THE HYDRODYNAMIC FORCES AND SHIP MOTIONS IN OBLIQUE WAVES
(DE HYDRODYNAMISCHE KRACHTEN EN SCHEEPSBEWEGINGEN IN SCHUIN INKOMENDE GOLVEN)

by

DR. IR. J. H. VUGTS

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VOORWOORD

Gedurende een periode van ongeveer vijf jaar, heeft de schrijver van dit rapport zich bezig gehouden met een uitgebreid onderzoek waarvan het uiteindelijk doel was om een betrouwbare methode te ontwikkelen voor het voorspellen van de krachten op en de bewegingen van een schip veroorzaakt door golven.

Het probleem van een lichaam in golven kan met de bestaande hydrodynamische theorie geformuleerd worden, maar het vinden van een numerieke oplossing voor het algemene drie-dimensionale geval bij voorwaarzegde snelheid zou zeker nog jaren van nauwgezette inspanning vergen en zelfs dan is het nog geenszins zeker dat de resultaten dit rechtvaardigen. Daarom werd er, om op niet te lange termijn een praktisch bruikbare oplossing te verkrijgen, de voorkeur aan gegeven om een vereenvoudigde methode toe te passen. Hiervoor werd de „strip theorie“, in feite een quasi drie-dimensionale methode, uitgekozen.

Het genoemde onderzoek programma bestond uit een theoretisch deel, waarin het wiskundig model werd afgeleid en drie reeksen modelverslagen om de resultaten te verifiëren.

De eerste stap van dit programma, die het twee-dimensionale geval omvatte, is reeds gerapporteerd in twee publikaties van het Scheepsstudiecentrum: de rapporten no. 112 S en no. 115 S. Gedetailleerde rapporten over de tweede en de derde stap, waarin het drie-dimensionale geval, respectievelijk zonder en met voorwaartse snelheid, wordt behandeld, zullen in de nabije toekomst gepubliceerd worden.

Het voor U liggende rapport, dat tevens het proefschrift van de auteur is, geeft een algemeen overzicht van het gehele onderzoek.

Zoals de auteur met nadruk vermeldt, is de gevonden oplossing niet meer dan een praktische benadering van de werkelijkheid. Desondanks mag gesteld worden, zelfs wanneer de beperkingen van deze methode in de beschouwing betrokken worden, dat er met deze oplossing en het rekenprogramma dat daarmede is ontwikkeld nu een uiterst nuttige methode beschikbaar is voor het opstellen van een prognose van alle scheepsbewegingen (schrikken uitgezonderd).

Van de experimenten werd de eerste serie, met cilindrische vormen, uitgevoerd bij het Laboratorium voor Scheepsbouwkunde aan de Technische Hogeschool te Delft, de tweede en derde serie, met een scheepsmodel, werden uitgevoerd in de Zeevangstank van het Nederlands Scheepsbouwkundig Proefs station te Wageningen. Voor deze lastigenomen proeven werd door het N.S.P. een subsidie beschikbaar gesteld, hetgeen hier met dank vermeld wordt.

PREFACE

During a period of about five years, the author of this report has been engaged on an extensive research programme, with the ultimate goal to obtain a reliable prediction method for the forces on and the motions of a ship caused by waves.

The problem of a body in waves can be formulated with the existing hydrodynamic theory, but finding a numerical solution of the general three-dimensional case at forward speed would certainly require years of elaborate effort and even then it is by no means certain that the results will justify this. Therefore, to obtain a practical solution at a relatively short notice, it has been preferred to use a simplified method. For this the “strip theory”, in fact a quasi three-dimensional method, has been selected.

The research programme mentioned consisted of a theoretical part in which the mathematical model was derived and three series of model experiments to verify these results.

The first step of this programme, treating the two-dimensional case, has already been reported in two publications of the Ship Research Centre: the reports no. 112 S and no. 115 S. Detailed reports on the second and the third step, in which the three-dimensional case, at zero speed and with forward speed respectively, were investigated, will be published in the near future.

The present report, that is also the author’s doctors thesis, gives a general survey of the results of all investigations.

As the author stresses, the solution found is no more than a practical approximation of reality. Yet, even when the restrictions of the method are taken into consideration, it may be stated that with this solution and the computer program that has been developed with it, a very powerful means to predict all ship motions (except surge) in oblique waves is now available.

The first series of experiments, with cylinder shapes, has been performed at the Shipbuilding Laboratory of the Delft University of Technology, the second and third series, with a ship model, were performed in the Seakeeping Basin of the Netherlands Ship Model Basin at Wageningen. For these latter experiments a grant was obtained from the N.S.M.B., which is gratefully acknowledged here.

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NOMENCLATURE

1 General conventions
- The subscript $a$ with a harmonically varying variable indicates that the amplitude of that variable is meant.
- The subscripts $i$, $j$ or $k$ are used for a direction or a degree of freedom in a Cartesian system; in general they vary from 1 to 6, unless specified otherwise.
  For the hydrodynamic coefficients also letters are used instead of figure subscripts; then $x = 1$, $y = 2$, $z = 3$, $\phi = 4$, $\theta = 5$ and $\psi = 6$.
- An accent ' indicates that the quantity concerned must be taken per unit length of the body; a double accent '"' means the quantity per unit of area of the body surface.
- The phase difference of a harmonic variable $a$ with respect to another harmonic variable $b$ is denoted by $\epsilon_{ab}$; positive when $a$ leads $b$.

2 Coordinate systems

$OX_1X_2X_3$ space fixed system of Cartesian coordinates (fixed to the earth); the ship's track coincides with $OX_1$; the vertical axis is positive downwards.

$ox_1x_2x_3$ system parallel with $OX_1X_2X_3$, but translating with the ship's speed $V$; $ox_1x_2$ coincides with the water surface at rest; the origin $o$ is situated in the longitudinal plane of symmetry at mid-length of the body (see figure 4.1).

$Gxyz$ system parallel with $ox_1x_2x_3$ with its origin in the ship's centre of gravity (see figure 4.1).

3 List of symbols

Symbols not included in the list below are only used at a specific place and are explained where they occur.

$a$ scale factor in the conformal transformation of a section contour to the unit circle.

$a_{2n+1}$ transformation coefficients of the conformal transformation of a section contour to a semi-circle.

$a_{ij}$ hydrodynamic coefficient of the mass term in the $i$-th force equation due to motion in the $j$-th mode; expressed in the system $ox_1x_2x_3$ and for $V \neq 0$.

$b_{ij}$ hydrodynamic coefficient of the damping term in the $i$-th force equation due to motion in the $j$-th mode; expressed in the system $ox_1x_2x_3$ and for $V \neq 0$.

$c_{ij}$ hydrostatic restoring coefficient in the $i$-th force equation due to a static displacement in the $j$-th mode at $V = 0$; expressed in the system $ox_1x_2x_3$.

$a_{ij}^0, b_{ij}^0$ the coefficients $a_{ij}, b_{ij}$ at zero forward speed.

$b$ half width of a section at the waterline.

$f_j$ generalized direction cosine on $S_0$, defined in (3.1.12).

$g$ acceleration of gravity.

$k$ wave number.

$m$ mass of the body.

$p$ pressure at a point of the body surface.

$p_j$ pressure $p$ resulting from the body motion $s_j$.

$p_{ij}, q_{ij}, r_{ij}$ the coefficients $a_{ij}, b_{ij}, c_{ij}$ after transformation from $ox_1x_2x_3$ to $Gxyz$.

$p_{2m}, q_{2m}$ expansion coefficients in the multipole expansion for the velocity potential of the oscillation problem.

$s_j$ $j$-th mode of motion of the body.

$u, v, w$ wave orbital velocities in $x$-, $y$- and $z$-direction, respectively.

$(x_{1b}, x_{2b}, x_{3b})$ coordinates of a point on the body surface $S_0$ in $ox_1x_2x_3$.

$(x_{1g}, o, x_{3g})$ coordinates of the centre of gravity in $ox_1x_2x_3$.

$x$ surge motion.

$y$ sway motion.

$z$ heave motion.
\( A \)  
\( A_{WL} \) cross sectional area.  
\( A_{ij} \) waterline area of the body.  
\( B_{ij} \) coefficient of the hydrodynamic force in the \( i \)-th direction which is in phase with the body acceleration in the \( j \)-th direction.  
\( C_{ij} \) coefficient of the hydrostatic force in the \( i \)-th direction due to a displacement in the \( j \)-th direction.  
\( B \) breadth of the body; centre of buoyancy.  
\( C_0 \) immersed contour of a cross section in its middle or rest position.  
\( F \) force or moment in the system \( o x_1, x_2, x_3 \).  
\( F_n = V_{j}/(gL) \) Froude-number.  
\( G \) centre of gravity of the body.  
\( \bar{G} \) metacentric height.  
\( I_{ij} \) mass moment or product of inertia.  
\( I_{WL} \) longitudinal moment of inertia of the waterline.  
\( K, M, N \) moment components in \( Gxyz \).  
\( K_w, M_w, N_w \) wave moments in the system \( Gxyz \).  
\( L \) length of the body.  
\( L_{pp} \) length between perpendiculars.  
\( L_o \) immersed length of the body in its middle or rest position.  
\( M \) metacentre.  
\( \bar{O}B \) distance of centre of buoyancy below the waterline.  
\( S \) instantaneous immersed surface of the body.  
\( S_0 \) immersed surface of the body in its middle or rest position.  
\( S_{WL} \) static longitudinal moment of the waterline.  
\( T \) draught of the body.  
\( T^* \) effective mean draught of a section used in the calculation of wave exciting forces (see section 5.1 and 5.3).  
\( V \) constant forward speed of the body.  
\( X, Y, Z \) force components in \( Gxyz \).  
\( X_{w}, Y_{w}, Z_{w} \) wave forces in the system \( Gxyz \).  
\( X_{wk} \) wave exciting force in the \( k \)-th direction, expressed in the system \( o x_1, x_2, x_3 \).  
\( (X_{wk})_{F.K} \) Froude-Kriloff part of \( X_{wk} \), neglecting the body-wave interaction.  
\( (X_{wk})_{d} \) diffraction part of \( X_{wk} \), resulting from the body-wave interaction.  
\( X_{wkc} \) cosine component of \( X_{wk} \) (see eq. (5.3.2)).  
\( X_{wks} \) sine component of \( X_{wk} \) (see eq. (5.3.2)).  
\( \alpha_w \) wave slope.  
\( \gamma = \omega V/g \) wave-form parameter.  
\( \varepsilon \) phase angle of \( j \)-th mode of motion.  
\( \varepsilon_{wk} \) phase angle of wave exciting force in the \( k \)-th direction.  
\( \zeta \) wave elevation.  
\( \theta \) pitch motion; Eulerian angle.  
\( \lambda \) wave length.  
\( \mu \) angle of wave incidence.  
\( \varrho \) specific mass of water.  
\( \phi \) roll motion; Eulerian angle.  
\( \Phi(x_1, x_2, x_3; t) \) velocity potential.  
\( \Phi_d(x_1, x_2, x_3; t) = \Phi(x_1, x_2, x_3)e^{-i\omega t} \) diffraction potential of the incident waves about the restrained body.  
\( \Phi_j(x_1, x_2, x_3; t) = \Phi_j(x_1, x_2, x_3)e^{-i\omega t} \) velocity potential for the oscillatory motion \( s_j \) in still water.  
\( \varphi_j \) real part of \( \varphi_j \).  
\( \varphi_{ji} \) imaginary part of \( \varphi_j \).  
\( \varphi_0(x_1, x_2, x_3) \) disturbance potential which accounts for the form of the body in steady movement in still water (see eq. (4.4.1)).  
\( \Phi_s(x_1, x_2, x_3) = \varphi_s(x_1, x_2, x_3) \) steady velocity potential for the body moving at constant speed \( V \) in still water.
\( \Phi_w(x_1, x_2, x_3; t) = \phi_w(x_1, x_2, x_3) e^{-i \omega t} \) velocity potential of the incident wave system.

\( \varphi_{2m} \) multipole potential.

\( \psi \) yaw motion; Eulerian angle.

\( \omega \) circular frequency of motion; wave frequency.

\( \omega_e \) circular frequency of encounter.

\( V \) volume of displacement; gradient.

\( \Delta \) Laplace operator.

\( \mathfrak{b} \) principal value integral.
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Summary

It is the intention of this report to come to a practical solution of the problem of the hydrodynamic forces and ship motions in waves in which all essential features are maintained and which produces quantitative results of sufficient accuracy for most practical applications. After a historical survey of the subject the introduction deals with the conversion of the physical problem into a mathematical model. This modelling includes a statistical method which is capable of describing the state of the sea and of the induced ship motions. The simplifications required and the consequences of the formulation are indicated. The motions are obtained by superimposing the responses to individual sine waves. The equations of motion for this stationary harmonic process are derived and their interpretation is discussed.

The problem of reacting forces induces in the ship's oscillations in still water and secondly the wave exciting forces on the restrained ship. Before discussing each sub-problem in detail, the subject is viewed from the standpoint of hydrodynamics. A general solution for a body moving at a certain forward speed has not been found so far. But solving the problem some relations can be derived which have to be satisfied by the approximate solution which is to be constructed. The hydrodynamic coefficients of the fluid reactive forces are determined in three steps. The first is a solution for two-dimensional oscillations of infinitely long cylinders. The second combines these values to coefficients for a three-dimensional body at zero forward speed, while the third generalizes this process to include the effects of speed of advance; both in an approximate way. Toward this end a strip theory is used. Except for the longitudinal surge all degrees of freedom are covered. In each stage the theoretical results are compared against experiments, specially conducted to verify them. The equations found at forward speed differ in several respects from the equations for the heave and pitch coefficients, which have been derived earlier.

The calculation of the wave exciting forces proceeds from the relations found by classical hydrodynamics. To obtain practical results for the forces in oblique waves again strip theory is used. It is shown that the resulting equations coincide with the usual expressions, derived by the so-called relative motion hypothesis, as long as the forward speed is zero. But at speed of advance they differ fundamentally. Experiments have been conducted to verify the calculations at zero speed, but at speed experimental data is scarce. The subject is concluded by a discussion of some special topics. Where possible the results obtained are compared with available information, while further the author gives his views on the influence of certain effects and on the use of the method described in practical applications.

1 Introduction

1.1 The historical development of ship motion studies

"In a broad sense the laws of nature are Gaussian". Under this heading St. Denis and Pierson [1.1] began their considerations on the statistical description of sea waves and ship motions, which were published in 1953 and opened a new and modern era of seakeeping research.

In fact the history of the study of ship motions dates back as far as the first part of the 18th century. At that time Euler carried out a study on the motions of ships in still water. This study was not published until when it appeared in Latin in his work "Scientia Navalis" [1.2], edited in two volumes in St. Petersburg in 1749. The first volume has a typical mathematical framework with lemma's, corollaries and propositions. In the second things are elucidated in a more descriptive style. At about the same time, 1746, the French scientist Bouguer published a comprehensive book "Traité Du Navire, De Sa Construction Et De Ses Mouvements" [1.3], which contained similar considerations on ship motions. He noted that he was familiar with the fact that Euler had been working on the subject, but that he had not yet been able to lay hands on his results. In waves, motions were examined for the first time by Daniel Bernoulli [1.4]. In 1757 he won the Prix De l'Académie Des Sciences in Paris for a study on "la manière de diminuer le plus qu'il est possible le roulis et le tanguage d'un navire, sans qu'il perde sensiblement par cette diminuation aucune de ses bonnes qualités que sa construction soi lui donner." The present author has not succeeded in tracing this work. It seems [1.5] that Bernoulli examined forced oscillations in
waves, including the phenomenon of resonance. Unfortunately his ideas on wave motion were wrong so that he arrived at incorrect conclusions. For a long time his work was, however, considered as classic. When one reads the preface of Bouguer’s book, it is not strange that the subject was initiated by mathematicians and physicists. The controversy between practical shipbuilders and people who want to introduce theoretical considerations and analytical methods to give a solid foundation to naval science, replacing experience and working at hazard, is at least two hundred years old. The great men of the 18th century merely applied their universal minds and strong mathematical tools as a sport to numerous physical problems. Among these, ship motions was only one example.

Many, however, consider the actual beginning of the investigations to be in 1861 when William Froude [1.6] published his paper “On The Rolling Of Ships”. The ship was assumed sailing broadside on to the waves and was treated as a tiny raft or cork, which had to follow the wave slope and the orbital motion of the wave particles. This formulation holds true only for waves which are very long in relation to the ship's transverse dimensions. Or, stated otherwise: it is only correct in the range of very low motion frequencies. This is an important fact to note, for it is typical of most engineering approaches in that they are only valid in certain ranges of practical interest. Froude’s work was continued by his son R. E. Froude with a paper in 1896 “The Non-Uniform Rolling Of Ships” [1.7], where the motion in regular and irregular waves was examined in a ‘modern’ way, in a manner which resembles the general analysis of to-day in several aspects, up to the principle of linear superposition for harmonic waves and for the forced part of the rolling motion!

A large step forward was next made by Kriloff with his famous papers in 1896 [1.8] and 1898 [1.9]. The latter dealt with six degrees of freedom. He stated that he could approximate the actual pressure on the ship's surface by the corresponding pressure in the wave structure, not being disturbed by the presence of the ship. Implicitly the same assumption had been made by W. Froude in his earlier and more restricted work. Since then this hypothesis has been known as the Froud->Kriloff hypothesis and has dominated nearly all work in ship motion theory up to 1953, and in many cases even more recently. Speaking in modern terms Kriloff computed the wave exciting forces and the restoring forces, including the hydrostatic coupling effects between heave and pitch. In addition he introduced a damping (resistance) to the motions in an additional, estimated term. So essentially he only left out the hydrodynamic mass and the hydrodynamic coupling between the various motions. Both of these effects are very important. The first became known at an early date, but the influence of the coupling terms would only be established some sixty years later.

W. Froude already realized that an apparent increase in mass was necessary, which caused a longer natural roll period. The concept of added mass is one of classical hydrodynamics, but the presence of the free fluid surface, making the contributions frequency dependent, and the complexity of the ship’s geometry restrict its numerical application greatly. In the first fifty years of the 20th century some attempts have been made to get more information on the added mass and damping of ships or ship’s sections. No major development in the actual hydrodynamic properties was achieved, however, except for the limiting case of infinitely high motion frequencies. Yet there was a distinct desire to proceed, as evidenced in a comprehensive survey of available knowledge by Weinblum and St. Denis in 1950 [1.10]. In the absence of a better basis they treated uncoupled motions in regular waves using the Froude-Krioll hypothesis. They tried to allow for hydrodynamic mass and damping effects; omitting the couplings was not of so great importance in their study because they considered a mathematically described, simple and also longitudinally symmetric ship form. The greatest limitation they felt at that moment was the restriction to regular waves. On actual sea conditions they could only quote a saying of Lord Rayleigh: “The basic law of the seaway is the apparent lack of any law”. Only three years later this was drastically changed by the first paper mentioned [1.1]. For although oceanographers had indicated the technique of spectral analysis of sea waves some 6 years earlier in a paper by Barber and Ursell [1.11] it was not before 1953 that St. Denis and Pierson made this theory known to the shipbuilding world. At that time it was again suggested that the sea could be described by assuming that the wave elevation is the sum of a large number of simple sinusoidal waves, each having its own length, height and direction and that the ultimate motion of a vessel at sea is the corresponding sum of the responses to each individual wave component. Since the phases of those wave components are randomly distributed only statistical information can be obtained. The superposition of harmonic quantities makes the performance in regular waves to basic elements of motion studies. It meant an enormous stimulation for the research of ships in waves. And in a relatively short time great progress was made.

In the meantime Ursell [1.12] had found an analytic solution for the boundary value problem of a cylinder heaving in the surface of an ideal fluid, which is very well usable in the generally required range of motion
frequencies. Finally Korvin-Kroukovsky and Jacobs [1.13] presented in 1957 a possibility to apply this knowledge to three-dimensional shipforms by a strip theory. Now the basic elements were available to obtain engineering solutions for the coupled heave and pitch in waves. Much improvement was necessary and has subsequently been achieved. The notable interplay of theory and experiment has been very fruitful in this respect. The accidental agreement, but on the other hand often large differences between calculated and measured motions, revealed that some important phenomena were not well understood and this led to more basic investigations. Especially the fundamental experimental methods of oscillation techniques in still water and the measurement of wave forces on the restrained model provided information which stimulated theoreticians to touch up their work or to develop new, more realistic mathematical models. Along these lines Grim [1.14], Tasai [1.15] and Gerritsma [1.16, 1.17] have greatly contributed to the knowledge of ship motions. The theoretical work of e.g. Ogilvie and Newman, to which reference will often be made, is typical for the continuous adaptation of the assumptions, on which the theory was built, to the results of experiments. In this way a rigorous mathematical framework has been obtained.

In the present report it will be the task to verify and further improve the existing methods for heave and pitch on the one hand, and on the other to generalize, modify or complete the procedures to obtain a similar technique, as far as possible, for coupled roll, sway and yaw. The findings will be verified continually and necessary adaptations or additions will be introduced. Before proceeding to an actual formulation of the problem considered, attention is drawn to some consequences of the method used. A phenomenon which is changing with time can be described deterministically or statistically. In the first way the various quantities are functions of time and at each point of time their specific values can be stated. The formulation of a problem generally leads to relations between the respective quantities in the form of differential and/or integral equations. A pendulum or a mass-spring system are simple mechanical examples. In the second way the explicit time dependence is not considered. A certain quantity is only known as an average or as a probability of occurrence. At a specific point of time its value is unknown because the whole time history is unknown. The problem is formulated in distributions of the relevant quantities over the independent variables. Many examples are found in nature. The latter way is chosen when a time history is not important or when it is too difficult or even impossible to obtain it. A relation between the two formulations must exist. Especially when the actual phenomenon consists of a linear superposition of harmonic processes it is generally not too difficult to indicate that relation. In the case of a ship sailing on the surface of a sea, circumstances change to some extent because the free fluid surface represents a ‘memory’ of the system. Each occurrence is, in fact, dependent on all preceding occurrences.

In the consideration of the performance of ships at sea both methods are applied. The manouevring is for a large part studied in the absence of waves and it is important that the time history of various quantities is known. So it is formulated deterministically. Ship motions are logically characterized statistically because the sea waves in which they find their origin can only be handled in that way. Yet a time history may be important, for instance in combinations of motions and manoeuvres while approaching a harbour entrance. A good review of a description in the time domain and in the frequency domain has been given by Ogilvie [1.18] in 1964, together with the possibilities and difficulties of passing from one to the other.

It is further interesting to note that many of the earlier works dedicated to ship motions originated from a strength or vibration analysis. The motions were necessary information to obtain the mass inertia forces but did rarely play an important role in themselves. There is one exception in this respect. The rolling motion continued to attract a lot of attention after the paper by W. Froude had appeared. This becomes clear when one considers that around 1850 not only wood was slowly being replaced by steel but that also sails disappeared in favour of mechanical propulsion devices. The latter change was accompanied by the introduction of awkward rolling phenomena. Before then sails had had a large stabilizing influence in most courses. Investigations into the effect of bilge keels, water chambers, U-tube tanks and other special devices to reduce the rolling motion, such as gyroscopes and fins, appeared regularly from then on. It was a revolutionary time. This period of the second half of the 19th and the beginning of the 20th century is a tremendous source of ideas. In many respects we, with our much better theoretical and experimental possibilities, are now elaborating and realizing what was begun at that time.

