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THE INFLUENCE OF FRICTION ON THE THEORETICAL STRENGTH OF PIN-LOADED HOLES ORTHOTROPIC PLATES

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SUMMARY

Stresses have been calculated for infinite orthotropic plates with a circular hole loaded by a perfectly fitting rigid pin. The effect of friction at the interface between pin and plate material is evaluated.

The calculations are based on the analytical method of complex stress functions. A numerical approach was used for satisfying the boundary conditions of the contact area between pin and hole.

Stress concentration factors based on the nominal bearing stress are presented graphically for five laminates of carbon fibre reinforced plastic. A quadratic failure criterion was used to predict the bearing stress at which first significant damage occurs.

The results indicate that the presence of friction has a significant influence on the stress distribution around a pin-loaded hole.
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1. INTRODUCTION

In the analysis of mechanically fastened joints in orthotropic materials the prediction of the stress distribution around the fastener hole is of fundamental interest for the prediction of strength. Most published work on stress distributions around pin-loaded holes in anisotropic materials is related to problems in which the hole is loaded frictionless by the pin. From Reference [1] it is known that the radial pressure distribution between pin and hole and the tangential stresses are affected considerably by the presence of friction. Since strength predictions based on frictionless theoretical analysis are generally conservative compared to experimental results the role of friction cannot be neglected. Although the stresses in the net-section may not be affected strongly by the presence of friction in the contact area between pin and hole it is obvious that the prediction of the bearing strength of composites requires an accurate description of both the radial, tangential and shear (friction) stresses along the edge of the hole.

Another important effect is the friction between the plate surfaces resulting from the clamping force of the fastener. This friction will relieve the load transmitted by shear in the fastener. In Reference [2] is shown that joint strength depends strongly on the clamping force; in certain cases the strength can be increased by a factor 2 to 3 by a proper tightening of the bolt. Composite laminates however may behave visco-elastically in the direction perpendicular to the laminate plane, i.e. in the direction of the clamping force. The possible visco-elasticity will relieve the clamping force and it is therefore questionable whether strength predictions may rely on it. In this report the effect of clamping will not be considered.

Reference [3] deals with stress-distributions around pin-loaded holes in orthotropic plates with arbitrary load direction. Pin friction is not considered. However the applied method of analysis can simply be modified in order to include friction in the calculations, as will be illustrated in this report. The intention is to present numerical results for some carbon fibre reinforced plastic (C.F.R.P.) laminates.

These results may be useful as a theoretical background for experimental work on the subject of mechanical fasteners in composites. The method of Reference [3] is restricted to infinite plates with a single, pin-loaded hole. For a parametric study on the influence of the coefficient of friction the restriction is unimportant, provided that the results are not used for plates with rows of narrowly spaced pin-loaded holes or for finite width plates with too small width over hole diameter ratio's. These plates show failure modes completely different from plates with large spacing or large width.

In this report extensive calculations will not be given. The method of analysis is worked out in detail in Reference [3]. A resume of the theory is presented in Chapter 2 in order to make this report self-contained. In Chapter 3 a rather extended description of the contact problem between pin and hole is given. In Chapter 4 it is shown that the theory of Reference [3] is not modified fundamentally by the presence of friction.

Numerical results are presented for a number of C.F.R.P. laminates. The engineering elastic constants and strength values are listed in Table 1. The calculations have been made for coefficients of friction 0, 0.2 and 0.4. For the $(90^\circ/\pm45^\circ)_g$-laminate a wider range of coefficients has been investigated, including the cases of no-slip over the entire contact region and of no-slip over a part of that region. Since in the method of Reference [3] the load direction is arbitrary the laminate with the highest directional sensitivity,
the \((90^\circ/\pm 45^\circ)\)_s-laminate, is investigated for three angles between the load direction and the principal material axes of the laminate.

From the results the general conclusion can be drawn that the value of the coefficient of friction is of paramount influence on the stress-distribution around a pin-loaded hole. However, the ultimate bearing strength changes substantially only when the type of first significant damage of the laminate is changed by the present of friction or when the coefficient of friction has a high value.
2. SOLUTION OF THE FRICTIONLESS CONTACT PROBLEM

A schematic presentation of the stress problem round a pin-loaded hole in an infinite orthotropic plate as treated in Reference [3] is given in Figure 1. The pin is assumed to be infinitely rigid. The expression for the radial load distribution on the edge of the hole has to fulfill the next requirements:

\[
P_r = p_0 \sum_{n=1,2}^\infty a_n \sin n\theta \quad \text{for} \quad 0 \leq \theta \leq \pi
\]
\[
P_r = 0 \quad \text{for} \quad \pi < \theta < 2\pi
\]  

(2.1.)

This expression can be found by multiplying a sine series

\[
p_0 \sum_{n=1,2}^\infty a_n \sin n\theta
\]

continuous on the whole contour by a step function

\[
\frac{1}{2} + \frac{2}{\pi} \sum_{m=1,3}^\infty \frac{\sin m\theta}{m} = \begin{cases} 
1 & \text{for} \quad 0 < \theta < \pi \\
0 & \text{for} \quad \pi < \theta < 2\pi
\end{cases}
\]  

(2.2.)

resulting in

\[
P_r = p_0 \left[ \frac{1}{2} \sum_{n=1,2}^\infty a_n \sin n\theta + \frac{1}{\pi} \left\{ \sum_{n=1,3}^\infty \frac{a_n}{n} + \sum_{m,n=2,4}^\infty \frac{a_n}{n^m} \left( \frac{1}{n-m} + \frac{1}{n+m} \right) \cos m\theta \right\} \right]
\]  

(2.3.)

in which

\[
\sum = \sum_{m,n=1,3}^\infty + \sum_{m=2,4}^\infty \sum_{n=2,4}^\infty
\]

Expression (2.3.) is continuous on the entire contour of the hole. The terms with odd values of \( n \) represent the symmetric part of the load with respect to the \( Y \)-axis, the terms with even \( n \) the asymmetric part.

The solution of Reference [3] is based on the method of complex stress functions of Lekhnitskii, Reference [4]. The general expressions for these functions in the case of a pin-loaded hole in an infinite orthotropic plate with zero stresses at infinity are

\[
\phi_k(z_k) = A_k \ln \zeta_k + \phi_k^0(z_k) \quad k = 1,2
\]  

(2.4.)

in which
\[ \zeta_k = \frac{z_k + \sqrt{z_k^2 - s_k\varphi - 1}}{1 - is_k\varphi} \] (2.5.)

\[ z_k = x + s_k\varphi y \] (2.6.)

\[ s_k\varphi = \frac{s_k \cos \varphi - \sin \varphi}{s_k \sin \varphi + \cos \varphi} \] (2.7.)

\( \varphi \) is the angle between the material symmetry axes and the coordinate axes. According to Reference [4] in Figure 1 it has a negative value. \( s_k \) can be solved from

\[ s_1^2 s_2^2 = S_{22}/S_{11} \] (2.8.)

\[ s_1^2 + s_2^2 = -(2S_{12} + S_{66})/S_{11} \] (2.9.)

where \( S_{ij} \) are the material compliances in the principal material directions.

In (2.4.) the functions \( A_k \ln \zeta_k \) are multi-valued. Their presence is required since the resultant forces on the edge of the hole are not zero. According to Reference [3] is

\[ A_k = \frac{1}{2\pi i(s_k\varphi - s_k\varphi)} \left\{ \frac{1}{2} (R_x + s_k\varphi R_y) + \frac{(S_{12} - S_{11} s_k s_\varphi) S_{11}(s_k\varphi - s_k\varphi)(\bar{s}_k\varphi - s_k\varphi)(R_x + s_k\varphi R_y)}{4S_{11} s_k^2 (s_k + s_\varphi)^2} \right\} \] (2.10.)

\[ k = 1, 2 \quad \lambda = 3 - k \]

where \( R_x \) and \( R_y \) are the components of the resultant force in \( X \)- and \( Y \)-direction respectively. With

\[ X = \frac{p_x \cos \theta = p_o \cos \theta}{\frac{1}{2} \sum_{n=1,2} a_n \sin n\theta} + \frac{1}{\pi} \left\{ \sum_{n=1,3} a_n \left( \frac{1}{n + m} + \frac{1}{n + m} \right) \cos m\theta \right\} \] (2.11.)

\[ Y = \frac{p_x \sin \theta = p_o \sin \theta}{\frac{1}{2} \sum_{n=1,2} a_n \sin n\theta} + \frac{1}{\pi} \left\{ \sum_{n=1,3} a_n \left( \frac{1}{n - m} + \frac{1}{n + m} \right) \cos m\theta \right\} \] (2.12.)

the components become

\[ R_x = \int_0^{2\pi} X \, ds = \int_0^{2\pi} p_x \cos \theta \, d\theta = 2p_o \sum_{n=2,4} a_n \frac{n\pi}{n^2 - 1} \] (2.13.)
\[ R_y = \int_0^{2\pi} Y \, ds = \int_0^{2\pi} P_r \sin \theta d\theta = p_0 \frac{\pi}{2} a_1 \] (2.14)

or, with \( a_1 \) chosen unity

\[ R_y = p_0 \frac{\pi}{2} \]

