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Generation of Time-Limited Signals in the Nonlinear Fourier Domain via b-Modulation

Sander Wahls

Delft Center for Systems and Control, TU Delft, Delft, The Netherlands. s.wahls@tudelft.nl

Abstract *Current modulation techniques for the nonlinear Fourier spectrum do not offer explicit control over the pulse duration in the time domain. To address this issue, it is proposed to modulate the b-coefficient instead of the reflection coefficient.*

Introduction

The normalized nonlinear Schrödinger equation

$$j\partial_x q + \partial_t^2 q + 2q^2 q^* = 0 \tag{1}$$

describes the propagation of a complex envelope $q(x,t)$ in an ideal focusing single-mode fiber without losses or noise[Fn. 1]. Here, x denotes the location in the fiber, t is retarded time, j is the imaginary number and the * denotes complex conjugation. The propagation of a fiber input $q(0,t)$, which is complicated in the time domain, becomes trivial when a suitable nonlinear Fourier transform (NFT) is applied. The nonlinear Fourier spectrum evolves in a amazingly simple way that is reminiscent of propagation in linear channels when described in the conventional frequency domain. There is currently much interest in utilizing the NFT for fiber-optic information transmission, and the principal feasibility of the concept has by now been demonstrated in many experiments. The reader is referred to[1] for a recent survey.

One of the open problems when data is modulated in the nonlinear Fourier domain (NFD) is that the effective duration of the corresponding time domain pulse, which has to fit into the processing window of the transmitter, is difficult to control[2,3]. In order to address this problem in a principled way, it has been proposed in[4] to use the NFT for periodic signals instead of the usual NFT for vanishing signals. A practical demonstration of this concept has been presented in[5] based on a special family of pulses. However, no general way to generate signals with a fixed desired period is known at the moment. The recent proposal[6] also does not offer exact control over the pulse duration. In this paper, we propose another approach to modulation in the NFD that allows us to control the duration of the fiber input exactly.

The Nonlinear Fourier Transform

The NFT of a time domain signal $q(t)$ is now defined following[7]. First, the scattering problem

$$\partial_t\phi_1 = -j\xi\phi_1 + q\phi_2, \quad \partial_t\phi_2 = -q^*\phi_1 + j\xi\phi_2, \tag{2}$$

$$\lim_{t\to-\infty} e^{j\xi t}\phi_1(t,\xi) = 1, \quad \lim_{t\to-\infty} \phi_2(t,\xi) = 0 \tag{3}$$

is introduced, where $\xi \in \mathbb{R}$. Second, one defines $a(\xi) := \lim_{t\to\infty} e^{j\xi t}\phi_1(t,\xi)$ and $b(\xi) := \lim_{t\to\infty} e^{-j\xi t}\phi_2(t,\xi)$. These functions satisfy

$$|a(\xi)|^2 + |b(\xi)|^2 = 1 \quad \forall\xi\in\mathbb{R}. \tag{4}$$

Under suitable decay assumptions on $q(t)$, $a(\xi)$ and $b(\xi)$ extend into functions $a(\lambda)$ and $b(\lambda)$ that are analytic in the upper half plane, $\Im\lambda > 0$. The NFT of $q(t)$ consists of the reflection coefficient $r(\xi) = b(\xi)/a(\xi)$, $\xi\in\mathbb{R}$, and the discrete spectrum (λ_k,ρ_k), $k = 1,\ldots,K$. Here, the bound states λ_k are given by the roots $a(\lambda_k) = 0$ of $a(\lambda)$ in the upper half plane $\Im\lambda > 0$. Assuming that all zeros of $a(\lambda)$ in $\Im\lambda > 0$ are simple, the residues are given by $\rho_k := b(\lambda_k)/\frac{da}{d\lambda}(\lambda_k)$. It may happen that there are no bound states at all, $K = 0$.

In this paper, we are concerned with time-limited signals, i.e., signals that satisfy

$$q(t) = 0 \quad \forall t \notin \left[-\frac{T}{2},\frac{T}{2}\right], \quad 0 < T < \infty. \tag{5}$$

A known necessary condition for (5) is that the inverse Fourier transform of $b(\xi)$ satisfies[8–10,Fn. 2]

$$B(\tau) := \int_{-\infty}^{\infty} b(\xi)e^{j\xi\tau}\frac{d\tau}{2\pi} = 0 \quad \forall\tau\notin[-T,T]. \tag{6}$$

Since our goal is to generate time-limited signals, the question arises whether this condition is also sufficient. For the NFT that corresponds to the Korteweg-de Vries equation (instead of the nonlinear Schrödinger equation considered here),

[1] The proposed modulator also applies in the defocusing case, but in the interest of brevity we do not discuss this case.

