are generated by breaking wind waves. These waves have wave periods in the order of minutes, while the wind waves have wave periods of about 10 seconds.

The aim of this work is to develop a model which describes the generation of these low-frequency waves due to wave groups and breaking waves. This model is tested with laboratory and field measurements.

The developed model is linear in the low-frequency wave amplitude. It assumes that the wind-wave spectrum is narrow and that the propagation direction of the wind waves is shore normal. Also the bottom topography varies only in the cross-shore direction. New in the model is that it calculates in the frequency domain and uses only the variance spectrum of the wind waves. This means that no relative-phase information of the wind-wave frequency components will be used. (The wind-waves are assumed to have random phases.)

To compare the model with laboratory measurements a realistic sea state has to be created in the laboratory flume. To this end the control signal for the wave board has to be correct up to second order in the wind-wave steepness. A new method to obtain this control signal was developed, which makes use of the method of multiple scales.

The model results compared satisfactorily with the field measurements, but the comparison with the laboratory data showed that the model overpredicted the energy densities of the low-frequency waves with the highest frequencies. The reason for this overprediction is not completely clear; it seems that bottom friction on a sloping bottom cannot be neglected.
CHAPTER 1

INTRODUCTION

This thesis deals with low-frequency waves, which propagate normal to the shore and are generated by breaking wind waves. These low-frequency waves, with wave periods in the order of minutes, are called surf beat. The 'beat' refers to the fact that the frequencies of these waves are equal to the beat frequencies of the wind-wave groups. This correlation of wind-wave groups and surf beat is no coincidence: It turns out that the group variation of the wind waves generates and amplifies the surf beat.

The importance of surf beat cannot be overestimated. First of all, the wind waves are so steep that they will break on the beach when approaching the water line while the surf-beat waves have moderate wave steepnesses and will reflect on the water line almost fully. As a consequence of this the energy density of the wind waves decreases rapidly towards the water line while that of the surf beat increases. The wave-energy content at surf-beat frequencies is therefore higher than that at wind-wave frequencies close to the water line.

Secondly, surf beat is proven to be important for sediment transport and sand-bar migration. This can easily be understood if one realizes that the wavelengths of the surf-beat waves are comparable to the distances between along-shore bars and between the bars and the shore line.

Thirdly, the frequencies of the surf-beat waves are comparable to the natural frequencies of small harbours, thus producing resonant harbour oscillations and problems for moored ships.

And last but not least: a scientific approach. Surf beat is measured, but where does it come from?
The aim of this work is to develop a model for the generation of low-frequency waves due to wind-wave groups and breaking wind waves. This model has to be verified against laboratory and field measurements. The outline of this thesis, which is the result of this work, is as follows.

In the next chapter some background information of the phenomenon of surf beat is provided. We will elaborate on the importance of surf beat and the physical mechanism of the low-frequency wave generation due to wind-wave groups and breaking of wind waves. The existing models for the generation of surf beat will be studied and the position of the present model among them will be established.

In chapter 3 a new theoretical model for the surf-beat generation will be explained in detail. The importance of this new model is that it calculates the low-frequency energy-density spectrum directly from the energy-density spectrum of the wind waves. So the phase information of the wind waves is not needed: the specific properties of an individual realisation of the wind-wave field with a certain energy-density spectrum have no influence on the energy-density spectrum of the low-frequency waves. In contrast, all existing models (to the knowledge of this author) start from specific time series of the incident waves.

The model will be tested with laboratory measurements and existing field data. The laboratory experiments are conducted in a flume which is equipped with a wave maker. The control signal for the wave board has to be such that the waves which enter the flume are generated correctly up to second order. Also low-frequency waves which are reflected from the beach have to be absorbed by the wave board to simulate the situation in nature, where these reflected waves travel to the open sea without rereflection to the beach.

A new method to derive the control signal for the wave board is given in chapter 4. Multiple-Scales Analysis is used to obtain an expression for the control signal which is less complex than the pre-existing ones. In this method it is assumed that the wind waves can be represented to first order by a sinusoidal water motion with a slowly varying amplitude (the wave groups). The variation of the amplitude acts on a longer time and length scale than the primary motion so different time and length scales are introduced, which explains the name of the method. The new control signal is tested in the flume and is shown to work satisfactorily.

Chapter 4 also deals with the data analysis. It is explained how the envelope of a measured random wave signal is calculated and how the low-frequency wave field can be decomposed in bound and free components. It was found that the signal-to-noise
ratio in the flume experiments was low, which has a considerable influence on the decomposition. Principal-Component Analysis was used to reduce the influence of noise on the data. The data from different sensors (wave-height meters and current meters) are expanded in (orthogonal) eigenfunctions of the cross-spectral matrix, which is the most efficient representation of the data. Because the noise is independent of the signal (by definition) it will show up in a different eigenvector. If we assume that the signal is concentrated in the eigenvectors which explain most of the variance of the data, the noise can easily be filtered out.

In chapter 5 the results of the comparison with the measurements are presented. First we deal with the laboratory experiments. It is shown that correlations between low-frequency waves and the envelope of the wind waves are predicted correctly, but the energy-density spectra of the low-frequency waves are not predicted satisfactorily. It was found the the energy densities of the low-frequency waves with the highest frequencies are strongly overpredicted by the model.

Secondly, the comparison with field measurements is given. In this case both the correlations between the envelope of the wind waves and the low-frequency waves, and the energy-density spectra are predicted correctly. This seems to be in contradiction with the laboratory case.

To investigate this difference in behaviour of the model, the model was run in the laboratory case with a deeper and a less deep seaward boundary. It was found that the deeper the seaward boundary, the worse the model performance. Indeed, the field measurements had a relatively shallow seaward boundary. The reason for this behaviour of the model is not entirely clear; it is discussed in section 5.3, in which also an nonlinear extension of the present model is given.

Conclusions from the present work are presented in chapter 6.

The thesis contains an appendix which deals with the derivation of the first-order low-frequency wave equation from the fully nonlinear time- and depth-averaged Navier-Stokes equations. Nonlinearity parameters are identified and it is shown in detail how an expansion in these parameters leads to the first-order low-frequency wave equation.
CHAPTER 2

BACKGROUND

In this chapter we focus on the recent developments in the literature concerning low-frequency wave generation. First we deal with some early observations of surf beat and determine the main characteristics of the phenomena. Then we elaborate on the importance of the study of surf beat. This is followed by a few physical mechanisms, which have been proposed to explain the observations. Finally a short review will be given of the existing models and the position of the present model in this group is shown.

Some forty years ago Munk (1949) detected the low-frequency waves and found a linear relation between their amplitudes and the amplitudes of the wind waves. The low-frequency waves had periods in the order of minutes while the wind waves have periods in the order of ten seconds. The low-frequency wave energy was surprisingly high compared to that of deep water waves. Looking at the time records of both types of waves he found that the low-frequency waves lagged behind the wind wave envelope with about 140 seconds, which he attributed to the time needed for the wind-wave envelope to travel to the surf zone and for the low-frequency waves to travel back to the wave recorder as free waves. So the low-frequency waves were probably generated in the surf zone and Munk termed the waves surf beat.

Tucker (1950) reported the same kind of measurements, but he used correlograms to analyze his data. He also found that the low-frequency waves are lagging behind the wind-wave envelope, and that the largest absolute value of the correlation coefficient occurs when this coefficient is negative. This means that the wind-wave envelope and the low-frequency waves are 180 degrees out of phase, apart from travel considerations. These two observations, the linear dependence of the low-frequency wave energy on that of the wind waves and the negative correlation are the main characteristics of surf beat.
The importance of surf beat.

The importance of surf beat has been clearly shown in the last twenty-five years. An important factor is the energy content of the surf beat compared to that of the wind waves. The spectra of run-up elevation (the vertical motions of the waterline) and the inner-surf-zone elevations (close to the waterline) are almost entirely dominated by low-frequency waves (e.g. Guza et al., 1982, Wright et al., 1982, Holman and Bowen, 1984, Bowen and Huntley, 1984). This can be clearly understood: The wind-wave energy decreases rapidly to the waterline due to wave breaking while the low-frequency waves reflect almost for 100% thus producing standing waves with their elevation maxima on the waterline.

Another important fact is the impact of low-frequency waves on sediment transport. Short (1975) reported measurements at the north Alaskan coast of low-frequency wave patterns and sand-bar positions. He suggested that the peaks in the low-frequency wave spectra can be attributed to standing low-frequency waves. The predicted positions of the antinodal and nodal points of the standing waves agreed with the measured spacing of the bar crests. Sallenger and Holman (1987) reported measurements in which a dominant peak in the low-frequency spectrum was clearly related to a bar position during the first day of a storm, but later on this relationship vanished. Both articles are examples of extensively reported observations of relations between low-frequency wave motion and positions of bar crests. However, this gives no clear evidence of the formation of bars by low-frequency waves, maybe the bars were there first and the low-frequency waves fit themselves in the bar pattern.

Boczar-Karakiewicz and Davidson-Arnott (1987) showed that a nonlinear wave model, in which the interaction between the first-order waves and their superharmonics is the main physical mechanism, is able to predict bar formation. On the other hand, Aagaard (1988, 1989) showed that in his field experiments the migration of bars is dominated by low-frequency waves. Also Beach and Sternberg (1988) showed that low-frequency waves are dominant for sediment transport. They found that concentration gradients of suspended sediment appear to be associated with diffusive processes and develop and decay on time scales of the low-frequency motions rather than those of the wind waves. Roelvink and Stive (1988) developed a model for bar formation in which low-frequency waves were shown to be essential. These observations clearly show that low-frequency wave motions are important for sediment transport.

Another field in which surf beat is of importance is shipping. Due to the low natural frequencies of moored offshore ships they are greatly affected by low-frequency wave motion. Also resonances in harbours can be due to surf beat (see for instance Bowers
Finally, a purely scientific reason for studying low-frequency waves is that they are observed and it is of interest to look into the mechanism of their generation.

The mechanism of surf beat.

A few mechanisms have been proposed to explain the observed phenomena related to surf beat. Especially the negative correlation puzzled the scientists for many years. Biesel (1952) proved theoretically that groups of short waves are accompanied by nonlinear low-frequency waves, bound to these groups. However he did not consider the consequences of this finding for surf beat. Longuet-Higgins and Stewart (1960, 1962 and 1964) clarified the theoretical result of Biesel with the concept of radiation stress, which is the mean momentum flux induced by the wind waves. Due to this nonlinear effect, water is pushed away from areas with high waves and accumulated under low waves. This results in low-frequency waves, bound to the wind-wave groups. An important feature is that the bound low-frequency waves are 180 degrees out of phase with the wind-wave envelope. A way to explain the observations by Munk (1949) and Tucker (1950) qualitatively is to suppose that the bound low-frequency waves are reflected in the surf zone, and propagate seaward as free low-frequency waves. Since the paper pays no attention to the mechanism of reflection, it is difficult to compare the theoretical results with experiments. However, the amplitudes of the low-frequency waves generated by this mechanism are proportional to the incident wave height squared, while the observations showed a linear relation.

Symonds et al. (1982) concentrated on the generation of low-frequency waves in the surf zone. Due to wave breaking the radiation stress of the wind waves decreases towards the shore line. This is counteracted by a rise of the mean water level towards the waterline. The rise in mean water level is called set up and has been verified in field and laboratory experiments (see for instance Bowen et al. 1968 and Kostense 1984). Higher waves will produce a greater set up than lower waves. When wind-wave groups approach a beach and start breaking, the sequence in wave heights will give rise to a time-varying breakpoint which will produce a time-varying set up. In this way free low-frequency waves are produced which travel seawards directly from the breakpoint or travel from the breakpoint to the waterline to be reflected back seawards. The low-frequency wave energy outside the surf zone is determined by the relative phases of the two kinds of low-frequency waves.

Another proposed mechanism for the surf-beat phenomena is bore-bore capture (Bradshaw, 1980 and others), in which the low-frequency waves are due to the capture
of a breaking wave by another breaking wave. This mechanism would produce no incoming surf-beat energy and a gradual red-shifting of the low-frequency wave spectrum as more bores are captured. Guza and Thornton (1985) reported that their field experiments did not show this behaviour.

**Models for surf beat.**

Quantitative models were lacking until 1971. Gallagher (1971) presented a model for two-dimensional propagation in which surf beat is due to nonlinear resonant excitation of edge waves travelling alongshore. He treated the surf zone very crudely and he used only one damping coefficient, which describes both bottom friction and radiation of low-frequency energy to deep water. Despite these approximations he could reproduce the order of magnitude of the low-frequency wave data from field experiments. Foda and Mei (1981) used the multiple-scales method (see chapter 4) to derive a third-order evolution equation for edge waves. The magnitude of these edge waves was assumed to be of the same order as that of the wind waves. Wave breaking was included empirically in the model. They showed that resonant excitation of the edge waves is not needed for growing amplitudes, but the growth rate is small in the case of nonresonance. Bottom friction is necessary to keep the edge-wave amplitudes finite in the case of resonance. They did not compare their model with experiments, but the order of magnitude of the low-frequency wave energy and the periods of the low-frequency waves is estimated to be comparable to those observed by Munk and Tucker.

Both models described above can predict the correct order of magnitude of the characteristics of the surf beat, but both contain parameters of which the magnitude is not easily assessed. Another similarity is the two dimensionality of the models, which automatically results in an explanation of the surf beat in terms of edge waves. However, Huntley and Kim (1984) showed that the cross-shore velocity component of the surf beat is predominantly due to so-called leaky waves; these are waves which can travel back to the open sea. These observations give confidence in one-dimensional models in which no edge waves are present. The models that we treat now are all one dimensional.

The relatively simple case of one-dimensional bichromatic high-frequency waves is addressed first here. Lo (1981) used a stream-function model which calculates in the time domain and which can explain the linear relation between the low-frequency wave amplitude and the wind-wave envelope height by taking the generation of low-frequency waves due to the breaking of the high-frequency waves, which appear in periodic wave groups, into account. He found that the surf-zone region was very important for low-frequency wave generation, and that the smaller the beach slope, the higher the surf beat wave heights. He argued that this is due to the fact that more time is available for the
energy transfer from the wave groups to the low-frequency waves. Lo (1981) did not mention how the forcing of the low-frequency waves due to wave breaking was modelled.

Symonds et al. (1982), apparently being unaware of the work of Lo, showed that the generation of bound low-frequency waves by wave groups is not the only mechanism to produce low-frequency waves. They investigated the forcing of low-frequency waves by the time variation of the location of the break point. As explained above, this variation gives rise to a time-varying set up and hence produces free low-frequency waves. The analysis was performed for bichromatic short waves with periodic wave groups. The difference between this model and that of Lo (1981) is that the calculations are done in the frequency domain and that the bound low-frequency waves are ignored. Kostense (1984) tested this model in a wave flume and found good qualitative agreement.

Schäffer and Svendsen (1988) presented a bichromatic model in which they allow for wave-height modulations after the waves are broken by taking a fixed (in space) break point. The model uses both the bound low-frequency waves and the low-frequency wave generation due to wave breaking, but because of the wave modulations inshore of the breakpoint bound low-frequency waves are generated in this region too. In a later paper by Schäffer and Jonsson (1990) the model is compared with the Kostense (1984) data. An overestimation of the outgoing low-frequency wave amplitude was attributed to bottom friction, which is not included in the model. This is the only model which allows for wave height modulations of broken high-frequency waves inside the surf zone and the consequent generation of bound low-frequency waves in this region. Important to note is that the wave-height modulation in the surf zone is chosen such in the final model (Schäffer, 1990), that high waves break so strongly that they appear as low waves in the surf zone and vice versa.

The calculation of one-dimensional surf beat due to random wind waves was first addressed by Lo (1981). His model uses the generation of bound low-frequency waves due to wave groups and due to wave breaking and calculates in the time domain. A comparison of his model with field data by Goda (1975), using a wind-wave spectrum as input, showed the right order of magnitude for the low-frequency spectral components.

List (1992) also proposed a model which uses random wind waves as input. The model closely resembles that of Lo. It uses as input measured wind-wave envelope time series from which low-frequency wave time series are calculated. A comparison with field data showed results for the low-frequency wave energy levels which are 5 to 50% off, and good qualitative results for the correlation between wind-wave envelope and low-frequency surface elevation.
The model presented in this thesis also deals with low-frequency wave generation by random wind waves. The main difference with the models of Lo (1981) and List (1992) is that the calculations are performed in the frequency domain. The input of the model is the energy spectrum of the wind waves, the idea being that it is sufficient to describe the low-frequency waves in terms of statistical properties, in particular the energy-density spectrum. The low-frequency energy-density spectrum is not dependent on one specific realization of the process described by the energy spectrum of the wind waves. In chapter 3 this model is described in detail.
CHAPTER 3

THE SURF BEAT MODEL

In this chapter a model for the generation of low-frequency waves due to breaking wind waves will be presented. Only the shore-normal propagating low-frequency waves will be considered. The motivations to develop the model are explained in chapter 2. The most important reason is to allow the calculation of the variance spectrum of the low-frequency waves (to first order) entirely in the frequency domain from the variance spectrum of the wind waves without the necessity of time series. This means that no relative-phase information of the wind-wave frequency components will be used. However, it should be noted that the model uses the two-dimensional Rayleigh distribution for the wind-wave envelope which assumes the wind waves to have random relative phases. Another reason is that field measurements are usually presented in the form of variance spectra, in which the relative-phase information is not available.

The plan of the chapter is as follows. The first part deals with the theoretical model. In this section the basic assumptions of the model will be given together with its input and output. Then the model itself will be treated systematically. Some analytic results will be presented for cases with simple bottom profiles.

The second part provides the numerical scheme, which is used to solve the model on more complex bottom profiles. This numerical model is tested for a few cases for which analytical solutions are known. The model results will be compared with experiments in chapter 5.

The last part of this chapter deals with an extension of the model to two dimensions. So the assumption of shore-normal propagation of the waves is relaxed. However, the bottom topography will be assumed to be one dimensional. It will be shown that the extension is relatively simple and straightforward. The two-dimensional model is not compared with experiments.
3.1 THE THEORETICAL MODEL

3.1.1 Introduction

In this section the theoretical model for the generation of surf beat is presented. The basic assumptions in the model are the following:
- The bottom topography and the incoming wind-wave field are assumed to vary only in the cross-shore direction.
- The model is linear in the low-frequency wave surface elevation.
- The wind-wave envelope is assumed to be Rayleigh distributed, which means that the wind waves have random relative phases. A result of this is that nonlinear wave-wave interactions have no influence on the forcing of the low-frequency waves (to first order).
- Furthermore, the model assumes that the energy propagation of the wind waves can be represented by one group velocity. This last assumption only makes sense if the wind-wave variance spectrum is narrow.
- Finally, it is assumed that the wind waves will be completely destroyed due to wave breaking; reflection of these waves is not accounted for.

The input parameters of the model are the wind-wave variance spectrum and the bottom topography. The model predicts the variance spectrum of the low-frequency waves. Also correlations between e.g. wind-wave envelope and low-frequency surface elevation are calculated.

The plan of this section is as follows. In section 3.1.2 the low-frequency wave equation in the frequency domain will be derived from that in the time domain and the boundary conditions are formulated. The seaward boundary condition is chosen such that all incoming low-frequency waves are bound. The landward boundary condition is such that the low-frequency waves are fully reflected by the beach.

In section 3.1.3 the forcing will be calculated. Because the forcing is a complex quantity, both its phase and its absolute value have to be determined. Especially the calculation of the absolute value is rather complex. Use will be made of ideas from wave-group statistics to calculate the joint probability density of two values of the wave envelope which differ a time lag. Section 3.1.4 deals explicitly with the determination of this probability density.

Finally, in section 3.1.5 some analytic solutions to the low-frequency wave equation are given for a few bottom profiles.
3.1.2 The low-frequency wave equation

The basic equations are the linearized shallow-water equations. After elimination of the velocity a second-order partial differential equation results for the low-frequency surface elevation. (For the derivation of this equation see Appendix A.) This low-frequency wave equation reads:

\[
\frac{1}{g} \zeta_t - h_z \zeta_x - h \zeta_{xx} = \frac{1}{g \rho} S_{xx}\tag{3.1.1}
\]

in which \( \zeta \) is the low-frequency surface elevation, \( h \) the mean-water depth, \( g \) the acceleration of gravity and \( S \) the radiation stress. The indices denote partial differentiation to the index variable. The right-hand-side of this equation is the second space derivative of the radiation stress, which is the driving force for the low-frequency waves. Because the wind waves are described by a continuous variance spectrum the radiation stress is randomly varying.

As stated before, our aim is to calculate the variance spectrum of the low-frequency waves from that of the wind waves. This is the motivation to expand the variables in a Fourier-Stieltjes (FS) integral (see e.g. Mardia, 1972). We can not use a Fourier integral because this integral is not defined in this case. The reason is that the variables are in principle non zero over the entire time axis, so the Fourier integral diverges. The Fourier-Stieltjes integral, which belongs to a quantity \( a \) is given by

\[
a = \int_0^\infty e^{i \omega t} dA \tag{3.1.2}
\]

in which \( dA \) is a random increment function for an interval \( (\omega, \omega + d\omega) \). It can be seen as the complex amplitude of the waves in a frequency interval \( (\omega, \epsilon) \). It is related to the variance spectrum \( G_{aa} \) of \( a \) by

\[
G_{aa} d\omega = \frac{1}{2} <dAdA^*> \tag{3.1.3}
\]

where \( <..> \) indicates ensemble averaging and the asterisk denotes the complex conjugate. The FS-transformed low-frequency wave equation reads
\[ \frac{\omega^2}{g} dZ(x, \omega) + h_x dZ_x(x, \omega) + h_d Z_{xx}(x, \omega) = -d\phi(x, \omega) \] (3.1.4)

in which \( dZ \) is the FS transform of the low-frequency surface elevation \( \zeta \) and \( d\phi \) is the FS transform of the forcing (apart from the double space derivative). This is the equation, subject to boundary conditions on both outer ends, which we will solve in this chapter.

The boundary conditions can be found as follows. On the landward boundary, at which we choose the \( x \)-coordinate to be zero, we assume fully reflected waves. If we assume the bottom slope to be nonzero on this boundary, the condition of vanishing horizontal velocity is not appropriate. Instead of this condition we will use the fact that the depth close to the water line will be so small that the third term on the left hand side of equation (3.1.4) can be neglected compared to the other terms on the left-hand-side (see also section 3.1.5). Due to breaking the wind waves are saturated close to this boundary so the forcing of the low-frequency waves will also be small compared to the other terms so we end up with

\[ \frac{\omega^2}{g} dZ + h_x dZ_x = 0 \] for \( x < 0 \) (3.1.5)

The seaward boundary has to be such that the incoming low-frequency waves are bound and the outgoing low-frequency waves are free. Because the bottom slope does not have that much effect on the magnitude of the bound low-frequency waves in regions of greater depth, we assume that the term \( h_x dZ_x \) can be neglected compared to the other two terms on the left-hand-side of the low-frequency wave equation. With this assumption we can simply use the expression for the bound low-frequency waves for a horizontal bottom which can easily be calculated. We now have

\[ dZ = dZ_b + dZ_f \] (3.1.6)

and

\[ dZ_x = dZ_{b_x} + dZ_{f_x} \] (3.1.7)

in which the subscripts \( b \) and \( f \) denote bound and free respectively. The free wave component can be eliminated by putting
\[ dZ_x = -ikdZ_f \]  \hspace{1cm} (3.1.8)

in which \( k \) is the wavenumber of the outgoing free wave, which is given by

\[ k = \frac{\omega}{\sqrt{gh}} \]  \hspace{1cm} (3.1.9)

as follows from the low-frequency wave equation. The low-frequency velocity for the free waves is given by

\[ c = \sqrt{gh} \]  \hspace{1cm} (3.1.10)

When we combine (3.1.6), (3.1.7) and (3.1.8) we obtain:

\[ dZ_x + \frac{i\omega}{\sqrt{gh}} dZ_b x + \frac{i\omega}{\sqrt{gh}} dZ_b \]  \hspace{1cm} \at \ x = x_0 \]  \hspace{1cm} (3.1.11)

in which \( x_0 \) is the position of the seaward boundary. Note that both boundary conditions are linear in \( dZ \).

It should be noted that equation (3.1.4) is a second-order ordinary differential equation, which is much easier to solve than the original partial differential equation given in (3.1.1). For certain bottom profiles equation (3.1.4) admits an analytic solution. For instance in the case of a plane sloping bottom the solution can be found as a combination of integrals over Bessel functions of the first and second kind. However, the calculation of the forcing term becomes more complicated than in the time domain. In the next section it will be explained how the forcing can be calculated.

3.1.3 The forcing

Because the FS transform of the forcing is a complex quantity, both its absolute value and its phase have to be calculated. We will deal with the calculation of the phase first, after which the calculation of the absolute value follows. The phase of the forcing will be space and frequency dependent. The frequency dependence of the phase cannot be determined because only the wind-wave variance spectrum is given. However, we want to calculate the variance spectrum of the low-frequency waves so this
frequency dependence is of no importance. The reason for this is that the variance spectral density of the low-frequency waves is proportional to the absolute value of \( dZ \) squared, so it does not depend on the relative phase of the frequency components of the low-frequency waves.

To calculate correlations between the wind-wave envelope and the low-frequency waves we only need the phase relation between the forcing and the low-frequency waves and not the phase relation of low-frequency waves of different frequencies. So again the frequency dependence of the phases is not important.

