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Theoretical investigation on discharge-induced river-bank erosion

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E. Mosselman
THEORETICAL INVESTIGATION

ON

DISCHARGE-INDUCED RIVER-BANK EROSION

by

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ABSTRACT

Bank erosion is incorporated in one-dimensional and two-dimensional horizontal models for river morphology. The banks are assumed to consist of a fraction of cohesive material, which becomes washload after being eroded, and a fraction of granular material, with the same properties as the material of the bed. The banks are taken to be eroded by discharge flow causing lateral entrainment of lower parts of the banks and near-bank bed degradation, both inducing mass failure of upper parts of the banks.

Theoretical analyses are performed in order to reveal the influence of bank erosion on the morphological system. From an analysis of characteristics of the one-dimensional model it is concluded that generally river widths cannot be stabilized by protecting certain carefully chosen bank sections only, and that computations of river planimetry can be decoupled from the computations of flow and bed topography. A linear analysis of the one-dimensional model is used to clarify the interactions between bank and bed disturbances, whereas a linear analysis of the two-dimensional model is used to demonstrate that the input of bank erosion products decreases transverse bed slopes, but hardly influences the wave lengths and damping lengths of flow and bed topography in natural rivers with moderately migrating banks.
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INTRODUCTION

Land use planning in alluvial river valleys and the choice of locations for bridges and hydraulic structures require predictions of future river planform changes and, consequently, knowledge of river-bank erosion and river meandering. Of particular interest is the problem of stabilizing a river planform by constructing protection works at certain carefully chosen bank sections only. Such a discontinuous bank protection obviously yields an economic solution, but is also desirable from an environmental point of view, as natural banks appear to be very important for riverine ecosystems. For this reason, some channelized streams in the FRG have been changed back into more natural ones by partial removal of bank protection works (Keller and Brookes, 1983; Kern and Nadolny, 1986). For rivers in the Netherlands, De Bruin et al. (1986) have developed similar ideas, parts of which have been incorporated in the Dutch government's policy for town and country planning (Ministerie van VROM, 1988).

River flow, bed topography and planform are interrelated and as a consequence they are all affected by changes in bank erodibility. A numerical two-dimensional model for the morphology of rivers with erodible banks, RIPA, will be developed as a tool for the prediction of planform changes and the associated morphological consequences. The present theoretical investigation serves as a preparatory study, aimed at gaining insight in the physical processes and their interactions.

River bank erosion is determined by flow, bed topography, sediment transport, bank properties and water quality. The processes involved are outlined in Section 2.1.1. In the present investigation only discharge induced river-bank erosion is taken into account. The banks are assumed to consist of a fraction of cohesive material, which becomes washload after being eroded, and a fraction of granular material, with the same properties as the material of the bed.

Two main manifestations of bank erosion are river widening and meandering. They often occur together, thus complicating their analysis. River width establishment and meandering are discussed in Sections 2.1.2 and 2.1.3 respectively. The basic processes of bank erosion are taken to be lateral entrainment of lower parts of the bank and near-bank bed degradation, both inducing mass failure of upper parts. The corresponding bank erosion model is presented in Section 2.2.

The mathematical model and some analyses are presented in Chapter 3. A one-dimensional version of the model is used to study river widening, and a two-dimensional version to study the development of meanders.

Bank disturbances are found to be non-propagating, which implies that river widths cannot be stabilized by protecting certain carefully chosen bank sections only, unless these sections are so closely interspaced that other effects, not included in the model, become significant. An example of the latter is the use of groynes for bank protection. This finding does not mean, however, that river migration cannot be stopped by a discontinuous bank protection.
It is also found that bank disturbances do not influence the propagation of flow and bed disturbances, which allows computations of river planimetry to be decoupled from the computations of flow and bed topography. The input of bank erosion products appears to decrease the transverse bed slopes in curved rivers, but it hardly affects the wave lengths and damping lengths of flow and bed topography in rivers with banks that migrate only moderately. The analyses can also be used to explain that easily eroded banks may lead to a braided river.
2 BANK EROSION

2.1 Qualitative description

2.1.1 Processes involved

River-bank erosion is a complex phenomenon in which many factors play a role. The rate of bank retreat is determined by flow, bed topography, sediment transport, bank properties and water quality.

Flow exerts shear stresses that can remove particles from the banks. The near-bank flow pattern is determined by discharge, ships and wind. Groundwater flow can also cause bank erosion, as will be commented upon when discussing bank properties. In the present investigation only the influence of discharge will be taken into account.

Near-bank bed topography affects bank erosion in two ways. Indirectly, it determines flow velocities in the bank region and hence shear stresses. Directly, it determines the total bank height, which is an important parameter for bank stability. An increase of bank height decreases stability.

Bank erosion products participate in the sediment transport process. They influence the sediment balance as far as they don’t disintegrate into fine material transported as washload. Murphey Rohrer (1983) finds a correlation between migration rates and a sediment imbalance, defined as the difference between sediment transport capacity and the actual sediment flux. Neill (1987) estimates the limits of channel migration from sediment transport rates. In an earlier publication (Neill, 1983), he describes a bend in the Tanana River (Alaska), in which there is a more or less complete exchange of bedload, all incoming bedload being deposited on the inner point-bar and being replaced by material from outer bank erosion. It results in an extremely high migration rate of about 50 m/year. Humphrey (1978) has identified some enhancement of migration rates downstream of meander cutoffs, which is ascribed to a local increase in sediment supply.

Banks erode by either entrainment of individual particles or mass failure under gravity with subsequent removal of slumped debris. Many bank properties are significant for the resistance to erosion: bank material weight and texture, shear strength and cohesive strength, physio-chemical properties, bank height and cross-sectional shape, groundwater level and permeability, stratigraphy, tension cracks, vegetation and constructions.

River-banks are predominantly cohesive. The erosion of cohesive soils is a complicated topic, because resistance to erosion is determined mainly by physio-chemical interparticle forces that result from residual electrical charges at the surfaces of clay mineral sheets. These forces depend on temperature and electro-chemical properties of the pore and eroding fluids (Arulananandan et al., 1980). Furthermore, the erodibility of cohesive sediments can be influenced by living organisms. Small animals can disrupt sediments, while micro-organisms can have a stabilizing effect by secreting substances that bind sediments (cf. Montague, 1984).
An additional complicating factor is that cohesive material undergoes structural changes during the process of erosion, transport and sedimentation. Disturbed bank debris resulting from mass failure is less resistant to erosion than the original bank. Erosion products often disintegrate into fine washload, but flocculation may occur with consequent settling on the river bed. The muddy structure of this clay deposition is very dissimilar from the structure of the original consolidated bank.

The stability of a bank with respect to mass failure depends on the balance of forces on the most critical potential failure surface. Mass failure can be triggered by removal of particles at the toe, leading to lowering of the bed or oversteepening of the bank, but also by other causes, such as the development of tension cracks and their filling with water (Springer et al., 1985; Ullrich et al., 1986) or the generation of high pore water pressures. The most favourable conditions for high pore water pressures are during rapid drawdown in the river following a high flow stage.

A significant mechanism of bank erosion is caused by groundwater seepage, which can be induced after flooding, but is also related to land use and precipitation (Hagerty, 1983). In this mechanism, bank structure is an important factor. Due to the mode of formation of an alluvial valley, alluvial river-banks are usually composed of a series of more or less horizontal layers of varying permeability, resulting in a poor drainage in vertical direction and piping in pervious layers. The latter can cause removal of silt and fine sands with failure of more cohesive overlying layers and related upper bank collapse (e.g. Henkel, 1967).

Vegetation can both increase and decrease the stability of river banks. Grasses and shrubs of low biomass usually improve the resistance to erosion. They reduce near-bank flow velocities, they cover the soil and their roots and rhizomes reinforce the soil and introduce extra cohesion. Whether trees increase or decrease bank stability depends on a number of factors. Thorne and Osman (1988b) give a thorough qualitative description of how type, age, health and density of trees influence bank stability. For computations, the best way at present to take effects of bank vegetation into account is to incorporate them into the parameters used to represent bank material characteristics (Thorne and Osman, 1988b).

As cohesive sediments are affected by electrochemical properties of the eroding fluid, water quality strongly influences the erodibility of cohesive banks. Vegetation also depends heavily on water quality. The gradual loss of reed-beds along river banks in Norfolk (UK) for instance can partly be ascribed to eutrophication (Boar et al., 1984; Brooke and Ash, 1988).

In view of the many factors that influence bank erosion and river meandering it is interesting to note that the main planimetric properties of meanders seem to be determined by flow parameters only. Empirical meander geometry relations derived from laboratory streams and natural rivers appear to hold for other streams as well, such as meltwater streams on glacier ice, which do not bear sediment, density currents, and the Gulf Stream, which is not confined by any boundaries (Leopold and Wolman, 1960; Zeller, 1967).
2.1.2 River width

It has been recognized for a long time that width and other geometrical properties of a river are correlated with river discharge. The higher the volume of water passing through a cross-section per unit of time, the wider the river will be. This promoted the formulation of sets of empirical relations for equilibrium river geometries by Lacey (1929), Leopold and Maddock (1953) and many others. These empirical relations are referred to as 'regime theory'. Later attempts to derive similar relationships by using descriptions of the fundamental processes involved have been termed 'rational regime theory' (Ranette, 1979). However, the fundamental equations for water flow and sediment transport need an additional relation for closure and as yet there is no consensus on what fundamental relationship should be used to determine river width.

Some researchers adopt an extremal concept as additional relationship, such as the theory of minimum stream power or the one of minimum rate of energy dissipation, stating that a river tends to adjust its hydraulic geometry in such a way that its stream power or rate of energy dissipation is a minimum. An example is Chang's (1982) width predictor based on the minimum stream power concept.

Though extremal theories were initially presented as mere hypotheses, it has been attempted later on to justify them mathematically by using the theory of calculus of variations. This theory identifies the minimization of a functional with the solution of an associated steady-state differential equation. Yang and Song (1979) show along these lines that the velocity distribution that satisfies a linearized momentum equation without inertia terms, is the one that minimizes the total rate of energy dissipation. Chen (1980) argues that their derivation is only valid for flow regions bounded by a closed surface, and, as a consequence, cannot be applied to open-channel flow. In addition, Lambert (1988) reveals contradictions between statements derived from extremal hypotheses and well established opinions on river behaviour. Apparently extremal theories only hold for a very restricted class of problems.

Another approach uses the assumption that river width is controlled by erodibility of the banks. Narrow rivers are considered to widen until the critical shear stresses of the banks are no longer exceeded and until near-bank bed degradation, which can induce bank failure, no longer occurs. These ideas are followed in the present investigation.

It should be noted that there is some discussion on whether bed degradation leads to wider rivers. Chang (1983, 1984) finds that degrading streams tend to assume a narrower width, while aggrading streams tend to widen. The latter phenomenon has also been observed in experimental studies by Fujita and Muramoto (1982). Contrarily, Thorne and Osman (1988a) find that bed degradation leads to channel widening, which complies with field observations of streams with high, steep banks (Thorne et al., 1981).

Bank erodibility is not the only limiting factor for river widening. When a river becomes very wide and shallow, its cross-sectional shape may become unstable and develop into a number of separate, narrower channels, thus transforming into a braided or an unabranching river. The effect of channel widening on the
transition from a meandering to a braided river is demonstrated quantitatively by Friedkin (1965), while a relation presented by Struiksma and Klaassen (1988) yields a possibility to quantify this effect.

The actual establishment of the cross-sectional shape of a river does not result from bank erosion alone, but from a balance between the opposing mechanisms of bank erosion and accretion. Parker (1978) treats this problem by considering a lateral sedimentary equilibrium in which bank material moves as lateral bed load towards the channel centre and lateral diffusion of suspended sediment, generated by the non-uniform distribution of suspended sediment across the width, overloads the near-bank flow and causes deposition. Previously, this mechanism had been suggested qualitatively in Van Bendegom's (1975) lecture notes. Bank accretion can also be attributed to the development of a point bar during high discharges, emerging from the water level when the discharges are low.

