AN INITIAL-VALUE PROBLEM FOR THE MOTION OF A SHIP MOVING WITH CONSTANT MEAN VELOCITY IN AN ARBITRARY SEAWAY

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Abstract

The motion of a freely floating or submerged body, which is moving with a constant average forward speed and oscillating arbitrarily in any of the six degrees of freedom, is formulated as an initial-value problem. The seaway is assumed to be arbitrary. The body is assumed to be 'smooth', but no symmetry of the body is required. The fundamental assumption is that both the free-surface disturbance due to forward motion of the body and the oscillations are small enough so that the problem may be linearized. By an approach similar to that of Wehausen (1965), it is shown how the present treatment of the problem leads also to Ogilvie's (1965) modified results of Cummins' (1962) decomposition of the velocity potential for the case of an oscillating body with a constant average forward speed. The linearized equations of motion of the body are then derived as a set of six integro-differential equations. Existence and uniqueness theorems are not established either for the boundary-value problem or for the integral equation which is constructed.

I. Introduction

In the study of the problem of ship motions, it is desirable to be able to write down equations of motion which are valid whatever the nature of the seaway. This means that the validity of these equations should not require the forcing functions to depend sinusoidally upon time. Cummins (1962) made an important advance toward this goal by considering a certain decomposition of the velocity potential resulting from 'forced motion' with no waves present. In the present work we shall consider an initial-value problem for ship motions with forward velocity and show how Cummins' (1962) results can also be derived from this treatment of the problem.

We shall consider the motion of a freely floating or submerged body which is moving with a constant forward speed and oscillating arbitrarily (not necessarily periodically) in any of the six degrees of freedom. The body is supposed to have zero average translational and angular velocity in the oscillatory motion in some appropriate sense of the word 'average'. Essentially all that this means is that there is a definite surface moving forward at a given speed such that the oscillatory motion can be referred to this surface with only small error. The precise meaning will be explained in Chapter II.

The position and velocity of both body and the free surface are assumed to be known at some fixed instant of time which we shall take to be $t = 0$. Besides, we allow the possible presence of waves which may diffract upon the body and cause it to move. The incident waves are assumed to be 'known'. Otherwise, the nature of the seaway is supposed to be arbitrary. Our fundamental assumption is that the problem can be linearized, in the sense that the oscillations are small and that the disturbance of the free surface due to the forward motion of body is also small. In order to achieve the last requirement realistically, the body may be, for example, either thin, slender, flat, or deeply submerged. Aside from this, we require only that the form of the body be 'smooth'; no symmetry of the body is assumed.

As usual, we shall assume the fluid to be incompressible and inviscid and the motion of the fluid to be irrotational. The analytical method which is used in the present work was first introduced by Volterra (1934) for solution of certain initial-value problems for water waves and was later extended by Finkelstein (1957). Wehausen (1965) later showed how such a technique can be modified to solve a class of problems in ship motions and, in particular, how the decomposition of the velocity potential of Cummins' type may be conveniently made by this treatment. The present work is an extension of that of Wehausen (1965) to include forward motion.

For the purpose of the linearization of the boundary conditions at both the free surface and the hull-fluid interface, we keep two perturbation parameters in mind: ε_S , measuring the smallness of the free-surface disturbance caused by the forward motion, and ε_M, measuring the smallness of the oscillatory motion. However, in order to have a development which is simultaneously applicable to thin or slender ships, and to deeply submerged bodies, we do not follow the traditional scheme of linearization of introducing separate boundary-value problems for each of the order $\mathcal{E}_s$, $\mathcal{E}_M$, $\mathcal{E}_s \mathcal{E}_M$ etc., respectively. The boundary condition at the wetted hull is linearized by means of a Taylor's series expansion of the potential function, in which all the terms of orders $\mathcal{E}_s$, $\mathcal{E}_M$ and $\mathcal{E}_s \mathcal{E}_M$ are kept. The result is the condition which is also called the Timman-Newman boundary condition. A somewhat different derivation of this boundary condition is presented in Chapter II of the present work so that additional insight into the nature of the derivation may be gained. As to the free-surface boundary condition, terms of order higher than $\mathcal{E}_s \mathcal{E}_M$ are discarded, so that the traditional homogeneous free-surface boundary condition is obtained. For the justification and limitation of the applicability of such a development, we refer to the discussions in the following papers: Timman and Newman (1962), Newman (1965), and Ogilvie (1964).

The problem is first formulated for the general case of unsteady average forward speed and its appropriate integral equation for the velocity potential is obtained. However, in order that the decomposition of the velocity potential for the 'forced motion' may be conveniently made, it is necessary to assume the average forward speed to be constant. Cummins' (1962) development for the case of constant forward speed was later modified by Ogilvie (1965) in order that it should satisfy the Timman-

Newman boundary condition. Hence we refer often to Ogilvie's (1965) work, and effort is also made to preserve the same notation whenever it is convenient to do so, so that cross reference between the two may be easily made.

Uniqueness is not established either for the boundary-value problem or for the integral equation which is constructed. This may be shown in both senses if there is no forward motion. It would be desirable to establish this in the present case also. Furthermore, no existence theorems have been established for solutions of the integral equations. Thus in a certain sense the work is purely formal. However, if one is willing to concede that both uniqueness and existence should be provable, the final equations show the proper form of the linearized equations of motion and the nature of their ingredients. In particular, one should note that they are a set of six coupled integro-differential equations. We have not attempted to find any solution corresponding to a special geometry. This would be a reasonable next step.

II. Mathematical Formulation

Coordinate systems. It will be convenient to consider simultaneously three right-handed cartesian coordinate systems. Let $\bar{O}\bar{x}\bar{y}\bar{z}$ be fixed in space in such a way that $\bar{O}\bar{x}$ is in the direction of the forward motion of the body; $\bar{O}\bar{y}$ is directed oppositely to the force of gravity and the $(\bar{x}, \bar{z})$-plane coincides with the undisturbed free surface. The coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$ will be taken to be fixed in the body in such a way that when the body is at rest, the axis $\hat{O}\hat{y}$ is directed oppositely to the force of gravity with the center of gravity of the body lying on the line of $\hat{y}$-axis, $\hat{O}\hat{x}$ towards the bow and $\hat{O}\hat{z}$ to the starboard; and when the body is at rest the $(\hat{x}, \hat{z})$-plane coincides with the undisturbed free surface. Finally, we introduce the coordinate system $Oxyz$, moving at a speed equal to the average forward speed of the body such that when the body is at rest the two systems, $\hat{O}\hat{x}\hat{y}\hat{z}$ and $Oxyz$ coincide with each other. Hence the two systems, $\bar{O}\bar{x}\bar{y}\bar{z}$ and $Oxyz$ are always parallel to each other; Ox coincides with $\bar{O}\bar{x}$ and the (x, z)-plane coincides with the $(\bar{x}, \bar{z})$-plane. In particular, if we assume that at the initial instant the two systems $Oxyz$ and $\bar{O}\bar{x}\bar{y}\bar{z}$ coincide, then at any later instant t we have

$$\left.\begin{aligned} \bar{x} &= x + \int_0^t C(\tau)\,d\tau , \\ \bar{y} &= y , \\ \bar{z} &= z , \end{aligned}\right\} \qquad (1)$$

where $C(t)$ is the average translatory speed of the body.

In conformity with the assumption of the small oscillatory motion of the body, we shall assume that the displacements of the coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$ from $Oxyz$ are small.

Let (x_1, y_1, z_1) and $(\theta_1, \theta_2, \theta_3)$ describe the linear and angular displacements of $\hat{O}\hat{x}\hat{y}\hat{z}$ from $Oxyz$, where $x_1 = x_1(t)$, $y_1 = y_1(t), \cdots, \theta_3 = \theta_3(t)$. Thus they describe, respectively, the surging, heaving, sway, rolling, yawing and pitching motions of a ship. Note that at any instant t, the position of the origin $\hat{O}$ is given by (x_1, y_1, z_1) in the $Oxyz$ system.

Suppose that $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ are the three unit coordinate vectors of the $Oxyz$-frame and $\hat{\underline{e}}_1$, $\hat{\underline{e}}_2$ and $\hat{\underline{e}}_3$ are those of the $\hat{O}\hat{x}\hat{y}\hat{z}$-frame. If P is a point in the body with coordinates $(\hat{x}, \hat{y}, \hat{z})$ and (x, y, z) when referred to the $\hat{O}\hat{x}\hat{y}\hat{z}$- and the $Oxyz$-frames, respectively, then, since $\underline{\hat{O}P} = \underline{OP} - \underline{O\hat{O}}$, we have

$$\hat{x}\,\hat{\underline{e}}_1 + \hat{y}\,\hat{\underline{e}}_2 + \hat{z}\,\hat{\underline{e}}_3 = (x - x_1)\,\underline{e}_1 + (y - y_1)\,\underline{e}_2 + (z - z_1)\,\underline{e}_3 .$$

Suppose that at an instant t the body frame $\hat{O}\hat{x}\hat{y}\hat{z}$ has angular displacements $(\theta_1, \theta_2, \theta_3)$ relative to the $Oxyz$-frame. Then, without assuming smallness of the angular displacements, one can establish the following transformation between the two coordinate systems:

$$\begin{pmatrix} x-x_1 \\ y-y_1 \\ z-z_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} . \qquad (2)$$

If we now assume that θ_1 , θ_2 and θ_3 are all small enough so that one may assume $\cos\theta_1 \cong 1$, $\sin\theta_1 \cong \theta_1$, etc., then (2) simplifies to

$$\left. \begin{aligned} x - x_1 &= \hat{x} + \theta_2 \hat{z} - \theta_3 \hat{y} \\ y - y_1 &= \hat{y} + \theta_3 \hat{x} - \theta_1 \hat{z} \\ z - z_1 &= \hat{z} + \theta_1 \hat{y} - \theta_2 \hat{x} \end{aligned} \right\} \qquad (3)$$

or

$$X_i - X_{1i} = \hat{X}_i + \varepsilon_{ijk} \theta_j \hat{X}_k \;, \quad i, j, k = 1, 2, 3 . \qquad (3a)$$

In (3a) we have used the usual cartesian tensor notations and the repeated indices imply summation. If we now solve (3) for $\hat{x}$, $\hat{y}$ and $\hat{z}$ and assume that x_1 , y_1 and z_1 have the same order of smallness as θ_1 , θ_2 and θ_3 , then one may discard terms of the higher order and obtain

$$\left.\begin{aligned} \hat{x} &= x - x_1 - \theta_2 z + \theta_3 y, \\ \hat{y} &= y - y_1 - \theta_3 x + \theta_1 z, \\ \hat{z} &= z - z_1 - \theta_1 y + \theta_2 x, \end{aligned}\right\} \tag{4}$$

or $$\hat{x}_i = x_i - x_{1i} - \varepsilon_{ijk}\,\theta_j\, x_k \, . \tag{4a}$$

Note that the unit coordinate vector $\underline{e}_1$, for example, has the coordinate $(x, y, z) = (1,0,0)$ in the $Oxyz$-frame. If we let $\underline{e}_1 = \hat{x}\,\hat{\underline{e}}_1 + \hat{y}\,\hat{\underline{e}}_2 + \hat{z}\,\hat{\underline{e}}_3$, then from (4) one can easily find that $\hat{x} = 1$, $\hat{y} = -\theta_3$ and $\hat{z} = \theta_2$, where we have set $x_1 = y_1 = z_1 = 0$. Hence $\underline{e}_1 = \hat{\underline{e}}_1 + (-\theta_3)\hat{\underline{e}}_2 + \theta_2\,\hat{\underline{e}}_3$, from which one obtains the following result which will be useful later:

$$\underline{e}_1 \cdot \hat{\underline{e}}_1 = 1, \quad \underline{e}_1 \cdot \hat{\underline{e}}_2 = -\theta_3 , \quad \underline{e}_1 \cdot \hat{\underline{e}}_3 = \theta_2 \quad .$$

Similarly one may further obtain that $\underline{e}_2 \cdot \hat{\underline{e}}_1 = \theta_3$, $\underline{e}_2 \cdot \hat{\underline{e}}_2 = 1$, $\underline{e}_2 \cdot \hat{\underline{e}}_3 = -\theta_1$; $\underline{e}_3 \cdot \hat{\underline{e}}_1 = -\theta_2$, $\underline{e}_3 \cdot \hat{\underline{e}}_2 = \theta_1$ and $\underline{e}_3 \cdot \hat{\underline{e}}_3 = 1$. Or in tensor notation we write this result as

$$\hat{\underline{e}}_i \cdot \underline{e}_j = \delta_{ij} + \varepsilon_{ijk}\,\theta_k \, . \tag{5}$$

<u>Geometrical description of the ship</u>. The surface of the body will be given at all instants in the body coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$ by the equation

$$\hat{F}(\hat{x}, \hat{y}, \hat{z}) = 0. \tag{6}$$

By the transformation (4) this function can be written as

$$\hat{F}(\hat{x}, \hat{y}, \hat{z}) = \hat{F}(x_i - x_{1i} - \varepsilon_{ijk}\theta_j x_k) = F(x, y, z, t).$$

Thus $\hat{F}(\hat{x}, \hat{y}, \hat{z}) = 0$ and $F(x, y, z, t) = 0$, respectively, describe the same body surface at its instantaneous position in the body frame $\hat{O}\hat{x}\hat{y}\hat{z}$ and the translating frame $Oxyz$. Let us denote this body surface by S. Note that with reference to the $Oxyz$-frame there is an imaginary surface S_0 given by

$$S_0: \hat{F}(x, y, z) = 0, \tag{7}$$

which is stationary with respect to the $Oxyz$-frame and coincides exactly with the body surface S when the latter is in its undisturbed position. This imaginary surface S_0 will be called the "reference" surface.

<u>Description of the sea</u>. The form of the free-surface $\bar{O}\bar{x}\bar{y}\bar{z}$ will be described either in the space reference frame by the equation

$$\bar{y} = \bar{Y}(\bar{x}, \bar{z}, t), \tag{8}$$

or in the translating reference frame $Oxyz$ by the equation

$$y = Y(x, z, t) , \tag{8a}$$

where we let

$$\bar{Y}(\bar{x}, \bar{z}, t) = \bar{Y}\left(x + \int_0^t c\, d\tau, z, t\right) = Y(x, z, t) .$$

<u>Potential functions and their preliminary decompositions</u>. As usual we shall assume the fluid to be heavy, incompressible, and inviscid, and the flow to be irrotational, so that a potential function may be defined. Let $\phi(\bar{x}, \bar{y}, \bar{z}, t)$ be the potential function such that its gradient equals the velocity vector of a fluid particle with respect to the space reference frame $\bar{O}\bar{x}\bar{y}\bar{z}$. This velocity will be referred to as 'the absolute' velocity of a fluid particle. We shall also write

$$\phi(\bar{x}, \bar{y}, \bar{z}, t) = \phi\left(x + \int_0^t c\, d\tau, y, z, t\right) = \varphi(x, y, z, t) .$$

The relative velocity of a fluid particle with respect to the translating reference frame $Oxyz$ then should be given by

$$\underset{\sim}{V} = \nabla\left(-cx + \varphi(x, y, z, t)\right) = \left(-c + \varphi_x , \varphi_y , \varphi_z\right) . \tag{9}$$

Note that $\phi_{\bar{x}} = \varphi_x$, $\phi_{\bar{y}} = \varphi_y$, $\phi_{\bar{z}} = \varphi_z$ etc., but $\phi_t = \varphi_t - c\varphi_x$. $\nabla\varphi$ still gives the absolute velocity but is expressed in terms of the variables of the $Oxyz$ system.

In the subsequent development we shall suppose that the resultant fluid motion of our problem is composed of the superposition of two parts: 1) the disturbance due to the translation of the body fixed in its undisturbed position with the forward speed $C(t)$ into otherwise undisturbed fluid region, and 2) the fluid motion due to the oscillatory motion of the body and the oncoming waves. Hence we shall write

$$\varphi(x,y,z,t) = \varphi_0(x,y,z,t) + \varphi_1(x,y,z,t), \tag{10}$$

where φ_0 and φ_1, respectively, represent the fluid motions due to the first and the second parts mentioned above.