The space dependence of the phase of the forcing will be important. The forcing is related to the wind-wave envelope squared via the radiation stress. Because it takes the wind-wave envelope some time to pass through the surf zone, a certain distance corresponds to a phase shift in the forcing. A way to calculate this phase shift is to assume that the wind-wave envelope travels with the group velocity \( C_g \) of the wind waves. Implicit in this assumption is that the wind-wave variance spectrum is narrow. The phase shift due to the finite travel time of the wind-wave envelope now becomes:

\[
\Delta(\psi(x)) = \arg(d\phi) = \int_{x_0}^{x} \frac{\omega}{C_g} \, dx
\]  
(3.1.12)

When the wind waves approach the shoreline the group velocity changes due to depth changes, but also due to changes in the shape of the variance spectrum of the wind waves. In the model presented here the influence of the change in shape of the variance spectrum of the wind waves on the group velocity will be neglected. So when the bottom profile is known, this integral can be determined and the phase of the forcing is known throughout the surf zone. Let us now turn to the absolute value of the forcing.

The absolute value of the forcing term in equation (3.1.4) will be calculated from the variance spectrum of the forcing which in turn will be determined from the variance spectrum of the wind waves. As mentioned above, the forcing term is proportional to the second space derivative of the radiation stress. The radiation stress is related to the wind-wave energy density \( E \) as

\[
S = (2n - \frac{1}{2})E
\]  
(3.1.13)

in which \( n \) is a depth-dependent quantity given by
\[ n = \frac{1}{2} \frac{kh}{\sinh(2kh)} \]  

(3.1.14)

with \( k \) a characteristic wavenumber of the wind waves. The local and instantaneous wind-wave energy density is given by

\[ E = \frac{1}{2} \rho g R^2 \]  

(3.1.15)

in which \( R \) is the local and instantaneous value of the wind-wave envelope. So the variance spectrum of the envelope squared has to be determined to obtain the variance spectrum of the forcing.

It should be noticed here that relation (3.1.13) between radiation stress and energy density is derived for the case of sinusoidal waves. Due to wave shoaling this relation is not strictly valid. However, because (3.1.13) is an integral relation it is not sensitive to the exact shape of the waves. Studies which use this relation well inside the surf zone have proven its applicability in this region (Bowen, 1969, Longuet-Higgins, 1970a,b and Thornton, 1970).

A way to calculate the variance spectrum of the wind-wave envelope squared is to start from the covariance of the square of the envelope:

\[ C(R_1^2, R_2^2) = E(R_1^2 R_2^2) - E(R_1^2)E(R_2^2) \]  

(3.1.16)

in which \( C \) is the covariance, \( E \) is the expected-value operator, \( R_1 = R(t) \) and \( R_2 = R(t+\tau) \). The Fourier transformation of the covariance gives the required variance spectrum.

The expected values in equation (3.1.16) can be found from the joint probability density \( p(R_1, R_2) \) of the values of the envelope separated by a time lag \( \tau \) in the following way:

\[ E(R_1^2 R_2^2) = \int_0^\infty \int_0^\infty R_1^2 R_2^2 p(R_1, R_2) dR_1 dR_2 \]  

(3.1.17)

and

\[ E(R_1^2) = R_{\text{rms}}^2 = \int_0^\infty \int_0^\infty R_1^2 p(R_1, R_2) dR_1 dR_2 = \int_0^\infty R_1^2 p(R_1) dR_1 \]  

(3.1.18)
Note that the probability density is a functional of the time lag $\tau$, so for each time lag $p(R_1, R_2)$ is a two-dimensional function of $R_1$ and $R_2$. To calculate the probability density we need four ingredients: (a) the energy-density spectrum of the wind waves, (b) the bottom profile, (c) a breaking criterion for the wind waves and finally (d) some statistics of the wave envelope to determine the dependence on the time lag $\tau$. Before we will continue with the calculation of the probability density and the expected values, which is described in the next section 3.1.4, let us state the line of reasoning.

Suppose all the four ingredients just mentioned are at our disposal. Then we are able to calculate the covariance of the square of the envelope for each time lag. This covariance is multiplied by a depth dependent value to obtain the covariance of the radiation stress:

$$C(S_1, S_2) = \frac{(2n-1)^2}{16} \rho^2 g^2 C(R_1, R_2)$$  \hspace{1cm} (3.1.19)

This covariance is then Fourier transformed to obtain the variance spectrum of the radiation stress, which in turn enables us to calculate the absolute value of the FS transform of the (normalized) radiation stress:

$$|d\phi| = \sqrt{d\phi d\phi^*} = \sqrt{|d\phi|^2} = \frac{1}{\rho g} \sqrt{G_{n, s}(x, \omega)} d\omega$$  \hspace{1cm} (3.1.20)

Now we have found both the absolute value of the FS transform of the radiation stress (equation 3.1.20) and its phase (equation 3.1.12), so the complex FS transform of the radiation stress is known. The forcing term in the FS transformed low-frequency wave equation (3.1.4) can now be found as:

$$d\phi_{xx} = |d\phi|_{xx} + 2i \frac{\omega}{C_g} |d\phi| \left( \frac{-\omega^2 C_s}{C_g^2} - i \frac{\omega C_s}{C_g^2} \right) |d\phi| e^{i \frac{\omega dx}{C_t}}$$  \hspace{1cm} (3.1.21)

Now that we have this expression for the forcing we can solve the differential equation for $dZ$.

The method outlined above is also used to calculate correlation coefficients, for instance the correlation between low-frequency surface elevation and wind-wave envelope. At first sight this may seem impossible because the frequency dependence of the phases of the spectral components of e.g. low-frequency surface elevation is not known. However, to calculate this
correlation only the phase difference between the FS transform of the low-frequency surface elevation and the FS transform of the wind-wave envelope at each frequency has to be known, not the phase difference between different spectral components. The required phase differences at each frequency follow directly from differential equation (3.1.4).

3.1.4 The joint probability density

In this section we concentrate on the calculation of the joint probability density of two values of the wave envelope separated by a time lag. As mentioned in the preceding section we need four ingredients at each location: (a) the energy density spectrum of the wind waves, (b) the still-water depth, (c) a breaking criterion for the wind waves and finally (d) statistics of the wave envelope distribution to determine the dependence of the probability density on the time lag \( \tau \).

For non-breaking waves the statistics of the wave envelope can be determined from that of the wind waves. The surface elevation of the wind waves is assumed to be Gaussian distributed. It can easily be shown that the corresponding wave envelope is Rayleigh distributed. Rice (1944, 1945) has shown that for a narrow-band Gaussian process the joint probability density of two values of the wave envelope is the two-dimensional Rayleigh distribution. This density has been used successfully by Kimura (1980) in his description of wave groups.

In the case of breaking waves the probability density has to be modified because the wave height is limited to a critical value, mainly determined by the local depth.

We will first concentrate on the influence of breaking on the one-dimensional probability distribution of the wave envelope. We will use the formulation of Battjes and Jansen (1978), who developed a model for the spatial variation of the root-mean-square wave height and the time-mean set up due to random breaking waves. The reasons to concentrate on their model are twofold. First, in their model a truncated one-dimensional wave-envelope density is used to describe the breaking wind waves. In the present model a two-dimensional version of this idea is needed. Secondly, their model is used in the present model to calculate the evolution of the root-mean-square wave envelope from deep water to the water line, which we need to calculate the joint probability density.

The basic idea of the Battjes-Jansen model is that the probability of breaking at each location is mainly controlled by the local depth and the local wave-energy. To elaborate this idea they assume that an individual wave breaks as soon as its height reaches a certain depth-limited value \( H_c \), or equivalently, as soon as the wave envelope reaches a certain depth-limited value
$R_e$. This means that as soon as the height of an individual wave reaches $H_e$ the wave has a probability of one to break.

They realize that this assumption is a simplification. Not all the heights of broken waves passing a certain point are equal, nor are they all necessarily larger than those of the non-broken waves. However, their purpose is the derivation of mean square wave-height values from a distribution of wave heights, which are somehow limited by breaking.

In absence of breaking the probability distribution of $R$ is given by the one-dimensional Rayleigh distribution. Under the assumption that at each depth a maximum value of the wave envelope $R_e$ can be defined, which is equal to the value of the wave envelope at breaking or broken waves, the probability distribution is given by

$$F(R) = Pr(R \leq R) = 1 - e^{-\frac{R^2}{R_m^2}} \quad \text{for } 0 \leq R \leq R_e \quad (3.1.22)$$

$$= 1 \quad \text{for } R \geq R_e \quad (3.1.23)$$

in which the underlined quantities are the stochastic variables. The parameter $R_m$ in this expression is related to the fraction of broken waves at a certain location as will be shown below. In the case of non-breaking waves it is equal to the modal value of the wave envelope. (Battjes and Jansen described their model in terms of wave heights, but because the present model is described in terms of the wave envelope, which is a continuous quantity, their formulation is adapted to the wave-envelope description.) The fraction of broken waves $Q$ is given by the fraction of waves with height equal to the critical wave height or by the fraction of envelope values equal to the critical wave-envelope value:

$$Q = Pr(R = R_e) = e^{-\frac{R_e^2}{R_m^2}} \quad (3.1.24)$$

The parameter $R_m$ can be eliminated in the following way. The probability density is obtained from the probability distribution as
\[ p(R) = \frac{\partial F(R)}{\partial R} = 2 \frac{R^2}{R_m^2} e^{-\frac{R^2}{R_m^2}} \]

for \( R < R_c \) \hspace{1cm} (3.1.25)

\[
\left[ 1 - 2 \frac{R_c}{R_m^2} e^{-\frac{R^2}{R_m^2}} \right] \delta(R - R_c) \]

for \( R = R_c \) \hspace{1cm} (3.1.26)

\[ = 0 \]

for \( R > R_c \) \hspace{1cm} (3.1.27)

From this expression we can calculate the root-mean-square value of the wave envelope as

\[ R_{rms} = \left[ \int_0^\infty R^2 p(R) dR \right]^{1/2} = \left[ (1 - Q)R_m^2 \right]^{1/2} \]

(3.1.28)

If we eliminate \( R_m \) from this equation and the expression (3.1.24) for the fraction of broken waves we obtain a transcendental equation for \( Q \):

\[ \frac{1 - Q}{\log Q} = \frac{R_{rms}^2}{R_c^2} \]

(3.1.29)

which gives \( Q \) as function of \( R_{rms}/R_c \).

In the Battjes and Jansen (1978) paper the maximum value for the wave envelope is chosen as

\[ R_c = \frac{0.44}{k} \tanh \left( \frac{\gamma k h}{0.88} \right) \]

(3.1.30)

in which \( k \) is a representative local wavenumber.

The variation of \( R_{rms} \) (and hence \( Q \)) can be found from the energy-balance equation. Battjes and Jansen realized that the fraction of broken waves is related to the energy-dissipation rate.
The energy-dissipation rate $D$ in a broken wave is estimated from that of a bore of corresponding height as

$$D = \alpha Q \tilde{f} \rho g R_e^2 \quad (3.1.31)$$

in which $\alpha$ is a constant of order 1 and $\tilde{f}$ is a representative frequency, for instance the mean frequency. The set of equations is closed with the energy-balance equation, which relates the variation of the root-mean-square value of the wave envelope to the energy-dissipation rate. It reads

$$\frac{\partial C E}{\partial x} + D = 0 \quad (3.1.32)$$

in which $E$ is the mean wave energy density given by

$$E = \frac{1}{2} \rho g R_{rms}^2 \quad (3.1.33)$$

and $D$ is the time-mean dissipated energy per unit area. The time-mean set-up is found from the time-averaged equation (3.1.1). For the details of the model the reader is referred to the Battjes and Jansen (1978) paper.

Before we turn to the determination of the joint probability density let us recapitulate one ingredient in the Battjes-Jansen model which is of great concern for this determination. Battjes and Jansen started from the probability density of the wave envelope in the case of non-breaking waves and determined the probability distribution for non-breaking waves. Then they introduced a local depth-dependent maximum value for the wave envelope and adapted the probability distribution accordingly. In the following we will follow the same route, with the exception that our probability distribution and density are two-dimensional.

The probability density for non-breaking waves is the two-dimensional Rayleigh distribution as given by Rice (1944, 1945):

$$p(R_1, R_2) = \frac{4R_1 R_2}{R_m^4 (1 - \kappa^2)} \exp \left[ -\frac{R_1^2 + R_2^2}{R_m^2 (1 - \kappa^2)} \right] I_0 \left[ \frac{2\kappa R_1 R_2}{R_m^2 (1 - \kappa^2)} \right] \quad (3.1.34)$$
in which \( R_m \) is equal to the mean value for the wave envelope in the case of non-breaking waves. The parameter \( \kappa \) depends on the time lag \( \tau \) and the energy spectrum of the underlying process according to

\[
m_0^2 \kappa^2 = \left| \int_0^\tau G(\omega)e^{i\omega \tau} d\omega \right|^2 \tag{3.1.35}
\]

in which \( m_0 \) is the total variance of the wind-wave surface elevation, \( G(\omega) \) is the spectral density of the surface elevation of the wind waves and \( \omega \) is the angular frequency. The characteristics of this distribution can be found in e.g. Battjes (1971). He showed that \( \kappa^2 \) equals the auto correlation coefficient of the wind-wave envelope squared.

In the case of breaking waves this probability density has to be modified. The density can be calculated from the cumulative distribution or probability of non-exceedence \( P \) defined as:

\[
P = \text{Probability} \ (R_{1\leq R_{1\leq R_{2\leq R}}}) \tag{3.1.36}
\]

which can be expressed in terms of the two-dimensional Rayleigh distribution with the same assumptions as Battjes and Jansen as

\[
P = \int_{R_{1\leq R_{2\leq R}}} p_{nb}(R_1, R_2) dR_1 dR_2 \quad 0 < R_1, R_2 < R_c
\]

\[
P = Pr(R_{1\leq R_{2\leq R}}) = \int_{0}^{R_1} p_{nb}(R_1) dR_1 \quad 0 < R_1 < R_c, \quad R_2 = R_c
\]

\[
P = 1 \quad R_1 = R_2 = R_c
\]

in which the subscript \( nb \) denotes non breaking, which means that the probability density \( p_{nb}(R_1, R_2) \) has the same shape as the two-dimensional Rayleigh distribution, just like in the non-breaking case. However, the parameters in this distribution will not be the same as in the non-breaking case, they still have to be defined. We will make the following assumptions about them.

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First, we assume that the marginal probability density is equal to that used by Battjes and Jansen (1978). This means that \( R_m \) fulfills the following relation:

\[
Q = e^{-\frac{R_m^2}{R_e^2}} \tag{3.1.38}
\]

in which \( Q \) the fraction of broken waves. This gives us again the possibility to eliminate \( R_m \) in favor of \( R_{rms} \). The cross-shore variations of \( Q \) and \( R_{rms} \) are found from the Battjes-Jansen model (1978).

Secondly, we assume the relation (3.1.35) between \( \kappa \) and the wind-wave energy spectrum to hold throughout the surf zone. At this stage a problem arises. It is in general not known how a certain wind-wave spectrum evolves during wave breaking. In this model the influence of the change in spectral shape due to wave breaking on \( \kappa^2 \) is neglected. The only influence of the breaking waves on the probability density of the non-breaking waves is thus via \( R_m \), or \( Q \).

The joint probability density for two values of the wave envelope separated by a time lag is found from the probability distribution by differentiation:

\[
p(R_1, R_2) = \frac{\partial^2 P(R_1 \leq R_1, R_2 \leq R_2)}{\partial R_1 \partial R_2} \tag{3.1.39}
\]

so the probability density becomes with help of (3.1.37):
\[ p_b(R_1, R_2) = p_{nb}(R_1, R_2) \]

\[ p_{nb}(R_1) = \int_0^{R_2} p_{nb}(R_1, R_2') dR_2' \]

\[ p_b(R_1, R_2) = \lim_{dR_2 \to 0} \frac{\int_0^{R_2} p_{nb}(R_1, R_2') dR_2' - \int_0^{R_2} p_{nb}(R_1, R_2') dR_2'}{dR_2} = \lim_{dR_2 \to 0} \frac{\int_0^{R_2} p_{nb}(R_1, R_2') dR_2' - \int_0^{R_2} p_{nb}(R_1, R_2') dR_2'}{dR_2} = \]

\[ = \int_0^{R_2} p_{nb}(R_1, R_2') dR_2' \delta(R_2 - R_c) \]

\[ 0 < R_1 < R_c, R_2 = R_c \]

\[ = \int_0^{R_2} p_{nb}(R_1, R_2') dR_2' \delta(R_1 - R_c) \delta(R_2 - R_c) \]

\[ R_1 = R_2 = R_c \]

\[ p_b(R_1, R_2) = 0 \]

\[ R_1, R_2 \geq R_c \]

To summarize, the procedure to obtain the joint probability density of two values of the wave envelope which differ a time lag is as follows. First, for given maximum value of the wave envelope \( R_c(x) \), the Battjes-Jansen model is run to provide the variation of the root-mean-square value of the wave envelope \( R_{rms}(x) \), and the percentage of broken waves \( Q(x) \) with location \( x \). Then the two-dimensional probability distribution of \((R_1, R_2)\) is calculated as indicated above, and finally the corresponding joint probability density using equation (3.1.40).

Now that we have determined the probability density of the breaking waves, we are able to calculate the expected values of the wave-envelope squared:
\[
E(R_1^2 R_2^2) = \int_0^{R_1} \int_0^{R_2} p_d(R_1, R_2) \, dR_1 \, dR_2 = \\
= \int_0^{R_c} R_1^2 R_2^2 p_{nb}(R_1, R_2) \, dR_1 \, dR_2 + 2R_c^2 \int_0^{R_c} R_1^2 p_{nb} dR_1 \, dR_2 + \\
+ R_c^4 \int_0^{R_c} p_{nb} dR_1 \, dR_2 = \\
= \int_0^{R_c} (R_1^2 - R_c^2)(R_2^2 - R_c^2) p_{nb} \, dR_1 \, dR_2 + R_c^2 (2R_{rms}^2 - R_c^2) \\
E(R_1^2) = E(R_2^2) = \int_0^{R_1} R^2 p_d(R) \, dR = R_{rms}^2
\]

The double integrals in equation (3.1.41) can be expressed as a one dimensional integral if we substitute the expression of the modified two-dimensional Rayleigh distribution. We then obtain

\[
E(R_1^2 R_2^2) = \frac{(1+\kappa^2) R_{rms}^4 - R_c^2 R_{rms}^2}{(1-Q)^2} \left( -\frac{(1+\kappa^2) R_{rms}^4 + 2R_c^2 R_{rms}^2}{(1-Q)^2} \right) \exp(-2b) I_0(2\kappa b) + \\
+ \frac{2\kappa R_c^2 R_{rms}^2}{(1-Q)} \exp(-2b) I_1(2\kappa b) + \\
+ 4R_c^2 \frac{(1+\kappa^2) R_{rms}^2}{(1-Q)(1-\kappa^2)} \int_0^1 x \exp(-bx^2) I_0(2\kappa bx) \, dx
\]

in which

\[
b = \frac{1-Q}{(1-\kappa^2)} \frac{R_c^2}{R_{rms}^2}
\]

The covariance of the wave envelope squared can now be found from equation (3.1.16). Now
we have completed the task of calculating the forcing, and the low-frequency wave equation can be integrated to find the low-frequency wave energy-density spectrum as function of position. In the next section the integration is performed analytically for some simple bottom profiles.

3.1.5 Solutions for the low-frequency wave equation

In this section analytical solutions for the low-frequency wave equation will be found for certain bottom profiles. This is done to obtain some idea about the solutions. The results will also be used to test the numerical model. We assume the FS transform of the forcing to be known. The bottom profiles which will be dealt with are a plane horizontal bottom, a plane sloping bottom and bottoms in which the depth is another power of the horizontal coordinate.

The horizontal bottom

In this case the low-frequency wave equation (3.1.4) reduces to

\[
\frac{\omega^2}{g}dZ + h \frac{dZ}{x} = -d\phi_{xx}
\]  

(3.1.44)

The homogeneous solution to this equation is given by

\[
dZ^h = A \cos(Kx) + B \sin(Kx)
\]  

(3.1.45)

in which \( K \) is the wavenumber of the waves, given by

\[
K = \frac{\omega}{\sqrt{gh}}
\]  

(3.1.46)

The homogenous solution describes free low-frequency waves with phase velocity \( \sqrt{gh} \). The bound low-frequency waves can be found as the particular solution. With the method of variation of parameters we find:

\[
dZ^p = \int \left[ \sin(Kx)\cos(Kx')d\phi(x',x',x') - \cos(Kx)\sin(Kx')d\phi(x',x',x') \right] dx'
\]  

(3.1.47)

Note that, if dissipation can be neglected, the absolute value of the forcing remains constant,
but its phase will vary with the horizontal coordinate. If we use the expression for the phase of the forcing as obtained in section 3.1.3 (equation 3.1.12) we find:

\[
d\phi_{xx} = -\frac{\omega^2}{C_g^2} |d\phi| e^{i\Delta\psi}
\]  

(3.1.48)

This expression can be used to reduce the expression for the bound long waves, because the group velocity is constant in constant depth. This means that the integrations in that expression can be performed to obtain finally

\[
dZ' = -\frac{gd\phi}{gh-C_g^2}
\]  

(3.1.49)

This is exactly the expression that Longuet-Higgins and Stewart obtained for the bound low-frequency waves in their 1962 paper. Note that the bound waves are 180 degrees out of phase with the radiation stress. This leads to the physical interpretation that the bound low-frequency waves arise due to the excess momentum flux under high wind waves, which pushes water to the lower wind waves so that a bound low-frequency wave results.

If we use the relation between the Fourier-Stieltjes transform and the spectral density (see equation 3.1.3) we obtain

\[
G_{\zeta\zeta} = \frac{g^2 G_{ss}}{(gh-C_g^2)^2}
\]  

(3.1.50)

in which \( G_{\zeta\zeta} \) and \( G_{ss} \) are the one-sided energy densities of the bound low-frequency waves and the normalized radiation stress respectively. The normalization factor is again \( \rho g \). The energy density of the forcing can be evaluated as

\[
G_{ss} = \frac{(2n-1)^2}{4} \int_0^\infty C(R_1, R_2^2) e^{-i\omega \tau} d\tau
\]  

(3.1.51)

in which \( C(R_1, R_2^2) \) is the covariance of the wave-envelope squared at a time lag \( \tau \). In the case of non-breaking wind waves the covariance can easily be calculated with help of the relations given in 3.1.4. We then find

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\[ C(R_1^2, R_2^2) = \kappa^2(E(R^4) - E^2(R^2)) \] \hspace{1cm} (3.1.52)

in which \( \kappa^2 \) is the correlation coefficient of the wave-envelope squared, which is given by

\[ \kappa^2(\tau) = \frac{\int_0^\infty G(\omega) e^{i\omega \tau} d\omega}{\left( \int_0^\infty G(\omega) d\omega \right)^2} \] \hspace{1cm} (3.1.53)

When we use equations (3.1.51), (3.1.52) and (3.1.53) in equation (3.1.3) we obtain

\[ G_{\zeta\zeta}(\omega) = g^2 \left( \frac{2n - \frac{1}{2}}{(gh - C_s^2)^2} \right) \int_0^\infty G(\omega')G(\omega' - \omega) d\omega' \] \hspace{1cm} (3.1.54)

Notice that the energy density of the bound low-frequency waves is obtained by taking a convolution of the energy density of the wind waves. This convolution is also found in full nonlinear theories as in Ottesen-Hansen (1978) and Laing (1986). However, the factor preceding the convolution is different from the factors in the nonlinear theories. Here the assumption of the narrow-bandedness of the wind-wave energy density shows up. The factor matches the factor given by Longuet-Higgins and Stewart (1962).

**The plane sloping bottom**

First we will look at the behaviour of the bound waves, after which we will solve the complete differential equation (3.1.4).

