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Analysis and prediction potential of new design formulas for granular filters

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Analysis and prediction potential of new design formulas for granular filters

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ABSTRACT:
This report deals with the load at the interface between a filter and a base, as caused by the flow in an open channel. The load is characterized by the shear stress parallel to the average direction of the flow. New design formulas for granular filters have been developed applying a by Koenders derived equation for the shear stress at the interface between top layer material and base material. The derived equation by Koenders has been examined critically with respect to assumptions and boundary conditions. Finally, the prediction potential of the new filter formulas has been determined.
The main conclusion is that the new formulas can not be recommended, because the driving force, e.g. the hydraulic gradient in the flow above the filter, is neglected at the interface to layer/base material. Therefore, it is recommended to derive a new shear stress equation taking into account the hydraulic gradient.

Reference
order AK 315004302 dated November 16, 1998
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A Munich note of Koenders

B Derivation Munich-formulas
List of Symbols

A \ [s^{-1}] \quad \text{laminar term in the force in the filter}

a \ [s/m] \quad \text{coefficient in Forchheimer equation}

B \ [m^{-1}] \quad \text{turbulent term in the force in the filter}

b \ [s^2/m^2] \quad \text{coefficient in Forchheimer equation}

C \ [m^{-9/2}] \quad \text{Chézy-coefficient}

C_0 \ [-] \quad \text{damping coefficient in B/K-formula}

c_{a_0}, c_b \ [-] \quad \text{constants in Forchheimer equation}

c_1 \ldots c_d \ [-] \quad \text{coefficients in } C_0

c_{o_0}, c_7 \ [-] \quad \text{coefficients in Forchheimer equation}

D \ [m] \quad \text{thickness granular layer}

D_b \ [m] \quad \text{diameter basematerial}

D_{bx} \ [m] \quad \text{diameter base material exceeded by (100-x)\% (weight percentages)}

D_f \ [m] \quad \text{diameter filter material}

D_{fx} \ [m] \quad \text{diameter filter material exceeded by (100-x)\% (weight percentages)}

D_t \ [m] \quad \text{diameter toplayer material}

D_{tx} \ [m] \quad \text{diameter toplayer material exceeded by (100-x)\% (weight percentages)}

D_{50cr} \ [m] \quad \text{critical diameter toplayer material}

c \ [-] \quad \text{parameter in the B/K-formula}

F \ [N/m^3] \quad \text{force per unit volume}

g \ [m/s^2] \quad \text{gravity acceleration}

i \ [-] \quad \text{energy gradient}

n \ [-] \quad \text{pososity}

R \ [m] \quad \text{largest eddy or hydraulic radius}

r \ [-] \quad \text{discrepancy ratio}

u \ [m/s] \quad \text{flow velocity}

u_i \ [m/s] \quad \text{flow velocity at the interface}

u_0 \ [m/s] \quad \text{depth-averaged flow velocity}

u_{base} \ [m/s] \quad \text{flow velocity in a granular layer not influenced by the velocity in the flow}

u_{50c} \ [m/s] \quad \text{critical flow velocity}

u(z) \ [m/s] \quad \text{flow velocity at a depth } z

x \ [m] \quad \text{longitudinal coordinate}

x_m \ [m] \quad \text{prediction potential}

y \ [m] \quad \text{transversal coordinate}

z \ [m] \quad \text{depth from the transition of flow and top layer}

\alpha \ [-] \quad \text{damping coefficient}
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Unit</th>
<th>Description</th>
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<tr>
<td>$\alpha_c$</td>
<td>[-]</td>
<td>ratio between standard deviation and average value of strength (grading factor)</td>
</tr>
<tr>
<td>$\alpha_{c,b}$</td>
<td>[-]</td>
<td>ratio between standard deviation and average value of strength for base material (grading factor)</td>
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<tr>
<td>$\alpha_{c,t}$</td>
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<tr>
<td>$\alpha_l$</td>
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<td>$\beta$</td>
<td>[m$^{-1}$]</td>
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<td>[m$^2$/s]</td>
<td>laminar contribution to kinematic turbulence</td>
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<tr>
<td>$\nu_t$</td>
<td>[m$^2$/s]</td>
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<td>[N/m$^2$]</td>
<td>critical value strength according to Grass</td>
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<td>[N/m$^2$]</td>
<td>mean value load</td>
</tr>
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<td>[N/m$^2$]</td>
<td>characteristic mean value strength at the transition</td>
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<tr>
<td>$\tau_{0,k,\text{transition}}$</td>
<td>[N/m$^2$]</td>
<td>characteristic mean value load at the transition</td>
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<td>$\tau_{\text{base}}$</td>
<td>[N/m$^2$]</td>
<td>characteristic value at a depth $z$</td>
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<td>[N/m$^2$]</td>
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<tr>
<td>$\psi_t$</td>
<td>[-]</td>
<td>Shields parameter for top layer material</td>
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1 Introduction

1.1 Reason of the study

WL Delft Hydraulics has been commissioned by the Road and Hydraulic Engineering Division of Rijkswaterstaat (order AK 315004302 dated November 16, 1998) to evaluate design formulas for granular filters on the basis of a shear stress equation developed by Koenders. The evaluation has to be done in the same manner as in a recent study with respect to the Bakker/Konter-formula and the Hoffmans/Grass-formula (Verheij, 1998).