In the present investigation, dominant bed load and constant discharge are assumed, which means that suspended-sediment diffusion and point-bar emergence cannot be accounted for. Therefore, only erosion of banks will be considered. This might seem an unacceptable shortcoming, but still leaves validity for many problems of considerable practical relevance, such as maintenance of a deep and consequently narrow cross-section for shipping, and protection of land, lives and properties.

The problem of meandering, in which both bank erosion and accretion play a role, can even be treated in this way by taking a constant width beforehand. Then bank accretion is simply assumed to balance the erosion of the opposite bank. This approach is successfully used in the meander migration models of Ikeda et al. (1981) and Crosato (1987).
2.1.3 River meandering

The development of meanders is probably the most intriguing phenomenon related to river-bank erosion. Yet for a long time the cause of river meandering was poorly understood, as is reflected by the fact that several explanations persisted to be in circulation. Among these theories were: earth rotation, secondary flow, excessive slope and energy, discharge variability, shear stress variations, transverse seiches, bed instability and bank-line irregularities. Research in the last decade, however, has definitely increased the understanding of the phenomenon.

In the seventies, many investigators share the opinion that the development of alternate bars due to instability of the bed causes local bank erosion by a local increase of near-bank flow velocities and water depths, thus transforming an initially straight channel into a sinuous one. Olesen (1983), however, argues that in view of the large propagation speed of alternate bars and the generally low erodibility of the banks, a steady bed deformation offers a more adequate explanation for the formation of meanders.

As opposed to these 'bar theories', Ikeda et al. (1981) introduce a 'bend theory' of river meanders, describing the lateral bend growth in a mildly curved channel with erodible banks. Bank erosion is related to the near-bank value of the main-flow perturbation, which is induced by channel curvature. Blondeaux and Seminara (1985) extend this analysis and demonstrate that the bend growth of the bend theory is associated with a steady bed deformation of the alternate-bar type, thereby unifying the bar and bend theories. They explain that the propagating alternate bars are bed disturbances that exhibit the maximum rate of amplification when no forcing from any external cause is present, and that the steady alternate bars are caused by resonance, forced by channel curvature. The wave lengths of the steady alternate bars are found to be about three times as large as the ones of the propagating alternate bars.

Blondeaux and Seminara (1985) do not consider the occurrence of steady alternate bars in straight channels, and hence they do not offer an explanation for the initiation of meandering. Such an explanation, however, is given by Struiksma et al. (1985), who show that a steady undulation of the alternate-bar type may develop in a straight channel as a dynamic response to the redistribution of water and sediment motion after an upstream flow disturbance. This implies that meandering can be initiated by any steady flow disturbance, such as an obstacle or the exit of a channel bend.

A synthesis is given by Crosato (1987). She presents a meander migration model that describes both the initiation of meandering due to an upstream flow disturbance and the continuation of meandering due to channel curvature. Bank erosion is caused by a local increase of the near-bank longitudinal flow velocity as a result of a steady bed deformation. The inherent transverse bed slopes are determined by transverse shear stresses, counteracting gravity. These transverse shear stresses are caused by secondary flow and redistribution of the main flow. The first is generated by curvature, the latter by an upstream flow disturbance, which
might include a sudden change of curvature along the channel. More details on Crosato's model are given in Section 3.5.

The initial bank erosion creates channel curvature, spatially oscillating with the same wavelength as the steady bed undulation. As channel curvature forces in its turn the deformation of the bed, resonance is met, which was first recognized by Blondeaux and Seminars (1985). Bank migration rates will consequently increase during the development of sinuosity. At higher sinuosities, however, the oscillations of bed deformation and curvature may become out of phase, thus decreasing migration rates. Indeed, observations suggest that an optimal channel curvature exists at which migration rates reach a maximum (Hickin and Nanson, 1984; Begin, 1986). This is a topic of current research (Crosato, 1989).

Now that a satisfactory meander theory seems to have been attained, it is interesting to reconsider the earlier theories on the cause of river meandering. Bed deformation appears to play a central role, but some of the other explanations might fit in with the theory as well.

Secondary flow generated by channel curvature contributes to the deformation of the bed and thereby influences the development of meanders. It should be noted that the vertical shear stresses exerted on the banks by secondary flow hardly affect bank erosion, as they are small with respect to the longitudinal shear stresses exerted by the main flow, which are related to the deformation of the bed. Only in case of non-alluvial rivers with rectangular cross-sections, the influence of secondary flow on bank erosion might become dominant (cf. Kitanidis and Kennedy, 1984).

Coriolis forces due to the rotation of the earth also generate secondary currents, but the effect is small in shallow rivers (cf. Kalkwijk and Booj, 1986; Larsson, 1986; Booj, 1988). Actually, the relative insignificance of Coriolis forces in most rivers has been recognized for many years, but theories ascribing meander initiation to earth rotation persisted to be in circulation, which can be explained from the fact that secondary currents were believed to be the main cause of meandering, and that hence some explanation was needed for the occurrence of secondary flow in a straight channel.

Excessive slope and energy cause high flow velocities and hence bank erosion, but this does not explain the formation of meanders as widening is another possible way of reducing the flow velocities and the energy expenditure per unit width.

Discharge variability cannot be an explanation for meandering, because experiments show that meanders also develop when the discharge is constant (e.g. Friedkin, 1945). Nevertheless, discharge variations may strongly affect the phenomenon actually observed in natural rivers. This is discussed at the end of this section.

Shear stress variations along the banks obviously play an important role. They result from the spatially oscillating flow and bed deformation.

The idea of transverse standing waves is conceptually correct, but the relations presented by Werner (1951) and Anderson (1967) do not take sediment motion into account and consequently they do not comply with modern theories.

Finally, any bank-line irregularity may form a flow disturbance, inducing the steady bed undulation that initiates meandering.
The interpretation of field observations is often complicated by the discontinuous character of river migration. Natural river banks can seem stable or only little migrating for decades, and then suddenly experience substantial erosion. Such an event is not necessarily related to an extremely high flood, since apart from discharge variability, also fluctuations over time of bank geometry and river width play a role. The mechanisms causing this discontinuous nature of river migration are discussed below.

Morphological changes are mainly caused by high discharge events. Quasi-steady flow is usually assumed in morphological computations, but the actual formation of the bed may be strongly affected by unsteadiness of the flow. Furthermore, discharge fluctuations influence bank stability. A rapid drawdown after flooding can leave a poorly drained bank saturated with water, resulting in a larger weight and reduced strength, which might lead to mass failure. This water can also induce seepage out of the bank (piping), with internal erosion of sand layers and resulting failure of overlying bank alluvium. High-flood periods also play an important role in meander cutoffs, which are a dramatic form of river migration.

During the retreat of a cohesive river-bank, bank geometry fluctuates. Following mass failure slump, debris accumulates at the bank toe. The debris is removed by lateral erosion prior to further bank oversteepening or bed degradation generating further mass failures. These periodical bank geometry changes cause apparent variations in bank erodibility, thus complicating erosion laws. Osman and Thorne (1988) present a geomechanical river-bank model that accounts for some of these changes.

Nanson and Hickin (1983) describe a cycle in which consecutive floods of similar magnitudes cause different rates of bank erosion, depending on the stage within a sequence of river width fluctuations. A river bend can experience a flood flow which causes rapid and substantial erosion of the outer bank. If the sediment supply to this bend is small, the point-bar deposition cannot keep up with the outer bank erosion, and as a consequence, the channel width is enlarged considerably. During the next few floods of similar magnitude, very little bank erosion occurs because of the reduced velocities in the overwide bend. Meanwhile, however, lateral accretion of the point-bar continues at the inner bank, eventually reducing the channel width to its original value. The inner bank becomes vegetated and the cycle is complete. The next major flood will cause large cutbank erosion again. Worth noting is that for this kind of rivers, point-bar removal might be used as a means of bank protection.
2.2 Bank erosion model

In the present investigation only bank erosion due to discharge is taken into account. Shear stresses exerted by discharge flow may cause erosion at the toe of the bank, which can be subdivided into lateral fluvial entrainment, $\Delta n$, and near-bank bed degradation, $\Delta z_b$. Both types of erosion may induce mass failure, as they decrease bank stability, cf. Fig. 2.1.

![Diagram of river-bank erosion](image)

**Fig. 2.1** River-bank erosion due to lateral fluvial entrainment, $\Delta n$, and near-bank bed degradation, $\Delta z_b$, both potentially inducing mass failure. $H$ = total bank height, $h_{tc}$ = depth of tension cracks, $\varphi$ = bank slope.

Bed degradation results from gradients in sediment transport capacity, which can be determined from the flow field by using an appropriate sediment transport formula. The lateral erosion can be determined with a simple but generally used relation for the erosion of cohesive soils (e.g. Ariathurai and Arulanandan, 1978):

\[
\frac{\partial n}{\partial t} = E \left( \frac{r_{bank}}{r_c} - 1 \right) \quad \text{for } r_{bank} \geq r_c
\]  \hspace{1cm} (2-1)

\[
\frac{\partial n}{\partial t} = 0 \quad \text{for } r_{bank} < r_c
\]  \hspace{1cm} (2-2)

in which $\partial n/\partial t$ is the erosion rate, $E$ is an erodibility coefficient, $r_{bank}$ is the flow shear stress on the bank and $r_c$ is a critical shear stress below which no erosion occurs.

Arulanandan et al. (1980) give relations to determine the erodibility coefficient and the critical shear stress of a cohesive soil. Osman and Thorne (1988) consider the approach of Arulanandan et al. to be one of the most promising of the currently available methods, because calculation of erodibility and critical shear stress is based on the electrochemical properties of the soil, pore water and eroding fluid.
Though the near-bank flow field is essentially three-dimensional, it can be represented well by the longitudinal shear stress in case of mildly curved flow, as continuity implies that the vertical component of the flow field close to the banks is driven by the perturbation of the longitudinal component, and is relatively small with respect to the latter (Blondeaux and Seminara, 1985). So \( \tau_{\text{bank}} \) can be taken to be the longitudinal shear stress on the banks. It can be related to the longitudinal bed shear stress, \( \tau_{\text{bx}} \), by

\[
\tau_{\text{bank}} = \alpha_L \tau_{\text{bx}}
\]

in which \( \alpha_L = 0.75 \) for width-to-depth ratios above 5 (Lane, 1953, cf. Figure 2.2).

![Graph showing shear stress distribution](image)

**Fig. 2.2** Shear stress distribution according to Lane (1953).

For cohesive banks, mass failure is not a continuous process that immediately follows the erosion at the toe, but a discontinuous one, active only during discrete events whenever a critical stability condition is exceeded. The time-average behaviour, however, can be modelled well by an immediate response to toe erosion, as will be adopted here. It implies that time-average bank migration rates are not influenced by bank stability characteristics with respect to mass failure. They are determined entirely by fluvial entrainment of material at the toe. A similar conclusion is drawn by Osman and Thorne (1988) for the migration process they term 'parallel bank retreat', in which the bank slope remains constant. It complies with field observations of meandering rivers by Hickin and Nanson (1984), who find the relationship between grain sizes at the outer bend toe and bank migration resistance to be very similar to Shields' diagram. They conclude that bank migration is primarily determined by fluvial entrainment of basal sediments, after which cohesive upper sediments erode by the collapse of cantilevered overhangs.
The banks are assumed to consist of a fraction $\omega$ of cohesive material, which becomes washload after being eroded, and a fraction $1-\omega$ of granular material, with the same properties as the material of the river bed. Hence, the volume, $\Delta V_1$, of bank erosion products per unit length of river to be accounted for in the sediment balance after bank retreat due to lateral erosion, $\Delta n$, can be expressed as

$$\Delta V_1 = (1-\omega) \cdot H \cdot \Delta n \quad \text{for } \Delta n \geq 0$$  \hspace{1cm} (2-4)

in which $H$ denotes the total bank height, i.e. the elevation difference between the top of the bank and the bed level at the toe.