From expression (3.1.49) it is clear that the solution of Longuet-Higgins and Stewart (1962) breaks down in very shallow water, in which the group velocity (nearly) equals \( \sqrt{gh} \). So this solution is not an good first guess at very shallow water on a sloping bottom. As we will see in chapter 5, the measurements show that the phase shift of 180 degrees between the bound low-frequency waves and the wave envelope changes to a 90 degrees shift in shallow water. This happens even before breaking, so the breakpoint forced waves of Symonds e.a. (1982) are not yet formed. To investigate this phenomenon we turn to the FS transformed low-frequency wave equation (3.1.4):
\[ \frac{\omega^2}{g} dZ(x, \omega) + h_x dZ_x(x, \omega) + h dZ_{xx}(x, \omega) = -d\phi(x, \omega) \] (3.1.55)

We have pointed out before that the space dependence of the FS transforms \( dZ \) and \( d\phi \) contains of two parts: The space dependence of the absolute value and the space dependence of the phase of the quantities. Before breaking the space dependence of the absolute value of the low-frequency wave surface elevation is much smaller than that of the phases (see for instance figure 5.9), so in a first approximation we can neglect the space variation of that absolute value. We then obtain:

\[ \left[ (gh - C_h^2) - i \left( \frac{g C_s h_x}{\omega} - \frac{g C_s h}{\omega} \right) \right] dZ = \left[ \frac{2 g C_s^2}{\omega^2} - g \cdot \frac{g C_s C_{xx}}{\omega^2} - 2 i \left( \frac{g C_s}{\omega} \right) \right] d\phi \] (3.1.56)

In shallow water this reduces to

\[ dZ = \frac{q_L^2 \left[ \frac{1}{2} h_x^2 + q_s^2 \left( \frac{3}{4} h_x^2 - q_L^2 \right) \right] - i h_x q_L \left[ q_s^2 q_L^2 + \frac{1}{2} (q_L^2 - \frac{3}{4} h_x^2) \right]}{h \left( q_s^4 + \frac{1}{4} q_L^2 h_x^2 \right)} d\phi \] (3.1.57)

in which the subscripts \( L \) and \( s \) denote long and short waves respectively, and \( q = kh \). The difference with the solution of Longuet-Higgins and Stewart (equation 3.1.49) is that the phase shift of the low-frequency waves compared to the forcing is not \( \pi \), but dependent on the bottom slope and the relative depth of the short waves \( (q_s) \) and that of the long waves \( (q_L) \). The relative phase of \( dZ \) compared to \( d\phi \) depends on the ratio of the two terms in the numerator of equation (3.1.57). This ratio depends on the ratio of \( h_x^2 \) and \( q_L^2 \), and on the ratio between \( q_s \) and \( q_L \). From the measurements (see chapter 5) we find that this latter ratio is about 4. We then find

\[ \text{arg}(dZ) = \text{arg}(d\phi) + \arctan \left( \frac{\left( x - \frac{1}{6} \right) \left( x + \frac{1}{6} \right)}{x^3 - \frac{1}{4}} \right) \] (3.1.58)

in which

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\[
\frac{q_t^2}{h_x^2}
\]

Coming from deeper water, the phase shift is first about \( \pi \), but at smaller depths (when \( x=1.5 \)) a 90 degrees phase difference between the bound low-frequency waves and the wave envelope occurs. At even smaller depths, when \( x=0.18 \), the imaginary term becomes zero, which indicates that the wind-wave envelope and the low-frequency wave are in phase. Because this is not observed in the experiments, we can conclude that or the bound waves are released before this happens, or the assumptions underlying this simple analysis are no longer valid.

The complete problem is solved as follows. The water depth is given by

\[
h = x \tan \beta \tag{3.1.59}
\]

in which case the homogeneous solution of (3.1.4) becomes

\[
dZ^h = AJ_0(u) + BY_0(u) \tag{3.1.60}
\]

in which \( J_0 \) and \( Y_0 \) are Bessel functions of the first respectively second kind. \( u \) is given by

\[
u = \frac{2\omega}{\sqrt{g} \tan \beta} \tag{3.1.61}
\]

The particular solution is given by

\[
dZ^p = \frac{2\pi}{\tan \beta} \int x \left[ J_0(u)Y_0(u')d\phi_{\lambda\lambda}(x') - Y_0(u)J_0(u')d\phi_{\lambda\lambda}(x') \right] dx' \tag{3.1.62}
\]

This solution was also found by Symonds et al (1982). Close to the shore line all wind waves are broken so the forcing is negligible and the particular solution will be negligible too. Because for small arguments

\[
Y_0(z) = \frac{2}{\pi} \log z \tag{3.1.63}
\]

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the constant $B$ in (3.1.60) must be equal to zero. We then have close to the shore line

$$dZ = AJ_0 \left( \frac{2\omega}{\sqrt{\tan \beta}} \right)$$

(3.1.64)

This solution denotes a standing wave close to the shore line, as one would expect.

The power-law variation of the depth

The bottom profile can in this case be written as

$$h = bx^\lambda \quad \text{with} \quad \lambda > 0$$

(3.1.65)

in which $b$ and $\lambda$ constants. The low-frequency wave equation (3.1.14) now becomes

$$\frac{\omega^2}{g} dZ + \lambda bx^{\lambda-1} dz_x + bx^\lambda dz_{xx} = -d\phi_{xx}$$

(3.1.66)

An easy way to solve this equation is to perform a coordinate transformation

$$u = x^\beta$$

(3.1.67)

in which $\beta$ is another constant which we can still choose. We then find

$$\frac{\omega^2}{g} dx + (\lambda + \beta - 1) \beta bu^{1+\frac{\lambda-2}{\beta}} dZ_u + b\beta^2 u^{2+\frac{\lambda-2}{\beta}} dZ_{uu} = -d\phi_{xx}$$

(3.1.68)

Now we choose $\beta$ such that

$$\beta = 1 - \lambda$$

(3.1.69)

so that the term with $dZ_u$ vanishes. Note that this is meaningful only when $\lambda$ is not equal to 1. However, with that value of $\lambda$ we are back to the plane sloping bottom case which is described above.

We now arrive at
\[
\frac{\omega^2}{g} \frac{dZ}{b(1-\lambda)^2} \frac{1}{\gamma} \frac{dZ}{u} \frac{1}{1-\lambda} \frac{d\phi_x}{du} = -d\phi_x \quad (\lambda \neq 1)
\]

(3.1.70)

The advantage of this form is that the solution to the homogeneous equation can be found in reference books. Gradshteyn and Ryzhik (1980) give as homogeneous solution:

\[
dZ^h = \sqrt{u} \left[ AJ_1 \left( \frac{\gamma u}{2\delta} \right) + BY_1 \left( \frac{\gamma u}{2\delta} \right) \right]
\]

(3.1.71)

in which

\[
\delta = \frac{2-\lambda}{2(1-\lambda)}
\]

(3.1.72)

and

\[
\gamma = \frac{1}{\sqrt{g}} \frac{2\omega}{(2-\lambda)}
\]

(3.1.73)

The functions \( J_1 \) and \( Y_1 \) are Bessel functions of fractional order. The particular solutions are given by

\[
dZ^p = dZ^h_1 \int_{\nu_0}^{\nu} \frac{\pi u}{2b(1-\lambda)^2} d\nu - dZ^h_2 \int_{\nu_0}^{\nu} \frac{\pi u}{2b(1-\lambda)^2} d\nu \quad (\lambda \neq 1)
\]

(3.1.74)

in which \( dZ^h_i \) (i=1,2) are the two independent homogeneous solutions.

Note that the solution in (3.1.71) is not meaningful when \( \lambda = 0, \lambda = 1 \) and \( \lambda = 2 \). The first two cases are already treated, but the last case needs some extra attention. In that case we have

\[
u^2 dZ_{uu} + \frac{\omega^2 g}{b} dZ = -\frac{1}{b} d\phi_x
\]

(3.1.75)

of which the homogeneous solution is
\[ dZ^h = Ax \frac{1}{2(\sqrt{1+a}+1)} + Bx \frac{1}{2(\sqrt{1+a}-1)} \]  

(3.1.76)

in which

\[ a = \frac{4\omega^2}{gb} \]  

(3.1.77)

The particular solution of equation (3.1.75) is given by

\[ dZ^p = dZ_1^h \int_{x_0}^{x} \frac{2x^2dZ_2^h d\phi_{za} dx}{b/1+a} - dZ_2^h \int_{x_0}^{x} \frac{2x^2dZ_1^h d\phi_{za} dx}{b/1+a} \]  

(3.1.78)

Let us now have a look at the solutions close to the water line. Again, nearly all wind waves will be broken, so the forcing will be negligible and the particular solutions can be neglected compared to the homogeneous solutions. For \( \lambda \neq 1 \) and \( \lambda \neq 2 \) the Bessel functions can be approximated for small arguments by

\[ J_\nu(z) = \left(\frac{1}{2}\pi\right)^{\nu} \frac{1}{\Gamma(\nu+1)} = Cx^{1-\lambda} \]  

(3.1.79)

and

\[ Y_\nu(z) = \frac{\Gamma(\nu)}{\pi\left(\frac{1}{2}\pi\right)^{\nu}} = Cx^\lambda \]  

(3.1.80)

for \( \nu \neq 0 \). We thus see that the homogeneous solution with the Bessel function of the second kind either must vanish to keep the solution finite at the shoreline or dies out, while the other homogeneous solution behaves just the other way around. Note that also the homogeneous solution for the case \( \lambda = 2 \) has this behaviour. This means that the physically realistic solution, which must be finite at the shore line, would vanish close to the shore line. However, one would expect a standing low-frequency wave close to the water line, so the solutions for the power-law varying depth are not realistic at the water line.
Mathematically, the absence of standing waves can be explained as follows from the low-frequency wave equation (3.1.66). Close to the waterline all wind waves are broken and the forcing can be neglected. If \( \lambda > 1 \) we see that the second and the third term in (3.1.66) decrease with decreasing \( x \) because the derivatives \( dZ_x \) and \( dZ_{xx} \) must be finite. To fulfil the equation close to the water line \( dZ \) must vanish there too, so no standing waves emerge. If \( 0 < \lambda < 1 \) the third term in (3.1.66) will decreases with decreasing \( x \), but the factor \( x^{1-\lambda} \) in the second term increases without limit with decreasing \( x \). To keep this term finite \( dZ_x \) has to decrease at least as fast as \( x^{1-\lambda} \). This means that \( dZ \) will at least decrease as fast as \( x^{2-\lambda} \), so it will vanish at the shoreline i.e. no standing waves. In the case that \( \lambda = 1 \) the third term in (3.1.66) decreases with decreasing \( x \) and we and up with

\[
\frac{\omega^2}{g} dZ + b dZ_x = 0
\]

which can be integrated to

\[
dZ = A e^{-\frac{\omega x}{g^b}}
\]

which does give standing-wave solutions.

Physically the absence of standing waves if \( \lambda \neq 1 \) means that the solution breaks down close to the shoreline. Of course this does not come as a surprise because close to the shore line the model assumption that the low-frequency wave is small compared to the depth is violated. In this sense the plane sloping bottom solution is not consistent with the assumption of linearity close to the water line.

We will close this section with the special solutions which occur when

\[
\frac{1}{2\beta} = n + \frac{1}{2}
\]

or

\[
\lambda = \frac{2n+1}{6n+1}
\]
in which \( n \) is an integer. In this case the Bessel functions can be written as spherical polynomials:

\[
J_{n+\frac{1}{2}}(z) = \sqrt{\frac{2z}{\pi}} z^n \left( -\frac{1}{z} \frac{d}{dz} \right)^n \frac{\sin z}{z} \quad (3.1.85)
\]

\[
Y_{n+\frac{1}{2}}(z) = -\sqrt{\frac{2z}{\pi}} z^n \left( -\frac{1}{z} \frac{d}{dz} \right)^n \frac{\cos z}{z} \quad (3.1.86)
\]

and the solution can be written entirely in terms of elementary functions.
3.2 THE NUMERICAL MODEL

3.2.1 Introduction

In this section we will deal with the numerical integration of the low-frequency equation. The numerical scheme uses central differences because of simplicity and stability.

3.2.2 The numerical scheme

The low-frequency wave equation can be discretized as

\[
\frac{\omega^2}{g} \frac{dZ^n}{dx} + \frac{h}{2\Delta x} (dZ^{n+1} - dZ^{n-1}) + \frac{h}{\Delta x^2} (dZ^{n+1} - 2dZ^n + dZ^{n-1}) = -d\phi_x
\]  

(3.2.1)

in which \( \Delta x \) is the step size in the cross-shore direction and \( n \) denotes the number of the grid point.

The discretization of the seaward boundary condition (3.1.6) reads

\[
\frac{dZ^2 - dZ^1}{\Delta x} + \frac{i\omega}{\sqrt{gh}} dZ^1 = -dZ^*_b + \frac{i\omega}{\sqrt{gh}} dZ^*_b
\]  

(3.2.2)

The FS transform of the bound waves can be found from equation (3.1.48).

The landward boundary condition (3.1.5) can be discretized as

\[
\frac{\omega^2}{g} \frac{dZ^N}{dx} + \frac{h}{\Delta x} (dZ^N - dZ^{N-1}) = 0
\]  

(3.2.3)

in which \( N \) denotes the last point of integration. It is the last "wet" point in the grid, and this "wetness" is determined from the still-water level instead of the mean water level due to the set up. This is consistent in the model because the low-frequency wave amplitudes, which include the set up, are neglected compared to the local still-water depth.

The integration of the low-frequency wave equation now boils down to the following set of linear equations, written as a linear operator working on the vector \( \overline{dZ} \)
\[ \overline{M} \cdot dZ = \overline{F} \]  
\hspace{1cm} (3.2.4)

in which

\[
\overline{M} = \begin{pmatrix}
\frac{1}{\Delta x} & \frac{i \omega}{\sqrt{gh}} & \frac{1}{\Delta x} & 0 & \cdots & 0 \\
0 & \cdots & \frac{h_x}{2\Delta x} & \frac{h}{\Delta x^2} & \omega^2 - \frac{2h}{\Delta x^2} & \frac{h_x}{2\Delta x} & \frac{h}{\Delta x^2} & \cdots & 0 \\
0 & \cdots & 0 & \frac{h_x}{2\Delta x} & \frac{h}{\Delta x^2} & \frac{h_x}{2\Delta x} & \frac{h}{\Delta x^2} & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \frac{1}{\Delta x} & \frac{1}{\Delta x} & 0 & \cdots & 0 \\
\end{pmatrix},
\]  
\hspace{1cm} (3.2.5)

\[ \overline{dZ} = \begin{pmatrix} dZ^1 \\
\vdots \\
\vdots \\
\vdots \\
dZ^N \end{pmatrix} \quad \text{and} \quad \overline{F} = \begin{pmatrix} 
\frac{dZ_{bx} + i \omega}{\sqrt{gh}} \\
-\frac{d\phi^{x_2}}{} \\
\vdots \\
-\frac{d\phi^{x_N}}{} \\
\vdots \\
-\frac{d\phi^{x_{N-1}}}{\Delta x} \\
0 \\
\end{pmatrix},
\]  
\hspace{1cm} (3.2.6)

The linear operator \( \overline{M} \) is a tri-diagonal matrix. The set of linear equations is solved with the subroutine TRIDAG (Press, 1986), by standard techniques.

### 3.2.3 Test of the numerical scheme

In this section some tests on the numerical scheme are described. The tests are for a horizontal bottom and for a plane sloping bottom.

In the horizontal bottom case the test consisted of a calculation of the energy densities of the
bound low-frequency waves. The solution for the bound waves is given in section (3.1.5), equation (3.1.53) as

\[
G_{\zeta z} = \frac{g^2(2n-1)}{2gh-C_g^2} \int_0 \left( \frac{G(\omega') \frac{d}{d\omega'} \omega'}{G(\omega'-\omega) \frac{d}{d\omega'} \omega'} \right) d\omega'
\]

(3.2.7)

From a given wind-wave spectrum the bound low-frequency wave spectrum is calculated directly with equation (3.2.7). The result of this is compared with the result from the numerical computation.

The correctness of the numerical model is checked as follows. Because the analytical model is linear in the low-frequency wave amplitude, discrepancies will exist between the results of the analytical model and real low-frequency waves. The ratio of the discrepancies in low-frequency wave amplitude to the theoretical values of the wave amplitudes, which we will call the relative error, will be of the order of the nonlinearity parameter \( kA \) in which \( k \) is a characteristic wavenumber of the wind waves and \( A \) is a characteristic amplitude of the wind waves. (For more details on this see appendix A.) This nonlinearity parameter is a measure of the expected correctness of the theoretical model. So a reasonable relative error in the numerical model is at most of the same order as this nonlinearity parameter. In the following we will concentrate on energy densities of the low-frequency waves, which are proportional to the low-frequency wave amplitude squared. This means that the allowable relative error between numerical result and theoretical result is at most twice the nonlinearity parameter.

The input spectrum of the wind waves used was a Jonswap spectrum with a peak frequency of .63 Hz, a peak enhancement factor of 3.3 and a root-mean-square wave height of 4.0 cm. The boundary condition 'upstream' was that only bound waves could enter the domain of calculation (given by equation (3.2.2)) and the 'downstream' condition was such that only bound waves could leave the domain.

The results of the calculation are given in figure (3.1). The test was performed with three depths: 42, 30 and 20 cm. In the last case about one percent of the waves was breaking, so this case is just on the limit of the applicability of the derived equation (3.2.7). The deviations of the model curves from the theoretical curves, derived from equation (3.2.7) are 5 to 10%. The nonlinearity parameter is in all cases about 5%, and the allowable relative error is about 10%, which is the case here. This means that the error in the numerical model is just lower than the error we expect the analytical model will have, which is satisfactory.
The discrepancies between the numerical model and the analytical solution probably arise from the fact that the covariance of the wave envelope squared is actually a so-called circular covariance. This means that due to the discrete Fourier transform the covariance is mirrored in half the largest time lag, so that the obtained values for the covariances are a summation of the actual value at that time lag and the mirrored value at a much larger time lag.

The other tests are performed with the plane sloping bottom. In these tests the forcing was put equal to zero, so we concentrate fully on the propagation of the low-frequency waves. The correctness of the numerical model cannot be stated in terms of the discrepancy between analytical-model results and real low-frequency waves because the forcing due to wind waves is absent here. We will, somewhat arbitrarily, accept the numerical model when the deviations between the numerical model results and the analytical expressions is less than 5%.

First we consider the case in which a low-frequency wave travels from the seaward boundary to the water line, is reflected there and travels back to the seaward boundary. We will again
concentrate on the energy density of the low-frequency waves. In figure (3.2) the model results are compared with the analytical solution in this case, which is a Bessel function of the first kind squared:

\[ G_{\xi \xi} = A J_0^2(u) \]  \hspace{1cm} (3.2.8)

in which \( u \) is given by

\[ u = \frac{2\omega}{\sqrt{g \tan\beta}} \sqrt{x} \]

This solution was found in section (3.1.5). The deviations of the energy densities calculated by the numerical model from those of the analytical solution are less than 0.1%, so the numerical model behaves satisfactorily.
To concentrate more on the propagation of the waves which travel to open water a test is included in which the waves are produced on the water line and travel to the seaward boundary. The analytical solution is given by

\[ G_{zz} = A \left[ J_0^2(u) + Y_0^2(u) \right] \] (3.2.9)

![Graph showing G (mm^2/Hz) vs. h (m) with curves for 0.1 Hz, 0.2 Hz, and 0.3 Hz.](image)

**Figure 3.3:** One-sided spectral densities for outgoing waves versus depth, with frequencies as indicated. Drawn line: analytical results, dashed line: numerical results.

The results of the comparison are given in figure (3.3). The deviations between the numerical-model results and the analytical results are about 5%, except for the lowest frequency. The reason for the discrepancy at the lowest frequency is probably that the gradients close to the water line become too large for the numerical model. A finer grid should solve this problem. However, such steep gradients are not likely to occur in real cases because in nature the low-frequency waves are reflected at the water line. So this matter is not pursued further.

Concluding we can say that the numerical scheme performs satisfactorily. The bound low-frequency waves are overestimated somewhat by the numerical model compared to the
analytical model, probably due to the fact that the covariance of the wave envelope squared in the numerical model is circular. However, the error in the analytical model due to linearization, compared to nature will be of the same order of magnitude. This means that the numerical model behaves good enough. Also the model is not able to reproduce steep gradients correctly, but a finer grid will probably solve this problem. However, these steep gradients are not expected to occur in real cases.
3.3 AN EXTENSION TO TWO DIMENSIONS

3.3.1 Introduction

In this section we shortly describe an extension of the surf-beat model to two dimensions.

We limit ourselves to the case of a two-dimensional bottom topography so that the water depth is constant in the along-shore direction. We also assume that the wind-wave energy-density spectrum is narrow in both the frequency and the directionality domain. This means that we can describe the wind-wave envelope with one propagation velocity $C_g$ in the mean direction $\bar{\theta}$ of the energy-density spectrum. These assumptions are the same as in the one-dimensional case.

The low-frequency wave equation will describe three kinds of waves. First, the waves bound to the wave groups, which are essentially the same as in the one-dimensional case. Second, the free waves which can travel back to the open sea, so called leaky modes. Also these waves have their equivalent in the one-dimensional case. Finally the edge waves, which are free waves unable to reach the open sea due to refraction.

The plan of this section is as follows. First, the low-frequency wave equation will be Fourier-Stieltjes transformed in time and in the along-shore direction to obtain again an ordinary differential equation in the cross-shore direction. This equation has much resemblance with that of the one-dimensional case. Then the boundary conditions are formulated. The seaward boundary is a little more complex than in the one-dimensional case due to the non-zero angle of incidence of the wind waves. We will choose the seaward boundary such that the edge waves have died out because these waves are part of the problem to be solved.

The next section will deal with the forcing. It will turn out that most of section (3.1.3) can be copied to this section.

Finally, in section (3.3.4) an analytic solution to the full problem will be calculated for a bottom profile with a constant slope.

3.3.2 The governing equations

If the bottom topography is only varying in the cross-shore direction, the linearized low-frequency wave equation becomes
\[ \frac{1}{g} \zeta_x - (h \zeta_x)_x - h \zeta_{yy} = \frac{1}{g \rho} (S_{xx} + 2S_{xy} + S_{yy}) \]  

(3.3.1)

in which \( x \) varies in the cross-shore direction (\( x=0 \) on the still-water line) and \( y \) varies in the along-shore direction. This equation has to describe bound waves, edge waves and free leaky-mode waves. It can again be Fourier-Stieltjes transformed. Because the wave motion is homogeneous in the along-shore direction, the FS transform can also be applied in the \( y \) direction. The FS transform of a quantity \( a \) is now defined by

\[ a = \int_{0}^{L} e^{i \omega t - i y \beta} dA \]  

(3.3.2)

in which \( L \) is the \( y \) component of the wavenumber of the waves. (Note that by performing this FS transform we extract low-frequency phenomena with angular frequency \( \omega \) and \( y \) component of the wave vector \( l \).) We then obtain for the low-frequency wave equation

\[ \left( \frac{\omega^2}{g} - hl^2 \right) dZ_x + (hdZ_x)_x = -(d\phi_x - 2ild\phi_x - l^2 d\phi) \]  

(3.3.3)

in which \( d\phi \) is the FS transform of the normalized radiation stress.

The boundary conditions are a little more complicated than in the one-dimensional case. On the landward boundary, which we choose at \( x=0 \), we again assume fully reflecting low-frequency waves. The wind waves are completely broken, so the forcing will be zero on this boundary. We thus find

\[ \frac{\omega^2}{g} dZ_x + \zeta_x dZ_x = 0 \]  

for \( x>0 \)  

(3.3.4)

The seaward boundary again has to be such that the incoming low-frequency waves are bound to the wind-wave groups and the outgoing waves are free. Also in this case we will neglect the term \( \zeta_x dZ_x \) compared to the other terms because at these depths the bottom slope will be less important. We thus have (see chapter 3)

\[ dZ = dZ_x + dZ_f \]  

(3.3.5)
and

\[ dZ_z = dZ_b_z + dZ_f_z \quad (3.3.6) \]

in which the subscripts b and f denote bound and free respectively. The free wave component can be eliminated by putting

\[ dZ_f_z = -i k dZ_f \quad (3.3.7) \]

in which \( k \) is the x component of the wavenumber of the outgoing free wave, which is given by

\[ k = \frac{\omega}{\sqrt{gh}} \cos \alpha \quad (3.3.8) \]

So we assume the low-frequency velocity for the free waves to be given by

\[ c = \sqrt{gh} \quad (3.3.9) \]

which is consistent with the low-frequency equation. When we combine these equations we obtain the same equation as in the one-dimensional case:

\[ dZ_z + i \frac{\omega}{\sqrt{gh}} dZ = dZ_b_z + i \frac{\omega}{\sqrt{gh}} dZ_b \quad (3.3.10) \]

Note that both boundary conditions are linear in \( dZ \).

3.3.3 The forcing

Again, the FS transform of the forcing is a complex quantity with absolute value and phase. Let us first concentrate on the phase.

The phase of the forcing will be space and frequency dependent. For the same reason as given in chapter 3, the frequency dependence is of no importance. We only need the variation of phase over space. This variation is due to the fact that the forcing is related to the wind-wave envelope squared via the radiation stress. Because the wind-wave envelope needs some time to travel a certain distance, this distance corresponds to a phase shift. Just
like in the one-dimensional case we assume the wind-wave envelope to travel with the group velocity of the wind waves. A problem is now that the wind-wave energy also has a directional component. The simplest approximation is to deal with the direction of the wave energy flux only, which is given by

$$\bar{\theta} = \arctan \frac{F_y}{F_x}$$  \hspace{1cm} (3.3.11)$$

in which $F_x$ and $F_y$ the components of the energy-flux vector $\vec{F}$, given by

$$\vec{F} = \int \int_0^{\pi} \rho g G(\omega, \bar{\theta}) C_\theta(\omega) \bar{e} \ d\theta d\omega$$  \hspace{1cm} (3.3.12)$$

with the unit vector $\bar{e}$ given by

$$\bar{e} = \begin{pmatrix} \sin \bar{\theta} \\ \cos \bar{\theta} \end{pmatrix}$$  \hspace{1cm} (3.3.13)$$

With this assumption the phase shift due to the finite travel time becomes

$$\Delta(\psi(x)) = \int_{\bar{s}_0}^{x} \frac{\omega}{C_s} \cos \bar{\theta} \ dx$$  \hspace{1cm} (3.3.14)$$

When the bottom profile is known this integral can be determined so the phase of the forcing is known throughout the surfzone. We will now deal with the absolute value of the forcing.