The project has been carried out by WL Delft Hydraulics with technical assistance of Delft Geotechnics. Ir. H.J. Verheij of WL Delft Hydraulics carried out the study and was in charge of the project management; Dr. H. den Adel of Delft Geotechnics provided technical contributions. Dr. G.J.C.M. Hoffmans was the representative of the Road and Hydraulic Engineering Division of Rijkswaterstaat.

1.2 Objective and approach

Koenders (1998) solved within the framework of the Filter and Erosion Research Club FERC, a second order differential equation for the stress stress in a granular filter (see Appendix A). With some assumptions and boundary conditions the result at the filter/base interface z = -d it results in the equation:

$$\tau(-d) = \tau_0 \frac{2e^{2d}}{e^{2d} + 1}$$  \hspace{1cm} (1)

It is assumed that the derivation of this equation for the shear stress is correct taking into account the boundary conditions and various assumptions.

The equation will be applied by developing new design formulas for granular filters.

The objectives of the study are:
1. To consider critically the derived shear stress equation with respect to assumptions, boundary conditions, etcetera,
2. To develop filter design formulas comparable with the Bakker/Konter and Hoffmans/Grass formula,
3. To determine the prediction potential of the new formulas.

The afore-mentioned objectives will be the items of the successive chapters respectively. In the last chapter conclusions will be presented.
2 Analysis of the derived formula for shear stresses in filters

The Navier-Stokes equation for uniform flow in filters reads:

\[ \rho v_t \frac{\partial^2 u}{\partial z^2} + F + \rho g i = 0 \]  \hspace{1cm} (2)

With \( F = -\rho \left( Au + Bu_i^2 \right) \) \hspace{1cm} (3)
and \( A = \frac{c_b u_t}{D_i^2} \) and \( B = \frac{c_b}{D_f} \) follows:

\[ \rho v_t \frac{\partial^2 u}{\partial z^2} - \rho Au - \rho Bu_i^2 + \rho g i = 0 \] \hspace{1cm} (4)

The shear stress is related to the flow velocity by the Boussinesq hypothesis of which the full equation reads:

\[ \tau = \rho (v + v_t) \frac{\partial u}{\partial z} \] \hspace{1cm} (5)

Assuming the laminar contribution is much smaller than the turbulent term \( v_t \) (or the eddy viscosity) and subsequently may be neglected, the Boussinesq hypothesis may be simplified to:

\[ \tau = \varepsilon \frac{\partial u}{\partial z} \text{ with } \varepsilon = \rho v_t \] \hspace{1cm} (6)

Also neglecting the term \( \rho g i \), thus assuming a situation without a driving force, equation (4) can be changed into:

\[ \frac{\partial \tau}{\partial z} - \rho A u - B \rho u_i^2 = 0 \] \hspace{1cm} (7)

Let \( v_i \) be proportional to \( u \) itself: \( v_i = \alpha_i u \) (case II of Koenders), then:

\[ \tau = \alpha_i \rho u \frac{\partial u}{\partial z} \] \hspace{1cm} (8)

Substituting eq.(8) into eq.(7) and after solving the differential equation with boundary conditions at the interfaces (\( \tau = \tau_0 \) at the flow/top layer, \( u = 0 \) at the top layer/base material, and assuming turbulent conditions e.g. \( A = 0 \)) Koenders find an equation describing the value of the shear stress as a function of the depth \( z \):

\[ \tau(z) = \frac{\tau_0 e^{-\lambda z} \left( e^{2\lambda z} + 1 \right)}{e^{2\lambda z} + 1} \text{ with } \lambda = \frac{2B}{\sqrt{\alpha_i}} \] \hspace{1cm} (9)

With \( B = \frac{c_b}{D_f} \) follows:

\[ \lambda = \frac{2c_b}{\sqrt{\alpha_i D_f}} \] \hspace{1cm} (10)
Eq.(9) implies that the shear stress diminishes with increasing values of $z$. It is assumed that the derivation of the shear stress equation by Koenders as presented above is correct.

At the interface filter-base material $z = -d$ (see Figure 2.1), thus:

$$
\tau(-d) = \tau_0 \frac{2e^{\lambda d}}{e^{2\lambda d} + 1} \quad \text{with} \quad \lambda = \frac{2c_s}{\sqrt{\alpha_D D_f}}
$$

(11)

![Figure 2.1](image)

Damping of shear stress according to Hoffmans (1997)

The term \( \frac{2e^{\lambda d}}{e^{2\lambda d} + 1} \) may also be written as:

$$
\frac{2e^{\lambda d}}{e^{2\lambda d} + 1} = \frac{1}{(e^{2\lambda d} + 1)/2e^{\lambda d}} = \frac{1}{\frac{1}{2} e^{\lambda d} + \frac{1}{2} e^{-\lambda d}} = \frac{1}{\cosh(\lambda d)} = \sec h(\lambda d)
$$

(12)

For large values of $\lambda d$ (in general $\lambda d >> 1$) the term can be written as:

$$
\frac{2e^{\lambda d}}{e^{2\lambda d} + 1} \approx 2e^{-\lambda d}
$$

which means that the shear stress will become zero. There will be no base flow as was assumed by Shimizu.