Analogously, the volume, $\Delta V_2$, of bank erosion products per unit length of river to be accounted for in the sediment balance after bed degradation, $\Delta z_b$, can be expressed as

$$\Delta V_2 = (1-\omega) \cdot H \frac{-\Delta z_b}{\tan \varphi} \quad \text{for } \Delta z_b \leq 0$$  \hspace{1cm} (2-5)

provided that $|\Delta z_b| \ll H$ and with $\varphi$ denoting the bank slope. The bank slope, $\varphi$, changes during bed degradation, as failure plane slopes depend on bank height. Here the variability of $\varphi$ is not taken into account.

Note that $\Delta n$ is positive for both left bank and right bank erosion, whereas erosion of the bed corresponds to a negative value of $\Delta z_b$.

The sediment balance reads:

$$\frac{\partial z_b}{\partial t} + \frac{\partial s_x}{\partial x} + \frac{\partial s_y}{\partial y} = 0$$  \hspace{1cm} (2-6)

in which $t$ denotes time, $x$ and $y$ denote coordinates in longitudinal and transverse direction respectively, $z_b$ is the bed level and $s_x$ and $s_y$ are sediment transport rates per unit width in $x$ and $y$ direction respectively. The transverse sediment transport rate, $s_y$, is made up of various contributions: transport due to a transverse component of the bed shear stress exerted by the flow, transport due to gravity acting along a sloping bed and transport due to lateral input of bank erosion products. The direction of the bed shear stress differs from the depth-averaged flow direction due to the influence of secondary flow. The equation for the transverse sediment transport rate can be written as

$$s_y = s_x \frac{v}{u} - s_x A \frac{h}{R_x} - s_x \frac{1}{f(\theta)} \frac{\Delta z_b}{\partial y} + s_{\text{bank}}$$  \hspace{1cm} (2-7)

in which $u$ and $v$ are depth-averaged flow velocities in $x$ and $y$ direction respectively, $h$ is the water depth, $R_x$ is an 'effective' radius of streamline curvature related to the intensity of secondary flow (cf. Struikema et al., 1985), $A$ is a coefficient
which weights the influence of secondary flow, depending on the eddy viscosity model applied, $f(\theta)$ is a function which weights the influence of a transverse bed slope, and $s_{\text{bank}}$ represents the transverse transport of bank erosion products. For the first three right-hand terms of Eq. (2.7) reference is made to Koch and Flokstra (1980).

In order to enable the analyses in Chapter 3, the physically not very realistic assumption is made that the transverse transport rate of bank erosion products, $s_{\text{bank}}$, decreases linearly from its maximum value at the source bank to zero at the opposite bank. It implies that bank erosion products are assumed to be distributed evenly over a cross-section:

$$\frac{\partial s_{\text{bank}}}{\partial y} = -\frac{1}{B} \frac{\partial (V_1 + V_2)}{\partial t}$$

(2-8)

in which $B$ denotes river width. The negative sign originates from the fact that $s_{\text{bank}}$ is directed off the eroding bank, as can be verified easily by integrating the equation with respect to $y$.

With Eqs. (2-4) and (2-5) the relation becomes

$$\frac{\partial s_{\text{bank}}}{\partial y} = \frac{(1-\omega) \cdot H}{B} \left( \frac{\partial n}{\partial t} + \frac{1}{\tan \phi} \frac{\partial z_b}{\partial t} \right)$$

(2-9)

Further elaboration of the sediment balance depends on the nature of the problem under consideration. Here, two special cases will be investigated: the case in which two identical banks both erode and the case in which only one steep bank erodes.

**Two identical eroding banks**

The concept of two identical eroding banks is convenient in one-dimensional analyses, where physical quantities are represented by one value per cross-section.

As $\partial B_b/\partial t$ has the same value near both banks, it does not contribute to width changes of the bed. Bed degradation leads to a vertical shift of cross-sections, but not to their deformation, cf. Fig. 2.3. Hence width changes of the bed, $\partial B_b/\partial t$, can be related to bank migration rates by

$$\frac{\partial B_b}{\partial t} = 2 \frac{\partial n}{\partial t}$$

(2-10)
However, the degradation of the bed may involve a change of the water depth and hence a change of the depth-averaged river width.

\[
\frac{\partial B_w}{\partial t} = \frac{\partial n}{\partial t} + \frac{2}{\tan \phi} \frac{\partial h}{\partial t} \tag{2-11}
\]

Consequently, changes of the depth-averaged river width are given by

\[
\frac{\partial B}{\partial t} = 2 \cdot \frac{\partial n}{\partial t} + \frac{1}{\tan \phi} \frac{\partial h}{\partial t} \tag{2-12}
\]

As two banks erode, Eq. (2-9) must be transformed into

\[
\frac{\partial s_{\text{bank}}}{\partial y} = 2 \cdot \frac{1 - \omega}{B} \left( - \frac{\partial n}{\partial t} + \frac{1}{\tan \phi} \frac{\partial z_b}{\partial t} \right) \tag{2-13}
\]

Combination of Eqs. (2-6), (2-7), (2-12) and (2-13) yields

\[
(1 + 2\gamma) \frac{\partial z_b}{\partial t} + \frac{\partial s_x}{\partial x} + \frac{\partial s_{\text{tana}}}{\partial A} + \gamma \frac{\partial h}{\partial t} - 2 \cdot \frac{1 - \omega}{B} \frac{\partial B}{\partial t} = 0 \tag{2-14}
\]

in which

\[
\gamma = \frac{H \cdot (1 - \omega)}{B \cdot \tan \phi} \tag{2-15}
\]

and

\[
\tan \alpha = \frac{\nu}{u} - A \frac{h}{R_s} - \frac{1}{f(\theta)} \frac{\partial z_b}{\partial y} \tag{2-16}
\]

where \( \alpha \) can be identified with the sediment transport direction when no bank erosion occurs.
The fact that here bed degradation does not contribute to width changes of the bed seems to contradict the findings of other researchers. As yet, however, no general agreement exists on whether degrading streams become narrower or wider. Chang (1983, 1984) finds that degrading streams tend to assume a narrower width, while aggrading streams tend to widen. Contrarily, Thorne and Osman (1988a) find that bed degradation leads to channel widening. It should be noted, however, that their conclusion is based on a computation in which the banks are eroded by lateral entrainment as well, thus obscuring the actual contribution of bed degradation.

One steep eroding bank

Whether bed degradation in case of one eroding bank results in a width change of the bed depends on whether the bed degrades over the full width or in a near-bank region only. This problem can be avoided by assuming that bank slopes are close to 90°, so that terms with $\gamma$ vanish. The assumption is realistic for migrating rivers, where the eroding cohesive banks are often very steep. Accordingly, width changes are related to bank migration rates by

$$\frac{\partial B}{\partial t} = \frac{\partial n}{\partial t} \quad (2-17)$$

Combination of Eqs. (2-6), (2-7), (2-9) and (2-17) yields, neglecting the $\tan^{-1} \phi$ term

$$\frac{\partial z_b}{\partial t} + \frac{\partial s_x}{\partial x} + \frac{\partial s_x \tan \alpha}{\partial y} - \frac{(1-\omega) \cdot H}{B} \frac{\partial B}{\partial t} = 0 \quad (2-18)$$

with $a$ given by Eq. (2-16).

Fig. 2.5 One steep eroding bank.
3.1 Basic equations

The mathematical model presented below is two-dimensional horizontal. Its derivation from the fully three-dimensional equations by means of integration over depth is based on a similarity hypothesis, stating that the vertical profiles of the main and the secondary velocities are self-similar (cf. De Vriend, 1981). This similarity hypothesis is assumed to hold for shallow, mildly curved channels, where most of the flow is not influenced by the banks. In large natural rivers, these conditions are usually satisfied.

Though channel curvature is an important feature of natural rivers, mainly straight channels will be considered here, because their analysis is already believed to reveal essential properties of the phenomena involved. An extension to curved channels will be made at the end of this chapter.

The bed material and the hydraulic roughness are assumed to be uniform.

---

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial z_b}{\partial x} + g \frac{\partial h}{\partial x} + \frac{\tau_{bx}}{\rho h} = 0 \]  

\[ \text{Longitudinal momentum equation:} \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial z_b}{\partial y} + g \frac{\partial h}{\partial y} + \frac{\tau_{by}}{\rho h} = 0 \]  

\[ \text{Transverse momentum equation:} \]

in which \( t \) denotes time, \( x \) and \( y \) denote coordinates in longitudinal and transverse direction respectively, \( u \) and \( v \) are depth-averaged flow velocities in \( x \) and \( y \) direction respectively, \( z_b \) is the bed level, \( h \) is the water depth, \( g \) is the acceleration due to gravity, \( \rho \) is the mass density of water and \( \tau_{bx} \) is the bed shear stress in \( x \) direction.

in which \( \tau_{by} \) denotes the bed shear stress in \( y \) direction.
Continuity equation for water motion:

\[
\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 \tag{3-3}
\]

A flow disturbance may curve the streamlines. The centripetal acceleration of the water particles is then provided by a transverse slope of the water surface, which yields a transverse pressure gradient that is distributed uniformly over the depth. As, on the contrary, the vertical distributions of flow velocity and, hence, centrifugal acceleration are non-uniform, a spiral motion is induced. This secondary flow needs a certain distance to adapt to a changing curvature, which can be described by (cf. Rozovskii, 1957, and De Vriend, 1981):

\[
\lambda_\tau \frac{\partial h}{\partial x} + \frac{h}{R_\tau} = \frac{h}{R_{s1}} \tag{3-4}
\]

in which \( \lambda_\tau \) is the adaptation length, \( R_\tau \) is an 'effective' local radius of streamline curvature related to the intensity of developing secondary flow (cf. Struiksma et al., 1985) and \( R_{s1} \) is the local radius of streamline curvature for fully developed secondary flow, determined from the flow field by

\[
\frac{1}{R_{s1}} = \frac{1}{u} \frac{\partial v}{\partial x} \tag{3-5}
\]

However, as the adaptation length, \( \lambda_\tau \), is small compared with other length scales in the model, an instantaneous adaptation of the secondary flow will be assumed here:

\[
\frac{1}{R_\tau} = \frac{1}{u} \frac{\partial v}{\partial x} \tag{3-6}
\]

Relation for lateral fluvial erosion presented in Section 2.2:

\[
\frac{\partial n}{\partial t} = E \left( \frac{\tau_{\text{bank}}}{\tau_c} - 1 \right) \quad \text{for } \tau_{\text{bank}} \geq \tau_c \tag{3-7}
\]

\[
8n = 0 \quad \text{for } \tau_{\text{bank}} < \tau_c \tag{3-8}
\]

in which \( \partial n/\partial t \) denotes the bank migration rate due to lateral fluvial entrainment, \( E \) is a bank erodibility coefficient, \( \tau_{\text{bank}} \) is the longitudinal flow shear stress on the bank and \( \tau_c \) is a critical shear stress below which no bank erosion occurs.
The bank shear stress, \( r_{\text{bank}} \), can be related to the longitudinal bed shear stress, \( r_{\text{bx}} \), by

\[
r_{\text{bank}} = \alpha_L r_{\text{bx}}
\]  
(3-9)

in which \( \alpha_L = 0.75 \) for width-to-depth ratios above 5 (Lane, 1953).

A general power law is adopted for the sediment transport formula

\[
s_x = m \cdot u^b \cdot (1 - e^{-\frac{sz_b}{\delta x}})
\]  
(3-10)

in which \( s_x \) is the sediment transport rate per unit width in longitudinal direction, \( m \) is a coefficient, \( b \) is an exponent and \( e \) is a factor accounting for the longitudinal slope effect on the transport rate. The latter will be neglected, however, yielding

\[
s_x = m \cdot u^b
\]  
(3-11)

from which it follows that

\[
\frac{\delta s_x}{\delta x} = \frac{b \cdot s_x \cdot \frac{\delta u}{\delta x}}{u^b}
\]  
(3-12)

Equations (3-10) and (3-11) imply that sediment transport rates depend on local hydraulic conditions only, which holds well only in case of dominant bed load.