As in the one-dimensional case the absolute value of the forcing will be calculated from the variance spectrum of the forcing, which in turn will be determined from the variance spectrum of the wind waves. The radiation stress is in the two-dimensional case related to the wind-wave energy as

$$S_\psi = [(n-\frac{1}{2}) + ne_i e_j] E$$  \hspace{1cm} (3.3.15)$$

in which the indices i and j denote the components of tensor $\vec{S}$ and the vector $\vec{e}$. The depth-
dependent quantity $n$ is given by

$$n = \frac{1}{2} + \frac{kh}{\sinh(2kh)} \quad (3.3.16)$$

To calculate the forcing the same procedure as in sections (3.1.3) and (3.1.4) can be followed. Again, the variance density of the forcing is found from the covariance of the wind-wave envelope squared. This covariance can be obtained via the joint probability density of the values of the square of the envelope separated by a time lag $\tau$, which is taken to be the two-dimensional Rayleigh distribution for the non-breaking waves. The percentage of broken waves $Q$ is found from the Battjes-Jansen model (1978), which is slightly modified due to the non-zero angle of incidence. Again we choose the parameter $k^2$ to be independent of depth.

With these assumptions the forcing term in the FS transformed low-frequency wave equation becomes

$$d\phi_x - i\ld\phi_x - l^2 d\phi = \left[ d\phi|_{\infty} + i(2k_b - D)d\phi|_{\infty} + \left( -k_b^2 + i^2 + i k_b \right) d\phi \right] e^{-\frac{\omega}{C_*} \cos \theta} dx \quad (3.3.17)$$

in $k_b$ is the wavenumber of the bound low-frequency waves in the cross-shore direction, given by

$$k_b = -\frac{\omega}{C_*} \cos \theta \quad (3.3.18)$$

With this equation the differential equation for $dZ$ can be solved.

3.3.4 An analytical solution

When a horizontal bottom or a plane sloping bottom is present an analytical solution for the low-frequency wave equation can be found.

In the case of a horizontal bottom for instance, the solution is simply given by the solution found in the one-dimensional case. The solution only looks different due to a rotation of the wavenumbers over an angle $\theta$. 

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In the case of a plane sloping bottom we have

\[ h = x \tan \beta \]  \hspace{1cm} (3.3.19)

in which \( \beta \) is the bottom-slope angle.

In the case \( l=0 \) the homogeneous solution is given by (see Eckart 1951)

\[ dZ^h = Ae^{iy}e^{-ix}L_n(2lx) \]  \hspace{1cm} (3.3.20)

in which \( L_n \) is the Laguerre polynoom of order \( n \) (\( n=0,1,2.. \)). The dispersion relation for these waves reads

\[ \omega^2 = gl(2n+1)\tan \beta \]  \hspace{1cm} (3.3.21)

These equations describe edge waves which are confined to the shoreline. In the cross-shore direction they are standing waves with exponentionally decaying amplitudes and in the along-shore direction they may be standing or progressive. Physically the waves are trapped to the shore by refraction. The particular solution to equation (3.3.3) can be expressed in terms of confluent hyper-geometric functions. We will not elaborate on these solutions any further.

In the case that \( l=0 \) we find the one-dimensional solution again. These waves are called leaky-mode waves in this context.
CHAPTER 4
LABORATORY EXPERIMENTS

In this chapter we describe laboratory experiments which were conducted to test the surf-beat model. We elaborate on the control signal for the wave board, the experimental conditions and the data analysis. It turned out that the noise level was high compared to the signal level which means that the data analysis has to be performed with great care.

The amplitudes of the low-frequency waves in nature which are bound to wind-wave groups are proportional to the radiation stress, which is a second order effect as described in chapter 3. If the wave generation by a wave board in a flume would be first order, bound low-frequency waves are generated by the radiation-stress gradients. However, these bound low-frequency waves do not fulfill the boundary condition at the wave board, so free low-frequency waves are generated. This does not happen in nature. Therefore, the generation of the waves in the flume has to be correct up to second order. In the next section the control signal for the wave board will be derived. This control signal should be such that the correct first-order waves are generated together with the bound low-frequency waves, also called the subharmonics.

The conventional method for the generation of bound low-frequency waves requires the evaluation of convolution-type integrals which are unduly time consuming and thus expensive. A new method is presented in section 4.1 which makes use of the method of multiple scales. The convolutions are replaced by multiplications which are faster and thus cheaper. The new control signal has been tested and shown to work satisfactorily by Van Leeuwen and Klopman (1992).

In section 4.2 the experimental equipment and the experimental conditions are described in detail. Important to note is that an artificial bar could be placed at different positions from the water line to study resonances.

The third section of this chapter deals with the data analysis. The data collection and analysis
will be described. As mentioned above, the data contain a lot of noise. The influence of the noise has to be reduced and we used the Complex-Harmonic Principal-Component Analysis to do the job. This method is described in some detail in section 4.3.3.3.

Finally, the last section describes the verification of the control signal for the wave board.
4.1 LOW-FREQUENCY WAVE GENERATION IN A FLUME

4.1.1 Introduction

The generation of random waves correctly up to second order is important for laboratory experiments in which the problem under investigation is perceptive to second-order effects in the wave field. For instance, second-order subharmonics are important for studies of the surf-beat mechanism, the generation and evolution of sand bars and the slow-drift motion of moored vessels. The second-order superharmonics sharpen the wave crests and flatten the wave troughs and are important for sand transport, among others. In this section we will concentrate on the correct generation of the second-order subharmonics.

Sand (1982) and Barthel e.a. (1983) calculated the second-order wave-board motion for the correct generation of second-order subharmonics, the bound long waves. They base their work on the transfer function for these low-frequency waves in absence of a wave board, as first given by Ottesen-Hansen (1978). This transfer function is exact to second order. To obtain the second-order spectrum a convolution-type integral has to be performed. The integration is in the frequency domain and the integrand is the product of the energy-density spectra of the first-order waves at two different frequencies and the transfer function. In this way the nonlinear interactions of all first-order spectral components are taken into account.

A first look at the resulting equations for the wave-board movement reveals the disadvantage of their use: they are very complex, and it requires considerable computing time to obtain a second-order signal for the wave board. When the first-order spectrum is narrow, this procedure seems unduly expensive. This is because only the frequency components near the peak frequency will give rise to substantial sub- and superharmonics.

Hence Kloppman and Kostense (private communication) suggested to make a new start, and to use the method of multiple scales. The same method is used by Mei (1989) to calculate the second-order waves in absence of a wave board. The water surface is assumed to oscillate with the peak frequency, with a modulated amplitude. The modulation acts on a longer time and length scale than the first-order waves. To incorporate these slow modulations new time and length scales are introduced to describe these phenomena. So a cascade of new variables is introduced, hence the name of the method.

In this method the calculation of the second-order surface elevations is reduced to a few multiplications in the time domain. In principle, the theory is valid for narrow first-order spectra, but it can even be applied to a Pierson-Moskowitz spectrum (see Kloppman and van Leeuwen, 1990).
In this section we derive the control signal for the wave board for the generation of second-order subharmonic waves in a flume. This control signal should be such that, away from the wave board, it will reproduce the second-order surface elevations found by Mei (1989) exactly. The use of the multiple-scales method to find the wave-board motion to second order resembles the use of the same method by Agnon and Mei (1985), who determine the slow-drift motion of two-dimensional bodies in beam seas to second order.

First, the governing equations are given. Secondly, the perturbation method of multiple scales will be explained. Then the first-order problem will be solved, after which the second-order control signal for the subharmonic waves will be derived. In order to obtain realistic sea waves in a flume, waves that are reflected on a beach or some other construction in the flume have to be absorbed by the wave board. Otherwise rereflection of these waves from the wave board will disturb the wave pattern. For this reason a system which actively absorbs waves which travel to the wave board is installed. This system has influence on the wave generation, so the control signal for the wave board has to be adjusted, which will be done in section 4.1.5. Finally, the new control signal is tested in section 4.1.6.

The comparison of the model and the experiments is described in chapter 5.

![Figure 4.1: Definition sketch.](image)

4.1.2 The governing equations

In figure (4.1) a sketch of the situation is given. The flume is equipped with a translating wave board. The water depth in absence of waves is $h$. 

The basic equations for the velocity potential are the following. The continuity equation reads

$$\Delta \phi = 0 \quad (4.1.1)$$

in which $\Delta$ is the Laplace operator and $\phi$ the velocity potential. This equation follows from the conservation of mass, with the assumptions of incompressibility and irrotationality. The kinematic free-surface boundary condition reads

$$\zeta_x + \phi_x \zeta_x = \phi_z \quad \text{on } z = \zeta \quad (4.1.2)$$

in which $\zeta$ is the surface elevation and the lower index indicates partial differentiation to the index variable. The equation states that the particles at the surface follow the surface displacements. The dynamic free-surface boundary condition is given by

$$g \zeta + \phi_t + \frac{1}{2} (\phi_x^2 + \phi_z^2) = 0 \quad \text{on } z = \zeta \quad (4.1.3)$$

It states that the pressure at the surface should equal the atmospheric pressure, which is taken zero here. The boundary condition at the bottom is given by

$$\phi_z + h_x \phi_x = 0 \quad \text{on } z = -h \quad (4.1.4)$$

and states that the velocity of the water particles perpendicular to the bottom is zero. The boundary condition at the wave board reads

$$\phi_x = \frac{dX}{dt} \quad \text{on } x = X \quad (4.1.5)$$

in which $X$ is the wave-board position. This condition states that the particles next to the wave board should follow the wave-board motions. Finally, far from the wave board the solution must describe the first-order waves with the bound second-order waves as given by Mei (1989). Free second-order waves should not occur.

Because the free-boundary conditions are nonlinear, perturbation techniques are used to reduce the nonlinear boundary-value problem to a set of linear boundary-value problems. To this end the surface elevation $\zeta$, the velocity potential $\phi$ and the wave-board position $X$ are expanded in a small non-linearity parameter $\epsilon$, which is equal to the wave steepness.
Because the free surface is part of the problem to be solved it is not known a-priori where the boundary condition at the free surface has to be applied. Because the amplitude of the surface elevation is finite but small, Taylor series expansions will be carried out at this boundary.

4.1.3 The method of analysis

In this section the method of multiple scales is explained in short. The method is compared briefly with the conventional spectral method to calculate second-order effects.

The objective of the perturbation method is to find expansions for the potential and the surface elevation which are valid for small-but-finite amplitude motions. It is convenient to introduce a small dimensionless parameter \( \varepsilon \) which describes the order of the amplitude of the motion. In our case \( \varepsilon \) is the wave steepness \( kA \), in which \( k \) is the wavenumber and \( A \) is the wave amplitude. Next, it is assumed that the wave potential, the surface elevation and the wave-board position can be represented by the following expansions

\[
\phi = \sum_{n=1}^{\infty} \varepsilon^n \phi_n \quad (4.1.6)
\]

\[
\zeta = \sum_{n=1}^{\infty} \varepsilon^n \zeta_n \quad (4.1.7)
\]

\[
X = \sum_{n=1}^{\infty} \varepsilon^n X_n \quad (4.1.8)
\]

The expansions are substituted in the basic equations (4.1.1)-(4.1.5). Because the \( \phi_n \), \( \zeta_n \) and \( X_n \) are independent of \( \varepsilon \), the coefficients for each power of \( \varepsilon \) are set equal to zero. This leads to \( n \) sets of linear equations, one for each order. Of course \( \phi_n/\phi_{n-1} \) must be bounded in order to have a consistent perturbation scheme.

In the conventional spectral methods the first-order quantities \( \zeta_1 \) and \( \phi_1 \) are decomposed into Fourier series of the form

\[
\zeta_1 = \sum_{n=0}^{N-1} a_n \cos \omega_n t + b_n \sin \omega_n t \quad (4.1.9)
\]
with $\omega_n = 2\pi n / T$, in which $T$ the time interval on which we observe $\zeta_1$.

Then the Fourier series of the second-order quantities $\zeta_2$ and $\phi_2$ are found from the second-order equations. In this way the nonlinear interactions of all first-order Fourier components are taken into account to obtain the second-order Fourier components. This results in very complex expressions for the second-order quantities, especially for the wave-board motion. A way to decrease the number of calculations is to take only those nonlinear interactions into account which occur between first-order Fourier components which contain more energy than a certain minimum. However, the resulting equations are still very complex.

In the case of a narrow first-order spectrum a stronger assumption can be used. The assumption is that $\zeta_1$ varies sinusoidally with a modulated amplitude. However, two problems occur when the straightforward expansions from equations (4.1.6)-(4.1.8) are used in the governing equations.

First, the solution turns out to have a limited range of validity. So-called secular terms, such as

$$et \sin \omega t$$

appear in the second-order terms. Thus, it can be seen that $\phi_n / \phi_{n-1}$ is not bounded as $t$ increases, which means that the expansion is not uniformly valid as $t$ increases. The problem arises from the fact that the (angular) frequency $\omega$ is wave-amplitude dependent in a nonlinear system. A way to circumvent this problem is to introduce new variables which take care of the frequency changes. These new variables are used to eliminate the secular terms, where use will be made of the orthogonality conditions. The orthogonality conditions are called solvability conditions in this case.

Secondly, nonlinear interactions of a sinusoidal wave with itself will only produce superharmonics in the straightforward expansion, while subharmonics are clearly present in nature. Also here the introduction of new variables, describing different time and length scales, will solve the problem.

In the method of multiple scales the expansions for the velocity potential, the surface elevation and the wave-board position are considered to be functions of multiple independent variables, or scales. These new variables are introduced as

$$t_n = \mu^n t$$  \hspace{1cm} (4.1.10)
\[ x_n = \mu^n x \]  

(4.1.11)

in which \( \mu \) is a small parameter, which describes the modulation of the first-order wave amplitude. So the variables with \( n > 0 \) will describe the slow modulation of the primary-wave amplitude and will be called slow variables hereafter. Because the slow variables will be used to describe the wave-amplitude variations, \( \mu \) will be of the order \( \omega_{\text{group}}/\omega_{\text{wave}} \). The derivatives with respect to \( x \) and \( t \) become

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \mu \frac{\partial}{\partial t_1} + O(\mu^2)
\]

(4.1.12)

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial x_0} + \mu \frac{\partial}{\partial x_1} + O(\mu^2)
\]

so that a derivative to a slow variable is of a lower order than a derivative to a fast variable. In the following, the index 0 will be omitted for brevity.

In the case of a narrow-banded energy-density spectrum of the primary waves the method of multiple scales can be applied. The assumption is that the water surface oscillates with the peak frequency of the spectrum with a modulated wave amplitude. The second-order effects arise from the nonlinear interactions of the primary waves. The second-order superharmonics have twice the frequency of the primary waves, and produce sharper peaks and flatter troughs and thus give rise to wave asymmetry. The second-order subharmonics arise from the wave-amplitude modulations and appear as long waves, bound to wave groups.

Motivated by the multiple scales we introduce the following notation:

\[
\zeta_n = \sum_{m=-n}^{m=n} \zeta_{nm} e^{-im\omega t}
\]

(4.1.13)

in which \( \omega \) is the angular frequency of the first-order waves and \( \zeta_{n,-m} = \zeta_{n,m}^{*} \) to keep \( \zeta_n \) real. Note that \( \zeta_{nm} \) depends only on the slow variables. The same notation will be used for \( \phi_n \) and \( X_n \).

As the solution method is posed so far, two small parameters are introduced, \( \varepsilon \) and \( \mu \). In order to solve the problem these two parameters have to be related; otherwise the order of
the terms in the equations is not uniquely determined. A standard way to do this in perturbation analysis is

\[ \mu = e^\lambda \]  

(4.1.14)

in which \( \lambda \) is a parameter of order one. We will assume

\[ \lambda = 1 \]  

(4.1.15)

so the wave steepness is equal to the modulation parameter. A motivation for this choice comes from the characteristic shape of the first-order spectrum which we usually encounter, such as a Jonswap spectrum or a Pierson-Moskowitz spectrum. For such a spectrum the magnitude of the parameter \( \mu \) can be taken as the spectral width at half the spectral value of the peak frequency. As magnitude of \( e \) we can take the product of the wavenumber at the peak frequency and half the significant wave height. If we choose these values for the parameters they are of the same order of magnitude.

Now that the slow time and length scales are introduced together with the perturbation series for the potential, the surface elevation and the wave-board motion, the first- and second-order boundary-value problems can be formulated.

### 4.1.4 The first-order solution

The perturbation series for the wave potential, the surface elevation and the wave-board displacement plus the fast and slow time scales will be incorporated in equations (4.1.1)-(4.1.5). Then Taylor series expansions can be carried out in the vertical direction around \( z=0 \), and in the horizontal direction around \( x=0 \). The expansions lead to the following first-order problem:

\[ \Delta \phi_i = 0 \]  

(4.1.16)

\[ \phi_{iz} + g\phi_{iz} = 0 \]  

on \( z=0 \)  

(4.1.17)

\[ \phi_{iz} = 0 \]  

on \( z=0 \)  

(4.1.18)

\[ \phi_{ix} = \frac{dX_i}{dt} \]  

on \( x=0 \)  

(4.1.19)
with

\[ \zeta_1 = -\frac{1}{g} \phi_1 \]

on \( z=0 \) \hspace{1cm} (4.1.20)

To solve the first-order problem we use as further input that the first-order surface elevation far from the wave board is given by

\[ \zeta_1 = \frac{1}{2} (A e^{i\psi} + \ast) = \zeta_{11} e^{-i\omega t} + \zeta_{11}^* e^{i\omega t} \]

\hspace{1cm} (4.1.21)

in which \( \psi = kx - i\omega t \), \( A = A(t_1, x_1) \) is the complex amplitude of the first-order waves. The magnitude of \( A \) is equal to the envelope \( R \) of the surface elevation in a time simulation based on the first-order energy-density spectrum which we want to have in the flume. Its phase is equal to the phase of the surface elevation at the begin time of the simulation. The asterisk denotes the complex conjugate of the preceding term. Note that we used the notation defined in equation (4.1.13).

The wave-board which we will use has a piston-type motion with no \( z \)-variation, so the wave-board position is given by

\[ X_{11} = ia \]

\hspace{1cm} (4.1.23)

in which \( a(t_1) \) is the slowly varying amplitude of the wave-board stroke. Although \( X_{10} \) is a first-order quantity, it does not show up in the first-order problem because the wave-board position is differentiated to a fast variable (see equation (4.1.19)) and \( X_{10} \) only depends on the slow variables. It will produce the subharmonics as we will see in the second-order problem.

Bie\'sel (1951) found the solution to the first-order problem as

\[ \phi_{11} = -B \frac{\cosh Q_{1a} a x}{\cosh q} + \sum_{j=1}^n C_j \frac{\cos p_j}{\cos p_j} e^{-t x a} \]

\hspace{1cm} (4.1.24)
in which

\[ B = \frac{\omega}{nk} \quad (4.1.25) \]

\[ C_j = -\frac{\omega}{n_f l_j} \quad (4.1.26) \]

with

\[ Q = k(z+h) \quad P_j = l_j(z+h) \quad q = kh \quad P_j = l_j h, \]

\[ n = \frac{1}{2} + \frac{q}{\sinh(2q)} \quad n_j = \frac{1}{2} + \frac{P_j}{\sin(2P_j)} \quad (4.1.26a) \]

\[ k \] is the positive root of \( \omega^2 = gk \tan kh \) and \( l_j \) is the positive and real root of \(-\omega^2 = gl_j \tan l_j h\)

with \((j-1/2)\pi < l_j h < j\pi\) for \(j = 1, 2, 3, \ldots\)

Note that \(\phi_{10}\) does not show up in the first-order solution because it only depends on the slow variables. It was shown by Mei (1989, p. 615) to represent the bound low-frequency waves, arising from the amplitude modulation of the primary waves. It will show up in the second-order problem.

The first-order surface elevation can be calculated from equation (4.1.20) as

\[ \zeta_{11} = \frac{\omega B}{2g} e^{il_j a} + \frac{\omega}{2g} \sum_{j=1}^{\infty} C_j e^{-il_j a} \quad (4.1.27) \]

and

\[ \zeta_{10} = 0 \quad (4.1.28) \]

The first term on the right-hand-side of equation (4.1.27) represents the progressive waves and the second term represents the evanescent modes. We will elaborate on these waves shortly. First we identify the quantity \(a\).
Far from the wave board, where only progressive waves are present because the evanescent modes have died out, \( \zeta_{11} \) should satisfy equation (4.1.21):

\[
\zeta_{11} = \frac{1}{2} A e^{ix},
\]

(4.1.29)

so that using (4.1.27) far from the wave board we can identify

\[
a = \frac{gA}{\omega B}
\]

(4.1.30)

The first-order wave-board position is now given by

\[
X_{11} = \frac{g}{2 \omega B} - iA
\]

(4.1.31)

and the first-order potential becomes

\[
\phi_{11} = -\frac{g}{2 \omega} \cosh \frac{Q}{2 \omega} A e^{ix} + \frac{g}{2 \omega} \sum_{j=1}^{\infty} \frac{C_j}{B \cosh \frac{Q}{2 \omega}} e^{-ijx} A
\]

(4.1.32)

**Figure 4.2:** The velocity profile due to the wave-board motion \( X \) is equal to the velocity profile of the progressive wave and that of the evanescent modes.
The first term on the right in equation (4.1.32) is identical to that obtained by Mei (1989, p. 611). It represents the free waves travelling in the positive x-direction. The second term describes standing waves in the z-direction, with amplitudes decaying in the x-direction, the so-called evanescent modes. They arise because the wave board does not produce the correct velocity profile over the water depth, but only an approximation. So the progressive waves produced by the wave board do not fulfill the boundary condition exactly. Evanescent modes are generated so that the sum of the progressive waves and the evanescent modes fulfils the boundary condition to first order (see figure (4.2)). The influence of the wave-board boundary condition is via the factors $B$ and $C_j$, which indeed only occur in the third term.

The first-order surface elevation becomes

$$\zeta_{11} = \frac{1}{2} Ae^{ux} + \frac{1}{2} \sum_{j=1}^\infty \frac{C_j e^{-j\pi A}}{B}$$  \hspace{1cm} (4.1.33)

4.1.5 The second-order solution

In this section we solve the boundary-value problem for the second-order subharmonic wave-board motion. For the full second-order solution the reader is referred to Van Leeuwen and Klopman (1991, 1992). The subharmonic wave generation is relatively easy to handle because the subharmonic waves far from the wave board have no z-dependence, so the equations can be integrated over the depth.

The boundary-value problem for $\phi_{10}$ is given by

$$\Delta \phi_{10} = 0$$  \hspace{1cm} (4.1.34)

$$\phi_{10} = 0$$  \hspace{1cm} on $z=0$  \hspace{1cm} (4.1.35)

$$\phi_{10} = 0$$  \hspace{1cm} on $z=-h$  \hspace{1cm} (4.1.36)

$$\phi_{10} = 0$$  \hspace{1cm} on $x=0$  \hspace{1cm} (4.1.37)

The solution to this problem is that $\phi_{10}$ does not depend on the fast variables. We need higher-order equations to determine the dependence of $\phi_{10}$ on the slow variables. Let us now
have a look at the problem for \( \phi_{20} \).

The boundary-value problem for \( \phi_{20} \) reads

\[
\Delta \phi_{20} = 0
\]

\[
\phi_{20} = (\zeta_{11} \phi_{11z} + \ast) \bigg|_x \quad \text{on } z = 0 \tag{4.1.38}
\]

\[
\phi_{20} = 0 \quad \text{on } z = -h \tag{4.1.39}
\]

\[
\phi_{20} = X_{10_t} - \phi_{10_t} - (X_{11}^* \phi_{11x} + \ast) \bigg|_x \quad \text{on } x = 0 \tag{4.1.40}
\]

This last equation can immediately be integrated to

\[
\int_{-h}^{0} \phi_{20} \, dz = hX_{10_t} - h\phi_{10_t} - (\zeta_{11} \phi_{11x} + \ast) \bigg|_x = 0 \tag{4.1.41}
\]

in which we used the equations (4.1.17)-(4.1.20). The fastest way to find \( X_{10} \) is to ignore the solution for \( \phi_{20} \) and start from the continuity equation. It reads after integration over depth

\[
\zeta_t + [(h + \zeta) \phi_x]_z = 0 \tag{4.1.42}
\]

The second-order subharmonic continuity equation is given by

\[
\int_{-h}^{0} \phi_{20} \, dz + (\zeta_{11} \phi_{11x} + \ast) = C \tag{4.1.43}
\]

Note that \( \zeta_{10_t} = 0 \) from equation (4.1.28). \( C \) depends on the slow variables and is determined by the fact that far from the wave board the subharmonic velocity potentials describe the bound low-frequency waves which are not dependent on the fast variable \( x \). So, far from the wave board \( \phi_{20} \) must vanish and we thus find

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\( \mathbf{C} = \left[ \zeta_{11} \Phi^{*}_{11z^{+*}} \right]_{x=L} \) \hspace{1cm} (4.1.44)

in which \( x=L \) denotes a place far from the wave board where the evanescent modes have died out. The same equation can be found via Green's theorem for \( \Phi_{10} \) and \( \Phi_{20} \) (see Agnon and Mei, 1985).

If we combine equations (4.1.41), (4.1.43) and (4.1.44) we find for the wave-board motion

\[ X_{10_{1}} = \left[ \Phi_{10_{n}} + \frac{1}{h} \left( \zeta_{11} \Phi^{*}_{11z^{+*}} \right)_{x=L} \right] \] \hspace{1cm} (4.1.45)

Note that we did not have to solve for \( \Phi_{20} \). Via the continuity equation we are able to link the wave-board position to the variables far from the wave board. Recall that \( \Phi_{10} \) does not depend on the fast variables. This result can be interpreted as that the evanescent modes give rise to low-frequency motion close to the wave board, which are described by that part of \( \Phi_{20} \) which depends on the fast space coordinate.