1.2 A general description of the problem and its simplification

In its most general form the problem of a ship at sea is that of the dynamic equilibrium of forces and moments in and on an elastic body moving in the inter-
face of two different media. The present work only concerns itself with the external loads on the underwater part of the vessel and with the motions resulting therefrom. So two restrictions will be made right from the beginning: the vessel is considered as a rigid body and the air environment in itself is fully neglected though due recognition will be given to the free surface. As long as no structural or vibrational problems are to be dealt with, the first restriction can be made without hesitation. The borderland of hydrodynamic flutter is not likely to enter the shipbuilding field other than in very special circumstances. The reason for the second restriction is the fact that the density of air is roughly only one thousandth of that of water. So it is clear that for many practical problems aerodynamic forces may be neglected with respect to hydrodynamic forces. Yet it is not always obvious that the hydrodynamic loads on a vessel and its motions, which are interrelated, will not be affected by air and wind. It is especially hard to believe so during adverse weather. But also in much less severe conditions especially the equilibrium in and about the longitudinal direction of the ship may be influenced. Due to the ship's dimensions perpendicular to this axis the hydrodynamic contributions in these two directions are of an order smaller than in the other four. So ships having large superstructures or e.g. tall masts with radar antennae may suffer from aerodynamic forces as well, which may alter the ship's surging and rolling and thereby the whole motion pattern more or less.

The question remaining is difficult enough, however. At present it is commonly accepted that for the consideration of ship motions the problem may be regarded as linear, while the fluid may be idealized. Both at least for a practical solution of the majority of practical problems. So the superposition principle plays an important role in the efforts towards solution. This is beyond the hope of anybody viewing the occurrences in nature. But several investigations have been undertaken to verify this basic linearity assumption and each time it has been confirmed surprisingly well. Ogilvie gives a good survey on this point in his earlier mentioned paper [1.18]. As far as fluid idealization is concerned the facts point in two directions. While it is practically sure that the restriction to a homogeneous, incompressible fluid, free of surface tension is not a serious limitation the viscosity leads to complications. On the one hand water may and must be considered inviscid in dealing with ship motions in waves. It is a logic consequence of the validity of the linearization, for if viscosity would have a great influence then linearity would be impaired as well. On the other hand flow separation and consequent eddy formation are distinctly perceptible, especially with the lateral ship motions. It makes itself primarily felt in additional damping and in a change in the coefficients which couple the motions mutually. In both respects, linearity and neglect of viscous phenomena, ship motion investigations are in a much more favourable position than in the manoeuvring field, where they are dominant in many cases.

With a linear superposition of motions in regular sine waves under the above simplifications the problem has been reduced greatly. What remains is the study of the harmonic oscillations of a rigid body, moving at forward speed in or in the proximity of the surface of an ideal fluid under the action of gravity waves. That problem can be solved in principle by a combination of rigid body and classical fluid dynamics. But only in a strictly numerical sense and with the aid of a computer of a capacity only very recently available. This does not offer much possibility of improving the existing knowledge basically or to gain experience in actual problems. Therefore analytical methods, if need be approximate methods, remain of great importance.

In this work an attempt is made to formulate the problem and next to reduce it to manageable proportions so that a practical solution can be obtained. Within its framework of validity it must exhibit all essential features of the actual occurrences correctly and be capable of producing quantitative information to an acceptable degree of accuracy. The effects governed by rigid body characteristics and by hydrodynamics must be incorporated separately, since they are controlled by different parameters. It must not only be possible to obtain the state of motion in all relevant degrees of freedom, but also detailed information on the external loads on the structure. Finally the solution must be as transparent and easy to apply to numerical computations as possible. To this end the process will be described in a general, mathematical formulation. This is a powerful tool, but relating to physical phenomena in nature it is good to remember Poincaré's words: "Mathematics can never tell what is, it can only say what would be if".

1.3 Presentation of the results of the investigations
The present work is the result of a both theoretical and experimental study, performed over a period of about five years. It presents the overall findings and possibilities of usage without going too much into the details of the various separate investigations made in this period.

In the study three major sub-problems were distinguished. In the first place the hydrodynamic coefficients, the wave exciting forces and the motions resulting therefrom, all as a two-dimensional problem; that is for infinitely long cylindrical bodies in beam waves.
The results of this investigation have earlier been reported by the author [1.19, 1.20]. In the second place the hydrodynamic coefficients in all degrees of freedom, except surge, for ship-shaped vessels. A detailed report on this subject is to appear shortly. In the third place the wave exciting forces on ship-shaped vessels due to waves of arbitrary direction. On this point another report will be issued at a future date. Both reports to be published hereafter will appear in the series of the Netherlands Ship Research Centre TNO. They will present the full experimental results which have been obtained in the course of the study, while developing the subject to the present state.

2 A formulation of the problem

2.1 General considerations

As explained in section 1.2 the problem consists of the dynamic equilibrium of forces and moments acting on the ship and of its resulting motions in a stationary train of regular sine waves. The external forces* on the body depend on the incoming waves, on their length, height and direction, and on the fluid reactions to the motions of the body as to type of motion, frequency and amplitude. Due to the supposed linearity of the system the two contributions of different nature can be separated entirely. In a block diagram this procedure can be visualized as in figure 2.1.

![Diagram of the problem](image)

Fig. 2.1 Schematic representation of the problem considered.

Thereby two subproblems arise, each of which is more easily tractable than the complete problem:

1. the forces induced by the harmonic oscillations of a rigid body, moving in the undisturbed surface of an ideal fluid;
2. the forces produced by waves coming in on the restrained ship.

So the first deals with motions in the absence of waves, the second with waves in the absence of motions. Both are hydrodynamic problems, which serve as input for the motion problem which is treated from the viewpoint of rigid body dynamics.

General hydrodynamic considerations will first be given in chapter 3, followed by a detailed discussion of each subproblem in the chapters 4 and 5. The subject is concluded by a discussion of the results obtained in chapter 6. Finally, in chapter 7 the procedure will be applied to an example.

Before actually starting a formal formulation of the problem three further restrictions will be made. Firstly, the ship's form is transversely symmetric with respect to the vertical centre plane. Longitudinal symmetry will not be assumed. Secondly, at rest the ship is floating upright in stable equilibrium, and thirdly it is following a straight track at a constant mean forward speed in the above plane of symmetry. In principle these simplifications are not essential to the problem, but they facilitate the formulation greatly and they will be supposed throughout. It is emphasized that what follows does not cover the case of wave excited motions during arbitrary manoeuvres, nor those manoeuvres themselves in the absence of waves.

In view of the fact that the input of a regular train of sea waves has a definite direction it is clear that a system of axes with an orientation which is fixed in space is best suited for a description of the wave problem. A right handed space fixed system, which for the present case will be taken as fixed to the earth, will be denoted $OX_1X_2X_3$ with capital letters. A system which will always remain parallel but translates with the constant forward speed $V$ of the body is indicated by $\alpha_1x_2x_3$. The track coincides with the $OX_1=\alpha x_1$ direction.

The six possible motions are defined as three translations of the centre of gravity along the axes and as three rotations about them. They are indicated by:

- displacements:
  \[ s_j = s_{ja}\cos(\omega t + \epsilon_j) = Re(s_{ja}e^{-i(\omega t + \epsilon_j)}) \]
- velocities:
  \[ \dot{s}_j = -\omega s_{ja}\sin(\omega t + \epsilon_j) = Re(-i\omega s_j) \]
- accelerations:
  \[ \ddot{s}_j = -\omega^2 s_{ja}\cos(\omega t + \epsilon_j) = Re(-\omega^2 s_j) \]

\[ j = 1, 2, ..., 6 \] (2.1.1)

The motions being harmonic, the reactive forces of the fluid will be harmonic as well with the same period. And by the imposed linearity their magnitude will be directly proportional to the motion amplitudes. The reactive forces will be divided into two parts: hydrostatic and hydrodynamic forces. The former are defined as the fluid forces acting on the body at a static di-
placement from the rest position floating with zero forward speed in still water. Therefore they are determined by the geometry of the body only. They can be expressed by:

\[ F_i = - \sum_{j=1}^{6} C_{ij} \delta_j \quad (i = 1, \ldots, 6) \]  \hspace{1cm} (2.1.2)

Due to the presence of the free surface a phase difference with respect to the body velocity will develop for the hydrodynamic forces. So they can always be thought of as having one component in phase with the body velocity and another in phase with the body acceleration. Or:

\[ F_i = - \sum_{j=1}^{6} (A_{ij} \delta_j + B_{ij} \delta^2) \quad (i = 1, \ldots, 6) \]  \hspace{1cm} (2.1.3)

In these formulae \( A_{ij} \), \( B_{ij} \) and \( C_{ij} \) are certain coefficients. \( A_{ij} \) and \( B_{ij} \) will be functions of the form of the body, the forward speed and the frequency of motion, but not of the motion amplitude or the time; \( C_{ij} \) is a function of the shape only. When \( i = j \) the fluid reaction in the same direction as the motion is indicated, when \( i \neq j \) the induced coupling force in the \( i \)-th direction due to motion in the \( j \)-th mode is indicated.

The wave forces on the restrained vessel are harmonic as well and will show a phase difference with respect to the incoming waves. Let the waves forces be indicated by a subscript \( w \), then:

\[ X_{wi} = X_{wia} \cos(\omega t + \epsilon_{wi}) = Re\{X_{wia}e^{-i(\omega t + \epsilon_{wi})}\} \quad (i = 1, \ldots, 6) \]  \hspace{1cm} (2.1.4)

Since the equilibrium position has been taken as the zero level for forces and for motions the ship’s resistance, being balanced by the thrust, and the ship’s weight, balanced by the displacement, will not enter the equations. So the total external force on the vessel is given by:

\[ F_i = - \sum_{j=1}^{6} (A_{ij} \delta_j + B_{ij} \delta_j + C_{ij} \delta_j) + X_{wi} \quad (i = 1, \ldots, 6) \]  \hspace{1cm} (2.1.5)

2.2 The equations of motion in a space fixed system

The state of motion of an arbitrary rigid body under the action of certain external disturbances is completely described by Newton’s law of dynamics. It is a vector equation and it can be formulated for translations and rotations, respectively, as:

\[ \frac{d}{dt}(mV) = F \]  \hspace{1cm} (2.2.1)

\[ \frac{d}{dt}(P) = M \]

Where \( m \) is the mass of the body, \( V \) the instantaneous velocity of the centre of gravity, \( P \) the instantaneous angular momentum about it and \( F \) and \( M \) represent the resulting external force and moment on the body acting in and about the centre of gravity. The corresponding equations in scalar form are strongly dependent on the choice of the coordinate axes. They will be developed below. The mass \( m \) and its distribution in the body will be considered constant. This, of course, is not strictly true due to the consumption of fuel and supplies or to ballasting underway, but the differences during a voyage are generally very small. Anyway they are negligible during an interval which is large with respect to the period of the motions. When necessary the resulting equations can be interpreted as being valid for a restricted length of time only.

Let \( OX_1X_2 \) coincide with the undisturbed water surface while \( OX_3 \) points vertically downwards. The body moves forward with a speed \( V \) along \( OX_1 \). The system \( ox_1x_2x_3 \), translating with the vessel, has its origin in the centre of gravity \( G \) when at rest. It is parallel with \( OX_1X_2X_3 \). The relations between the two systems are:

\[ X_1 = x_1 + V t \]  \hspace{1cm} (2.2.2)

\[ X_2 = x_2 \]

\[ X_3 = x_3 + a \]

where \( a \) is the distance of \( G \) below the water surface.

The velocity of and the momentum about the centre of gravity is further given by:

\[ V = (\dot{x}_1 + V, \dot{x}_2, \dot{x}_3) \]

\[ P = \{(I_{11}\dot{x}_4 - I_{12}\dot{x}_5 - I_{13}\dot{x}_6), (-I_{21}\dot{x}_4 + I_{22}\dot{x}_5 - I_{23}\dot{x}_6), \\
(-I_{31}\dot{x}_4 - I_{32}\dot{x}_5 + I_{33}\dot{x}_6)\} \]  \hspace{1cm} (2.2.3)

The equations of motion are obtained by substitution of (2.1.5) and (2.2.3) into (2.2.1). In the equation for the translations \( m \) is constant, but in that for the rotations the inertia characteristics \( I_{ij} \) are functions of time, since the position and orientation of the body with respect to the axes change continually. The unit vectors are time invariable, however. Supposing the motions
to be small, which is also a requirement for the linearization of the hydrodynamic subproblems, and observing that the mean value of $I_{12} = I_{21} = I_{23} = I_{32} = 0$ due to the symmetry of the body, extensive simplifications are possible in a first order approximation. The following result is obtained in scalar form:

\[
m\ddot{s}_1 + \sum_{j=1}^{6} (A_{ij} \ddot{s}_j + B_{ij} \dot{s}_j + C_{ij} \dot{s}_j) = X_{w1}
\]

\[
m\ddot{s}_2 + \sum_{j=1}^{6} (A_{ij} \ddot{s}_j + B_{ij} \dot{s}_j + C_{ij} \dot{s}_j) = X_{w2}
\]

\[
m\ddot{s}_3 + \sum_{j=1}^{6} (A_{ij} \ddot{s}_j + B_{ij} \dot{s}_j + C_{ij} \dot{s}_j) = X_{w3}
\]

\[(I_{11} \ddot{s}_4 - I_{13} \ddot{s}_6) + \sum_{j=1}^{6} (A_{ij} \ddot{s}_j + B_{ij} \dot{s}_j + C_{ij} \dot{s}_j) = X_{w4}
\]

\[(I_{22} \ddot{s}_5 + I_{33} \ddot{s}_6) + \sum_{j=1}^{6} (A_{ij} \ddot{s}_j + B_{ij} \dot{s}_j + C_{ij} \dot{s}_j) = X_{w5}
\]

\[(-I_{31} \ddot{s}_4 + I_{33} \ddot{s}_6) + \sum_{j=1}^{6} (A_{ij} \ddot{s}_j + B_{ij} \dot{s}_j + C_{ij} \dot{s}_j) = X_{w6}
\]

(2.2.8)

Because of the symmetry of the body a motion in the longitudinal plane cannot produce any force perpendicular to that plane, so:

\[A_{ij} = B_{ij} = C_{ij} = 0\]

for $i = 2, 4$ or $6$ and $j = 1, 3$ or $5$ (2.2.5)

The reverse is also true by the same reason:

\[A_{ij} = B_{ij} = C_{ij} = 0\]

for $i = 1, 3$ or $5$ and $j = 2, 4$ or $6$ (2.2.6)

By the definition of the hydrostatic forces they can only be buoyancy forces, originating from vertical movements and acting vertically. Therefore:

\[C_{11} = C_{12} = C_{16} = 0\] for all $i$

and

\[C_{1j} = C_{2j} = C_{6j} = 0\] for all $j$ (2.2.7)

Finally the following equations remain:

\[(m + A_{11}) \ddot{s}_1 + B_{11} \dot{s}_1 + A_{13} \ddot{s}_3 + B_{13} \dot{s}_3 + A_{15} \ddot{s}_5 + B_{15} \dot{s}_5 = X_{w1}\]

\[(m + A_{22}) \ddot{s}_2 + B_{22} \dot{s}_2 + A_{24} \ddot{s}_4 + B_{24} \dot{s}_4 + A_{26} \ddot{s}_6 + B_{26} \dot{s}_6 = X_{w2}\]

\[(m + A_{33}) \ddot{s}_3 + B_{33} \dot{s}_3 + A_{35} \ddot{s}_5 + B_{35} \dot{s}_5 + A_{33} \ddot{s}_3 + B_{33} \dot{s}_3 = X_{w3}\]

2.3 A transformation of the equations in terms of the Eulerian angles

In mechanics it is usual to introduce a set of three independent angular displacements, the so-called Eulerian angles. They can be chosen in various ways. Here they will be defined as three subsequent rotations; the first, $\psi$, being about the absolutely vertical axis $\alpha x_1$; the second, $\theta$, about the rotated position of the $\alpha x_1$-axis which remains in the horizontal plane; the third, $\phi$, about the position of the $\alpha x_1$-axis after the previous two rotations. Only the latter axis coincides with a body axis. The three rotations are called yawing, pitching and rolling, respectively. The angles are illustrated in figure 2.2. The advantage of this particular choice is that, due to the character of the motions of surface ships, the first and second rotation will always be small. Yaw and pitch amplitudes will rarely exceed 0.15 rad, so that within a one per cent error $\sin \psi \approx \psi$, $\sin \theta \approx \theta$ and $\cos \psi \approx \cos \theta \approx 1$. Only rolling angles may become appreciably larger.

\[\psi = \phi \cos \theta \cos \psi - \theta \sin \psi \approx \phi\]

\[s_2 = \phi \cos \theta \sin \psi + \theta \cos \psi \approx \theta\]

\[s_6 = -\phi \sin \theta + \psi \approx \psi\]

(2.3.1)

Fig. 2.2 The Eulerian angles.

The rotational vectors $\phi$, $\theta$, $\psi$ are not directed along $\alpha x_1, x_2, x_3$. The relation between $s_4, s_5, s_6$ on the one hand and $\phi$, $\theta$, $\psi$ on the other is obvious from the introduction of the Eulerian angles:

\[s_4 = \phi \cos \theta \cos \psi - \theta \sin \psi \approx \phi\]

\[s_2 = \phi \cos \theta \sin \psi + \theta \cos \psi \approx \theta\]

\[s_6 = -\phi \sin \theta + \psi \approx \psi\]

(2.3.1)
So in considering small motion amplitudes and linearizing consequently the Eulerian angles coincide with the angular displacements about the space fixed axes. Further it is customary to replace \( s_1, s_2 \) and \( s_3 \) by \( x, y \) and \( z \). Finally, the numbered indices are likewise substituted by letters for the corresponding motions. This is easily changed in the equations (2.2.8).

In the fourth and sixth of these equations appears the product of inertia \( I_{13} = I_{31} \), the only one which is not equal to zero because in general the \( x_1 \) and \( x_3 \) axes will not be principal axes. It is clear, however, that for ordinary vessels \( I_{13} \) will be very small and that terms containing it will be often of the same order as those neglected by the linearization. The two terms concerned can then be dropped without objections. For the sake of completeness they have been retained in the equations (2.2.8). It is noted that it requires a separate consideration on the orders of magnitude involved to remove them, for in fact they remain present in a linear formulation.

2.4 Discussion of the equations of motion

The equations of motion which have been derived in the preceding sections have the appearance of a set of simultaneous, linear differential equations of the second order. In form they are entirely analogous to the equations for well-known mechanical systems as a pendulum or a mass-spring system. But there is a very essential difference. While the coefficients in the equations for said systems are constant the \( A_{ij} \) and \( B_{ij} \) are functions of the frequency of motion. This is caused, as has already been remarked, by the presence of the free surface. Returning to section 2.1 it is recalled that \( A_{ij} \) and \( B_{ij} \) have merely been presented as coefficients to denote two components of the hydrodynamic force resulting from a stationary harmonic oscillation of the body: one in phase with the velocity and one in phase with the acceleration. The use of \( \delta_j \) and \( \delta_i \) only allows a condensed formulation of that force in one expression. Consequently, the equations of motion presented here are only a formal representation. They may not be interpreted as actual differential equations in time, the solution of which produces the time history of the relevant quantities. They do not pretend to be anything else but a set of twelve algebraic equations, fixing the amplitudes and phases of the six harmonic oscillations of a rigid body in the surface of a fluid under the action of a train of regular sine waves at one specified frequency.

The analogy with well-known mechanical systems is tempting, however. The more so since this physical model speaks much better to the practical naval architect than the complicated mathematical formulation which becomes necessary otherwise. There is no objection at all in considering the ship in waves to be a true mechanical system as long as one only investigates the stationary case in regular waves and accepts that the system characteristics will be different in another train of waves. There is only one case where the description given transforms into an actual differential equation of time. That is the free oscillation of the vessel in still water in one degree of freedom: pure heaving, pure rolling or pure pitching. So the mathematical model cannot do justice to the physical reality in full. But with the proper interpretation and the proper restrictions it is of great value for a thorough study. For a more elaborate discussion of a time and a frequency description reference is made to Tick [2.1], Cummins [2.2] and Ogilvie [2.3].

Accepting the mechanical analogy \( A_{ij} \) may be considered as “added mass” or “hydrodynamic mass” (mass moment of inertia, respectively) and \( B_{ij} \) as a damping coefficient. In the same way \( A_{ij} \) and \( B_{ij} \) are usually named mass and damping coupling coefficient, respectively. This, however, is also only a formal nomenclature.

Due to the symmetry of the body and the linearization of the problem the motions fall apart into two groups. Coupled surge, heave and pitch in the longitudinal centre plane, and coupled sway, roll and yaw perpendicular to it. The two groups are entirely independent of one another.

3 A general hydrodynamic approach

3.1 Introduction

Intuitively one feels that physically the body and liquid movements in a prescribed wave condition and for a given state of a solid body will be uniquely determined. The theoretical formulation of this problem should equally have a unique solution. Yet, to the knowledge of the author, a mathematical proof of this fact for the most general case has never been given. Under some restrictive assumptions uniqueness theorems have among others been derived by John [3.1] and Stoker [3.2]. It is most likely that one single solution will exist in all relevant cases. That will be assumed anyway.

In the following a hydrodynamic formulation will be presented from which certain relations can be obtained without actually solving the problem. These relations provide useful material for a comparison with the formulae which will be derived in chapter 4 and 5 as practical approximations. The results follow directly from the properties of potential theory and most of them have been derived before by other authors. For details and for a rigorous mathematical treatment reference will be made to their papers. The description is given in a space-fixed coordinate system and the assumptions of chapter 2 will be used. As usual
the fluid is considered ideal and irrotational, while for the present only the case of zero forward speed will be treated.