The contribution of bank erosion products to the sediment balance depends on the nature of the problem under consideration. It has been elaborated in Section 2.2 for two special cases, viz. the case in which two identical banks erode and the case in which only one steep bank erodes.

When two identical banks erode, the sediment balance reads

\[
(1 + 2\gamma) \frac{\delta z_b}{\delta t} + \frac{\delta s_x}{\delta x} + \frac{\delta s_R \cdot \tan\phi}{\delta y} + \gamma \frac{\delta h}{\delta t} - \frac{(1-\omega) \cdot H \cdot \delta B}{B} = 0
\]  
(3-13)

with

\[
\gamma = \frac{H \cdot (1-\omega)}{B \cdot \tan\phi}
\]  
(3-14)

in which \( H \) is the bank height, \( \omega \) is a washload factor, \( B \) is the river width, \( \phi \) is the bank slope and \( \tan\phi \) is given by

\[
\tan\phi = \frac{v}{u} - \frac{h}{R_x} - \frac{1}{\delta z_b} \frac{1}{f(\theta)} \frac{\delta y}{\delta y}
\]  
(3-15)

in which \( A \) is a coefficient which weighs the influence of secondary flow, depending on the eddy viscosity model applied, and
$f(\theta)$ is a function which weighs the influence of a transverse bed slope. The sediment transport direction equals $\alpha$ when no bank erosion occurs.

Width changes are related to bank migration rates by

$$\frac{\partial B}{\partial t} = 2 \frac{\partial n}{\partial t} + \frac{1}{\tan \varphi} \frac{\partial h}{\partial t}$$  \hfill (3-16)

When only one steep bank erodes, the sediment balance becomes

$$\frac{\partial s_b}{\partial t} + \frac{\partial s_x}{\partial x} + \frac{\partial s_x \tan \alpha}{\partial y} - \frac{(1 - \omega) H}{B} \frac{\partial B}{\partial t} = 0$$  \hfill (3-17)

with $\alpha$ given by Eq. (3-15).

Here, width changes are related to migration rates by

$$\frac{\partial B}{\partial t} = \frac{\partial n}{\partial t}$$  \hfill (3-18)
3.2 Characteristic celerities

The analysis of characteristics is used to reveal essential properties of the mathematical system. It also provides clue to whether the computations of flow, bed topography and river planimetry can be decoupled, i.e. whether they can be executed in separate computational steps. Here the characteristics are determined from a one-dimensional model, which can be obtained from the two-dimensional model by considering a channel with rectangular cross-sections ($\bar{z}_b/\bar{y} = 0$) and by neglecting transverse velocity components ($\nu = 0$, $1/R_x = 0$). The set of equations presented in Section 3.1 can then be written as

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial z_b}{\partial x} + g \frac{\partial h}{\partial x} \frac{r_{bx}}{\rho h} = 0 \quad (3.19)$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \quad (3.20)$$

$$\frac{\partial n}{\partial t} = E \left( \frac{\alpha_L \cdot r_{bx}}{r_c} - 1 \right) \quad \text{for } \alpha_L \cdot r_{bx} \geq r_c \quad (3.21)$$

$$\frac{\partial n}{\partial t} = 0 \quad \text{for } \alpha_L \cdot r_{bx} < r_c \quad (3.22)$$

$$\frac{\partial s_x}{\partial t} = b \cdot s_x \frac{\partial u}{\partial x} \quad (3.23)$$

$$(1 + 2\gamma) \frac{\partial z_b}{\partial t} + \frac{\partial s_x}{\partial x} + \gamma \frac{\partial h}{\partial t} - \frac{(1-\omega) \cdot H \cdot \partial B}{B \cdot \partial t} = 0 \quad (3.24)$$

$$\frac{\partial B}{\partial t} = - \frac{\partial n}{\partial t} + \frac{1}{\tan \phi} \frac{\partial h}{\partial t} \quad (3.25)$$

in which it has been assumed that left and right banks have the same properties.

It is assumed that $\alpha_L \cdot r_{bx} \geq r_c$, so that Eq. (3.22) can be omitted.

Integration of Eqs. (3.19), (3.20) and (3.24) over width yields

$$\frac{\delta (Bhu)}{\delta t} + \frac{\delta (Bhu^2)}{\delta x} + gBh \left( \frac{\delta z_b}{\delta x} + \frac{\delta h}{\delta x} \right) + B \frac{r_{bx}}{\rho} = 0 \quad (3.26)$$
\[
\frac{\partial (Bh)}{\partial t} + \frac{\partial (Bhu)}{\partial x} = 0 \tag{3-27}
\]

\[
B(1 + 2\gamma) \frac{\partial z_b}{\partial t} + \frac{\partial (Bsz_h)}{\partial x} + B\gamma \frac{\partial h}{\partial t} - (1-\omega) \cdot H \cdot \frac{\partial B}{\partial t} = 0 \tag{3-28}
\]

The bed shear stress, \( r_{bx} \), can be expressed as

\[
r_{bx} = \frac{\rho g u_x^2}{C_x^2} \tag{3-29}
\]

in which \( C \) is the Chézy coefficient for hydraulic roughness. Hence

\[
r_{\text{bank}} = \alpha_L \cdot \frac{\rho g u^2}{C^2} \tag{3-30}
\]

and analogously

\[
r_c = \alpha_L \cdot \frac{\rho g u_c^2}{C_c^2} \tag{3-31}
\]

where \( u_c \) is a critical flow velocity.

Consequently, Eqs. (3-21) and (3-25) can be combined into

\[
\frac{\partial B}{\partial t} = 2E \cdot \left(\frac{u^2}{u_c^2} - 1\right) + \frac{1}{\tan \varphi} \frac{\partial h}{\partial t} \tag{3-32}
\]

For convenience two dimensionless parameters are introduced. The sediment transport parameter, \( \psi \), is defined by

\[
\psi = b \frac{g_z}{hu} \tag{3-33}
\]

The relative effective bank height is given by

\[
\eta = \frac{(1-\omega) \cdot H}{h} \tag{3-34}
\]

Hence Eqs. (3-26), (3-27), (3-28) and (3-32) can be written as

\[
\frac{\partial (Bhu)}{\partial t} + \frac{\partial (Bhu^2)}{\partial x} + gBh \left(\frac{\partial z_b}{\partial x} + \frac{\partial h}{\partial x}\right) + \frac{Bgu^2}{C^2} = 0 \tag{3-35}
\]

\[
\frac{\partial (Bh)}{\partial t} + \frac{\partial (Bhu)}{\partial x} = 0 \tag{3-36}
\]
\[ (1 + 2\gamma) \frac{\partial u}{\partial t} + \frac{\partial s}{\partial x} \frac{\partial B}{\partial x} + \frac{\partial h}{\partial x} + \gamma \frac{\partial h}{\partial t} - \eta \frac{\partial B}{\partial t} = 0 \]  

\[ \frac{\partial B}{\partial t} = 2E \left( \frac{u^2}{u_c^2} - 1 \right) + \frac{1}{\tan \phi} \frac{\partial h}{\partial t} \]  

The characteristics of this set of equations can be obtained by adding the expressions for the total differential, e.g.

\[ du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx \]  

Similar relations hold for \( dh, dz_b \) and \( dB \). The resulting system can be written in matrix notation as

\[ A \dot{f} = g \]  

with

\[ f = \begin{pmatrix} u & \frac{\partial u}{\partial t} & \frac{\partial u}{\partial x} & \frac{\partial g}{\partial t} & \frac{\partial g}{\partial x} & \frac{\partial g}{\partial \phi} & \frac{\partial g}{\partial \zeta} & \frac{1}{\partial B} & \frac{\partial u}{\partial B}^T \end{pmatrix} \]  

\[ g = \begin{pmatrix} -\frac{g u}{h C^2}, 0, 0, \frac{2E}{B} \left( \frac{u^2}{u_c^2} - 1 \right), \frac{1}{\frac{u^2}{u_c^2}}, \frac{1}{h C}, \frac{1}{h C}, \frac{1}{h C}, \frac{1}{B C} \end{pmatrix} \]  

\[ A = \begin{pmatrix} 1 & 2 & \frac{u^2}{B} & 1 & 1 & 1 & 0 & 1 & 1 \ 0 & 1 & \frac{Fr^2}{2} & 0 & 0 & 1 & 0 & 0 & 1 \ 0 & \psi & Fr^2 & 0 & (1+2\gamma)Fr^2 & 0 & -\eta & \psi/b & 0 \ 0 & 0 & \frac{N}{N} & 0 & 0 & 0 & -\eta & 0 & 0 \ 1 & \phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & Fr^2 & 0 & 0 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & Fr^2 & 0 & 0 & 0 & 0 \ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]  

in which the Froude number, \( Fr \), and the relative celerity of disturbances, \( \phi \), are defined by

\[ Fr = \frac{u}{\sqrt{gh}} \]  

\[ \phi = \frac{1}{\frac{dx}{dt}} \]  

Discontinuities in the solution of Eq. (3-40) can only exist when the determinant of the matrix \( A \) equals zero

\[ \det(A) = 0 \]
This is equivalent to

\[
\begin{pmatrix}
2 - \psi & 1 + Fr^2 - Fr^2\psi & 1 & 1 - \psi \\
1 & Fr^2 - Fr^2\psi & 0 & 1 - \psi \\
\psi & -\gamma Fr^2\psi & -(1+2\gamma)Fr^2\psi & \psi b + \eta \psi \\
0 & -\gamma Fr^2\psi & 0 & \eta \psi
\end{pmatrix} = 0 \quad (3\text{-}47)
\]

in which odd-numbered columns of matrix A have been eliminated by using the total differential relations in the lower four rows.

Elaboration of this expression yields four celerities, \( \psi \), corresponding to four families of characteristics along which discontinuities or disturbances in the solution propagate.

The fact that all elements of the fourth row contain a factor \( \psi \) implies that \( \psi = 0 \) is a solution, which can be identified with the celerity of the banks. Bank disturbances are found to be non-propagating, as is already evident from the bank erosion equation (3-38). The dependence of bank erosion on flow velocities rather than on flow velocity gradients causes bank disturbances to exhibit an essentially different behaviour compared to bed disturbances.

An important implication is that the equilibrium location of a bank is not influenced by the banks in other cross-sections. This implies that river widths cannot be stabilized by protecting certain carefully chosen bank sections only, unless these sections are so closely interspaced that other effects, not included in the present model, become significant. An example of the latter is the use of groyes as bank protection. Groyes guide the main flow in such a way that its erosive action is kept away from the unprotected bank sections, thus reducing the widening at these sections and inhibiting channel migration. This guiding of the main flow may be enhanced by the occurrence of an eddy between each pair of groyes (cf. Jansen et al., 1979).

The conclusion that river widths cannot be stabilized by protecting certain bank sections only does not imply that river migration cannot be stopped in this way. Meander migration models such as the one of Crosato (1987) are a promising tool to investigate this type of planform stabilization.

After dividing out \( \psi \) in the fourth row, Eq. (3-47) can be simplified into

\[
\begin{pmatrix}
1 - \psi & 1 & 1 & 0 \\
1 & Fr^2(1-\psi) & 0 & 1 - \psi \\
\psi & 0 & -(1+2\gamma)Fr^2\psi & \psi b \\
0 & -\gamma Fr^2 & 0 & \eta
\end{pmatrix} = 0 \quad (3\text{-}48)
\]

which yields the cubic equation

\[
\psi^3 - 2\psi^2 + \frac{(1 - \frac{\psi Fr^{-2}}{1+2\gamma})\psi + \frac{\psi Fr^{-2}}{1+2\gamma}} {\eta + \gamma} = 0 \quad (3\text{-}49)
\]

For \( \gamma = 0 \) this reduces to the cubic equation derived by De Vries (1959). By analyzing this equation, he demonstrated that the quasi-steady flow assumption in morphological computations holds well for small to moderate Froude numbers (Fr < 0.6 to 0.8).
When quasi-steady flow is assumed beforehand, Eq. (3-48) reduces to

\[
\begin{bmatrix}
1 & Fr^{-2} & 1 & 0 \\
1 & 1 & 0 & 1 \\
\psi & 0 & -(1+2\gamma)Fr^2\psi & \psi/b \\
0 & -\gamma & 0 & \eta
\end{bmatrix} = 0
\]  
(3-50)

which yields an expression for the celerity of bed disturbances

\[
\varphi = \frac{1 + (1-b^{-1})\gamma/\eta}{1 - Fr^2(1+\gamma/\eta)} \frac{\psi}{1+2\gamma}
\]  
(3-51)

The occurrence of the bank parameter, \( \gamma \), in this relation corresponds to two counteracting effects. In the factor \((1+2\gamma)^{-1}\) it represents the influence of the input of bank erosion products due to bed degradation, which is found to decrease the celerity of bed disturbances. To the contrary, \( \gamma/\eta \) represents the influence of an increase of the depth-averaged river width due to an increase of the water depth, which appears to increase the celerity of bed disturbances.