Summarizing: Koenders derived an equation for the shear stress at the interface top layer - base material with the following assumptions:

- The energy gradient $i$ which is the driving force for the water flow is neglected, although this is not necessary. Eq.(11) will also result if the term $\rho g i$ is not neglected.
- The flow velocity at the filter/base interface is assumed to be neglectable, instead of a base flow $u(z) = u_0 + (u_s - u_0) \exp[\beta z]$
• Boussinesq hypothesis with respect to the shear stress is applicable.
• Uniform flow in x direction only (neglectable flow in y and z direction).
• Turbulent flow in the filter.
• The eddy viscosity $v_i$ is proportional to the flow velocity $u$.

The presence of the term $\rho gi$ is essential, because it is responsible for the average flow, not only in the water above the filter but also inside the filter. It creates an extra driving force which is independent of the depth and responsible for the turbulent flow conditions in the filter near the interface. In reality, the laminar term $\rho Au$ will never become dominant, which may occur when neglecting the term $\rho gi$. 
3 Development of new design formulas

3.1 Modified Bakker/Konter formula

In Verheij (1998) the Bakker/Konter-formula, B/K-formula for short, in the 1997-version (de Groot et al, 1997) is presented:

\[
\frac{D_{1/5}}{D_{50}} = \frac{2.2 \psi_k \Delta_b}{\epsilon^2 C_0 \psi_i \Delta_i D_{150}} \frac{R}{D_{150}} \quad (13)
\]

\[
C_0 = c_1 \left[ 1 + c_2 \exp \left( -c_3 \frac{d}{D_{150}} \right) \right] \left( \frac{R}{D_{150}} \right)^{c_4} \quad (14)
\]

\[c_1 = 3.3 \]
\[c_2 = 0.8 \quad (0.5 \text{ to } 1.0)\]
\[c_3 = 0.8 \quad (0.5 \text{ to } 1.0)\]
\[c_4 = 0.25 \quad (0.2 \text{ to } 0.3)\]

The expression \(c_2 \exp \left( -c_3 \frac{d}{D_{150}} \right)\) is responsible for the damping of the load as function of the depth in the filter layer. Thus the B/K-formula can be changed simply into a B/K-modified formula by substituting equation (11) into the equation for \(C_0\) replacing the exponential term, resulting in:

\[
C_0 = c_1 \left[ 1 + c_2 \left( \frac{2e^{2d}}{e^{2d} + 1} \right) \left( \frac{R}{D_{150}} \right)^{c_4} \right] \quad (15)
\]

Note that \(\exp \left( -c_3 \frac{d}{D_{150}} \right)\) is based on the reduction of the flow velocity in the filter, whereas eq.(15) is based on reduction of the shear stress. Since \(c_3\) is determined by fitting the experimental data, there is no error introduced by assuming the damping of the velocity instead of the shear stress.

3.2 Munich-formulas

The Hoffmans/Grass approach takes into account statistical distributions of load and strength. The same approach as presented in Verheij (1998) will be followed for the next derivation. Stability of load and strength at the interface between filter and base material reads as:

\[
\tau_{0, \text{transition}} = \tau_{c, \text{transition}} \quad (16)
\]
Two approaches will be followed: Munich-approach with shear stress defined as:

$$\tau(-d) = \tau_0 \frac{2e^{ld}}{e^{2ld} + 1}$$  \hspace{1cm} (17)$$

and a modified Munich-approach with a shear stress that takes into account a base term (see Appendix B):

$$\tau(-d) = \tau_0 \frac{2e^{ld}}{e^{2ld} + 1} + \tau_{\text{base}} \left(1 - \frac{2e^{ld}}{e^{2ld} + 1}\right)$$  \hspace{1cm} (18)$$

with $\tau_{\text{base}} = \eta \tau_0$

Eq.(18) assumes that there is no interaction between the shear stress of the base term and the usual term as derived by Koenders. This type of assumption is usually valid for linear phenomena like elastic resposns or laminar flow conditions. For non-linear phenomena this type of assumption is void.

Shortly, both approaches will be discussed. Details are presented in the Appendix B.

**Munich-approach**

Eq.(17) will be used to estimate the load at the interface of filter and base material.

Applying Grass' concept, so replacing $\tau_0$ by $\tau_0 + \gamma_t \sigma_0$ (see Figure 2.1) and substituting $\tau_{0,k,\text{transition}}$ for $\tau(-d)$ eq.(17) can be written as:

$$\tau_{0,k,\text{transition}} = (\tau_0 + \gamma_t \sigma_0) \frac{2e^{ld}}{e^{2ld} + 1}$$  \hspace{1cm} (19)$$

The strenght of the base material may be written as:

$$\tau_{c,k,\text{transition}} = \tau_{c,G} - \gamma_b \sigma_{c,b} = \tau_{c,G} - \gamma_b \alpha c,b \tau_{c,G} = \tau_{c,G} (1 - \gamma_b \alpha c,b) = \psi_{c,G,b} \Lambda b \rho g D_{50} (1 - \gamma_b \alpha c,b)$$  \hspace{1cm} (20)$$

Substituting (19) and (20) into (16) re-arranging and assuming $\alpha_{c,b} = \alpha_c$ and $\gamma_b = \gamma$ results into:

$$\frac{D_{f15}}{D_{50}} = \frac{D_{f15} e^{2ld} + 1 \psi_{c,G,b} \Lambda b}{D_{f50} 2e^{ld} \psi_{c,G,i} \Lambda i}$$  \hspace{1cm} (21)$$

This equation is a general formula with the implicit assumptions of Gaussian distributions of load and strength, and a relationship based on uniform flow.