\( \phi_k^O(z_k) \) in (2.4.) are functions, holomorphic outside and on the edge of the hole. They can be solved from the boundary condition formulae for the loads on the edge of the hole

\[ 2 \text{Re} \sum_{k=1,2} \left\{ A_k \ln \zeta_k + \phi_k^O(z_k) \right\} = \int_0^S Y \, ds \]

\[ 2 \text{Re} \sum_{k=1,2} \left\{ s_{k\theta} A_k \ln \zeta_k + s_{k\theta} \phi_k^O(z_k) \right\} = -\int_0^S X \, ds \]

or

\[ \sum_{k=1,2} \left\{ \phi_k^O(z_k) + \phi_k^O(z_k) \right\} = \int_0^S Y \, ds - 2 \text{Re} \sum_{k=1,2} \left\{ A_k \ln \zeta_k \right\} = f_2 \] (2.15)

\[ \sum_{k=1,2} \left\{ s_{k\theta} \phi_k^O(z_k) + s_{k\theta} \phi_k^O(z_k) \right\} = -\int_0^S X \, ds - \]

\[ 2 \text{Re} \sum_{k=1,2} \left\{ s_{k\theta} A_k \ln \zeta_k \right\} = f_1 \] (2.16)

In (2.15.) and (2.16.) the integrals contain multi-valued parts eliminating the multi-valued expressions \(-2 \text{Re} \sum_{k=1,2} \left\{ A_k \ln \zeta_k \right\} \) and \(-2 \text{Re} \sum_{k=1,2} \left\{ s_{k\theta} A_k \ln \zeta_k \right\} \) respectively. So \( f_1 \) and \( f_2 \) are single-valued; they can be evaluated from (2.11.) and (2.12.). It is noted that all integration constants, not relevant for the solution of the elasticity problem, have been omitted.

The functions \( \phi_k^O(z_k) \) may be represented by power series of \( z_k \) with negative powers only. These series are replaced by series of \( \zeta_k \) for which on the edge of the hole

\[ \zeta_1 = \zeta_2 = \sigma = \cos \theta + i \sin \theta \]

As a result (2.15.) and (2.16.) contain only one variable \( \sigma \) on the edge of the hole. (2.15.) and (2.16.) then reduce to
\[(s_{k\phi} - s_{k\phi}) \phi_k^0(\sigma) + (s_{k\phi} - s_{k\phi}) \phi_k^0(\sigma) + (s_{k\phi} - s_{k\phi}) \phi_k^0(\sigma) = s_{k\phi} \frac{f_2 - f_1}{2} \quad k = 1, 2 \quad \ell = 3-k \quad (2.17)\]

\(\phi_k^0(\sigma)\), continuous on the edge of the hole, are the boundary values of functions \(\phi_k(\zeta_k)\), holomorphic outside and on the edge of the hole, while \(\phi_k^0(\infty) = 0\).

Applying Cauchy's integral for the infinite region

\[
\frac{1}{2\pi i} \oint \frac{\phi_k^0(\sigma)}{\sigma - \zeta_k} = \phi_k^0(\zeta_k)
\]

\[
\frac{1}{2\pi i} \oint \frac{\phi_k^0(\sigma)}{\sigma - \zeta_k} = 0
\]

to expression (2.17.) results in

\[
\phi_k^0(\zeta_k) = \frac{p_0}{2\pi i(s_{k\phi} - s_{k\phi})} \left\{ \sum_{m,n} \frac{2n (m^2 - n^2 + 1 + 2i s_{k\phi} m)}{N_{m,n}} \zeta_k^{-m} - \frac{\pi i}{4} \sum_{n=0,1,2} a_n \left[ (1 + is_{k\phi}) + a_{n+2} (1 - is_{k\phi}) \right] \zeta_k^{-n-1} \right\} \quad (2.18)
\]

where

\[
\Sigma = \sum_{m,n} \Sigma_{m=1,3} \Sigma_{n=1,3} \Sigma_{m=2,4} \Sigma_{n=2,4}
\]

\[
a_0 = 0
\]

\[
N_{m,n} = m(m+1)^2 - n^2 \left\{ (m-1)^2 - n^2 \right\} \quad (2.20)
\]

Except for the coefficients \(a_n\) the complex stress functions \(\phi_k(\zeta_k)\) are known.

In Reference [3] these coefficients are solved from a displacement boundary condition for a number of points on the contact area between pin and hole.

For an arbitrary point of this area the displacement consists of two parts:

- a part equal to the displacement of the pin as a rigid body. In Reference [3] the components of this displacement are made zero in X-direction and \(v_1\) in Y-direction. \(v_1\) is the displacement of the point corresponding to \(\theta = 90^\circ\).
- a displacement relative to the pin with components \(u_R\) and \(v_R\). For the contact area the radial displacement of the plate material with respect to the pin must be zero. For small deformations this requirement results in

\[
\frac{u_R}{v_R} = -tg \theta
\]
or, with \( u = u_r \) and \( v = v_i + v_r \).

\[
u \cos \theta + (v - v_i) \sin \theta = 0 \quad \text{(2.21.)}
\]

(2.21.) is the displacement boundary condition which is imposed on the loaded part of the edge of the hole. Expressions for \( u \), \( v \) and \( v_i \) can be found by substituting \( \phi_k(z_k) \) into the displacement formulae:

\[
u = 2 \Re \sum_{k=1,2} \{ u_{k\varphi} \phi_k(z_k) \} \quad \text{(2.22.)}
\]

\[
v = 2 \Re \sum_{k=1,2} \{ v_{k\varphi} \phi_k(z_k) \} \quad \text{(2.23.)}
\]

in which

\[
u_k = S_{11\varphi} s_{k\varphi}^2 + S_{12\varphi} - S_{16\varphi} s_{k\varphi}
\]

\[
v_k = S_{12\varphi} s_{k\varphi} + S_{22\varphi}/s_{k\varphi} - S_{26\varphi}
\]

\( S_{ij\varphi} \) are the material compliances in the coordinate directions. Formulae (2.22.) and (2.23.) substituted into condition (2.21.) yield

\[
a_1 \theta + \sum_{n=2,3} a_n \sin n\theta = 0 \quad \text{(2.26.)}
\]

in which \( a_1 \theta \) and \( a_n \theta \) are known coefficients. They are worked out in detail in Reference [3] where (2.21.) is imposed to 22 points of the contact area. The 22 values of \( \theta \), defining those points give in (2.26.) 22 equations from which 22 unknown coefficients \( a_n \) may be solved by inversion of the matrix \([a_n\theta]\) in

\[
[a_n\theta] \{ a_n \} = \{-a_1 \theta\}
\]

With the coefficients \( a_n \) solved the complex stress functions \( \phi_k(z_k) \) are completely known, as well as the radial edge stress given by

\[
\sigma_r = -p_o \sum_{n=1,2} a_n \sin n\theta \quad \text{(2.28.)}
\]

The numerical results of Reference [3] indicate that the convergence of the series of 22 coefficients \( a_n \), together with the known \( a_1 = 1 \), is sufficient in the cases considered.

The stresses can be calculated with the stress formulae
\[ \sigma_x = 2 \text{Re} \sum_{k=1,2} \left\{ s_{k\rho} \phi_k'(z_k) \right\} \]  
(2.29.)

\[ \sigma_y = 2 \text{Re} \sum_{k=1,2} \left\{ \phi_k'(z_k) \right\} \]  
(2.30.)

\[ \tau_{xy} = -2 \text{Re} \sum_{k=1,2} \left\{ s_{k\rho} \phi_k'(z_k) \right\} \]  
(2.31.)

The complex stress functions \( \phi_k(z_k) \) are solved as functions of \( \zeta_k \). Before substitution into formulae (2.29.) - (2.31.) the functions must be differentiated with respect to \( z_k \):

\[ \phi_k'(z_k) = \frac{d}{dz_k} \left\{ \phi_k(z_k) \right\} = \frac{d\left\{ \phi_k(\zeta_k) \right\}}{d\zeta_k} \frac{d\zeta_k}{dz_k} \]

\[ = \phi_k'(\zeta_k) \frac{r_k}{\sqrt{z_k^2 - s_{k\rho}^2 - 1}} \]  
(2.32.)

The stresses are made dimensionless with the bearing stress \( p \) according to the classic definition

\[ p = \frac{R}{D t} \]

With hole diameter \( D = 2 \) and plate thickness \( t = 1 \) this results in

\[ p = \frac{1}{2} \sqrt{\frac{2}{R_x} + \frac{2}{R_y}} = \frac{1}{2} P_0 \sqrt{4 \left\{ \frac{22}{\pi n^2} + \frac{4}{n^2 - 1} \right\} + \frac{\pi^2}{4}} \]  
(2.33.)

In general the direction of the load resultant will not coincide with the displacement vector of the pin. Since the pin is given a known displacement, the direction of the load resultant follows from the calculation. With known \( R_y \) and calculated \( R_x \) the angle \( \delta \) between the resultant \( R \) and the \( Y \)-axis is easily found. In all relevant figures of Reference [3] this angle \( \delta \) is shown.