In practice, however, these effects will only be noticeable in rather narrow or incised streams. For wide rivers in alluvial plains, \( \gamma \) can be neglected. Then the fourth column in Eq. (3-47) does not play a role in further elaboration of the determinant, which means that terms with \( \partial B/\partial t \) and \( \partial B/\partial x \) have no effect on the celerities of flow and bed topography. As a consequence, computations of river planimetry can be decoupled from the computations of flow and bed topography. This leads to a procedure consisting of three computational steps for cases in which the computations of flow and bed topography can be decoupled as well (quasi-steady flow). In the first step the flow field is computed while keeping the bed and bank configuration fixed. Sediment transport rates and bank migration rates are calculated from the flow field. In the second step bed level changes are computed from the sediment transport gradients and the input of bank erosion products. Finally, bank-line changes are calculated from the bank migration rates in the third step.

Such an approach has already been used in the meander migration models of Ikeda et al. (1981) and Crosato (1987), where both flow and bed topography are assumed to be steady when computing bank migration rates.
3.3 Linear analysis of the one-dimensional model

The aim of linearization is to obtain a simplified version of the mathematical model that still retains its essential properties and may allow analytical solution. Here the one-dimensional equations of Section 3.2 are linearized in order to study river widening.

For quasi-steady flow and wide rivers (or narrow rivers with steep banks), Eqs. (3.35) to (3.38) can be written as

\[
\frac{1}{B} \frac{\partial (Q_0)}{\partial x} + gh \left( \frac{\partial z_b}{\partial x} + \frac{\partial h}{\partial x} \right) + \frac{gu^2}{C^2} = 0
\]  

(3.52)

\[ Q = Bh = \text{constant} \]  

(3.53)

\[
\frac{\partial z_b}{\partial t} + \frac{\psi}{B} \frac{\partial u}{\partial x} + s_x \frac{\partial B}{\partial x} - h \frac{\partial B}{\partial t} = 0
\]  

(3.54)

\[
\frac{\partial B}{\partial t} = 2E \left( \frac{u^2}{u_c^2} \right) - 1 \quad \text{if } u \geq u_c
\]  

(3.55)

Every quantity is assumed to be represented by the sum of two terms: a zero-order term, corresponding to steady uniform flow without morphological changes, plus a first-order perturbation term, e.g.

\[ u = u_0 + u' \quad \text{with } u' \ll u_0 \]  

(3.56)

Similar expressions hold for \( z_b, h, B \) and \( s_x \). The essence of the simplification lies in the neglect of higher-order terms that originate from products of first-order terms.

As the zero-order solution applies to situations in which no morphological changes occur, the zero-order flow velocity, \( u_0 \), may not exceed the critical flow velocity, \( u_c \). Here they are taken to be equal

\[ u_c = u_0 \]  

(3.57)

It means that the river width just complies with the zero-order solution.

Linearization results in:

\[
\frac{u_0}{g} \frac{\partial u'}{\partial x} + \frac{h'}{h_0} \frac{\partial z_{bo}}{\partial x} + \frac{\partial z_b'}{\partial x} + \frac{\partial h'}{\partial x} - \frac{2u_0 u'}{h_0 C^2} = 0
\]  

(3.58)

\[
\frac{u'}{u_0} + \frac{h'}{h_0} + \frac{B'}{B_0} = 0
\]  

(3.59)
\[
\frac{\delta z_{o'}}{\delta t} + \psi_{o} b \frac{\delta u'}{\delta x} + \frac{s_{x o}}{B_o} \frac{\delta B'}{\delta x} - \eta_{o} \frac{h_{o} \delta B'}{B_o \delta t} = 0
\]  
(3-60)

\[
\frac{\delta B'}{\delta t} = -4E \frac{u'}{u_{o}} \quad \text{if } u' \geq 0
\]  
(3-61)

The linearized equation of bank erosion is identical to the one used in their linear bend theory of river meanders.

The zero-order slope satisfies Chézy's relation for steady uniform flow

\[
\frac{\delta z_{b o}}{\delta x} = -i_{o} - \frac{u_{o}^2}{h_{o} c_o^2}
\]  
(3-62)

In the following it is assumed that Froude numbers are small, which implies that the convective term can be neglected. This assumption and the relations for \(i_{o}\) expressed by Eq. (3-62) are used to rewrite the linearized momentum equation into

\[
\frac{\delta z_{b'}}{\delta x} + \frac{\delta h'}{\delta x} + \frac{2 i_{o} u'}{u_{o}} - \frac{i_{o} h'}{h_{o}} = 0
\]  
(3-63)

From Section 3.2 it follows that the (dimensional) celerity, \(c_{o}\), of bed disturbances is given for small Froude numbers by

\[
c_{o} = \varphi_{o} u_{o} = \psi_{o} u_{o} = \frac{b_{o} s_{x o}}{h_{o}}
\]  
(3-64)

Hence the linearized sediment balance can be written as

\[
\frac{\delta z_{b'}}{\delta t} + c_{o} \frac{h_{o} \delta u'}{u_{o} \delta x} + \frac{h_{o} c_{o}}{B_o} \frac{\delta B'}{\delta x} - \eta_{o} \frac{h_{o} \delta B'}{B_o \delta t} = 0
\]  
(3-65)

Furthermore eliminating \(u'\) by using the linearized continuity equation, the set of equations becomes

\[
\frac{\delta z_{b'}}{\delta x} + \frac{\delta h'}{\delta x} - \frac{3 i_{o} h'}{h_{o}} - \frac{2 i_{o} B'}{B_{o}} = 0
\]  
(3-66)

\[
\frac{\delta z_{b'}}{\delta t} - c_{o} \frac{\delta h'}{\delta x} - \frac{h_{o} b - 1}{B_{o}} \frac{\delta B'}{\delta x} - \eta_{o} \frac{h_{o} \delta B'}{B_o \delta t} = 0
\]  
(3-67)

\[
\frac{\delta B'}{\delta t} = -4E \left( \frac{h'}{h_{o}} + \frac{B'}{B_{o}} \right) \quad \text{if } \frac{h'}{h_{o}} + \frac{B'}{B_{o}} \leq 0
\]  
(3-68)
The solutions are assumed to be given by

\[ z_b' = z_b \exp(ikx + rt) \quad (3.69) \]

\[ h' = h \exp(ikx + rt) \quad (3.70) \]

\[ B' = B \exp(ikx + rt) \quad (3.71) \]

in which \( k \) is a wavenumber, \( r \) is a complex frequency and \( i \) is the imaginary unit defined by \( i^2 = -1 \). The real and imaginary parts of \( r \) can be interpreted to represent the diffusion and propagation character of the solution respectively. This can be expressed after Vreugdenhil (1982) as

\[ r = -ikc_e - k^2D_e \quad (3.72) \]

where \( c_e \) is the effective celerity and \( D_e \) is the effective diffusion.

Substitution of the postulated solutions results in a set of equations, which yields nontrivial solutions if the determinant of the coefficient matrix equals zero

\[
\begin{vmatrix}
  ik & ik - \frac{c_o}{D_o} & -2 \frac{c_o}{3D_o} \\
  r & -ikc_o & -ik \left( \frac{b-1}{b} \right) c_o - \eta_0r \\
 0 & 4E & 4E + B_o r
\end{vmatrix} = 0 \quad (3.73)
\]

where \( D_o \) is given by

\[ D_o = \frac{c_o^2 \eta_0}{3\eta_o} \quad (3.74) \]

This is the diffusion coefficient for cases in which backwater effects and bank erodibility can be neglected (De Vries, 1973).

For convenience some dimensionless parameters are introduced. The Péclet number is defined by

\[ P = \frac{c_o}{kD_o} \quad (3.75) \]

The bank erodibility coefficient, \( \epsilon \), is defined by

\[ \epsilon = \frac{4E D_o}{B_o \ c_o^2} \quad (3.76) \]
The dimensionless complex frequency is given by

\[ \rho = \frac{D_0}{c_0} \frac{r}{\epsilon} \]  

(3-77)

Elaboration of the determinant yields with these notations the characteristic equation

\[ (p^2 - i\epsilon)\rho^2 + \left(1 + \frac{\epsilon}{3} p^2 - i(1-\eta_0)\epsilon P\right)\rho + \frac{\epsilon}{b} = 0 \]  

(3-78)

with

\[ \rho = -1 \frac{1}{P} \frac{c_e}{c_0} - \frac{1}{P^2} \frac{D_e}{D_0} \]  

(3-79)

For \( \epsilon = 0 \), i.e. for fixed banks, the characteristic equation reduces to

\[ \rho = \frac{-1}{P^2 - iP} = \frac{-1 - iP}{1 + P^2} \]  

(3-80)

which is the relation given by Vreugdenhil (1982).

Solutions of Eqs. (3-78) and (3-79) are shown in Figures 3.2 to 3.5, where \( c_e/c_0 \) and \( D_e/D_0 \) are given as functions of the Péclet number, \( P \).

Fig. 3.2 Relative effective celerity, \( c_e/c_0 \), as function of the Péclet number, \( P \), for \( \epsilon = 10^{-6} \), \( \eta_0 = 0 \) and \( b = 5 \). The graph coincides with the one for \( \epsilon = 0 \).
Based Lines correspond to $c = 0$, $n_0 = 0$ and $b = 5$.

The Becker number, $P$, for $\epsilon = 1$, $n_0 = 0$ and $b = 5$.

Relative effective rate, $c/c_0$, as function of.

Fig. 3.3
Fig. 3.5 Relative effective diffusion, $D_e/D_o$, as function of the Péclet number, $P$, for $\epsilon = 1$, $\eta_o = 0$ and $b = 5$. Dashed lines correspond to $\epsilon = 0$.

The graphs show that for small Péclet numbers (i.e. short disturbances) the solutions are independent from bank erodibility. The effective diffusion vanishes and only two waves remain, one steady ($c_o = 0$) and one propagating with celerity $c_w = c_o$. This complies with the results from the analysis of characteristics in Section 3.2, which applies to infinitely small Péclet numbers as the theory of characteristics describes the behaviour of infinitely short disturbances.

At large Péclet numbers, the effective celerity vanishes in case of fixed banks, but becomes negative in case of erodible banks, which corresponds to a propagation in upstream direction. Two nonzero values of the effective diffusion, $D_e$, emerge, one of them considerably exceeding $D_o$ at higher values of $\epsilon$, thus reflecting that bank erosion instead of sediment transport and bed resistance becomes dominant in providing diffusion.