<u>The linearized kinematic and dynamical boundary conditions on the free surface</u>. A systematic linearization of the mathematical expressions for the fact that the free surface is a material surface and the assumption that the pressure everywhere on the free-surface is constant will lead, respectively, to the following conditions:

$$\bar{Y}_t(\bar{x},\bar{z},t) - \phi_{\bar{y}}(\bar{x},0,\bar{z},t) = 0, \tag{11}$$

and

$$g\,\bar{Y}(\bar{x},\bar{z},t) + \phi_t(\bar{x},0,\bar{z},t) = 0. \tag{12}$$

They are, respectively, the kinematic and dynamical boundary conditions on the free-surface and are to be satisfied at the undisturbed free surface $\bar{y} = 0$. Elimination of $\bar{Y}$ from (11) and (12) gives

$$\phi_{tt}(\bar{x}, 0, \bar{z}, t) + g\,\phi_{\bar{y}} = 0 \qquad \text{on} \qquad \bar{y} = 0. \tag{13}$$

The relationships $\phi(\bar{x}, \bar{y}, \bar{z}, t) = \varphi(x, y, z, t)$, $\bar{Y}(\bar{x}, \bar{z}, t) = Y(x, z, t)$, $\phi_{\bar{x}} = \varphi_x$ etc. and $\phi_t = \varphi_t - c\,\varphi_x$ give us easily the counterparts of (11), (12), and (13) in the $Oxyz$ system as follows:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) Y(x, z, t) - \varphi_y(x, 0, z, t) = 0, \tag{14}$$

$$g\,Y(x, z, t) + \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varphi(x, 0, z, t) = 0, \tag{15}$$

and

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)^2 \varphi(x, 0, z, t) + g\,\varphi_y = 0, \tag{16}$$

which are to be satisfied on the undisturbed free surface $y = 0$.

<u>Linearization of the boundary condition at the hull-fluid interface</u>. The boundary condition at the hull-fluid interface, like the case of the free-surface boundary condition, can be formulated either in the space reference frame $\bar{O}\bar{x}\bar{y}\bar{z}$ or in the translating reference frame $Oxyz$. It will be convenient in the future if we work here with the translating reference frame $Oxyz$.

Let us first work out an expression for the velocity of the ship hull relative to the translating reference frame $Oxyz$ according to the transformation (3) and (5) valid for the small-oscillations approximation. Let Q be a typical point on the surface of the body with coordinates $(\hat{x}, \hat{y}, \hat{z})$ in the body reference frame $\hat{O}\hat{x}\hat{y}\hat{z}$ and (x, y, z) in the $Oxyz$-frame. Then the vectors $\underline{\hat{X}}' = \hat{x}\,\underline{\hat{e}}_1 + \hat{y}\,\underline{\hat{e}}_2 + \hat{z}\,\underline{\hat{e}}_3 = \underline{\hat{O}Q}$ and $\underline{X} = x\,\underline{e}_1 + y\,\underline{e}_2 + z\,\underline{e}_3 = \underline{OQ}$ are the position vectors of Q in $\hat{O}\hat{x}\hat{y}\hat{z}$ and $Oxyz$ respectively. It is easily verified from (5) that

$$\underline{\hat{e}}_i = \underline{e}_i + \varepsilon_{ijk}\,\theta_k\,\underline{e}_j \,. \tag{17}$$

Hence we may write

$$\begin{aligned}\underline{\hat{X}}' &= \hat{x}\,\underline{\hat{e}}_1 + \hat{y}\,\underline{\hat{e}}_2 + \hat{z}\,\underline{\hat{e}}_3 \\ &= \hat{x}\,\underline{e}_1 + \hat{y}\,\underline{e}_2 + \hat{z}\,\underline{e}_3 + (\theta_2\hat{z} - \theta_3\hat{y})\,\underline{e}_1 + (\theta_3\hat{x} - \theta_1\hat{z})\,\underline{e}_2 \; + \\ &\qquad + (\theta_1\hat{y} - \theta_2\hat{x})\,\underline{e}_3 \,.\end{aligned}$$

Let us define $\underline{\hat{X}} = \hat{x}\,\underline{e}_1 + \hat{y}\,\underline{e}_2 + \hat{z}\,\underline{e}_3$; then we have

$$\underline{\hat{X}}' = \underline{\hat{X}} + (\theta_1, \theta_2, \theta_3) \times \underline{\hat{X}} \,. \tag{18}$$

Obviously $\underline{\hat{X}}$ defined above is not a position vector of the point Q with reference to any of the two reference frames considered here. We may now write

$$\underline{X} = \underline{X}_1 + \underline{\hat{X}}' = \underline{X}_1 + \underline{\hat{X}} + (\theta_1, \theta_2, \theta_3) \times \underline{\hat{X}} \,. \tag{19}$$

The formula (19) gives immediately the velocity of the point Q relative to the $Oxyz$-frame as follows:

$$\underline{V}(Q;t) = \underline{\dot{X}}(Q;t) = \underline{\dot{X}}_1(t) + (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3) \times \underline{\hat{X}} \, , \tag{20}$$

where we take $\dot{\underline{\hat{X}}} = 0$ since the body is to be assumed rigid. If the vector $\underline{V}$ in (20) is resolved along the $Oxyz$-frame we may then write

$$V_i\,(Q;t) = \dot{X}_{1i}(t) + \varepsilon_{ijk}\,\dot{\theta}_j\,\hat{X}_k \, . \tag{20a}$$

Let $\underline{n}$ be the unit normal vector to the hull surface pointing into the body. Let $\underline{n}$ be resolved along both reference frames so that

$$\underline{n} = \sum_{i=1}^{3} \hat{n}_i\,\underline{\hat{e}}_i = \sum_{j=1}^{3} n_j\,\underline{e}_j \, . \tag{21}$$

Suppose that the function $\hat{F}$ given in (6) which describes the body surface is so chosen that the inward normal is given by the formula

$$\hat{n}_i\,(\hat{x},\hat{y},\hat{z}) = \hat{F}_{,i}\,(\hat{x},\hat{y},\hat{z}) \Big/ \sqrt{\hat{F}_{,k}\hat{F}_{,k}} \tag{22}$$

where $\hat{F}_{,i}$ represents the partial derivative of the function $\hat{F}$ with respect to its i-th variable. Then from (21) and (5) we obtain easily that

$$n_j(\hat{x},\hat{y},\hat{z}) = \hat{n}_j(\hat{x},\hat{y},\hat{z}) + \varepsilon_{jki}\,\theta_k\,\hat{n}_i$$

$$= \left\{\hat{F}_{,j} + \varepsilon_{jki}\,\theta_k\,\hat{F}_{,i}\right\} \Big/ \sqrt{\hat{F}_{,k}\hat{F}_{,k}} \; . \tag{23}$$

The boundary condition on the body is

$$\underline{n} \cdot \nabla(-cx + \varphi) = \underline{n} \cdot \underline{V} \; ,$$

i.e., the normal component of the fluid velocity at a point Q on the body surface equals that of the surface. In component form this can be written as

$$\left[\hat{n}_i(\hat{x},\hat{y},\hat{z}) + \varepsilon_{ijk}\,\theta_j\,\hat{n}_k\right]\cdot\left[-c\,\delta_{1i} + \varphi_{,i}(x,y,z,t)\right]$$

$$= \left[\hat{n}_i(\hat{x},\hat{y},\hat{z}) + \varepsilon_{ijk}\,\theta_j\,\hat{n}_k\right]\cdot\left[\dot{X}_{1i} + \varepsilon_{ijk}\,\dot{\theta}_j\,\hat{X}_k\right] . \tag{24}$$

Note that in (24) variables of both coordinate systems are involved. Transformation (3) can now be used in writing

$$\varphi(x,y,z,t) = \varphi\left(\hat{x} + x_1 + \theta_2\hat{z} - \theta_3\hat{y},\ \hat{y} + y_1 + \theta_3\hat{x} - \theta_1\hat{z},\ \hat{z} + z_1 + \theta_1\hat{y} - \theta_2\hat{x},\ t\right).$$

Since the quantities X_{1i} and θ_i are small compared to the quantities $\hat{X}_i$, it is reasonable to assume that the following Taylor's series expansion for the potential function $\varphi_{,i}$ is possible:

$$\varphi_{,i}(x,y,z,t) = \varphi_{,i}(\hat{x},\hat{y},\hat{z},t) + [X_{1\ell} + \varepsilon_{\ell mn}\theta_m x_n]\varphi_{,i\ell} + \cdots . \tag{25}$$

This is now to be substituted back into (24). One may at this point introduce several perturbation parameters, for example, say, ε_S measuring the smallness of the free-surface disturbance due to forward motion and ε_M measuring the smallness of the oscillatory motion. Then by following usual scheme of linearization one can deduce from (25) and (24) linearized conditions for separate boundary-value problems of the orders ε_s , ε_M and $\varepsilon_s\varepsilon_M$, etc., respectively. However, the goal of the present work will be better served by following a slightly different approach. We shall not be so specific about the introduction of the perturbation parameters but rather shall discard whatever terms of order $\theta_j\theta_k$ or $X_{1i}\theta_j$ appear in (24), for they are clearly terms of the order ε_M^2. (Terms of order $\varepsilon_s\varepsilon_M$ will always be retained.) If this is done then we have, from (24) and (25),

$$\begin{aligned}\hat{F}_{,i}(\hat{x},\hat{y},\hat{z})\,\varphi_{,i}(\hat{x},\hat{y},\hat{z},t) + \varepsilon_{ijk}\theta_j\hat{F}_{,k}\varphi_{,i} + \hat{F}_{,i}[X_{1\ell} + \varepsilon_{\ell mn}\theta_m\hat{x}_n]\varphi_{,i\ell} -\\ - c\,\delta_{1i}[\hat{F}_{,i} + \varepsilon_{ijk}\theta_j\hat{F}_{,k}]\\ = \hat{F}_{,i}(\hat{x},\hat{y},\hat{z})[\dot{X}_{1i} + \varepsilon_{ijk}\dot{\theta}_j\hat{x}_k],\end{aligned}$$

where the expression (22) has been used for $\hat{n}_i$. If we now drop all the circumflexes over the variables in the last expression, we have

$$\hat{F}_{,i}(x,y,z)\,\varphi_{,i}(x,y,z,t)+\varepsilon_{ijk}\,\theta_j\,\hat{F}_{,k}\,\varphi_{,i}+\hat{F}_{,i}\,[X_{1l}+\varepsilon_{lmn}\,\theta_m X_n]\,\varphi_{,il}-$$
$$-\,c\,\delta_{1i}\,[\hat{F}_{,i}+\varepsilon_{ijk}\,\theta_j\,\hat{F}_{,k}]$$
$$=\hat{F}_{,i}(x,y,z)\,[\dot{X}_{1i}+\varepsilon_{ijk}\,\dot{\theta}_j\,X_k]\,. \tag{26}$$

Obviously, (26) is now a condition for the potential function φ to be satisfied at a point on the surface defined by the equation $\hat{F}(x,y,z)=0$ which is precisely the imaginary surface S_o defined by (7).

Although certain physical interpretations of the implication of (26) are possible, we shall not do this but rather refer to the original paper of Timman and Newman (1962) and to the discussion on p. 39 of Ogilvie (1964). However, we should like to remark here that the fact that (26) is to be satisfied at an imaginary boundary S_o comes out naturally as the result of linearization and the way φ is expanded into Taylor's series in (25). The question of whether S_o represents the mean position of the oscillating surface S or not is immaterial. In fact, S_o here will seldom be the mean position of S, since we are considering an arbitrary oscillatory motion of a body in forward motion.

Let us now put $\varphi=\varphi_o+\varphi_1$ in (26) and assume that φ_1, and its derivatives have the same order of magnitude as θ_j or X_{1i}. Hence, terms like $\varepsilon_{ijk}\,\theta_j\,\hat{F}_{,k}\,\varphi_{1,i}$ and $(X_{1l}+\varepsilon_{lmn}\,\theta_m X_n)\,\varphi_{1,il}$ are to be discarded. Then from (26) we have

$$\hat{F}_{,i}(x,y,z)\left[\varphi_{0,i}(x,y,z,t)+\varphi_{1,i}(x,y,z,t)\right] + \varepsilon_{ijk}\,\theta_j\,\hat{F}_{,k}\,\varphi_{0,i} +$$
$$+ \hat{F}_{,i}\left[X_{1\ell}+\varepsilon_{\ell mn}\,\theta_m X_n\right]\varphi_{0,i\ell} - c\,\delta_{1i}\left[\hat{F}_{,i}+\varepsilon_{ijk}\,\theta_j\,\hat{F}_{,k}\right]$$
$$= \hat{F}_{,i}(x,y,z)\left[\dot{X}_{1i} + \varepsilon_{ijk}\,\dot{\theta}_j\,X_k\right]. \qquad (27)$$

Suppose that there is no oscillatory motion (i.e., we set $\varepsilon_M=0$), so that $X_{1i}=0$, $\theta_i=0$, and $\varphi_1 \equiv 0$, $\varphi = \varphi_0$. Then from the last condition we have

$$\hat{F}_{,i}(x,y,z)\,\varphi_{0,i}(x,y,z,t) - c\,\delta_{1i}\,\hat{F}_{,i} = 0.$$

Hence $n_{0i}\,\varphi_{0,i} = c\,n_{01}$ on $S_0 : \hat{F}(x,y,z)=0$, where we have written $n_{0i}(Q_0) = \hat{F}_{,i}(Q_0)/\sqrt{\hat{F}_{,k}\hat{F}_{,k}}$, for $Q_0 \in S_0$. Or we write

$$\varphi_{0n}(x,y,z,t)\Big|_{S_0} = \underline{n}_0 \cdot \nabla(-cx). \qquad (28)$$

With this result another condition for φ_1 can now be obtained from (27). Thus we have, from (27) and (28),

$$\hat{F}_{,i}(x,y,z)\,\varphi_{1,i}(x,y,z,t) + \varepsilon_{ijk}\,\theta_j\,\hat{F}_{,k}\,\varphi_{0,i} + \hat{F}_{,i}\left[X_{1\ell}+\varepsilon_{\ell mn}\,\theta_m X_n\right]\varphi_{,i\ell} -$$
$$- c\,\delta_{1i}\,\varepsilon_{ijk}\,\theta_j\,\hat{F}_{,k}$$
$$= \hat{F}_{,i}\left[\dot{X}_{1i} + \varepsilon_{ijk}\,\dot{\theta}_j\,X_k\right].$$

Let us again write $n_{oi}(x,y,z)$ for $\hat{F}_{,i}(x,y,z)/\sqrt{\hat{F}_{,k}\hat{F}_{,k}}$ and rearrange terms in the last expression. Then we have

$$n_{oi}\,\varphi_{1,i} = n_{oi}\left[\dot{X}_{1i} + \varepsilon_{ijk}\,\theta_j\,X_k\right] + n_{oi}\Big\{-\left[X_{1\ell} + \varepsilon_{\ell mn}\,\theta_m\,X_n\right]\varphi_{o,i\ell} -$$
$$- \varepsilon_{ijk}\,\theta_k\left(-c\,\delta_{1j} + \varphi_{o,j}\right)\Big\}. \tag{29}$$

For convenience let us introduce a vector $\underset{\sim}{A}$ defined by

$$\underline{A} = \underline{X}_1 + (\theta_1, \theta_2, \theta_3)\times\underline{X}\,. \tag{30}$$

Or in component form, $A_i = X_{1i} + \varepsilon_{ijk}\,\theta_j\,X_k$. From (30) we have, then,

$$\frac{\partial\underline{A}}{\partial t} = \dot{\underline{X}}_1 + (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)\times\underline{X}\,, \tag{31}$$

which is a vector evaluated at a point $Q_o = (x,y,z)$ on S_o. Note that the difference between the vector given by (31) and that of (20) is of the second order. Hence (31) may be regarded as the first-order approximation of the velocity vector of a point Q on the actual body surface.