In Reference [3] it is assumed that the contact area, i.e. the load transmitting part of the edge is between \( \theta = 0^0 \) and \( \pi \), independent of the direction of the elastic principal axes. In all cases considered the contact angle \( \gamma_C \), see Figure 2, proved to be smaller than \( 180^0 \) which is in agreement with the results of References [5] and [6]. The contact angle is initially unknown and must be a part of the solution of the problem; it can be determined with an iteration procedure by the requirement that the contact pressure must be positive everywhere. Since an iteration increases the computing work considerably in Reference [3] a standard contact angle yielding sufficient accuracy has been adopted.
3. THE CONTACT PROBLEM WITH FRICTION

As in Reference [3] the problem is treated in plane stress. The pin is considered to be a rigid circular inclusion of the same diameter as the hole which is given a translation $v_1$ in the $Y$-direction only, thereby transmitting a load on the surrounding plate. When friction is taken into account the contact area is divided into a region of slip of the plate material over the pin and a region of no-slip. This adds a second unknown angle, the angle of no-slip $\gamma_{NS}$ (see Figure 2) to the problem. The angle $\gamma_{NS}$ must be a part of the solution of the problem as well.

For the edge of the hole the following boundary conditions can be formulated:

- in the region of separation the edge is free of normal and shear stresses.
- in the region of contact, defined by $\gamma_C$, there are both normal and shear stresses of unknown magnitude. $\gamma_C$ can be determined iteratively by the requirement that tractions at the ends of the contact area are (physically) impossible. The contact area is divided into

1. A region of slip. The radial displacement of the points of this part of the edge with respect to the pin must be zero. In addition the shear load $P_{T0}$ on the edge must be equal to the product of the radial contact pressure and the coefficient of friction $\mu$. Its direction must oppose the direction of the relative displacement of the points of this part of the contact region. For the present this requirement will be written as $|P_{T0}| = \mu P_r$.

2. A region of no-slip, defined by $\gamma_{NS}$. The displacements of the points of this part of the edge must be equal to the displacements of the pin, i.e. $u = 0$ and $v = v_1$. The angle $\gamma_{NS}$ can be determined iteratively by the requirement that in the no-slip area $|P_{T0}| \leq \mu P_r$.

N.B. 1. It is conceivable that the contact area has gaps existing of small zones of clearance between pin and plate material. In these zones the displacement condition would result in physically impossible tractions. However, in none of the cases investigated in this report these tractions occur.

N.B. 2. The "angle of incidence" between the material principal axes and the edge of the hole varies over the contact area. Therefore the coefficient of friction probably is not a constant. In this report however, it is assumed to be constant over the entire contact area.

As has been indicated three unknown quantities, the angle of no slip $\gamma_{NS}$, the angle of contact $\gamma_C$ and the radial pressure $P_r$ have entered into the problem. The angles $\gamma_{NS}$ and $\gamma_C$ must be determined iteratively, starting with initial values until the requirements mentioned above are met. A complicating fact is that in cases where the material principal axes and coordinate axes do not coincide the contact area and no-slip area are not symmetric with respect to the $Y$-axis. Hence the first and last point of both areas must be determined separately. An extensive computing effort is required to obtain the correct values of $\gamma_{NS}$ and $\gamma_C$.

A second complication is that a poor estimate of the initial value of $\gamma_{NS}$ may result in a discontinuity in $P_{T0}$ since in the separation points between the slip and no-slip region, where $|P_{T0}| = \mu P_r$, both requirements with respect to the displacements must be fulfilled. When an initial value $\gamma_{NS} = 0$ is chosen (the whole contact area is slip-region) the friction force $P_{T0}$ will change of sign discontinuously at the point where the relative displacement of the edge with respect to the pin changes of sign. This will result in non-convergent series representing $P_r$ and $P_{T0}$; as a result the iteration procedure is impossible.
In order to circumvent these difficulties a third zone is introduced inside the contact area and on both sides of the no-slip region. It is called the "release-area" defined by $\gamma_\tau$ (see Figure 2) and $\gamma_{NS}$. In the two parts of this area the plate material is still assumed to slip over the pin; the requirement $|P_{r\theta}| = \mu P_r$ is released however. The "release-area" has no physical relevance; it makes the contact problem mathematically more tractable since it yields smooth, continuous functions for $P_r$ and $P_{r\theta}$ between slip and no-slip region or between regions with opposite slip direction. An important condition for the continuity of $P_r$ and $P_{r\theta}$ is that the no-slip area or the point where the relative displacement changes of sign is inside the "release area". Since their position is dependent on the angle $\phi$ between material axes and coordinate axes the position of the "release-area" must be made dependent on $\phi$ as well.

The following formulation for the contact problem now will be adopted, yielding sufficient accuracy as preliminary calculations have shown:

- For the contact area a standard angle $\gamma_C$ and fixed first and last point is taken.
- For the "release-area" a standard angle $\gamma_\tau$ has been adopted; its position is made dependent on the angle $\phi$ between material and coordinate axes.
- A no-slip area may be defined inside the "release-area". In this report this has been done for only one laminate and $\phi = 0$.

It is noted that the "release-area" can be reduced iteratively until the requirement $\gamma_{NS} = \gamma_\tau$ is met. In this report this has not been done because the computational effort does not warrant correspondingly improved results.
4. MATHEMATICAL DEFINITION OF FRICTION FORCE $P_{r\theta}$

The positive directions of loads, shear stresses and the relative tangential displacement $v_t$ are chosen as shown in Figure 3. It is noted that a negative relative displacement produces a positive friction load $P_{r\theta}$; a positive $P_{r\theta}$ causes a negative shear stress $\tau_{r\theta}$ on the edge of the hole.

As in Reference [3] a sine series is adopted for the radial pressure $P_r$

$$P_r = P_0 \sum_{n=1,2}^{\infty} a_n \sin n\theta \quad 0 \leq \theta \leq \pi$$

$$P_r = 0 \quad \pi < \theta < 2\pi$$

The terms with odd coefficients $n$ represent the symmetric part of the load with respect to the $Y$-axis; the terms with even $n$ the asymmetric part, in Figure 4 indicated with $\Delta P_r$.

In the slip region the symmetric part of the radial load yields relative tangential displacements, see Figure 4A

$$v_t \begin{cases} < 0 \quad \text{for} \quad 0 < \theta < \pi/2 \\ > 0 \quad \text{for} \quad \pi/2 < \theta < \pi \end{cases}$$

According to the sign-convention this results in the requirement for the symmetric part of $P_{r\theta}$:

$$P_r = \begin{cases} \mu P_r \quad \text{for} \quad 0 < \theta < \pi/2 \\ -\mu P_r \quad \text{for} \quad \pi/2 < \theta < \pi \end{cases}$$

and for its components in $X$- and $Y$-direction

$$X_f \begin{cases} < 0 \quad \text{for} \quad 0 < \theta < \pi/2 \\ > 0 \quad \text{for} \quad \pi/2 < \theta < \pi \end{cases}$$

$$Y_f > 0 \quad \text{for} \quad 0 < \theta < \pi$$

As indicated in Figure 4B the asymmetric part of the radial load yields the following requirements for the components of the asymmetric part $\Delta P_{r\theta}$ of the friction force

$$X_f < 0 \quad \text{for} \quad 0 < \theta < \pi$$
\[
Y_F \begin{cases} 
> 0 & \text{for } 0 < \theta < \pi/2 \\
< 0 & \text{for } \pi/2 < \theta < \pi 
\end{cases}
\]

Additional requirements for the mathematical formulation of \( P_{r\theta} \) are:

- in the point defined by \( \theta = 90^\circ \) the component of the friction force in Y-direction must be zero.
- inside the "release-area" (or no-slip area) there is a point where \( P_{r\theta} = 0 \);
  for \( \phi = 0 \) this point is defined by \( \theta = 90^\circ \)
  for \( \phi \neq 0 \) it may be a different point.

Formulations for the components of the friction force obeying all the requirements mentioned above are

\[
X_F = -P_o \sin \theta \sum_{n=1,2}^\infty b_n \sin n\theta \\
Y_F = P_o \cos \theta \sum_{n=1,2}^\infty b_n \sin n\theta
\]

(4.1.) and (4.2.) are continuous on the entire edge of the hole including the non-loaded area. After multiplication by the step-function (2.2.) they become

\[
X_F = -P_o \sin \theta \left[ \frac{1}{2} \sum_{n=1,2}^\infty b_n \sin n\theta + \frac{1}{\pi} \sum_{n=1,3}^\infty b_n \left( \frac{1}{n-m} + \frac{1}{n+m} \right) \cos m\theta \right]
\]

(4.3.)

\[
Y_F = P_o \sin \theta \left[ \frac{1}{2} \sum_{n=1,2}^\infty b_n \sin n\theta + \frac{1}{\pi} \sum_{n=1,3}^\infty b_n \left( \frac{1}{n-m} + \frac{1}{n+m} \right) \cos m\theta \right]
\]

(4.4.)

yielding

\[
P_{r\theta} = P_o \sum_{n=1,2}^\infty b_n \sin n\theta \quad \text{for } 0 \leq \theta \leq \pi
\]

\[
P_{r\theta} = 0 \quad \text{for } \pi < \theta < 2\pi
\]

The symmetric part of \( P_{r\theta} \) with respect to the Y-axis is represented by the terms with even \( n \) and the asymmetric part by the terms with odd \( n \). The components of the friction force in X- and Y-direction become

\[
R_{X_F} = \int_{0}^{2\pi} X_F \, d\theta = -P_o \frac{\pi}{2} b_1
\]

(4.5.)
\[ R_{y_F} = \int_{0}^{2\pi} Y_F \, d\theta = 2p_o \sum_{n=2,4}^{\infty} \frac{nb_n}{n^2 - 1} \]  

(4.6.)