The interactions between bank and bed disturbances can be clarified further by introducing additional simplifications. It is assumed that the distances, $x$, under consideration are small, so that the hydraulic friction terms (both terms with $i_o$ in Eq. 3-63) can be neglected. This effectively means that the behaviour at small Péclet numbers is studied. The linear momentum equation then reduces to

$$\frac{\partial z_{b}'}{\partial x} + \frac{\partial h'}{\partial x} - \frac{\partial z_w'}{\partial x} = 0$$  \hspace{1cm} (3.81)
Hence the water level disturbance, \( z_w' \), is constant

\[
z_w'(x,t) = \text{constant} \tag{3-82}
\]

This is equivalent with a 'rigid-lid approximation', from which it follows that

\[
\frac{\partial z_b'}{\partial t} = -\frac{\partial h'}{\partial t} \tag{3-83}
\]

Using this relation to eliminate \( z_b' \), the set of equations becomes

\[
\frac{\partial h'}{\partial t} + c_o \frac{\partial h'}{\partial x} = -\frac{h_o}{B_o} \left( \eta_o \frac{\partial B'}{\partial t} + \frac{b-1}{b} \frac{c_o}{\partial x} \frac{\partial B'}{\partial x} \right) \tag{3-84}
\]

\[
\frac{\partial B'}{\partial t} + 4E \frac{B'}{B_o} = -4E \frac{h'}{h_o} \quad \text{if} \quad \frac{h'}{h_o} + \frac{B'}{B_o} \leq 0 \tag{3-85}
\]

The left-hand terms in the first equation represent a simple wave with celerity \( c_o \). The right-hand terms act as a source, implying that bank erosion products and width disturbances may generate a propagating bed wave. The source terms do not influence the celerity of the bed wave. The second equation shows that bank disturbances do not propagate. However, the two equations are coupled via the source terms, thus forming a hyperbolic system in which both bed and bank disturbances have two celerities, one equal to zero and one equal to \( c_o \).
3.4 Linear analysis of the two-dimensional model

The initiation of meandering is studied with a linear analysis of the two-dimensional model. It is assumed that only one bank per cross-section erodes, either the left one or the right one, and that the eroding bank is steep. Furthermore assuming quasi-steady flow, the set of equations presented in Section 3.1 can be written as

\[
\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial h}{\partial x} + g \frac{\partial z_b}{\partial x} + \frac{r_{bx}}{\rho h} = 0
\]  

\[
\frac{\partial v}{\partial x} + u \frac{\partial v}{\partial y} + g \frac{\partial h}{\partial y} + g \frac{\partial z_b}{\partial y} + \frac{r_{by}}{\rho h} = 0
\]  

\[
\frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0
\]  

\[
\frac{\partial B}{\partial t} = E \left( \frac{r_{bank}}{r_c} - 1 \right) \quad \text{for } r_{bank} \geq r_c \]  

\[
\frac{\partial B}{\partial t} = 0 \quad \text{for } r_{bank} \leq r_c
\]  

\[
r_{bank} = a_{L\cdot r_{bx}} \text{(near-bank)}
\]  

\[
\frac{\partial z_b}{\partial t} + b_s \frac{\partial u}{\partial x} + \frac{\partial(s \cdot \tan \alpha)}{\partial y} - \frac{(1 - \omega) \cdot H}{B} \frac{\partial B}{\partial t} = 0
\]  

\[
\tan \alpha = \frac{v}{u} + A \frac{h}{u} \frac{\partial v}{\partial x} - \frac{1}{\dot{f}(\theta)} \frac{\partial z_b}{\partial y}
\]  

The bed shear stresses are given by

\[
r_{bx} = \frac{\rho g}{G^2} \frac{u^2 + v^2}{u}
\]  

\[
r_{by} = \frac{\rho g}{G^2} \frac{u^2 + v^2}{v}
\]
while the critical bank shear stress is defined as

$$\tau_c = a_L \rho B \frac{u_c^2}{G^2}$$  \hspace{1cm} (3-96)

where $u_c$ is a critical near-bank flow velocity. Consequently

$$\frac{\partial B}{\partial t} = E \left( \frac{u_{nb}^2 + v_{nb}^2 u_{nb}^2}{u_c^2} - 1 \right) \quad \text{if} \quad u_{nb}^2 + v_{nb}^2 u_{nb} \geq u_c^2$$  \hspace{1cm} (3-97)

in which $u_{nb}$ and $v_{nb}$ denote near-bank flow velocities in $x$ and $y$ direction respectively.

The water level, $z_w$, is given by

$$z_w = z_b + h$$  \hspace{1cm} (3-98)

The equations can be simplified by introducing a rigid-lid approximation. It is assumed that water level variations are negligible compared to water depth. Bed level variations can then simply be related to water depth variations by

$$\frac{\partial z_b}{\partial t} = -\frac{\partial h}{\partial t} ; \quad \frac{\partial z_b}{\partial x} = -\frac{\partial h}{\partial x} ; \quad \frac{\partial z_b}{\partial y} = -\frac{\partial h}{\partial y}$$  \hspace{1cm} (3-99)

However, this is not applied to the momentum equations because water level gradients have significant dynamic effects. They act as pressure gradients, so, under the rigid-lid approximation, $z_w$ represents the piezometric head rather than the water level.

Following the same procedure as in Section 3.3, it is assumed that every quantity is represented by the sum of two terms: a zero-order term, corresponding to steady uniform flow without morphological changes, plus a first order perturbation term:

$$u = u_o + u'' \quad \text{with} \quad u'' \ll u_o$$  \hspace{1cm} (3-100)

$$v = 0 + v'' \quad \text{with} \quad v'' \ll u_o$$  \hspace{1cm} (3-101)

$$h = h_o + h'' \quad \text{with} \quad h'' \ll h_o$$  \hspace{1cm} (3-102)

$$z_w = z_{wo} + z_w'' \quad \text{with} \quad z_w'' \ll z_{wo}$$  \hspace{1cm} (3-103)

$$B = B_o + B' \quad \text{with} \quad B' \ll B_o$$  \hspace{1cm} (3-104)

$$s_x = s_{xo} + s_x'' \quad \text{with} \quad s_x'' \ll s_{xo}$$  \hspace{1cm} (3-105)

$$\tan \alpha = 0 + \alpha'' \quad \text{with} \quad \alpha'' \ll \pi/2$$  \hspace{1cm} (3-106)

A double prime is used to indicate perturbed quantities that vary with both $x$ and $y$. This notation will prove convenient later on. Again it is assumed that the river width just complies with the zero-order solution, implying that the zero-order flow velocity, $u_o$, equals the critical near-bank flow velocity, $u_c$

$$u_c = u_o$$  \hspace{1cm} (3-107)
Linearization yields the following set of equations

\[
\begin{align*}
    u'_o & \frac{\partial u''}{\partial x} + g \frac{\partial z''}{\partial x} + \frac{2g u'_o}{h_0 c^2} u'' - \frac{g u'_o^2}{h_0^2 c^2} h'' = 0 \\
    u'_o & \frac{\partial v''}{\partial x} + g \frac{\partial z''}{\partial y} + \frac{g u'_o}{h_0 c^2} v'' = 0 \\
    u'_o & \frac{\partial h''}{\partial x} + h_0 \frac{\partial u''}{\partial x} + h_0 \frac{\partial v''}{\partial y} = 0
\end{align*}
\]

\[\begin{align*}
    \frac{\partial B'}{\partial t} & = 2E \frac{u'_o b'}{u'_o} \\
    \frac{\partial h''}{\partial t} & - b s_{xo} \frac{\partial u''}{\partial x} - s_{xo} \frac{\partial a''}{\partial y} + \frac{(1-\omega) \cdot H}{B_0} \frac{\partial B'}{\partial t} = 0 \\
    a'' & = \frac{v''}{u'_o} + A \frac{h_0}{u'_o} \frac{\partial v''}{\partial y} + \frac{1}{f(\theta_o)} \frac{\partial h''}{\partial y}
\end{align*}\]

The pressure terms can be eliminated by cross-differentiation of the two momentum equations, yielding

\[
\frac{\partial^2 u''}{\partial x \partial y} + \frac{2g u''}{h_0 c^2} \frac{\partial u''}{\partial y} - \frac{g u'_o h''}{h_0^2 c^2} \frac{\partial^2 v''}{\partial y^2} - \frac{g u''}{h_0 c^2} \frac{\partial^2 v''}{\partial x^2} - \frac{g}{h_0 c^2} \frac{\partial v''}{\partial x} = 0
\]

A steady-state analysis of the bed is appropriate for an investigation of the incipient meander, as bank migration is usually much slower than the propagation of bed disturbances (Olesen, 1983). Hence the bed is assumed quasi-steady, i.e. \( \frac{\partial h''}{\partial t} = 0 \). It should be noted that this assumption also excludes the shallowing effect of river widening, which means that only the very beginning of meandering is described, or that bank retreat is assumed to be balanced instantly by accretion of the opposite bank.

Eqs. (3-111) to (3-113) can then be combined into

\[
\begin{align*}
    b \frac{\partial u''}{\partial x} + \frac{1}{u'_o} \frac{\partial v''}{\partial y} + A \frac{h_0}{u'_o} \frac{\partial^2 v''}{\partial x \partial y} + \frac{1}{u'_o} \frac{\partial^2 h''}{\partial y^2} - \frac{2E h_0}{s_{xo} B_0} \eta_o u'_o = 0
\end{align*}
\]

if again \( \eta_o = (1-\omega) \cdot H/B_0 \) is used.
The perturbations are assumed sinusoidal in transverse direction. A sine is chosen for perturbations that should vanish when being integrated over width, and a cosine is chosen for perturbations that should vanish at the banks \( y = \pm B/2 \) because of the impermeability condition.

\[
\begin{align*}
u'' &= u' \sin(\pi y/B) \\
v'' &= v' \cos(\pi y/B) \\
h'' &= h' \sin(\pi y/B) 
\end{align*}
\] (3.116) (3.117) (3.118)

Substitution into the set of equations yields

\[
\frac{\pi}{B_o} \frac{\partial u'}{\partial x} + \frac{2g}{h_0} \frac{\pi}{B_o} u' - \frac{g u_o}{h_0} \frac{\pi}{B_o} h' - \frac{g}{h_0} \frac{\partial v'}{\partial x} = 0 \quad (3.119)
\]

\[
\frac{\partial u'}{\partial x} - \frac{\pi}{B_o} v' + \frac{u_o}{h_0} \frac{\partial h'}{\partial x} = 0 \quad (3.120)
\]

\[
\frac{b}{u_o} \frac{\partial u'}{\partial x} - \frac{\pi}{B_o} v' - A \frac{h_o}{u_o} \frac{\partial v'}{\partial x} - \frac{1}{f(\theta_0)} \frac{\pi^2}{B_o^2} h' \sin(\pi y/B_o) + \frac{2E}{s_{xo}} \frac{h_o}{\eta_o} \frac{u'_{nb'}}{u_o} = 0 \quad (3.121)
\]

Note that width changes do not play a role. The only effect of bank erosion that remains is the inclusion of bank erosion products in the sediment balance.

The sediment equation is evaluated along the eroding bank. In case of left-bank erosion:

\[
u'_{nb'} = u''(y = +B_o/2) = u'\sin(\pi/2) \quad (3.122)
\]

and for right-bank erosion:

\[
u'_{nb'} = u''(y = -B_o/2) = u'\sin(-\pi/2) \quad (3.123)
\]

Both cases lead to

\[
\frac{b}{u_o} \frac{\partial u'}{\partial x} - \frac{\pi}{B_o} v' - A \frac{h_o}{u_o} \frac{\partial v'}{\partial x} - \frac{1}{f(\theta_0)} \frac{\pi^2}{B_o^2} h' - \frac{2E}{s_{xo}} \frac{h_o}{\eta_o} \frac{u'}{u_o} = 0 \quad (3.124)
\]

It might seem strange to evaluate the linear sediment balance at the left bank in one river reach and at the right bank in the other, but the exact position in transverse direction has become immaterial, as the equations are reduced to one dimension by prescribing transverse shapes. Bank erosion is then merely related
to the degree of cross-sectional deformation, and as bank erosion products are taken to be distributed evenly over a cross-section, it is irrelevant from which bank they originate.