Condition (29) can be put into a compact form if the following vector identity is used:

$$\nabla\times\left[\underline{A}\times\underline{\lambda}\right] = (\underline{\lambda}\cdot\nabla)\underline{A} - (\underline{A}\cdot\nabla)\underline{\lambda} + \underline{A}\,\nabla\cdot\underline{\lambda} - \underline{\lambda}\,\nabla\cdot\underline{A}\,. \tag{32}$$

Let $\underline{\lambda} = \nabla(-cx+\varphi_0) = -c\underline{e}_1 + \nabla\varphi_0$ and $\underline{A}$ be as defined by (30); then

1) $\nabla\cdot\underline{\lambda} = \nabla\cdot\nabla(-cx+\varphi_0) = \nabla^2(-cx+\varphi_0) = 0,$

2) $\nabla\cdot\underline{A} = \nabla\cdot\{\underline{X}_1 + (\theta_1,\theta_2,\theta_3)\times\underline{X}\}$

$$= X_{1i,i} + \varepsilon_{ijk}\theta_j\delta_{ki} = \varepsilon_{iji}\theta_j = 0,$$

3) $(\underline{\lambda}\cdot\nabla)\underline{A} = \{[\nabla(-cx+\varphi_0)]\cdot\nabla\}\{\underline{X}_1 + (\theta_1,\theta_2,\theta_3)\times\underline{X}\}$

$$= (-c\delta_{1\ell} + \varphi_{0,\ell})[X_{1i} + \varepsilon_{ijk}\theta_j X_k]_{,\ell}$$

$$= (-c\delta_{1\ell} + \varphi_{0,\ell})[\varepsilon_{ijk}\theta_j\delta_{k\ell}]$$

$$= (-c\delta_{1\ell} + \varphi_{0,\ell})[\varepsilon_{ij\ell}\theta_j]$$

$$= -\varepsilon_{ijk}\theta_k(-c\delta_{1j} + \varphi_{0,j}),$$

4) $-(\underline{A}\cdot\nabla)\underline{\lambda} = -\{[\underline{X}_1 + (\theta_1,\theta_2,\theta_3)\times\underline{X}]\cdot\nabla\}\nabla(-cx+\varphi_0)$

$$= -\{X_{1i} + \varepsilon_{ijk}\theta_j X_k\}[-c\delta_{1\ell} + \varphi_{0,\ell}]_{,i}$$

$$= -\{X_{1i} + \varepsilon_{ijk}\theta_j X_k\}\varphi_{0,i\ell}.$$

Note that 3) and 4) are precisely the same as the second and the third term in the right-hand side of the equality in (29). Hence with the use of the identity (32), (29) can now be written as

$$\underline{n}_o \cdot \nabla \varphi_1 = \underline{n}_o \cdot \frac{\partial \underline{A}}{\partial t} + \underline{n}_o \bullet \nabla \times \left\{ \underline{A} \times \nabla (-cx + \varphi_o) \right\}. \qquad (33)$$

As was remarked before, to the first order approximation we have $\partial \underline{A} / \partial t = \underline{V}$, the velocity of a point Q on the instantaneous position of the hull surface relative to the $Oxyz$-frame. We shall then rewrite (33) as

$$\varphi_{1n}(\underline{x};t)\Big|_{S_o} = V_n(\underline{x};t) + \underline{n}_o(\underline{x}) \bullet \nabla \times \left\{ \underline{A}(\underline{x};t) \times \underline{V}_o(\underline{x};t) \right\}, \qquad (34)$$

where $\underline{V}_o = \nabla\left(-cx + \varphi_o(x,y,z,t)\right)$, and

$$\underline{X} = (x, y, z) \in S_o .$$

Note that (34) is precisely the condition originally derived by Timman and Newman (1962). It might be interesting to note that in (34) an additional second term is needed in a condition derived from the kinematic boundary condition on the hull surface S . However, it should not be surprising that this should be the case, since S_o , being an imaginary boundary, is not a material surface. Therefore, even though $\varphi_{1n} = V_n$ is the appropriate condition for a material surface, it need not hold on S_o , and, in fact, when a body is translating as well as oscillating, it does <u>not</u> hold.

Finally, if the condition (34) is specialized to thin-, flat- or slender-ship approximations, it is found that (34) in fact represents a combination of several boundary conditions for separate boundary-value problems. Moreover, in this case, the additional second term contains terms of higher order in perturbation parameters. Such terms should be discarded if a strictly first-order expression of the boundary condition is required. Nevertheless, the present form of the boundary condition in (34) will be used in the present work because there seems to be no harm in keeping some of these higher-order terms, provided it is understood that the accuracy is good only to the first-order approximation.

III. Solution of the Initial-Value Problem by the Method of Green's Function.

The initial-value problem. In the subsequent development we shall need only the translating reference frame $Oxyz$. Henceforth, the surface $S_0 : \hat{F}(x,y,z) = 0$ will be restricted to the part of the surface which is below $y = 0$, i.e., S_0 coincides with the wetted part of the hull surface when the ship is in its undisturbed position. We shall be looking for an unsteady velocity potential $\varphi_1(x,y,z,t)$ satisfying the following equations and boundary conditions:

$$\varphi_{1xx} + \varphi_{1yy} + \varphi_{1zz} = 0 \quad , \quad y < 0 ; \tag{35}$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)^2 \varphi_1 + g\,\varphi_{1y} = 0 \tag{36}$$

on the undisturbed free surface $y = 0$;

$$\varphi_{1n}(\underline{x};t)\Big|_{S_0} = V_n(\underline{x};t) + \underline{n}_0(\underline{x}) \cdot \nabla \times \left\{ \underline{A}(\underline{x};t) \times \underline{V}_0(\underline{x};t) \right\} , \tag{37}$$

where $\underline{x} = (x,y,z)$ in S_0, and $\underline{V}_0(\underline{x};t) = \nabla\left(-cx + \varphi_0(x,y,z,t)\right)$ is already prescribed;

$$\varphi_{1n}(\underline{x};t)\Big|_{B} = 0 \quad , \tag{38}$$

where B is the bottom. If the fluid is infinitely deep, the last condition is replaced by

$$\lim_{y \to -\infty} \varphi_{1y}(x,y,z,t) = 0 \; . \tag{38a}$$

In addition, φ_1, φ_{1t} and their first derivatives are assumed to be uniformly bounded at ∞.

The initial position and velocity of the ship are assumed known. Moreover, at the initial instant both Y_1 and Y_{1t} on the free surface are prescribed:

$$\left.\begin{aligned} Y_1(x,z,0) &= f_1(x,z) \; , \\ Y_{1t}(x,z,0) &= f_2(x,z) \end{aligned}\right\} \tag{39}$$

where f_1 and f_2 are given functions of x and z.

From (39) and (14) we see that $\varphi_{1y}(x,0,z,0)$ is also determined. Thus at $t=0$, φ_{1n} is given on all boundaries and is bounded at infinity and therefore, $\varphi_1(x,y,z,0)$ can be obtained as the solution to a Neumann problem. We also know from (15) that $Y_1(x,z,t)$, the free-surface elevation due to the fluid motions associated with φ_1, is given in linearized theory by

$$Y_1(x,z,t) = -\frac{1}{g}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varphi_1(x,0,z,t) \; . \tag{40}$$

Hence, from the initial values given in (39), $\varphi_{,t}(x,0,z,0)$ and $\varphi_{,tt}(x,0,z,0)$ can also be found.

<u>The time-dependent Green's function and use of Green's theorem.</u> The Green's function $G(x,y,z;\xi,\eta,\zeta;t)$ in question is required to be a solution of Laplace's equation in the variables ξ,η,ζ with a singularity like $1/r =$ $= [(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{-1/2}$ at (x,y,z) but otherwise harmonic in ξ,η,ζ in the region $y \le 0$, $\eta \le 0$. Thus

$$G(x,y,z;\xi,\eta,\zeta;t) = 1/r + H(x,y,z;\xi,\eta,\zeta;t) \tag{41}$$

where $H(x,y,z;\xi,\eta,\zeta;t)$ is harmonic everywhere in the domain of definition. We shall suppose that G is a symmetric function of t satisfying the following boundary and initial conditions:

$$\left(\frac{\partial}{\partial t} - c(t)\frac{\partial}{\partial \xi}\right)^2 G(\underline{x};\xi,0,\zeta;t) + g\, G_\eta = 0 , \tag{42a}$$

$$G_\nu(\underline{x};\underline{\xi};t)\Big|_B = 0 \qquad \text{or} \qquad \lim_{\eta\to-\infty} G_\eta = 0 , \tag{42b}$$

$$G(\underline{x};\xi,0,\zeta;0) = 0 , \qquad G_t(\underline{x};\xi,\eta,\zeta;0) = 0 , \tag{42c}$$

$$G = O(R^{-2}) , \qquad G_R = O(R^{-3}) \qquad \text{as } R\to\infty , \tag{42d}$$

where $R = \left[(x-\xi)^2 + (z-\zeta)^2 \right]^{1/2}$, and

$$\frac{\partial}{\partial \nu} \equiv n_1(\underline{\xi}) \frac{\partial}{\partial \xi} + n_2 \frac{\partial}{\partial \eta} + n_3 \frac{\partial}{\partial \zeta} .$$

A method of construction of such Green's function can be found either in Stoker [1957, pp. 188-191] or in Wehausen and Laitone [1960, pp. 491-495]. It is a property of this Green's function that

$$G(\underline{x};\underline{\xi};t) = G(\underline{\xi};\underline{x};t) \tag{43}$$

where $\underline{x} = (x,y,z)$ and $\underline{\xi} = (\xi,\eta,\zeta)$ is a point on the boundary of the fluid region considered.

With Green's function described above we may proceed now to set up an integral equation for the function φ_1. To achieve this goal we start, in the usual fashion, with applying Green's theorem to the Green's function G and to φ_{1t} in the fluid region bounded by the undisturbed free surface F, the reference surface S_o, the bottom B (if any), and a large sphere Ω of radius a centered at the origin of the translating reference frame. Note that only parts of F, B and Ω will serve as bounding surfaces and we shall call these parts F', B' and Ω'. Then

$$\varphi_{1t}(\underline{x};t) = \frac{1}{4\pi} \iint_{F'+S_o+B'+\Omega'} \left\{ G(\underline{x};\underline{\xi};t-\tau)\, \varphi_{1t\nu}(\underline{\xi};t) - \varphi_{1t}\, G_\nu \right\} dS, \tag{44}$$

where the normal vector is taken to be exterior to the fluid region considered. The surface integral over Ω' vanishes as the sphere Ω extends to infinity because of the boundedness of $\varphi_{1t\nu}$ and φ_{1t} and the behavior of G at ∞. The integral over B' also vanishes since both $\varphi_{1\nu}$ and G_ν are zero on B. After letting $a \to \infty$, (44) then becomes

$$\varphi_{1t}(\underline{x};t) = \frac{1}{4\pi} \iint_F \left\{ G(\underline{x};\xi,0,\zeta;t-\tau)\,\varphi_{1t\eta}(\xi,0,\zeta,t) - \varphi_{1t} G_\eta \right\} d\xi\, d\zeta$$

$$+ \frac{1}{4\pi} \iint_{S_0} \left\{ G(\underline{x};\underline{\xi};t-\tau)\,\varphi_{1t\nu}(\underline{\xi};t) - \varphi_{1t} G_\nu \right\} dS. \qquad (45)$$

Interchanging t with τ in (45) and observing the fact that $G(\underline{x};\underline{\xi};t-\tau) = G(\underline{x};\underline{\xi};\tau-t)$, let us now integrate both sides of (45) with respect to τ from 0 to t. Then we have

$$4\pi\varphi_1(\underline{x};t) - 4\pi\varphi_1(\underline{x};0)$$

$$= \int_0^t d\tau \iint_F \left\{ G(\underline{x};\xi,0,\zeta;t-\tau)\,\varphi_{1\tau\eta}(\xi,0,\zeta;\tau) - \varphi_{1\tau} G_\eta \right\} d\xi\, d\zeta$$

$$+ \int_0^t d\tau \iint_{S_0} \left\{ G(\underline{x};\underline{\xi};t-\tau)\,\varphi_{1\tau\nu}(\underline{\xi};\tau) - \varphi_{1\tau} G_\nu \right\} dS$$

$$= I_F + I_{S_0}. \qquad (46)$$

We may proceed further with (46) by making use of the boundary conditions as follows. From (42a) we have now, on the undisturbed free-surface F ,

$$G_\eta(\underline{x};\xi,0,\zeta;t-\tau) = -\frac{1}{g}\left[\left(\frac{\partial}{\partial t} - c(t-\tau)\frac{\partial}{\partial \xi}\right)^2 G(\underline{x};\xi,0,\zeta;t-\tau)\right] \quad (47)$$

where

$$\frac{\partial}{\partial t} G(\underline{x};\underline{\xi};t-\tau) \equiv G_t(\underline{x};\underline{\xi};t-\tau) = -\frac{\partial}{\partial \tau} G(\underline{x};\underline{\xi};t-\tau)$$

is the derivative of G with respect to its seventh variable. With further use of the free-surface boundary condition for the potential function φ_1 , it is not difficult to verify that the following identity holds for $\underline{\xi} = (\xi,0,\zeta)$ on F .

$$\left\{G(\underline{x},\xi,0,\zeta;t-\tau)\,\varphi_{1t\eta}(\xi,0,\eta,\tau) - \varphi_{1t}G_\eta\right\}$$

$$\left.\begin{aligned}
&= \frac{\partial}{\partial \tau}\left\{G\left[\mathcal{D}Y_1(\xi,\zeta,\tau)\right] + Y_1\left[\mathcal{D}G(\underline{x};\xi,0,\zeta;t-\tau)\right]\right\} \\
&\quad -\frac{1}{g}\frac{\partial}{\partial \tau}\left\{\varphi_1(\xi,0,\zeta,\tau)\left[c(\tau)+c(t-\tau)\right]\frac{\partial}{\partial \xi}\mathcal{D}G(\underline{x};\xi,0,\zeta;t-\tau)\right\} \\
&\quad +\frac{1}{g}\frac{\partial}{\partial \tau}\frac{\partial}{\partial \xi}\left\{\varphi_1(\xi,0,\zeta,\tau)\left[c(\tau)(\mathcal{D}G)\right]\right\} \\
&\quad +\frac{1}{g}\varphi_1(\cdot,0,\zeta,\tau)\frac{\partial}{\partial \xi}\left\{c(\tau)\,\mathcal{D}^2G - \left[3c(\tau)+c(t-\tau)\right]\frac{\partial}{\partial t}(\mathcal{D}G) + \right. \\
&\qquad \left. + \left[\dot{c}(\tau) - \dot{c}(t-\tau)\right](\mathcal{D}G) + c(\tau)\left[c(t-\tau)-c(\tau)\right]\frac{\partial}{\partial \xi}(\mathcal{D}G)\right\} +
\end{aligned}\right\} \quad (48)$$

$$+\frac{1}{g}\frac{\partial}{\partial\xi}\Big[\varphi_1(\xi,0,\zeta,\tau)\Big\{2c(\tau)(D^2G)+c(\tau)[c(\tau)+2c(t-\tau)]\frac{\partial}{\partial\xi}(DG)-$$

$$-\dot{c}(\tau)\,(DG)\Big\}\Big]$$

$$-\frac{1}{g}\frac{\partial}{\partial\xi}\Big\{\varphi_{1\xi}(\xi,0,\zeta,\tau)[c^2(\tau)(DG)]\Big\}+\varphi_{1\eta}(\xi,0,\zeta,\tau)\Big\{c(t-\tau)\frac{\partial}{\partial\xi}G\Big\},$$

where

$$DY_1(\xi,\zeta,t)\equiv\left(\frac{\partial}{\partial t}-c(t)\frac{\partial}{\partial\xi}\right)Y_1(\xi,\zeta,t)$$

and

$$DG(\underline{x};\underline{\xi};t-\tau)\equiv\left(\frac{\partial}{\partial t}-c(t-\tau)\frac{\partial}{\partial\xi}\right)G(\underline{x};\underline{\xi};t-\tau).$$