Note that this expression does not depend on $\alpha_c$ and so this relation is independent of the gradation of either toplayer or base material as long as these gradations are the same or nearly the same.
modified Munich-approach

Similar as above the following equation can be derived for the situation with a base term:

\[
\tau_{0,k,\text{transition}} = \eta \tau_0 + (\tau_0 + \gamma_c\sigma_0 - \eta \tau_0) \frac{2e^{2d}}{e^{2d} + 1}
\]  

(22)

Substitution of eq.(22) into (16) together with (20) and re-arranging results in:

\[
\frac{D_{f/15}}{D_{50}} = \frac{D_{f/15}}{D_{50}} \frac{1 + \alpha_0\gamma_t}{\eta + (1 + \alpha_0\gamma_t - \eta) \frac{2e^{2d}}{e^{2d} + 1}} \frac{\psi_{e,G,b} \Lambda_b}{\psi_{e,G,a} \Lambda_t}
\]  

(23)

Note that now a base term for the shear stress is introduced into (22), while in the derivation of the term \( \frac{2e^{2d}}{e^{2d} + 1} \) it was assumed that there was no base flow.
4 Prediction potential

4.1 Value of $\lambda d$

For the coefficients $c_a$ and $c_b$ the following values may be derived. Therefore, eq.(3) may also be written as:

$$F = -\rho g (au + bu^2)$$  \hspace{1cm} (24)

with $a = \frac{c_0 D_f^2 (1 - n)}{n^3 g D_f^{15}}$ \hspace{1cm} with \hspace{0.5cm} c_0 = 160

and $b = \frac{c_7}{n^2 g D_f^{15}}$ \hspace{1cm} with \hspace{0.5cm} c_7 = 2.2

Comparing the equations (3) and (24) results in $A = g.a$ and $B = g.b$ and after substituting the relevant equations for $a$, $A$, $b$ and $B$ results in:

$$c_a = c_0 \frac{(1 - n)^2}{n^3} \hspace{1cm} with \hspace{0.5cm} c_0 = 160 \hspace{1cm} (25)$$

$$c_b = \frac{c_7}{n^2} \hspace{1cm} with \hspace{0.5cm} c_7 = 2.2$$

This results in:

$$\tau(-d) = \tau_0 \frac{2e^{2d}}{e^{2d} + 1} \hspace{1cm} with \hspace{0.5cm} \lambda = \sqrt{\frac{2c_7}{\alpha_t n^2 D_f}} \hspace{1cm} (26)$$

Eq.(26) results for the term $\lambda d$ into:

$$\lambda d = \sqrt{\frac{2d^2 c_7}{\alpha_t n^2 D_f^{15}}} \hspace{1cm} (27)$$

Den Adel (1995) relates $\epsilon$ to $u$ according: $\epsilon = \alpha D_f^{15} \rho u$ (in analogy with Shimizu).

Comparing this with the applied relation by Koenders (see eq.(8)): $\epsilon = \alpha_t \rho u$ it may be concluded that: $\alpha_t = \alpha D_f^{15}$

Substituting $\alpha D_f$ in eq.(27) instead of $\alpha_t$ results in:

$$\lambda d = \frac{d}{D_f^{15}} \sqrt{\frac{2c_7}{\alpha n^2}} \hspace{1cm} (28)$$

Now a comparison can be made of eq.(26) with (Den Adel, 1995):
\[ u(z) = (u_z - u_0) e^{\alpha z} \] with \[ \beta = \sqrt{\frac{bg}{2\alpha D_{f15}}} = \frac{1}{D_{f15}} \sqrt{\frac{1,1}{\alpha n^2}} \] (29)

If \( \lambda d > 2 \), then eq.(26) can be simplified to: \( \tau(-d) = \tau_0 2e^{-2\lambda d} \) or

\[ \frac{\tau(-d)}{\tau_0} = 2e^{-\lambda d} = 2 \exp \left[ -\frac{d}{D_{f15}} \sqrt{\frac{4,4}{\alpha n^2}} \right] = 2 \exp \left[ -\frac{2d}{D_{f15}} \sqrt{\frac{1,1}{\alpha n^2}} \right] \] (30)

whereas (29) reads:

\[ \frac{u(-d)}{u_z - u_0} = e^{-\beta d} = \exp \left[ -\frac{d}{D_{f15}} \sqrt{\frac{1,1}{\alpha n^2}} \right] \] (31)

Note that the same values of \( \alpha \) are used. Both exponents differ a factor 2. This is no error or mistake, since Koenders assumes eq.(8), \( \tau \) will be proportional to \( u^2 \).