Expressions (4.3.) and (4.4.) are almost equal to expressions (2.11.) and (2.12.) for the components of the radial pressure on the edge of the hole. It is obvious that the same method as resumed in Chapter 2 and worked out in detail in Reference [3] results in the complex stress functions for the friction load. They are

\[
\phi_k(z_k) = A_k \ln \zeta_k + \frac{p_o}{2\pi i(s_k - s_k)} \left[ \sum_{m,n}^{2\nu b \{s_k (m^2 - n^2 + 1) - 2im\}} \frac{N_{m,n}}{\zeta_k^{-m}} \right. \\
- \frac{\pi i}{4} \sum_{n=0,1,2}^{\infty} \frac{(b_n (s_k - i) + b_{n+2} (s_k + i))}{n+1} \zeta_k^{-n-1} \left. \right] 
\]

(4.7.)

in which \( b_o = 0 \).
5. DETERMINATION OF THE CONSTANTS $a_n$ AND $b_n$

Since the problem treated in this report is a problem of linear elasticity the complex stress functions of the pin-loaded hole problem with friction can be found by superimposing the stress functions of Reference [3] on those evaluated in Chapter 4, resulting in

$$\phi_k(z_k) = A_k \ln \zeta_k$$

\[+ \frac{p_0}{2\pi i (s_{k_\theta} - s_{k_\theta})} \left[ \sum_{m,n}^\infty \frac{2\pi (m^2 - n^2 + 1)(a_n + b_n s_{k_\theta}) + 2\text{im}(a_n s_{k_\theta} - b_n)}{N_{m,n}} \right] \zeta_k^{-m}

- \frac{\pi}{4} \sum_{n=0,1,2}^\infty \left( \frac{(a_n - ib_n)(1 + is_{k_\theta}) + (a_{n+2} + ib_{n+2})(1 - is_{k_\theta})}{n + 1} \right) \zeta_k^{-n-1} \]

(5.1.)

in which $a_0 = b_0 = 0$, $k = 1,2$, and $\lambda = 3-k$.

For $A_k$ expression (2.10.) can be used; it is noticed however that $R_x$ and $R_y$ occurring in (2.10.) are now composed of both the radial and friction forces; hence, resulting from (2.13.), (2.14.), (4.5.) and (4.6.)

$$R_x = p_0 \left\{ 2 \sum_{n=2,4}^{\infty} \frac{na_n}{n^2 - 1} - \frac{\pi}{2} b_1 \right\}$$

(5.2.)

$$R_y = p_0 \left\{ \frac{\pi}{2} a_1 + 2 \sum_{n=2,4}^{\infty} \frac{nb_n}{n^2 - 1} \right\}$$

(5.3.)

In Reference [3] coefficient $a_1$ has been chosen unity yielding $R_y = \frac{\pi}{2} p_0$. In this report the same value for $R_y$ will be chosen. This implies that $a_1$ has become an unknown coefficient. Together with the even coefficients $b_n$ it has to comply with the equation according to (5.3.)

$$\frac{\pi}{2} a_1 + 2 \sum_{n=2,4}^{\infty} \frac{nb_n}{n^2 - 1} = \frac{\pi}{2}$$

or

$$a_1 = 1 - \frac{4}{\pi} \sum_{n=2,4}^{\infty} \frac{nb_n}{n^2 - 1}$$

(5.4.)

Substitution of (5.1.) in (2.22.) and (2.23.) yields expressions for the displacements $u$ and $v$. For the edge of the hole these expressions are given in the Appendix.

The boundary conditions formulated in Chapter 3 will be imposed on a number of points of the contact area arranged symmetrically with respect to the $Y$-axis. The displacement boundary conditions are not relevant in the point defined by $\theta = 90^\circ$. Therefore this point must be excluded. Then the number of points used for the boundary conditions will be even, resulting in an even number of equations from which unknown coefficients can be solved. With (5.4.) added the total number of equations is odd. So an odd number of coefficients will be solved, for instance $N+1$ coefficients $a_n$ and $N$ coefficients $b_n$. 
For the slip-area the displacement boundary condition and the requirement concerning the value of $P_r\theta$ are

$$u \cos \theta + (v - v_1) \sin \theta = 0$$  \hspace{1cm} (5.5.)

$$\sum_{n=1,2}^{N+1} b_n \sin n\theta = \pm \mu \sum_{n=1,2}^{N} a_n \sin n\theta$$  \hspace{1cm} (5.6.)

In (5.6.) the $+$ sign should be chosen for the part of the slip area to the right of the $Y$-axis, the $-$ sign for the part to the left.

With expressions (A.1.), (A.2.) and (A.3.) for the displacements (5.5.) can be written as

$$a_{1\theta} a_1 + \sum_{n=2,3}^{\infty} a_n \sin n\theta + \sum_{n=1,2}^{\infty} b_n \sin n\theta = 0$$  \hspace{1cm} (5.7.)

in which $a_{1\theta}$, $a_n\theta$ and $b_n\theta$ are known coefficients depending on $\theta$. They are presented in the Appendix. Substitution of (5.4.) in (5.7.) and (5.6.) yields, with $N$ chosen even:

$$\sum_{n=2,3}^{N+1} a_n \sin n\theta + \sum_{n=1,2}^{N} b_n \sin n\theta = \sum_{n=2,4}^{N} \left( b_n + \frac{4}{n^2-1} a_1 \theta \right) \sin \theta = 0$$  \hspace{1cm} (5.8.)

$$\sum_{n=2,3}^{N+1} a_n \sin n\theta + \sum_{n=1,2}^{N} b_n \sin n\theta = \sum_{n=2,4}^{N} \left( b_n + \frac{4}{n^2-1} a_1 \theta \right) \sin \theta = 0$$  \hspace{1cm} (5.9.)

For points of the "release-area" the friction force and the radial pressure are released from condition (5.6.). So only condition (5.5.) is imposed on points of this area, resulting in equation (5.8.).

For the no-slip area the relative displacements must be zero, hence the displacement conditions are

$$u = 0$$  \hspace{1cm} (5.10.)

$$v - v_1 = 0$$  \hspace{1cm} (5.11.)

With the expressions for the displacements presented in the Appendix (5.10.) and (5.11.) can be written as

$$c_{1\theta} a_1 + \sum_{n=2,3}^{\infty} c_n \sin n\theta \sin n\theta + \sum_{n=1,2}^{\infty} d_n \sin n\theta = 0$$

$$c_{1\theta} a_1 + \sum_{n=2,3}^{\infty} c_n \sin n\theta \sin n\theta + \sum_{n=1,2}^{\infty} d_n \sin n\theta = 0$$
or, with (5.4.)

\[
\begin{align*}
\sum_{n=2,3}^{N+1} c_{n\theta} a_n + \sum_{n=1,3}^{N-1} d_{n\theta} b_n + \sum_{n=2,4}^{N} \left( d_{n\theta} - \frac{4}{\pi} \frac{n}{n^2 - 1} c_{1\theta} \right) b_n &= -c_{1\theta} \quad (5.12.) \\
\sum_{n=2,3}^{N+1} c_{n\theta} a_n + \sum_{n=1,3}^{N-1} d_{n\theta} b_n + \sum_{n=2,4}^{N} \left( d_{n\theta} - \frac{4}{\pi} \frac{n}{n^2 - 1} c_{1\theta} \right) b_n &= -c_{1\theta}^* \quad (5.13.)
\end{align*}
\]

The coefficients \(c_{n\theta}^*, c_{n\theta}, d_{n\theta}^*, d_{n\theta}\) are also known. They are given in the Appendix as functions of \(\theta\).

In general \(N-m\) values of \(\theta\) defining points of the slip area including the "release-area" are chosen for equations (5.8.) and \(N-m\) different values defining points of the slip area without the "release-area" for equations (5.9.). On the no-slip area \(m\) points are chosen for the set of equations (5.12.) and (5.13.). Hence the total number of equations is \(2N\) from which \(N\) coefficients \(a_n\) \((n = 2 \ldots N+1)\) and \(N\) coefficients \(b_n\) \((n = 1 \ldots N)\) are solved. Coefficient \(a_1\) can be calculated from (5.4.).

With all coefficients solved the complex stress functions as well as the radial and friction shear stress along the edge of the hole are known.

\[
\begin{align*}
\sigma_r &= -p_0 \sum_{n=1,2}^{N+1} a_n \sin n\theta \\
\tau_{r\theta} &= -p_0 \sum_{n=1,2}^{N} b_n \sin n\theta
\end{align*}
\]

After differentiation of the complex stress functions with respect to \(z_k\) the tangential stress \(\sigma_\theta\) can be calculated with

\[
\sigma_\theta = 2 \text{Re} \sum_{k=1,2} \left( (1 + \frac{2}{s_{k\theta}}) \phi_k'(z_k) \right) - \sigma_r
\]
6. NUMERICAL CALCULATIONS

For all investigated laminates, angles $\phi$ and coefficients of friction a standard contact area $11^0 < \theta < 169^0$ has been adopted. The "release-area" is defined by

$$75(1 - c_4) < \theta < 105(1 - c_4)$$

so the first and last point are made dependent on $\phi$ by the factor $1 - c_4$ in which

$$c_4 = S_{11}(s_1 s_2 + 1) \sin \phi \cos \phi \text{Im}(s_1 + s_2) \quad \text{(see (A.4.))}$$

The factor $(1 - c_4)$ is chosen rather arbitrarily on physical grounds because it accounts for material rotation, being zero for $\phi = 0^0$ and $90^0$ and having its maximum for $\phi = 45^0$. In the two cases with arbitrary load direction the results indicate that the point where the friction shear changes of sign is near the middle of the "release-area".