Let \( \Phi_j(x_1, x_2, x_3; t) \) be the velocity potential for the oscillatory motion \( s_j \) in still water. \( \Phi_j \) satisfies the Laplace equation in the entire fluid region \( R \) outside the body, the free surface condition, the boundary conditions on the bottom \( B \) of the fluid domain and on the body surface \( S \), and appropriate conditions at the other boundaries of the fluid domain, if present, or at infinity. Due to the linearization the free surface condition may be satisfied on \( x_3 = 0 \), while the body surface \( S \) may be taken in its middle or rest position \( S_0 \). It will be assumed that the fluid domain is very large in all directions. Then:

\[
\begin{align*}
\Phi_j(x_1, x_2, x_3; t) &= \phi_j(x_1, x_2, x_3)e^{-i\omega t} \\
\Delta \phi_j &= \frac{\partial^2 \phi_j}{\partial x_1^2} + \frac{\partial^2 \phi_j}{\partial x_2^2} + \frac{\partial^2 \phi_j}{\partial x_3^2} = 0 \quad \text{(in } R) \\
k \phi_j + \frac{\partial \phi_j}{\partial x_3} &= 0 \quad \text{(on } x_3 = 0) \\
\frac{\partial \phi_j}{\partial n} &= 0 \quad \text{(on } B) \\
\frac{\partial \phi_j}{\partial n} e^{-i\omega t} &= f_j^k, \quad \text{or: } \frac{\partial \phi_j}{\partial n} = -i\omega f_j s_j \quad \text{(on } S_0) \\
\lim_{r \to \infty} r^k \left\{ \frac{\partial \phi_j}{\partial r} + ik \phi_j \right\} &= 0 \quad \text{(radiation condition)}
\end{align*}
\]

where \( s_j \) and \( \phi_j \) will usually be complex; \( k \) is the wave number: \( k = \omega / \sqrt{\mu} \), and \( f_j \) the generalized direction cosines on \( S_0 \) defined by:

\[
\begin{align*}
f_1 &= \cos(n, x_1) \\
f_2 &= \cos(n, x_2) \\
f_3 &= \cos(n, x_3) \\
f_4 &= x_2 \cos(n, x_3) - x_3 \cos(n, x_2) \\
f_5 &= x_3 \cos(n, x_1) - x_1 \cos(n, x_3) \\
f_6 &= x_1 \cos(n, x_2) - x_2 \cos(n, x_1)
\end{align*}
\]

(3.1.2)

Let further the incident wave system be given by the potential \( \phi_w(x_1, x_2, x_3) \) and the diffraction of waves about the restrained body by \( \phi_d(x_1, x_2, x_3; t) \). Both potentials satisfy the Laplace equation and the boundary conditions at the free surface and the bottom. Their sum satisfies the condition at the body surface, while only \( \phi_d \) meets the requirements at infinity given by the radiation condition. Thus:

\[
\begin{align*}
\phi_w(x_1, x_2, x_3; t) &= \phi_w(x_1, x_2, x_3)e^{-i\omega t} \\
\phi_d(x_1, x_2, x_3; t) &= \phi_d(x_1, x_2, x_3)e^{-i\omega t} \\
\Delta \phi_w &= \Delta \phi_d = 0 \quad \text{(in } R) \\
k \phi_w + \frac{\partial \phi_w}{\partial x_3} &= k \phi_d + \frac{\partial \phi_d}{\partial x_3} = 0 \quad \text{(on } x_3 = 0) \\
\frac{\partial \phi_w}{\partial n} &= \frac{\partial \phi_d}{\partial n} = 0 \quad \text{(on } B) \\
\frac{\partial \phi_w}{\partial n}(\phi_w + \phi_d) &= 0 \quad \text{(on } S_0) \\
\lim_{r \to \infty} r^k \left\{ \frac{\partial \phi_d}{\partial r} + ik \phi_d \right\} &= 0 \quad \text{(radiation condition)}
\end{align*}
\]

(3.1.3)

Again \( \phi_w \) and \( \phi_d \) will be complex. All these potentials are considered to be known. The fluid flow during the body’s oscillations in waves is thus determined by:

\[
\phi(x_1, x_2, x_3; t) = \phi_w + \phi_d + \sum_{j=1}^{6} \phi_j
\]

(3.1.4)

From the Bernoulli-equation the unsteady pressure on the body follows from:

\[
\begin{align*}
p &= -\frac{\partial \phi}{\partial t} = i \omega \varphi e^{-i\omega t} = \\
&= i \omega \left\{ \phi_w + \phi_d + \sum_{j=1}^{6} \phi_j \right\} e^{-i\omega t} \quad \text{(on } S_0)
\end{align*}
\]

(3.1.5)

while the hydrodynamic force in the \( k \)-th direction is:

\[
F_k = -\int_{S_0} p f_k dS = \\
=-i \omega e^{-i\omega t} \int_{S_0} \left( \phi_w + \phi_d \right) f_k dS + \sum_{j=1}^{6} \int_{S_0} \phi_j f_k dS
\]

(3.1.6)

Next consider the problem of forced oscillation in still water in one mode of motion only. Adjust the time reference so that \( s_j \) is real in:

\[
s_j = s_j e^{-i\omega t}
\]

(3.1.7)

The time-invariable part of \( \phi_j \) remains complex. The potential can be written as follows:

\[
\phi_j(x_1, x_2, x_3; t) = \phi_j e^{-i\omega t} = \\
\left\{ \phi_j(x_1, x_2, x_3) + i \varphi j(x_1, x_2, x_3) \right\} e^{-i\omega t}
\]

(3.1.8)

The dynamic pressure on the body surface is:

\[
p_j(x_1, x_2, x_3; t) = i \omega \varphi j e^{-i\omega t} \quad \text{(on } S_0)
\]

(3.1.9)
and the hydrodynamic reactive force in the k-th direction resulting from the motion $s_j$ is obtained by:

$$F_{kj}(t) = -\iint_{S_0} p_j f_k dS = -i \omega a e^{-iat} \iint_{S_0} \varphi_j f_k dS$$  \hspace{1cm} (3.1.10)$$

As indicated in (2.1.4) this force has also been written formally as:

$$F_{kj}(t) = -A_{kj} \ddot{s}_j - B_{kj} \delta_j = (s_{ja} \omega^2 A_{kj} + is_{ja} \omega B_{kj}) e^{-iat}$$  \hspace{1cm} (3.1.11)$$

Upon equating the real and imaginary parts of (3.1.10) and (3.1.11) the following expressions for the so-called hydrodynamic coefficients $A_{kj}$ and $B_{kj}$ result:

$$A_{kj} = \frac{\theta}{s_{ja}} \iint_{S_0} \varphi_j f_k dS$$  \hspace{1cm} (3.1.12)$$
$$B_{kj} = \frac{-\theta}{s_{ja}} \iint_{S_0} \varphi_j f_k dS$$

The consideration of the other subproblem of the restrained body in waves produces the wave exciting forces. The force in the k-th direction appears from (3.1.6):

$$X_{wk} = -i \omega a e^{-iat} \iint_{S_0} (\varphi_w + \varphi_d) f_k dS$$  \hspace{1cm} (3.1.13)$$

In what follows it is useful to eliminate $f_k$ with (3.1.1). The substitution $f_k = i/(s_{ka} \partial \varphi_k/\partial n)$ leads to the formula:

$$X_{wk} = \frac{q}{s_{ka}} e^{-iat} \iint_{S_0} (\varphi_w + \varphi_d) \frac{\partial \varphi_k}{\partial n} dS$$  \hspace{1cm} (3.1.14)$$

When the forward speed of the ship is no longer zero the problem is substantially more difficult. This is caused by interactions of the incident waves and of the unsteady fluid motion due to the oscillations of the body with the steady flow of the translating ship. These interactions arise since two boundary conditions are essentially nonlinear: that on the moving body and that on the free surface. Therefore a consequent approximation of the combined problem need not generally coincide with a superposition of the steady and unsteady problems after linearization in themselves. In some cases nonlinear interaction terms then are of the same order as the linear effects and can no longer be neglected. For this reason most approaches have been restricted to conditions where the disturbance by the forward motion is small so that a linearized free surface condition also holds for the complete problem. This includes deeply submerged bodies, thin or slender ships and low frequency oscillations of other bodies at slow speeds. A general treatment of the case with forward speed has not been given so far. Where information on any point is available this will be mentioned in the following sections.

3.2 Relations between corresponding hydrodynamic coefficients

Suppose the body to oscillate in still water in the j-th mode at zero forward speed. The velocity potential is $\Phi_j$. The hydrodynamic coefficients are given by (3.1.12). Let the body next perform oscillations in the k-th mode. The velocity potential is $\Phi_k$ and the corresponding hydrodynamic coefficients are obtained by interchanging $k$ and $j$ in (3.1.12). It will be shown that a simple relation between them exists. Therefore Green's second identity:

$$\iint_{R} (\varphi \Delta \psi - \psi \Delta \varphi) dR = \oint_{A} \left( \frac{\partial \psi}{\partial n} - \frac{\partial \varphi}{\partial n} \right) dA \hspace{1cm} (3.2.1)$$

is applied to $\varphi = \varphi_j$ and $\psi = \varphi_k$. Since both potentials satisfy the Laplace equation it follows that:

$$\oint_{A} \frac{\partial \varphi_j}{\partial n} dA = \oint_{A} \frac{\partial \varphi_k}{\partial n} dA \hspace{1cm} (3.2.2)$$

$A$ is a closed surface consisting of the body surface $S_0$, the free surface $FS$, the bottom of the fluid domain $B$ and a circular cylinder $C$ about the $x_1$-axis with large radius $r$. On substitution of the appropriate conditions (3.1.1) for the surfaces $FS$, $B$ and $C$ these contributions appear to vanish, so that only the surface $S_0$ remains:

$$\iint_{S_0} \frac{\partial \varphi_j}{\partial n} dS = \iint_{S_0} \frac{\partial \varphi_k}{\partial n} dS \hspace{1cm} (3.2.3)$$

With the boundary condition on the body then:

$$s_{ja} \iint_{S_0} \varphi_j f_k dS = s_{ja} \iint_{S_0} \varphi_k f_j dS \hspace{1cm} (3.2.4)$$

or:

$$\frac{1}{s_{ja} S_0} \iint_{S_0} \varphi_j f_k dS = \frac{1}{s_{ja} S_0} \iint_{S_0} \varphi_k f_j dS \hspace{1cm} (3.2.5)$$

Of course, the real and imaginary parts of (3.2.5) are equal as well. Therefore (3.1.12) results in:

$$A_{kj} = A_{jk}$$  \hspace{1cm} (3.2.6)$$
$$B_{kj} = B_{jk}$$

When the fluid domain is not only restricted in depth but also in the $x_1$- and $x_2$-directions by walls or the
like, the radiation condition at infinity must be replaced by the stipulation that the normal velocity components at the boundaries are zero. The closing surface \( C \) is then analogously replaced by these walls and the whole derivation remains valid. Thus at zero forward speed (3.2.6) is true for the vessel in any arbitrary environment, provided of course that the underlying assumptions remain valid.

At forward speed the determination of similar relations becomes much more difficult. Timman and Newman [3.3] attacked the problem of the damping coefficients of a symmetric ship. They arrived at the following conclusions; the damping coefficients \( B_{jj} \) and the coupling coefficients of surge-pitch and sway-roll are even functions of the forward speed \( V \), or \( B_{15} = B_{31} \), \( B_{24} = B_{42} \); the other coupling coefficients are odd functions of \( V \), or \( B_{kj} = -B_{jk} \) for all other \( j \neq k \). Newman [3.4] derived the same result again in a different and more general way. It consequently applies to the entire complex force on the understanding that it is only valid for its dynamic and frequency-dependent part. Other force-components such as the hydrostatic forces and forces arising from the movement of the vessel in the steady flow field do contribute to the complete force acting on the body but are not incorporated in the representation of the harmonically varying forces by \( A_{kj} \) and \( B_{kj} \). In Newman’s derivation the longitudinal symmetry has not been supposed explicitly any longer. But due to certain properties of the reverse-flow relations which he uses the body may only have slight asymmetry. Of course, for distinctly nonsymmetrical bodies the above relations at forward speed may be disturbed. Perhaps the behaviour as even or odd functions in \( V \) will then approximately be valid after subtraction of the zero speed value of \( A_{kj} = A_{jk} \neq 0 \) and \( B_{kj} = B_{jk} \neq 0 \). In both works discussed above, the usual further assumption has been made of a small disturbance of the free surface due to the forward motion.

Another correspondence in the hydrodynamic coefficients will exist between \( A_{kj} \) and \( B_{kj} \) mutually, since they represent the real and imaginary part of one and the same function. This means that, observing certain rules, knowledge of a damping coefficient is sufficient to determine the corresponding mass coefficient and vice versa. Such relations are known under the name Kramers-Kronig relations. In the shipbuilding field they were first derived by Kotik and Mangulis [3.5] for the added mass and damping in heaving at zero forward speed. Their expectation that the same or similar relations would be more generally valid for all modes of motion and regardless of forward speed was confirmed by Ogilvie [3.6], who derived them in a universal way. They read:

\[
A_{kj}(\omega) - A_{kj}(\infty) = \frac{2}{\pi} \int_0^\infty \{B_{kj}(\xi) - B_{kj}(\infty)\} \frac{d\xi}{\xi^2 - \omega^2}
\]

\[
B_{kj}(\omega) - B_{kj}(\infty) = -\frac{2\omega^2}{\pi} \int_0^\infty \{A_{kj}(\xi) - A_{kj}(\infty)\} \frac{d\xi}{\xi^2 - \omega^2}
\]

(3.2.7)

Ogilvie gives the second relation as:

\[
B_{kj}(\omega) - B_{kj}(\infty) = -\frac{2}{\pi} \int_0^\infty \{A_{kj}(\xi) - A_{kj}(\infty)\} \frac{\xi^2 d\xi}{\xi^2 - \omega^2}
\]

(3.2.8)

The approach in the two works cited is somewhat different, but when the Fourier-transforms of the functions used by the authors both exist, the expressions are identical. In all normal cases considered in this report this is true, so that it is implied that:

\[
\int_0^\infty \{A_{kj}(\omega) - A_{kj}(\infty)\} d\omega = 0
\]

(3.2.9)

The practical use of these equations is limited by the fact that one of the coefficients must be known over the entire frequency range. Relatively simple approximations are generally only valid in the range of interest. An inaccurate representation outside this range may immediately lead to large errors in the coefficient which has to be derived. It is further remarked that it is an essential requirement for the validity of these relations that the system remains stable.

3.3 Peculiarities of the coefficients at forward speed

The wave pattern created by an oscillating pressure point at forward speed depends on the wave-form parameter \( \gamma = \omega V/g \). Waves arising from the forward motion alone are determined by the Froude-number \( Fn = V/\sqrt{(gL)} \), where \( L \) is some length, and those from the oscillation alone by the frequency number \( \xi_L = \omega/\sqrt{(gL)} \). The combined wave disturbance is controlled by both quantities together, which is generally expressed by the speed \( Fn \) and the product \( \gamma = \xi_L \cdot Fn = \omega V/g \). From the investigations of several authors it has appeared that \( \gamma = \frac{1}{\xi_L} \) represents a critical case. Vossers [3.7] gives a review of these investigations. Physically it means that for \( \gamma > \frac{1}{\xi_L} \) the waves of an oscillating pressure point are confined to a sector behind the point so that it always proceeds in still water. For \( \gamma < \frac{1}{\xi_L} \) the waves run faster and come ahead of it. At \( \gamma = \frac{1}{\xi_L} \) a wave front perpendicular to the course proceeds at the same speed as the pressure point. It may be expected, and has been shown, that this parameter will also play a role when finite bodies oscillate at
forward speed. So the value of $\gamma$ in relation to the width of a canal and the length of the vessel determines whether wave reflections from the walls will interfere with the oscillations or not. Further it appears that both hydrodynamic coefficients $A_{kj}$ and $B_{kj}$ may represent irregularities around $\gamma = \frac{1}{2}$.

An interesting study has been performed by Newman [3.8]. He investigated the damping coefficients of an oscillating ellipsoid with three unequal axes in six degrees of freedom. The body was fully submerged but close to the free surface. He found a mathematical singularity at $\gamma = \frac{1}{2}$ which caused the $B_{ij}$ to become infinite for surge, heave and pitch. But for sway, roll and yaw the damping coefficients are not singular and remain bounded for all speeds and frequencies. So there seems to be a fundamental difference between the so-called longitudinal motions in the plane of symmetry of the body and the lateral motions perpendicular to that plane.

Probably it is not a bold statement that something similar will hold for shiplike bodies in the free surface. In fact the same singularity has been found in computations of the heave and pitch damping coefficients for the Todd Series Sixty hull forms [3.9]. These computations, however, were based upon Newman’s thin ship theory and it is doubtful whether this produces correct results. Any attempt to obtain an experimental verification of the singular behaviour of the coefficients around $\gamma = \frac{1}{2}$ will suffer from difficulties in the measurements, which show a large scatter in this range.

Anyway it may be expected that the coupled surge-heave-pitch motion of ships and the corresponding mass and damping coefficients show peculiar effects about $\gamma = \frac{1}{2}$. In sway and yaw oscillation experiments have been performed by Van Leeuwen [3.10]. He did not experience difficulties in measuring the hydrodynamic forces at any value of $\gamma$ and the resulting $A_{kj}$ and $B_{kj}$ were fully regular. But the sway damping distribution and the stability indices were subject to large changes around $\gamma = \frac{1}{4}$. However, it is not clear how far his results have been influenced by waves reflected against the tank’s walls which struck the model for $\gamma \approx 0.37$ and below. The fact that the changes were also influenced and shifted from $\gamma = \frac{1}{4}$ by the presence of the rudder and by the action of the propeller suggests at least that other phenomena come into play as well and that they are not typical for $\gamma = \frac{1}{4}$.

Anyhow it is understandable that system characteristics may be different below and above this point, for in the first case the ship proceeds in already disturbed water. From his own experience with experiments the author can confirm the difficulties in measuring heave and pitch coefficients and the large scatter in the results, especially in the damping. His sway, roll and yaw tests with a shipmodel have been performed in a wide basin to avoid wall influence as far as possible. If the distance over which the waves had to travel was not still enough to damp the wall interference it can only have appeared for $\gamma$ below 0.30. No peculiarities in these modes have been observed during the tests or in the analysis, neither in the value of the coefficients nor in their distribution over the length.

Another interesting feature of Newman’s study [3.8] is the occurrence of negative damping in some modes at high forward speeds. This could imply that some motions may not always be stable. It then also implies that a source of energy must be present apart from the harmonically oscillating forces. At zero forward speed there is no such source but when moving ahead the body’s propulsion system produces energy to overcome the resistance. This may act as an additional source. Newman goes a little further into this subject. It is, of course, questionable if negative damping will actually occur physically, for eventually viscous losses will be positive. The computations seem to indicate possible negative damping at high velocities in surge, heave and pitch. So again in the symmetric, longitudinal motions. According to Newman no such cases have been found in sway, roll or yaw.

### 3.4 Relations between the wave exciting forces and the damping coefficients

The wave exciting force at zero forward speed was found in (3.1.14) to be:

$$X_{wk} = \frac{\rho}{s_{ka}} e^{-\int_0^t} \int_{S_0} (\varphi_w + \varphi_d) \frac{\partial \varphi_k}{\partial n} dS$$

(3.4.1)

When Green’s formula (3.2.1) is applied to $\varphi = \varphi_k$ and $\psi = \varphi_d$ there results after a similar reduction as in section 3.2:

$$\int_{S_0} \varphi_k \frac{\partial \varphi_k}{\partial n} dS = \int_{S_0} \varphi_k \frac{\partial \varphi_k}{\partial n} dS$$

(3.4.2)

And since on $S_0$ is $\partial \varphi_d / \partial n = -(\partial \varphi_w / \partial n)$ (see 3.1.3)):

$$\int_{S_0} \varphi_d \frac{\partial \varphi_k}{\partial n} dS = - \int_{S_0} \varphi_k \frac{\partial \varphi_w}{\partial n} dS$$

(3.4.3)

The latter result makes it possible to eliminate the diffraction potential from the wave exciting force. So it is possible to avoid the diffraction problem altogether. It suffices to solve for the potential of the incoming waves and the oscillation potential in still water:

$$X_{wk} = \frac{\rho}{s_{ka}} e^{-\int_0^t} \int_{S_0} \left\{ \varphi_k \frac{\partial \varphi_k}{\partial n} - \varphi_k \frac{\partial \varphi_w}{\partial n} \right\} dS$$

(3.4.4)
This result is originally due to Haskind [3.11] and the relations obtained are therefore called Haskind-relations. As appears from the above derivation they are valid for both infinite and restricted waterdepth.

Further the complete determination of \( \varphi_k \) on the body can be circumvented as well by applying Green’s theorem once again to \( \varphi = \varphi_w \) and \( \psi = \varphi_k \). Then one obtains:

\[
\int_S \left( \frac{\partial \varphi_k}{\partial n} - \varphi_k \frac{\partial \varphi_w}{\partial n} \right) dS = -\int_C \left( \varphi_w \frac{\partial \varphi_k}{\partial n} - \varphi_k \frac{\partial \varphi_w}{\partial n} \right) dC \quad (3.4.5)
\]

Since \( \varphi_w \) does not satisfy a radiation condition at infinity the contribution over \( C \) now not vanishes. So that finally \( X_{wk} \) may be obtained from the well-known wave potential of regular sinusoidal waves and the asymptotic behaviour of the oscillation potential at infinity, which generally is much easier established:

\[
X_{wk} = -\frac{\partial}{\partial \zeta} e^{-ist} \int_C \left( \varphi_w \frac{\partial \varphi_k}{\partial n} - \varphi_k \frac{\partial \varphi_w}{\partial n} \right) dC = -\frac{\partial}{\partial n} \int_0^{2\pi} \int_0^\infty \left( \varphi_w \frac{\partial \varphi_k}{\partial r} - \varphi_k \frac{\partial \varphi_w}{\partial r} \right) r \, dr \, dx_3 \quad (3.4.6)
\]

where polar coordinates are used in the latter expression; \( dC = r \, dx \, dx_3 \) and \( \partial/\partial n = -\partial/\partial r \).

Newman made the Haskind-relations known and used them [3.12], while he also extended them for the case that the forward speed is no longer zero [3.4]; this discussion will be continued in chapter 5. Since the oscillation potential \( \varphi_k \) at a large distance from the vessel is associated with the energy radiation by outgoing surface waves, obviously a relation will exist between the damping coefficients \( B_{kk} \) and the wave exciting forces \( X_{wk} \). As the former is a function of the frequency of motion only and the latter of the frequency and of the direction in which the waves travel with respect to the ship, it may be expected that such a relation will involve the integral of the exciting forces over all angles of incidence. Accordingly Newman [3.12] derived for an arbitrary three-dimensional floating or submerged body at zero forward speed in deep water:

\[
B_{kk} = \frac{\omega^3}{4\pi \rho g^2} \int_0^{2\pi} \left[ X_{wk}(\mu) \right]^2 d\mu \quad (3.4.7)
\]

An inverse relation would be much more desirable because the damping coefficients are comparatively easy to obtain from the far field oscillation potential or by methods discussed in the next chapter. But at the same time it is clear that this will not be possible in general. The two exceptions being bodies with rotational symmetry about a vertical axis and two-dimensional bodies. For symmetrical bodies it is easily verified that (3.4.7) results in:

\[
X_{wka} = \zeta_a \left[ \frac{4\rho g^3}{\omega^3} B_{kk} \right] \cos \mu \quad (k = 1, 5)
\]

\[
X_{wka} = \zeta_a \left[ \frac{4\rho g^3}{\omega^3} B_{kk} \right] \sin \mu \quad (k = 2, 4)
\]

\[
X_{w3a} = \zeta_a \left[ \frac{2\rho g^3}{\omega^3} B_{33} \right] \quad (3.4.8)
\]

\[
X_{w6a} = 0
\]

The two-dimensional case of beam waves on infinitely long cylinders leads to [3.12]:

\[
X_{wka} = \zeta_a \left[ \frac{\rho g^2}{\omega} B_{kk} \right] \quad (k = 2, 3, 4) \quad (3.4.9)
\]

It is very attractive to avoid the diffraction problem but from the discussion above it is clear that the results only have a limited applicability. Only in some special cases can the exciting force magnitude be found and it is stressed that its phase relation cannot be obtained at all. However, when the above relations can be used they provide simple results of the same accuracy as obtained for the oscillation problem: no further approximations have been introduced.

4 The determination of the fluid reactive forces; the value of the hydrodynamic coefficients

4.1 General observations

In this chapter only the reactive forces of the fluid on the body due to the oscillations in still water will be considered. Thus a solution will be sought for the problem of the harmonic oscillations of a rigid body, moving at forward speed on the surface of an ideal fluid, as formulated in section 1.2. This is a linear boundary-value problem in potential theory, for which the conditions have been stated in the preceding chapter, while some properties of the solution have been derived as well. It has also been remarked already that in principle the problem can be solved numerically. However, a computer programme for three-dimensional cases including the free surface effects does not yet exist. And it has never been possible to obtain exact, analytic solutions for other forms than a sphere or an ellipsoid. Actual computations which have been published for such elementary bodies are generally even more restricted in the sense that they only apply
to zero forward speed or to a limited range of motion frequencies.