Now we will eliminate \( \Phi_{10} \) in favor of the second-order surface elevation. To do this we need the third-order subharmonic continuity equation far from the wave board, where the evanescent modes have died out. It is given by

\[ \zeta_{20_{1}} + h \Phi_{10_{n}} + \left( \zeta_{11} \Phi^{*}_{11z^{+*}} \right)_{x_{1}} = 0 \] \hspace{1cm} (4.1.46)

The surface elevation \( \zeta_{20} \) describes the bound waves and the propagation velocity of these waves is the group velocity of the first-order waves \( C_{g} \). We thus have

\[ \zeta_{20_{1}} = -C_{g} \zeta_{20_{1}} \] \hspace{1cm} (4.1.47)

With this equation the continuity equation (4.1.46) can be integrated with respect to \( x_{1} \) to obtain

\[ C_{g} \left[ \zeta_{20} - S_{1} \right] = h \Phi_{10_{n}} + \left( \zeta_{11} \Phi^{*}_{11z^{+*}} \right) \] \hspace{1cm} (4.1.48)
in which \( S_1 \) is an integration constant. The wave-board motion can now be expressed with equation (4.1.45) as

\[ X_{10} = \frac{C}{\theta} [\zeta_{20} - S_1] \quad (4.1.49) \]

This equation can be integrated to obtain the control signal for the wave board:

\[ X_{10} = \frac{C}{\theta} \int_0^1 (\zeta_{20} - S_1) \, dt_1 \quad (4.1.50) \]

and the constant \( S_1 \) is found by demanding that the wave-board stroke remains finite:

\[ S_1 = \overline{\zeta_{20}} \quad (4.1.51) \]

where the bar indicates the value of the quantity averaged over \( t_1 \). The equation for the wave-board motion (4.1.50) states that the volume flux due to this motion produces the surface elevation of the low-frequency waves, just as one would expect.

An expression for \( \zeta_{20} \) can be found from the boundary-value problem for \( \Phi_{30} \) or from the second-order Bernoulli equation. We will use the last method because its physical relevance is more clearly understood. The Bernoulli equation on the free surface reads

\[ g \zeta + \Phi_f + |\Phi_z|^2 + |\Phi_x|^2 = 0 \quad \text{on } z = \zeta \quad (4.1.52) \]

The second-order subharmonic version of this equation reads

\[ g \zeta_{20} + \Phi_{10} + |\Phi_{11_z}|^2 - \frac{\omega^2}{g^2} |\Phi_{11_z}|^2 = 0 \quad \text{on } z = 0 \quad (4.1.53) \]

To obtain this equation we made use of the first-order dynamic free-surface boundary condition and a Taylor expansion around \( z = 0 \). The third-order subharmonic continuity equation was given by (see 4.1.46)

\[ \zeta_{20} + h \Phi_{10} + (\zeta_{11} \Phi_{11_z} + \ast) \bigg|_{x_1} = 0 \]
Now we use the fact that the bound subharmonic waves travel with the group velocity \( C_g \) of the first-order waves and eliminate the velocity potential \( \phi_{10} \) by substitution of equation (4.1.53), and the use of equation (4.1.47) and a similar equation for \( \phi_{10} \). We then find for the surface elevation

\[
\zeta_{20} = \frac{1}{gh - C_g^2} \left[ h |\phi_{11}^*|^2 - \frac{\omega h}{g} |\phi_{11}|^2 - C_g \left( \zeta_{11} \phi_{11}^{*11} + \ast \right) \right]
\]  

(4.1.54)

The right-hand side can be evaluated with help of the expressions for the first-order quantities to obtain

\[
\zeta_{20} = \frac{g(2n - \frac{1}{2}) |A|^2}{gh - C_g^2}
\]  

(4.1.55)

in which \( n \) is given by equation (4.1.26a). This equation can be written as

\[
\zeta_{20} = \frac{S}{\rho(gh - C_g^2)}
\]  

(4.1.56)

in which \( S \) is the radiation stress of the first-order waves. Note that this expression was also obtained in chapter 3. If we use this in equation (4.1.50) for the slow wave-board motion we obtain

\[
X_{10} = \frac{gC_g \left( 2n - \frac{1}{2} \right) t_i}{h(gh - C_g^2)} \int \left( |A|^2 - |A|^2 \right) dt_i
\]  

(4.1.57)

Concluding we find that the total control signal for the generation of first-order waves plus the corresponding bound low-frequency waves is given by

\[
X = X_{11} + X_{10}
\]  

(4.1.58)

in which the separate control signals are given by equations (4.1.31) and (4.1.57).
Now that we have obtained the control signal for the wave board up to second order we will give the recipe for the generation of the complete control signal.

- First, the peak frequency of the target first-order energy-density spectrum has to be determined. From this frequency the wavenumber \( k \), the group velocity \( C_g \) and the quantities \( q \) and \( p_j \) have to be calculated.

- Secondly, a time series for the first-order surface elevation \( \zeta_{11} \) has to be generated from the target first-order energy-density spectrum. In this way \( A(t_j) \) is determined. We used the random-amplitude/random-phase method described by Tucker et al. (1984).

- Third, the control signal for the wave board has to be calculated in the time domain from equation (4.1.58). The time integration in equation (4.1.57) was performed with the modified midpoint rule and the time differentiations with central differences. The numerical accuracy of both operations was second order.

These three steps are sufficient to obtain the control signal for the wave board correct up to second order.

Let us finally compare the new method with the conventional spectral method in computation speed. To obtain the new signal an FFT and a few extra multiplications have to be performed. In the conventional method an FFT, a few extra multiplications and a convolution are needed. An FFT needs about \( 4pN \) multiply-add operations in which \( N \) is the number of time steps and \( p = 2 \log N \) (see for instance Bendat and Piersol, 1986). For the convolution in the conventional method \( N^2 \) operations are needed, as can be observed in Barthel et al. (1983) and Sand and Mansard (1986). If we neglect the extra multiplications (of which more are needed in the conventional approach) the gain in computational speed of the new method compared to the conventional method is

\[
\frac{4pN+N^2}{4pN} = 1 + \frac{N}{4p} \quad (4.1.59)
\]

The number of time steps will be of the order \( 10^4 \), so that the gain in speed is a factor 250, which is a big factor indeed.

Note that the equations can also be used to generate second-order bichromatic waves.
4.1.6 The influence of the absorption system

In this section the control signal for the wave board is derived when an active wave-absorption system is present.

The absorption system presently in use in the Fluid Mechanics Laboratory of Delft University of Technology measures the water elevation on the wave board, transforms this signal to a wave-board position, and compares this with the actual wave-board position. The system compensates for the difference between these two by changing the wave-board position. In this way the reflection coefficient of the wave board is greatly reduced. The system has been developed by Delft Hydraulics. The reflection coefficient of the wave board with the absorption system switched on has been measured for different frequencies in the low-frequency range. The results are given in figure (4.4).

The measured instantaneous water elevation on the wave board is transformed to a wave-board position signal in the following way

\[ X_c = -\sqrt{\frac{g}{h}} \int_0^t \zeta \, dt \]  \hspace{1cm} (4.1.60)

This signal consists of two contributions: one part, \( X_{dr} \), comes from the waves which travel from the flume to the wave board and the waves which are reflected off the wave board, the other part, \( X_{cd} \), arises directly from the waves generated by the wave board itself. The first part is the part we want, the second part has influence on the generation of the second-order waves. So the control signal has to be adjusted to compensate for this effect.

The resulting total wave-board motion which results when the absorption system is turned on is given by

\[ X = 2X_i + X_c = 2X_i + X_{dr} + X_{cd} \]  \hspace{1cm} (4.1.61)

in which \( X_i \) is the input signal one provides to the system and \( X \) the resulting wave-board position. Note that the input signal is multiplied by a factor two by the absorption system. We want the resulting wave-board position to be such that it produces the correct first- and second-order waves and compensates for the waves which travel back to the wave board. This means that it has to be equal to the wave-board position which produces the correct first- and second-order waves, given in section (4.1.4) and denoted \( X_T \) here, plus the wave
board motion described by $X_d$:

$$X = X_T - X_d$$  \hspace{1cm} (4.1.62)$$

If we combine both equations for the resulting wave-board motion we can solve for the input signal we have to provide to the system:

$$X_i = \frac{1}{2}X_T - \frac{1}{2}X_{cd} = \frac{1}{2}X_T + \frac{1}{2} \sqrt{\frac{g}{h}} \int_{t=0}^{t} \zeta_T dt$$  \hspace{1cm} (4.1.63)$$

in which $\zeta_T$ is the surface elevation at the wave board which would result from $X_T$ in absence of the absorption system. It is determined in the following way. As mentioned before, the wave-height meter is mounted on the wave board, which is moving around its mean position. The measured water elevation for small wave-board displacements can be expanded in a Taylor series as

$$\zeta(x-X) = \zeta(x=0) + X\zeta_x(x=0) + ...$$  \hspace{1cm} (4.1.64)$$

If the perturbation scheme is used one obtains

$$\zeta(x-X) = e\zeta_1(x=0) + e^2\zeta_2(x=0) + e^2X_1\zeta_1(x=0) + ...$$  \hspace{1cm} (4.1.65)$$

Let us first deal with the first-order waves. The surface elevation on the wave board is given by

$$\zeta_{11} = \frac{1}{2}A + \frac{1}{2} \sum_{j=1}^{n} \frac{C_j}{B} A$$  \hspace{1cm} (4.1.66)$$

so the compensation signal becomes

$$X_{cd_{11}} = -\frac{1}{2\omega} \sqrt{\frac{g}{h}} (A - \sum_{j=1}^{n} \frac{C_j}{B} A)$$  \hspace{1cm} (4.1.67)$$

and the input signal for the total system is obtained as
\[ X_{11} = \frac{1}{4\omega} \left[ \frac{2g}{B} \sqrt{\frac{g}{h}} \right] - \sqrt{\frac{g}{h} \sum_{j=1}^{\infty} \frac{C_j}{B}} A \]  \hspace{1cm} (4.1.68)

Now we turn to the second-order low-frequency waves. The second-order low-frequency surface elevation is given by (see 4.1.53)

\[ \zeta_{20} = \frac{1}{g} \left[ \Phi_{10} |^{2} - \frac{\omega^4}{g^2} |\Phi_{11}|^2 \right] \]  \hspace{1cm} (4.1.69)

which can be written as

\[ \zeta_{20} = \zeta_{20, e} + \zeta_{20, m} \]  \hspace{1cm} (4.1.70)

in which \( \zeta_{20, e} \) is the surface elevation of the low-frequency waves far from the wave board, as given by (4.1.56), and \( \zeta_{20, m} \) is the contribution to the total low-frequency surface elevation from the evanescent modes. This division of the surface elevation is possible because it turns out that the low-frequency contributions from all cross products between first-order progressive waves and first-order evanescent modes vanish. We thus find for the compensation motion

\[ X_{rd10} = \frac{\sqrt{gh}}{C_g} X_{10} - \sqrt{\frac{g}{h}} \int_{0}^{t} \zeta_{20, m} + (X_{11} \zeta_{11} * + *) dt + C_1 \]  \hspace{1cm} (4.1.71)

in which we used equation (4.1.50) to express \( \zeta_{20} \) in terms of \( X_{10} \), and in which \( C_1 \) is a constant which has to be such to keep the input wave-board position finite. The input signal for the low-frequency waves for the total system is given by

\[ X_{10} = \frac{1}{2} \left[ \frac{gC_g}{h(gh-C_g^2)} \left( 1 - \sqrt{\frac{gh}{C_g}} \right) - \sqrt{\frac{g}{h} \left( \frac{gk^2}{4\omega^2} + \frac{\omega^4}{4g \sum_{j=1}^{\infty} \frac{C_j}{B}} \right) } \right] \int_{0}^{t} \left[ |A|^2 - |A|^2 \right] dt_1 \]  \hspace{1cm} (4.1.72)

To obtain the total control signal for the wave board the three steps at the end of section (4.1.4) have to be followed in which equation (4.1.58) has to be replaced by

\[ X = X_{11} + X_{10} \]  \hspace{1cm} (4.1.73)
4.2 EXPERIMENTAL EQUIPMENT AND CONDITIONS

Wave flume

The experiments were conducted in a wave flume of 40 m length and .8 m width. The water depth was 42 cm. At 19 m from the wave board a 1 in 25 concrete slope began (see figure (4.3)). A concrete bar could be placed on the sloping bottom, thus forming an artificial sand bar. The bar has the shape of a Gaussian distribution with a height of 10 cm and a width on half the maximum height of 15 cm. The position of the artificial bar could be changed from nearly no wave breaking to nearly 100% wave breaking over the bar. The depth of the highest point of the bar ranged from 4 to 8 cm; during some experiments the bar was absent. In this way we are able to study the influence of the presence of the bar on low-frequency wave generation and especially look for resonances.

Figure 4.3: Experimental set up.

Reflection absorption

The flume is equipped with a hydraulically-driven wave board which was prepared to move in a piston-type motion only. The wave board has a device for active absorption of waves entering from the flume as explained in section 4.1.5. This is necessary to prevent these waves to be reflected by the wave board and reenter the flume. On a natural beach waves which are reflected from the beach travel back to deep water.

We performed a test of the absorption system on the wave board. To this end the slope was replaced by a vertical wall 38 m from the wave board. The reflection coefficient of the wave board was determined at a range of frequencies. This coefficient was obtained in the following way. The wave board produced waves of a certain frequency until a steady situation occurred with standing waves. A wave-height meter was placed at a surface-elevation maximum. Then the absorption system was switched on. After a transient phase the wave-height meter will experience partly-absorbed waves coming from the direction of the wave board and waves which are reflected from the wall. In this situation the waves from the wall will be higher than the partly-absorbed waves so a partly-standing wave field arises.
Then, after another transient phase, the wave-height meter measures the combined surface elevation due to partly-absorbed waves coming from the direction of the wave board and the partly-absorbed waves fully reflected from the wall. In this situation both kinds of waves are once partly absorbed by the wave board, so their amplitudes are the same and a standing-wave pattern is present. The reflection coefficient of the wave board is given by the ratio of the amplitude of the standing waves after the absorption system was turned on to that of the standing waves before the absorption system was turned on.

In figure (4.4) the variation of the reflection coefficient as function of frequency is given. For frequencies higher than .1 Hz the reflection coefficient is well below 10%, which is acceptable for our purpose. High-frequency waves will break on the beach and their reflection back to the wave board will be very small. Low-frequency waves will reflect nearly 100% on the beach. As long as the amplitudes of these waves is of second order when they approach the wave board, which is the case in our experiments, the reflection on the wave board will reduce them to third order. Because the surf-beat model is a second-order model and because the wave-board control signal will be correct up to second order this is acceptable.

![Graph](image)

**Figure 4.4:** Reflection coefficient of the wave board with active wave absorption versus frequency.

A problem is formed by the waves with frequencies lower than about .05 Hz. The reflection coefficient of the wave board is too high in this case. The reason for this high reflection is probably leakage of water below and along the sides of the wave board. In our case it means
that we can only test the model down to .05 Hz. However, surf-beat phenomena are usually above this frequency (after scaling up of course).

**Second-order wave generation**

The second-order wave-board control signal was tested with bichromatic and continuous first-order spectra. It is shown to work satisfactorily in section 4.4 (see also Van Leeuwen and Klopman (1992)).

In chapter 5 four laboratory experiments are evaluated in detail. The energy-density spectrum of the first-order waves was a Jonswap spectrum with a peak frequency of 0.63 Hz, a peak enhancement factor of 3.3 (which is the standard value) and a root-mean-square wave height of 4.0 cm.

The reflection coefficient of the first-order waves on the beaches was found to be below 10% for all frequencies so the influence of these reflected waves on the generation of the low-frequency waves is neglected.

Some experiments were performed with other peak frequencies (10% variation) and with Pierson-Moskowitz spectra, but the variation in the spectra was too small to give rise to significant changes in the low-frequency wave field. The magnitudes of the variations had to remain relatively small (10% variation in peak frequency) because of bad wave-board performance outside the variation interval.

**Instrumentation**

The surface elevations were collected with wave-height meters on the horizontal and the sloping bottom. On the sloping bottom current meters were placed next to the wave-height meters to be able to distinguish incoming and outgoing low-frequency waves.

The wave-height meters are of the conductance type. They were calibrated before and after each run by moving the wave-height meters up and down to 9 reference elevations in the flume. A least-squares fit through the calibration points showed that the linearity of the meters was better than 0.1 %. The accuracy of the meters was found to be about 0.3 mm.

Particle velocities were measured with disk-shape electromagnetic current meters. We used the calibration tables which were supplied with the meters. A test with sinusoidal waves with an amplitude of 3 cm showed that the errors in the meters were smaller than 1%, which agreed well with the tables. To check the values from the current meters, linear wave theory was used in combination with wave-height meter signals.
4.3 DATA ANALYSIS

4.3.1 Introduction

In this section we concentrate on data analysis. As will be shown, the method of analysis has a significant influence on the interpretation of the data.

We have to know the characteristics of the first-order wave field which are of importance for the generation of the low-frequency waves. Apart from the energy spectra we also need the wave envelope. The determination of the wave envelope is not easy when the waves become more and more nonlinear due to shoaling and breaking.

To test the theory developed in chapter 3 in detail we have to distinguish in the measurements free incoming, bound incoming and free outgoing low-frequency waves on the horizontal part of the flume. On the sloping part of the flume we can only distinguish incoming and outgoing waves. This difference is due to the fact that the phase velocities of the bound and the free low-frequency waves are so close in shallow water that they are inseparable. Apart from the energy densities of these waves we will also need their phases compared to that of the first-order wave envelope for the calculation of the correlations between the two.

The next section deals with the determination of the first-order wave envelope. Section 4.3.3 concentrates on the decomposition of the low-frequency wave field into its three components; the simple case of the sloping bottom is not treated separately. In section 4.3.4 some considerations on the noise level and error analysis are given.

4.3.2 Determination of the first-order wave envelope.

The envelope of a sinusoidal wave with a modulated amplitude can formally be determined by means of the Hilbert transform. This transform of a signal \( \eta(t) \) is defined as a convolution in the time domain:

\[
\eta_H(t) = \int_{-\infty}^{\infty} \eta(t') \frac{dt'}{\pi(t-t')} \quad (4.3.1)
\]

in which \( \eta_H \) is the Hilbert transform of \( \eta \). In the frequency domain this corresponds to a multiplication

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\[ X_H(f) = \begin{cases} 
-iX(f) & \text{for } f > 0 \\
iX(f) & \text{for } f < 0 
\end{cases} \quad (4.3.2) \]

in which \( X_H \) is the Fourier transform of \( \eta_H \). Note that \( X_H \) is not the Hilbert transform of \( X \). A useful way to understand the Hilbert transform is via the analytical signal \( z(t) \) associated with \( \eta(t) \), defined by

\[ z(t) = \eta(t) + i\eta_H(t) \quad (4.3.3) \]

in which \( i \) is the square root of \(-1\). This equation can also be written as

\[ z(t) = R(t)e^{i\theta(t)} \quad (4.3.4) \]

The envelope signal \( R(t) \) and the instantaneous phase signal \( \theta(t) \) are given by

\[ R(t) = \sqrt{\eta(t)^2 + \eta_H(t)^2} \quad (4.3.5) \]

\[ \theta(t) = \tan^{-1}\left[ \frac{\eta_H(t)}{\eta(t)} \right] \]

Figure (4.5) gives an example of this procedure.

When the first-order waves shoal and break, nonlinear superharmonics grow strongly and the wave envelope determined via the Hilbert transform will look rather jagged. Indeed, a description of the wave field by (4.3.4) in which \( R \) is the slowly-varying amplitude is not possible in this case. One can however low-pass filter the wave envelope to obtain a more realistic envelope. The cut-off frequency can be chosen as the cut-off frequency for the low-frequency waves, which was 0.4 Hz in our case. This frequency is in the spectral valley where nearly no wave energy is present.

List (1991) proposed another way to determine the wave envelope. In his method, first the absolute value of the surface elevation is calculated, which is then low-pass filtered via an FFT. The cut-off frequency is chosen the same as in the previous method. The resulting signal is multiplied by \( \pi/2 \) to obtain the wave envelope. The numerical advantage of this method is that only two FFT’s have to be performed while the method with the Hilbert
Figure 4.5: Determination of the wave envelope with use of the Hilbert transform. Drawn line: surface elevation, dashed line: Hilbert transform of surface elevation, heavy line: wave envelope.

Figure 4.6: Wave envelope determination. Thin line: wind-wave surface elevation, heavy line: envelope from Hilbert transform, which is coincident with that from the method of List.

transform needs four FFT’s.

In figures (4.6) and (4.7) a comparison is made between the two methods for nearly linear and highly nonlinear irregular first-order waves. Note that in the highly nonlinear case the calculated envelope does not follow the crests and troughs of the individual waves. Because the waves are nonlinear, this is impossible to accomplish. The calculated envelope depends heavily of the chosen cut-off frequency and is a compromise between a calculated envelope which does follow the crests exactly, but is rather jagged, and a smoother calculated
envelope. The resulting wave envelopes from the two methods described above are equal in the first case and nearly equal in the second. The discrepancies are so small that the method of List will be used in the following because of its numerical efficiency, with a cut-off frequency of 0.4 Hz.

4.3.3 Decomposition of the low-frequency wave field

In this section the low-frequency wave field is decomposed into incoming bound waves, incoming free waves and outgoing free waves. (Note that incoming waves are wave travelling to the beach and outgoing waves are waves travelling from the beach. This convention is chosen to avoid confusion in chapter 5.) The decomposition itself is rather simple, but the determination of accurate estimates of the Fourier components of the measured time series, which are needed in the decomposition, requires great care.

Let us first concentrate on the decomposition itself. To determine the amplitudes of the three independent low-frequency waves we need at least three wave-height meters.

First we perform a Fourier transform on the measured time series. To distinguish the bound and free waves we note that they have different phase velocities, or different wavenumbers. We thus have at every frequency

$$A(\omega, x) = Z_{\text{bin}}(\omega)e^{ikx} + Z_{\text{fin}}(\omega)e^{iku} + Z_{\text{fout}}(\omega)e^{-ikx}$$

(4.3.6)
in which $A(\omega, x)$ is the Fourier component of the time series measured at position $x$ and $Z_{\text{bin}}, Z_{\text{fin}}, Z_{\text{fout}}$ are the Fourier components at $x=0$ of the bound incoming, free incoming and free outgoing low-frequency waves respectively. The wavenumbers of the free waves and those of the bound waves follow from the dispersion relation for free waves and from the group velocity of the primary waves respectively:

$$k = \frac{\omega}{\sqrt{gh}} \quad \text{and} \quad \kappa = \frac{\omega}{C_g}$$  \hspace{1cm} (4.3.7)

For three equally spaced wave-height meters located at positions $-D$, 0 and $D$, labelled 1, 2 and 3, we then find after a little algebra (see Kostense (1984)):

$$Z_{\text{bin}} = \frac{\frac{1}{2}(A_1 + A_3) - A_2 \cos kD}{\cos kD - \cos kD},$$  \hspace{1cm} (4.3.8)

$$Z_{\text{fin}} = \frac{\frac{1}{2}(A_3 - A_1) + A_2 \sin kD - Z_{\text{bin}}(\sin kD + \sin kD)}{2 \sin kD}$$  \hspace{1cm} (4.3.9)

and

$$Z_{\text{fout}} = A_2 - Z_{\text{bin}} - Z_{\text{fin}}$$  \hspace{1cm} (4.3.10)

Because in our case more than three wave-height meters are present we used the least squares method to determine the $Z_i$ from the $A_i$.

An estimate of the energy densities of the different waves can now be found by

$$G_{\text{bin}} = \frac{2}{T} Z_{\text{bin}} Z_{\text{bin}}', \quad \text{etc.}$$  \hspace{1cm} (4.3.11)

in which $T$ is the length of the time series (or of a segment of that time series).

To determine the wavenumber of the bound low-frequency waves in equations (4.3.8)-(4.3.10) we need an expression for the group velocity. The group velocity can be determined in a number of ways. One method is to use a weighing with the variance density. We then have

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\[ C_g = \frac{\int_{\omega_1}^{\omega} C_g(\omega)G(\omega)d\omega}{\int_{\omega_1}^{\omega} G(\omega)d\omega} \]  \hspace{1cm} (4.3.12)

in which \( \omega_1 \) is a lower bound of the first order spectrum and \( G(\omega) \) the variance density at frequency \( \omega \).

Another method is to use the group velocity of the peak frequency. The peak frequency \( f_p \) can be determined from

\[ f_p = \frac{\int_{\omega_1}^{\omega_2} f G(\omega)d\omega}{\int_{\omega_1}^{\omega_2} G(\omega)d\omega} \]  \hspace{1cm} (4.3.13)

in which the integration bounds are determined as those angular frequencies were the energy density of the first-order waves \( G(\omega) \) is equal to 80% of the maximum energy density.