A strongly simplified version of the equations derived so far is considered now in order to introduce two important length scales after Struijsma (1983). The transverse velocity terms are neglected in the momentum equation, yielding

\[
\frac{\partial u'}{\partial x} + \frac{2g}{h_o c^2} u' = \frac{gu_o}{h_o^2 c^2} h' \quad (3-125)
\]

The continuity equation is used to eliminate \( v' \) from the sediment balance. Furthermore neglecting the influence of secondary flow (\( A = 0 \)) and bank erosion (\( E = 0 \)), one obtains

\[
\frac{\partial h'}{\partial x} + \frac{\pi^2}{\frac{B_o}{h_o^2} f(\theta_o)} h' = \frac{h_o (b-1)}{u_o} \frac{\partial u'}{\partial x} \quad (3-126)
\]

Both equations represent a retarded adaptation to a source (De Vriend and Struijsma, 1983). The characteristic length scales of the adaptations are

\[
\lambda_w = \frac{c^2}{2g} h_o \quad (3-127)
\]

for the main flow, and

\[
\lambda_s = \frac{1}{\frac{\pi^2}{\frac{B_o}{h_o^2} f(\theta_o)}} h_o \quad (3-128)
\]

for the bed respectively.

The complete linear equations become with these notations

\[
\lambda_w \frac{\partial u'}{\partial x} + u' - \lambda_w \frac{\pi B_o}{\pi} \frac{\partial^2 v'}{\partial x^2} - \frac{1}{2} \frac{\pi B_o}{\pi} \frac{\partial v'}{\partial x} - \frac{u_o}{2h_o} h' = 0 \quad (3-129)
\]

\[
\frac{\partial u'}{\partial x} - \frac{\pi}{B_o} v' + \frac{u_o}{h_o} \frac{\partial h'}{\partial x} = 0 \quad (3-130)
\]

\[
b \frac{\partial u'}{\partial x} - \frac{M}{\lambda_w} u' - \frac{\pi}{B_o} v' - \frac{\pi}{B_o} A \frac{\partial v'}{\partial x} - \frac{1}{\lambda_s} \frac{u_o}{h_o} h' = 0 \quad (3-131)
\]

in which a bank erosion coefficient, \( M \), has been introduced, defined as

\[
M = \frac{c^2 h_o u_o^2 h_o E}{g B_o s_{xo}} \quad (3-132)
\]
The bank erosion coefficient, $\epsilon$, used in Section 3.3 is related to $M$ by

$$\epsilon = \frac{4}{3\eta_{o} b \frac{Fr_{o}}{2} M}$$  \hspace{1cm} (3-133)

The perturbations are assumed harmonic in longitudinal direction

$$u' = \hat{u} \exp(ikx)$$ \hspace{1cm} (3-134)

$$v' = \hat{v} \exp(ikx)$$ \hspace{1cm} (3-135)

$$h' = \hat{h} \exp(ikx)$$ \hspace{1cm} (3-136)

in which $k$ is a complex wavenumber.

Insertion results in a set of equations, which yields nontrivial solutions if the determinant of the coefficient matrix equals zero

$$\begin{vmatrix}
1k\lambda_{w} + 1 & -1k \frac{B_{o}}{\pi} (1k\lambda_{w} + \frac{1}{2}) & -\frac{1}{2} \\
1k\lambda_{w} & -\frac{\pi}{B_{o}} \lambda_{w} & ik\lambda_{w} \\
1k\lambda_{w}b - M & -\frac{\pi}{B_{o}} (\lambda_{w} + ik\lambda_{w}A_{o}) & -\frac{\lambda_{w}}{\lambda_{S}}
\end{vmatrix} = 0$$ \hspace{1cm} (3-137)

Elaboration gives a fourth-degree characteristic polynomial

$$(k\lambda_{w})^4 [2b \left( \frac{1}{f(\theta_{o})} \frac{g}{C^2} \frac{\lambda_{S}}{\lambda_{w}} \right) +$$

$$+ 1(k\lambda_{w})^3 \left[ \frac{g}{C^2} (2A + \frac{1}{f(\theta_{o})} (2 + (b-2M)\frac{\lambda_{S}}{\lambda_{w}})) \right] +$$

$$+ (k\lambda_{w})^2 \left[ 1 + \frac{g}{C^2} (3A - \frac{1}{f(\theta_{o})} (1-M)\frac{\lambda_{S}}{\lambda_{w}}) \right] +$$

$$+ 1(k\lambda_{w}) \left[ \frac{b-3}{2} - \frac{\lambda_{w}}{\lambda_{S}} - \frac{\lambda_{w}}{\lambda_{S}} \frac{M}{2} \right] = 0$$ \hspace{1cm} (3-138)

For $M = 0$ this is the same polynomial as the one Struiksma et al. (1985) present. They show that for conditions prevailing in most natural rivers, the model can be simplified even further by neglecting transverse velocity contributions in the momentum equation, and by neglecting the influence of secondary flow ($A = 0$).
Application to Eqs. (3-129) to (3-131) gives, after eliminating \( v' \)

\[
\frac{\partial u'}{\partial x} + \frac{1}{\lambda_w} u' = \frac{1}{2\lambda_w} \frac{u_0}{h_0} h' \tag{3-139}
\]

\[
\frac{\partial h'}{\partial x} + \frac{1}{\lambda_s} h' = \frac{h_0}{u_0} (b-1) \frac{\partial u'}{\partial x} - \frac{h_0}{u_0} \frac{M}{\lambda_w} u' \tag{3-140}
\]

Again two relaxation equations are found with characteristic length scales \( \lambda_w \) and \( \lambda_s \). They are coupled via the source terms and can be combined into a homogeneous second-order differential equation, e.g. by eliminating \( u' \)

\[
\lambda_w^2 \frac{\partial^2 h'}{\partial x^2} + \lambda_w \left( \frac{\lambda_w}{\lambda_s} - \frac{b-3}{2} \right) \frac{\partial h'}{\partial x} + \frac{\lambda_w}{\lambda_s} + \frac{M}{2} = 0 \tag{3-141}
\]

Elimination of \( h' \) instead of \( u' \) yields an identical equation for \( u' \). This combined equation no longer describes solely a mere relaxation. For certain values of the coefficients it represents a damped undulation, which implies that overdeepening occurs when the bed adapts to changes of conditions along the channel (de Vriend and Struikema, 1983). In fact, the equation is similar to the equation for damped mechanical vibrations, but with \( x \) instead of \( t \) as the independent variable.

The water depth perturbation, \( h' \), is again assumed harmonic in longitudinal direction. Insertion of Eq. (3-136) into the second-order differential equation yields a second-degree characteristic polynomial

\[
(k \lambda_w)^2 - i(k \lambda_w) \left( \frac{\lambda_w}{\lambda_s} - \frac{b-3}{2} \right) \frac{\lambda_w}{\lambda_s} - \frac{M}{2} = 0 \tag{3-142}
\]

The two roots of this equation, \( (k \lambda_w)_1 \) and \( (k \lambda_w)_2 \), are complex. They satisfy the symmetry relations

\[
\text{Re}(k \lambda_w)_1 + \text{Re}(k \lambda_w)_2 = 0 \tag{3-143}
\]

\[
\text{Im}(k \lambda_w)_1 + \text{Im}(k \lambda_w)_2 = \frac{b-3}{2} \frac{\lambda_w}{\lambda_s} \tag{3-144}
\]

In Figures 3.6 to 3.8, the roots are depicted as a function of the 'interaction parameter', i.e. the ratio \( \lambda_s/\lambda_w \), and the erosion coefficient, \( M \). For reference, the solutions for \( M = 0 \) are represented by dashed lines in each figure. The left-hand graphs show \( \text{Re}(k \lambda_w) \), corresponding to a periodic behaviour of the solution, whereas the right-hand ones show \( \text{Im}(k \lambda_w) \), corresponding to an exponential (growing or damping) behaviour. Because \( \text{Re}(k \lambda_w)_1 = - \text{Re}(k \lambda_w)_2 \), only positive values of \( \text{Re}(k \lambda_w) \) have been drawn.
Fig. 3.6 Real part (left) and imaginary part (right) of the relative wavenumber, $k\lambda_w$, as function of the interaction parameter, $\lambda_0/\lambda_w$, for $M = 0.1$ and $b = 5$. Dashed lines correspond to $M = 0$.

Fig. 3.7 Real part (left) and imaginary part (right) of the relative wavenumber, $k\lambda_w$, as function of the interaction parameter, $\lambda_0/\lambda_w$, for $M = 1$ and $b = 5$. Dashed lines correspond to $M = 0$. 
Fig. 3.8  Real part (left) and imaginary part (right) of the relative wavenumber, $k\lambda_W$, as function of the interaction parameter, $\lambda_B/\lambda_W$, for $M = 10$ and $b = 5$. Dashed lines correspond to $M = 0$.

It appears that bank erosion only has an appreciable effect at rather large values of bank erodibility, and mainly at large values of the interaction parameter. Meandering rivers, however, are characterized by relatively small values of the interaction parameter (cf. Struikisma and Klaassen, 1988). Apparently, the input of bank erosion products does not need to be accounted for when determining the wavelengths and damping lengths of flow and bed topography in natural rivers with moderately migrating banks.

The above linear analysis of the two-dimensional model is thought to describe the initiation of meandering. When the process continues, some assumptions no longer hold, as the channel becomes curved and progressively wider, cf. Fig. 3.9. The influence of curvature is treated in the next section. Here some remarks are made on the influence of channel widening.

Fig. 3.9  Initiation of meandering causes a straight river to become curved and wider.
Channel widening is considered to continue until the critical shear stresses of the banks are no longer exceeded or until erosion of one bank is balanced by accretion of the opposite bank. Consequently, the equilibrium river width depends strongly on bank erodibility. The more erodible the banks are, the wider and shallower the cross-sections will be. Figures 3.6 to 3.8 offer an opportunity to explain qualitatively the effect of widening on the wavenumbers from the linear analysis. From the definitions of $\lambda_s$ and $\lambda_w$, it follows that the interaction parameter, $\lambda_s/\lambda_w$, is given by

$$\frac{\lambda_s}{\lambda_w} = \frac{2}{\pi^2} \frac{g}{C^2} \frac{b_o}{h_o} \frac{\theta_o^2}{f(\theta_o)} \tag{3-145}$$

It shows that widening increases the interaction parameter, leading to longer wave lengths (smaller $\text{Re}(k\lambda_w)$) and less damping (smaller $\text{Im}(k\lambda_w)$). Ultimately, the interaction parameter might reach values beyond the periodic range, which is thought to correspond to braided rivers (Struijsma and Klaassen, 1988). This complies with laboratory observations of Friedkin (1945), who concludes that a braided channel results when the banks are extremely easily eroded.
3.5 Curved channels

For simplicity only straight channels have been considered until now, but most natural rivers are curved. Crosato (1987) presents a linear analysis of a two-dimensional model in which channel curvature is accounted for. She arrives at an inhomogeneous second-order differential equation

\[ \lambda_w^2 \frac{\partial^2 h'}{\partial s^2} + \lambda_w \left( \lambda_w - \frac{b-3}{2} \right) \frac{\partial h'}{\partial s} + \frac{\lambda_w}{\lambda_s} h' = 
\]

\[ = A f(\theta_c) \frac{B_o}{\pi} \lambda_w \frac{h_o}{R_c} + f\left(\frac{1}{R_c}\right), \frac{\partial^2 (1/R_c)}{\partial s^2} \]  (3.146)

where \( R_c \) is the radius of curvature of the channel centre-line and \( s \) is the longitudinal coordinate in a curvilinear coordinate system in which the \( s \)-axis coincides with the centre-line.

Fig. 3.10 Curvilinear coordinate system.