A numerical solution of the general three-dimensional case at forward speed will no doubt require a great effort. It is possible with the aid of a finite element technique and a very large computer. Ultimately it will be the only correct solution of the mathematical model to which the physical problem has been reduced. But on present indications it is not at all obvious that the physical assumptions will produce results of such an accuracy that the labour involved in solving the numerical problem is justified. Nor has it been ascertained that other, more simplified methods intended for practical purposes will not be sufficient for the aims pursued. The information available on these points almost exclusively concerns the longitudinal heave and pitch motions. Besides it is not very likely that a computational procedure will be available shortly, while there is a definite need for qualitative and quantitative information on the dynamic performance of seagoing vessels now. Further numerical solution will neither permit the evaluation of certain effects such as the influence of forward speed or changes in the shape of the body explicitly. For these reasons a first and more or less analytic solution, approximate as it might be, is of great value. If successful it provides a much better possibility to gain insight in the occurrences and to ascertain trends, which can possibly be developed into methods to improve the properties of the system concerned.

Several of such approximations have been attempted in the past 15 years, again mainly exclusively for heave and pitch motions. But none of these approaches led to satisfactory results, either by serious departures from the physical reality as observed (for instance hydrodynamic forces being of higher order or not exhibiting resonance effects) or by insurmountable mathematical difficulties. The reasons why are indicated simply and clearly in [4.1].

It has been shown that for the longitudinal ship motions and the range of frequencies associated with them the so-called strip theory produces the most acceptable results. Therefore each cross section of the ship is considered to be part of an infinitely long cylinder. Each two-dimensional problem so constructed is solved separately and after that the solutions are combined in some way or another to yield a solution for the ship as a whole. It was originally introduced by Korvin-Kroukovsky (see [1.13]), analogous with the existing theory for air-ships. Many authors have contributed to a further development of the strip theory and equally many have criticized it for not being rationally founded. In what follows in this chapter also a strip method approach will be used. Where by a strip method or strip theory will simply be understood the stringing of a series of two-dimensional elements to construct an approximate solution for an actually three-dimensional problem. The assumptions which form the basis for the analysis will be stated explicitly at the beginning. A consequent derivation from there on along the lines of potential theory will ultimately lead to practical formulae for the hydrodynamic coefficients.

By the description of the concept strip theory it is clear that the longitudinal translation surge cannot be dealt with. This motion will be left out of consideration. In the two-dimensional problem taking place in a transverse plane the cross section can only perform swaying, heaving and rolling. A solution for the three-dimensional pitching and yawing can be obtained by making the hypothesis that locally these rotational motions are equivalent to a vertical and a transverse translatory motion of angle times the distance from the axis of rotation.

The criticism on strip theory is understandable. It certainly has the great drawback that it neglects the mutual interactions between the various cross sections. But it has been shown that for slender bodies strip theory results logically from the truly three-dimensional theory for high frequencies of motion. So it may be expected that the correctness of this neglect depends primarily on the range of frequencies involved in relation to the size of the body, or in physical terms on the relative length of the waves generated by the oscillations and the dimensions of the body. This can be understood easily. Short waves will not be affected distinctly by parts of the body being many wave lengths away and vice versa. But for long waves the same parts are close to the source of the disturbance and will directly attribute to the hydrodynamic phenomena. Naturally the basic principle of strip theory will break down at the ends of a body. Looking at the matter in this physical way another aspect will be formed by the phase relation of the motions of the various sections. A phase identity for all sections as with sway, heave and roll motions will resemble two-dimensional conditions, while a phase transition of 180 degrees at mid-length as with pitch and yaw will promote interference effects.

The above is only a qualitative evaluation; nobody can say where the limits of relatively high frequencies or of long waves will be, nor to what extent the end effects will influence the ultimate results. Therefore a thorough investigation is necessary, in which careful experiments are indispensable.

In the whole strip theory approach three major stages can be distinguished. Firstly, the solution of the two-dimensional problem of oscillating cylinders.
In this stage the elementary local values of the hydrodynamic coefficients must be determined. Secondly, the combination of these values to approximate the three-dimensional coefficients at zero forward speed. Here physically three-dimensional effects come into play, but only as far as due to the ship's form. The strip theory neglects them. In the third and final stage the same three-dimensional case is considered at forward speed. The speed adds an additional three-dimensional aspect to the occurrences which will be included in an approximate manner. It further leads to the interactions between the steady and the unsteady problems as mentioned in chapter 3. These interactions can only partly be accounted for. Each stage has been studied separately. The results will be presented in the following sections.

The problem is treated in the space fixed system of Cartesian coordinates \(ox_1x_2x_3\); \(ox_1x_2\) coincides with the undisturbed water surface, \(ox_1\) being the longitudinal direction, positive forward, and \(ox_2\) the transverse direction, positive to starboard. The vertical axis \(ox_3\) is vertical downwards. The origin is situated at mid-length in the line of intersection of the water surface and the longitudinal plane of symmetry. The system is illustrated in figure 4.1.

Fig. 4.1 The coordinate systems.

In chapter 2 the quantities \(A_{ij}\) and \(B_{ij}\) have been introduced formally as coefficients in a convenient representation of the hydrodynamic forces. The notation of these hydrodynamic coefficients will now be altered slightly to correspond with earlier publications. The \(A_{ij}\) and \(B_{ij}\) in the space fixed system will further be denoted \(a_{ij}\) and \(b_{ij}\) for the general three-dimensional case at forward speed. The two-dimensional coefficients per unit length are indicated by an accent \((a_{ij}^0\) and \(b_{ij}^0)\). When the forward speed is zero a superscript 0 is added: \(a_{ij}^0\), \(b_{ij}^0\) and \(a_{ij}^0\), \(b_{ij}^0\). For ease of working the indices \(i\) and \(j\) will be expressed in numbers during the process of derivation, but in the final results the numbers will be replaced by letters indicating the type of motion concerned: 1, 2, 3 becomes \(x, y, z\) and 4, 5, 6 becomes \(\phi, \theta, \psi\), respectively. The coefficients \(a_{ij}\), \(b_{ij}\), \(c_{ij}\) as given here are strictly associated with the geometrical choice of the coordinate system. For a certain underwater shape they are essentially unvarying. But in the equations of motion as presented in chapter 2 ultimately the value of the coefficients with respect to axes passing through the centre of gravity are needed. The coefficients in the system \(Gxyz\) (see figure 4.1) will be indicated by \(p_{ij}\), \(q_{ij}\) and \(r_{ij}\). As mentioned in section 2.3 the motions will be called \(x, y, z, \phi, \theta\) and \(\psi\). To obtain the coefficients a simple transformation of coordinates is necessary, which will be described in section 4.5. It has the advantage that when the position of the centre of gravity changes (especially in height) with different ways of loading the vessel, \(a_{ij}\) and \(b_{ij}\) remain unaltered and the computations need not be repeated. Moreover the influence of the position of \(G\) then appears explicitly in the ultimate hydrodynamic coefficients.

4.2 The two-dimensional case

If a strip theory ever has to produce correct results for the hydrodynamic coefficients, the calculation of the two-dimensional values must be rather accurate. Therefore an extensive study on this point was made by the author in 1967 and 1968. The experiments performed were the first to produce measured values of the coefficients in swaying and rolling, so that the calculations could be checked. The results have been published in [4.2] and more detailed in [4.3].

The three-dimensional potential problem at \(V = 0\) has been formulated in the equations (3.1.1). Two-dimensionally it is fully analogous. Only the radiation condition must be reformulated. In two dimensions it states that the potential for large \(x_3\) must be of such a form that a train of regular waves at constant amplitude progresses from the cylinder to infinity. It is worthwhile viewing the free surface condition a little closer. At very low frequencies of motion the first term is eliminated and the condition reduces to:

\[
\frac{\partial \varphi}{\partial x_3} = 0 \quad (x_3 = 0)
\]

which is the condition for a rigid wall. For very high frequencies the first term dominates the second and it degenerates into:

\[
\varphi_{ij} = 0 \quad (x_3 = 0)
\]

The fact that both terms are retained implies that they are supposed to be of the same order, or that \(k\) and \(1/L\) (with \(L\) some characteristic length) are comparable. This leads to \(\omega^2 L/g \approx O(1)\). When for \(L\) the half
beam of the section is chosen this means that the square of the non-dimensional frequency parameter $\omega/\sqrt{(B/2g)}$ has to be of $O(1)$. What the limits are is not clear. From experiments it turns out that the agreement with computations is generally good between 0.10 and 4 ($\omega/\sqrt{(B/2g)}$ between about 0.3 and 2). This is of $O(1)$ indeed. For frequencies outside this range reliable experimental results are not available.

In 1949 Ursell [4.4] found an analytic solution of the problem for the case of a circular cylinder heaving in the surface of a further unrestricted fluid domain, which was valid at a large range of frequencies. The potential is constructed of a source potential and a sum of multipole potentials both placed in the origin:

$$\Phi_3 = C \cdot \text{Re} \left[ \left\{ \varphi_{\text{source}} + \sum_{m=1}^{\infty} c_m \varphi_{2m} \right\} e^{-i\omega t} \right]$$  (4.2.1)

$C$ is a constant, while the coefficients $c$ are chosen in such a way that the boundary condition on the cylinder surface is satisfied. Both $\varphi_{\text{source}}$ and $c$ are complex, but $\varphi_{2m}$ is real. In a real notation the potential becomes:

$$\Phi_3 = C \left[ \left\{ \varphi_s + \sum_{m=1}^{\infty} p_m \varphi_{2m} \right\} \cos \omega t + \right.$$  
$$\left. + \left\{ \varphi_s + \sum_{m=1}^{\infty} q_m \varphi_{2m} \right\} \sin \omega t \right]$$  (4.2.2)

The method was soon extended to other forms by the introduction of a conformal transformation. It is well-known that a great variety of forms can be mapped to the unit circle by the transformation:

$$w = a \left\{ \zeta + \sum_{n=0}^{\infty} a_{2n+1} \zeta^{-(2n+1)} \right\}$$  (4.2.3)

where $w$ and $\zeta$ are complex; $w$ represents a point in the actual physical plane and $\zeta$ the corresponding point in the reference plane. $a_{2n+1}$ are the transformation coefficients, while $a$ is a scale factor. A class of sections given by a two parameter transformation ($n = 0$ and $n = 1$) has for long been known as the so-called Lewis-forms [4.5]. The extension of Ursell’s method to elliptic cylinders and to Lewis-forms was made by Tasai [4.6] and on a principle somewhat different from Ursell's by Grim [4.7; 4.8], while Porter [4.9] ultimately formulated the solution for heaving of an arbitrarily shaped cylinder by not imposing restrictions on $n$. In Porter’s work also a restricted waterdepth was accounted for.

For swaying and rolling the procedure is fully analogous. The source potential must now be replaced by a dipole because of the asymmetry of the flow for these motions. For the same reason the multipole potentials are a little different in form. Both Grim [4.10] and Tasai [4.11] have published data for elliptic and Lewis-form cylinders. But the generalization of Ursell's method to motion in all three degrees of freedom for arbitrarily shaped cylinders was not given until 1967 by De Jong in Delft [4.12]. An English and somewhat adapted version of this work will be published in 1972 [4.13]. Details of the mathematical formulation and of the actual computations have been given by the author in appendix 1 of [4.3] following entirely De Jong's work.

An extensive study on the correctness of the underlying assumptions and of the mathematical solution could now take place. Previous measurements had shown that for heaving the physical simplifications were quite acceptable, though even here the experimental evidence was scarce. But to the knowledge of the author not a single experimental check on swaying and rolling did exist. Therefore he performed a series of forced oscillation experiments with a number of cylindrical models in the towing tank of the Shipbuilding Laboratory of the Delft University of Technology. The experiments covered a range of frequencies wider than the wave induced motions of ordinary vessels at sea will exhibit. The cross section of the cylinders is shown in figure 4.2. The circular, the triangular and the rectangular section at three different $B/T$-
ratios were subjected to all three degrees of freedom. In heaving two additional cylinders were investigated to judge upon the importance of a close representation of section shape. They were derived from the circle by a conformal transformation with three parameters \( n = 0, 1 \) and \( 2 \). When these shapes are approximated by Lewis-forms \( n = 0 \) and \( 1 \) only they are identical with the circle. As mentioned before this study has been reported in detail in [4.3]. Here only the main conclusions will be summarized.

1. The two assumptions of fluid idealization and linearity are very well acceptable for the quantitative analysis of oscillations of ship-like sections in heave and sway at any frequency.

2. In roll the conclusion is less definite. On the one hand the experimental results are distinctly less accurate due to the mechanical difficulties of the tests and to the small forces to be measured. On the other the hydrodynamic quantities for the rotational motion are of a smaller order of magnitude than for both translations, so that physical departures from the assumptions are of much more importance. Yet it was shown that the same two simplifications were confirmed sufficiently well to be justifiable for a practical engineering solution. However, some viscous effects have to be taken into account additionally.

3. At the edges of relatively sharply edged sections separation occurs. This is a source of energy loss due to eddy formation which contributes to the damping coefficients \( b_{ij} \). Considering the magnitude of the wave radiation in swaying and heaving this effect is not of much importance for the translations. For rolling it cannot be neglected, however. Therefore such contributions have to be added to the coefficients \( b_{ij}^{0} \) and \( b_{ij} = b_{ij}^{0} \). Apparently the shedding of eddies does not seriously affect that part of the pressure distribution which is in phase with the body acceleration, because the mass coefficients \( a_{ij}^{0} \) are predicted fairly correctly by potential theory, at least for the sections tested. This conclusion may be invalidated for sections provided by appendices as bilge keels, stern tubes and the like.

4. For the sections investigated a close fit representation of the section contour did not greatly change the results computed for a Lewis-form approximation. As far as the accuracy of the tests permits the differences are fully confirmed by the experiments, however.

5. The formulation of the problem is sufficiently accurate to be a valuable basis for a further quantitative approach. The effects mentioned in conclusion 3 may be dealt with separately and can be included to improve the numerical accuracy.

By way of example the results for the cylinder with rectangular cross section are presented in figure 4.3 to 4.5. It is noted that these figures only serve as an illustration. They cannot prove the conclusions stated above without considering all the details of the investigation [4.3].

Quite another point of the above investigation does not concern the physics of the problem but the numerical aspects of Ursell’s method. The results of these considerations have not been reported previously. The velocity potentials are of the form of equation (4.2.2). In principle they contain an infinite series of multipole potentials. Ursell showed in [4.4] that for the heaving circular cylinder the expansion was rapidly convergent for \( \omega^2 (R/2g) < 1 \). In fact he actually proved that \( p_{2m} \) and \( q_{2m} \) then were of the order \( 1/m^3 \) and he concluded that it sufficed to take \( M = 6 \) as the last term. For higher frequencies he showed the convergence in [4.4a], although the expansion is not

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Fig. 4.3 Added mass and damping coefficient in heaving.

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Added mass and damping coefficient in swaying  

Fig. 4.4  Coupling coefficients of sway into roll

Added mass moment of inertia and damping coefficient in roll  

Fig. 4.5  Coupling coefficients of roll into sway
practical in this range. A proof that the series converges for all three modes of motion and for an arbitrarily shaped cylinder has not been given, but it is acceptable that this will be the case. While actually performing the computations it soon turned out, however, that for swaying and rolling $M$ generally had to be increased. The expansion coefficients $p_{2m}$ and $q_{2m}$ sometimes decreased irregularly and it was evidently not allowed to truncate the series so soon. So the convergence for the lateral motions may be considerably less than for the vertical motion. Much depends on the section shape and on the frequency of motion. It appeared that this was even true in heaving. So Ursell's findings may certainly not be generalized. The coefficients $p_{2m}$ and $q_{2m}$ are determined by satisfying the boundary condition on the body surface in a number of points $I \geq M$ by the method of least squares. It is recommended that the expansion coefficients are included in the output of the computer and that some trial computations are made with variable $M$ and $I$ to consider its effect.

The fact that there exists a fixed relation between the damping coefficients and the amplitude of the radiated waves:

$$b_i^0 = \frac{\partial \theta^2}{\omega^2} \left( \frac{\zeta_i}{s_i} \right)^2$$  \hspace{1cm} (4.2.4)

permits another check on the computations. It can easily be shown [4.3] that a function of various computational components must be essentially constant at any frequency. In defining an error function as the difference with this constant the numerical accuracy of the computations can be judged to some extent. In this way it turned out that the accuracy could not be considered to be sufficient in some cases, despite the fact that the expansion coefficients had become acceptably small. No definite reasons for this fact can be given. Sometimes deficiencies in $p_{2m}$, $q_{2m}$ and in the error function go hand in hand and sometimes they are evidently rather independent of one another. In general it may be expected that sections with a rather low and with a rather high $B/T$-ratio, and very low or very high frequencies of motion may lead to difficulties in one way or the other. However, generally these limitations are well beyond the range of interest for practical applications.

For the reasons stated above any computation must be considered carefully on both points before accepting the numerical values produced by the computer. The computations are too complicated to judge about the extent in which inaccuracies make themselves felt in the hydrodynamic coefficients. From experience it appears that the coefficients are less sensitive to the numerical details than the checks discussed. Of course the coupling coefficients are most liable to be influenced. The only thing which can be said is that the results will be correct when the expansion coefficients and the error function are considered satisfactory; the opposite need not be true. It seems that Tasai has not noticed the above sufficiently when he presented his computations for some classes of Lewis-forms in [4.11]. He did not give many numerical details in his report, but he stated that he followed the same procedure as Ursell with $M = 6$. This explains that the comparative calculations with the computer programme in Delft showed fairly large differences with Tasai's values in some cases, while the symmetry relations $a_{2m}^0 = a_{2m}^0$ and $b_{2m}^0 = b_{2m}^0$ were not always fulfilled in his results. In accordance with the general formulae (3.2.6) this should definitely be true.

Apart from numerical objections as to the series expansion of multipoles the conformal transformation method also has certain drawbacks. In the first place it is only possible to deal with simply connected and symmetrical sections having vertical tangents at the waterline and a horizontal tangent at the line of symmetry. In fact the conformal transformation maps one half of the underwater part of the section on all four quadrants so that a completely symmetrical body arises in the reference plane. In practice, however, the matter of the tangents does not constitute a real limitation to the user. But it is rather difficult to obtain the transformation coefficients $a_{2m+1}$ from the inverse of the relation (4.2.3). Some methods have been developed, which are discussed in [4.13] to [4.16].

Ursell has presented his multipole solution for an infinite fluid domain and so far only this case has been generalized and programmed. It is possible to adapt this method to include a restricted waterdepth but arbitrary boundaries of the fluid domain, especially when they are no longer symmetrical, will be extremely difficult. It is doubtful whether an analytic approach by a multipole expansion or otherwise remains possible. Then a direct numerical solution of the potential problem must be attempted. Such a solution has been given by Frank [4.17]. The section contour is approximated by a series of straight line elements, while the velocity potential is represented by a distribution of wave sources over the contour. The strength of the sources is constant along each segment and must be determined from an integral equation resulting from the boundary condition on the cylinder surface. So this approach avoids all conformal transformation techniques and the difficulties associated therewith. But it has another drawback, namely that the integral equation method fails to give a solution at certain frequencies, the so-called irregular frequencies. For many
cases these frequencies will be outside the range of interest for practical problems. And when they are not, a smoothing process applied to the hydrodynamic coefficients as functions of frequency can remove the singularities from the results. Therefore this phenomenon will not be a serious limitation to the applicability. Frank’s method as described in [4.17] has also only been worked out for symmetrical sections and an infinite fluid domain. But it is much more suited to be extended to unsymmetrical contours, to heeled sections, multiple hull configurations and to various boundaries of the fluid domain as the bottom, a wall or the like, although it will certainly require quite an effort to do so.

4.3 The three-dimensional case at zero forward speed

The mathematical formulation of this case has been presented in section 3. It is now required to find an (approximate) solution for \( \varphi_j \) from the known two-dimensional potentials. To this end three assumptions will be made. The first is that any change in a longitudinal direction is small with respect to changes in a transverse or vertical direction. Or explicitly:

\[
\begin{align*}
  f_1 &= e f_1, \\
  \frac{\partial}{\partial x_1} &= e \frac{\partial}{\partial \xi_1} \\
  x_1 - x_1(0) &= \frac{1}{e} \xi_1
\end{align*}
\]

(4.3.1)

where \( e \) is a small parameter and underlined quantities are of the same order as the corresponding quantities in the other directions; \( x_1(0) \) is the fixed and finite coordinate of some point. This assumption will generally be a fair approximation for ships and other fairly slender bodies. It is violated most at the ends of the vessel. The direction cosines \( f_j \) have been defined in (3.1.2). With the statement above the following simplifications must be accepted:

\[
\begin{align*}
  f_5 &= x_3 f_1 - x_1 f_3 \approx -x_1 f_3 \\
  f_6 &= x_1 f_2 - x_2 f_1 \approx x_1 f_2
\end{align*}
\]

(4.3.2)

since the other term is of an order \( e^2 \) smaller.

For each separate cross section of the body the potential is known for sway, heave and roll, when that section is considered to be part of an infinitely long cylinder. Since the sections are different in form, to each section belongs a different \( \varphi_j(x_2, x_3) \) with \( j = 2, 3, 4 \). When points on the contour \( S_0 \) are indicated by \((x_{1b}, x_2, x_3)\) therefore a parameter \( x_{1b} \) can be attached to the potentials: \( \varphi_j = \varphi_j(x_{1b}, x_2, x_3), j = 2, 3, 4 \). The functional relation in \( x_{1b} \) is not known, but for any \( x_{1b} \) the corresponding \( \varphi_j \) can be determined. The second assumption now is that in the absence of forward speed and for \( j = 2, 3, 4 \) the actual three-dimensional potential is sufficiently approximated by the successive two-dimensional potentials thus linked together. So the mutual interaction of sections is not accounted for. It is an essential limitation of strip theory. Of course, this assumption is again violated specially at the vessel’s ends, where the conditions deviate significantly from the two-dimensional situation. It will further be supposed that \( \varphi \) and its derivatives will be continuous in \( x_{1b} \). In practice a number of discrete \( \varphi_j \)'s will be known. The spacing can be made as small as desired, after which numerical procedures can be applied to obtain all relevant quantities.

The above process is only valid for \( j = 2, 3, 4 \). It has to be generalized for \( j = 5 \) and 6. Therefore the third and final assumption stipulates that in hydrodynamic respect pitching for any section is identical with local heaving at the same velocity and that yawing is identical with local swaying at the same velocity. This leads to the local potentials:

\[
\begin{align*}
  \varphi_5(x_{1b}, x_2, x_3) &= -x_{1b} \frac{\partial}{\partial x_3} \varphi_3(x_{1b}, x_2, x_3) \\
  \varphi_6(x_{1b}, x_2, x_3) &= x_{1b} \frac{\partial}{\partial x_3} \varphi_3(x_{1b}, x_2, x_3)
\end{align*}
\]

(4.3.3)

again indicating the longitudinal position of a section by \( x_{1b} \).

The parameter \( x_{1b} \) has been introduced purposely to distinguish the length-coordinate on the body from the general longitudinal coordinate \( x_1 \). This has been done to emphasize the specific character of the potentials obtained as above. A function \( \varphi_j(x_1, x_2, x_3) \) and the use of \( x_1 \) instead of \( x_{1b} \) in the expressions for \( \varphi_5 \) and \( \varphi_6 \) would suggest that the potential could be evaluated at any point in the flow field, for instance to obtain the wave pattern at a large distance from the body. In general this is certainly not possible. The potentials remain basically two-dimensional in nature. They may produce a fairly correct picture of the real flow next to the body, that is in any point \((x_1, x_2, x_3)\) for \( -\frac{1}{2} L < x_1 < +\frac{1}{2} L \), but they do not contain any information for other \( x_1 \). Therefore they will definitely fail to do so in the region before and behind the body. But that is not the objective of the process followed either. It may be expected that the potentials thus
constructed will represent the occurrences as good as possible very close to the vessel. Therefore they will only be used to evaluate the pressure distribution on the body surface and so to find the hydrodynamic coefficients.