Finally, the group velocity can be determined via the correlation of the wave envelopes in the time series at different wave-height meters. We take the time series from two wave-height meters a distance \( d \) apart and determine for both the wave envelope by the method described in section (4.3.2). Then the time lag \( \tau \) corresponding to the maximum correlation of the two wave-envelope time series is determined. The group velocity is found from

\[ C_g = \frac{d}{\tau} \]  \hspace{1cm} (4.3.14)

The last method is the most direct one and probably most reliable if an averaging is performed over different wave-height meter combinations, but it takes a lot of computational effort to find the group velocity in this way. We have used the first method because it gave more robust estimates of the averaged group velocity than the second method when the degrees of freedom of the estimate of the high-frequency spectrum were varied.
Let us now turn to the determination of accurate Fourier components \( A_i \) prior to the decomposition. Three methods to obtain more accurate estimates are evaluated. They range from a direct decomposition, were we use the raw Fourier transforms of the wave-height meter signals, to the Complex-Harmonic Principal-Component Analysis, which reduces the influence of noise. This last, rather elaborate, method was finally chosen for the decomposition here because the noise level is high.

4.3.3.1 Direct method

The first method which comes to mind is the following. First divide the time series in segments of equal length. This length must of course be so long as to contain the longest waves which are of importance. Then perform an FFT on each segment separately and use equations (4.3.8) to (4.3.10) directly to obtain the Fourier components \( Z \) of the low-frequency waves. Proceed by calculating the energy-density spectra of these waves as in (4.3.11) and average this result over the segments to increase reliability.

![Diagram](image)

**Figure 4.8:** One-sided spectra of bound low-frequency waves. Drawn line: method of Laing, triangles: direct method, boxes: cross-spectral method, crosses CHPCA method.

An example of the outcome of this procedure is given in figure (4.8) which is obtained from a laboratory experiment. The figure shows the energy-density spectra of the low-frequency
incoming bound waves for the three methods, which are used to obtain accurate estimates for the Fourier components, that we will treat here. The normalized random error in the estimates for the spectral densities is about 30%. In this experiment we used a Jonswap spectrum for the first-order waves and the wave-board control signal was second-order.

We see that the results which are obtained via the direct method are generally too large compared to the theoretical values, obtained from the measured first-order spectrum with the method of Laing (1986). Laing has developed a full non-linear method to determine the energy densities of the bound low-frequency waves from a first-order surface-elevation spectrum.

Furthermore, it was found that a slight change of for instance the distance between the wave-height meters in the calculations (not in the measurements) caused the energy densities of the decomposed low-frequency waves to increase dramatically. So probably the decomposition was incorrect.

Indeed, the spectral estimates for the $A_i$ are not very reliable because the normalized random error is 100%. To use these values in the formulas for the low-frequency waves gives of course unreliable estimates for these quantities, despite the frequency averaging. The values which are found in this way for the low-frequency wave amplitudes are very sensitive to the wavenumbers and hence to the velocities of the waves.

A way out of this is to try to reduce the normalized random error in the $A_i$. This is done in the next method.

4.3.3.2 Cross-spectral method

In this method the random error in the $A_i$ is reduced via segment averaging of the auto-spectra and the cross-spectra before using the decomposition relations. This will result in more reliable estimates for the low-frequency wave components.

Of course, segment averaging cannot be used on the $A_i$ because the phases of the spectral components are randomly varying. However, not the phases themselves are of interest but only the phases relative to the $A_i$ of the other wave-height meters. They can be found from the cross-spectra of the surface elevation signals. The cross-spectral densities are given by:

$$G_{ij} = rac{2}{N \Delta t} A_i A_j^* = C_{ij} e^{-iQ_{ij}}$$  \hspace{1cm} (4.3.15)
in which $C_y$ and $Q_y$ are the coincident and quadrature spectra respectively of the surface elevations at gauge locations $x_i$ and $x_j$. The relative phases can be found from

$$\phi_{ik}(\omega) = -\arctan\left(\frac{Q_{ik}(\omega)}{C_{ik}(\omega)}\right) \quad (4.3.16)$$

in which the index $k$ denotes the wave-height meter relative to which the phases of the low-frequency waves of the other wave-height meters are determined. The absolute values of the $A_i$ can be found from the auto-spectra. So the errors in the estimates for the $A_i$ can be reduced via segment averaging of the auto-spectra and the cross-spectra before using the decomposition relations.

An example of a decomposition with the cross-spectral method is also given in figure (4.8). The normalized random errors are about 25%. The energy density of the bound incoming waves is reduced compared to that obtained from the direct method. This means that the relatively large amplitude of the bound incoming waves found via the direct method is due to the fact that this method kept the random errors in the $A_i$ prior to the decomposition. The energy-density levels of the bound waves are still somewhat higher than their theoretical prediction at small and large frequencies. This was the motivation to consider Principal-Component Analysis.

4.3.3.3 Complex-harmonic Principal-Component Analysis

The cross-spectral method reduced the normalized random errors in the spectral estimates of the Fourier components of the wave-height meter signals and the results obtained for the bound low-frequency waves were much better. Because noise is by definition uncorrelated with the wave signals, by use of the cross-spectra we reduced the influence of noise on the phases of the low-frequency signals from the wave-height meters. However, the auto-spectra still contain noise contributions. Furthermore, the phases are biased to the wave-height meter which is used as basis. An error in the phase of this wave-height meter has a big influence on all other phases. This is the reason to turn to a method in which this bias is not present. In this method the cross-spectra, in which the noise level is greatly reduced due to segment averaging, are used to decrease the influence of noise in the auto-spectra. This will result in an even better estimate of the Fourier components before decomposition than the cross-spectral method.

The method is called Complex-Harmonic Principal-Component Analysis (CHPCA) in full.
It was developed by Wallace and Dickinson (1972) and has been widely used in meteorology and oceanography. The method is used to determine certain patterns in data sets which contain a lot of noise. The word pattern is used here to stress that the method can be used for more than sinusoidal water waves. Especially when more than one pattern is present at a certain frequency the method is advisable because the behavior of the different patterns can be studied, which is very difficult from ordinary spectral techniques. In our case we know the patterns of the waves and the method helps us to remove noise from the data.

An example of the use of the method in coastal engineering can be found in Tatavarti et al (1988). By use of the CHPCA method they were able to improve the estimates of the reflection coefficients of low-frequency waves. These were overestimated in the conventional method because noise tends to push the reflection coefficient to one.

In this subsection we will first explain the method in short and then apply it to our example. The reader is referred to Preisendorfer (1988) for a complete description.

Let us now turn to the method itself. Suppose \( m \) realizations of a random process are measured, for instance the surface elevation level \( \zeta_s \) at \( p \) places \( x_i \) and at \( n \) discrete times \( t_i, 1=0,1,2..n-1 \). So our data can be represented by

\[
\zeta_s(t, x_i)
\]

(4.3.17)

in which the index \( r \) numbers the realization and \( i \) denotes the wave-height meter. The Complex Harmonic part of the method means that we first Fourier transform the data and then concentrate on each frequency bin separately. So we have

\[
z_s(\omega, x_i) = \frac{1}{n} \sum_{i=0}^{n-1} \zeta_s(t_i, x_i) \exp(-i\omega t_i)
\]

(4.3.18)

in which the angular frequency \( \omega = 2\pi j/(t_{n-1} - t_0) \).

We now form the cross-spectral matrix

\[
P_{ik}(\omega) = \frac{1}{m} \sum_{r=0}^{m} z_s(\omega, x_i) z^*(\omega, x_k)
\]

(4.3.19)

which is hermitian. So far, the method is identical to the cross-spectral method. Note that the basis of the cross-spectral matrix is formed by the wave-height meters, or by distinct
points in physical space.

Now we look for the eigenvectors $e_k(\omega_p x_i)$ of the cross-spectral matrix, in which the index $k$ numbers the eigenvectors in order of the magnitude of the corresponding eigenvalues $\lambda_k(\omega)$. If the eigenvectors are taken as a basis for the cross-spectral matrix, this matrix will by a diagonal matrix, with the eigenvalues on the diagonal. These eigenvalues denote the variance of the corresponding eigenvectors, just as in the original cross-spectral matrix where the diagonal elements are the variances of the low-frequency waves at the wave-height meters.

The variance of an eigenvector is distributed over the locations of measurements, e.g. the wave-height meters. It is easy to see that the total variance at a certain point (wave-height meter) is the sum of the contributions of the variances of the eigenvectors at that point. Now we arrive at the heart of the method. The assumption is that the distribution of the variance of the wave signals over the wave-height meters can be described by only a few eigenvectors. The resulting eigenvectors are interpreted as describing the distribution of the variance of the noise. The reason that we need the eigenvectors is that they are orthogonal, and we know that noise is orthogonal to the wave signals.

We still have to determine which eigenvectors will represent the wave signals, and which will be attributed to noise. Recall that the eigenvalues denote the variance of the eigenvectors. Because the variances of the wave signals will be higher than that of the noise the eigenvectors with the biggest eigenvalues will describe the waves. Usually the eigenvalues can indeed be divided into two categories of different magnitudes. Hence only the largest eigenvalues may be considered important in the representation of the total variance of the wave signals. In our case, in which the wave patterns are so regular, we will only need the first and occasionally the second eigenvalue; the others are less than 1% of the first.

Now that we have explained the method we will put the words into equations. The cross-spectral matrix becomes in the new representation

$$P_{\omega}(\omega) = \begin{pmatrix}
\lambda_1(\omega) & 0 & \ldots & \ldots & 0 \\
0 & \ldots & 0 & \ldots & \ldots \\
\ldots & 0 & \lambda_k(\omega) & 0 & \ldots \\
\ldots & \ldots & 0 & \ldots & 0 \\
0 & \ldots & \ldots & 0 & \lambda_p(\omega)
\end{pmatrix} \quad (4.3.20)$$

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The eigenvectors are numbered according to the value of their eigenvalue: the highest eigenvalue corresponds to \( k = 1 \) etc.. Because the cross-spectral matrix is hermitian the eigenvalues are real, so this ordering is indeed possible.

The components of the Fourier transforms \( z_r(\omega_j, x_i) \) on the eigenvectors \( e_k(\omega_j, x_i) \) are given by

\[
y_{nk}(\omega_j) = \sum_{i=1}^{p} z_r(\omega_j, x_i)e_k(\omega_j, x_i)
\] (4.3.21)

in which \( r \) denotes the realization. Because the eigenvectors form a basis we also have

\[
z_r(\omega_j, x_i) = \sum_{k=1}^{p} y_{nk}(\omega_j)e_k^*(\omega_j, x_i)
\] (4.3.22)

The \( y_{nk}(\omega_j) \) are called the complex principal amplitudes. If we compare this equation with (4.3.18) we see that the function of the eigenvectors resembles that of the Fourier components. In the original PCA, which works in the time domain, this resemblance is even more striking. For a stationary process the two are even identical. Because the eigenvectors are orthogonal and completely determined by the experiment they are called also Empirical Orthogonal Functions (EOF's).

The part of the variance at a certain point \( x_i \) explained by eigenvector \( k \) is given by the coherence of \( z_r(\omega_j, x_i) \) and \( y_{nk}(\omega_j) \) times the variance at that point:

\[
Var_k(\omega_j, x_i) = \frac{\left(\frac{1}{m} \sum_{r=1}^{m} z_r(\omega_j, x_i)y_{nk}(\omega_j)\right)^2}{\left(\frac{1}{m} \sum_{r=1}^{m} |z_r(\omega_j, x_i)|^2\right)\left(\frac{1}{m} \sum_{r=1}^{m} |y_{nk}(\omega_j)|^2\right)} \times \left[\frac{1}{m} \sum_{r=1}^{m} |z_r(\omega_j, x_i)|^2\right] (4.3.23)
\]

\[
= \lambda_k(\omega_j) |e_k(\omega_j, x_i)|^2
\] (4.3.24)

and the total variance at that point becomes
\[ \text{Var}(\omega_p x_i) = \sum_{k=1}^{p} \text{Var}_k(\omega_p x_i) = \sum_{k=1}^{p} \lambda_k(\omega_p) |e_k(\omega_p x_i)|^2 \] (4.3.25)

The quantity \(|e_k(\omega_p x_i)|\) is called the gain of eigenvector \(k\). It shows how the variance of the eigenvector is distributed over the physical space.

The phase of the Fourier component of sensor \(i\) compared to that of the eigenvector \(k\) is given by the argument of the eigenvector at that point. So we have

\[ \phi_k(\omega_p x_i) = \arg[e_k(\omega_p x_i)] \] (4.3.26)

In this way we have found expressions for both the amplitude and the relative phases of the Fourier components of the time series which are less sensitive to noise. Hence we have reached our goal.

An example is used to gain some more insight in the way this method reduces the influence of noise. Suppose we have two wave-height meters which give a Fourier transformed signal \(X_1 + n_1\) and \(X_2 + n_2\)

in which the \(X_i\) denote the wanted wave signals and the \(n_i\) denote the noise. We assume that the two noise signals are uncorrelated. Note that the noise signals are uncorrelated with the wave signals by definition. This means that cross spectra with the noise signals are negligible after average averaging. We then have for the cross-spectral matrix

\[ P = \begin{pmatrix} |X_1|^2 + |n_1|^2 & X_1^* X_2 \\ X_1 X_2^* & |X_2|^2 + |n_2|^2 \end{pmatrix} \] (4.3.27)

The eigenvalues of this matrix can easily be found. For simplicity we take the noise signals to have approximately the same variance density. This is a reasonable assumption in the case of wave-height meters. We then find

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\[ \lambda_{1,2} = \frac{1}{2} \left( |X_1|^2 + |X_2|^2 \right) + |n|^2 \pm \sqrt{\left( |X_1|^2 + |X_2|^2 \right)^2 + (4 \gamma_{12}^2 - 2) |X_1|^2 |X_2|^2} \]  
(4.3.28)

in which the coherence between the two signals is given by \( \gamma_{12} \). The corresponding eigenvectors can be found as

\[ e_k = \frac{1}{\sqrt{|X_1^* X_2|^2 + (|X_1|^2 + |n|^2 - \lambda_k)^2}} \begin{pmatrix} -X_1^* X_2 \\ |X_1|^2 + |n|^2 - \lambda_k \end{pmatrix} \]  
(4.3.29)

The variance at point \( i \) explained by eigenvector 1 is given by

\[ \lambda_1 |e_{1i}|^2 = \frac{\lambda_1 \left| X_1^* X_2 \right|^2}{|X_1^* X_2|^2 + (|X_1|^2 + |n|^2 - \lambda_1)^2} \]  
(4.3.30)

which can be evaluated as

\[ \lambda_1 |e_{1i}|^2 = |X_1|^2 + |n|^2 \left[ \frac{\gamma_{12}^2 (1 - a + \sqrt{b}) + 2a \left| \frac{n}{X_1} \right|^2 \gamma_{12}^2 - a^2 - 1 - 2a(\gamma_{12}^2 - 1) + (a - 1)\sqrt{b}}{|X_1|^2 \left( 2a \gamma_{12}^2 - 1 \right) + a^2 + 1 - (a - 1)\sqrt{b}} \right] \]  
(4.3.31)

where

\[ a = \frac{|X_1|^2}{|X_2|^2} \]  
(4.3.32)

and

\[ b = a^2 + 1 + 2a(2\gamma_{12}^2 - 1) \]  
(4.3.33)
This complicated expression can be calculated analytically in two limiting cases. For $\gamma_{12}^2 = 0$ we have

$$\lambda_1 |e_{11}|^2 = \|X_i\|^2 + |n|^2$$

(4.3.34)

and for $\gamma_{12}^2 = 1$ we find

$$\lambda_1 |e_{11}|^2 = \|X_i\|^2 + \frac{1}{2} |n|^2$$

(4.3.35)

We thus see that the influence of the noise on the total signal is strongly reduced in the last case. So when we use this method our estimates for the amplitudes of the Fourier transforms of our measured time series are more reliable. Also note that the relative phases do not depend on noise at all in this case. This last result was also obtained with the standard cross-spectral analysis. In the case of the CHPCA method however the phases are not biased to the phase of one special wave-height meter.

Finally, figure (4.8) gives an example of how the method works for a real case. Compared to the cross-spectral method the energy densities of the bound incoming low-frequency waves are closer to the theoretical values as found with Laing (1986).

All in all, we can conclude that noise indeed has influence on the decomposition, and on the interpretation of the data. (Another example of this is given in the section 4.4.) Hence the CHPCA method will be used in the analysis of the data.

### 4.3.4 Noise level and error analysis

The noise level and the error analysis need special attention. We will first concentrate on the noise level and than on the uncertainties in the spectral estimates of the decomposed low-frequency waves.

Tests have been performed with a bichromatic first-order wave spectrum so that only low-frequency waves with one frequency, the difference frequency, appeared. The number of points in the FFT was chosen such that no leakage occurred. The parts of the decomposed low-frequency wave spectrum in which no energy was present theoretically did contain energy due to the whole process of sampling and processing. The mean energy level at these frequencies was attributed to noise, and the noise level thus found was about 0.3 mm. Increasing the measurement time, and hence the accuracy of the estimator of the amplitudes,
did not reduce this value. This noise level of 0.3 mm was also used in the continuous spectrum cases. It corresponds to an energy density of 0.9 mm²/Hz in our case.

The uncertainties in the estimators of the spectral densities of the energy density spectra before decomposition are the normalized random errors, defined as

\[
e_r = \frac{\sqrt{\text{variance(energy density)}}}{\text{energy density}} = \sqrt{\frac{2}{\text{dof}}} \tag{4.3.36}
\]

in which \textit{dof} is the number of degrees of freedom of the estimator (see for instance Bendat and Piersol 1986). The \textit{dof} in the estimations of the low-frequency energy densities cannot be too high because then the frequency bins become too broad in which case the wavenumbers of the low-frequency waves are not known accurately enough for the decomposition to make sense.

After decomposition the uncertainties in the spectral estimates are not precisely known. First, we do not know the influence of the CHPCA method on the uncertainties. Second, after the noise reduction, a least-squares method was used to obtain the different low-frequency wave components and Fourier components from the CHPCA method for the different wave-height meters are not independent; they all belong to the same principal component. So the standard techniques to obtain uncertainties of the spectral estimators cannot be applied.

Approximate error estimates were obtained in the following way. The bound low-frequency wave spectral densities are predicted correctly from first-order energy-density spectra by full second-order theories as that of Ottesen-Hansen (1978) and Laing (1986). (See for instance laboratory experiments by Kostense in 1984.) The errors in the estimates of the spectral densities of the bound waves are estimated to be the deviation from the calculated spectral densities from the theory of Laing (1986), based on the measured first-order energy-density spectrum. This first-order spectrum is taken equal to the measured spectrum around the peak frequency of the first-order waves. Of course these spectral densities contain their own uncertainties, but to obtain these spectra the degrees of freedom can be greatly increased. The error estimator will be denoted by \( e_L \) and is given by

\[
e_L = \left[ \frac{1}{N-1} \sum_{i=1}^{N} \left( \frac{G_{\xi_0 \xi_0}^{(0)} - G_{\xi_0 \xi_0}^{(0)}}{G_{\xi_0 \xi_0}^{(0)}} \right)^2 \right]^{1/2} \tag{24}
\]
in which $G_{\zeta_L}(i)$ the theoretical value for the bound-wave spectral density at frequency $i$, which is obtained by following Laing (1986). $G_{\zeta_L}(i)$ is the measured bound-wave spectral density at frequency $i$. The subscript $L$ denotes 'Laing' to remember that the error is calculated with the theoretical values of Laing as reference. In this way we obtained error estimates for the spectral estimates of 25%. The errors in the estimates of the other (free) components are taken equal to that of the bound components.
4.4 VERIFICATION OF THE CONTROL SIGNAL

In this section we will verify the theory for the second-order control signal for the wave board as derived in section 4.1. Our criterion for a correct control signal is that the free incoming low-frequency waves, which arise due to a bad second-order control signal or due to reflections of low-frequency waves on the wave board, have a magnitude which is at most third order, or which is smaller than the noise level.

To this end a full system test was performed with a Jonswap first-order spectrum with a peak frequency of 0.63 Hz, a peak-enhancement factor of 3.3 and a first-order rms wave height of 4 cm.

The energy spectra of the free low-frequency waves obtained with the three methods given in section 4.3.3 are given in figure 4.9. Note the enormous decrease in energy densities when the CHPCA method is used to analyze the data. This result clearly shows how noise influences the interpretation of the data, and the importance of the analysis method.

Let us now concentrate on the CHPCA result to interpret the behaviour of the wave board. The noise level is estimated to be around 0.9 mm² /Hz (see section 4.2), so the free low-frequency waves have spectral densities below the noise level and our criterion is met for frequencies above 0.1 Hz. It is not met at the lowest frequency, but remember that the reflection coefficient of the wave board is high in this region. The energy densities of the free incoming low-frequency waves are much lower than those of the bound incoming low-frequency waves as can be seen from a comparison with figure 4.8 (page 80). Only where the energy densities are very low this is not true, but in that case noise will disturb the picture.

However, the noise level, about 0.9 mm² /Hz, is much higher than the expected level of third order effects. Third-order effects are effects one order (e) higher than the second-order effects. e is of the order of the first-order wave steepness kA in which k is a mean wavenumber of the first-order waves and A is their mean amplitude. So the third-order waves are a factor 0.1 lower in amplitude than the second-order waves, which gives a factor 0.01 in energy density. This gives an estimate of the energy density of the third-order waves of about 0.02 mm² /Hz, which is much lower than the estimate for the noise level.

This is not satisfactory and a second test was performed in which the coherence between the first-order wave envelope and the free incoming low-frequency waves was calculated, see
Figure 4.9: One-sided spectra of incoming free low-frequency waves. Triangles: direct method, boxes: cross-spectral method, crosses: CHPCA method.

Figure 4.10. The idea is that if the second-order control signal is not correct in amplitude, the bound low-frequency waves generated by the wave board cannot fulfill the boundary conditions on the wave board. As a result free low-frequency waves are generated with an amplitude which is proportional to the amplitude of the first-order waves squared.

Because the coherence in figure 4.10 is lower than the 95% coherence interval on zero coherence we can conclude that the control signal was indeed correct up to second order.
Figure 4.10: Coherence between the high-frequency wave envelope and the free incoming low-frequency waves. The 95% confidence interval on zero coherence is given by the dotted line.
CHAPTER 5

MODEL VERIFICATION

In this chapter the results of the surf-beat model are compared with laboratory and field measurements.

The advantage in the laboratory is that the situation is under control. In a flume no edge waves can disturb the wave patterns, so the model can be tested in a situation which meets the assumptions in the model. Also, by using a concrete bottom one is able to concentrate fully on the wave motion without the complicating sediment-transport processes which change the bottom profile so that the wave field will respond etc. For instance the occurrence of resonances can be studied when bars are present. However, the bars in nature are movable so that a well-defined equilibrium state may not occur.

The advantage of data from the field is that a direct comparison can be made between the model and nature, so no scaling has to be performed. For instance, the interactions between the water and the bottom, producing bottom friction and sediment transport, are nonlinear processes which are difficult to scale properly in a laboratory. A disadvantage is that the assumptions which are the basis of the one-dimensional surf-beat model are almost never met in nature. As we will see, even in the case that the bottom profile is fairly two dimensional and the wind-waves are coming in from a nearly shore-normal direction the low-frequency wave field can be two dimensional with edge waves present.

The plan of this chapter is as follows. First we deal with the laboratory measurements. The model results are compared with experiments conducted in the wave flume as described in chapter 4. The comparison comprises variances of the low-frequency waves, variance spectra of the forcing and of the low-frequency waves and correlations of low-frequency waves with the envelope of the high-frequency waves. High-frequency waves
are waves with frequencies around the peak frequency of the incoming wave field. This expression will be used to distinguish these waves from the low-frequency waves and because the term wind wave is not appropriate for waves generated by a wave board.

Secondly, a comparison with a field measurement is performed. The comparison pertains to the same characteristics of the low-frequency waves as in the laboratory case. No bar was present during this field experiment.

Finally, we discuss the results and deal with possible improvements of the model. This is motivated by the fact that the model appears to be unable to predict the correct energy-density spectra in the laboratory case.
5.1 COMPARISON WITH LABORATORY DATA

5.1.1 Introduction

In this section we concentrate on the low-frequency wave characteristics in the laboratory experiments and compare the data with the numerical model.

The low-frequency wave field is decomposed in incoming bound waves, bound to the wave groups, incoming free waves and outgoing free waves on the horizontal part of the flume. On the sloping bottom only two sensors are available at each location, so there the wave field is split in incoming and outgoing waves. At small depths the difference between the phase velocities of bound and free waves becomes so small that this decomposition is justified. Besides, linear theory breaks down when the group velocity becomes comparable to the phase velocity of free low-frequency waves given by $\sqrt{gh}$.

The plan of this section is as follows. First a detailed comparison is performed for a plane slope. We start with a comparison of the forcing function for the low-frequency waves. Then we concentrate on the low-frequency-wave characteristics: spectra of incoming and outgoing waves and correlations of these waves with the envelope of the high-frequency waves. It will be shown that the energy-density spectra of the low-frequency waves are not predicted properly, but the results for the correlations are generally good.

Secondly, the influence of the spectral shape and the value of the peak frequency of the high-frequency waves was studied. A comparison between the low-frequency wave spectra from Jonswap and Pierson-Moskowitz high-frequency wave spectra with a 10% variation in peak frequency showed that the variations in high-frequency spectra were too small to have a significant influence on the low-frequency wave spectrum.

Thirdly, the influence of the position of the bar will be examined. We concentrate on the occurrence of possible resonances in parts of the low-frequency wave spectrum.