As mentioned in Chapter 3 a no-slip area has been considered in only one case. The results show only small differences with the corresponding case without no-slip area. Therefore in all other investigated cases the no-slip area has been omitted.

The calculations have been made for the following angles $\phi$ and coefficients of friction

<table>
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<td>$90^0/180^0$</td>
<td>$0, 0.2, 0.4$</td>
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<td>$15^0/30^0$</td>
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N.B. The case $\mu = \infty$ is calculated by considering the whole contact area $11^0 < \theta < 169^0$ as no-slip area. It cannot be extended to $0^0 < \theta < 180^0$ because then discontinuities in $P_{F0}$ will occur in the points defined by $\theta = 0^0$ and $180^0$. It is obvious that the series adopted for $P_{F0}$ cannot provide these discontinuities.

Generally $N$ is chosen 28 and $m$ zero, resulting in 29 coefficients $a_n$ and 28 coefficients $b_n$. As an example the coefficients of the $(90^0/180^0)_s$-laminate, $\mu = 0.4$ and $\phi = 0^0$, $15^0$ and $30^0$ are listed in Table 2.

The tangential, radial and shear stresses along the edge of the hole are shown in Figures 5-10. The stresses are made dimensionless with the bearing stress

$$p = \frac{1}{2} \sqrt{R_x^2 + R_y^2}$$
in which \( R_x \) and \( R_y \) are given in (5.2.) and (5.3.).

From the calculated stress pattern a strength prediction has been made by applying the Tsai-Hill failure criterion. The predicted bearing strength \( p \) is listed in Tables 3-5 together with the angle \( \theta \) where first significant damage occurs.
7. DISCUSSION OF RESULTS

- Evaluation of the solution.

As may be concluded from the numerical results the convergence of the series of coefficients \( a_n \) and \( b_n \) is sufficient. For the symmetric loading condition \( \varphi = 0 \) the even coefficients \( a_n \) and odd coefficients \( b_n \) should be zero. In all cases these coefficients show small values resulting from the limited accuracy of the solution. They do not significantly affect the results.

In general the coefficients yield smooth functions for \( P_r \) and \( P_{rs} \). In the "release-area" no values of the friction force are generated exceeding \( \mu \ P_r \). At the ends of the contact area no tractions occur.

- The influence of friction on the stress distributions.

The small coefficients of friction \( \mu = 0.2 \) and \( 0.4 \) significantly influence the stress distribution only in the vicinity of \( \theta = 90^\circ \) where both the radial and tangential stress decrease with increasing \( \mu \). In all cases the presence of friction yields compressive tangential stresses at the top of the hole. The tendency of friction to reduce the tangential stress has also been noted in Reference [7].

As shown in Figure 8 the higher values of \( \mu \) also influence the stresses in the vicinity of the net area. This is physically obvious since these values of \( \mu \) result in increased shear stresses on the sides of the hole, accordingly relieving the radial pressure on the top. As a result the point where the maximum tangential stress is developed moves from \( \theta = 5^\circ \) for \( \mu = 0 \) to \( \theta = 0^\circ \) for \( \mu > 1 \). Simultaneously the contact area increases.

As Figure 10 shows the difference between the stress-distributions in the non-rotated and the rotated \( (90^\circ/\pm45^\circ)_{5/2} \)-laminate is remarkably small. The peak stresses near the top of the hole are hardly influenced by the rotation. Only the places where they occur are rotated over an angle corresponding with the angle \( \varphi \). For \( \mu = 0 \) this phenomenon already was known from Reference [3]; for \( \mu = 0.4 \) used for Figure 10 the similarity of the stress distributions even seems to be stronger. The decrease of the maximum tangential stress as a result of the rotation is relatively small compared to the corresponding case with \( \mu = 0 \) of Reference [3].

For quasi-isotropic and isotropic materials a sine distribution for the radial pressure is often assumed in literature. From the results of this report it may be clear that this is allowable only for the frictionless case.

- The bearing strength.

The failure criterion used to predict the bearing strength neglects important aspects such as three-dimensional stresses and non-linear elasticity. Nevertheless the calculated value of the bearing strength is a good indication for the first significant damage and the place on the edge of the hole where it occurs.

Although the stress-distributions in the \( (0^\circ/90^\circ/\pm45^\circ)_{5/2} \)-, the \( (90^\circ/\pm45^\circ)_{5/2} \)- and the \( (90^\circ/\pm45^\circ)_{5/2} \)-laminate are almost equal, the calculated bearing strength values show great differences. It is obvious that these differences are caused by the different strength values of the laminates themselves.

All laminates considered have a maximum bearing strength for the relatively low coefficient of friction \( \mu = 0.2 \). Except for the \( (90^\circ/\pm45^\circ)_{5/2} \)-laminate the maximum
values are about 10% higher than the corresponding values for \( \mu = 0 \). The angle \( \theta \) defining the point where first significant damage occurs decreases with increasing coefficient of friction. Since the vicinity of the net area is considered as a dangerous zone for failure it may be concluded that the presence of too much friction has an unfavourable effect on the place where the initial damage occurs. For the \((90^\circ/\pm 45^\circ)_{s}\)-laminate this is clearly shown in Table 4. Therefore mechanical fasteners for composite laminates should have smooth surfaces yielding a low coefficient of friction in order to guarantee a high value of the bearing strength and a "safe" place where first damage occurs.

The \((90^\circ/\pm 45^\circ)_{s}\)-laminate shows a much greater difference between the bearing strength values for \( \mu = 0 \) and 0.2 than the 10% previously mentioned. For \( \mu = 0 \) first damage occurs in the point defined by \( \theta = 90^\circ \). This damage is dominated by shear failure in the \( \pm 45^\circ \)-layers and splitting of the \( 90^\circ \)-layers. Generally laminates dominated by unidirectional layers in one direction have a low resistance against cleavage. Therefore first damage of the \((90^\circ/\pm 45^\circ)_{s}\)-laminate in the frictionless case occurs at a relatively low value of the bearing stress. For \( \mu = 0.2 \) the first damage, occurring in the point defined by \( \theta = 50^\circ \), is dominated by shear failure in the \( 90^\circ \)-layers and compressive failure in the \( \pm 45^\circ \)-layers. So the presence of friction modifies the mode of failure and considerably improves the bearing strength of the laminate. Since in actual joints friction always occurs this may be an important reason for the discrepancy between experimental values and calculated values, based on the frictionless case, of some laminates.

As may be concluded from Table 3 the \((90^\circ/\pm 45^\circ)_{s}\)-laminate has a relatively high bearing strength as compared with the \((90^\circ/\pm 45^\circ)_{s}\)- and the \((90^\circ/\pm 45^\circ)_{s}\)-laminate. The reason is that the \( \pm 45^\circ \)-layers of the \((90^\circ/\pm 45^\circ)_{s}\)-laminate which was used to measure the engineering constants and strength values are composed of square fabrics whereas the other laminates consist of unidirectional layers only. The square fabrics yield relatively high values of the modulus of rigidity and shear strength which has a favourable effect on the bearing strength of the laminate. Table 4 indicates that high values of the coefficient of friction reduce the bearing strength essentially. For \( \mu = \infty \) failure almost takes place in the net area. It is governed by shear in the \( 90^\circ \)-layers and compressive failure in the \( \pm 45^\circ \)-layers.

As was discussed in a previous paragraph the stress distributions in the \((90^\circ/\pm 45^\circ)_{s}\)-laminate are not changed significantly by rotation of the material axes over the two angles considered. It is noted that this applies to the stresses only and not to the bearing strength. As Table 5 indicates the influence of the rotation on the bearing strength increases considerably with increasing coefficient of friction.
8. CONCLUSIONS

The effects of friction on the stress distribution round a pin-loaded hole in infinite orthotropic plates were investigated theoretically. Generally the pin was given a displacement in the direction of one of the principal material axes. Coefficients of friction 0, 0.2 and 0.4 were considered; for the \((90^\circ/\pm45^\circ)\_s\)-laminate a wider range of values including \(\infty\) was used. In the \((90^\circ/\pm45^\circ)\_s\)-laminate, having a high directional sensitivity, the pin was given a displacement under \(0^\circ\), \(15^\circ\) and \(30^\circ\) with one of the material axes.

- The presence of friction has a strong influence on the contact pressure between pin and plate and on the induced tangential stresses. The sine distribution for the contact pressure, often recommended in literature, is only valid for a few laminates in the frictionless case.

- Rotation of the \((90^\circ/\pm45^\circ)\_s\)-laminate over the two angles considered has only a modest influence on the stress distribution. The influence of the rotation on the bearing strength increases considerably with increasing coefficient of friction.