The left-hand terms represent a damped undulation, similar to the one arrived at in the previous section. The right-hand terms represent a source, expressing that channel sinuosity acts as a forcing function for the deformation of the bed. Under appropriate conditions this forcing function may cause large amplifications or even resonance, which was first recognized by Blondeaux and Seminara (1985). The cross-sectional flow and bed deformation may induce bank erosion and thereby increase channel curvature, i.e. increase the forcing function for cross-sectional deformation as long as sinuosity is not too large. These mechanisms are considered to form the essence of meandering.
Comparison with the homogeneous differential equation in the previous section, i.e. Eq.(3-141), indicates that the results from the two linear analyses of two-dimensional models can be combined into a general equation for river-bed deformation, in which both bank erosion products and channel curvature are accounted for.

\[
\lambda_w \frac{2}{\partial s^2} + \lambda_s \left( \frac{\lambda_w}{\lambda_s} \frac{b-3}{2} \frac{\partial h'}{\partial s} + \frac{\lambda_w}{\lambda_s} \frac{M}{2} h' \right) = \\
= A f(\theta_o) \frac{B_o \lambda_w h_o}{\pi \lambda_s R_c} + f(1/R_c) \frac{\partial^2 (1/R_c)}{\partial s^2}
\]

(3-147)

where \(M\) must be reduced if a constant width is assumed, because only the net result of bank erosion and bank accretion contributes to the sediment balance.

The influence of bank erosion on transverse bed slopes is clarified by considering the axi-symmetric solution, which describes the situation in an infinitely long bend with constant curvature. For a mildly curved bend it is fair to assume that curvature is not influenced by bank migration. As the axi-symmetric solution corresponds to the fully developed situation, all variables depend on local, cross-sectional conditions only and hence all derivatives in longitudinal direction vanish. The differential equation reduces to

\[
\pi \left( 1 + \frac{M \lambda_s}{2 \lambda_w} \right) h' = A f(\theta_o) \frac{h_o}{R_c}
\]

(3-148)

It follows that the input of bank erosion products decreases transverse bed slopes, which complies with the common observation that pool depths increase markedly at resistant or stabilized banks. This is of considerable practical importance as it implies that stopping the input of bank erosion products by bank stabilization increases transverse bed slopes and near-bank water depths, thus possibly destabilizing the bank.

A slightly different approach to reveal the influence of bank erosion on the axi-symmetric solution is suggested by Struiksma (1988). The bank erosion is kept independent from the other variables, which implies that it is thought to arise from external causes, such as ship waves or groundwater seepage. Then the linearized flow and sediment equations for a straight channel read

\[
\frac{\partial u'}{\partial x} + \frac{1}{\lambda_w} u' = \frac{1}{2\lambda_w} \frac{u_o}{h_o} h'
\]

(3-149)

\[
\frac{\partial h'}{\partial x} + \frac{1}{\lambda_s} h' = \frac{h_o (b-1)}{u_o} \frac{\partial u'}{\partial x} - \frac{h_o}{B_o s_{x_o}} \frac{\partial n}{\partial x}
\]

(3-150)
Combination and extension to curved channels yields

\[
\lambda_w \frac{\partial^2 h'}{\partial s^2} + \lambda_w \left( \lambda_s - \frac{b - 3}{2} \right) \frac{\partial h'}{\partial s} + \lambda_w h' =
\]

\[
- A f(\theta_o) \frac{B_o}{\lambda_s R_c} \frac{h_o}{h_o} + \frac{\partial (1/R_c)}{\partial s}, \frac{\partial^2 (1/R_c)}{\partial s^2} + \frac{\partial n}{\partial t} \frac{B_o}{s_{xo}} \frac{h_o}{h_o} \frac{\partial^2 n}{\partial s \partial t}
\]

This gives for the axi-symmetric case

\[
h' = A f(\theta_o) \frac{B_o}{\lambda_s R_c} \frac{h_o}{h_o} - \lambda_w \frac{h_o}{s_{xo}} \frac{\partial n}{\partial t}
\]

or, using the definition of \( \lambda_s \),

\[
h' = f(\theta_o) \left( A \frac{h_o}{B_o} - \frac{1}{\lambda_s} \frac{h_o}{h_o} \frac{\partial n}{\partial t} \right)
\]

Again it follows that the input of bank erosion products reduces transverse bed slopes, but the relation obtained here differs from the one for discharge-induced bank erosion.

In the derivations it has been assumed that bank erosion products are distributed evenly over a cross-section. In reality their distribution will only extend over a part of the cross-section, so the actual reduction of transverse bed slopes can be expected to be stronger in regions near eroding banks. As those are the regions where the largest transverse slopes are found, it seems reasonable to expect that the overall effect will also be a stronger reduction than predicted by the above analyses.

The uncertainties in the transverse distribution of bank erosion products stress the need of applying a fully two-dimensional model when assessing the influence of bank erosion on river bed topography.

Thorne and Osman (1988a) also give an explanation for the reduction of transverse bed slopes due to bank erodibility. They argue that the near-bank water depth in the outer bend is limited by a critical bank height for mass instability, and that banks migrate as long as the flow shear stresses exceed the bank material's critical shear stress for entrainment. In fact this means they describe a process of widening and associated shallowing instead of the influence of bank erosion products on transverse bed slopes. Besides, the concept of a limiting critical bank height is questionable, as this critical height may become larger when bank slopes become less steep after failure.
One-dimensional and two-dimensional models for river morphology are presented in which bank erosion is taken into account. The banks are assumed to consist of a fraction of cohesive material, which becomes washload after being eroded, and a fraction of granular material, with the same properties as the material of the bed. Bank erosion is taken to be caused by discharge flow. The basic processes are lateral entrainment of lower parts of the bank and near-bank bed degradation, both inducing mass failure of upper parts of the bank.

From an analysis of characteristics it is found that bank disturbances are non-propagating. This is essentially different from the behaviour of bed disturbances and can be explained by the fact that erosion of cohesive banks is taken to depend on flow velocities rather than on flow velocity gradients. An important implication is that river widths cannot be stabilized by protecting certain carefully chosen bank sections only, unless these sections are so closely interspaced that other effects, not included in the present model, become significant. An example of the latter is the use of groynes for bank protection. It should be noted that this conclusion does not imply that river migration cannot be stopped by a discontinuous bank protection. Furthermore, it is found that bank disturbances do not influence the propagation of flow and bed disturbances, which implies that computations of river planimetry can be decoupled from the computations of flow and bed topography.

A linear analysis of the one-dimensional model is used to clarify the interactions between bank and bed disturbances. It shows that short bank disturbances generate two independent waves, a propagating bed wave, and a steady wave at the location of the bank disturbance. This is in accordance with the results from the analysis of characteristics. The interactions become manifest at larger length scales. Bank erosion is found to provide additional diffusion to the system, as well as to cause an upstream propagation or extension of disturbances.

A linear analysis of the two-dimensional model shows that bank erosion only has an appreciable effect on the wave lengths and damping lengths of bed topography at rather large values of bank erodibility, and mainly at large values of the interaction parameter. Meandering rivers, however, are characterized by relatively small values of the interaction parameter. Apparently, the input of bank erosion products can be neglected when determining the characteristic length scales of moderately migrating meanders.

Bank erosion may cause widening of a river, which leads to longer wave lengths and less damping. Ultimately, when the banks are extremely easily eroded, the river may become braided.
The input of bank erosion products is found to decrease transverse bed slopes in curved channels, which complies with the commonly observed phenomenon of deeper pool depths near resistant or stabilized banks. This is of considerable practical importance as it implies that stopping the input of bank erosion products by bank stabilization increases transverse bed slopes and near-bank water depths, thus possibly even destabilizing the bank.

In the analyses of both one-dimensional and two-dimensional models it is assumed that bank erosion products are distributed evenly over a cross-section. The assumption is not very realistic and can be considered to be a major shortcoming of the present theoretical investigation. This stresses the need of applying a fully two-dimensional model when assessing the influence of bank erosion on river morphology.
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<th>Symbol</th>
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<td>A</td>
<td>secondary flow coefficient</td>
<td>2-7</td>
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<tr>
<td>A</td>
<td>coefficient matrix</td>
<td>3-43</td>
</tr>
<tr>
<td>B</td>
<td>depth-averaged river width</td>
<td>2-8</td>
</tr>
<tr>
<td>B_w</td>
<td>river width at the bed</td>
<td>2-10</td>
</tr>
<tr>
<td>B_w</td>
<td>river width at the water level</td>
<td>2-11</td>
</tr>
<tr>
<td>C</td>
<td>Chézy coefficient for hydraulic roughness</td>
<td>3-29</td>
</tr>
<tr>
<td>D_e</td>
<td>effective diffusion</td>
<td>3-72</td>
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<tr>
<td>D_0</td>
<td>zero-order diffusion coefficient</td>
<td>3-74</td>
</tr>
<tr>
<td>E</td>
<td>erodibility coefficient</td>
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<td>Fr</td>
<td>Froude number</td>
<td>3-44</td>
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<tr>
<td>H</td>
<td>total bank height</td>
<td>2-4</td>
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<tr>
<td>M</td>
<td>bank erosion coefficient</td>
<td>3-132</td>
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<tr>
<td>P</td>
<td>Péclet number</td>
<td>3-75</td>
</tr>
<tr>
<td>Q</td>
<td>discharge</td>
<td>3-53</td>
</tr>
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<td>R_c</td>
<td>radius of curvature of channel centreline</td>
<td>3-146</td>
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<tr>
<td>R_s</td>
<td>local radius of streamline curvature for fully developed secondary flow</td>
<td>3-5</td>
</tr>
<tr>
<td>R**</td>
<td>effective local radius of streamline curvature</td>
<td>2-7</td>
</tr>
<tr>
<td>V</td>
<td>volume of bank erosion products per unit length</td>
<td>2-4</td>
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<th>Symbol</th>
<th>Description</th>
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<td>b</td>
<td>exponent in sediment transport formula</td>
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<td>c_e</td>
<td>effective celerity of bed disturbances</td>
<td>3-72</td>
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<tr>
<td>c_o</td>
<td>zero-order celerity of bed disturbances</td>
<td>3-64</td>
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<tr>
<td>e</td>
<td>factor accounting for the influence of a longitudinal bed slope on sediment transport</td>
<td>3-10</td>
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<td>f(δ)</td>
<td>function weighing the influence of a transverse bed slope on sediment transport</td>
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<tr>
<td>g</td>
<td>acceleration due to gravity</td>
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<td>h</td>
<td>water depth</td>
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<td>i</td>
<td>imaginary unit</td>
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<td>k</td>
<td>complex wavenumber</td>
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<tr>
<td>m</td>
<td>coefficient in sediment transport formula</td>
<td>3-10</td>
</tr>
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<td>n</td>
<td>distance of bank retreat due to lateral fluvial entrainment</td>
<td>2-1</td>
</tr>
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<td>r</td>
<td>complex frequency</td>
<td>3-69</td>
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<td>coordinate in longitudinal direction in a curvilinear coordinate system (curved channels)</td>
<td>3-146</td>
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<td>transverse transport of bank erosion products</td>
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<td>s_x</td>
<td>sediment transport per unit width in x direction</td>
<td>2-6</td>
</tr>
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<td>s_y</td>
<td>sediment transport per unit width in y direction</td>
<td>2-6</td>
</tr>
<tr>
<td>t</td>
<td>time</td>
<td>2-6</td>
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<td>u</td>
<td>depth-averaged flow velocity in x direction</td>
<td>2-7</td>
</tr>
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<td>u_c</td>
<td>critical flow velocity</td>
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<tr>
<td>u_nb</td>
<td>near-bank flow velocity in x direction</td>
<td>3-97</td>
</tr>
<tr>
<td>v</td>
<td>depth-averaged flow velocity in y direction</td>
<td>2-7</td>
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<tr>
<td>v_nb</td>
<td>near-bank flow velocity in y direction</td>
<td>3-97</td>
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<td>x</td>
<td>coordinate in longitudinal direction in a rectangular coordinate system (straight channels)</td>
<td>2-6</td>
</tr>
<tr>
<td>y</td>
<td>coordinate in transverse direction in a rectangular coordinate system (straight channels)</td>
<td>2-6</td>
</tr>
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<td>z_b</td>
<td>bed level</td>
<td>2-5</td>
</tr>
<tr>
<td>z_w</td>
<td>water level</td>
<td>3-81</td>
</tr>
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</table>
Sub- and superscripts

\( u_0 \) zero-order solution

\( u_x \) first-order perturbation, varying with x only

\( u_{xy} \) first-order perturbation, varying with both x and y

\( \delta \) amplitude of first-order perturbation