5.1.2 Detailed comparison for a plane slope.

In this subsection a detailed comparison between model and measurements is given for the case without a bar. This situation can most easily be compared with the field experiment, which will be described in section 5.2. However, the beach slope is much steeper in the present case, which will have a considerable influence on the low-
frequency wave field.

First we deal with the forcing, which is the input in the low-frequency wave model. Then we concentrate on the energy-density spectra of the incoming and outgoing low-frequency waves. Finally, a comparison is made between the correlations of incoming and outgoing waves and the envelope of the high-frequency waves.

The experimental conditions were the following:
- A plane slope of 1:25.
- The still-water depth is 0.42 m in the horizontal part of the flume.
- The four measurement locations are 0.40 m, 2.23 m and 3.83 m from the still-water line and on the horizontal part of the flume (see figure 4.3), which corresponds to still-water depths of respectively 1.6 cm, 9.0 cm, 15.3 cm and 42 cm.
- The high-frequency energy-density spectrum was a Jonswap spectrum with a root-mean-square wave height of 4.2 cm and a peak frequency of .63 Hz.

The details of the experimental set up are treated in chapter 4 (see figure 4.3).

![Figure 5.1: One-sided spectra of the normalized radiation stress from the measurements. Depths are as indicated.](image-url)
The forcing

We will use the radiation-stress variance density, divided by $\rho g$, at different locations in the flume as a measure for the forcing. The radiation stress is calculated from the envelope of the high-frequency waves, which is calculated with the method described in chapter 4.

In figure 5.1 the variance-density spectra of the radiation stress are given. The calculated percentages of broken waves are 0, 12, 30 and 100% at the still-water depths of respectively 42 cm, 15.3 cm, 9.0 cm and 1.6 cm. We counted the number of broken and unbroken waves in the flume at these locations to check these calculations and found agreement within a few percents. It is readily seen that the variance density of the radiation stress at lower frequencies decreases much faster than that at higher frequencies when the waves are breaking. So the wave groups become shorter and shorter during the breaking process. The reason for this is probably the following: due to wave breaking the higher waves in a wave group are decreased in size. The result of this is that the wave groups are topped off, and the variance density of the envelope is partly destroyed due to breaking and partly transferred to higher frequencies due to the removal of the top of the group, resulting in a less sinusoidal shape of the wave envelope.

Figure 5.2 shows the model results for the variance-density spectra of the radiation stress. It shows the same behaviour as the measurements: the variance density at the lower frequencies decreases more strongly with wave breaking. A comparison with figure 5.1 shows that the model overpredicts the variance-density spectra of the radiation stress at the location where 12% of the high-frequency waves is broken, which corresponds to a still-water depth $h$ of 15.3 cm. However, where 30% of the waves is broken ($h=9.0$ cm) the model behaves satisfactorily.

This comparison learns that in nature the initial breaking process destroys the wave groups more strongly than the model predicts. One of the reasons for this discrepancy can be that the model assumes spilling breakers as far as the influence of breaking on wave groups is concerned. However, we observed in the flume that especially the higher waves, which broke far from the shore line, were more plunging. This means that the top of the highest parts of the wave envelope are cut off more rigorously which gives a stronger reduction of the wave-envelope variation and hence of the strength of the forcing.

From this model behaviour one would expect a little overprediction of the low-frequency
The low-frequency wave spectra

In figure 5.3a-d a comparison between the energy-density spectra of the low-frequency waves as calculated from the measurements and the model results is given. The four locations are at still-water depths of 42 cm, 15.3 cm, 9.0 cm and 1.6 cm, which correspond to approximately 0, 12, 30 and 100% of broken waves. At the first three locations the spectra concern the outgoing low-frequency waves and at the last location the total low-frequency wave spectrum is given. A decomposition at this last location is not possible because only one sensor, a wave-height meter, was present.

It is clear that the model strongly overpredicts the energy densities of the waves with higher frequencies. While the measurements show a maximum energy density around 0.12 Hz the model gives a much higher maximum around 0.25 Hz.
Figure 5.3a: One-sided spectra of the outgoing low-frequency waves on the horizontal bottom. Boxes: measurements, drawn line: model results.

Figure 5.3b: One-sided spectra of the outgoing low-frequency waves at a still-water depth of 0.153 m. Boxes: measurements, drawn line: model results.

Figure 5.3c: One-sided spectra of the outgoing low-frequency waves at a depth of 0.09 m. Boxes: measurements, drawn line: model results.

Figure 5.3d: One-sided spectra of all low-frequency waves at a still-water depth of 1.6 cm. Boxes: measurements, drawn line: model results.

Note that the energy spectrum of the low-frequency waves has a completely different shape at the location closest to the waterline (figure 5.3d) compared to the other locations (figures 5.3a-c). The reason is that the total low-frequency wave spectrum will show the characteristics of standing waves. The dip in energy density which is present in figure 5.3d at 0.16 Hz can be attributed to the presence of a surface elevation node of the standing waves of that frequency at this distance from the water line. This can be seen as follows: the surface elevation node corresponds to the first zero crossing of the Bessel function of the first kind because the low-frequency waves are standing waves this close to the water line. This gives the relation between frequency and distance from the waterline:

\[ J_0(\chi) = 0 \quad \Rightarrow \quad \chi = 2.405 \]  

(5.1.1)
in which \( \chi \) given by

\[
\chi = \frac{2\omega}{\sqrt{\tan \beta}} \sqrt{\chi} \tag{5.1.2}
\]

(see chapter 3, section 3.1.5). We thus find

\[
f = \chi \frac{\sqrt{\tan \beta}}{4\pi \sqrt{\chi}} \tag{5.1.3}
\]

The quantity \( x \) is the distance of the wave-height meter to the actual water line, which includes the time-averaged set up. The time-averaged set up can be calculated from the time average of equation (3.1.1), which reads after integration over the space coordinate \( x \):

\[
\overline{\zeta}_x = -\frac{1}{\rho gh} \overline{S}_x \tag{5.1.4}
\]

The radiation stress can in the surf zone be approximated by

\[
\overline{S} = \frac{3}{16} \rho g H_{rms}^2 - \frac{3}{16} \rho g \gamma_{rms}^2 h^2 \tag{5.1.5}
\]

in which \( \gamma_{rms} \) the ratio of the rms wave height to the local water depth. (Note that this is not the same quantity as the \( \gamma \) introduced in chapter 3, but its numerical value will be nearly the same inside the surf zone.) So equation (5.1.4) becomes

\[
\overline{\zeta}_x = -\frac{3}{8} \gamma_{rms}^2 h_x \tag{5.1.6}
\]

The time-averaged set up at the waterline now becomes

\[
\overline{\zeta}(x=0) = \frac{3}{8} \gamma_{rms}^2 h - \frac{3}{8} \gamma_{rms} H_{rms} \tag{5.1.7}
\]
in which the index b denotes the quantity evaluated just prior to breaking. In our case this can be approximated by the highest value of \( H_{\text{rms}} \), which is about 5 cm. \( \gamma_{\text{rms}} \) is about 0.4, so the set up will be about 0.75 cm. The distance over the slope which corresponds to this set up is 18.75 cm. For the measurements shown in figure 5.3d, the distance from the still-water line to the wave-height meter is 40 cm, the total distance of the wave-height meter to the actual mean water-line position is 58.75 cm. If we use this value in equation (5.1.3) we find a frequency of 0.16 Hz. The model shows this dip in energy density (figure 5.3d) and also the measurements show a small dip. However, this last dip is not significant.

The first maximum of the Bessel function appears for \( \chi = 3.963 \), which corresponds to a frequency of 0.26 Hz. Indeed, the model shows a (too) strong energy density, but this peak is not visible in the measurements.

The next frequency at which the energy-density has to show a minimum in figure 5.3d, which corresponds to a surface elevation node is at \( \chi = 5.520 \), which gives a frequency of 0.35 Hz, just outside the frequency range. Note that this is the second surface elevation node of the waves with this frequency at this distance from the water line. The model seems to predict this minimum too, although it is just outside the frequency range of calculation.

The reason for the bad behaviour of the model is not entirely clear. The model of List (1992), which differs from the present model in that it calculates in the time domain, also shows an overprediction of the low-frequency energy densities at higher frequencies (List, private communication). Also the bichromatic model of Schäffer (1991) predicts a monotonic increase of low-frequency wave energy with decreasing frequencies (see his figure 4.18), although a bichromatic model cannot be compared directly to a full-spectral model. The results of the other existing models in this respect are not known to this author. The matter will be discussed further in section 5.3.

The correlations

Roelvink and Stive (1988) have shown that the value of the correlation between the low-frequency waves and the high-frequency wave envelope at zero time lag is of great importance to sediment transport. The present laboratory experiment shows the following results. On the horizontal bottom the correlation is negative, as one would expect for the incoming bound low-frequency waves. The magnitude of the correlation is -0.45 which is lower than the theoretical value of -1, which arises from the theoretical result that the bound low-frequency waves are 180 degrees out of phase with the envelope of the
primary waves. However, also free outgoing low-frequency waves are present, which will reduce the correlation due to the fact that they are not significantly correlated with the envelope of the high-frequency waves at zero time lag.

At a depth of 15.3 cm, where about 12% of the high-frequency waves is breaking, the correlation does not change significantly, retaining a value of -0.54. At a depth of 9 cm, where about 30% of the high-frequency waves is breaking, the correlation is -0.33. This change is probably due to the fact that free low-frequency waves are generated by the time-varying break point, which are positively correlated with the envelope of the high-frequency waves. Finally, at a depth of 1.6 cm, where nearly all high-frequency waves are broken, the correlation turned to a positive value of 0.58. This high positive value is due to three mechanisms.

First, the water depth at this position is so small that the amplitudes of the low-frequency waves become of the same order of magnitude as this depth. This means that the water depth in which the high-frequency waves travel is modulated by the low-frequency waves, thus producing a positive correlation between the low-frequency waves and the high-frequency wave envelope.

Secondly, the bound low-frequency waves are not 180 degrees out of phase with the high-frequency wave envelope in shallow water with a sloping bottom, but this phase difference is close to 90 degrees (see section 3.1.5). This will bring the correlation at zero time lag to zero.

Finally, free waves generated by the time-varying break point will come into play. Incoming free waves of this kind are positively correlated with the high-frequency wave envelope (simply because higher breaking waves give rise to a bigger set up), while outgoing wave of this origin are negatively correlated to this envelope. So incoming low-frequency waves generated by the time-varying breakpoint mechanism travel shorewards with the high-frequency wave envelope, having nearly the same phase. This will push the correlation coefficient at zero time lag to +1.

Unfortunately no measurements are available between the position of 30% broken waves and the position of 100% broken waves. This is because the wave-height meters began to feel the bottom too strongly so that the measurements were not reliable any more (see chapter 4).
Figure 5.4a: Correlation between incoming low-frequency waves and high-frequency wave envelope on the horizontal bottom. The 95\% C.I. on R=0 is 0.23. Striped line: measurements, drawn line: model results.

Figure 5.5a: Correlation between the outgoing low-frequency waves and the high-frequency wave envelope on the horizontal bottom. The 95\% C.I. on R=0 is 0.17. Striped line: measurements, drawn line: model results.

Figure 5.4b: Same as above at a depth of 15.3 cm. The 95\% C.I. on R=0 is 0.24.

Figure 5.5b: Same as above at a depth of 15.3 cm. The 95\% C.I. on R=0 is 0.24.

Figure 5.4c: Same as above at a depth of 9 cm. The 95\% C.I. on R=0 is 0.23.

Figure 5.5c: Same as above at a depth of 9 cm. The 95\% C.I. on R=0 is 0.23.

Let us now turn to the cross-correlations in more detail, and focus on the value of the correlation coefficient as a function of the time lag between the low-frequency waves and
the envelope of the high-frequency waves. To get a better picture of the physics involved, the low-frequency waves are again split-up in incoming and outgoing waves.

The cross-correlations of incoming and outgoing waves with the high-frequency wave envelope as function of the time lag are given in figures 5.4a-c and 5.5a-c. The locations again concern the locations where 0, 12 and 30% of the waves is broken. A positive time lag denotes that the envelope is leading the low-frequency wave signal. The 95% confidence interval (C.I.) on zero correlation was about 0.20 in all cases. The exact values for the confidence intervals are given in the figure captions.

The 95% C.I. was calculated in the following way. Because neighboring points in an almost periodic time series are not independent a reduced number of points \( n \) is used. This number is found by Garrett and Toulany (1981) as

\[
\frac{1}{n} = \frac{1}{n} + 2 \sum_{j=1}^{m} (n-j) R_{xy}(j)
\]

in which \( n \) is the number of samples in the time series, \( R_{xy}(j) \) is the autocorrelation of the product of the two series to be correlated at time lag \( j \) and \( m \) is the time lag at which \( R_{xy}(j) \) experiences its first zero crossing.

In the figures 5.4a-c a strong negative peak can be observed near zero time lag which can be attributed to the bound low-frequency waves. The incoming low-frequency waves on the horizontal part of the flume are the bound low-frequency waves as predicted by Longuet-Higgins and Stewart (1962). In 5.4b and 5.4c, which deal with the cross correlation between the incoming low-frequency waves and the high-frequency wave envelope on the slope in shallow water, a strong negative peak can be observed at a small positive time lag and probably a smaller positive peak at a small negative time lag. Clearly, the phase difference between the bound low-frequency waves and the high-frequency wave envelope is shifting from 180 degrees to 90 degrees with the envelope leading. However, due to the time-variation of the breakpoint position a positive correlation has to be expected at zero time lag, simply because higher waves give rise to a larger set up. The total correlation signal is of course due to a combination of both effects, and the bound waves contain more energy in this case. The model clearly shows the same behaviour, which is promising.

The three figures which show the correlation between the outgoing waves and the envelope of the high-frequency waves (figures 5.5a-c) are more complex. One would
expect negative correlations at time lags which are equal to the time needed for the bound waves to travel to the waterline and for the free reflected waves to travel back to the sensor location. Indeed, the measurements show strong correlations around the expected time lags of 22 s, 12 s and 9 s respectively. In the measurements we can still recognize the 90 degrees phase shift which the low-frequency waves underwent during the shoaling of the high-frequency waves. The strong negative peak in figure 5.5b at a time lag of about 4.5 s cannot be attributed to free low-frequency waves which are produced by the time variation of the break-point position and travel directly to the sensor location. These waves will be negatively correlated with the wave envelope, but when we calculate the position of their release we find it to be at a (still-water) depth of 1.2 cm, where all high-frequency waves are broken. A more realistic position of the mean breakpoint (if it can be defined at all for this purpose) as the position, where about 50% of the high-frequency waves is broken, leads to a negative correlation at a time lag of 2.3 s, at which no significant correlation can be seen.

In the model results the time lags of strong correlations coincide with those of the measurements. However, the change in relative phase is nearly absent, only a little asymmetry can be observed. The model did show a stronger shift in relative phase for the incoming low-frequency waves (see figures 5.4b-c).

### 5.1.3 The influence of the spectral shape

The influence of the high-frequency wave spectrum on the low-frequency wave spectra was also to be studied. The high-frequency wave spectra differed in peak frequency and spectral shape. To have realistic high-frequency wave spectra which are such that the assumptions of the theoretical model are met, the shapes of these spectra were chosen to be standard Jonswap and Pierson-Moskowitz spectra. An upper bound on the peak frequency of the high-frequency waves was given by the wave board. It could not handle spectra with peak-frequencies above about 0.7 Hz. The high-frequency tail of the spectra forced the wave board to oscillate more quickly than it could move smoothly, so breaking waves occurred just in front of the board. On the other end, the reflection coefficient of the wave board increases rapidly if the frequency of the low-frequency waves is below 0.05 Hz (see figure 4.4). This gives a lower bound on the peak frequency of the high-frequency waves because if it gets too low the low-frequency wave band which can be studied becomes very narrow. So we did not perform experiments with spectra with a peak frequency lower than 0.6 Hz. In order to keep the stroke of the wave board within its limits and the low-frequency waves detectable from the noise the root-mean-square wave height of the high-frequency wave fields was kept on its maximum value 4.2 cm.

Because the variations in spectral shapes and magnitudes were too small to give
significant changes in the low-frequency wave spectra it was not possible to draw any conclusions on the influence of the spectral shape of the primary waves on the low-frequency wave spectra.

5.1.4 The influence of the bar position

In this subsection the influence of the presence and the position of a bar on the low-frequency wave spectra will be studied. The experimental conditions were as follows:
- The high-frequency spectrum was a Jonswap spectrum with a root-mean-square wave height of 4.2 cm and a peak frequency of 0.63 Hz.
- The distance of the bar from the still-water line ranged from 2.32 m to 3.21 m, which corresponds to a still-water level of 4 cm to 9 cm on the bar crest. At the first position nearly all waves broke before or on the bar and at the last position only the very highest waves broke on the bar. We will also use the results from the experiment without a bar. To show the results three bar positions are chosen, 2.32 m, 2.64 m and 3.05 m from the still-water line, corresponding to still-water depths of 4.0 cm, 5.5 cm and 7.4 cm on the bar.

We concentrate on the energy-density levels of the low-frequency waves on the horizontal part of the flume and close to the water line. In this way both sides of the bar are observed. Due to the bad model performance found in the last section, the model results for the varying bar positions are not given.

In figure 5.6 the energy-density spectra of the low-frequency waves are given at a still-water depth of 1.6 cm. We see here that the energy densities of the measurement without a bar are slightly higher around 0.19 Hz than those of the others. However, this feature is probably not significant. Note the dips in spectral density at 0.16 Hz, 0.27 Hz and 0.35 Hz. The first and the last dips can be understood in terms of standing wave surface elevation nodes as we have seen earlier. The reason for the dip at 0.27 Hz is unclear. A maximum in energy density is expected to be found at 0.22 Hz, which comes close to the observed peak at 0.19 Hz.

Symonds and Bowen (1984) find so called half-wave resonances in their model. The resonance condition reads

\[ \frac{1}{f} = 2 \int_{n_0}^{n_s} \frac{1}{\sqrt{gh}} \, dx \]
Figure 5.6: One-sided spectra of all low-frequency waves at a still-water depth of 1.6 cm. _- plane bottom, - - bar at 7.4 cm, ...bar at 5.5 cm, - - bar at 4.0 cm, all still-water depths.

and states that the distance from the bar crest (x=x_b) to the water line is equal to an integer number, n, of half wavelengths. In these cases a surface-elevation antinode exists over the bar. For our bar positions we obtain for n=1 resonance frequencies which range from 0.18 Hz to 0.20 Hz, which agrees with the frequency of the observed peak around 0.18 Hz. However, also the experiment in which no bar is present shows this behaviour so we cannot be conclusive about the presence of these resonances.

In figure 5.7 the energy-density spectra of the outgoing low-frequency waves in the horizontal part of the flume are given for the measurements. The spectra seem to form two categories. The spectrum with no bar present and that with the bar in deeper water show a shoulder at 0.13 Hz. The spectra with the bar in shallower water do not show this feature, but they show an upheaval at 0.18 Hz. It should be noted however that the errors in the spectral estimates are about 25%, so the features are just significant. The strong amplification of the spectral densities at lowest frequencies in the case with the bar on 4.0 cm still-water depth is striking however.

Symonds e. a. (1982) and Symonds and Bowen (1984) also find so-called quarter-wave resonances in their model. This resonance, at which the outgoing free wave has a maximum amplitude, occurs when a surface-elevation node is present at the mean
breakpoint position. This happens if the distance from the mean breakpoint position to the water line is about $2m+1$, in which $m$ is an integer, times a quarter of the wavelength. Let us try to identify these resonances. An assumption in Symonds e. a. (1982) and Symonds and Bowen (1984) is that the width of the zone in which the high-frequency waves start to break is small compared to the distance of the mean breakpoint to the waterline, which is not the case in our experiments. Furthermore, a direct comparison with their model is probably not justified because their model is dichromatic for the high-frequency waves. These limitations have to be borne in mind when studying the following.

The mean breakpoint position is not determined easily but it is estimated to be 4 m from the water line (from observations). If both resonance conditions are fulfilled, an extra amplification of the outgoing free low-frequency waves can be expected. For the experiment with a bar at 2.32 m from the still-water line (the experiment with the bar closest to the water line), the conditions give $m=n=2$, which results in a frequency of 0.19 Hz. Note that this frequency is also found in the half-wave resonance case described above. This can explain the peak observed in figure 5.7, but again, the assumptions underlying this calculation are doubtful. For other distances of the bar from the water
line these double-resonance cases occur at frequencies outside the frequency range of the figures.

The model results for the experiments with the bars (not shown here) show a strong decrease in spectral density at a frequency of 0.12 Hz, which can also be seen in figure 5.7 for the two experiments with the bar closest to the water line. This would mean a destructive interference between low-frequency waves reflected from the water line and seaward travelling low-frequency waves produced by the time varying breakpoint. However, the model does not predict the low-frequency spectra correctly as we have seen in section 5.1.2.

When we compare figures 5.6 and 5.7 the following can be said related to the model of Symonds et al (1982). One would expect the wave field close to the waterline to consist of standing low-frequency waves. The amplitudes of the outgoing low-frequency waves can be expected to be half that of the total signal. If these waves travel from the waterline to the horizontal part of the flume we do not expect that the shape of the energy-density spectrum would change because the waves can probably be treated as linear. Because the two figures don’t show the same spectral shapes, this means that something happens to these waves on their way to deeper water, next to shoaling effects. This is consistent with the statement made by Symonds and Bowen (1984) that the free outgoing wave is a superposition of waves reflected from the water line and those waves which are produced by the time-varying breakpoint and travel directly seaward. Note however that we can only say that something happens to the low-frequency waves when they travel seaward, not what happens to them.
5.2 COMPARISON WITH FIELD DATA

5.2.1 Introduction

In this section we concentrate on the low-frequency wave characteristics in the field observations and compare the data with the numerical model.

First we describe the data, how they were collected and what their general characteristics are. Then we will describe the low-frequency wave characteristics and compare the data with the model. The characteristics we study here are the variances of the incoming and outgoing low-frequency waves, the spectra of incoming and outgoing waves and correlations of these waves with the envelope of the high-frequency waves.

5.2.2 Data description

![Diagram of bottom profile with measurement locations](image)

Figure 5.8: Bottom profile, the arrows indicate the measurement locations.

The data were collected at the research facility of the American Society for Coastal Engineering at Duck, North Carolina, by J. Hubertz, R.A. Holman and R. Sternberg. They were given to the author by Jeff List, with approval of J. Hubertz and R.A. Holman.

The bottom profile was fairly two dimensional, although some irregularities existed. The
data were collected at two cross-shore arrays of instruments about 100 m apart. The bottom profile (fig 5.8) was measured between these two arrays. An instrument location consisted of diaphragm-type pressure sensors and Marsh-McBirney current meters. The current meters were arranged such that they measured currents directed cross-shore and along shore.

The incoming wind waves consisted of highly-grouped swell with a peak frequency of about 0.08 Hz and a root-mean-square wave height of about 40 cm. Although the beach was nearly two-dimensional and the swell was highly grouped the waves were not long crested.

At all measurement locations the group velocity of the wind waves is nearly equal to the phase velocity of free low-frequency waves. We will neglect the difference and decompose the wave field in incoming and outgoing waves only; the bound waves are not treated separately.

To test for the one-dimensionality of the low-frequency waves we calculated the ratios of the low-frequency (0.007 Hz < f < 0.03 Hz) variance of the along-shore current component to that of the cross-shore current component. This ratio ranged from 0.25 to 0.70 with an average value of 0.30. From this simple test we can conclude that the cross-shore motions are predominant, but that along-shore motions, perhaps including edge waves, are not negligible. This hampers a comparison between the data and the one-dimensional cross-shore model derived in this study.

A first look at the energy-density spectra of the low-frequency waves showed another problem. The energy-density spectra of the signals of all pressure sensors in the first array contained a single strong, narrow peak in the low-frequency wave band. These peaks were located at different frequencies for the different pressure sensors. They are at least suspicious because the instruments are located close to each other, which would mean that dramatic changes in the low-frequency wave climate would have occurred over a very short distance.

To eliminate these peaks the coherences between the wind-wave envelope and the low-frequency waves were calculated. Two bands of significant coherence could be found: One from 0.007 Hz to 0.032 Hz and another from 0.037 Hz to 0.052 Hz. The coherences in the gap from 0.032 Hz to 0.037 Hz were not significant for all wave-height meters. Hence we decided to choose one of the two bands for a comparison with the model. Because the peaks in the low-frequency wave energy densities occurred at 0.033 Hz, 0.055 Hz, we chose, still somewhat arbitrarily, the first band for the comparison. This means that we will consider
only the low-frequency wave characteristics in the frequency band ranging from 0.007 Hz to 0.030 Hz. All other low-frequency waves were filtered out.

![Figure 5.9: Root-mean-square wave heights of incoming low-frequency waves. The boxes indicate the measurements and the drawn line the model results.](image)

![Figure 5.10: Root-mean-square wave heights of outgoing low-frequency waves. The boxes indicate the measurements and the drawn line the model results.](image)

5.2.3 The low-frequency wave characteristics

First we concentrate on the variances of the incoming and outgoing low-frequency waves, then on the spectral energy of the outgoing low-frequency waves after which we deal with
correlations between incoming and outgoing waves on the one hand and the high-frequency wave envelope on the other.