With the use of the identity (48), the integral over the undisturbed free surface F in (46) now becomes

$$I_F=-\iint_F\Big\{G\,[DY_1(\xi,\zeta,0)]+Y_1[DG(\underline{x};\xi,0,\zeta;t)]\Big\}d\xi\,d\zeta$$

$$+\frac{1}{g}\iint_F\varphi_1(\xi,0,\zeta,0)\Big\{[c(0)+c(t)]\frac{\partial}{\partial\xi}DG(\underline{x};\xi,0,\zeta;t)\Big\}d\xi\,d\zeta$$

$$-\frac{1}{g}\oint_{\Gamma_0}\varphi_1(\xi,0,\zeta,0)\Big\{c(0)\,DG(\underline{x};\xi,0,\zeta;t)\Big\}d\zeta\;+$$

$$
\left.
\begin{aligned}
&+\frac{1}{g}\int_0^t d\tau \iint_F \varphi_1(\xi,0,\zeta,\tau)\frac{\partial}{\partial\xi}\Big\{c(\tau)D^2G(\underline{x};\xi,0,\zeta;t-\tau)-\\
&\qquad -[3c(\tau)+c(t-\tau)]\frac{\partial}{\partial t}(DG)+[\dot{c}(\tau)-\dot{c}(t-\tau)](DG)+\\
&\qquad +c(\tau)[c(t-\tau)-c(\tau)]\frac{\partial}{\partial\xi}(DG)\Big\}\,d\xi\,d\zeta\\
&+\frac{1}{g}\int_0^t d\tau \oint_{\Gamma_0} \varphi_1(\xi,0,\zeta,\tau)\Big\{2c(\tau)D^2G(\underline{x};\xi,0,\zeta;t-\tau)+\\
&\qquad +c(\tau)[c(\tau)+2c(t-\tau)]\frac{\partial}{\partial\xi}(DG)-\dot{c}(\tau)(DG)\Big\}\,d\zeta\\
&-\frac{1}{g}\int_0^t d\tau \oint_{\Gamma_0} \varphi_{1\xi}(\xi,0,\zeta,\tau)\Big\{c^2(\tau)\,DG(\underline{x};\xi,0,\zeta;t-\tau)\Big\}\,d\zeta\\
&+\int_0^t d\tau \iint_F \varphi_{1\eta}(\xi,0,\zeta,\tau)\Big\{c(t-\tau)\frac{\partial}{\partial\xi}G(\underline{x};\xi,0,\zeta;t-\tau)\Big\}\,d\xi\,d\zeta\,,
\end{aligned}
\right\}\quad (49)
$$

where

$$DG(\underline{x};\xi,0,\zeta;0)\equiv\left(\frac{\partial}{\partial t}-c(0)\frac{\partial}{\partial\xi}\right)G(\underline{x};\xi,0,\zeta;0),$$

and

$$DG(\underline{x};\xi,0,\zeta;t)\equiv\left(\frac{\partial}{\partial t}-c(t)\frac{\partial}{\partial\xi}\right)G(\underline{x};\xi,0,\zeta;t)\,.$$

In deriving (49) we have made use of the condition (42c), namely, $G(\underline{x};\xi,0,\zeta;0)=0$, $G_t(\underline{x};\xi,\eta,\zeta;0)=0$. It follows from this condition that

$$DG(\underline{x};\xi,0,\zeta;0)=G_t(\underline{x};\xi,0,\zeta;0)-c(0)G_\xi(\underline{x};\xi,0,\zeta;0)=0\,.$$

Note also that in obtaining (49), Gauss's theorem for the plane region has been used. Since the undisturbed free surface F may be considered as being bounded inwardly by Γ_0, the contour around the intersection of S_0 with F, and outwardly by Γ_∞

at infinity, one of the integrals, for example, becomes

$$\iint_F \frac{\partial}{\partial \xi} \left\{ \varphi_{1\xi}(\xi, 0, \zeta, \tau) \left[c^2(\tau) \mathcal{D} G \right] \right\} d\xi \, d\zeta$$

$$= \oint_{\Gamma_0} \varphi_{1\xi}(\xi, 0, \zeta, \tau) \left[c^2(\tau) \mathcal{D} G \right] d\zeta ,$$

where Γ_0 is oriented counterclockwise, and the integral around Γ_∞ vanishes because of the behavior of G and the boundedness of $\varphi_{1\xi}$ at infinity. The other contour integrals appearing in (49) are obtained in a similar fashion.

We shall next proceed with the integral over S_0 in (46) by making use of the boundary condition of the function φ_1 on S_0. Let us first introduce the following notations. Let

$$\underline{e}_k = \begin{cases} \underline{e}_k , & k = 1, 2, 3, \\ \underline{e}_{k-3} \times \underline{X} , & k = 4, 5, 6, \end{cases} \tag{50a}$$

$$n_{0k} = \begin{cases} \underline{n}_0(\underline{x}) \cdot \underline{e}_k , & k = 1, 2, 3, \\ \underline{n}_0(\underline{x}) \cdot \left[\underline{e}_{k-3} \times \underline{X} \right] , & k = 4, 5, 6, \end{cases} \tag{50b}$$

$$h_k = \begin{cases} \underline{n}_o(\underline{x}) \cdot \left\{ \nabla \times \left[\underline{e}_k \times \underline{V}_o(\underline{x};t) \right] \right\}, & k = 1,2,3, \\ \underline{n}_o(\underline{x}) \cdot \left\{ \nabla \times \left[\left(\underline{e}_{k-3} \times \underline{X} \right) \times \underline{V}_o(\underline{x};t) \right] \right\}, & k = 4,5,6, \end{cases} \tag{50c}$$

where

$$\underline{V}_o(\underline{x};t) = \nabla \left(-cx + \varphi_o(\underline{x};t) \right)$$

and

$$\alpha_1 = X_1, \ \alpha_2 = Y_1, \ \alpha_3 = Z_1 \ ; \ \alpha_4 = \theta_1 \ , \ \alpha_5 = \theta_2 , \ \alpha_6 = \theta_3 \ . \tag{50d}$$

Then the vector $\underline{A}$ defined in (30) can be written as

$$\begin{aligned} \underline{A}(\underline{x},t) &= \underline{X}_1(t) + (\theta_1, \theta_2, \theta_3) \times \underline{X} \\ &= \sum_{k=1}^{6} \alpha_k(t) \, \underline{e}_k(\underline{x}) \ . \end{aligned} \tag{51}$$

Thus (37), the boundary condition for φ_1 on S_o, can now be written in the following form:

$$\left. \varphi_{1n}(\underline{x};t) \right|_{S_o} = \sum_{k=1}^{6} n_{ok}(\underline{x}) \, \dot{\alpha}_k(t) + \sum_{k=1}^{6} h_k(\underline{x};t) \, \alpha_k(t) \ . \tag{52}$$

Taking the time derivative of (52), we have

$$\varphi_{1 n t}(\underline{x};t)\Big|_{S_0} = \sum_{k=1}^{6} n_{0k}(\underline{x})\,\ddot{\alpha}_k(t) + \sum_{k=1}^{6} h_k(\underline{x};t)\,\dot{\alpha}_k(t) + \sum_{k=1}^{6} \dot{h}_k(\underline{x};t)\,\alpha_k(t)\,, \tag{53}$$

where

$$\dot{h}_k(\underline{x};t) \equiv \partial h_k(\underline{x};t)/\partial t\,.$$

The condition (53) is now to be substituted into the integrand of the integral over S_0 in (46). If this is done, we have then

$$\left.\begin{aligned}
&\int_0^t d\tau \iint_{S_0} \left\{ G(\underline{x};\underline{\xi};t-\tau)\,\varphi_{1\tau\nu}(\underline{\xi};\tau) - \varphi_{1\tau}\,G_\nu \right\} dS \\
&= \int_0^t d\tau \iint_{S_0} G(\underline{x};\underline{\xi};t-\tau) \left\{ \sum_{k=1}^{6} n_{0k}(\underline{\xi})\,\ddot{\alpha}_k(\tau) + \sum_{k=1}^{6} h_k(\underline{\xi};\tau)\,\dot{\alpha}_k(\tau) + \sum_{k=1}^{6} \dot{h}_k(\underline{\xi};\tau)\,\alpha_k(\tau) \right\} dS \\
&- \int_0^t d\tau \iint_{S_0} \varphi_1(\underline{\xi};\tau)\,G_{\nu t}(\underline{x};\underline{\xi};t-\tau)\,dS \\
&+ \iint_{S_0} \varphi_1(\underline{\xi};0)\,G_\nu(\underline{x};\underline{\xi};t)\,dS - \iint_{S_0} \varphi_1(\underline{\xi};t)\,G_\nu(\underline{x};\underline{\xi};0)\,dS \\
&= I_{S_0}\,.
\end{aligned}\right\} \tag{54}$$

Note that (54) and (49) together equal the right-hand side of (46).

Define the operator $\mathcal{L}$ as follows:

$$
\left.
\begin{aligned}
\mathcal{L}\{\varphi_1\}(\underline{x};t) \equiv\ & 4\pi\,\varphi_1(\underline{x};t) + \iint_{S_0} \varphi_1(\underline{\xi};t)\, G_\nu(\underline{x};\underline{\xi};0)\, ds\ + \\
& + \int_0^t d\tau \iint_{S_0} \varphi_1(\underline{\xi};\tau)\, G_{\nu t}(\underline{x};\underline{\xi};t-\tau)\, ds \\
& - \frac{1}{g}\int_0^t d\tau \iint_F \varphi_1(\xi,0,\zeta,\tau)\frac{\partial}{\partial\xi}\Big\{C(\tau)\, D^2 G(\underline{x};\xi,0,\zeta;t-\tau)\ - \\
& \quad - [3C(\tau)+C(t-\tau)]\frac{\partial}{\partial t}(DG) + [\dot{C}(\tau)-\dot{C}(t-\tau)](DG)\ + \\
& \quad + C(\tau)[C(t-\tau)-C(\tau)]\frac{\partial}{\partial\xi}(DG)\Big\}\, d\xi\, d\zeta \\
& - \int_0^t d\tau \iint_F \varphi_{1\eta}(\xi,0,\zeta,\tau)\Big\{C(t-\tau)\frac{\partial}{\partial\xi} G(\underline{x};\xi,0,\zeta;t-\tau)\Big\}\, d\xi\, d\zeta \\
& - \frac{1}{g}\int_0^t d\tau \oint_{\Gamma_0} \varphi_1(\xi,0,\zeta,\tau)\Big\{2C(\tau)\, D^2 G(\underline{x};\xi,0,\zeta;t-\tau)\ + \\
& \quad + C(\tau)[C(\tau)+2C(t-\tau)]\frac{\partial}{\partial\xi}(DG) - \dot{C}(\tau)(DG)\Big\}\, d\zeta \\
& + \frac{1}{g}\int_0^t d\tau \oint_{\Gamma_0} \varphi_{1\xi}(\xi,0,\zeta,\tau)\Big\{C^2(\tau)\, DG(\underline{x};\xi,0,\zeta;t-\tau)\Big\}\, d\zeta\ .
\end{aligned}
\right\} \quad (55)
$$

With this definition of $\mathcal{L}$, the following integral equation for φ_1 can be obtained from (46), (49) and (54):

$$
\left.
\begin{aligned}
\mathcal{L}\{\varphi_1\}(\underline{x};t) &= 4\pi\,\varphi_1(\underline{x};0) + \iint_{S_0} \varphi_1(\underline{\xi};0)\,G_\nu(\underline{x};\underline{\xi};t)\,ds \\
&- \iint_F \left\{ G\left[D\Upsilon_1(\xi,0,\zeta,0)\right] + \Upsilon_1\left[DG(\underline{x};\xi,0,\zeta;t)\right]\right\} d\xi\, d\zeta \\
&+ \frac{1}{g}\left[c(0)+c(t)\right] \iint_F \varphi_1(\xi,0,\zeta,0)\left\{\frac{\partial}{\partial\xi} G(\underline{x};\xi,0,\zeta;t)\right\} d\xi\, d\zeta \\
&- \frac{1}{g}\, c(0) \oint_{\Gamma_0} \varphi_1(\xi,0,\zeta,0)\left\{DG(\underline{x};\xi,0,\zeta;t)\right\} d\zeta \\
&+ \int_0^t d\tau \iint_{S_0} G(\underline{x};\underline{\xi};t-\tau)\left\{\sum_{k=1}^{6} n_{0k}(\underline{\xi})\,\ddot{\alpha}_k(\tau) + \right. \\
&\qquad \left. + \sum_{k=1}^{6} h_k(\underline{\xi};\tau)\,\dot{\alpha}_k(\tau) + \sum_{k=1}^{6} \dot{h}_k(\underline{\xi};\tau)\,\alpha_k(\tau)\right\} ds.
\end{aligned}
\right\} \quad (56)
$$

IV. Motions of a Ship with Steady Average Forward Speed.

So far our development has been perfectly general in the sense that no restriction is imposed upon the average forward speed. Henceforth we shall, however, assume that c = const., i.e., the average forward speed of the ship is a steady one. As a consequence of this assumption we have now $\varphi_o = \varphi_o(\underline{x})$, $h_k = h_k(\underline{x})$, and $\underline{V}_o = \underline{V}_o(\underline{x})$, i.e., they become independent of time. Thus the only time-dependent functions are the Green's function $G(\underline{x};\underline{\xi};t)$, the unsteady velocity potential $\varphi_1(\underline{x};t)$ and the various displacements $\alpha_k(t)$. The integral operator defined in (55) and the integral equation (56) now become, respectively,

$$
\left.\begin{aligned}
\mathcal{L}\{\varphi_1\}(\underline{x};t) \equiv\; & 4\pi\,\varphi_1(\underline{x};t) + \iint_{S_o} \varphi_1(\underline{\xi};t)\,G_\nu(\underline{x};\underline{\xi};0)\,ds \\
& + \int_0^t d\tau \iint_{S_o} \varphi_1(\underline{\xi};\tau)\,G_{\nu t}(\underline{x};\underline{\xi};t-\tau)\,ds \\
& - \frac{c}{g}\int_0^t d\tau \iint_F \varphi_1(\xi,0,\zeta,\tau)\frac{\partial}{\partial\xi}\Big\{D^2G(\underline{x};\xi,0,\zeta;t-\tau) - \\
& \qquad - 4\frac{\partial}{\partial t}(DG)\Big\}\,d\xi\,d\zeta \\
& - c\int_0^t d\tau \iint_F \varphi_{1\eta}(\xi,0,\zeta,\tau)\Big\{\frac{\partial}{\partial\xi}G(\underline{x};\xi,0,\zeta;t-\tau)\Big\}\,d\xi\,d\zeta \\
& - \frac{c}{g}\int_0^t d\tau \oint_{\Gamma_o} \varphi_1(\xi,0,\zeta,\tau)\Big\{2\,D^2G(\underline{x};\xi,0,\zeta;t-\tau) + \\
& \qquad + 3c\frac{\partial}{\partial\xi}(DG)\Big\}\,d\zeta \\
& + \frac{c^2}{g}\int_0^t d\tau \oint_{\Gamma_o} \varphi_{1\xi}(\xi,0,\zeta,\tau)\Big\{DG(\underline{x};\xi,0,\zeta;t-\tau)\Big\}\,d\zeta ,
\end{aligned}\right\} \quad (57)
$$

and

$$\left.\begin{aligned}
\mathcal{L}\{\varphi_1\}(\underline{x};t) &= 4\pi\varphi_1(\underline{x};0) + \iint_{S_0} \varphi_1(\underline{\xi};0)\,G_\nu(\underline{x};\underline{\xi};t)\,ds \\
&- \iint_F \left\{ G\left[D\,\Upsilon_1(\xi,\zeta;0)\right] + \Upsilon_1\left[DG(\underline{x};\xi,0,\zeta;t)\right] \right\} d\xi\, d\zeta \\
&+ \frac{2c}{g} \iint_F \varphi_1(\xi,0,\zeta;0) \left\{ \frac{\partial}{\partial\xi} DG(\underline{x};\xi,0,\zeta;t) \right\} d\xi\, d\zeta \\
&- \frac{c}{g} \oint_{\Gamma_0} \varphi_1(\xi,0,\zeta,0) \left\{ DG(\underline{x};\xi,0,\zeta;t) \right\} d\zeta \\
&+ \int_0^t d\tau \iint_{S_0} G(\underline{x};\underline{\xi};t-\tau) \left\{ \sum_{k=1}^{6} n_{0k}(\underline{\xi})\,\ddot{\alpha}_k(\tau) \, + \right. \\
&\qquad\qquad \left. + \sum_{k=1}^{6} h_k(\underline{\xi})\,\dot{\alpha}_k(\tau) \right\} ds.
\end{aligned}\right\} \quad (58)$$

We see that integrals over the undisturbed free-surface F are involved in the expressions on both sides of the equality in (58). The behavior of Green's function G at infinity of our problem makes these integrals converge even though the domain of integration F actually extends to infinity. In case of zero average forward speed, i.e., c = 0, all the integrals over F and around Γ_0 in the definition of the integral operator $\mathcal{L}$ disappear. Hence equation (58) in this particular case can be made to yield an integral equation for a function defined only on the reference surface S_0 by letting the singular point $\underline{x} = (x, y, z)$ converge to a point of the surface S_0. However, with non-zero average forward speed, we shall inevitably deal with an integral operator involving both surfaces S_0 and F. Therefore, a considerably greater difficulty should be anticipated in solving an integral equation obtained from (58).