- In all investigated cases the theoretical bearing strength has a maximum value for coefficient of friction 0.2. Although there is a lack of knowledge of coefficients of friction the low value 0.2 may imply that for actual joints pins with smooth surfaces yielding low friction should be used. The maximum values of the bearing strength are about 10% higher than the theoretical values in the frictionless cases. Laminates being sensitive for cleavage however show a much better improvement. In these laminates the presence of only small friction between pin and plate changes the mode of failure essentially. Generally the angle \(\theta\) defining the point on the edge of the hole where first damage occurs decreases with increasing friction coefficient.
REFERENCES


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Table 1: The laminate elastic constants and strength values. Most values are measured; $G_{\alpha\beta}$ is calculated from unidirectional and $\pm45^\circ$-angle-ply values. Except for the $(90^\circ/\pm45^\circ)_s$-laminate, from which the $\pm45^\circ$ layers consist of square fabrics, all laminates are built up from unidirectional prepregs.
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<td>-0.102d-01</td>
<td>-0.383d-02</td>
<td>0.956d-02</td>
<td>0.147d-01</td>
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<tr>
<td>12</td>
<td>0.105d-06</td>
<td>-0.356d-02</td>
<td>0.522d-03</td>
<td>0.118d-01</td>
<td>0.123d-01</td>
<td>-0.446d-02</td>
</tr>
<tr>
<td>13</td>
<td>-0.222d-02</td>
<td>-0.123d-06</td>
<td>0.119d-01</td>
<td>-0.727d-03</td>
<td>-0.625d-02</td>
<td>-0.311d-02</td>
</tr>
<tr>
<td>14</td>
<td>-0.936d-07</td>
<td>0.290d-02</td>
<td>-0.176d-02</td>
<td>-0.544d-02</td>
<td>0.354d-02</td>
<td>0.757d-02</td>
</tr>
<tr>
<td>15</td>
<td>0.489d-02</td>
<td>0.939d-07</td>
<td>-0.290d-02</td>
<td>0.382d-02</td>
<td>0.643d-02</td>
<td>-0.891d-03</td>
</tr>
<tr>
<td>16</td>
<td>0.734d-07</td>
<td>0.106d-02</td>
<td>0.468d-02</td>
<td>0.313d-02</td>
<td>-0.156d-02</td>
<td>-0.248d-02</td>
</tr>
<tr>
<td>17</td>
<td>0.158d-02</td>
<td>-0.693d-07</td>
<td>0.237d-02</td>
<td>-0.278d-02</td>
<td>0.242d-03</td>
<td>0.389d-02</td>
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<tr>
<td>18</td>
<td>-0.463d-07</td>
<td>-0.425d-03</td>
<td>-0.217d-02</td>
<td>-0.496d-04</td>
<td>0.428d-02</td>
<td>-0.179d-03</td>
</tr>
<tr>
<td>19</td>
<td>0.139d-02</td>
<td>0.532d-07</td>
<td>0.900d-03</td>
<td>0.217d-02</td>
<td>-0.113d-02</td>
<td>0.964d-03</td>
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<tr>
<td>20</td>
<td>0.489d-07</td>
<td>0.214d-02</td>
<td>0.198d-02</td>
<td>-0.351d-03</td>
<td>0.570d-03</td>
<td>0.128d-02</td>
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<tr>
<td>21</td>
<td>0.252d-02</td>
<td>-0.292d-07</td>
<td>-0.402d-03</td>
<td>-0.738d-03</td>
<td>0.136d-02</td>
<td>0.392d-03</td>
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<tr>
<td>22</td>
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<td>-0.778d-03</td>
<td>-0.384d-03</td>
<td>0.703d-03</td>
<td>0.187d-03</td>
<td>-0.568d-03</td>
</tr>
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<td>23</td>
<td>0.624d-03</td>
<td>0.247d-07</td>
<td>0.850d-03</td>
<td>0.269d-03</td>
<td>-0.243d-03</td>
<td>0.546d-03</td>
</tr>
<tr>
<td>24</td>
<td>0.287d-07</td>
<td>0.160d-02</td>
<td>0.133d-03</td>
<td>-0.294d-03</td>
<td>0.760d-03</td>
<td>0.739d-04</td>
</tr>
<tr>
<td>25</td>
<td>0.208d-02</td>
<td>-0.667d-08</td>
<td>-0.230d-03</td>
<td>0.348d-04</td>
<td>-0.880d-04</td>
<td>0.904d-04</td>
</tr>
<tr>
<td>26</td>
<td>0.122d-08</td>
<td>-0.234d-03</td>
<td>0.554d-04</td>
<td>0.167d-03</td>
<td>-0.195d-04</td>
<td>0.898d-04</td>
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<tr>
<td>27</td>
<td>0.355d-03</td>
<td>0.820d-08</td>
<td>0.193d-03</td>
<td>-0.736d-04</td>
<td>0.148d-03</td>
<td>0.262d-04</td>
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<tr>
<td>28</td>
<td>0.117d-07</td>
<td>0.698d-03</td>
<td>-0.145d-03</td>
<td>-0.150d-05</td>
<td>-0.376d-04</td>
<td>-0.149d-04</td>
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<tr>
<td>29</td>
<td>0.972d-03</td>
<td>0.166d-04</td>
<td>-0.645d-05</td>
<td>-0.645d-05</td>
<td>-0.645d-05</td>
<td>-0.645d-05</td>
</tr>
</tbody>
</table>

Table 2: The coefficients a_n and b_n of the (90°/±45°)_s-laminate for three values of φ and coefficient of friction 0.4.
<table>
<thead>
<tr>
<th>coeff. of friction</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>laminate</td>
<td>$\bar{p}$ MPa</td>
<td>$\theta^\circ$</td>
<td>$\bar{p}$ MPa</td>
</tr>
<tr>
<td>$(0^\circ/90^\circ/\pm45^\circ)_s$</td>
<td>299</td>
<td>60</td>
<td>332</td>
</tr>
<tr>
<td>$(90^\circ/\pm45^\circ)_s$</td>
<td>437</td>
<td>55</td>
<td>471</td>
</tr>
<tr>
<td>$(90^\circ/\pm45^\circ)_s$</td>
<td>348</td>
<td>55</td>
<td>384</td>
</tr>
<tr>
<td>$(90^\circ/\pm45^\circ)_s$</td>
<td>371</td>
<td>60</td>
<td>413</td>
</tr>
<tr>
<td>$(90^\circ/\pm45^\circ)_s$</td>
<td>242</td>
<td>90</td>
<td>329</td>
</tr>
</tbody>
</table>

Table 3: Bearing strength values and angles $\theta$ defining the points on the edge of the hole where first significant damage occurs for five laminates of C.P.R.P. and three coefficients of friction.

<table>
<thead>
<tr>
<th>coeff. of friction</th>
<th>$\bar{p}$ MPa</th>
<th>$\theta^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>371</td>
<td>60</td>
</tr>
<tr>
<td>0.2</td>
<td>413</td>
<td>45</td>
</tr>
<tr>
<td>0.4</td>
<td>402</td>
<td>35</td>
</tr>
<tr>
<td>0.4*</td>
<td>408</td>
<td>35</td>
</tr>
<tr>
<td>0.6</td>
<td>373</td>
<td>30</td>
</tr>
<tr>
<td>0.8</td>
<td>342</td>
<td>30</td>
</tr>
<tr>
<td>1.0</td>
<td>321</td>
<td>25</td>
</tr>
<tr>
<td>no-slip</td>
<td>181</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 4: Bearing strength values of the $(90^\circ/\pm45^\circ)_s$ laminate and angles $\theta$ defining the points on the edge of the hole where first significant damage occurs for different values of the coefficient of friction.

* no-slip area $85^\circ < \theta < 95^\circ$
## Coefficient of friction

<table>
<thead>
<tr>
<th>Angle $\varphi$</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \text{ MPa}$</td>
<td>$\theta^\circ$</td>
<td>$p \text{ MPa}$</td>
<td>$\theta^\circ$</td>
</tr>
<tr>
<td>$0^\circ$</td>
<td>242</td>
<td>90</td>
<td>329</td>
</tr>
<tr>
<td>$15^\circ$</td>
<td>241</td>
<td>105</td>
<td>300</td>
</tr>
<tr>
<td>$30^\circ$</td>
<td>236</td>
<td>90,115</td>
<td>269</td>
</tr>
</tbody>
</table>

Table 5: Bearing strength values of the $(90^\circ_s/\pm45^\circ_s)$-laminate and angles $\theta$ defining the points on the edge of the hole where first significant damage occurs for three values of the coefficient of friction and three values of $\varphi$. $\varphi$ is the angle between the direction of the pin-displacement and one of the principal material axes.
Figure 1: Schematic representation of the problem of a pin-loaded hole in an infinite orthotropic plate. The $\alpha\beta$-axes are the material symmetry axes.