A consequence of the above is that it is impossible to check whether the various \( \varphi_j \)'s satisfy the mathematical conditions imposed on them three-dimensionally. In fact only the condition on \( S_0 \) can be considered straightforwardly. For, of course, on \( S_0 \), \( x_{1b} \) is identical with \( x_1 \). For the other requirements it would appear possible to take two attitudes: \( x_{1b} \) is entirely equivalent with \( x_1 \) (for instance in the region next to the body) or \( x_{1b} \) is a constant (for instance in the region forward and aft of it). It is, however, not very useful to do so. The potentials satisfy the ordinary two-dimensional requirements in a transverse plane and influences from outside that plane cannot be included.

The dynamic pressure in points on \( S_0 \) is given by the linearized Bernoulli-equation:

\[
p_j(x_{1b}, x_{2b}, x_{3b}; t) = i$q_0 \varphi_j(x_{1b}, x_{2b}, x_{3b}) e^{-i\omega t} (4.3.4)
\]

and the hydrodynamic forces by:

\[
F_k(t) = -\int_{S_0} p_j \varphi_k ds = -i$q_0 e^{-i\omega t} \int_{LoC_0} \varphi_j \varphi_k ds dx_{1b} (4.3.5)
\]

where \( C_0 \) is the immersed contour of a section in its middle position and \( ds \) a line element along \( C_0 \).

In the same way as the formulae (3.1.12) have been derived for the ship as a whole, the following local values will be obtained:

\[
a_{kj} = -\frac{q}{s_{kj}} \int_{C_0} \varphi_j \varphi_k ds (k = 2, 3, 4) \tag{4.3.6}
\]

\[
b_{kj} = -\frac{q}{s_{kj}} \int_{C_0} \varphi_j \varphi_k ds (j = 2, 3, 4)
\]

The results which have been derived above could have been anticipated. For sway, heave and roll and their mutual couplings the coefficients are simply obtained by a mere integration of the known two-dimensional coefficients along the length. The coupling coefficients of sway-yaw, roll-yaw and heave-pitch are determined by the integration of the same values multiplied with its distance from the origin while for the added mass and damping in pitch and yaw the elementary coefficients must be multiplied with this distance twice, once to transfer the rotation in a translatory motion and once to obtain a moment from a force.

It is worthwhile to compare exact three-dimensional solutions (as far as they do exist) with the strip theory approximations. This is only possible for bodies of ellipsoidal form. And it must be emphasized that the results for such forms are certainly not conclusive for ship-like vessels. The well rounded ends of ellipsoids are undoubtedly liable to three-dimensional effects which are not present with ships. Their pointed ends will favour the resemblance with two-dimensional conditions. It follows that if the agreement for ellipsoids is reasonable it will be good for ships, but if it is bad no conclusion can be drawn. In the opinion of the author a comparison of an exact solution and a strip theory approximation yet provides some insight into the effects of form which are necessarily neglected.

Since the validity of strip theory will depend on the relative length of the body it would be illustrative to consider a family of ellipsoids with varying ratio of axes. Unfortunately sufficient information to do so is not available. The most comprehensive numerical results until now have been published by Kim [4.18]. From his data a sphere, two spheroids and an ellipsoid of revolution have been selected. The three spheroids form a family with varying draught; \( L/B = 1 \) and \( B/T = 2, 4 \) and 8 respectively. The sphere and the ellipsoid are two members of a family with varying length; \( B/T = 2 \) and \( L/B = 1 \) and 4. A third member with \( L/B = 8 \) would be very desirable but results for this body do not exist. The poor amount of information is illustrated in table 4.1. The data are incomplete in other respects as well, so all modes of motion and a sufficiently large frequency range cannot be covered. The results for heaving and pitching are presented.

<table>
<thead>
<tr>
<th>Table 4.1 Family of ellipsoids</th>
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<tbody>
<tr>
<td>( L/B )</td>
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<tr>
<td></td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>8</td>
</tr>
</tbody>
</table>
graphically in the figures 4.6 to 4.9. From those figures it appears that the agreement is poor. The influence of \( B/T \)-ratio is not large. But an increase in \( L/B \) produces much improvement. Since most ships have ratios from 6 to 7 it would be most interesting to compare the case \( L/B = 8 \) as well. The limiting values correspond much better when the length increases and the frequency at which the three-dimensional solution degenerates into a strip theory is moving much more towards the range of practical interest. Besides, the degeneration in itself does only take place for slender bodies. For a model of the Todd Sixty Series, block coefficient 0.70, a comparison of the computed hydrodynamic coefficients has also been made with experimental values. The results are presented in the next section, together with experimental data at forward speed. The expectation that the three-dimensional effects will be much less pronounced than for the ellipsoids shown above turns out to be true. It will appear that the computed hydrodynamic coefficients for ships for \( V = 0 \) and in the range of frequencies involved in wave induced motions are confirmed well by the experiments, so that the hypotheses made form a reasonable basis for the extension to the case at forward speed.

### 4.4 The three-dimensional case with forward speed

The ship is now proceeding at a constant speed \( V \). Since the coordinate system \( ax_1, x_2, x_3 \) travels with the ship at the same speed the problem is that of a stationary body performing oscillations in a fluid flow with a homogeneous velocity distribution \( -V \) at infinity.

The potential for the steady problem is given by:

\[
\Phi(x_1, x_2, x_3) = \varphi(x_1, x_2, x_3) = -V x_1 + \varphi_0(x_1, x_2, x_3)
\]  
(4.4.1)
Fig. 4.8 Added mass and moment of inertia for heaving and pitching ellipsoids.

Fig. 4.9 Damping coefficients for heaving and pitching ellipsoids.

\( \varphi_0 \) is a disturbance potential due to the presence of the body; \( \Phi_s \) is independent of time. The requirements for \( \varphi_0 \) are clearly:

\[
\begin{aligned}
\frac{\partial^2 \varphi_0}{\partial x_1^2} + \frac{\partial^2 \varphi_0}{\partial x_2^2} + \frac{\partial^2 \varphi_0}{\partial x_3^2} &= 0 & \text{(in R)} \\
V \frac{\partial \varphi_0}{\partial x_1} + \frac{\partial \varphi_0}{\partial x_3} &= 0 & \text{(on } x_3 = 0) \\
\frac{1}{2}V^2 - V \frac{\partial \varphi_0}{\partial x_1} - g \frac{\varphi_0}{\partial x_3} &= 0 & \text{(4.4.2)}
\end{aligned}
\]

\[
f_1 \left( -V + \frac{\partial \varphi_0}{\partial x_1} \right) + f_2 \frac{\partial \varphi_0}{\partial x_2} + f_3 \frac{\partial \varphi_0}{\partial x_3} = 0,
\]

or:

\[
\frac{\partial \varphi_0}{\partial n} = Vf_1 \quad \text{(on } S_0) 
\]

\[
\lim_{r \to \infty} \varphi_0 = 0
\]

In many cases the velocity components due to the disturbance potential will be of an order smaller than \( V \). At high \( Fr \) this is certainly no longer true. Then the ship sailing in still water creates an impressive wave system which obviously cannot be neglected any more in the occurrences. Of course, things depend on the shape of the vessel and the assumption of a certain degree of slenderness will be favourable in this respect. But the question is more complicated than that. It is the relation of form and speed which determines the wave pattern, while it is also true that the visible disturbance of the free surface does not exclusively dictate pressure and velocity components on the body resulting from \( \varphi_0 \). An example of the wave profile along the hull of a Todd Series Sixty model at various speeds is shown in figure 4.10. In principle it will be possible to determine \( \varphi_0 \) for any shape and any speed.
in the form of a source distribution on the vessel’s surface. This is of no practical use, however. The best which can be done in the problem under consideration is therefore to omit \( \varphi_0 \). This will certainly impose restrictions on the validity of the solution of which the limits are not known either. Experiments will have to show the significance of this neglect. For the time being \( \varphi_0 \) is retained in the further formulation to allow as complete a specification as possible.

The ultimate problem is a combination of the steady one discussed above and the unsteady one of the preceding section. Let it be described by the sum of the potentials:

\[
\Phi(x_{1b}, x_2, x_3; t) = \Phi_s + \Phi = -Vx_1 + \varphi_0(x_1, x_2, x_3) + \varphi_j(x_{1b}, x_2, x_3)e^{-\text{int}}
\]

(4.4.3)

It has already been discussed in section 3.1 that two boundary conditions are nonlinear in nature: those on the free surface and on the body. Thereby interaction terms arise which in fact invalidate the superposition of the separate solutions. It will be assumed that the form of the potential (4.4.3) is correct, but that the component potentials \( \varphi_0 \) and \( \varphi_j \) may necessarily be adapted to include these interaction effects.

Timman and Newman [4.19] have shown that it is not correct for \( \Phi_j \) to satisfy the boundary condition on the body surface at rest. On the contrary, it must be satisfied in the actual instantaneous position. Here a part of the interaction effects emerges. They derived the following first order approximation to be met at \( S_0 \):

\[
\frac{\partial \varphi_j}{\partial n} e^{-\text{int}} = n \cdot \left[ \frac{\partial \varphi_j}{\partial t} + V \times (\mathbf{a}_j \times V \Phi_0) \right]
\]

(4.4.4)

where \( \mathbf{a}_j \) is the displacement vector from \( S_0 \) to \( S \) of any point on the surface. In terms of the motions \( s_j \) the following relations exist:

\[
\mathbf{a}_j = s_j \hat{j}_j \quad (j = 1, 2, 3)
\]

\[
\mathbf{a}_j = s_j(i_{j-3} \times r) \quad (j = 4, 5, 6)
\]

(4.4.5)

\[
\mathbf{a}_j \cdot n = f_j s_j = f_j s_0 e^{-\text{int}}
\]

With \( r \) the radius vector to a point on \( S_0 \); \( r = (x_{1b}, x_{2b}, x_{3b}) \). The first term in (4.4.4) represents the same boundary condition on \( S_0 \) as in the case with zero forward speed. The second is an additional term due to the interactions. It can easily be shown that for the separate modes of motion it results in:

\[
V \times (\mathbf{a}_j \times V \Phi_0) =
\begin{cases}
  s_j V(i_j \times V \varphi_0) & (j = 1, 2, 3) \\
  Vs_j i_{j-3} - V(s_j \cdot r \cdot V \varphi_0) \times i_{j-3} + & (j = 4, 5, 6)
\end{cases}
\]

(4.4.6)

When the disturbance potential is omitted this reduces to:

\[
V \times (\mathbf{a}_j \times V \Phi_j) =
\begin{cases}
  0 & (j = 1, 2, 3) \\
  Vs_j i_{j-3} & (j = 4, 5, 6)
\end{cases}
\]

(4.4.7)
The second group of terms is a steady pressure change, which may cause the body to obtain a different condition of equilibrium at forward speed than at rest. If necessary this new equilibrium can be taken as the middle position about which the dynamic oscillations are performed, but in general the differences will be small. The steady part in the pressure is not considered any further. In the dynamic part an interaction term appears which will only vanish by omitting $\phi_0$. Combined terms like this cannot easily be neglected otherwise, since that requires that the mutual order of magnitude of the parameters in both separate problems is fixed. The dynamic pressure on $S_0$ is now given by:

\[
p_f(x_{1b}, x_{2b}, x_{3b}; t) = \quad
\]

The potentials $\varphi_2$, $\varphi_3$ and $\varphi_4$ with and without forward speed are equal, at least as long as $\varphi_0$ may be neglected.

It is also true that $\varphi_0$ must satisfy the condition on $S$ rather than on $S_0$. But since $\varphi_0$ will be omitted entirely it is of no use considering this point further.

The potentials being known, the pressure can be obtained again from the Bernoulli equation. In its full form it reads:

\[
\frac{\partial \Phi}{\partial t} + \frac{1}{2} \left[ \left( \frac{\partial \Phi}{\partial x_1} \right)^2 + \left( \frac{\partial \Phi}{\partial x_2} \right)^2 + \left( \frac{\partial \Phi}{\partial x_3} \right)^2 \right] + \frac{p}{\rho} - g x_3 = 0
\]

Introducing (4.4.3) and neglecting second order terms with respect to the corresponding linear terms it results in:

\[
\frac{\rho g}{\frac{\partial \Phi}{\partial t}} = \left( -\frac{\partial \Phi}{\partial x_1} + V \frac{\partial \Phi}{\partial x_1} - \varphi_0 \cdot V \frac{\partial \Phi}{\partial x_1} e^{-i\omega t} \right) + \\
+ \left( -\frac{1}{2} V^2 + V \frac{\partial \Phi}{\partial x_1} + g x_3 \right)
\]

(4.4.12)
\[\begin{align*}
a'_{yy} &= a''_{yy} + V \frac{\partial b''_{yy}}{\partial x_{1b}} \\
b'_{yy} &= b''_{yy} - V \frac{\partial a''_{yy}}{\partial x_{1b}} \\
a'_\phi &= a''_\phi + V \frac{\partial b''_\phi}{\partial x_{1b}} \\
b'_\phi &= b''_\phi - V \frac{\partial a''_\phi}{\partial x_{1b}} \\
a'_y &= a''_y + x_{1b} V \frac{\partial b''_y}{\partial x_{1b}} \\
b'_y &= b''_y - x_{1b} V \frac{\partial a''_y}{\partial x_{1b}} \\
a'_\phi &= a''_\phi + x_{1b} V \frac{\partial b''_\phi}{\partial x_{1b}} \\
b'_\phi &= b''_\phi - x_{1b} V \frac{\partial a''_\phi}{\partial x_{1b}} \\
a'_y &= x_{1b} a''_y + V \frac{\partial b''_y}{\partial x_{1b}} = a''_y \\
b'_y &= x_{1b} b''_y - x_{1b} V \frac{\partial a''_y}{\partial x_{1b}} = b''_y \\
a'_\phi &= x_{1b} a''_\phi + x_{1b} V \frac{\partial b''_\phi}{\partial x_{1b}} = a''_\phi \\
b'_\phi &= x_{1b} b''_\phi - x_{1b} V \frac{\partial a''_\phi}{\partial x_{1b}} = b''_\phi \\
a'_y &= x_{1b} a''_y + 2 V \frac{\partial b''_y}{\partial x_{1b}} + x_{1b} V \frac{\partial b''_y}{\partial x_{1b}} - V^2 \frac{\partial a''_y}{\partial x_{1b}} \\
b'_y &= x_{1b} b''_y - 2 V a''_y - x_{1b} V \frac{\partial b''_y}{\partial x_{1b}} - V^2 \frac{\partial a''_y}{\partial x_{1b}} \\
a'_\phi &= x_{1b} a''_\phi + 2 V \frac{\partial b''_\phi}{\partial x_{1b}} + x_{1b} V \frac{\partial b''_\phi}{\partial x_{1b}} - V^2 \frac{\partial a''_\phi}{\partial x_{1b}} \\
b'_\phi &= x_{1b} b''_\phi - 2 V a''_\phi - x_{1b} V \frac{\partial b''_\phi}{\partial x_{1b}} - V^2 \frac{\partial a''_\phi}{\partial x_{1b}} \\
a'_y &= x_{1b} a''_y + 2 x_{1b} V \frac{\partial b''_y}{\partial x_{1b}} + x_{1b} V \frac{\partial b''_y}{\partial x_{1b}} - x_{1b} V^2 \frac{\partial a''_y}{\partial x_{1b}} = x_{1b} a''_y \\
b'_y &= x_{1b} b''_y - 2 x_{1b} V a''_y - x_{1b} V \frac{\partial b''_y}{\partial x_{1b}} - x_{1b} V^2 \frac{\partial a''_y}{\partial x_{1b}} = x_{1b} b''_y \\
a'_y &= x_{1b} a''_y + V \frac{\partial b''_y}{\partial x_{1b}} \\
b'_y &= b''_y - V \frac{\partial a''_y}{\partial x_{1b}} \\
a'_z &= -x_{1b} b''_z - x_{1b} V \frac{\partial b''_z}{\partial x_{1b}} = -x_{1b} a''_z \\
b'_z &= -x_{1b} b''_z + x_{1b} V \frac{\partial a''_z}{\partial x_{1b}} = -x_{1b} b''_z \\
a'_\theta &= -x_{1b} a''_\theta + 2 V \frac{\partial b''_\theta}{\partial x_{1b}} - x_{1b} V \frac{\partial b''_\theta}{\partial x_{1b}} + V^2 \frac{\partial a''_\theta}{\partial x_{1b}} \\
b'_\theta &= -x_{1b} b''_\theta + 2 V a''_\theta + x_{1b} V \frac{\partial b''_\theta}{\partial x_{1b}} + V^2 \frac{\partial a''_\theta}{\partial x_{1b}} \\
a'_\theta &= x_{1b} a''_\theta + 2 x_{1b} V \frac{\partial b''_\theta}{\partial x_{1b}} + x_{1b} V \frac{\partial b''_\theta}{\partial x_{1b}} - x_{1b} V^2 \frac{\partial a''_\theta}{\partial x_{1b}} = -x_{1b} a''_\theta \\
b'_\theta &= x_{1b} b''_\theta - 2 x_{1b} V a''_\theta - x_{1b} V \frac{\partial b''_\theta}{\partial x_{1b}} - x_{1b} V^2 \frac{\partial a''_\theta}{\partial x_{1b}} = -x_{1b} b''_\theta \end{align*}\]
transformed directly into the corresponding coefficients at \( V = 0 \) with the aid of (3.1.12). For \( j = 5 \) and 6 the expressions (4.4.10) must be substituted after which the same process can be applied. After somewhat lengthy algebraic operations it results in the relations 4.4.16 (page 34).

Again the coefficients for the whole ship will be obtained by integration along the length.

The results obtained above differ from the usual strip theory formulae, as they have been worked out for heave and pitch. From [4.21] for instance the following previous relations can be deduced:

\[
\begin{align*}
\alpha_{zz} &= \alpha_{zz}^0 \\
\beta_{zz} &= \beta_{zz}^0 - V \frac{\partial \alpha_{zz}^0}{\partial x_{1b}} \\
\alpha_{1\theta} &= -x_{1b} \beta_{izz}^0 - \frac{V}{\omega^2} \beta_{zz}^0 + \frac{V^2}{\omega^2} \frac{\partial \alpha_{zz}^0}{\partial x_{1b}} \\
\beta_{1\theta} &= -x_{1b} \beta_{izz}^0 + 2V \alpha_{zz}^0 + x_{1b} V \frac{\partial \alpha_{zz}^0}{\partial x_{1b}} \\
\alpha_{1} &= x_{1b}^2 \alpha_{zz}^0 + x_{1b} \frac{V}{\omega^2} \beta_{zz}^0 - x_{1b} \frac{V^2}{\omega^2} \frac{\partial \alpha_{zz}^0}{\partial x_{1b}} = -x_{1b} \alpha_{1\theta} \\
\beta_{1} &= x_{1b}^2 \beta_{zz}^0 - 2x_{1b} V \alpha_{zz}^0 - x_{1b}^2 V \frac{\partial \alpha_{zz}^0}{\partial x_{1b}} = -x_{1b} \beta_{1\theta} \\
\alpha_{1z} &= -x_{1b} \alpha_{zz}^0 = -x_{1b} \alpha_{1z} \\
\beta_{1z} &= -x_{1b} \beta_{zz}^0 + x_{1b} V \frac{\partial \alpha_{zz}^0}{\partial x_{1b}} = -x_{1b} \beta_{1z}
\end{align*}
\]

(4.4.17)

So locally only \( \alpha_{1z} \) and \( \beta_{1z} \) are identical. To check the correctness of either set of formulae the author has re-analyzed the heave and pitch experiments with a segmented model performed by Gerritsma and Beukelman in 1963 [4.22, 4.23], in which he has taken part as a student. The original experimental data have been taken and were transformed to a set of axes having their origin in \( o \) instead of the centre of gravity \( G \). Moreover the results were scaled to a 10 ft. model as used by the author for further experiments. The hull form has been chosen from the Todd Sixty Series, block coefficient 0.70.

For several of the coefficients the quantitative difference between the old and the new formulae is not very pronounced. In respect of the difficulties of the tests and the possible inaccuracies in the measurements the experiments are not conclusive in favour of one of them in these cases. It is also stressed that at the ship's ends any computation according to strip theory will fail seriously. This will become particularly evident from the sway-roll-yaw experiments to be discussed hereafter. Therefore no conclusions can be based on the quantitative results for the end sections. However, from the heave coefficients \( a_{zz}^0 \) and \( a_{1z} \) for \( Fn = 0 \) and 0.20, which are presented in the figures 4.11 and 4.12, it can be seen clearly that the new equations (indicated by a drawn line) give a distinct improvement over the relations used previously (indicated by a dotted line). The experimental results for the sections and their summation for the whole model are given by \( \bigcirc \), while the experimental results for the model as a whole are denoted by \( \Box \). For the sections 6 and 7 the tendency of the calculated coefficients vs. frequency is opposite to the curves obtained earlier and this effect is fully confirmed by the experiments.

When pitching the tendency in \( b_{1\theta} \) to increase instead of decrease for the lower frequencies in the sections 2 and 6 and for the whole model seems to be confirmed by the experiments also; see figure 4.14. On the other hand the \( a_{1\theta} \) in figure 4.13 gives the impression that for the most forward section the previous calculations present a better prediction. But this fact cannot be regarded as conclusive, as remarked before.

From the results it is also apparent that the differences in the two theories for the ship concerned as a whole only lead to distinctly different \( a_{1\theta} \) and \( b_{1\theta} \). So the improvement of the present formulae over the earlier ones acquire special importance for the distribution of the hydrodynamic forces, that is for the hydrodynamic loads on the structure. Yet in motion prediction the influences manifest themselves as well, in particular at the higher forward speeds [4.24].

It is not obvious that an approach as for heave and pitch is applicable to the lateral motions as well. Especially two points are questioned. The first one is fundamental: one sometimes doubts whether a similar potential solution for the problem at forward speed is valid at all. When performing lateral motions at speed the geometrical symmetry is lost and the body is placed in an oblique flow field at a varying angle of attack. In fact the problem is related to the non-stationary airfoil theory. Due to the presence of the free surface it is much more difficult, however, while the very low aspect ratios of the order of 0.05 to 0.10 make the problem significantly three-dimensional. A practical solution of this problem looks far away. The second point is that even when a similar construction appears to be possible it is questionable if viscous effects will not prevent us obtaining quantitatively correct results. No decisive answer to these questions can be obtained otherwise than by suitable experiments. Therefore the author has performed forced oscillation experiments in sway, roll and yaw at various forward speeds with the same model as used for the heave and pitch inves-
Fig. 4.11  Sectional added mass in heaving for $F_n = 0$ and 0.20.

Fig. 4.12  Sectional mass coupling coefficient of heaving into pitching for $F_n = 0$ and 0.20.
Fig. 4.13 Sectional added mass moment of inertia in pitching for $Fn = 0$ and 0.20.

Fig. 4.14 Sectional damping coefficient in pitching for $Fn = 0$ and 0.20.
tigation. Since the towing tank of the Shipbuilding Laboratory of the Delft University of Technology would produce unacceptable wall effects the tests were done at the seakeeping basin of the N.S.M.B. at Wageningen.