[2] Note that the condition is given for $\phi_2(\frac{T}{2},\xi)$ instead of $b(\xi)$ in the references, which is why the interval in (6) is different.

this is known to be true in the absence of bound states[11]. The arguments in[11] also seem to apply in the case considered here[Fn. 3], but a thorough investigation still needs to be carried out. Similarly, for the discrete-time version of the NFT considered here, a condition analog to (6) is also known to be sufficient in the absence of bound states[Fn. 4]. We hence assume that (6) implies (5) in the absence of bound states also in our case, and later justify this assumption numerically.

The b-Modulator

The b-modulator embeds $2N+1$ complex symbols in the coefficient $b(\xi)$ of a time-limited signal that satisfies (5). We remark that it was recently proposed[14] to embed information in the analytic extension $b(\lambda)$ of $b(\xi)$, but the methods in[14] do not lead to time-limited signals. Our b-modulator consists of three blocks, which will now be discussed.

The Mapper: The input of the mapper is a finite sequence $s_{-N},\dots,s_N$ of complex data symbols. The task of the mapper is to embed these data symbols in the coefficient $b(\xi)$, where $\xi\in\mathbb{R}$. Aiming at a fiber input $q(t)$ that satisfies (5), the mapper generates a $b(\xi)$ that satisfies (6). This can be achieved with a series expansion of the form

$$b(\xi)=As(\xi),\quad s(\xi):=\sum_{n=-N}^{N}s_n w_n(\xi),\qquad (7)$$

where the inverse Fourier transform $W_n(\tau)$ of each carrier $w_n(\xi)$ satisfies $W_n(\tau)=0$ for $\tau\notin[-T,T]$. As an example, we consider shifted raised cosine pulses $w_n(\xi)=\phi(\frac{\xi}{2\pi}-Bn)$, where

$$\phi(f)=\frac{\cos(\pi\beta f/B)}{1-(2\beta f/B)^2}\frac{\sin(\pi f/B)}{\pi f/B},\quad \begin{array}{l}\beta\in[0,1]\\ B>0\end{array}.$$

The inverse Fourier transform $\Phi(\tau)$ of $\phi(\frac{\xi}{2\pi})$ is known to be compactly supported on the interval $[-\frac{1+\beta}{2B},\frac{1+\beta}{2B}]$, so that we need to choose $B=(1+\beta)/(2T)$. The roll-off factor β remains as a free parameter. The real constant $A>0$ in (7) is chosen such that the energy $E:=\int_{-\infty}^{\infty}|q(t)|^2dt$ of the generated fiber input is equal to a desired value E_d. To find the A that leads to $E=E_d$, a binary search based on the formula[Fn. 5]

$$E=-\frac{1}{\pi}\int_{-\infty}^{\infty}\log\left(1-A^2|s(\xi)|^2\right)d\xi \qquad (8)$$

can be performed. A trivial lower bound for starting the binary search is $A\ge 0$. For an upper bound, we use that (4) implies $|b(\xi)|\le 1$ and thus $A\le 1/\sup_\xi|s(\xi)|=:u$. We observed that E converges towards a finite value in the limit $A\to u^-$ even though signals without bound states can have arbitrarily large energies[Fn. 6]. The choice of the carrier filters $w_n(\xi)$ in (7) influences the maximum energy that the b-modulator can achieve. To achieve arbitrary energies, other carrier filters need to be investigated in the future. Also note that the receiver needs to infer A from $b(\xi)$. This can be achieved e.g. using training symbols.

Recovery of $a(\xi)$: Next, $a(\xi)$ is recovered from $b(\xi)$. In the absence of bound states, it is given by the formula[p. 50 in 8] $a(\xi)=\exp\left\{\frac{1}{2\pi j}\int_{-\infty}^{\infty}\log(1-|b(\xi')|^2)/(\xi'-\xi)d\xi'\right\}$.

Inverse Scattering: This block recovers $q(t)$ from $a(\xi)$ and $b(\xi)$, which are now both known (the discrete spectrum is empty by construction). In principle, any inverse NFT algorithm can be used for this step because the reflection coefficient $r(\xi)=b(\xi)/a(\xi)$ is now known. See, e.g.,[15] and the references therein for algorithms.

Practical Implementation

We utilize the framework in[15]. In order to be able to compute D samples of a signal $q(t)$ that satisfies (5) from a given $b(\xi)$, a polynomial $\mathcal{B}(z)=\sum_{k=0}^{D-1}\mathcal{B}_k z^{-k}$, where $z=e^{2j\xi\epsilon}$ and $\epsilon=T/D$, has to be found such that $b(\xi)\approx z^{\frac{D+1}{2}}\mathcal{B}(z)$. This is not always possible because not every $b(\xi)$ corresponds to a time-limited signal with support (5). In our case, however, it is always possible to find such a polynomial since (6) holds. We write

$$b(\xi)\overset{(6)}{=}\int_{-T}^{T}B(\tau)e^{-j\xi\tau}d\tau\approx z^{\frac{D+1}{2}}\mathcal{B}(z).\qquad (9)$$

By applying a rectangular discretization to the integral in (9), one can show that the choice $\mathcal{B}_k=2\epsilon B(\tau_k)$, where $\tau_k=-T+(2k+1)\epsilon$, satisfies (9); it is of course possible to use better discretizations. The corresponding polynomial approximation of $a(\xi)$ and finally the samples of $q(t)$ are then

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[3]Note that the transmission coefficient is given by $t=1/a$, so that the ratio r/t considered is[11] is equal to b. The Paley-Wiener theorem[p. 375 in 12] provides the presentation (6) for the entire exponential type functions considered in[11].