The following identity may be established from (57):

$$\left.\begin{aligned}
\frac{\partial}{\partial t}\Big[\mathcal{L}\{\varphi_1\}(\underline{x};t)\Big] = {} & \mathcal{L}\{\varphi_{1t}\}(\underline{x};t) \\
& + \iint_{S_0} \varphi_1(\underline{\xi};0)\, G_{\nu t}(\underline{x};\underline{\xi};t)\, ds \\
& - \frac{c}{g}\iint_F \varphi_1(\xi,0,\zeta,0)\frac{\partial}{\partial \xi}\Big\{D^2 G(\underline{x};\xi,0,\zeta;t) - \\
& \qquad\qquad - 4\frac{\partial}{\partial t}(DG)\Big\}\, d\xi\, d\zeta \\
& - c\iint_F \varphi_{1\eta}(\xi,0,\zeta,0)\Big\{\frac{\partial}{\partial \xi} G(\underline{x};\xi,0,\zeta;t)\Big\}\, d\xi\, d\zeta \\
& - \frac{c}{g}\oint_{\Gamma_0} \varphi_1(\xi,0,\zeta,0)\Big\{2D^2 G + 3c\frac{\partial}{\partial \xi}(DG)\Big\}\, d\zeta \\
& + \frac{c^2}{g}\oint_{\Gamma_0} \varphi_{1\xi}(\xi,0,\zeta,0)\Big\{DG(\underline{x};\xi,0,\zeta;t)\Big\}\, d\zeta .
\end{aligned}\right\} \quad (59)$$

From (59) and (56) we find that φ_{1t} must satisfy

$$\left.\begin{aligned}
\mathcal{L}\{\varphi_{1t}\}(\underline{x};t) = &-\iint_F \Big\{ G_t[DY_1(\xi,\zeta,0)] \\
&\qquad + Y_1\left[\tfrac{\partial}{\partial t} DG(\underline{x};\xi,0,\zeta;t)\right]\Big\} d\xi\, d\zeta \\
&+ \frac{c}{g}\iint_F \varphi_1(\xi,0,\zeta,0)\frac{\partial}{\partial \xi}\left\{D^2G(\underline{x};\xi,0,\zeta,t) - 2\frac{\partial}{\partial t}(DG)\right\} d\xi\, d\zeta \\
&+ c\iint_F \varphi_{1\eta}(\xi,0,\zeta,0)\left\{\frac{\partial}{\partial \xi} G(\underline{x};\xi,0,\zeta;t)\right\} d\xi\, d\zeta \\
&- \frac{c^2}{g}\oint_{\Gamma_0} \varphi_{1\xi}(\xi,0,\zeta,0)\left\{DG(\underline{x};\xi,0,\zeta;t)\right\} d\zeta \\
&+ \frac{c}{g}\oint_{\Gamma_0} \varphi_1(\xi,0,\zeta,0)\Big\{D^2G(\underline{x};\xi,0,\zeta;t) + 3c\frac{\partial}{\partial \xi}(DG) - \\
&\qquad - \frac{\partial}{\partial t}(DG)\Big\} d\zeta \\
&+ \iint_{S_0} G(\underline{x};\underline{\xi};0)\left\{\sum_{k=1}^{6} n_{0k}(\underline{\xi})\ddot{\alpha}_k(t) + \sum_{k=1}^{6} h_k(\underline{\xi})\dot{\alpha}_k(t)\right\} dS \\
&+ \int_0^t d\tau \iint_{S_0} G_t(\underline{x};\underline{\xi};t-\tau)\left\{\sum_{k=1}^{6} n_{0k}(\underline{\xi})\ddot{\alpha}_k(\tau) + \sum_{k=1}^{6} h_k(\underline{\xi})\dot{\alpha}_k(\tau)\right\} dS
\end{aligned}\right\} \quad (60)$$

It would be desirable to establish the uniqueness of solution of the integral equations (58) and (60) and of equations to appear later which are of the same form, namely,

$$\mathcal{L}\{\varphi\} = f .$$

That is, we hsould like to prove that $\mathcal{L}\{\varphi\} = 0$ implies that $\varphi = 0$. Although this can be established for c = 0, we have not been able to prove it for $c \neq 0$. However, it seems very likely that this is so and we shall assume henceforth that it can be established.

From the physical situation we are considering here and from the linearity of the problem, it seems to be clear that the unsteady part of the velocity potential has the following constitution:

$$\varphi_1 = \varphi_F + \varphi_I + \varphi_D ,$$

where φ_F , φ_I , and φ_D represent, respectively, fluid motions due to a) forced oscillation of the body, b) the incoming waves and c) the diffracted waves. In order to consider a general situation, we do not wish to assume that the body starts oscillating from a state of rest relative to $Oxyz$ at the initial instant $t = 0$; on the other hand, it is desirable to do so for the convenience of the type of decomposition which we shall consider in the next chapter. As a possible approach to solving such a dilemma, we shall assume that the velocity potential φ_F may be further divided as follows:

$$\varphi_F = \varphi_{F0} + \varphi_{F1} ,$$

where φ_{F0} describes the fluid motion which would take place as a result of only the given motion of the body at the initial instant and φ_{F1} represents the fluid motion due to the oscillation of a body which has started from a state of rest and has achieved the given initial motion of the body instantly at $t=0$. Hence φ_{F0} and φ_{F1}, satisfy, respectively, the following boundary conditions:

$$\varphi_{F0n}(\underline{x};0)\Big|_{S_0} = \varphi_{1n}(\underline{x};0)\Big|_{S_0}, \quad \varphi_{F0n}(\underline{x};t)\Big|_{S_0} = 0 \quad \text{for } t>0;$$

and

$$\varphi_{F1n}(\underline{x};0)\Big|_{S_0} = 0, \quad \varphi_{F1n}(\underline{x};t)\Big|_{S_0} = \varphi_{1n}(\underline{x};t)\Big|_{S_0} \quad \text{for } t>0.$$

For convenience, let us henceforth write

$$\varphi_1 = \varphi_{F1} + \varphi_W, \tag{61}$$

where $\varphi_W = \varphi_{F0} + \varphi_I + \varphi_D$.

The two functions φ_{F1} and φ_W for $t>0$ are then defined, respectively, by the following equations:

$$\mathcal{L}\{\varphi_{F1}\}(\underline{x};t) = \int_0^t d\tau \iint_{S_0} G(\underline{x};\underline{\xi};t-\tau)\, \varphi_{1t\nu}(\underline{\xi};\tau)\, dS$$

$$= \int_0^t d\tau \iint_{S_0} G(\underline{x};\underline{\xi};t-\tau)\Big\{\sum_{k=1}^{6} n_{0k}(\underline{\xi})\,\ddot{\alpha}_k(\tau) +$$

$$+ \sum_{k=1}^{6} h_k(\underline{\xi})\,\dot{\alpha}_k(\tau)\Big\}\, dS, \tag{62}$$

and

$$\left.\begin{aligned}
\mathcal{L}\{\varphi_W\}(\underline{x};t) = {} & 4\pi\varphi_1(\underline{x};0) + \iint_{S_0} \varphi_1(\underline{\xi};0)\,G_\nu(\underline{x};\underline{\xi};t)\,ds \\
& - \iint_F \{G\,[DY_1(\xi,\zeta,0)] + Y_1\,[DG(\underline{x};\xi,0,\zeta;t)]\}\,d\xi\,d\zeta \\
& + \frac{2c}{g}\iint_F \varphi_1(\xi,0,\zeta,0)\left\{\frac{\partial}{\partial\xi}DG(\underline{x};\xi,0,\zeta;t)\right\}d\xi\,d\zeta \\
& - \frac{c}{g}\oint_{\Gamma_0} \varphi_1(\xi,0,\zeta,0)\,\{DG(\underline{x};\xi,0,\zeta;t)\}\,d\zeta\,.
\end{aligned}\right\} \tag{63}$$

φ_{F1} , and φ_W , respectively, satisfy the following initial and boundary conditions:

$$\varphi_{F1}(\underline{x};0) = 0\,, \qquad \varphi_{F1t}(x,0,z,0) = 0\,, \tag{64a}$$

$$\varphi_{F1n}(\underline{x};0)\Big|_{S_0} = 0\,, \tag{64b}$$

$$\begin{aligned}
\varphi_{F1n}(\underline{x};t)\Big|_{S_0} &= \varphi_{1n}(\underline{x};t)\Big|_{S_0} \\
&= \sum_{k=1}^{6} n_{0k}(\underline{x})\,\dot{\alpha}_k(t) + \sum_{k=1}^{6} h_k(\underline{x})\,\alpha_k(t)
\end{aligned} \tag{64c}$$

for $t>0$,

$$\varphi_W(\underline{x};0) = \varphi_1(\underline{x};0), \tag{65a}$$

$$\varphi_{Wt}(x,0,z,0) = \varphi_{1t}(x,0,z,0), \tag{65b}$$

$$\begin{aligned}\varphi_{Wn}(\underline{x};0)\Big|_{S_0} &= \varphi_{1n}(\underline{x};0)\Big|_{S_0} \\ &= \sum_{k=1}^{6} n_{0k}(\underline{x})\dot{\alpha}_k(0) + \sum_{k=1}^{6} h_k(\underline{x})\alpha_k(0),\end{aligned} \tag{65c}$$

$$\varphi_{Wn}(\underline{x};t)\Big|_{S_0} = 0 \qquad \text{for} \quad t > 0. \tag{65d}$$

Thus, φ_W describes that unsteady part of fluid motion which would take place with the given initial conditions and a body fixed relative to the translating reference frame.

Lastly, let us state in the following an important property of the integral operator defined in (57): Let $\beta(t)$ and $\psi(\underline{x};t)$ be any integrable functions and $\mathcal{L}$ be the integral operator defined in (57); then

$$\mathcal{L}\left\{\int_0^t du\,\beta(u)\,\psi(\underline{x};t-u)\right\} = \int_0^t \beta(u)\,\mathcal{L}\{\psi\}(\underline{x};t-u)\,du. \tag{66}$$

To prove this, let the following substitutions be made in (57):

$$\varphi_1(\underline{x};t) = \int_0^t \beta(u)\,\psi(\underline{x};t-u)\,du\ ,$$

$$\varphi_1(\underline{\xi};\tau) = \int_0^\tau \beta(u)\,\psi(\underline{x};\tau-u)\,du\ , \qquad \text{etc.}$$

Then from (57) we have

$$\mathcal{L}\left\{\int_0^t \beta(u)\psi(\underline{x};t-u)\,du\right\} \equiv 4\pi\int_0^t \beta(u)\psi(\underline{x};t-u)\,du$$

$$+\iint_{S_0} ds\int_0^t du\,\beta(u)\,\psi(\underline{\xi};t-u)\,G_\nu(\underline{x};\underline{\xi};0)$$

$$+\int_0^t d\tau\iint_{S_0} ds\int_0^\tau du\,\beta(u)\,\psi(\underline{x};\tau-u)\,G_{\nu t}(\underline{x};\underline{\xi};t-\tau)$$

$$-\frac{c}{g}\int_0^t d\tau\iint_F d\xi\,d\zeta\int_0^\tau du\,\beta(u)\,\psi(\xi,0,\zeta;\tau-u)\frac{\partial}{\partial\xi}\Big\{D^2G -$$

$$-4\frac{\partial}{\partial t}\big[DG(\underline{x};\xi,0,\zeta;t-\tau)\big]\Big\}$$

$$-c\int_0^t d\tau\iint_F d\xi\,d\zeta\int_0^\tau du\,\beta(u)\,\psi_\eta(\xi,0,\zeta,\tau-u)\Big\{\frac{\partial}{\partial\xi}G(\underline{x};\xi,0,\zeta;t-\tau)\Big\}$$

$$-\frac{c}{g}\int_0^t d\tau\oint_{\Gamma_0} d\zeta\int_0^\tau du\,\beta(u)\,\psi(\xi,0,\zeta,\tau-u)\Big\{2D^2G+3c\frac{\partial}{\partial\xi}(DG)\Big\}$$

$$+\frac{c^2}{g}\int_0^t d\tau\oint_{\Gamma_0} d\zeta\int_0^\tau du\,\beta(u)\,\psi_\xi(\xi,0,\zeta,\tau-u)\Big\{DG(\underline{x};\xi,0,\zeta;t-\tau)\Big\}$$

$$=$$

$$
\begin{aligned}
= &\int_0^t du\,\beta(u)\,4\pi\,\psi(\underline{x};t-u) + \int_0^t du\,\beta(u)\iint_{S_0}\psi(\underline{\xi};t-u)\,G_\nu(\underline{x};\underline{\xi};0)\,dS \\
&+\int_0^t d\tau\int_0^\tau du\,\beta(u)\iint_{S_0}\psi(\underline{\xi};\tau-u)\,G_{\nu t}(\underline{x};\underline{\xi};t-\tau)\,dS \\
&-\frac{c}{g}\int_0^t d\tau\int_0^\tau du\,\beta(u)\iint_F \psi(\xi,0,\zeta;\tau-u)\frac{\partial}{\partial\xi}\Big\{D^2G(\underline{x};\xi,0,\zeta;t-\tau)- \\
&\qquad\qquad -4\frac{\partial}{\partial t}(DG)\Big\}\,d\xi\,d\zeta \\
&-c\int_0^t d\tau\int_0^\tau du\,\beta(u)\iint_F \psi_\eta(\xi,0,\zeta;\tau-u)\Big\{\frac{\partial}{\partial\xi}G(\underline{x};\xi,0,\zeta;t-\tau)\Big\}\,d\xi\,d\zeta \\
&-\frac{c}{g}\int_0^t d\tau\int_0^\tau du\,\beta(u)\oint_{\Gamma_0}\psi(\xi,0,\zeta;\tau-u)\Big\{2D^2G+3c\frac{\partial}{\partial\xi}(DG)\Big\}\,d\zeta \\
&+\frac{c^2}{g}\int_0^t d\tau\int_0^\tau du\,\beta(u)\oint_{\Gamma_0}\psi_\xi(\xi,0,\zeta;\tau-u)\Big\{DG(\underline{x};\xi,0,\zeta;t-\tau)\Big\}\,d\zeta \,.
\end{aligned}
$$

Note that in obtaining the last expression the integral $\int_0^\tau du\,\beta(u)$ has been moved outside of the surface- and contour-integral signs. This operation is justified since the domains of integration S_0 , F and Γ_0 are independent of the parameters τ and u . Recall the following formula from analysis, sometimes called Dirichlet's formula: If $f(\tau,u)$ is continuous in the domain $0\le\tau\le t$, $0\le u\le t$, then

$$
\begin{aligned}
\int_0^t d\tau\int_0^\tau du\,f(\tau,u) &= \int_0^t du\int_u^t d\tau\,f(\tau,u) \\
&= \int_0^t du\int_0^{t-u} d\tau'\,f(\tau'+u,u)
\end{aligned}
$$

where we have put $\tau' \equiv \tau - u$.