Figure 2: Definition of the different regions of the contact area.
Figure 3: The sign-convention. Loads, displacements and shear stresses as presented have a positive sign.

\[ P_r = \rho_0 \sum_{n=1}^{\infty} a_n \sin n\theta \]

\[ \Delta P_r = \rho_0 \sum_{n=1}^{\infty} a_n \sin n\theta \]

Figure 4: Displacements and friction forces induced by the symmetric and the asymmetric part of the radial pressure.
Figure 5: The stress distribution around a pin-loaded hole in a $(0^\circ/90^\circ/\pm45^\circ)_s$-laminate of C.F.R.P. for three values of the coefficient of friction.
Figure 6: The stress distribution around a pin-loaded hole in a $(90^\circ/\pm45^\circ)_s$-laminate of C.F.R.P. for three values of the coefficient of friction.
Figure 7: The stress distribution around a pin-loaded hole in a $(90^0_2/\pm 45^0)_s$-laminate of C.F.R.P. for three values of the coefficient of friction.
Figure 8: The stress distribution around a pin-loaded hole in a $(90^\circ/\pm45^\circ)_s$-lamine of C.F.R.P. for four values of the coefficient of friction.
Figure 9: The stress distribution around a pin-loaded hole in a (90°/±45°)₈-laminate of C.F.R.P. for three values of the coefficient of friction.
Figure 10: The stress distribution around a pin-loaded hole in a $(90^\circ/\pm45^\circ)_s$-lamine of C.F.R.P. for three values of the angle $\varphi$ between the direction of the pin-displacement and the $\beta$-axis. The coefficient of friction is 0.4.
APPENDIX

Substitution of the complex stress functions \(\phi_k(z_k)\) as presented in (5.1.) in the displacement formulae (2.22.) and (2.23.) results in expressions for the displacements \(u\) and \(v\). For the hole boundary where

\[
\begin{align*}
\zeta_k &= \sigma = \cos \theta + i \sin \theta \\
\zeta_k^{n+1} &= \cos (n + 1) \theta + i \sin (n + 1) \theta \\
\zeta_k^m &= \cos m\theta + i \sin m\theta 
\end{align*}
\]

the displacements are in \(X\)- and \(Y\)-direction respectively:

\[
u = \frac{P_0}{4} \sum_{\frac{m,n}{m,n}} \frac{m(m^2-n^2+1)(c_4b_n-c_5a_n)+2mc_6a_n}{N_{m,n}} \cos m\theta \\
- \frac{8}{\pi} \sum_{\frac{m,n}{m,n}} \frac{n(m^2-n^2+1)c_6b_n-2m(c_5b_n+c_4a_n)}{N_{m,n}} \sin m\theta \\
- \sum_{n=0,1,2}^{\infty} \frac{b_n(c_6-c_5)+b_{n+2}(c_6+c_5)-c_4(a_n-a_{n+2})}{n+1} \cos (n+1) \theta \\
- \sum_{n=0,1,2}^{\infty} \frac{c_4(b_n+b_{n+2})-a_n(c_5-c_6)-a_{n+2}(c_5+c_6)}{n+1} \sin (n+1) \theta \tag{A.1.}
\]

\[
v = \frac{P_0}{4} \sum_{\frac{m,n}{m,n}} \frac{n(m^2-n^2+1)(c_4a_n-c_7b_n)+2mc_6b_n}{N_{m,n}} \cos m\theta \\
+ \frac{8}{\pi} \sum_{\frac{m,n}{m,n}} \frac{n(m^2-n^2+1)c_6a_n-2m(c_7a_n+c_4b_n)}{N_{m,n}} \sin m\theta \\
+ \sum_{n=0,1,2}^{\infty} \frac{a_n(c_6-c_7)+a_{n+2}(c_6+c_7)-c_4(b_n-b_{n+2})}{n+1} \cos (n+1) \theta \\
- \sum_{n=0,1,2}^{\infty} \frac{c_4(a_n+a_{n+2})-b_n(c_7-c_6)-b_{n+2}(c_7+c_6)}{n+1} \sin (n+1) \theta \tag{A.2.}
\]

In (A.1.) and (A.2.) is \(a_o = b_o = 0\).

The expression for the displacement \(v_1\) of the point defined by \(\theta = 90^\circ\) is
\[ v_i = \frac{P_0}{4} \left[ \frac{1}{2} (c_2 a_1 + c_4 b_1) - \frac{3}{2} c_6 a_1 + \frac{8}{\pi} \sum_{m=1,3}^{\infty} 2(c_2 a_1 + c_4 b_1) (-1)^{m+1} \right] \]

\[ + 2 \sum_{n=2,4}^{\infty} \left\{ \frac{(c_4 a_n - c_7 b_n)^2}{n^2 - 1} - \frac{2n c_6 b_n}{n^2 - 1} \right\} \left[ (-1)^{n} + \frac{2}{\pi (n^2 - 1)} \right] \]

\[ + \frac{4}{\pi} \sum_{m=2,4}^{\infty} \left\{ \frac{(c_4 a_n - c_7 b_n)^2 n (m^2 - n^2 + 1) (-1)^2}{N_{m,n}} \right\} \]

\[ + 2 \sum_{n=3,5}^{\infty} \left\{ \frac{c_7 a_n + c_4 b_n}{n^2 - 1} \right\} \left[ \frac{2n c_6 b_n}{n^2 - 1} \right] \left[ 2(-1)^{n-1} - n \right] \]

\[ + \frac{4}{\pi} \sum_{m=1,3}^{\infty} \left\{ \frac{(c_7 a_n + c_4 b_n)^2 2mn(-1)^2}{N_{m,n}} \right\} \] \hspace{1cm} (A.3.)

In (A.1.), (A.2.) and (A.3.) is

\[ c_4 = S_{11} (s_1 s_2 + 1) \sin \varphi \cos \varphi \Im(s_1 + s_2) \] \hspace{1cm} (A.4.)

\[ c_5 = S_{11} (\cos^2 \varphi - s_1 s_2 \sin^2 \varphi) \Im(s_1 + s_2) \] \hspace{1cm} (A.5.)

\[ c_6 = S_{12} - S_{11} s_1 s_2 \] \hspace{1cm} (A.6.)

\[ c_7 = S_{11} (\sin^2 \varphi - s_1 s_2 \cos^2 \varphi) \Im(s_1 + s_2) \] \hspace{1cm} (A.7.)

For materials with equal \( S_{11} \) and \( S_{22} \), resulting in \( s_1 s_2 = -1 \), it is easily seen that all constants \( c_4 - c_7 \) are invariant. So for these materials the constants do not depend on the angle \( \varphi \) between material axes and coordinate axes. This implies that the displacements \( u \) and \( v \) on the edge of the hole are independent of a rotation of these materials.

N.B. The derivation of expressions for the displacements in the frictionless case is given in detail in Reference [3].

The expressions for the displacements substituted in the displacement boundary condition for the slip area yield formula (5.8.):
\[ \sum_{n=2,3}^{N+1} a_{n\theta} a_n + \sum_{n=1,3}^{N-1} b_{n\theta} b_n + \sum_{n=2,4}^{N} \left( b_{n\theta} - \frac{4}{n^2} \frac{n}{n^2-1} a_{1\theta} \right) b_n = -a_{1\theta} \quad (5.8) \]

The coefficients \( a_{1\theta}, a_{n\theta} \) and \( b_{n\theta} \) can directly be used for programming purposes. They are given in detail below:

\[
a_{1\theta} = \frac{c_8}{2} \sin 2\theta + \frac{c_9}{2} \cos 2\theta + c_6 \left( \theta - \frac{\pi}{2} \right) \cos \theta - \frac{c_7}{2} \sin \theta
\]

\[
+ \frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{2c_9 \sin m\theta - c_8 m \cos m\theta - 2c_7(-1)^{\frac{m+1}{2}} \sin \theta}{m^2(m-2)(m+2)}
\]

\[
b_{1\theta} = \frac{c_8}{2} \cos 2\theta - \frac{c_9}{2} \sin 2\theta + c_6 \left( \theta - \frac{\pi}{2} \right) \sin \theta - \frac{c_6}{2} \cos \theta - \frac{c_4}{2} \sin \theta
\]

\[
+ \frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{2c_8 \sin m\theta + c_9 m \cos m\theta - 2c_4(-1)^{\frac{m+1}{2}} \sin \theta}{m^2(m-2)(m+2)}
\]

For \( n \) is even:

\[
a_{n\theta} = \frac{2}{n^2-1} \left[ c_{10} n \sin n\theta + c_{11} \cos n\theta + 2c_6 \sin n\theta - c_4(-1)^{\frac{n}{2}} \sin \theta
\]

\[
- \frac{c_6 n}{\pi} \left\{ \frac{4}{n^2-1} \cos \theta - (2\theta - \pi) \sin \theta \right\} \right] + \frac{8}{\pi} \sum_{m=2,4}^{\infty} \frac{2c_9 mn \sin m\theta - c_8 n(m^2-n^2+1) \cos m\theta - c_4 n(m^2-n^2+1)(-1)^{\frac{m}{2}} \sin \theta}{N_{m,n}}
\]

\[
b_{n\theta} = \frac{2}{n^2-1} \left[ c_{11} n \sin n\theta - c_{10} \cos n\theta - 2c_6 n \cos n\theta + c_4(-1)^{\frac{n}{2}} \sin \theta
\]

\[
- \frac{c_6 n}{\pi} \left\{ (2\theta - \pi) \cos \theta - 2\pi(-1)^{\frac{n}{2}} \sin \theta \right\} \right] + \frac{8}{\pi} \sum_{m=2,4}^{\infty} \frac{2c_8 mn \sin m\theta + c_9 n(m^2-n^2+1) \cos m\theta + c_4 n(m^2-n^2+1)(-1)^{\frac{m}{2}} \sin \theta}{N_{m,n}}
\]