[4]The discrete-time (DT) analog of $b(\xi)$ for a DT signal $Q[n]$ is of the form $\beta(w)=\sum_{k=-\infty}^{\infty}\beta[k]w^k$, where $|w|=1$. Note that $\beta(w)$ is the DT Fourier transform of the DT signal $\beta[k]$. It is known (see[13]) that, in the absence of bound states, $Q[n]=0$ for $n\notin[0,N]$ if and only if $\beta[k]=0$ for $k\notin[0,N]$.

[5]Eq. 8 follows by combining the definition $r=b/a$ with (4) and the nonlinear Parseval relation[p. 38 and 54f in 8], and finally using that $\log(1+x/(1-x))=\log(1/(1-x))=-\log(1-x)$.

[6]Choose, e.g., $s(\xi)$ in (8) to be a rectangle and let $A\to u^-$. The corresponding $q(t)$ however is no longer time-limited.

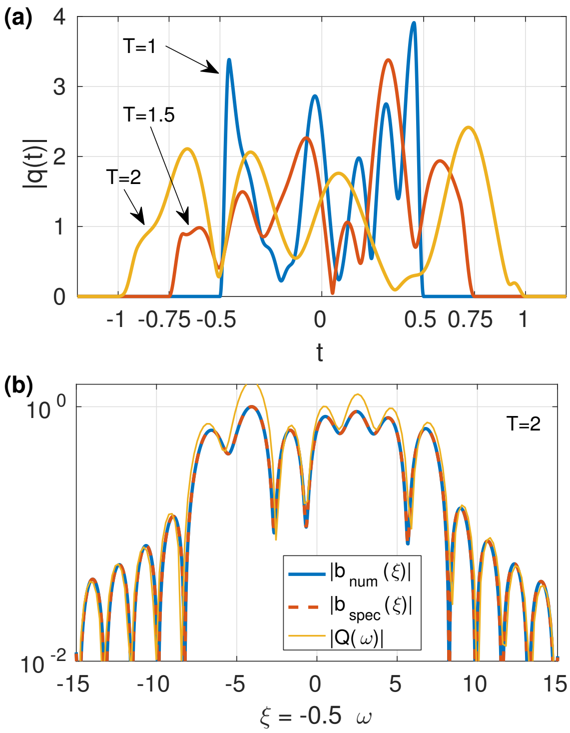

Fig. 1: Numerical example.

found as in the first algorithm in[15].

Numerical Example

We now validate the b-modulator numerically. We consider blocks of $2N+1=9$ symbols that are chosen randomly from the alphabet $\mathfrak{A}=\{1\pm i,-1\pm i\}$. These symbols modulate fiber inputs $q(t)$ that are time-limited to $[-\frac{T}{2},\frac{T}{2}]$ and have a desired energy of $E_d=3.5$. The series (7) was implemented using raised cosines as discussed earlier, with a roll-off factor of $\beta=0.05$. In order to determine the scaling factor A in (7), 20 steps of the binary search with lower bound zero and upper bound one were performed, where the integral in (8) was computed using Matlab's `integral` command. We generated 4096 samples in the interval $[-\frac{T}{2},\frac{T}{2}]$ as described earlier.

Fig. 1(a) shows the generated fiber inputs for different values of T, where new symbols were chosen in each run. Since the samples were generated inside the intervals $[-\frac{T}{2},\frac{T}{2}]$, the signals have been extended with zeros. Fig. 1(b) shows the magnitude of $b(\xi)$ for $T=2$ that has been computed numerically from the corresponding signal in Fig. 1(a) together with the specified, analytically known value. The numerically computed $b(\xi)$ matches its specification well, with an relative error $\|b_{\text{spec}}-b_{\text{num}}\|_2/\|b_{\text{spec}}\|_2$ of 0.001. Fig. 1(c) also shows the linear Fourier transform in order to illustrate that we are no longer in the linear regime, where both curves would be very close.

Conclusions

A simple modulation scheme operating in the nonlinear Fourier domain that generates time-limited signals has been proposed and validated numerically. The design of carrier filters for high-power transmission and the inclusion of bound states are open topics for future research.

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