Den Adel also estimated a value of 0.9 for \( \alpha \). Substituting this value into the equation for \( \lambda d \) together with \( c_7 = 2.2 \) and \( n = 0.4 \) gives:

\[ \lambda d = 5.53 \frac{d}{D_{f15}} \] (32)

Suppose \( d/D_{f15} = 3 \) (\( d/D_{f0} = 2 \) and \( D_{f0}/D_{f15} = 1.5 \)) then it follows:

- Munich-approach: \( \tau(-d) = \tau_0 \frac{2e^{\lambda d}}{e^{2\lambda d} + 1} = 0.000000125 \tau_0 \)

- modified Munich-approach:

\[ \tau(-d) = \tau_0 \frac{2e^{\lambda d}}{e^{2\lambda d} + 1} + 0.007 \tau_0 = 0.000000125 \tau_0 + 0.007 \tau_0 = 0.007000125 \tau_0 \]

For the Hoffmans/Grass-formula can be derived:

\[ \tau(-d) = \tau_0 e^{-3.7d/D_{f15}} + \text{base term} = 0.0006 \tau_0 + \text{base term} \]

with \( \text{base term} = \tau_{\text{base}} \left( 1 - e^{-3.7d/D_{f15}} \right) = \eta \tau_0 \left( 1 - e^{-3.7d/D_{f15}} \right) \) with \( \eta = 0.007 \)

resulting in: \( \tau(-d) = \tau_0 (0.0006 + 0.007) = 0.0076 \tau_0 \)

The result for Shimizu with \( \alpha = 0.9 \) reads:

\[ \tau(-d) = \tau_0 e^{-\beta d} = \tau_0 e^{-\left[ \frac{1,1}{\alpha n^2} \right] \frac{d}{D_{f15}}} = 0.00025 \tau_0 \]

Finally, the original Bakker/Konter-formula (see eq.(14)): 
\[ \tau(-d) = \tau_0 e^{-0.8d/D_{150}} = 0.202 \tau_0 \]

It can be concluded that the differences are large; the absence of the base term is the reason.

4.2 Accuracy of the new formulas

In this section values predicted with the modified Bakker/Konter formula (eq.13 and 15) and the two Munich-formulas (eq.21 and 23 respectively) will be compared with experimental results as mentioned in Van Huijstee et al (1991) as far as they concern simultaneous instability of filter and base material. The results are obtained with \( \gamma = 0.625, \eta = 0.007 \) and \( \alpha_0 = 0.4 \). With respect to the term \( \lambda d \) calculations are carried out with:

\[ \lambda d = 5.5 \frac{d}{D_{15}} \]

The predicted results with respect to \( D_{15}/D_{650} \) are given in Table 4.1 and partly presented in Figures 4.1 and 4.2. The table and figures clearly indicate that the modified Bakker/Konter-formula as well as the modified Munich-formula predict the measured values within a range of 1/3 to 3 times the line of perfect agreement. This result is not achieved for the Munich-formula, and therefore the results are not shown in a figure.

It should be noted that the values calculated with the Bakker/Konter-formula are based on the values of the coefficients mentioned in Section 3, which were originally the result of a fit carried out by Den Adel (1995). This implies that the prediction accuracy may be less.
<table>
<thead>
<tr>
<th>Experiment</th>
<th>R [m]</th>
<th>d [mm]</th>
<th>D_{450} [mm]</th>
<th>D_{115} [mm]</th>
<th>D_{750} [mm]</th>
<th>Ψ_a [-]</th>
<th>Ψ_f [-]</th>
<th>D_{750,c} [mm]</th>
<th>u_{4c} [m/s]</th>
<th>D_{115}/D_{750}</th>
<th>measured</th>
<th>BK-mod</th>
<th>Munich</th>
<th>Mun-mod</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q572 - T2</td>
<td>0.4</td>
<td>45</td>
<td>0.151</td>
<td>20</td>
<td>30</td>
<td>0.047</td>
<td>0.035</td>
<td>29</td>
<td>1.4</td>
<td>132</td>
<td>108</td>
<td>106000</td>
<td>160</td>
<td></td>
</tr>
<tr>
<td>Q572 - T3</td>
<td>0.2</td>
<td>45</td>
<td>0.151</td>
<td>20</td>
<td>30</td>
<td>0.047</td>
<td>0.035</td>
<td>25</td>
<td>1.1</td>
<td>132</td>
<td>64</td>
<td>106000</td>
<td>160</td>
<td></td>
</tr>
<tr>
<td>Q572 - T6a</td>
<td>0.4</td>
<td>30</td>
<td>0.151</td>
<td>15</td>
<td>20</td>
<td>0.047</td>
<td>0.035</td>
<td>17</td>
<td>1.15</td>
<td>99</td>
<td>147</td>
<td>30150</td>
<td>179</td>
<td></td>
</tr>
<tr>
<td>Q572 - T6b</td>
<td>0.4</td>
<td>30</td>
<td>0.285</td>
<td>15</td>
<td>20</td>
<td>0.028</td>
<td>0.035</td>
<td>17</td>
<td>1.25</td>
<td>53</td>
<td>88</td>
<td>17960</td>
<td>107</td>
<td></td>
</tr>
<tr>
<td>Q572 - T9</td>
<td>0.4</td>
<td>30</td>
<td>0.285</td>
<td>15</td>
<td>20</td>
<td>0.028</td>
<td>0.035</td>
<td>10</td>
<td>0.9</td>
<td>53</td>
<td>88</td>
<td>17960</td>
<td>107</td>
<td></td>
</tr>
<tr>
<td>Q572 - T11</td>
<td>0.4</td>
<td>40</td>
<td>0.151</td>
<td>15</td>
<td>20</td>
<td>0.047</td>
<td>0.035</td>
<td>18</td>
<td>1.2</td>
<td>99</td>
<td>147</td>
<td>1179540</td>
<td>180</td>
<td></td>
</tr>
</tbody>
</table>

| M633 - q   | 1     | 150    | 1.25       | 50          | 83          | 0.022   | 0.035   | 115            | 2.7          | 40             | 47       | 2773800 | 68     |
| M1012 - T2 | 1.25  | 150    | 0.14       | 21          | 34          | 0.043   | 0.035   | 65             | 2.6          | 150            | 212      | 136     |        |
| M1012 - T3 | 1.25  | 150    | 0.14       | 30          | 38          | 0.043   | 0.035   | 80             | 2.8          | 214            | 195      | 173     |        |