With the use of this formula, the last expression becomes

$$\int_0^t du\,\beta(u)\, 4\pi\,\psi(\underline{x}; t-u) + \int_0^t du\,\beta(u) \iint_{S_0} \psi(\underline{\xi}; t-u)\, G_\nu(\underline{x}; \underline{\xi}; 0)\, ds$$

$$+ \int_0^t du\,\beta(u) \int_0^{t-u} d\tau' \iint_{S_0} \psi(\underline{\xi}; \tau')\, G_{\nu t}(\underline{x}; \underline{\xi}; t-u-\tau')\, ds$$

$$- \frac{c}{g} \int_0^t du\,\beta(u) \int_0^{t-u} d\tau' \iint_F \psi(\xi, 0, \zeta; \tau') \frac{\partial}{\partial \xi} \Big\{ D^2 G(\underline{x}; \xi, 0, \zeta; t-u-\tau') -$$

$$- 4 \frac{\partial}{\partial t}(DG) \Big\}\, d\xi\, d\zeta$$

$$- c \int_0^t du\,\beta(u) \int_0^{t-u} d\tau' \iint_F \psi_\eta(\xi, 0, \zeta; \tau') \Big\{ \frac{\partial}{\partial \xi} G(\underline{x}; \xi, 0, \zeta; t-u-\tau') \Big\}\, d\xi\, d\zeta$$

$$- \frac{c}{g} \int_0^t du\,\beta(u) \int_0^{t-u} d\tau' \oint_{\Gamma_0} \psi(\xi, 0, \zeta; \tau') \Big\{ 2 D^2 G + 3c \frac{\partial}{\partial \xi}(DG) \Big\}\, d\zeta$$

$$+ \frac{c^2}{g} \int_0^t du\,\beta(u) \int_0^{t-u} d\tau' \oint_{\Gamma_0} \psi_\xi(\xi, 0, \zeta; \tau') \Big\{ DG(\underline{x}; \xi, 0, \zeta, t-u-\tau') \Big\}\, d\zeta$$

$$= \int_0^t du\,\beta(u)\, \mathcal{L}\{\psi\}(\underline{x}; t-u),$$

which completes the proof of the proposition.

V. Decomposition of the Velocity Potential φ_{F1}.

Recall that in (62) we wrote down an integral equation to be satisfied by φ_{F1} which can also be written as

$$\mathcal{L}\{\varphi_{F1}\}(\underline{x};t) = \sum_{k=1}^{6}\int_0^t \ddot{\alpha}_k(\tau)\iint_{S_o} n_{ok}(\underline{\xi})\,G(\underline{x};\underline{\xi};t-\tau)\,ds$$

$$+\sum_{k=1}^{6}\int_0^t \dot{\alpha}_k(\tau)\iint_{S_o} h_k(\underline{\xi})\,G(\underline{x};\underline{\xi};t-\tau)\,ds\,. \qquad (67a)$$

Let us next consider in the following two boundary-value problems. Suppose that $\varphi_{F1}^{(1)}$ and $\varphi_{F1}^{(2)}$ are two potential functions which, in addition to conditions similar to those satisfied by the potential function φ_1 on the undisturbed free-surface F, on the bottom B and at infinity (i.e., those stated in (35), (36), and (38) or (38a)), satisfy the following initial and boundary conditions:

$$\varphi_{F1}^{(\ell)}(\underline{x};0) = 0\,,\quad \varphi_{F1t}^{(\ell)}(x,0,z,0) = 0\,,\quad \ell = 1,2\,; \qquad (67b)$$

$$\varphi_{F1n}^{(1)}(\underline{x};t)\Big|_{S_o} = \sum_{k=1}^{6} n_{ok}(\underline{x})\,\dot{\alpha}_k(t) \qquad \text{for } t>0\,; \qquad (67c)$$

$$\varphi_{F1n}^{(2)}(\underline{x};t)\Big|_{S_o} = \sum_{k=1}^{6} h_k(\underline{x})\,\alpha_k(t) \qquad \text{for } t>0\,. \qquad (67d)$$

By precisely the same analysis used in deriving (58), the integral equation for the function φ_1, we may obtain the following two equations satisfied by $\varphi_{F1}^{(1)}$ and $\varphi_{F1}^{(2)}$, respectively:

$$\mathcal{L}\{\varphi_{F1}^{(1)}\}(\underline{x};t) = \sum_{k=1}^{6}\int_0^t d\tau\, \ddot{\alpha}_k(\tau)\iint_{S_0} n_{0k}(\underline{\xi})G(\underline{x};\underline{\xi};t-\tau)\,ds, \tag{67e}$$

and

$$\mathcal{L}\{\varphi_{F1}^{(2)}\}(\underline{x};t) = \sum_{k=1}^{6}\int_0^t d\tau\, \dot{\alpha}_k(\tau)\iint_{S_0} h_k(\underline{\xi})G(\underline{x};\underline{\xi};t-\tau)\,ds. \tag{67f}$$

Since the operator $\mathcal{L}$ is linear, adding (67e) and (67f) together gives the equation

$$\begin{aligned}\mathcal{L}\{\varphi_{F1}^{(1)} + \varphi_{F1}^{(2)}\}(\underline{x};t) &= \sum_{k=1}^{6}\int_0^t d\tau\, \ddot{\alpha}_k(\tau)\iint_{S_0} n_{0k}(\underline{\xi})G(\underline{x};\underline{\xi};t-\tau)\,ds \\ &+ \sum_{k=1}^{6}\int_0^t d\tau\, \dot{\alpha}_k(\tau)\iint_{S_0} h_k(\underline{\xi})G(\underline{x};\underline{\xi};t-\tau)\,ds.\end{aligned} \tag{67g}$$

If, as we have assumed, integral equations of the form

$$\mathcal{L}\{\varphi\} = f$$

have unique solutions, then we can conclude that

$$\varphi_{F1} = \varphi_{F1}^{(1)} + \varphi_{F1}^{(2)}. \tag{67h}$$

Define the two functions $\mathcal{K}_k^{(1)}$ and $\mathcal{K}_k^{(2)}$ to be the solutions of the following integral equations, respectively:

$$\mathcal{L}\{\mathcal{K}_k^{(1)}\}(\underline{x};t) = \iint_{S_o} n_{ok}(\underline{\xi})\, G(\underline{x};\underline{\xi};t)\, ds, \qquad k = 1, 2, \cdots, 6, \tag{68a}$$

and

$$\mathcal{L}\{\mathcal{K}_k^{(2)}\}(\underline{x};t) = \iint_{S_o} h_k(\underline{\xi})\, G(\underline{x};\underline{\xi};t)\, ds, \qquad k = 1, 2, \cdots, 6. \tag{68b}$$

With the assumption of the uniqueness of the solutions of the equations (68a) and (68b) it can be shown easily that $\mathcal{K}_k^{(1)}$ and $\mathcal{K}_k^{(2)}$ satisfy, respectively, the following conditions:

$$\mathcal{K}_{kn}^{(1)}(\underline{x};t)\Big|_{S_o} = n_{ok}(\underline{x}), \qquad \text{for} \quad t > 0, \tag{69a}$$

and

$$\mathcal{K}_{kn}^{(2)}(\underline{x};t)\Big|_{S_o} = h_k(\underline{x}), \qquad \text{for} \quad t > 0, \qquad k = 1, 2, \cdots, 6. \tag{69b}$$

Besides, they satisfy also the same free-surface condition, conditions on the bottom and at infinity, as those of φ_1 .

We now claim that the following decompositions of the potential functions $\varphi_{F1}^{(1)}$ and $\varphi_{F1}^{(2)}$ satisfy, respectively, the integral equations (67e) and (67f):

$$\varphi_{F1}^{(1)}(\underline{x};t) = \sum_{k=1}^{6} \int_0^t \ddot{\alpha}_k(\tau)\, \mathcal{K}_k^{(1)}(\underline{x}; t-\tau)\, d\tau , \tag{70a}$$

and

$$\varphi_{F1}^{(2)}(\underline{x};t) = \sum_{k=1}^{6} \int_0^t \dot{\alpha}_k(\tau)\, \mathcal{K}_k^{(2)}(\underline{x}; t-\tau)\, d\tau . \tag{70b}$$

To verify this statement, let us apply the operator $\mathcal{L}$ to both sides of, say, (70a). Then we have

$$\begin{aligned}
\mathcal{L}\{\varphi_{F1}^{(1)}\}(\underline{x};t) &= \sum_k \mathcal{L}\left\{\int_0^t \ddot{\alpha}_k(\tau)\, \mathcal{K}_k^{(1)}(\underline{x}; t-\tau)\, ds\right\} \\
&= \sum_k \int_0^t \ddot{\alpha}_k(\tau)\, \mathcal{L}\{\mathcal{K}_k^{(1)}\}(\underline{x}; t-\tau)\, d\tau \\
&= \sum_k \int_0^t d\tau\, \ddot{\alpha}_k(\tau) \iint_{S_0} n_{0k}(\underline{\xi})\, G(\underline{x};\underline{\xi};t-\tau)\, dS ,
\end{aligned}$$

where we have made use of the property of $\mathcal{L}$ stated in (66) and equation (68a). From (67h), (70a), and (70b) it is clear now that the function φ_{F1} has the following decomposition:

$$\left.\begin{aligned}
\varphi_{F1}(\underline{x};t) &= \sum_{k=1}^{6} \int_0^t \ddot{\alpha}_k(\tau)\, \mathcal{K}_k^{(1)}(\underline{x}; t-\tau)\, d\tau \\
&+ \sum_{k=1}^{6} \int_0^t \dot{\alpha}_k(\tau)\, \mathcal{K}_k^{(2)}(\underline{x}; t-\tau)\, d\tau .
\end{aligned}\right\} \tag{70c}$$

We now show that the decomposition (70c) toegether with the boundary conditions (69a), (69b) and those satisfied by $\mathcal{K}_k^{(1)}$ and $\mathcal{K}_k^{(2)}$ on the undisturbed free surface, the bottom, and at infinity in fact is essentially that of Cummins (1962) as it was later modified by Ogilvie (1964). Henceforth, reference will be made to Ogilvie's (1964) paper.

After integrating by parts, the decomposition (70c) takes the form

$$\left.\begin{aligned}\varphi_{F1}(\underline{x};t) &= \sum_{k=1}^{6}\dot{\alpha}_k(t)\mathcal{K}_k^{(1)}(\underline{x};+0) + \sum_{k=1}^{6}\alpha_k(t)\mathcal{K}_k^{(2)}(\underline{x};+0)\\ &+\sum_{k=1}^{6}\int_0^t \mathcal{K}_{kt}^{(1)}(\underline{x};t-\tau)\dot{\alpha}_k(\tau)\,d\tau + \sum_{k=1}^{6}\int_0^t \mathcal{K}_k^{(2)}(\underline{x};t-\tau)\alpha_k(\tau)\,d\tau\\ &-\sum_{k=1}^{6}\dot{\alpha}_k(0)\mathcal{K}_k^{(1)}(\underline{x};t) - \sum_{k=1}^{6}\alpha_k(0)\mathcal{K}_k^{(2)}(\underline{x};t)\,.\end{aligned}\right\}\tag{71}$$

On the other hand, Ogilvie introduced two sets of functions and proposed the following decomposition:

$$\left.\begin{aligned}\varphi_{F1}(\underline{x};t) &= \sum_{k=1}^{6}\dot{\alpha}_k(t)\psi_{1k}(\underline{x}) + \sum_{k=1}^{6}\alpha_k(t)\psi_{2k}(\underline{x})\\ &+\sum_{k=1}^{6}\int_{-\infty}^{t}\chi_{1k}(\underline{x};t-\tau)\dot{\alpha}_k(\tau)\,d\tau\\ &+\sum_{k=1}^{6}\int_{-\infty}^{t}\chi_{2k}(\underline{x};t-\tau)\alpha_k(\tau)\,d\tau\,.\end{aligned}\right\}\tag{72}$$

Thus, the decomposition (71) differs from that of Ogilvie in only inessential ways. Instead of starting with initial data at $t=0$, Ogilvie's decomposition starts from a state of rest at $t=-\infty$. In fact, the functions $\psi_{1k}(\underline{x})$, $\psi_{2k}(\underline{x})$, $\chi_{1k}(\underline{x};t)$, and $\chi_{2k}(\underline{x};t)$ can be identified, in that order, with $\mathcal{K}_k^{(1)}(\underline{x};+0)$, $\mathcal{K}_k^{(2)}(\underline{x};+0)$, $\mathcal{K}_{kt}^{(1)}(\underline{x};t)$ and $\mathcal{K}_{kt}^{(2)}(\underline{x};t)$, respectively. If this is done, then according to the conditions imposed upon the functions $\psi_{jk}(\underline{x})$ and

$\chi_{jk}(\underline{x};t)$ by Ogilvie, our present functions $\mathcal{K}_k^{(1)}$ and $\mathcal{K}_k^{(2)}$ must satisfy the following conditions:

$$\mathcal{K}_k^{(1)}(x,0,z,+0) = 0\ , \quad \mathcal{K}_k^{(2)}(x,0,z,+0) = 0 \qquad \text{on} \quad F\ ; \tag{73a}$$

$$\mathcal{K}_{kn}^{(1)}(\underline{x};+0)\Big|_{S_0} = n_{0k}(\underline{x})\ , \quad \mathcal{K}_{kn}^{(2)}(\underline{x};+0)\Big|_{S_0} = h_k(\underline{x})\ ; \tag{73b}$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)^2 \mathcal{K}_{kt}^{(\ell)}(x,0,z;t) + g\,\mathcal{K}_{kty}^{(\ell)} = 0,\ \ell = 1,2\ ; \tag{73c}$$

$$\mathcal{K}_{ktn}^{(1)}(\underline{x};t)\Big|_{S_0} = 0\ , \quad \mathcal{K}_{ktn}^{(2)}(\underline{x};t)\Big|_{S_0} = 0\ ; \tag{73d}$$

$$\mathcal{K}_{kt}^{(1)}(\underline{x};+0) = 0\ , \quad \mathcal{K}_{kt}^{(2)}(\underline{x};+0) = 0\ ; \tag{73e}$$

$$\mathcal{K}_{ktt}^{(1)}(x,0,z,+0) = -g\,\mathcal{K}_{ky}^{(1)}\ , \quad \mathcal{K}_{ktt}^{(2)}(x,0,z,+0) = -g\,\mathcal{K}_{ky}^{(2)}\ . \tag{73f}$$

Let us show in the following that these conditions are indeed satisfied. We see that (73b) and (73c) are already satisfied, for they are the same as those conditions originally satisfied by $\mathcal{K}_k^{(1)}$ and $\mathcal{K}_k^{(2)}$ in (69a) through (69c). The conditions (73d) follow immediately from (69a) and (69b) since both n_k and h_k are independent of time here. In order to verify (73a), we need the equations to be satisfied by $\mathcal{K}_k^{(1)}(\underline{x};+0)$ and $\mathcal{K}_k^{(2)}(\underline{x};+0)$. They may be obtained from (68a) and (68b) by setting $t = +0$. Thus from (68a), (68b), and from the operator $\mathcal{L}$ defined in (57) we have

$$4\pi \mathcal{K}_k^{(1)}(\underline{x};+0) + \iint_{S_0} \mathcal{K}_k^{(1)}(\underline{\xi};+0)\, G_\nu(\underline{x};\underline{\xi};0)\, ds = \iint_{S_0} n_{0k}(\underline{\xi})\, G(\underline{x};\underline{\xi};0)\, ds\,, \tag{74a}$$

$$4\pi \mathcal{K}^{(2)}(\underline{x};+0) + \iint_{S_0} \mathcal{K}_k^{(2)}(\underline{\xi};+0)\, G_\nu(\underline{x};\underline{\xi};0)\, ds = \iint_{S_0} h_k(\underline{\xi})\, G(\underline{x};\underline{\xi};0)\, ds. \tag{74b}$$