During the experiments the transverse forces were measured on 9 sections of equal length, while on each section the forces were determined in two locations differing in height. The set-up is shown in figure 4.15.

In this way nine points of the distribution along the length of the hydrodynamic force and moment about the $x_1$-axis could be found. The tests covered four speeds of advance ($F_n = 0, 0.10, 0.20$ and $0.30$) and a fairly large range of frequencies ($\omega \sqrt{B/2g}$) between $0.15$ and $1.20$, so that a detailed comparison with the corresponding computations can be made. As amplitude of motion was chosen $s_{2a} = 0.05$ m ($s_{2a}/B = 0.115$) for swaying and $s_{4a} = s_{6a} = 0.10$ rad. for rolling and yawing, while some additional tests were carried out for $s_{2a} = 0.03$ and $0.07$ m ($s_{2a}/B = 0.069$ and $0.161$), for $s_{4a} = 0.20$ rad. and for $s_{6a} = 0.06$ and $0.14$ rad.

![Fig. 4.15 Principle set-up for the forced oscillation experiments with a segmented model in sway, roll and yaw.](image)

![Fig. 4.16 Sectional damping coefficient in swaying for $F_n = 0$ and 0.20.](image)
Fig. 4.17 Sectional mass coupling coefficient of swaying into yawing for $Fn = 0$ and 0.20.

Fig. 4.18 Sectional damping coupling coefficient of swaying into rolling for $Fn = 0$ and 0.20.
Fig. 4.19 Sectional damping coefficient in rolling for $F_n = 0$ and 0.20.

Fig. 4.20 Sectional added mass moment of inertia in yawing for $F_n = 0$ and 0.20.
Apart from the experiments with the segmented model the whole model was also forcibly oscillated at \( V = 0 \) while being placed transversely in the tank. As an example of the results the figures 4.16 to 4.20 are presented, again for \( F_h = 0 \) and 0.20. For sway and yaw formulae analogous to (4.4.17) can easily be derived. In fact they are identical when the indices are changed and \(-x_{1b}\) is replaced by \(+x_{1b}\). The curves corresponding with these previous formulae are indicated in the figures as well. The conclusions differ in some respects from those in the case of heave and pitch. In general the quantitative difference between the two theories in horizontal motion is not large for the sections. Anyway too little to make a definite choice. But for the whole model the \( u_{0v} \) in figure 4.17 shows the right tendency compared to the experiments, while the formula previously used leads to contrary results.

An important fact which appears from the investigation is that the computations deviate from the experiments over the forward third of the length, but especially for the foremost section. By comparing the results for \( F_h = 0 \) and 0.20 mutually it is seen that the tendencies of the speed effects are predicted well by the computations, but that there is an additional effect which has to be included for the forward part of the vessel. Quantitatively this phenomenon is important over a third of the length only. It contributes considerably to the coefficients for the ship as a whole, however, and cannot be considered as a mere refinement of the present theory. In analogy with the airfoil theory it could be considered that a horizontal circulation of variable strength existed about the lateral plane of the vessel. But it seems unlikely that in the presence of the free surface and at the very low aspect-ratio of a shipbody a distinct pressure peak at the leading edge would develop. It looks more reasonable to expect from a circulation a rather smooth pressure distribution over the length so that all sections will be influenced. Since this is not the case another effect appears more probable. It is thought that a real three-dimensional approach to the forward part of the body which is combined with the strip theory approach to the rest of it may solve the problem. Possibly it is sufficient for the effects in question to have only a very rough approximation of the shape of the vessel, for instance by adding the solution for an oscillating sphere, ellipsoid or similar body. The changing position of the stagnation point and the influence of \( \varphi_0 \) in this region will probably contribute to the problem, so that it will presumably be necessary to add the solution for a three-dimensional body at forward speed to the results obtained above. It is questionable whether the fact that a simple body may be considered is of much help in this matter. It is most likely that a numerical solution will be the only practical way and in that case it may turn out that a numerical solution of the complete problem for the ship as a whole is an effort of comparable magnitude.

There is one other difference between the vertical motions on the one hand and the lateral motions on the other. For heave and pitch the computations according to strip theory fail in the low frequency range, even for \( V = 0 \). This is due to the fact that the frequency parameter \( \omega \sqrt{(B/2g)} \) has to be of order unity, as discussed in section 4.2. For sway, roll and yaw this requirement is apparently of no importance, for at \( V = 0 \) the computations and the experiments agree reasonably in the entire range. From the experiments with the segmented model it further became obvious that non-linearities in the first harmonic part of the hydrodynamic forces caused by the amplitude of motion are small and not of importance for practical applications in the field of ship motions.

Several authors have tried to include three-dimensional and forward speed effects in some form of strip theory. Also a combination of a strip theory and a slender body theory to cover the whole range of frequencies as good as possible has been proposed. Especially Grim has done much work in the field and he has reached many conclusions by physical reasoning which have been confirmed afterwards. The reader who is interested in the subject will find details in the references [1.13, 1.18, 4.8, 4.21 and 4.24 to 4.28]. The approach has been very different and in the opinion of the present author it is not so useful to discuss all these works here. In this report he has followed a systematic development along lines which are consistent in strip theory. To his knowledge at the time of deriving the formulation of the problem as presented, a similar treatment had not been given before. But apparently Semjenow-Tjen-Schanski, Blagoveshchensky and Holodiin have reached the same results as appears from their book [4.29], published in Leningrad in 1969 in Russian. They solely discuss the coefficients at forward speed for heave and pitch.

4.5 Transformation of the coefficients from \( ox_1x_2x_3 \) to \( Gxyz \)

For the dynamics of the body the equations of motion are most easily expressed in a coordinate system with its origin in the centre of gravity. Therefore the hydrodynamic coefficients have to be transformed to a parallel system of axes passing through \( G \). The motions in the two systems are related by:
\[ s_1 = x - x_{3y} \cdot \theta \]
\[ s_2 = y + x_{3y} \cdot \phi - x_{1y} \cdot \psi \]
\[ s_3 = z + x_{1y} \cdot \theta \]
\[ s_4 = \phi \]
\[ s_5 = \theta \]
\[ s_6 = \psi \] \hspace{1cm} (4.5.1)

\[ \begin{align*}
F_1 &= X \\
F_2 &= Y \\
F_3 &= Z
\end{align*} \]
\[ F_4 = K - x_{3y} \cdot Y \\
F_5 = M + x_{3y} \cdot X - x_{1y} \cdot Z \\
F_6 = N + x_{1y} \cdot Y \] \hspace{1cm} (4.5.2)

with \( X, Y, Z, K, M \) and \( N \) the force and moment components in \( Gxyz \). Considering that:
\[ F_1 = - \sum_{j=1}^{6} (a_{ij} \delta_{j} + b_{ij} \delta_{j}) \] \hspace{1cm} (4.5.3)

and:
\[ \begin{align*}
X &= -p_{xx} \ddot{x} - q_{xx} \ddot{z} - p_{zx} \ddot{z} - q_{zx} \ddot{z} - p_{x0} \ddot{\theta} - q_{x0} \ddot{\theta} \\
Y &= -p_{yy} \ddot{y} - q_{yy} \ddot{y} - p_{y0} \ddot{\phi} - q_{y0} \ddot{\phi} - p_{y0} \ddot{\psi} - q_{y0} \ddot{\psi} \\
Z &= -p_{zz} \ddot{z} - q_{zz} \ddot{z} - p_{z0} \ddot{\theta} - q_{z0} \ddot{\theta} - p_{zz} \ddot{x} - q_{zz} \ddot{x} \\
K &= -p_{\phi0} \ddot{\phi} - q_{\phi0} \ddot{\phi} - p_{\phi0} \ddot{\psi} - q_{\phi0} \ddot{\psi} - p_{\phi0} \ddot{\tilde{\phi}} - q_{\phi0} \ddot{\tilde{\phi}} \\
M &= -p_{\theta0} \ddot{\theta} - q_{\theta0} \ddot{\theta} - p_{\theta0} \ddot{\psi} - q_{\theta0} \ddot{\psi} - p_{\theta0} \ddot{\tilde{\phi}} - q_{\theta0} \ddot{\tilde{\phi}} \\
N &= -p_{\psi0} \ddot{\psi} - q_{\psi0} \ddot{\psi} - p_{\psi0} \ddot{\theta} - q_{\psi0} \ddot{\theta} - p_{\psi0} \ddot{\phi} - q_{\psi0} \ddot{\phi} \\
\end{align*} \]
\hspace{1cm} (4.5.4)

Substitution of all these equations into (4.5.2) leads to the following relations:
\[ p_{xx} = a_{xx} \]
\[ q_{xx} = b_{xx} \]
\[ p_{x0} = a_{x0} - x_{3y} \cdot a_{xx} + x_{1y} \cdot a_{zz} \]
\[ q_{x0} = b_{x0} - x_{3y} \cdot b_{xx} + x_{1y} \cdot b_{zz} \]
\[ p_{y0} = a_{y0} - x_{3y} \cdot a_{yy} + x_{1y} \cdot a_{zy} + x_{3y} \cdot a_{x0} + x_{1y} \cdot a_{z0} + x_{3y} \cdot a_{zz} - x_{3y} \cdot a_{x0} + x_{1y} \cdot a_{z0} + x_{3y} \cdot a_{zz} - x_{3y} \cdot a_{x0} + x_{1y} \cdot a_{z0} + x_{3y} \cdot a_{zz} \]
\[ q_{y0} = b_{y0} - x_{3y} \cdot b_{xx} + x_{1y} \cdot b_{zz} + x_{3y} \cdot b_{z0} + x_{1y} \cdot b_{z0} + x_{3y} \cdot b_{zz} \\
- x_{3y} \cdot b_{x0} + x_{1y} \cdot b_{z0} + x_{3y} \cdot b_{zz} + x_{1y} \cdot b_{z0} + x_{3y} \cdot b_{zz} \]
\[ p_{\phi0} = a_{\phi0} + x_{3y} \cdot a_{\psi0} + x_{3y} \cdot a_{\phi0} + x_{3y} \cdot a_{\psi0} + x_{3y} \cdot a_{\phi0} \]
\[ q_{\phi0} = b_{\phi0} + x_{3y} \cdot b_{\psi0} + x_{3y} \cdot b_{\phi0} + x_{3y} \cdot b_{\psi0} \]
\[ p_{\theta0} = a_{\theta0} - x_{1y} \cdot a_{y0} \]
\[ q_{\theta0} = b_{\theta0} - x_{1y} \cdot b_{y0} + x_{1y} \cdot b_{0y} + x_{1y} \cdot b_{0y} \]
\[ p_{\psi0} = a_{\psi0} - x_{1y} \cdot a_{y0} + x_{3y} \cdot a_{y0} - x_{1y} \cdot x_{3y} \cdot a_{yy} \]
\[ q_{\psi0} = b_{\psi0} - x_{1y} \cdot b_{y0} + x_{3y} \cdot b_{y0} - x_{1y} \cdot x_{3y} \cdot b_{yy} \] \hspace{1cm} (4.5.5)

For the hydrostatic coefficients the following results are valid:
\[ r_{xx} = c_{xx} = qgA_{WL} \]
\[ r_{yy} = c_{yy} + x_{1y} \cdot c_{xx} \] \hspace{1cm} (4.5.6)
\[ r_{zz} = c_{zz} + x_{1y} \cdot c_{xx} \]
\[ r_{\phi0} = c_{\phi0} + x_{3y} \cdot qgV \]
\[ r_{\theta0} = c_{\theta0} + x_{1y} \cdot c_{x0} + x_{1y} \cdot c_{x0} + x_{1y} \cdot c_{x0} + x_{3y} \cdot qgV \]
\[ r_{\psi0} = c_{\psi0} + x_{3y} \cdot qgV + qgV(x_{3y} - \overline{OB}) = qgV \cdot \overline{GM} \]

In the notation as adopted the motions are ultimately determined by the following equations:
\[ (m + p_{ex}) \ddot{x} + q_{vx} \dot{x} + p_{xz} \dot{z} + q_{xz} \dot{z} + p_{z0} \phi \dot{\phi} + q_{z0} \phi \dot{\phi} = X_w \]
\[ (m + p_{ey}) \ddot{y} + q_{vy} \dot{y} + p_{vy} \phi \dot{\phi} + q_{vy} \phi \dot{\phi} + p_{y0} \psi \dot{\psi} + q_{y0} \psi \dot{\psi} = Y_w \]
\[ (m + p_{ez}) \ddot{z} + r_{zx} \dot{x} + p_{z0} \dot{\phi} + q_{z0} \dot{\phi} + r_{z0} \theta + p_{z0} \dot{\phi} + q_{z0} \dot{\phi} + r_{z0} \theta + \dot{z} = Z_w \]
\[ (I_{xx} + p_{x0}) \ddot{\phi} + q_{x0} \phi \dot{\phi} + r_{x0} \theta \dot{\phi} + (-I_{xx} + p_{x0}) \ddot{\psi} + q_{x0} \psi \dot{\psi} + -I_{xx} + p_{x0}) \ddot{\psi} + q_{x0} \psi \dot{\psi} + \dot{z} = M_w \]
\[ (I_{yy} + p_{y0}) \ddot{\phi} + q_{y0} \phi \dot{\phi} + r_{y0} \theta \dot{\phi} + (-I_{yy} + p_{y0}) \ddot{\psi} + q_{y0} \psi \dot{\psi} + \dot{z} = N_w \]

(4.5.7)

Generally the surge coefficients are neglected so that the relations for the heave and pitch coefficients will be simplified somewhat in practice. The formulae are rather complicated in form but they result logically from a geometrical transformation of axes. Since the solution of the motion problem has to be carried out by a computer this does not offer any special complication. The influence of the longitudinal and vertical position of the centre of gravity is illustrated clearly. Quantitatively it is especially marked in the coefficients for the rolling motion, which are composed of various small contributions.

5 Calculation of the wave exciting forces in oblique waves

5.1 The forces at zero forward speed

The wave exciting force in the k’th-direction has been found by the so-called Haskind-relations in equation (3.4.4):

\[ X_{wk} = \frac{g}{s_k} e^{-i\omega t} \int_{S_0} \left\{ \varphi_w \frac{\partial \varphi_k}{\partial n} - \varphi_k \frac{\partial \varphi_w}{\partial n} \right\} dS \]

(5.1.1)

The first term is the force exerted by the pressure in the wave without taking account of the disturbance by the presence of the ship. So this term represents the whole force under the assumption of the Froude-Kriloff hypothesis. The second term is a contribution resulting from the diffraction of the waves about the body. It will turn out that this is a very important part of the force which certainly cannot be neglected. Especially not in the lateral forces and not even for the nearly statical case of very long waves. The equations will be derived in the same space fixed system \( \alpha x_1 x_2 x_3 \) as used in chapter 4.

The Froude-Kriloff force is easily computed when \( \frac{\partial \varphi_k}{\partial n} \) is replaced by the original \( -i \omega \zeta_k s_k \). Then:

\[ (X_{wk})_{F,K} = -i \omega \zeta_k e^{-i\omega t} \int_{S_0} \varphi_w s_k dS \]

(5.1.2)

The potential of incoming waves on deep water, of amplitude \( \zeta_k \) and travelling in a direction which makes an angle \( \mu \) with the positive \( x_1 \)-axis is:

\[ \varphi_w(x_1, x_2, x_3) = \frac{i \zeta_k}{\omega} e^{-k x_2} e^{i(k x_1 \cos \mu + k x_2 \sin \mu)} \]

(5.1.3)

So that:

\[ (X_{wk})_{F,K} = -i \omega \zeta_k e^{-i\omega t} \int_{S_0} e^{i(k x_1 \cos \mu + k x_2 \sin \mu)} dS \]

(5.1.4)

again indicating points on the body surface \( S_0 \) by \( (x_{1b}, x_{2b}, x_{3b}) \) and dividing the surface integral in an integration along the contour of a section and a subsequent integration along the length of the body. In real notation this becomes:

\[ (X_{wk})_{F,K} = \omega \zeta_k \cos \omega t \]

(5.1.5)

For numerical computation this equation is exact and in principle sufficient for use by a computer. However, it still requires a lot of labour to evaluate it. Therefore in general some simplifications are introduced, namely that the waves are long with respect to the beam and the draught of the vessel so that \( k x_2 \ll 1 \) and \( k x_3 \ll 1 \). Then \( \cos(k x_2 \sin \mu) \approx 1 \) and \( \sin(k x_2 \sin \mu) \approx k x_2 \sin \mu \), while \( e^{-k x_2} \) may be replaced by a constant mean value \( e^{-k^2T^2} \). Later on for \( T \) some value must be chosen; compare section 5.3. It sometimes equals half the draught of the ship or the distance of the centre of buoyancy below the waterline. As a consequence of the above simplifications it can easily be shown [5.2] that in the Froude-Kriloff force \( T^2 \) can be approximated by:

\[ T^2 = -\frac{1}{k} \ln \left( 1 - \frac{k T}{b} \right) e^{-k x_2^2} x_{2b} d x_{3b} \]

In fact the assumption of the relative lengths of waves, beam and draught is not satisfied too well in the important range of motion frequencies. For instance for \( \omega, (B/2g) \approx 1 \) and \( B/T = 2.5 \), is \( k B/2 = 1 \) and \( k T = 0.8 \). If desired the next term in the series expansions for the cos, the sin and the exponential function can
be retained to improve the accuracy or the assumption can be abandoned completely.

The diffraction force is apparently:

$$ (X_{wk})_d = -\frac{\varrho}{s_{ka}} e^{-i\omega t} \int_S \varphi_k \frac{\partial \varphi_w}{\partial n} dS $$  \hspace{1cm} (5.1.6)  

The normal derivative of the wave potential is:

$$ \frac{\partial \varphi_w}{\partial n} e^{-i\omega t} = (n \cdot \nabla \varphi_w) e^{-i\omega t} = \left( f_1 \frac{\partial \varphi_w}{\partial x_1} + f_2 \frac{\partial \varphi_w}{\partial x_2} + f_3 \frac{\partial \varphi_w}{\partial x_3} \right) e^{-i\omega t} $$  

$$ = f_1 \cdot u + f_2 \cdot v + f_3 \cdot w $$  \hspace{1cm} (5.1.7)  

where by definition, $u$, $v$, and $w$ indicate the orbital velocities. Again splitting the integral in two parts then holds:

$$ (X_{wk})_d = -\frac{\varrho}{s_{ka} L_0} \int_{C_0} \left( \varphi_k f_1 \cdot u + \varphi_k f_2 \cdot v + \varphi_k f_3 \cdot w \right) dx $$  

$$ \hspace{1cm} (5.1.8) $$  

On the other hand from (3.1.10) follows:

$$ F_{k}(t) = -i \varrho e^{-i\omega t} \int_S \varphi_k f_j dS = -i \varrho e^{-i\omega t} \int_{C_0} \varphi_k f_j ds $$  

and from (3.1.11):

$$ F_{k}(t) = (s_{ka} \omega^2 a_{k}^{0} + is_{ka} \omega b_{k}^{0}) e^{-i\omega t} $$  

$$ = s_{ka} e^{-i\omega t} \int_{C_0} (\omega^2 a_{k}^{0} + i \omega b_{k}^{0}) ds $$  

where $a_{k}^{0}$ and $b_{k}^{0}$ indicate the hypothetical added mass and damping coefficient per unit of area associated with a surface element of $S_0$. So that:

$$ \frac{\varrho}{s_{ka}} \varphi_k f_j = i \omega a_{k}^{0} - b_{k}^{0} $$  \hspace{1cm} (5.1.9)  

Upon substitution in (5.1.8) there appears:

$$ (X_{wk})_d = \int_{C_0} \left[ \left( -i \omega a_{k}^{0} + b_{k}^{0} \right) u + \left( a_{k}^{0} v + b_{k}^{0} w \right) \right] ds $$  

or:

$$ (X_{wk})_d = \int_{C_0} \left[ (a_{k}^{0} u + b_{k}^{0} v) + (a_{k}^{0} w + b_{k}^{0} v) \right] ds $$  \hspace{1cm} (5.1.10)  

This is a remarkable result for it entirely justifies a different approach to the problem of calculating exciting forces, which has often been followed in ship motion theory, see for example [5.1] and [5.2]. In that way of viewing the matter one considers that due to the orbital velocities in the wave structure there exists a relative motion between the ship's hull and the wave particles. The effects of this motion are put equal to those of the reverse problem, namely that of forced oscillation in still water. Physically it is clear that this will account at least for a part of the influence of the body on the waves. However, in the opinion of the author it is not at all obvious that the relative motion concept completely covers the diffraction problem. The derivation given here proves that this is the case. That is: as long as the forward speed is zero. The detailed expressions which will be worked out in section 5.3 will show that for the general case at $V \neq 0$ it is no longer true.

Since the orbital velocities differ from point to point the forces must be considered locally. Doing so one arrives at an expression for the wave forces as given here by way of example for the vertical force:

$$ (X_{w_3})_d = \int_{C_0} (X_{w_3}' + X_{w_3}'' + X_{w_3}''') ds $$  \hspace{1cm} (5.1.11)  

with

$$ X_{w_{3,1}}'' = -pf_3 $$  

$$ X_{w_{3,2}}'' = +b_{2k}^{0} w $$  

$$ X_{w_{3,3}}'' = +a_{2k}^{0} \dot{w} $$  

$X_{w_{3,1}}$ is the local Froude-Kriloff force, while $X_{w_{3,2}}''$ and $X_{w_{3,3}}''$ are the hydrodynamic contributions due to the relative motion, equal in magnitude but opposite in sign to the case of forced oscillation in still water. Equation (5.1.11) is identical with (5.1.5) plus (5.1.10), when the coupling of heave and surge is neglected. So the relative motion hypothesis is fully analogous with a solution of the diffraction problem. It is often stated that the relative motion hypothesis breaks down for the shorter waves. But the above results show that this is not true in principle. It is correct within the same framework as is usual in strip theory. So deviations from experiments in short waves must be attributed to three causes: the inaccuracies by the assumptions $k x_{k} \ll 1$ and $k x_{k} \ll 1$ (which are nearly always made), the neglect of the coupling with surge (which is not essential for the relative motion theory) or the fact that the local hydrodynamic coefficients, calculated two-dimensionally, are not sufficiently representative for the actual three-dimensional values. Further, of course, viscous contributions play a much more important role in short waves, where the forces arising from the potential flow are rather small and where the orbital velocities are relatively high.
Equation (5.1.10) is valid for all \( k \). The three groups of terms in the integral are never present together since a number of coupling coefficients vanish by symmetry. It is consistent with the foregoing development to neglect the first group altogether as the influence of surge in the heave-pitch problem cannot be accounted for in a strip theory. Thereby the following relations for the exciting forces at zero forward speed are obtained:

\[
(X_{wa})_{v,K} = \rho g \zeta_w \cos \alpha t \\
\left[ \int \frac{\cos(kx_{1b} \cos \mu) dx_{1b}}{L_0} e^{-kx_{1b}} \sin (kx_{2b} \sin \mu) f_4 ds \\
- \sin (kx_{1b} \cos \mu) dx_{1b} \left[ e^{-kx_{1b}} \cos (kx_{2b} \sin \mu) f_4 ds \right] \right] \\
- \rho g \zeta_w \sin \alpha t \\
\left[ \int \frac{\cos(kx_{1b} \cos \mu) dx_{1b}}{L_0} e^{-kx_{1b}} \cos (kx_{2b} \sin \mu) f_4 ds \\
- \sin (kx_{1b} \cos \mu) dx_{1b} \left[ e^{-kx_{1b}} \sin (kx_{2b} \sin \mu) f_4 ds \right] \right]
\]

\[
X_{w2} = (X_{w2})_{v,K} + \int \frac{dx_{1b}}{L_0} \left( a''_2 \phi_0 + b''_2 \phi_0 \right) ds \\
X_{w3} = (X_{w3})_{v,K} + \int \frac{dx_{1b}}{L_0} \left( a''_3 \phi_0 + b''_3 \phi_0 \right) ds \\
X_{w4} = (X_{w4})_{v,K} + \int \frac{dx_{1b}}{L_0} \left( a''_4 \phi_0 + b''_4 \phi_0 \right) ds \\
X_{w5} = (X_{w5})_{v,K} + \int \frac{dx_{1b}}{L_0} \left( a''_5 \phi_0 + b''_5 \phi_0 \right) ds \\
X_{w6} = (X_{w6})_{v,K} + \int \frac{dx_{1b}}{L_0} \left( a''_6 \phi_0 + b''_6 \phi_0 \right) ds
\]

The fact that \( f_5 \approx -x_{1b} f_1 \) and \( f_6 \approx +x_{1b} f_2 \) allows for the Froude-Kriloff components to be written:

\[
(X_{w5})_{F,K} = -\int \frac{x_{1b}(X_{w3})_{F,K} dx_{1b}}{L_0} \\
(X_{w6})_{F,K} = +\int \frac{x_{1b}(X_{w2})_{F,K} dx_{1b}}{L_0}
\]

while the coupling coefficients of heave and pitch can be expressed as minus the location of the section \( x_{1b} \) times the heave coefficients, and those of sway and yaw as \( x_{1b} \) times the sway coefficients. Therefore it is obviously sufficient to consider only \( k = 2, 3 \) and \( 4 \) since the pitch and yaw moments are easily obtained from the corresponding vertical and transverse force components.