For \( n \) is odd:

\[
a_{n\theta} = \frac{2}{n^2-1} \left[ c_{10} n \sin n\theta + c_{11} \cos n\theta + 2c_6 \sin n\theta
\]

\[
\left\{ \frac{n-1}{2} - c_7(-1)^{\frac{n-1}{2}} \right\} \sin \theta \right] + \frac{8}{\pi} \sum_{m=2,4}^{\infty} \frac{2c_6(-1)^{\frac{n-1}{2}} + c_7 n(-1)^{\frac{n-1}{2}} \sin \theta}{N_{m,n}}
\]
\[
\frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{2c_g mn \sin m\theta - c_g n(m^2-n^2+1) \cos m\theta - 2mn c_n(-1)^{\frac{m+1}{2}} \sin \theta}{N_{m,n}}
\]

\[
b_{n\theta} = \frac{2}{n^2-1} \left[ c_{11} n \sin n\theta - c_{10} \cos n\theta - 2c_6 n \cos n\theta + nc_6 \cos \theta
\right] +
\]

\[
\frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{2c_g mn \sin m\theta + c_g n(m^2-n^2+1) \cos m\theta - 2mn c_n(-1)^{\frac{m+1}{2}} \sin \theta}{N_{m,n}}
\]

For the highest value of m in the series any large number giving sufficient accuracy may be taken. In the coefficients is

\[
c_8 = c_5 \cos \theta - c_4 \sin \theta
\]

\[
c_9 = c_4 \cos \theta - c_7 \sin \theta
\]

\[
c_{10} = c_8 \cos \theta - c_9 \sin \theta
\]

\[
c_{11} = -c_8 \sin \theta - c_9 \cos \theta
\]

The expressions for the displacements substituted in the displacement boundary conditions for the no-slip area yield formulae (5.12.) and (5.13.):

\[
N+1 \sum_{n=2,3}^{n_{\theta}} c_{n\theta} a_n + \sum_{n=1,3}^{N-1} d_{n\theta} b_n + \sum_{n=2,4}^{N} d_{n\theta} \left( -\frac{4}{\pi} \frac{n}{n^2-1} \right) c_{1\theta} b_n = -c_{1\theta}
\]

(5.12.)

\[
N+1 \sum_{n=2,3}^{n_{\theta}} c_{n\theta} a_n + \sum_{n=1,3}^{N-1} d_{n\theta} b_n + \sum_{n=2,4}^{N} d_{n\theta} \left( -\frac{4}{\pi} \frac{n}{n^2-1} \right) c_{1\theta} b_n = -c_{1\theta}
\]

(5.13.)

in which:

\[
c_{1\theta} = (c_5 - 2c_6) \sin \theta \cos \theta + \frac{c_4}{2} \cos 2\theta + c_6 \left( \theta - \frac{\pi}{2} \right)
\]

\[
+ \frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{2c_4 \sin m\theta - mc_5 \cos m\theta}{m^2(m-2)(m+2)}
\]

\[
c_{1\theta}^* = (2c_6 - c_7)(1 - \sin^2 \theta) - \frac{c_4}{2} \sin 2\theta
\]

\[
+ \frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{mc_4 \cos m\theta - 2c_7 \sin m\theta + 2c_7(-1)^{\frac{m-1}{2}}}{m^2(m-2)(m+2)}
\]
\[
\begin{align*}
d_{n}^{\theta} &= \frac{1}{\pi} (c_{5} - c_{6}) + (2c_{5} - c_{6}) \sin^{2} \theta - \frac{c_{4}}{2} \sin 2\theta \\
&+ \frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{mc_{4} \cos m\theta + 2c_{5} \sin m\theta}{m^{2}(m-2)(m+2)} \\
d_{n}^{x} &= \frac{1}{2} (c_{7} - 2c_{6}) \sin 2\theta - c_{4} \cos^{2} \theta + c_{6} \left(\theta - \frac{n}{2}\right) \\
&+ \frac{8}{\pi} \sum_{m=1,3}^{\infty} \frac{2c_{4}(-1)^{m-1} - 2c_{4} \sin m\theta - mc_{7} \cos m\theta}{m^{2}(m-2)(m+2)}
\end{align*}
\]

For \( n \) is even:

\[
c_{n}^{\theta} = \frac{c_{6} \cos n\theta - c_{12} \cos n\theta + 2c_{6} \cos n\theta \cos \theta - n \cos n\theta \sin \theta) - \frac{4}{\pi} \frac{n}{n^{2}-1} c_{6}}{n^{2} - 1}
\]

\[
+ \frac{4}{\pi} \sum_{m=2,4}^{\infty} \frac{2c_{4} \cos m\theta - c_{5} \cos m\theta}{m^{2} - n^{2} + 1} \cos m\theta
\]

\[
c_{n}^{x} = \frac{-c_{9} \cos n\theta + c_{13} \cos n\theta + 2c_{6} \cos n\theta \cos \theta + n \cos n\theta \sin \theta) + n \left(\frac{2\theta}{n^{2}} - 1\right) c_{6} - c_{4} (-1)^{2}}{n^{2} - 1}
\]

\[
+ \frac{4}{\pi} \sum_{m=2,4}^{\infty} \frac{c_{4} \cos m\theta - c_{5} \cos m\theta}{m^{2} - n^{2} + 1} \cos m\theta
\]

For \( n \) is odd:

\[
d_{n}^{\theta} = \frac{-c_{12} \cos n\theta - c_{6} \cos n\theta + 2c_{6} \cos n\theta \cos \theta - n \cos n\theta \sin \theta) - n \left(\frac{2\theta}{n^{2}} - 1\right) c_{6}}{n^{2} - 1}
\]

\[
+ \frac{4}{\pi} \sum_{m=2,4}^{\infty} \frac{2c_{5} \cos m\theta + c_{4} \cos m\theta}{m^{2} - n^{2} + 1} \cos m\theta
\]

\[
d_{n}^{x} = \frac{c_{13} \cos n\theta + c_{9} \cos n\theta + 2c_{6} \cos n\theta \cos \theta + n \cos n\theta \sin \theta) + c_{7} (-1)^{2} + 2n c_{6} (-1)^{2}}{n^{2} - 1}
\]

\[
+ \frac{4}{\pi} \sum_{m=2,4}^{\infty} \frac{-c_{7} \cos m\theta - 2c_{4} \cos m\theta}{m^{2} - n^{2} + 1} \cos m\theta
\]
\[
\begin{align*}
c_{n\theta} &= \frac{c_{9} n \sin n\theta - c_{12} \cos n\theta + 2c_{6}(\sin n\theta \cos \theta - n \cos n\theta \sin \theta)}{n^2 - 1} \\
&\quad + \frac{4}{\pi} \sum_{m=1,3}^{\infty} \frac{2c_{4} mn \sin m\theta - c_{5} n(m^2 - n^2 + 1) \cos m\theta}{N_{m,n}}
\end{align*}
\]

\[
\begin{align*}
c_{n\theta}^* &= \frac{-c_{9} n \sin n\theta + c_{13} \cos n\theta + 2c_{6}(n \cos n\theta \cos \theta + \sin n\theta \sin \theta) - (2c_{6} + nc_{7})(-1)^{\frac{m-1}{2}}}{n^2 - 1} \\
&\quad + \frac{4}{\pi} \sum_{m=1,3}^{\infty} \frac{c_{4} n(m^2 - n^2 + 1) \cos m\theta - 2c_{7} mn \sin m\theta + 2c_{7} mn(-1)^{\frac{m-1}{2}}}{N_{m,n}}
\end{align*}
\]

\[
\begin{align*}
d_{n\theta} &= \frac{-c_{12} n \sin n\theta - c_{8} \cos n\theta - 2c_{6}(n \cos n\theta \cos \theta + \sin n\theta \sin \theta) + nc_{6}}{n^2 - 1} \\
&\quad + \frac{4}{\pi} \sum_{m=1,3}^{\infty} \frac{2c_{5} mn \sin m\theta + c_{4} n(m^2 - n^2 + 1) \cos m\theta}{N_{m,n}}
\end{align*}
\]

\[
\begin{align*}
d_{n\theta}^* &= \frac{c_{13} n \sin n\theta + c_{9} \cos n\theta + 2c_{6}(\sin n\theta \cos \theta - n \cos n\theta \sin \theta) - c_{4} n(-1)^{\frac{m-1}{2}}}{n^2 - 1} \\
&\quad + \frac{4}{\pi} \sum_{m=1,3}^{\infty} \frac{-c_{7} n(m^2 - n^2 + 1) \cos m\theta - 2c_{4} mn \sin m\theta + 2c_{4} mn(-1)^{\frac{m-1}{2}}}{N_{m,n}}
\end{align*}
\]

\[
\begin{align*}
c_{12} &= c_{5} \sin \theta + c_{4} \cos \theta \\
c_{13} &= c_{4} \sin \theta + c_{7} \cos \theta
\end{align*}
\]