In figures 5.9 and 5.10 the variances of the incoming and outgoing low-frequency waves are given as function of distance from the water line. The model does a very good job, only the initiation of breaking shows too much low-frequency wave energy in figure 5.10. This is ascribed to the sudden variation in root-mean-square wave height by breaking in the model, which causes strong gradients and hence a strong low-frequency wave production. A model calculation in which the spatial grid size in the model was reduced by a factor two showed exactly the same results, which shows that this is a 'real' model result.

![Figure 5.11: One-sided spectra of outgoing low-frequency waves at a depth of 4.33 m, station 7. --- measurements, --- model results.](image)

![Figure 5.12: One-sided spectra of outgoing low-frequency waves at a depth of 2.01 m, station 4. --- measurements, --- model results.](image)

![Figure 5.13: One-sided spectra of outgoing low-frequency waves at a depth of 1.52 m, station 2. --- measurements, --- model results.](image)
In figures 5.11-5.13 the energy-density spectra of the measured outgoing low-frequency waves are compared with the model results. The wave breaking starts just shoreward of the position for which figure 5.12 gives the spectra. The model results are much better than in the laboratory case. The main differences with the laboratory case are that in this case the beach slope is much lower, and that even at the seaward end the group velocity of the wind waves is nearly equal to the phase velocity of free low-frequency waves. We will elaborate on this in section 5.3.

Interesting to note is the change in the trend in the spectrum in the model results, which show a larger energy density in the waves with higher frequencies seaward of the surf zone (figures 5.11 and 5.12) and a larger energy density in the waves with lower frequencies shoreward (figure 5.13) of the initial break point. This transition region can be identified as the region with high energy in figures 5.9 and 5.10. It thus seems likely that the strong radiation-stress gradients are responsible, but the precise mechanism is unclear.

In figures 5.14 and 5.15 the correlations between respectively incoming and outgoing waves on the one hand and the wind-wave envelope on the other are given. The 95% confidence interval on zero correlation was 0.19 in both cases.

The strong negative correlation at zero time lag in figure 5.14 must again be attributed to the bound low-frequency waves. The low magnitude of the minimum observed correlation can perhaps be explained by edge waves or other along-shore wave motions, which were clearly present in the measurements. An important feature in figure 5.14 is
that the bound low-frequency waves seem to lag behind the envelope of the wind waves. We noticed the same thing in the laboratory case on the sloping bottom. However, in this case the wave breaking has not begun yet, so no break-point forced incoming free waves are present. Hence figure 5.14 clearly shows us that bound low-frequency waves on a sloping bottom are slightly lagging behind the envelope of the wind waves. This mechanism can thus be used to explain part of figure 5.4b and 5.4c. The model does not show this behaviour at this location, but closer to the water line, still outside the surf zone, it does (not shown here). Interesting in this respect is that the model of List (1992) also finds that the bound low-frequency waves are slightly lagging behind the wind-wave envelope on a sloping bottom. (Incidentally, this is probably the reason why the bound low-frequency waves, which become unrealistically high on shallow horizontal bottoms, behave realistically in the model with a shallow sloping bottom.) In figure 5.15, which gives the correlation of the outgoing waves and the wind-wave envelope, a significant correlation at a time lag of 100 s can be observed. This corresponds to the time needed for the bound waves to travel to the beach and for the free reflected wave to travel back to the pressure sensor. A significant correlation in the measurements can be observed at a time lag of 130 s, but this correlation is probably not real but due to the small amount of points used to calculate the correlation at such large time lags.

The model predicts the 100 s time lag satisfactorily, but the magnitudes of the correlations are overpredicted, which is ascribed to the presence of edge waves or other along-shore wave motions in the field measurements. The model also predicts a strong positive correlation at about 85 s which is attributed to a combination of that part of the bound waves which lag behind the wind-wave envelope and the positive correlation from the breakpoint-forced waves, both of which are reflected from the water line and have travelled back to the measurement location. A negative peak could be expected at a time lag of 60 s from the low-frequency waves which are produced by the time-varying breakpoint and have travelled directly to the measurement location. However, this peak is absent in both measurements and model results.
5.3 DISCUSSION

In this section we discuss the model performance and we elaborate on possible improvements. From the previous section we can conclude that the model does not predict the energy-density levels of the low-frequency waves correctly in the laboratory experiments. Let us briefly recapitulate the facts.

- The model overpredicts the energy densities at higher frequencies in the laboratory experiments.
- At the same time the energy densities at the lower frequencies in the laboratory experiments are slightly underpredicted.
- The laboratory experiments were done in a flume with a steep slope and in which the high-frequency waves reach deep water at the seaward boundary. In the field experiments the wind waves are still in shallow water at the seaward boundary of the domain of computation.
- The correlations between the low-frequency waves and the envelope of the wind waves are generally well predicted, although some significant features in the laboratory data are not predicted correctly by the model.

To investigate the influence of the depth at the seaward boundary the model was run for the two laboratory cases in which the depth in the horizontal part of the 'flume' varied from 0.2 m to 0.6 m. The same bottom slope was used as in the previous experiments. The results are given in figure 5.16. The energy densities at the higher frequencies increase dramatically with increasing depth of the horizontal part of the flume, while that at the energy densities at the lower frequencies decrease. Indeed, as could be expected from the model results for the field experiments, the deeper the water at the seaward boundary, the stronger the overprediction of the energy-density levels at the higher frequencies.

These results suggest that the travel of the outgoing waves to deeper water is not modelled correctly. However, in chapter 3 we have seen that these outgoing waves follow the analytical solution perfectly.

A possible factor is also bottom friction. One difference between the laboratory case and
the field measurements is that the long waves with higher frequency travel about 6 wave lengths over the sloping bottom in the laboratory experiments, while they travel about 2 wave lengths over a sloping bottom in the field situation. This means that bottom friction will be more important in the laboratory case and, because bottom friction is not included in the model, gives an overprediction in the laboratory situation. An explanation for the graphs in figure 5.16 can than be that in the case of a seaward depth of .2 m the long waves with higher frequencies travel about 4 wave lengths over the sloping bottom, which decreases the effect of bottom friction.

To test the influence of bottom friction in the present model some experiments were performed with a linear bottom friction. It should be noted here that these experiments do not give a good description of reality because in the model the frequency components are treated separately while bottom friction will act on the velocity of all frequency components together.

As can be expected, the energy densities decreased in these model experiments and this decrease was stronger at low-frequency waves with higher frequency. It was found impossible to decrease the energy density at higher frequencies to lower values than the
energy density at lower frequencies, which is a distinct feature of the measurements. However, again, these model experiments don’t give a good description of reality. A more realistic incorporation of bottom friction in the present theory will be discussed below.

It is important to note here that the model of List also shows a strong increase of energy density of the outgoing low-frequency waves with higher frequencies (private communication). His model performs even a little worse than the present model in predicting the outgoing wave spectrum. Also the bichromatic model of Schäffer (1991) predicts an increase of energy density with the long-wave frequency (the difference frequency of the two primary waves in this model), but the results of a bichromatic model can perhaps not be transferred that easily to a full spectral model. Interesting is that Schäffer compared his model to laboratory data from Kostense (1984) and found an overprediction of the model of about a factor 2 for the higher frequency waves, which he attributed to the absence of bottom friction in his model.

List performed some experiments with his model run in a bichromatic mode. He found that the bound low-frequency wave part was mainly responsible for the strong increase of energy density at higher frequencies, and not the waves produced by the time-varying breakpoint. But again, a bichromatic model may behave differently from a full spectral model.

Another major point in the present model related to this problem can be the linearity of the low-frequency wave model. Perhaps the assumption of linearity is too strong to be able to predict the correct energy-density levels. A full non-linear model has been developed by Roelvink (private communication). A first model run simulating the laboratory experiment with a plane beach showed the correct outgoing wave spectrum. So linearity can be the problem. (Roelvink notes a correct description of bottom friction is vital in his model.)

To investigate the influence of the assumption of linearity a little further let us have a look at the terms which are left out. The nonlinear low-frequency wave equation reads (see appendix A)

$$\zeta_x - g[h\zeta_x]_x = \frac{1}{\rho} - S_{xx} - [\zeta U]_x + [hUU_x]_x - \frac{1}{\rho} \left[ \frac{\zeta S_x}{h + \zeta} \right]_x$$

(5.3.1)

If we neglect the nonlinear terms and also the surface elevation compared to the local still-water depth we arrive at (see appendix A)
\[ \zeta_{1w} - g[h\zeta_{1w}]_x = \frac{1}{\rho} S_{1w} \]  \hspace{1cm} (5.3.2)

This equation is the basis of the present theory as developed in chapter 3 and tested in this chapter. Let us now have a look at the neglected terms.

- The term \([\zeta U]_x\), which describes nonlinear interactions between low-frequency waves. It arises from the continuity equation and can be interpreted as the gradient of the extra mass flux of the low-frequency waves due to the phase relation between surface elevation and particle velocity. As we have seen in the experiments, in deeper water the outgoing waves have an energy density which are a factor 10 or so higher than that of the incoming waves. This means that the outgoing waves will contribute strongly to this term in deeper water, which results in a mass flux seawards. However, the influence on the details of the energy-density spectrum of this term is not clear.

- The term \([hUU]_x\), which also describes nonlinear interactions between low-frequency waves. The term arises from the momentum equation and can be interpreted as the gradient of the convection of low-frequency momentum. Also the influence of this term on the energy-density spectrum of the low-frequency waves cannot be understood easily.

- The term \(-\frac{1}{\rho} \left[ \frac{\zeta S_x}{h + \zeta_x} \right]\), which describes the nonlinear interaction between the low-frequency waves and the forcing. Because the radiation stress has a variance-density spectrum which has higher energy densities at lower frequencies this can be an important term for the improvement.

The present low-frequency wave model can be improved to incorporate these nonlinear effects up to second order. The second-order low-frequency wave equation, as can be obtained from equation 5.3 reads (see appendix A):

\[ \zeta_{2w} - g[h\zeta_{2w}]_x = \frac{1}{\rho} S_{2w} - [\zeta U]_x + [hUU]_x = \frac{1}{\rho} \left[ \frac{\zeta_1 S_1}{h} \right]_x \]  \hspace{1cm} (5.3.3)

The first term on the right is the second-order forcing. Physically it contains the variation of the local mean water depth for the wind waves due to the low-frequency waves. Note
that the next approximation of the radiation stress due to the wind waves is at fourth order in wind-wave steepness, so it will be contained in $S_3$ (see appendix A). The influence of the low-frequency waves on the local mean water depth can be written as

$$S_2 = -\frac{\partial S_1}{\partial h} \zeta_1$$  \hspace{1cm} (5.3.4)

To use equations (5.3.3) and (5.3.4) we first solve the linear model as described in chapter 3. Then this first-order solution is used to calculate the forcing of the second-order problem. Note that the nonlinear terms will turn out to be convolutions in the frequency domain: The nonlinear interactions of all first-order low-frequency components have to be taken into account.

Important is also that we can incorporate bottom friction more realistically in this nonlinear model. Roelvink (private communication) found that the correct representation of bottom friction is vital for his model. The bottom friction can be represented as

$$\tau_b = -\frac{1}{2} \rho_f f_w |U| U$$  \hspace{1cm} (5.3.5)

in which $f_w$ is the friction coefficient. The magnitude of the low-frequency wave velocity can be calculated as

$$|U| = \sqrt{2 \int G_{ww} d\omega}$$  \hspace{1cm} (5.3.6)

in which $G_{ww}$ is the variance density of the velocity of the low-frequency waves. In our case this variance density will be calculated from the first-order solution. The bottom friction can be used as a sink term in the Fourier-Stieltjes transformed form of equation 5.3.3 as

$$-\frac{1}{2} f_w (|U| dU_2)_x$$  \hspace{1cm} (5.3.7)

From Mei (1989, p. 415) we can estimate that $f_w$ ranges from 0.01 to 1, so this will be an important term.

We thus have seen that the linear model can be extended in theory to handle nonlinear
effects, and also bottom friction can be incorporated. An implementation of this nonlinear model is beyond the scope of this thesis, but it seems a valuable extension.
CHAPTER 6

SUMMARY AND CONCLUSIONS

The main aim of this thesis was to develop a model for the generation of low-frequency waves due to wind waves, which are shoaling and breaking on a not too steep beach. In nature these low-frequency waves have a wave period of the order of a few minutes. We concentrated on shore-normal wave motion, but it is shown that an extension to three-dimensional wave motions on a beach can be made. In the following a summary of the thesis is presented which is followed by some concluding remarks and a recommendation.

The developed model is new in that it uses as input the energy spectrum of the wind waves and calculates in the frequency domain. In this way it is shown that a specific realization in the time domain of the wind-wave field with a certain energy-density spectrum has no influence on the energy-density spectrum of the low-frequency waves. Important aspects of the model are that it is linear in the low-frequency wave surface elevation and that the wind-wave spectrum is assumed to be narrow.

The results of the model are compared to laboratory and field data. To perform the laboratory experiments a new method to calculate the control signal for the wave board for the generation of random waves up to second order has been developed. The method uses the theory of multiple scales. The advantage of this new method over the conventional method is the increase in computational speed and hence a lower cost price.

The data from the laboratory experiments and from the field measurements are analyzed with the help of complex harmonic principal components to reduce the influence of noise. It is shown that this method for data analysis works satisfactorily.

The comparison with the laboratory data showed that the low-frequency waves with higher frequencies are strongly overpredicted. The reason for this overprediction is not
entirely clear. It appears that the error in the model originated in that part of the flume where the primary waves were shoaling from deeper water. Two possible courses are the neglect of bottom friction and the linearity of the low-frequency wave model. The correlations of the incoming and outgoing low-frequency waves with the envelope of the primary waves was reproduced correctly by the model, although some significant features in the data were not resolved by the model.

The comparison with the field data showed good results, both for the energy-density spectra of the low-frequency waves and for the correlations of the low-frequency waves with the envelope of the wind waves. However, it should be noted that some current meters showed particle velocities in the along-shore direction which were only slightly lower than those in the cross-shore direction, which hampered the comparison with the one-dimensional model.

The difference in model performance in laboratory and field cases can be due to the fact that long waves with higher frequencies travel a distance of about 6 wave lengths over the sloping bottom in the laboratory case, while they travel only 2 wave lengths in the field case. This means that bottom friction will have more influence in the laboratory case than in the field case, which can result in an overprediction of the outgoing low-frequency wave energy densities in the former.

Concluding one can say that the model predicts the correlations between the low-frequency waves and the envelope of the wind waves correctly, which shows that the main physical phenomena are present. The model does predict the correct order of magnitudes for the energy-density spectra of the low-frequency waves compared with the field experiment, but not compared with the laboratory experiments.

To increase the predictability nonlinear models with bottom friction are needed. A good example of such a model is that of Roelvink (1991, private communication). It is shown that the present model can be extended to incorporate both effects.
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APPENDIX A

DERIVATION OF THE LOW-FREQUENCY WAVE EQUATION

In this appendix a short derivation of the low-frequency wave equation is given. We start from the continuity equation and the momentum equation as given by Phillips (1979). He obtained these equations by integrating the Navier-Stokes equations over the instantaneous water depth and averaging the resulting equations over a mean high-frequency wave period. He then finds:

\[ \zeta_t + \frac{\partial}{\partial x_i} [(\zeta + h) U_i] = 0 \]  \hspace{1cm} (A.1)

and

\[ [(\zeta + h) U_i]_t + \frac{\partial}{\partial x_j} [(\zeta + h) U_j U_i] + g(\zeta + h) \frac{\partial}{\partial x_i} \zeta = -\frac{1}{\rho} \frac{\partial}{\partial x_i} S \]  \hspace{1cm} (A.2)

in which \( U_i \) is the total averaged horizontal velocity in direction \( i \) of the waves, which can be written as

\[ U_i = \frac{1}{\zeta + h} \int_{-h}^{\eta} u_i \, dz \]  \hspace{1cm} (A.3)

in which \( u_i \) is the horizontal velocity in direction \( i \) of the water parcels. The other variables in the conservation equations (A.1) and (A.2) are the instantaneous surface elevation \( \eta \), the mean surface elevation \( \zeta \), which is the low-frequency wave surface elevation in this case, radiation stress \( S \), the density \( \rho \) and the acceleration of gravity \( g \). The radiation stress is
the stress tensor given by

\[ S_y = \int_{-h}^{h} (\rho u_i u_j + p \delta_{ij})\,dz - \frac{1}{2} \rho g (\zeta + h)^2 \delta_{ij} - \rho (\zeta + h) U_{\eta_i} U_{\eta_j} \]  

(A.4)

in which \( U_{\eta_i} \) arises from the mass flux of the high-frequency waves and is given by

\[ U_{\eta_i} = \frac{1}{\zeta + h} \int_{-h}^{h} u_{\eta_i} \,dz \]  

(A.5)

Equations (A.1) and (A.2) can be used to obtain

\[ \zeta u - g \frac{\partial}{\partial x_i} \left( h \frac{\partial \zeta}{\partial x_j} \right) = \frac{1}{\rho} \frac{\partial^2 S_y}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_i} \left( \zeta U_j \right) + \frac{\partial}{\partial x_i} \left( h U_j \frac{\partial U_i}{\partial x_j} \right) - \frac{1}{\rho} \frac{\partial}{\partial x_i} \left[ \zeta \frac{\partial S_y}{\partial x_j} \right] \]  

(A.6)

which is more convenient for us as we will see. Equation (A.6) shows that the high and low-frequency waves are in close interaction. Also the radiation stress contains, apart from the high-frequency terms, contributions from the low-frequency waves.

In this thesis we deal with the linearized version of (A.6) which can be obtained as follows. First we introduce dimensional constants which indicate the order of magnitude of the quantities in equation (A.6). Then we will form non-dimensional quantities which indicate the order of magnitude of the terms in the equation. We proceed by introducing perturbation expansions of the physical quantities after which first- and higher-order equations can be derived.

We introduce a characteristic velocity scale \( V \), a low-frequency surface elevation scale \( A_L \), a high-frequency surface elevation scale \( A_s \), a high-frequency wavenumber scale \( k \), a low-frequency wavenumber scale \( K \), a low-frequency frequency scale \( \Omega \) and a depth scale \( D \) according to

\[ U = V U' \]
\[ \zeta = A_L \zeta' \]
\[ \eta = A_s \eta' \]
\[ h = D h' \]
\[ S = \rho g A_s^2 S' \]  
(A.7)

The prime indicates the dimensionless scaled variables. To couple the length and time scales we use the phase velocity of the low-frequency waves, which is of the order \( \sqrt{gD} \). We then have

\[ \Omega = \sqrt{gD} K \]  
(A.8)

and

\[ V = \sqrt{gD} \frac{A_L}{D} \]  
(A.9)

We then find (omitting the primes)

\[
e_{L} \zeta \nu - e_{L} \frac{\partial \zeta}{\partial x_i} \delta_{L} \frac{\partial \zeta}{\partial x_i} \frac{\partial h}{\partial x_i} = e_s \delta_{L} \frac{\partial S_y}{\partial x_i} - e_{L} \delta_{L} \frac{\partial (\zeta U_i)}{\partial x_i} + e_{L} \delta_{L} h \frac{\partial (U_i \frac{\partial U_i}{\partial x_j})}{\partial x_j} +
\]

\[
+ \delta_{L}^2 h \frac{\partial (U_i \frac{\partial U_i}{\partial x_j})}{\partial x_j} - e_s \delta_{L} \frac{\partial S_y}{\partial x_i} \frac{\partial x_i}{\partial x_i} \frac{\partial x_i}{\partial x_j} \frac{\partial x_i}{\partial x_j} +
\]

\[
- e_{L} \delta_{L} \frac{\partial S_y}{\partial x_i} \frac{\partial x_i}{\partial x_i} \frac{\partial x_i}{\partial x_j} \frac{\partial x_i}{\partial x_j} +
\]

\[
+ \frac{\zeta \partial S_y}{\partial x_j} \frac{\partial x_j}{\partial x_j} \frac{\partial x_j}{\partial x_j} \frac{\partial x_j}{\partial x_j} \frac{\partial x_j}{\partial x_j} +
\]

\[
+ \frac{\delta_{L}^2 h}{\rho (\delta_{L} \zeta + h)^2} \frac{\partial S_y}{\partial x_j} \frac{\partial x_j}{\partial x_j} \frac{\partial x_j}{\partial x_j} \frac{\partial x_j}{\partial x_j} \frac{\partial x_j}{\partial x_j}
\]

(A.10)

in which we introduced the following order-of-magnitude parameters

\[ e_s = K A_s \]
\[ e_L = K A_L \]
\[ \delta = \frac{A_S}{D} \]
\[ \delta_L = \frac{A_L}{D} \]  \hspace{1cm} (A.11)

In the situations studied here equation (A.10) can be approximated to the first four terms. This can easily be verified (see chapter 5): for the field experiment we have at a depth of 2 m:

\[ e_s = 0.03 \]
\[ e_L = 0.0003 \]
\[ \delta = 0.1 \]
\[ \delta_L = 0.01 \]
\[ |\nabla h| = 0.03 \]

and for the laboratory experiments we find at a depth of 20 cm:

\[ e_s = 0.01 \]
\[ e_L = 0.001 \]
\[ \delta = 0.2 \]
\[ \delta_L = 0.02 \]
\[ |\nabla h| = 0.04 \]

We then obtain in physical variables

\[ \zeta + \frac{\partial}{\partial x_i} \left( gh \frac{\partial \zeta}{\partial x_i} \right) = \frac{1}{\rho} \frac{\partial^2 S_y}{\partial x_i \partial x_j} \]  \hspace{1cm} (A.12)

in which the radiation stress is now given by

\[ S_y = \int_{-h}^{h} (\rho u \mu \rho \delta_y) dz - \frac{1}{2} \rho gh^2 \delta_y \]  \hspace{1cm} (A.13)

This equation shows that in the linearized case the radiation stress is determined only by the
high-frequency waves. The expression can be evaluated as

\[ S_y = \frac{1}{2} \rho g R^2 \left[ \left( n - \frac{1}{2} \right) \delta_y + n \frac{k k_i}{k_i} \right] \]  
(A.14)

which reduces to

\[ S = \left( 2n - \frac{1}{2} \right) \frac{1}{2} \rho g R^2 \]  
(A.15)

in the one-dimensional case. The one-dimensional form of equation (A.12) is the basis of the low-frequency wave model which is developed in chapter 3.

To obtain higher order equations for the low-frequency wave surface elevation we expand the surface elevation, the velocity and the radiation stress as

\[ \zeta = \sum_{n=1}^{\infty} \lambda^n \zeta_n \]  
(A.16)

\[ U = \sum_{n=1}^{\infty} \lambda^n U_n \]  
(A.17)

\[ S_y = \sum_{n=1}^{\infty} \lambda^n S_{y_n} \]  
(A.18)

in which \( \lambda \) is a nonlinearity parameter. Now we have to choose the order of magnitude of all the nonlinearity parameters \( e_s, e_L, \delta, \delta_L, \lambda \) and also \( h_x \). In the light of the present experiments and the field data we choose

\[ \lambda = \mathcal{O}(\delta) \]
\[ \delta_L, e_s, h_x = \mathcal{O}(\delta^3) \]
\[ e_L = \mathcal{O}(\delta^3 - 4) \]

We then find for the first-order low-frequency wave equation

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\[ \zeta_{1y} - \frac{\partial}{\partial x_i} \left( gh \frac{\partial \zeta_1}{\partial x_i} \right) = \frac{1}{\rho} \frac{\partial^2 S_{1y}}{\partial x_i \partial x_j} \]  

(A.19)

which is equation (A.12). For the second-order equation we find

\[ \zeta_{2y} - \frac{\partial}{\partial x_i} \left( gh \frac{\partial \zeta_2}{\partial x_i} \right) = \frac{1}{\rho} \frac{\partial^2 S_{2y}}{\partial x_i \partial x_j} - \frac{\partial}{\partial x_i} \left( \zeta_1 U_i \right) + \frac{\partial}{\partial x_i} \left( h U_i \frac{\partial}{\partial x_j} U_i \right) - \frac{1}{\rho} \frac{\partial}{\partial x_i} \left[ \zeta_1 \frac{\partial S_{1y}}{\partial x_j} \right] \]  

(A.20)
Curriculum vitae


Vervolgens studeerde ik sterrenkunde aan de Rijks Universiteit te Leiden. Na het behalen van kandidaats in sterrenkunde, natuurkunde en natuurkunde met scheikunde in 1984 vervolgde ik de studie sterrenkunde aan de Universiteit van Amsterdam. Uiteindelijk studeerde ik daar in 1987 af in de theoretische natuurkunde. Het afstudeeronderwerp was het mechanisme van de volumeviscositeit in relativistische gassen en in mengsels van straling en materie.

In november 1987 begon ik als onderzoeker in opleiding bij de werkgemeenschap Meteorologie en Fysische Oceanografie (MFO) van de Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO). Ik was gestationeerd aan de Technische Universiteit Delft onder de supervisie van prof. dr. ir. J.A. Battjes. Dit werk is nu afgerond met dit proefschrift.

Momenteel ben ik werkzaam als postdoc bij de Stichting Ruimte Onderzoek Nederland (SRON), ook een onderdeel van NWO. Mijn werkkamer is bij het Instituut voor Marien en Atmosferisch Onderzoek (IMAU) van de Rijks Universiteit Utrecht. Het onderzoek behelst data-assimilatie van grootschalige oceaanstromingen, waarbij ik me vooral richt op het afsnoeringsproces van grootschalige wervels bij instabiele oceaanstromingen.