Recall that in (42b) and (43) we had $G(\underline{x};\xi,0,\zeta;0)=0$ and $G(\underline{x};\underline{\xi};t) = G(\underline{\xi};\underline{x};t)$, so that, for $\underline{X} = (x,0,z)$ we have

$$G(x,0,z;\xi,\eta,\zeta;0) = G(\xi,\eta,\zeta;x,0,z;0) = 0 \tag{75}$$

for all ξ,η,ζ. But then also

$$G_\xi(x,0,z;\xi,\eta,\zeta;0) = G_\eta = G_\zeta = 0 \quad \text{for } \underline{X} \text{ on } F. \tag{76}$$

Hence

$$G_\nu(x,0,z;\xi,\eta,\zeta;0) = n_{01} G_\xi + n_{02} G_\eta + n_{03} G_\zeta = 0. \tag{77}$$

Therefore, for $\underline{X} = (x,0,z)$ on F, the equations for $\mathcal{K}_k^{(1)}(\underline{x};+0)$ and $\mathcal{K}_k^{(2)}(\underline{x};+0)$, i.e., (74a) and (74b) reduce to $\mathcal{K}_k^{(1)}(x,0,z;+0)=0$ and $\mathcal{K}_k^{(2)}(x,0,z;+0)=0$. Thus (73a) is also satisfied. There remain the last two conditions to be verified, namely, (73e) and (73f). For this purpose, we need equations to be satisfied by $\mathcal{K}_{kt}^{(1)}(\underline{x};+0)$ and $\mathcal{K}_{kt}^{(2)}(\underline{x};+0)$. They may be obtained by differentiating both sides of the equations (68a) and (68b) with respect to t and the identity (59). For instance, the equation for $\mathcal{K}_{kt}^{(1)}$ is obtained as follows:

$$\mathcal{L}\{\mathcal{K}_{kt}^{(1)}\}(\underline{x};t) = \iint_{S_0} n_{0k}(\underline{\xi})\, G_t(\underline{x};\underline{\xi};t)\, ds$$

$$- \iint_{S_0} \mathcal{K}_k^{(1)}(\underline{\xi};+0)\, G_{\nu t}(\underline{x};\underline{\xi};t)\, ds$$

$$+ \frac{c}{g} \iint_F \mathcal{K}_k^{(1)}(\xi,0,\zeta,+0) \frac{\partial}{\partial \xi}\left\{D^2 G(\underline{x};\xi,0,\zeta;t) - 4\frac{\partial}{\partial t}(DG)\right\} d\xi\, d\zeta$$

$$+ c \iint_F \mathcal{K}_{k\eta}^{(1)}(\xi,0,\zeta,+0)\left\{\frac{\partial}{\partial \xi} G(\underline{x};\xi,0,\zeta;t)\right\} d\xi\, d\zeta$$

$$+ \frac{c}{g} \oint_{\Gamma_0} \mathcal{K}_k^{(1)}(\xi,0,\zeta,+0)\left\{2 D^2 G + 3c\frac{\partial}{\partial \xi}(DG)\right\} d\zeta$$

$$- \frac{c^2}{g} \oint_{\Gamma_0} \mathcal{K}_{k\xi}^{(1)}(\xi,0,\zeta,+0)\left\{D G(\underline{x};\xi,0,\zeta;t)\right\} d\zeta\, .$$

But we have just shown that $\mathcal{K}_k^{(1)}(\xi,0,\zeta,+0) = 0$ (see also (73a)). Hence the last equation simplifies to

$$\left.\begin{aligned} \mathcal{L}\{\mathcal{K}_{kt}^{(1)}\}(\underline{x};t) &= \iint_{S_0} n_{0k}(\underline{\xi})\, G_t(\underline{x};\underline{\xi};t)\, ds \\ &- \iint_{S_0} \mathcal{K}_k^{(1)}(\underline{\xi};+0)\, G_{\nu t}(\underline{x};\underline{\xi};t)\, ds \\ &+ c \iint_F \mathcal{K}_{k\eta}^{(1)}(\xi,0,\zeta,+0)\left\{\frac{\partial}{\partial \xi} G(\underline{x};\xi,0,\zeta;t)\right\} d\xi\, d\zeta . \end{aligned}\right\} \quad (78)$$

Similarly, for $\mathcal{K}_{kt}^{(2)}$ we have

$$\left.\begin{aligned}\mathcal{L}\left\{\mathcal{K}^{(2)}_{kt}\right\}(\underline{x};t) &= \iint_{S_0} h_k(\underline{\xi})\, G_t(\underline{x};\underline{\xi};t)\, ds \\ &- \iint_{S_0} \mathcal{K}^{(2)}_k(\underline{\xi};+0)\, G_{\nu t}(\underline{x};\underline{\xi};t)\, ds \\ &+ c\iint_F \mathcal{K}^{(2)}_{k\eta}(\xi,0,\zeta,+0)\left\{\frac{\partial}{\partial\xi} G(\underline{x};\xi,0,\zeta;t)\right\} d\xi\, d\zeta.\end{aligned}\right\} \quad (79)$$

For $t=+0$, since $G(\underline{x};\xi,0,\zeta;0) = G_t(\underline{x};\xi,\eta,\zeta;0) = 0$, (78) and (79) reduce to

$$4\pi\,\mathcal{K}^{(1)}_{kt}(\underline{x};+0) + \iint_{S_0} \mathcal{K}^{(1)}_{kt}(\underline{\xi};+0)\, G_\nu(\underline{x};\underline{\xi};0)\, ds = 0, \quad (80)$$

and

$$4\pi\,\mathcal{K}^{(2)}_{kt}(\underline{x};+0) + \iint_{S_0} \mathcal{K}^{(2)}_{kt}(\underline{\xi};+0)\, G_\nu(\underline{x};\underline{\xi};0)\, ds = 0. \quad (81)$$

Obviously, $\mathcal{K}^{(1)}_{kt}(\underline{x};+0)=0$ and $\mathcal{K}^{(2)}_{kt}(\underline{x};+0)=0$ are solutions of (80) and (81), respectively. That they are also the only solutions is shown in Appendix II of Wehausen (1965).

The last condition of (73), i.e., (73f), follows easily from the free-surface boundary condition. Since the free-surface condition holds for all t, at $t=+0$, in particular, we have

$$\mathcal{K}^{(\ell)}_{ktt}(x,0,z,+0) - 2c\,\mathcal{K}^{(\ell)}_{ktx} + \mathcal{K}^{(\ell)}_{kxx} + g\,\mathcal{K}^{(\ell)}_{ky} = 0,$$

where $\ell = 1, 2$ and $k = 1, 2, \cdots, 6$. But for $t = +0$, we have $\mathcal{K}^{(\ell)}_{ktx}(x, 0, \zeta, +0) = 0$ and $\mathcal{K}^{(\ell)}_{kxx}(x, 0, \zeta, +0) = 0$ which follow immediately from (73e) and (73a), respectively. Hence, from the free-surface condition we have

$$\mathcal{K}^{(\ell)}_{ktt}(x, 0, \zeta, +0) = -g\mathcal{K}^{(\ell)}_{ky}, \quad \ell = 1, 2; \; k = 1, 2, \cdots, 6.$$

Thus $\mathcal{K}^{(1)}_k$ and $\mathcal{K}^{(2)}_k$ satisfy all the conditions (73a) through (73f).

VI. The Forces and Moments Acting on the Ship Hull.

In the following we shall first derive expressions for the forces and moments acting on the ship hull in terms of the steady and unsteady velocity potential, φ_0 and φ_1. The various types of decomposition which were made previously for the potential φ_1 will then be substituted for φ_1.

In principle, the only correct way of calculating the forces and moments acting on the ship hull is to integrate the pressure around the actual wetted surface. However, by an approach similar to that used in obtaining the linearized boundary condition on the hull-fluid interface, it is found that one may, in practice, integrate the pressure around the steady reference surface S_0 provided a proper correction is made.

The present calculation of forces and moments is similar to that of Appendix A of Ogilvie (1964) in the following senses: a) the same assumption is made about the wall-sidedness of the ship hull in the vicinity of the equilibrium water line where the undisturbed free-surface intersects with the undisturbed ship hull; b) the same criterion is used in discarding terms of higher order. An effort will also be made to preserve the same use of notation so that any cross reference between the present work and that of Ogilvie (1964) may be easily made.

To begin with, we shall consider simultaneously the body reference frame $\hat{O}\hat{x}\hat{y}\hat{z}$ and the steadily translating reference frame $Oxyz$. Toward the end, however, only the $Oxyz$ - system will be needed.

As before, let $S: \hat{F}(\hat{x},\hat{y},\hat{z})=0$, in the body reference frame $\hat{O}\hat{x}\hat{y}\hat{z}$, describe the actual ship surface which is below the equilibrium water line. Then the equation $\hat{F}(x,y,z)=0$ in the steadily translating reference frame $Oxyz$ describes the reference surface S_o which coincides exactly with S when the latter is in its undisturbed position. Note that both S and S_o are defined by the same function $\hat{F}$ but in two different reference frames.

For convenience, we shall define the following term: a point Q_o is the 'image point' of a point Q on the ship surface S if the coordinates of Q_o in the $Oxyz$-frame have precisely the same values as those of Q in the $\hat{O}\hat{x}\hat{y}\hat{z}$ frame. Thus Q coincides precisely with Q_o when S coincides with S_o. Suppose that $\underline{n}$ is the unit inward normal to the ship surface S and

$$\hat{n}_i(\hat{x},\hat{y},\hat{z}) \equiv \underline{n}\cdot\hat{\underline{e}}_i = \hat{F}_{,i}(\hat{x},\hat{y},\hat{z})\Big/\sqrt{\hat{F}_{,k}\hat{F}_{,k}}\ ,$$

$$i,k = 1,2,3.$$

If $\underline{n}_o$ is the unit inward normal to S_o, then due to the way S_o is related to S we have

$$n_{oi}(x,y,z) \equiv \underline{n}_o\cdot\underline{e}_i = \hat{F}_{,i}(x,y,z)\Big/\sqrt{\hat{F}_{,k}\hat{F}_{,k}}\ ,$$

$$i,k = 1,2,3.$$

Thus $\hat{n}_i$ and n_{oi} are in fact given by the same function. Hence if $Q \varepsilon S$ has the coordinates (a, b, c) in the $\hat{O}\hat{x}\hat{y}\hat{z}$ frame and $Q_o \varepsilon S_o$ is the image point of Q, i.e., Q has the same coordinates (a, b, c) but in a different reference frame, $Oxyz$, then, clearly,

$$n_{oi}(Q_o) = \hat{n}_i(Q) = \hat{F}_{,i}(a,b,c)/\sqrt{\hat{F}_{,k}\hat{F}_{,k}}\,, \tag{82}$$

$$i, k = 1, 2, 3.$$

The force on the ship hull is given by

$$\underline{X} = \iint_{S_W} p\,\underline{n}\,ds\,, \tag{83}$$

and the moment with respect to the point $\hat{O}$, the origin of the body reference frame $\hat{O}\hat{x}\hat{y}\hat{z}$ is given by

$$\underline{\mathfrak{M}} = \iint_{S_W} p\,\underline{\hat{X}}' \times \underline{n}\,ds \tag{84}$$

where $\underline{\hat{X}}' = \hat{x}\,\hat{\underline{e}}_1 + \hat{y}\,\hat{\underline{e}}_2 + \hat{z}\,\hat{\underline{e}}_3$ is the position vector of a point $Q = (\hat{x}, \hat{y}, \hat{z})$ on the ship hull with reference to the body reference frame $\hat{O}\hat{x}\hat{y}\hat{z}$ and S_W is the actual wetted surface of the ship hull.

In order to use the results of the present calculation of force and moment to write down equations of motion, we shall resolve the force and moment along the steadily translating reference frame $Oxyz$. For convenience, let us consider first the decomposition of the force vector along. From (83) we have

$$
\begin{aligned}
X_j &= \underline{X} \cdot \underline{e}_j = \iint_{S_W} p \, (\underline{n} \cdot \underline{e}_j) \, ds \\
&= \iint_{S_W} p \left\{ \sum_{i=1}^{3} \hat{n}_i \; \underline{\hat{e}}_i \cdot \underline{e}_j \right\} ds \\
&= \iint_{S_W} p \left\{ \hat{n}_j + \varepsilon_{jki} \, \theta_k \hat{n}_i \right\} ds \\
&= \iint_{S} p \left\{ \hat{n}_j + \varepsilon_{jki} \, \theta_k \hat{n}_i \right\} ds \\
&\quad + \iint_{S_1} p \left\{ \hat{n}_j + \varepsilon_{jki} \, \theta_k \hat{n}_i \right\} ds
\end{aligned}
\tag{85}
$$

where $S_W = S \cup S_1$, and $S_1 = S_W - S$.

Note that in obtaining (85) we have used the result from (5), namely, $\underline{\hat{e}}_i \cdot \underline{e}_j = \delta_{ij} + \varepsilon_{ijk} \theta_k$.

Let $Q_0 \varepsilon S_0$ be the image point of $Q \varepsilon S$. Suppose that $Q_0 \varepsilon S_0$ has the coordinates (x, y, z) in the $Oxyz$-frame; then Q has the same coordinates (x, y, z) in the $\hat{O}\hat{x}\hat{y}\hat{z}$-frame. The coordinates of $Q \varepsilon S$ in the $Oxyz$-frame are then given according to the transformation (3) as $Q = (X_i + x_{1i} + \varepsilon_{ijk} \theta_j x_k)$.

We shall assume that the function p can be expanded into Taylor's series as follows:

$$p(x+x_1+\theta_2 z-\theta_3 y,\ y+y_1+\theta_3 x-\theta_1 z,\ z+z_1+\theta_1 y-\theta_2 x,\ t)$$
$$= p(x,y,z,t) + [x_{1\ell}+\varepsilon_{\ell mn}\theta_m x_n]\, p_{,\ell}(x,y,z,t) + O(\varepsilon_M^2),$$

which means precisely that

$$p(Q;t) = p(Q_o;t) + [x_{1\ell}+\varepsilon_{\ell mn}\theta_m x_n]\, p_{,\ell}(Q_o;t) + O(\varepsilon_M^2). \quad (86)$$

Thus from (86) and (82) we may write the integrand of (85) as follows:

$$\left.\begin{aligned} &\{\hat{n}_j(Q)+\varepsilon_{jki}\theta_k \hat{n}_i(Q)\}\, p(Q;t) \\ &= \{n_{oj}(Q_o)+\varepsilon_{jki}\theta_k n_{oi}(Q_o)\}\, p(Q_o;t) + \\ &\quad + n_{oj}(Q_o)\,[x_{1\ell}+\varepsilon_{\ell mn}\theta_m x_n]\, p_{,\ell}(Q_o;t) + O(\varepsilon_M^2). \end{aligned}\right\} \quad (87)$$

Although the expression obtained on the right-hand side of the equality in (87) is a more complicated one, it has the advantage of being evaluated on a prescribed reference surface S_o which is stationary with respect to the $Oxyz$-frame. It follows immediately from (87) that

$$\left.\begin{aligned}
&\iint_S \{\hat{n}_j(Q) + \varepsilon_{jki}\theta_k \hat{n}_i(Q)\}\, p(Q;t)\, ds \\
&= \iint_{S_o} \{n_{oj}(Q_o) + \varepsilon_{jki}\theta_k \hat{n}_{oi}\}\, p(Q_o;t)\, ds \\
&\quad + \iint_{S_o} n_{oj}(Q_o)\,[x_{1\ell} + \varepsilon_{\ell mn}\theta_m x_n]\, p_{,\ell}(Q_o;t)\, ds \;+ O(\varepsilon_M^2),
\end{aligned}\right\} \qquad (88)$$

which shows how an integral over the surface S may be approximated by integrals over the stationary (relative to the $Oxyz$-frame) reference surface S_o.