Explanation:
BK-mod = Bakker/Konter modified
Munich = Munich-formula
Mun-mod = Munich-formula modified

1) estimated with $D_{750,c} = \frac{u_{4c}^2}{C^2\Psi_f}$ (C calculated with $D_{750}$)
Comparison Bakker/Konder-modified with measurements

Figure 4.1 Comparison of measured data and calculated data with the modified Bakker/Konder-formula

Comparison Munich-modified with measurements

Figure 4.2 Comparison of measured data and calculated data with the modified Munich-formula
In order to make the prediction potential $x_m$ (defined as the ratio of the sum of the calculated values to the sum of the measured values) and the variance of the results more clear, also the discrepancy ratio $r$ (defined as the ratio of the number of figures within the interval to those outside; see Figure 4.3) are determined, because a first order moment does not give an indication of the variance of the results, and therefore, a percentage has been calculated representing the number of results within a defined area. The upper and lower limit of this area are determined arbitrarily.

![Graph showing discrepancy ratio](image)

**Figure 4.3** Definition discrepancy ratio

The results are shown in Table 4.2 (for comparison the results for the Hoffmans/Grass-formula are also shown).

<table>
<thead>
<tr>
<th>predictor</th>
<th>$x_m$</th>
<th>$1/3&lt;r&lt;3$</th>
<th>$1/2&lt;r&lt;2$</th>
<th>$2/3&lt;r&lt;3/2$</th>
<th>$3/4&lt;r&lt;4/3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>B/K-modified</td>
<td>1.13</td>
<td>100%</td>
<td>67%</td>
<td>67%</td>
<td>33%</td>
</tr>
<tr>
<td>Munich</td>
<td>$4.5 \times 10^{13}$</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>mod. Munich</td>
<td>1.30</td>
<td>100%</td>
<td>78%</td>
<td>44%</td>
<td>44%</td>
</tr>
<tr>
<td>H/G-formula</td>
<td>0.98</td>
<td>100%</td>
<td>100%</td>
<td>89%</td>
<td>56%</td>
</tr>
</tbody>
</table>

Table 4.2 Prediction potential for data in Table 4.1

Considering the results in Table 4.2 the prediction potential of the three new formulas is not better and the variance larger than the Hoffmans/Grass-formula. With respect to Munich-formula the reason is the absence of the base term. This effect became already clear with the value of the shear stress at the interface $0.125 \times 10^{-6}$.

In the modified Munich-formula the base term has been introduced later (see eq.(22)), but putting a base term in may be considered as artificial because it does not play a role in the derivation by Koenders of the equation for the decreasing shear stress term.

The base term plays an implicit role in the modified Bakker/Konter-formula in which coefficients were fitted to original test results.

Summarizing, the new formulas will not be recommended.
5 Conclusions

The results of this evaluation of new design formulas for granular filters based on a new shear stress equation derived by Koenders allow the following conclusions:

- Neglecting the base term, e.g. the flow in the granular top layer, is not allowed.
- The prediction potential of both Munich-formulas and a modified Bakker/Konter-formula using the shear stress equation as derived by Koenders is less than the prediction potential of the Hoffmans/Grass-formula.

It is recommended to derive an equation for the shear stress that takes into account the base flow in the filter, thus without neglecting the driving force $\rho gi$, but also the laminar flow term $\rho Au$. 
References


Consider a base layer with a filter layer on top of it. Water flows (turbulently) over the filter layer. The problem is specified by the average velocity $U$ and the fluctuations in the water velocity at the top of the filter layer.

Simplifying assumption: horizontal set-up.

Horizontal force equilibrium of a box in the filter layer:

\[(\tau(y + dy) - \tau(y))dx dz + (p(x + dx) - p(x))dy dz + \text{body force} = 0\]

Because of the assumption of a horizontal flow, the body force vanishes. Thus:

\[\frac{\partial \tau}{\partial y} + \frac{\partial p}{\partial x} = 0\]

Using Forcheimer:

\[\frac{\partial \tau}{\partial y} - Av - Bv^2 = 0\]
The shear stress must therefore increase with $y$, that is decrease with depth. Now put forward the hypothesis that:

$$\tau = \varepsilon \frac{\partial v}{\partial y}$$

The dimension of $\varepsilon$ is the same as a viscosity.

The boundary conditions of the problem are:

1. Shearing stress $\tau$ given at the top of the filter layer: $\tau(0) = \tau_0$
2. Velocity at the filter/base interface is approximately zero: $v(-H) = 0$

**Case I:** let the fluid viscosity be $\mu$, then the laminar expression for $\varepsilon$ is:

$$\varepsilon = \alpha_0 \mu$$

where $\alpha_0$ is a constant.