5.2 The forces at forward speed

The hydrodynamics of waves coming in on a sailing vessel have not yet been treated, so that it is necessary to start from the actual definition of the wave exciting forces. In a coordinate system moving with the vessel at a speed \( V \) in the positive \( x_1 \)-direction is:

\[
X_{wa} = -\int \frac{p(f_4)}{S_0} ds \tag{5.2.1}
\]

with

\[
p = i \varphi (\varphi_w + \varphi_y) e^{-i\omega t} \tag{5.2.2}
\]

where \( \omega \) represents the actual wave frequency and \( \omega_e \) the frequency of encounter of the vessel with the waves given by:

\[
\omega_e = \omega - kV \cos \mu \tag{5.2.3}
\]

The boundary condition on \( S_0 \) is from (4.4.4) and subsequent equations:

\[
\frac{\partial \varphi_k}{\partial n} = -i \omega_e f_k s_{ka} \quad \text{for} \quad (k = 2, 3, 4) \tag{5.2.4}
\]

\[
\frac{\partial \varphi_k}{\partial n} = -(i \omega_e + \frac{V}{x_{1b}}) f_k s_{ka} \quad \text{for} \quad (k = 5, 6) \tag{5.2.5}
\]

with the frequency of motion, of course, taken equal to the frequency of encounter. Substitution of \( p \) and \( f_k \) in equation (5.2.1) then results in:

\[
X_{wa} = \frac{\varphi \omega}{\omega_m s_{ka}} e^{-i\omega t} \int \frac{(\varphi_w + \varphi_y)}{S_0} \frac{\partial \varphi_k}{\partial n} ds \quad \text{for} \quad (k = 2, 3, 4) \tag{5.2.6}
\]

\[
X_{wa} = \frac{\varphi \omega}{\omega_m s_{ka}} e^{-i\omega t} \int \frac{(\varphi_w + \varphi_y)}{S_0} \frac{\partial \varphi_k}{\partial n} ds \quad \text{for} \quad (k = 5, 6) \tag{5.2.5}
\]

In the second relation for \( k = 5, 6 \) \( \varphi_k \) can be expressed in the corresponding heave and sway potentials by (4.4.10):

\[
\varphi_k = (-1)^m \frac{s_{ka}}{\omega_m} \left( \omega_e - \frac{iV}{x_{1b}} \right) \frac{x_{1b}}{\omega_e} \varphi_m \quad \text{for} \quad (k = 5, m = 3) \tag{5.2.7}
\]

\[
\varphi_k = (-1)^m \frac{s_{ka}}{\omega_m} \left( \omega_e - \frac{iV}{x_{1b}} \right) \frac{x_{1b}}{\omega_e} \varphi_m \quad \text{for} \quad (k = 6, m = 2) \tag{5.2.8}
\]

With the slenderness assumption of the preceding chapter the normal derivative of \( \varphi_k \) transforms simply into the normal derivative of \( \varphi_m \) so that \( X_{wa} \) becomes:

\[
X_{wa} = (-1)^m \frac{\varphi \omega}{\omega_m s_{ma}} e^{-i\omega t} \int \frac{(\varphi_w + \varphi_y)}{S_0} \frac{\partial \varphi_m}{\partial n} dS \quad \text{for} \quad \begin{cases} k = 5, m = 3 \\ k = 6, m = 2 \end{cases} \tag{5.2.6}
\]

therefore it is again sufficient to consider only the sway and heave force and the roll moment from (5.2.5). Then the pitch and yaw moments are known as well.
In [5.3] Newman shows from Green's theorem that for small disturbances of the free surface and for bodies which are not too largely asymmetric in a longitudinal direction the following expression holds:

$$\int_{S_0} \varphi^+_\omega \frac{\partial \varphi^-_\omega}{\partial n} dS = \int_{S_0} \varphi^-_\omega \frac{\partial \varphi^+_\omega}{\partial n} dS$$

where the superscript $\pm$ corresponds with the direction of motion. The boundary condition on $S_0$ is:

$$\frac{\partial}{\partial n}(\varphi^+ + \varphi^-) = 0$$

so that:

$$\int_{S_0} \varphi^+ \frac{\partial \varphi^-}{\partial n} dS = -\int_{S_0} \varphi^- \frac{\partial \varphi^+}{\partial n} dS$$

The first equation of (5.2.5) has been derived for motion in a forward direction, so a $+$ must be attached to all potentials. However, since the boundary condition on $S_0$ at speed for $k = 2, 3$ and 4 is equal to that at zero speed also holds:

$$\frac{\partial \varphi^+_k}{\partial n} = \frac{\partial \varphi^+_k}{\partial n} = \frac{\partial \varphi^-_k}{\partial n} \quad (k = 2, 3, 4)$$

Insertion of the above relations in (5.2.5) gives:

$$X_{wk} = \frac{\varphi^+_\omega}{\varphi^-_\omega} e^{-lae} \int_{S_0} \left\{ \frac{\partial \varphi^-_\omega}{\partial n} - \frac{\partial \varphi^+_\omega}{\partial n} \right\} dS$$

$$(k = 2, 3, 4)$$

(5.2.7)

This result is analogous with (5.1.1). The wave potential has to be replaced by the potential in the moving coordinate system, which is of no influence on the form of the time invarying part of it, and the oscillation potential must be exchanged for the corresponding potential at a negative forward speed. This influences the potential $\varphi^+_\omega$ itself, but for $k = 2, 3, 4$ not its normal derivative. Finally the time dependent factor $e^{-lae}$ transforms into the factor $(\omega/\omega_e) e^{-lae}$.

The first term of (5.2.7) is again the Froude-Kriloff force, for the computation of which it is easier to substitute $f_1$ back for $\varphi^-_\omega/\partial n$. Thus:

$$(X_{wk})_{F.K.} = -i \varphi^+ \omega e^{-lae} \int_{S_0} \varphi^-_\omega f_1 dS \quad (k = 2, 3, 4)$$

(5.2.8)

$\varphi^+_\omega$ in the moving system is identical with (5.1.3) for $V = 0$, so that the whole Froude-Kriloff contribution with speed is also identical with (5.1.5) at zero speed except for the time dependent factor $e^{-lae}$ which has to be replaced by $e^{-lae}$. The diffraction force is:

$$(X_{wk})_d = -\frac{\varphi^+_\omega}{\omega_e \varphi^+_\omega} e^{-lae} \int_{S_0} \varphi^-_\omega \frac{\partial \varphi^+_\omega}{\partial n} dS \quad (k = 2, 3, 4)$$

(5.2.9)

Here as well the moving system does not have any influence on the normal derivative of $\varphi^+_\omega$, or with (5.1.7):

$$(X_{wk})_d = -\frac{\varphi^+_\omega}{\omega_e \varphi^+_\omega} \int_{S_0} \varphi^-_\omega \frac{\partial \varphi^+_\omega}{\partial n} dS$$

$$(k = 2, 3, 4)$$

(5.2.10)

In this equation $\varphi^-_\omega$ stands for the actual three-dimensional potential associated with the oscillatory motion of the body in still water at a speed $-V$. In the preceding chapter a method of approximating this potential from the corresponding two-dimensional potentials at $V = 0$ has been derived. In (4.4.13) the pressure at a forward speed $+V$ was found to be:

$$p^+_k = \rho \left( i \omega \varphi_k + V \frac{\partial \varphi_k}{\partial x_{1b}} \right) e^{-lae}$$

Denoting formally:

$$p^+_k = -\frac{\partial \varphi_k}{\partial t} = i \omega \omega \varphi^+_k e^{-lae}$$

$\varphi^+_k$ can be expressed by:

$$\varphi^+_k = \varphi_k - i V \frac{\partial \varphi_k}{\partial x_{1b}}$$

Of course the frequency of motion in the oscillation problem is $\omega_e$ and time derivatives have to be formed with $\omega_e$ as well. Equivalently is:

$$\varphi^-_k = \varphi_k + i V \frac{\partial \varphi_k}{\partial x_{1b}}$$

(5.2.11)

Substitution in (5.2.10) gives:

$$(X_{wk})_d =$$

$$= -\frac{\varphi^+_\omega}{\omega_e \varphi^+_\omega} \int_{S_0} \left\{ \varphi_k f_1 + \frac{i V}{\omega_e} \frac{\partial \varphi_k}{\partial x_{1b}} \right\} u +$$

$$+ \left\{ \varphi_k f_2 + \frac{i V}{\omega_e} f_2 \frac{\partial \varphi_k}{\partial x_{1b}} \right\} v + \left\{ \varphi_k f_3 + \frac{i V}{\omega_e} f_3 \frac{\partial \varphi_k}{\partial x_{1b}} \right\} w$$

$$ds$$
\[ X_{w3} = \begin{cases} & (X_{w3})_{F,K} + \int_{L_0}^{Lx} \int_{C_0}^{C_1} \
& \left\{ \left( a_{33}^{\prime 0} - \frac{V}{\omega_e} \frac{\partial b_{33}^{0}}{\partial x_{1b}} \right) \dot{w} + \frac{\omega}{\omega_e} \left( b_{33}^{\prime 0} + V \frac{\partial d_{33}^{0}}{\partial x_{1b}} \right) \right\} ds \
& (k = 2, 3, 4) \end{cases} \]

\[ X_{w4} = \begin{cases} & (X_{w4})_{F,K} + \int_{L_0}^{Lx} \int_{C_0}^{C_1} \
& \left\{ \left( a_{44}^{\prime 0} - \frac{V}{\omega_e} \frac{\partial b_{44}^{0}}{\partial x_{1b}} \right) \dot{v} + \frac{\omega}{\omega_e} \left( b_{44}^{\prime 0} + V \frac{\partial d_{44}^{0}}{\partial x_{1b}} \right) v \right\} ds \
& (k = 2, 3, 4) \end{cases} \]

\[ X_{w5} = -\int_{L_0}^{Lx} (X_{w3})_{F,K} ds \]

\[ X_{w6} = +\int_{L_0}^{Lx} (X_{w4})_{F,K} ds \]  

(5.2.13)

5.3 A reformulation of the equations for the exciting forces

The ultimate form of the preceding section will now be simplified as far as possible. It is emphasized that in principle only a few assumptions have been made to obtain them so that their validity is not at all restricted to a strip theory analysis. In fact the only essential assumptions have been that the disturbance of the free surface is small so that even at forward speed a linearized free surface condition holds and that the body is only slightly asymmetric longitudinally. The transverse symmetry of the body, its slenderness and the neglect of the surge coefficients do not influence the results in essence. When the local, real three-dimensional hydrodynamic coefficients were to be known they could be inserted without further simplifications to be necessary. It is also stressed that the Haskind-relations are valid irrespective of the depth of the water. Consequently the basic expressions for the wave exciting forces also hold for restricted water. Their further formulation as given in this chapter is only valid for deep water, however.

In the following the equations (5.2.13) will be used to work out as simple relations as possible. The full form of the Froude-Kriloff term has been given. The diffraction terms can easily be written out with the orbital velocities and accelerations in points of $S_0$:

\[ v = \frac{\partial \phi^+}{\partial x_2} e^{-i\omega t} = i\zeta_0 \omega \sin \mu e^{-k_{3b} \phi^+} e^{(k_{1b} \cos \mu + k_{2b} \sin \mu \cdot \omega t)} \]

\[ \dot{\phi} = i\zeta_0 \omega^2 \sin^2 \mu e^{-k_{3b} \phi^+} e^{(k_{1b} \cos \mu + k_{2b} \sin \mu \cdot \omega t)} \]

\[ w = \frac{\partial \phi^+}{\partial x_3} e^{-i\omega t} = -\zeta_0 \omega e^{-k_{3b} \phi^+} e^{(k_{1b} \cos \mu + k_{2b} \sin \mu \cdot \omega t)} \]

\[ \dot{w} = i\zeta_0 \omega^2 e^{-k_{3b} \phi^+} e^{(k_{1b} \cos \mu + k_{2b} \sin \mu \cdot \omega t)} \]  

(5.3.1)

Splitting the relations in a real and an imaginary part and taking the real part only the full form of the diffraction forces is found as well. The procedure is laborious and the resulting equations are lengthy, but

* For the waves differentiation has to take place in a fixed system, so that the wave frequency emerges and not the frequency of encounter.
when accurate results are needed for short waves too it is recommended to do so. In this report the resulting expressions will only be given under the usual assumptions \( kx_{2b} \ll 1 \) and \( kx_{1b} \ll 1 \), although this is not so correct for the higher range of motion frequencies. With this simplification it is possible to write:

\[
\begin{align*}
\cos(kx_{2b}\sin\mu) &\approx 1 \\
\sin(kx_{2b}\sin\mu) &\approx kx_{2b}\sin\mu \\
\frac{e^{i(kx_{2b}\cos\mu)}}{e^{ikx_{1b}\sin\mu}} &\approx 1 + ikx_{1b}\sin\mu \\
e^{-kx_{1b}} &\approx e^{-kT^*}
\end{align*}
\]

with \( T^* = T^* (x_{1b}, \omega) \) an equivalent mean draught of a section for a certain wave. The value chosen may be the same as the approximation used in the evaluation of the Froude-Kriloff terms (5.1.5). It is not possible to obtain a \( T^* \) from integrals of the type:

\[
\int e^{-kx_{2b}a_{kj}^{(0)} ds}_c \quad \text{or} \quad \int e^{-kx_{2b}a_{kij}^{(0)} x_{2b} ds}_c
\]

Only in the case of pressure integration over the ship’s hull is the introduction of \( T^* \) not necessary. Otherwise some choice must be made. The roughest approximation is to take \( T^* \) as a constant for the whole ship in which case it can be placed outside both the integral over \( C_0 \) and over \( L_0 \). Further it is clear that:

\[
\int a_{kj}^{(0)} ds = a_{kj}^{(0)} \quad \text{and} \quad \int b_{kij}^{(0)} ds = b_{kij}^{(0)}
\]

while:

\[
\int_0^2 ds = \begin{cases} 0 & \text{for} \quad (k = 2, 4) \\ 2b & \text{for} \quad (k = 3) \end{cases}
\]

\[
\int_0 b_{kij}^{(0)} x_{2b} ds = \begin{cases} A & \text{for} \quad (k = 2) \\ 0 & \text{for} \quad (k = 3) \\ 2b^3 - A \cdot OB & \text{for} \quad (k = 4) \end{cases}
\]

\[
\int_0^2 b_{kij}^{(0)} x_{2b} ds = 0
\]

due to the transverse symmetry.

After the necessary algebraic operations ultimately the following equations will be found:

\[
X_{wk} = X_{wkc} \cos \omega_4 t + X_{wks} \sin \omega_4 t
\]

\[
X_{wkc} = \frac{\cos(ks_{1b} \cos \mu)}{\sin(ks_{1b} \cos \mu)} \int_0^{L_0} \frac{P_k}{e^{-kT^*}} ds
\]

\[
+ \frac{\zeta_\omega \omega^2}{\omega_4} \sin(ks_{1b} \cos \mu) \int_0^{L_0} \frac{a_{kij}^{(0)} e^{-kT^*}}{\sin(ks_{1b} \cos \mu)} ds
\]

\[
\mp \frac{\zeta_\omega \omega^2}{\omega_4} \sin(ks_{1b} \cos \mu) \int_0^{L_0} \frac{b_{kij}^{(0)} e^{-kT^*}}{\sin(ks_{1b} \cos \mu)} ds
\]

\[
- \frac{\zeta_\omega \omega^2}{\omega_4} \sin(ks_{1b} \cos \mu) \int_0^{L_0} \frac{\partial b_{kij}^{(0)}}{\partial x_{1b}} e^{-kT^*} \sin(ks_{1b} \cos \mu) ds
\]

\[
(k = 2, 4)
\]

\[
X_{w3c} = X_{w3s}
\]

\[
\pm 2 \frac{\zeta_\omega \omega^2}{\omega_4} \sin(ks_{1b} \cos \mu) \int_0^{L_0} b_{kij}^{(0)} e^{-kT^*} \sin(ks_{1b} \cos \mu) ds
\]

\[
\mp \frac{\zeta_\omega \omega^2}{\omega_4} \sin(ks_{1b} \cos \mu) \int_0^{L_0} \frac{b_{kij}^{(0)} e^{-kT^*}}{\sin(ks_{1b} \cos \mu)} ds
\]

\[
- \frac{\zeta_\omega \omega^2}{\omega_4} \sin(ks_{1b} \cos \mu) \int_0^{L_0} \frac{\partial b_{kij}^{(0)}}{\partial x_{1b}} e^{-kT^*} \sin(ks_{1b} \cos \mu) ds
\]

\[
\pm \frac{\zeta_\omega \omega^2}{\omega_4} \sin(ks_{1b} \cos \mu) \int_0^{L_0} \frac{\partial b_{kij}^{(0)}}{\partial x_{1b}} e^{-kT^*} \sin(ks_{1b} \cos \mu) ds
\]

\[
X_{w5c} = \int_0^{x_{1b}} \frac{X_{w3c}}{X_{w3s}} dx_{1b}
\]

\[
\int_0^{x_{1b}} \frac{X_{w5c}}{X_{w5s}} dx_{1b}
\]

\[
X_{w6c} = \int_0^{x_{1b}} \frac{X_{w2c}}{X_{w2s}} dx_{1b}
\]

\[
X_{w6s} = \int_0^{x_{1b}} \frac{X_{w2c}}{X_{w2s}} dx_{1b}
\]

\[
(k = 2, 4)
\]

\[
P_k = \begin{cases} A & \text{for} \quad (k = 2) \\ \frac{2b^3}{3} - A \cdot OB & \text{for} \quad (k = 4) \end{cases}
\]

Taking the wave elevation at the origin of the moving axes as a reference for the phase relation one finds:

\[
\zeta = -i \omega \sin \omega_4 t = \frac{1}{g} \frac{\partial \phi_{ww}}{\partial t} (0, 0, 0) = \frac{X_{w2c}}{X_{w2s}}
\]

\[
\tan(\epsilon_{\omega_4} \pi) = \frac{X_{wkc}}{X_{wks}}
\]

The equations (5.3.2) differ in three respects from the usual strip theory results, as for instance given in [5.2] for heave and pitch. Firstly, the terms involving \( b_{kij}^{(0)} \) have the factor \( \omega^2/\omega_4 \) instead of \( \omega \). Secondly, the speed
terms involving $\partial a^0_w / \partial x_{1b}$ are opposite in sign and also have the factor $\omega^2 / \omega_s$ instead of $\omega$. Thirdly, the second speed term with $\partial b^0_w / \partial x_{1b}$ is not present in the previous results. The last fact has also been observed with the hydrodynamic coefficients in chapter 4. The sign opposition results from the Haskind-relations at forward speed, which contain the oscillation potential at a reversed direction of motion (see the discussion of eq. (5.2.7)); this has been found by Newman [5.3]. It may be expected that the ratio $\omega / \omega_s$ is specially important for the higher forward speeds and for wave directions within a sector of e.g. 45 degrees from the ship's track.

5.4 Transformation of the wave exciting forces from $ox'_1x'_2x'_3$ to $Gxyz$

For the same reason as the hydrodynamic coefficients the wave exciting forces have to be expressed in a coordinate system with its origin in $G$. Analogous with (4.5.2) is:

$$Y_w = X_{w2}$$  
$$Z_w = X_{w3}$$  
$$K_w = X_{w4} + x_{3g'} X_{w2}$$  
$$M_w = X_{w5} + x_{1g'} X_{w3} = - \int_{L_0}^{(x_{1b} - x_{1g})} X_{w3} \, dx_{1b}$$  
$$N_w = X_{w6} - x_{1g'} X_{w2} = + \int_{L_0}^{(x_{1b} - x_{1g})} X_{w2} \, dx_{1b}$$  

Attention has to be paid to the fact that the above relations are vectorial equations. Both amplitude and phase will be changed. Since the reference point for the wave elevation now also moves from $o$ to $G$ the phase relation has to be corrected also for the phase lag of the wave in $G$ with respect to the wave in $o$

in $o$:

$$\zeta = -i \zeta_o e^{-\omega_o t} = -\zeta_o \sin \omega_o t$$

in $G$:

$$\zeta = -i \zeta_o e^{i(kx_{1g} \cos \mu - \omega_o t)} = -\zeta_o \sin (\omega_o t - kx_{1g} \cos \mu)$$

Therefore:

$$\begin{align*}
\text{(phase of the exciting force} & \quad \text{with respect to the wave at } G) \\
\text{(phase of the exciting force} & \quad \text{with respect to the wave at } o) + kx_{1g} \cos \mu
\end{align*}$$

(5.4.2)

5.5 An experimental verification of the results obtained

Calculations have been made according to the equations (5.3.2) to (5.3.4) for the two-dimensional case of infinitely long cylinders and for a shipmodel at zero forward speed. The results have been compared to experiments.

For the cylinders the measured wave forces per unit length have been published in [5.4]. By way of example the results for a rectangular section with $B/T = 4$ and for a triangular section have been reproduced in fig. 5.1. For comparison two calculations are given, one according to the diffraction method of equation (3.4.9) and one according to the relative motion theory of the equations (5.3.2). The agreement with experiments is good, especially of the former. It has been shown that these methods are equivalent and the differences between them in this case can only originate from the numerical approximations $k/x_{2b} \ll 1$ and $k/x_{3b} \ll 1$. For the long waves this relation is satisfied and the results coincide indeed. For the range of medium and short wave lengths the inaccuracies resulting therefrom are certainly not negligible. Considering the difficulties of phase measurements the agreement with the calculations is also satisfactory. The fact that the phase angles $\epsilon_{yv}$ and $\epsilon_{kk}$ are nearly equal for the rectangle and differ 180 degrees for the triangle is e.g. correctly predicted.

It is a great advantage of the relative motion method over the diffraction method that it is capable of producing phase angles. To improve the accuracy of the calculations it can be suggested to use the very simple equation (3.4.9) for the amplitudes of the wave forces and (5.3.2) only for the corresponding phase relations. Generally the body motions are not too sensitive for the phases in the right hand side of equations of motion, but they are approximately proportional to the magnitude of the forces.

For infinitely long cylinders the vertical force expression in (5.1.11) reduces to:

$$Z_w = agB \cdot \zeta + b^0_{zz} w + a^0_{zz} \dot{w} = agB \cdot \zeta + b^0_{zz} \dot{w} + a^0_{zz} \dot{w}$$

(5.5.1)

where $\zeta$ denotes the wave elevation at a certain effective depth. Analogously it can be supposed that the roll moment in beam waves can be expressed by:

$$K_w = agA \cdot GM \cdot \dot{\alpha}_w + b^0_{\phi\phi} \dot{\alpha}_w + a^0_{\phi\phi} \ddot{\alpha}_w + b^0_{\phi\phi} \dot{\alpha}_w + a^0_{\phi\phi} \ddot{\alpha}_w$$

or:

$$K_w = agA \cdot GM \cdot \dot{\alpha}_w + \left( b^0_{\phi\phi} - b^0_{\phi\phi} / k \right) \ddot{\alpha}_w + \left( a^0_{\phi\phi} - a^0_{\phi\phi} / k \right) \ddot{\alpha}_w$$

(5.5.2)

with $\alpha_w$ the effective wave slope.

Sometimes this equation is reduced to what is thought to be the leading terms in the operational range of ships:

$$K_w \approx agA \cdot GM \cdot \dot{\alpha}_w + b^0_{\phi\phi} \ddot{\alpha}_w + a^0_{\phi\phi} \ddot{\alpha}_w$$

(5.5.3)
Fig. 5.1a

Fig. 5.1b

Fig. 5.1c

Fig. 5.1d

Fig. 5.1 Wave exciting forces and moments on two cylinders.
The correct equation for the roll moment can be derived algebraically from (5.3.2) to:

\[ K_w = \frac{\rho g A \cdot \overline{G} M \cdot \alpha_w}{k} \left( \frac{h}{k} \right) - \frac{a \cdot \rho g \beta_w}{k} \left( \frac{h}{k} \right) \]  

So the hydrodynamic roll coefficients do not belong to the expression at all.

For the triangular section the moment calculation according to (5.5.3) has also been given in fig. 5.1c. The centre of gravity is thought to be situated in the waterline and then it is easily shown that:

\[ \overline{K} M = \frac{4}{3} B \sqrt{3} \]

\[ \overline{K} G = T = \frac{1}{2} B \sqrt{3} \]

\[ \overline{G} M = -\frac{1}{18} B \sqrt{3} \]

So \( M \) lies below the waterline and \( \overline{G} M \) is negative. But this fact only changes the phase relation of \( K_w \) with respect to the waves and not the moment amplitude. For the very long waves (\( \omega \to 0 \)) the second and third term in (5.5.3) vanish and the non-dimensional moment amplitude becomes:

\[ \frac{K_w}{\rho g (B^3/12) k_c} = \frac{12 A - \overline{G} M}{B^3} = \frac{3B^2/3}{B^3} \cdot \frac{1}{18} B \sqrt{3} = \frac{1}{2} \]

Fig. 5.2 Lateral wave force and moments on a shipmodel for \( \mu = 30 \text{ deg.} \) at zero speed of advance.
It is evident from figure 5.1c that this method is greatly in error for the nearly static case of very long waves. But also for the important operational zone of many ships around \( \omega \sqrt{(B/2g)} = 0.50 \). An approximation of the result of (5.5.2) can be obtained by adding the curves corresponding to (5.5.3) and (5.5.4) and subtracting 0.50. This is not unreasonable in the range indicated. However, the amount of agreement will depend on the section considered since (5.5.2) and (5.5.4) differ basically.

The wave forces on a ship model at zero forward speed have been measured by the author for the same model as used in chapter 4, a parent form of the Todd Sixty Series with a block coefficient of 0.70. The results together with the calculations according to the equations (5.3.2) to (5.4.3) are given in the figures 5.2 and 5.3 for \( \mu = 30^\circ \) and in the figures 5.4 and 5.5 for \( \mu = 120^\circ \). The agreement is satisfactory. Due to the way of measuring, the yaw moments may be affected most by experimental inaccuracies. In the roll moments about the longitudinal axis of the ship clearly an additional component may be present, which will probably be of viscous origin. This is similar to the possible contributions in the hydrodynamic roll coefficients. Since during the experiments a wave reference could not be obtained simultaneously with the forces, all phase relations have been measured with respect to the horizontal force, after which the calculated \( \epsilon_{\text{RC}} \) has been added to the measured values to obtain the phase angle with respect to the wave in the center of gravity. For small angles of wave incidence and consequently small transverse forces this signal is not very reliable as a phase reference. This can clearly be seen in the figures for \( \mu = 30^\circ \). It was, however, the best that could be done and the agreement is still reasonable.

At forward speed only measurements for \( \mu = 180 \) deg. are available from [5.2]. They are reproduced in figure 5.6, together with the calculations according to the existing form of strip theory and to the form presented here. The differences between the two and the differences with the experiments are not sufficient to allow a conclusion to be drawn. The measurements of the pitch moment may show a tendency to fit the tendency of the present calculations slightly better, but it is marginal. To have an ultimate check on the equations as derived in this chapter careful experiments of the wave exciting forces at forward speed in oblique waves are needed.

![Graphs showing wave force and moment](image)

**Fig. 5.3** Vertical wave force and moment on a ship model for \( \mu = 30 \) deg. at zero speed of advance.
6 Discussion and conclusion

6.1 A general discussion of the results obtained

The problem of a vessel's behaviour at sea is a complicated one, especially when not only wave induced motions have to be considered but also the effect of wind and current. The next section will go into this subject a little further. The present study has only dealt with the hydrodynamics of wave induced motions. It has been performed in three stages: the two-dimensional one, an approximation of the three-dimensional at zero speed of advance and finally including the forward speed. Of course, the solution found is only a practical approximation of reality. Therefore the theoretical solution has been checked with experiments in each subsequent stage. If inadmissible inaccuracies had appeared in the first or second stage there would have been no sense in continuing. It has been shown that this is not the case, but also that the last step leaves something to be desired. Yet the method developed is directly applicable to ships like vessels keeping station as weather ships, oceanographic research ships, drilling vessels and the like.

Actual computations may have to be completed
Fig. 5.5 Vertical wave force and moment on a ship model for $\mu = 120^\circ$ deg. at zero speed of advance.

Fig. 5.6 Vertical wave force and moment on a ship model for $\mu = 180^\circ$ deg. at various speeds of advance.
with, for instance, the special hydrodynamic effects of appendages taken into account. Their possible lift action is truely three-dimensional and cannot be incorporated in a strip theory. These contributions have to be considered separately and must be added to the coefficients. An example of this procedure is presented in [6.1]. The same holds when the effect of appendages is mainly based on viscous phenomena. However, as far as projections from the hull contribute to the potential flow they can fully be taken into account in principle.

With a practical solution as by strip theory a few shortcomings have to be accepted. In the first place, the fact that the hydrodynamic coefficients for the vertical motions are not correct in the low frequency range (sections 4.2 and 4.4). In practice this does not appear serious since this range is well below the resonance zone in heave and pitch for ordinary surface ships. Yet one has to be careful when in special designs, such as semi-submarines, exceptionally long natural periods occur. In the horizontal modes of motion sway-yaw and in roll the above effect is not present. In the second place, the fact that the calculations for the longitudinal rotations pitch and yaw may be less accurate than for sway, heave or roll. The effects produced by these rotations have been constructed from the corresponding translations in a transverse plane and are more liable to deviations from the actual occurrences than for sway, heave or roll (section 4.1).

The influence of the vessel's ends does not dominate the integrated pressure due to two-dimensional potential flow for the usual ratios of ship's length to beam of 6 or above and as long as there is no forward speed. For shorter vessels deviations may be marked. When there is speed of advance the inclusion of speed terms as indicated leads to qualitatively correct results. The quantitative errors depend on the magnitude of the velocity and on the frequency of motion and are specifically located along the forward third of the length. This is probably due to a combination of three-dimensional and of speed effects which cannot be solved easily (section 4.4). In principle the afterbody is liable to the same influences, but as appears from the experiments it is of much less significance. By separation of the flowlines aft a distinct stagnation point with variable position is not present and the very confused local fluid motion apparently reasonably resembles two-dimensional conditions. This looks like being the major difference between the forward and after end of the vessel.

From the present investigation it does not seem likely that the speed contributions for the horizontal modes of motion result from a circulation about the body analogous to that about airfoils.

For high speeds the physical assumptions made are probably inadequate; the interaction terms between the velocity potentials for the steady and the unsteady motions, the disturbance potential for the presence of the shipbody in the uniform stream and the variable area for pressure integration then can no longer be neglected. Thereby the problem has actually become nonlinear. There can hardly be hope that a feasible solution for the most general case will be possible. A different engineering approach could be as follows. Determine the wave profile along the length at the required speed experimentally. Next solve the potential problem in two dimensions for each section at a local draught corresponding with the wave elevation measured and add the hydrodynamic coefficients thus found in the usual way. This accounts for a part of the phenomena discussed above. This procedure is practicable, though rather digressive.

In the opinion of the author the theoretical and experimental study as performed has shown that from an investigator's view it will be desirable to approach the actual three-dimensional case numerically. The physical assumptions leading to a formulation as a boundary value problem in potential theory are sufficient for the vast majority of practical applications.

6.2 An outline of the possibilities to apply the calculations to more complicated cases of a vessel's behaviour

A vessel at sea has to cope with other forces than the harmonic wave loads. The most obvious are current and wind forces. Since the equations used are a practical formulation in the frequency domain and do not represent actual differential equations of time varying quantities the motions under the combined action of wave, current and wind cannot be found by solving the analogous system of equations in which the appropriate external forces have merely been added to the right hand sides. Here a deterministical and a statistical description must be linked.

Accepting linearity an approximate solution for the combined problem is sought in splitting the disturbances in two parts, one as a real function of time and one as a stationary harmonic load. Under the action of the first part the system is considered deterministically and position and orientation of the vessel are found with respect to time. Under the action of the second part the problem is treated as described in the preceding chapters. In a confused sea it leads to a statistical variation of motions. To obtain an insight in the behaviour under the influence of the combined loads the resulting motions are superimposed. So the solutions of the separate problems are to be added rather than
the external forces. Of course, by following this procedure a true deterministic description cannot be obtained. It always remains a statistical variation about a mean position and a mean orientation changing with time.

External loads which are functions of the frequency of motion can be added directly to the wave forces in the right hand sides of the equations of motion. Such a group of forces is, for instance, the action of passive or active roll damping devices, at least in their linear range. The input for their action is the ship motion itself in one form or another and so their output is a frequency dependent force.

Entirely the same process as applied to the hydrodynamic forces can be used to obtain the aerodynamic coefficients and the wind forces. The aerodynamic coefficients associated with oscillations of the body in still air can undoubtedly be neglected with respect to the corresponding hydrodynamic coefficients. The exception being a possible contribution to the surge and roll damping of vessels with large constructions above water. The exciting forces caused by variable winds could be unraveled in a wind force spectrum and be added to the frequency domain description.

The constant part of the wind forces (zero shift) should be added to the other contributions in the time domain. Other such forces are current forces, drift forces (zero shift of the oscillatory wave forces), rudder forces and the like. The solution of this time domain part of the combined problem is a very difficult one as appears from the effort in the manoeuvring field.

6.3 The effect of the neglect of viscosity

Viscous contributions appear for two reasons: skin friction and flow separation. Skin friction is proportional to a velocity gradient and will contribute to the damping only, while separation changes the flow pattern about the body to a certain extent so that it may be felt in both the damping and the added mass. Any viscous component will be of a nonlinear nature. In the opinion of the author skin friction can be left out of consideration since it will even be small with respect to flow separation, although in unsteady fluid motions large velocity gradients and consequently large shear forces may occur. In cases where the damping due to wave radiation is small the influence of separation cannot be neglected, however. Such cases are e.g. the rolling motion, the hydrodynamic forces at the ends of the vessel in transverse motion and the local forces on appendages as bilge keels, rudders, stern tubes and the like.

From the experiments with two-dimensional cylinders discussed in section 4.2 it appears that eddy formation does not seriously affect the pressure distribution in phase with the body acceleration. Therefore it suffices to correct \( b_{\phi}^{0} \) and possibly \( b_{\phi}^{0} = b_{\phi}^{0} \). At least this is a fair approximation for the type of sections tested corresponding with the shape of ordinary shiplike vessels. It does certainly not mean that keels and similar projections from the hull can be neglected in the consideration of the mass coefficients for they do have a potential contribution. See e.g. Wendel [6.2], who has performed calculations for rectangles with and without keels in an infinite fluid domain without free surface. The result of the experiments demonstrates, however, that the potential part and the viscous part can be separated reasonably. Therefore it is suggested that the potential problem is solved for the sections including appendages to account for the contribution of the latter in the mass coefficients and in the wave damping and next to add a measured or estimated viscous damping part.

For the motions of shiplike vessels a procedure as described will only be important in rolling. The viscous contributions at the ends of the vessel in swaying and yawing may locally be significant but they are negligible with respect to the magnitude of the total wave damping in the range of frequencies associated with wave-induced motions. These statements do certainly not apply to the nearly steady movements of the vessel while manoeuvring. In that case the viscous part is no longer small with respect to the potential part and it may change the picture entirely. One also has to be careful in the determination of the external loads for a structural analysis because then the local viscous forces may become important as well.

6.4 The influence of the Froude-Kriloff hypothesis

In chapter 5 both the Froude-Kriloff terms and the diffraction terms in the wave exciting forces have been derived. The diffraction part is again divided in an added mass and a damping contribution. It is the added mass term which is the most important of the two. From equation (5.3.2) it can be seen that for the transverse force and for the yaw moment the Froude-Kriloff term and the added mass term are always in phase, while they are in counterphase for the heave force and the pitch moment. For the roll moment the phase relation depends on the mutual signs of \( P_{4} \) and \( a_{24}^{0} \). For very long waves (\( \omega \rightarrow 0 \)) the transverse force and the roll moment for each section reduce approximately to:

\[
1 + \frac{a_{24}^{0}}{\xi P_{4}} \times \text{the hydrostatic force} \quad (k = 2, 4)
\]

The second term is finite and generally also of \( O(1) \).
The vertical force for each section reduces to about
\[
\left(1 - \frac{\omega^2 a_{33}^0}{2 g b}\right) \times \text{the hydrostatic heave force}
\]
The second term now vanishes for \( \omega \to 0 \) although \( a_{33}^0 \) becomes infinitely large according to a factor \( \ln \omega \).
Therefore it is clear that the Froude-Kriloff hypothesis is not an unreasonable approximation for a first approach to heave and pitch in very long waves, but it is unacceptable for a consideration of sway, yaw or roll moments. The diffraction of long waves about a vessel apparently influences the horizontal force components especially. For shorter waves a comparison is much more complicated and the result will no doubt depend largely on the shape of the vessel.

It is not unusual to dimension roll stabilizers and the like on the basis of external loads on the vessel calculated according to the Froude-Kriloff hypothesis for lack of better information. This may lead to an underestimation of the moments by a factor 2 or even 3 and consequently to mal-dimensioning.

6.5 The influence of the accuracy of the various coefficients on the calculation of forces and motions

On considering the influence of errors introduced by the simplification of the physical problem to the mathematical one, a distinction must be made between the influence on local forces and on the resulting motions of the body. It will be clear that any error in the pressure distribution results in a directly proportional error in the corresponding local external load. For the motions, however, the integrated pressure distribution counts and it becomes effective only via a response function of the ship.

So the calculation of forces is the most critical. Very few experiments have been performed to compare a measured and a calculated pressure distribution directly. Some results may be found in [6.3], [6.4] and [6.11] for heaving. For swaying and rolling such pressure measurements have never been made, at least to the knowledge of the author. But when the calculation for both the added mass and damping and for the coupling coefficients is reasonably confirmed by experiments the possibility of errors in the pressure distribution is rather limited. Besides it must be realized that the uncertainties in the calculation only concern the dynamic part of the fluid forces. In most cases the hydrostatic contribution will be of the same magnitude or even larger than the dynamic one; it is computed very easily and very accurately. And the total external load on the structure does not present the whole picture either. In a structural analysis the load condition has to be completed by the inertial forces, which may assume fairly large values as well.

As to the effect on the wave induced motions two different groups of motions must be considered. In the first place surge, sway and yaw without a restoring term and in the second place heave, roll and pitch with restoration and thereby liable to resonance phenomena. The first group is largely controlled by the mass term. Therefore these motions are not very sensitive for the hydrodynamic coefficients. For frequencies which are low with respect to the natural frequency the motions of the second group are nearly static and are exclusively determined by the restoring term. For relatively high frequencies again the mass term dominates the motion response. So only in a restricted range about the natural frequency the hydrodynamic coefficients, and especially the damping coefficient, actually control the motion pattern. Therefore only in this range is it important for practical applications that the hydrodynamic theory is accurate. The qualitative discussion above is based on the well-known one degree of freedom system. The mutual coupling terms between the motions complicate this picture. A small term may suddenly play an important role because of an unfavourable phase difference between the motions considered. In general existing experience indicates, however, that fairly large differences in the coefficients may occur without the result of a significant change in the motions. In the opinion of the author a deliberate use of the methods and the calculations used in this report will provide results of sufficient accuracy for many practical applications.

6.6 An experimental check on the calculated motions

When comparing measured and calculated motions an over-all check is provided on the mathematical model. It must be the final proof that the simplifications are justified for this purpose.

The motions of an infinitely long cylinder in beam waves have been reported by the author in [6.5]. It was shown that the calculations and the experiments agreed quite satisfactorily, except for the magnitude of roll at resonance. An increase of the roll damping to allow for viscous effects reduces the peak value to reasonable proportions, but leaves the calculations unaffected otherwise. Therefore it was concluded that the mathematical model is correct for this case and that for the effect of elements which are lacking can be compensated by the introduction of semi-empirical or experimental data.

For ships models the calculations have been compared to motion experiments in quite a number of cases for heave and pitch; see e.g. [6.6, 6.7, 6.8]. Reference is
also made to the example presented in chapter 7. But for sway, roll and yaw measurements are scarce. For the Todd Sixty Series model with a block coefficient of 0.70 such experiments have been reported in [6.9] and [6.10].

In order to calculate the rolling motion, correct ship data such as the absolute positions of the metacentre and of the centre of gravity and the mass moments of inertia in air are generally not easily obtained.

If experiments in oblique waves are done use is sometimes made of free running models. These self-propelled models suffer from the influence of radio controlled rudder motions, which may be clearly perceptible in the rolling recorded. Therefore some well controlled motion tests in oblique waves, performed for the purpose of comparing with the computational procedure are badly needed. Ultimately full scale measurements will be desirable, but since sway and yaw will be very difficult to measure accurately and since the external circumstances of sea and wind cannot be controlled, careful model experiments in a wide basin are preferred. The tests should be done at various forward speeds and in regular waves of various incidence.

7 Application of the theory to the case of heave and pitch in head waves

By way of example the theoretical procedure has been applied to the motions of a ship in head waves ($\mu = 180$ deg.) for three speeds of advance. The ship is the parent form of the Todd Sixty Series with a block coefficient of 0.70 which has been used before. The forward speeds correspond to a Froude-number of 0.20 (somewhat below the service speed to be expected for this type of hull) and 0.30. So the higher speed is in fact rather far beyond the practical range. For comparison the motions without speed of advance are given as well. The longitudinal radius of gyration is chosen at 0.25 $L_{pp}$ and the centre of gravity is situated at the load waterline, 0.5% of $L_{pp}$ forward of the midship section.

The calculations are shown against the results of earlier calculations by strip theory and against experimental data, both obtained at the Shipbuilding Laboratory of the Delft University of Technology. This information was published in [6.7]. All results are shown in figure 7.1. The way of plotting is different from [6.7], but the data are identical.

Observing the curves as calculated, it is clear that the influence of forward speed is not insignificant. It is further noteworthy that the pitch response is obviously in error for long waves. It is evident that the pitch amplitude over wave slope should actually approach 1 when the wave length increases. The discrepancy is caused by the fact that the vertical hydrodynamic coefficients are not correct for low frequencies of motion. The same fact influences the wave exciting moment in that range. Exactly the same reasons apply to the theory used previously, but actual calculations in that range have rarely been performed with it. On physical grounds the exciting moment coefficient was drawn to the value 1.0 for $\omega \to 0$ and correspondingly $\theta / k_{w} c_{w}$ was extrapolated to 1.0 as well. However, both the present and the previous theory indicate that $M_{d} g l_{w} k_{w}$ increases indefinitely for $\omega \to 0$. The pitch response increases analogously. So in the range of very low frequencies of motion the theory obviously leads to erroneous results. Therefore the pitch calculations below a certain frequency have to be rejected and the physical extrapolation to a 1.0 pitch response factor has to be performed by hand. The magnitude of the frequency parameter at this point depends on the speed of advance. It is noted that this phenomenon will only be observed for pitch. It does not occur for the heave response (which is heave amplitude over wave amplitude instead of over wave slope) and neither for the lateral and roll motions. For these motions the low frequency discrepancy does not exist.

The mutual difference between the earlier and the new calculations are of the same order as the differences between the experiments and between the experiments and the calculations. A definite evaluation is therefore not possible. The more so since accurate motion experiments in waves are difficult and may in themselves not be interpreted as incontestable data. Apart from the physical assumptions which are necessary to formulate the problem and which are the same for both methods of calculation, the circumstances during experiments depart more or less from the circumstances to which the theories apply. This is partly due to purely experimental difficulties, such as the generation of correct artificial waves and the measuring accuracy, and partly to incidental circumstances for a certain ship, such as the strong flaring of sections in the bow and the stern, the shipment of water or slamming. Such effects cannot be incorporated in the theory.

The differences between the two calculations are most marked around resonance. However, it is just there that the above reasons are most valid. It is noted that the shipping of water actually occurred. The experimental results for the two wave heights for heaving around resonance may suggest that this fact has influenced the measurements indeed. They differ rather much, the one for the lowest wave height giving the largest heave response. Even at the lowest wave amplitude used of 1/100th of the length of the model still water was shipped. Of course the other reasons
Fig. 7.1 Vertical motions in head waves for a model of the Todd Sixty Series, $C_{Fh} = 0.70$.

mentioned may be of influence as well and the tendency of increasing relative heave with lower wave height may not be extrapolated. It is not impossible, however, that if the shipping of water is avoided the heave response at resonance is still larger. From the experimental results it would appear that in this case heaving is influenced more by the effects discussed than pitching. However, it is difficult to establish since heaving and pitching are rather strongly coupled.

As shown, from the experimental data available no conclusion on the possible improvement in the prediction of heave and pitch in head waves according to the present theory compared with the previous one can be obtained. For the other motions in waves of various incidence and at various forward speeds, calculations have very rarely been attempted. Suitable motion experiments are hardly present either. Therefore this is a field for future investigations. The results discussed in this chapter suggest that a very careful comparison of calculations with experiments in very low regular waves, remaining as close to the theory as possible, is the best approach. This does not form a serious limitation to practical applications, for it has been shown several times [1.18] that in the composition of the regular components to an irregular pattern at sea linearity is much better maintained than in any individual component of increasing height itself. For the further opinions of the author reference is made to the discussions and conclusions in chapter 6.
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