Thus:

$$\alpha_0 \frac{\partial^2 v}{\partial y^2} - Av - Bv^2 = 0 \quad \Rightarrow \quad \frac{\alpha_0}{2} \left( \frac{\partial v}{\partial y} \right)^2 - \frac{A}{2} v^2 - \frac{B}{3} v^3 = \text{const} \tan t$$

Or:

$$\int_0^v \frac{dv}{\left( c + \frac{A}{2} v^2 + \frac{B}{3} v^3 \right)^{1/2}} = \pm \frac{2}{\alpha_0} (y + H)$$

**Case II:** let $\varepsilon$ be proportional to $v$ itself:

$$\varepsilon = \rho \alpha_1 v$$

where $\alpha_1$ is a lengthscale.

Thus:

$$\frac{\alpha_1 \rho}{2} \frac{\partial^2 v}{\partial y^2} - Av - Bv^2 = 0 \quad \Rightarrow \quad \frac{\alpha_1 \rho}{4} \left( \frac{\partial v^2}{\partial y} \right)^2 - \frac{2A}{3} v^3 - \frac{B}{2} v^4 = \text{const} \tan t$$

Or:

$$\int_0^v \frac{dw}{\sqrt{\frac{2A}{3} w^{3/2} + \frac{B}{2} w + c}} = \pm \frac{2}{\sqrt{\alpha_1 \rho}} (y + H)$$

**Case III:** let $\varepsilon$ be proportional to a (constant) average velocity $V$ itself:

This is the same as Case I, but now $c = \rho \alpha_3 V$, with $\alpha_3$ a lengthscale.
Look at a typical solution. Take Case II and neglect the laminar term in the permeability $A \to 0$. Then:

$$\tau(y) = \frac{\tau_0 e^{-\lambda y \left( e^{2\lambda (y+H)} + 1 \right)}}{e^{2\lambda H} + 1} \quad \text{with} \quad \lambda = \sqrt{\frac{2B}{\alpha_1 \rho}}$$

For erosion stability the shear stress at $y = -H$ is relevant:

$$\tau(-H) = \frac{2\tau_0 e^{\lambda H}}{e^{2\lambda H} + 1}$$

**A few notes:**

The approximation $A \to 0$ is not correct near the base/filter interface, exactly where one wants to know the shearing stress.

The elementary solution obtained here is not present in the literature, though the same approximations are often made.

*All this shows that considerably more development work needs to be carried out.*

It is my ambition to calculate Case II and Case III with no approximations. The integrals are known in terms of Elliptical Functions.
B Derivation Munich-formulas

Munich-approach

At the interface filter/base material equilibrium is assumed of load and strenght:

\[
\tau_{0,k,\text{transition}} = \tau_{c,k,\text{transition}} \quad (B.1)
\]

The Koenders equation reads:

\[
\tau(-d) = \tau_0 \frac{2 e^{2d}}{e^{32d} + 1} \quad (B.2)
\]

which can be written by applying Grass' concept, so replacing \(\tau_0\) by \(\tau_0 + \gamma'_i \sigma_0\) and substituting \(\tau_{0,k,\text{transition}}\) for \(\tau(-d)\) as:

\[
\tau_{0,k,\text{transition}} = (\tau_0 + \gamma'_i \sigma_0) \frac{2 e^{2d}}{e^{32d} + 1} \quad (B.3)
\]

The strength of the base material may be written as:

\[
\tau_{c,k,\text{transition}} = \tau_{c,G} - \gamma_b \sigma_{c,b} = \tau_{c,G} - \gamma_b \alpha_{c,b} \tau_{c,G} = \tau_{c,G} (1 - \gamma_b \alpha_{c,b}) = \psi_{c,G,b} \rho g D_{b50} (1 - \gamma_b \alpha_{c,b}) \quad (B.4)
\]

Substituting (B.3) and (B.4) into (B.1) results in:

\[
(\tau_0 + \gamma'_i \sigma_0) \left( \frac{2 e^{2d}}{e^{32d} + 1} \right) = \psi_{c,G,b} \rho g \Delta_b D_{b50} (1 - \gamma_b \alpha_{c,b}) \quad (B.5)
\]

and after re-arranging:

\[
\frac{1}{D_{b50}} = \psi_{c,G,b} \rho g \Delta_b (1 - \gamma_b \alpha_{c,b}) \frac{1}{(\tau_0 + \gamma'_i \sigma_0) \left[ \frac{2 e^{2d}}{e^{32d} + 1} \right]} \quad (B.6)
\]

Multiplying both sides with \(D_{50}\):

\[
\frac{D_{150}}{D_{b50}} = D_{50} \psi_{c,G,b} \rho g \Delta_b (1 - \gamma_b \alpha_{c,b}) \frac{1}{(\tau_0 + \gamma'_i \sigma_0) \left[ \frac{2 e^{2d}}{e^{32d} + 1} \right]} \quad (B.7)
\]

Substituting for \(D_{150}\):
\[ \Delta_t D_{50} = \frac{\tau_0 + \gamma_t \sigma_0}{\psi_{c,G,t} \rho \gamma_t (1 - \alpha_c \gamma_t)} \]  

(B.8)

at the right hand side of (B.8) and performing some re-arranging, finally results in an explicit general relationship:

\[ \frac{D_{150}}{D_{b50}} = \frac{e^{2\gamma_d} + 1}{2e^{2\gamma_d}} \frac{\psi_{c,G,b} \Delta_b}{1 - \gamma_t \alpha_c \Delta_t} \]  

(B.9)

This equation is a general formula with the implicit assumptions of Gaussian distributions of load and strength, and a relationship based on uniform flow.

Assuming \( \alpha_c, b = \alpha_c, t \) and \( \gamma_b = \gamma_t \) results in:

\[ \frac{D_{150}}{D_{b50}} = \frac{e^{2\gamma_d} + 1}{2e^{2\gamma_d}} \frac{\psi_{c,G,b} \Delta_b}{\psi_{c,G,t} \Delta_t} \]  

(B.10)

Note that this expression does not depend on \( \alpha \) and so this relation is independent of the gradation of either toplayer or base material as long as these gradations are the same or nearly the same.

Formula (B.10) can easily be changed by multiplying both sides with \( D_{150}/D_{b50} \) in:

\[ \frac{D_{150}}{D_{b50}} = \frac{D_{115}}{D_{b50}} \frac{e^{2\gamma_d} + 1}{2e^{2\gamma_d}} \frac{\psi_{c,G,b} \Delta_b}{\psi_{c,G,t} \Delta_t} \]  

(B.11)

For a one layer system (toplayer directly on base material) the subscript \( t \) may be replaced by \( f \) resulting finally in the new H/G-like formula:

\[ \frac{D_{f15}}{D_{b50}} = \frac{D_{f15}}{D_{b50}} \frac{e^{2\gamma_d} + 1}{2e^{2\gamma_d}} \frac{\psi_{c,G,b} \Delta_b}{\psi_{c,G,t} \Delta_t} \]  

(B.12)

modified Munich-approach

Now a base term is assumed (as was done earlier by Hoffmans, 1997; see Figure 2.1):

\[ \tau(-d) = \tau_{base} + \left( \tau_0 - \tau_{base} \right) \frac{2e^{2\gamma_d}}{e^{2\gamma_d} + 1} \]  

(B.13)

or

\[ \tau(-d) = \tau_0 \frac{2e^{2\gamma_d}}{e^{2\gamma_d} + 1} + \tau_{base} \left( 1 - \frac{2e^{2\gamma_d}}{e^{2\gamma_d} + 1} \right) \]  

(B.14)

with \( \tau_{base} = \eta \tau_0 \)
Applying the Grass concept: \( \tau_0 = \tau_0 + \gamma, \sigma_0 \) and \( \tau_{0, k, \text{transition}} = \tau(-d) \):

\[
\tau_{0, k, \text{transition}} = \eta \tau_0 + (\tau_0 + \gamma, \sigma_0 - \eta \tau_0) \frac{2e^{2d}}{e^{2d} + 1} \tag{B.15}
\]

and subsequently:

\[
\frac{D_{150}}{D_{50}} = \frac{D_{150}}{\psi_{e,G,b} \rho_b \Delta_b \left(1 - \gamma, \alpha_{e,b}\right)} \frac{1}{\eta \tau_0 + (\tau_0 + \gamma, \sigma_0 - \eta \tau_0) \frac{2e^{2d}}{e^{2d} + 1}} \psi_{e,G,b} \Delta_b \tag{B.16}
\]

Substituting \( D_{50} \) (see equation B.8) gives:

\[
\frac{D_{150}}{D_{50}} = \frac{\tau_0 + \gamma, \sigma_0}{\eta \tau_0 + (\tau_0 + \gamma, \sigma_0 - \eta \tau_0) \frac{2e^{2d}}{e^{2d} + 1}} \frac{1 - \gamma, \alpha_{e,b} \psi_{e,G,b} \Delta_b}{\psi_{e,G,t} \Delta_t} \tag{B.17}
\]

This equation is a general formula with the implicit assumptions of Gaussian distributions of load and strength, and a relationship based on uniform flow.

Assuming \( \sigma_0 = \alpha_0 \tau_0 \) and assuming \( \alpha_{e,b} = \alpha_{e,t} \) and \( \gamma_b = \gamma_t \) results in:

\[
\frac{D_{150}}{D_{50}} = \frac{1 + \gamma, \alpha_0}{\eta + (1 + \gamma, \alpha_0 - \eta) \frac{2e^{2d}}{e^{2d} + 1}} \frac{\psi_{e,G,b} \Delta_b}{\psi_{e,G,t} \Delta_t} \tag{B.18}
\]

Note that this expression does not depend on \( \alpha_e \) and so this relation is independent of the gradation of either toplayer or base material.

Formula (B.18) can easily be changed by multiplying both sides with \( D_{15} / D_{50} \) in:

\[
\frac{D_{15}}{D_{50}} = \frac{D_{15}}{D_{50}} \frac{1 + \gamma, \alpha_0}{\eta + (1 + \gamma, \alpha_0 - \eta) \frac{2e^{2d}}{e^{2d} + 1}} \frac{\psi_{e,G,b} \Delta_b}{\psi_{e,G,t} \Delta_t} \tag{B.19}
\]

For a one layer system (toplevel directly on base material) the subscript \( t \) may be replaced by \( f \) resulting finally in the new H/G-like formula:

\[
\frac{D_{15}}{D_{50}} = \frac{D_{15}}{D_{50}} \frac{1 + \alpha_0 \gamma_f}{\eta + (1 + \alpha_0 \gamma_f - \eta) \frac{2e^{2d}}{e^{2d} + 1}} \frac{\psi_{e,G,b} \Delta_b}{\psi_{e,G,f} \Delta_t} \tag{B.20}
\]
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