We need next an appropriate expression for p. From Bernoulli's equation we have

$$\Phi_t(\underline{x};t) + \frac{p}{\rho} + gy + \frac{1}{2}(\nabla\Phi)^2 = \text{const.},$$

where we let

$$\Phi(\underline{x};t) = -cx + \varphi_o(\underline{x}) + \varphi_1(\underline{x};t). \qquad (89)$$

Assuming $p = 0$, $\nabla\Phi \to -c$ and $\Phi_t \to 0$ at $x = +\infty$, $y = 0$, we have

$$\left.\begin{aligned} p(x,y,z,t) &= -\rho g y - \rho \Phi_t(\underline{x};t) - \tfrac{1}{2}\rho(\nabla\Phi)^2 + \tfrac{1}{2}\rho c^2 \\ &= -\rho g y - \rho\varphi_{1t} + \rho c\varphi_{0x} + \rho c\varphi_{1x} - \\ &\quad -\rho(\nabla\varphi_0\cdot\nabla\varphi_1) - \tfrac{1}{2}\rho(\nabla\varphi_0)^2 - \tfrac{1}{2}\rho(\nabla\varphi_1)^2. \end{aligned}\right\} \quad (90)$$

We shall follow the usual practice of linearized theory of discarding the last two quadratic terms in (90). However, one should realize that omission of the term $\frac{1}{2}\rho(\nabla\varphi_0)^2$ may not be proper for the case of a deeply submerged body. Let us then write

$$\begin{aligned} p(\underline{x};t) &\doteq -\rho g y + \rho c\varphi_{0x} - \\ &\quad -\rho\left\{\varphi_{1t}(\underline{x};t) - c\varphi_{1x} + \nabla\varphi_0\cdot\nabla\varphi_1\right\}. \end{aligned} \quad (91)$$

This is now to be substituted into (88) in place of p. If this is done, then we have from (88) and (91) that

$$\iint_S p(Q;t)\left\{\hat{n}_j(Q) + \varepsilon_{jki}\theta_k\hat{n}_i(Q)\right\}ds$$

$$\left.\begin{aligned} &= \rho\iint_{S_0}\Big\{-gy[n_{0j} + \varepsilon_{jki}\theta_k n_{0i}] - g n_{0j}[y_1 + \theta_3 x - \theta_1 z] - \\ &\quad - n_{0j}\left[\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varphi_1(\underline{x};t) + \nabla\varphi_0(\underline{x})\cdot\nabla\varphi_1\right] \\ &\quad + c\left[(n_{0j} + \varepsilon_{jki}\theta_k n_{0i})\varphi_{0x} + n_{0j}(x_1 + \varepsilon_{lmn}\theta_m x_n)\varphi_{0x,l}\right]\Big\}ds. \end{aligned}\right\} (92)$$

In (92) we have discarded terms like $\varepsilon_{jki}\,\theta_k\,n_{oi}\,[\varphi_{1t} - c\varphi_{1x} + \nabla\varphi_o\cdot\nabla\varphi_1]$, etc., which are of the order $O(\varepsilon_M^2)$.

Next, let us consider the integral over S_1 in (85). Similar to the relationship between S_o and S, let S_{o1} be the area on the undisturbed position of the ship hull which has the same size and form as S_1. The area S_{o1} can be determined from S_1 in the same manner as finding the point Q_o on S_o after a point Q on S is assigned. The free-surface elevation $y = Y(x,z,t)$ can be expressed in terms of the ship coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$ as follows:

$$\hat{y} + y_1 + \theta_3\hat{x} - \theta_1\hat{z} = Y(\hat{x}+x_1+\theta_2\hat{z}-\theta_3\hat{y},\ \hat{z}+z_1+\theta_1\hat{y}-\theta_2\hat{x},\ t)$$

$$= Y(\hat{x},\hat{z},t) + [x_1+\theta_2\hat{z}-\theta_3\hat{y}]\,Y_x(\hat{x},\hat{z},t) + [z_1+\theta_1\hat{y}-\theta_2\hat{x}]\,Y_z(\hat{x},\hat{z},t) + \text{h.o.t.},$$

where we have used the transformation (3) and expanded the function Y into Taylor's series. Thus from the last expression we have

$$\left.\begin{aligned} \hat{y} &= Y(\hat{x},\hat{z},t) - y_1 - \theta_3\hat{x} + \theta_1\hat{z} + [x_1+\theta_2\hat{z}-\theta_3\hat{y}]\,Y_x + \\ &\quad + [z_1+\theta_1\hat{y}-\theta_2\hat{x}]\,Y_z + \text{h.o.t.} \\ &\equiv \hat{Y}(\hat{x},\hat{z},t). \end{aligned}\right\} \qquad (93a)$$

Thus, in the $\hat{O}\hat{x}\hat{y}\hat{z}$ system S_1 is defined as that part of the hull surface $\hat{F}(\hat{x},\hat{y},\hat{z}) = 0$ bounded by the equilibrium water line $\hat{y} = 0$ and the free-surface elevation $\hat{y} = \hat{Y}(\hat{x},\hat{z},t)$

according to (93a). Hence, in the $Oxyz$-system, S_{01} is obtained as that part of the reference surface $\hat{F}(x,y,z)=0$ bounded by the undisturbed free-surface $y=0$ and the wavy surface

$$
\begin{aligned}
y &= \hat{Y}(x,z,t) \\
&= Y(x,z,t) - y_1 - \theta_3 x + \theta_1 z + [x_1 + \theta_2 \hat{z} - \theta_3 \hat{y}]\, Y_x \\
&\qquad + [z_1 + \theta_1 \hat{y} - \theta_2 \hat{x}]\, Y_z + \text{h.o.t.}
\end{aligned}
\tag{93b}
$$

The relationship (87) permits us, as in (88), to replace the integral over S_1 by integrals over S_{01} as follows:

$$
\left.
\begin{aligned}
&\iint_{S_1} p(Q;t)\left\{\hat{n}_j(Q) + \varepsilon_{jki}\theta_k \hat{n}_i(Q)\right\} ds \\
&= \iint_{S_{01}} p(Q_0;t)\left\{n_{0j}(Q_0) + \varepsilon_{jki}\theta_k n_{0i}(Q_0)\right\} ds \\
&+ \iint_{S_{01}} n_{0j}(Q_0)\left[X_{1\ell} + \varepsilon_{\ell mn}\theta_m X_n\right] p_{,\ell}(Q_0;t)\, ds + O(\varepsilon_M^2).
\end{aligned}
\right\}
\tag{94}
$$

For simplicity, let us further assume that the ship hull is wall-sided near the equilibrium water line so that the integrals over S_{01} in (94) can be written as the following iterated integrals. The first integral on the right-hand side of the equality in (94) then becomes

$$
\left.
\begin{aligned}
&\iint_{S_{o1}} p(x,y,z,t)\left\{n_{oj}(x,y,z) + \varepsilon_{jki}\theta_k n_{oi}\right\} ds \\
&= \int_{\Gamma_o} d\lambda \left\{n_{oj}(x,0,z) + \varepsilon_{jki}\theta_k n_{oi}\right\} \int_0^{\hat{Y}(x,z,t)} p(x,y,z,t)\, dy \\
&= \int_{\Gamma_o} d\lambda \left\{n_{oj}(x,0,z) + \varepsilon_{jki}\theta_k n_{oi}\right\} \int_0^{\hat{Y}} \Big\{ -\rho g y + \rho c \varphi_{ox}(x,y,z) - \\
&\qquad - \rho\left[\left(\tfrac{\partial}{\partial t} - c\tfrac{\partial}{\partial x}\right)\varphi_1(\underline{x};t) + \nabla\varphi_o(\underline{x})\cdot\nabla\varphi_1\right]\Big\} dy,
\end{aligned}
\right\} \quad (95)
$$

where Γ_o is the contour where the undisturbed free-surface intersects with the reference surface S_o , and $\hat{Y}$ is given by (93b). Also, in (95) we have used the expression for p from (91) and set $n_{oj}(x,y,z) = n_{oj}(x,0,z)$ due to the wall-sidedness of the ship hull near the equilibrium waterline. The integration with respect to the variable y in the last expression of (95) may be carried out as follows:

$$
\int_0^{\hat{Y}} \Big\{ -\rho g y + \rho c \varphi_{ox}(x,y,z) - \rho\left[\left(\tfrac{\partial}{\partial t} - c\tfrac{\partial}{\partial x}\right)\varphi_1(x,y,z,t) + \nabla\varphi_o\cdot\nabla\varphi_1\right]\Big\} dy
$$

$$
\left.
\begin{aligned}
&= -\tfrac{1}{2}\rho g \hat{Y}^2(x,z,t) + \rho c \hat{Y} \varphi_{ox}(x,\mu\hat{Y},z) - \\
&\qquad - \rho\hat{Y}\cdot\left[\left(\tfrac{\partial}{\partial t} - c\tfrac{\partial}{\partial x}\right)\varphi_1(x,\mu'\hat{Y},z,t)\right] + O(\varepsilon_M^2) \\
&\qquad\qquad \text{where} \quad 0 \le \mu, \mu' \le 1 .
\end{aligned}
\right\} \quad (96a)
$$

Since

$$
Y(x,z,t) = Y_o(x,z) + Y_1(x,z,t) = \frac{c}{g}\varphi_{ox}(x,0,z) - \frac{1}{g}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varphi_1(x,0,z,t),
$$

we have from (93b) that

$$\hat{Y}(x,\zeta,t) = \frac{c}{g}\varphi_{ox}(x,0,\zeta) - \frac{1}{g}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varphi_1(x,0,\zeta,t) - y_1 - \theta_3 x + \theta_1 \zeta +$$

$$+ \frac{c}{g}[x_1 + \theta_2\zeta - \theta_3 y]\varphi_{oxx}(x,0,\zeta) + \frac{c}{g}[\zeta_1 + \theta_1 y - \theta_2 x]\varphi_{ox\zeta} + O(\varepsilon_M^2). \quad (96b)$$

We may further expand those functions, $\varphi_{ox}(x,\mu Y,\zeta)$, $\varphi_{1t}(x,\mu'\hat{Y},\zeta,t)$, and $\varphi_{1x}(x,\mu'\hat{Y},\zeta,t)$, etc., into Taylor's series as follows:

$$\left.\begin{aligned}\varphi_{ox}(x,\mu\hat{Y},\zeta) &= \varphi_{ox}(x,0,\zeta) + \mu\hat{Y}(x,\zeta,t)\varphi_{oxy}(x,0,\zeta) + O(\varepsilon_M^2),\\ \varphi_{1t}(x,\mu'\hat{Y},\zeta,t) &= \varphi_{1t}(x,0,\zeta,t) + \mu'\hat{Y}(x,\zeta,t)\cdot\varphi_{1ty}(x,0,\zeta,t) + O(\varepsilon_M^2),\end{aligned}\right\} \quad (96c)$$

etc.

Substituting (96c) and (96b) into (96a) we have

$$\int_0^{\hat{Y}}\left\{-\rho g y + \rho c\varphi_{ox}(x,y,\zeta) - \rho\left[\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varphi_1(x,y,\zeta,t) + \nabla\varphi_0\cdot\nabla\varphi_1\right]\right\}dy$$

$$= -\frac{1}{g}\rho c\varphi_{ox}(x,0,\zeta)\left[\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varphi_1(x,0,\zeta,t)\right] + O(\varepsilon_M^2).$$

This result is now to be substituted back into the last expression of (95). Then we have

$$\iint_{S_{o1}} p(x,y,\zeta,t)\left\{n_{oj}(x,y,\zeta) + \varepsilon_{jki}\theta_k n_{oi}\right\}ds$$

$$= -\rho c\int_{\Gamma_o} n_{oj}(x,0,\zeta)\varphi_{ox}(x,0,\zeta)\left\{\frac{1}{g}\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varphi_1(x,0,\zeta,t)\right\}ds + O(\varepsilon_M^2). \quad (97)$$

Following the same procedure as above, one may obtain the following result from the second integral in (94):

$$\iint_{S_{01}} n_{0j}(x,y,z)\left[x_{,\ell}+\varepsilon_{\ell mn}\theta_m x_n\right] p_{,\ell}(x,y,z,t)\,dS$$

$$= -\rho c\int_{\Gamma_0} n_{0j}(x,0,z)\,\varphi_{0x}(x,0,z)\left[y_1+\theta_3 x-\theta_1 z_1\right] ds + O(\varepsilon_M^2)\,. \tag{98}$$

The appropriate first-order expression for the force component is now obtained by adding together (98), (97), and (92) as follows:

$$\left.\begin{aligned} X_j' = \rho\iint_{S_0}\Big\{&-gy\left[n_{0j}+\varepsilon_{jki}\theta_k n_{0i}\right]-g\,n_{0j}\left[y_1+\theta_3 x-\theta_1 z\right]- \\ &-n_{0j}\left[\left(\tfrac{\partial}{\partial t}-c\tfrac{\partial}{\partial x}\right)\varphi_1(x,y,z,t)+\nabla\varphi_0(x,y,z)\cdot\nabla\varphi_1\right] \\ &+c\left[\left(n_{0j}+\varepsilon_{jki}\theta_k n_{0i}\right)\varphi_{0x}+n_{0j}\left(x_{1\ell}+\varepsilon_{\ell mn}\theta_m x_n\right)\varphi_{0x,\ell}\right]\Big\}\,dS \\ &-\rho c\int_{\Gamma_0} n_{0j}(x,0,z)\,\varphi_{0x}(x,0,z)\Big\{\tfrac{1}{g}\left(\tfrac{\partial}{\partial t}-c\tfrac{\partial}{\partial x}\right)\varphi_1(x,0,z,t)+ \\ &\qquad\qquad +y_1+\theta_3 x-\theta_1 z\Big\}\,ds\,, \\ &\qquad\qquad i,j,k=1,2,3. \end{aligned}\right\} \tag{99}$$

Note that (99) corresponds to the expression for X_j' on p. 99 of Ogilvie (1964) except that X_j in (99) is now resolved directly along the steadily translating reference frame $Oxyz$. For interpretation of terms appearing in (99) and remarks about the correction term, i.e., the contour integral around Γ_0 in (99), we refer to Ogilvie's (1964) discussion following immediately his expression for X_j'. However, we call attention here to the fact that the correction term is of order not lower than $\varepsilon_S \varepsilon_M$ and vanishes entirely when the body is completely submerged.

We may proceed in the same manner to obtain an expression for the moment components $\mathcal{M}_j$. As before, let Q be a typical point on S and Q_o be its image point on S_o. Define

$$\underline{l}_o = \underline{X} \times \underline{n}_o = \sum_{j=1}^{3} n_{o\,j+3}\, \underline{e}_j \,, \tag{100}$$

where $\underline{X} = x\,\underline{e}_1 + y\,\underline{e}_2 + z\,\underline{e}_3$ is the position vector of the point $Q_o \varepsilon S_o$ in the $Oxyz$-frame. Then the position vector of the image point $Q \varepsilon S$ in the $\hat{O}\hat{x}\hat{y}\hat{z}$-frame is given by $\hat{\underline{X}}' = x\,\hat{\underline{e}}_1 + y\,\hat{\underline{e}}_2 + z\,\hat{\underline{e}}_3$. Let us define also that

$$\underline{l} = \hat{\underline{X}}' \times \underline{n} = \sum_{j=1}^{3} \hat{n}_{j+3}\, \hat{\underline{e}}_j \,. \tag{101}$$

Thus from (101), (100) and the fact that $Q_o \varepsilon S_o$ is the image point of $Q \varepsilon S$ we may again establish the relationship

$$\hat{n}_{j+3}(Q) = n_{o\,j+3}(Q_o)\,, \quad j = 1, 2, 3\,. \tag{102}$$

We shall again decompose the moment acting on the ship hull about the origin of the ship coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$ directly along the steadily translating frame $Oxyz$ as follows:

$$\mathcal{M}_j \equiv \underline{\mathcal{M}} \cdot \underline{e}_j = \iint_{S_W} p\,(\hat{\underline{x}}' \cdot \underline{n}) \cdot \underline{e}_j \, ds$$

$$= \iint_{S_W} p \left\{ \sum_{i=1}^{3} \hat{n}_{i+3} \, (\hat{\underline{e}}_i \cdot \underline{e}_j) \right\} ds$$

$$= \iint_{S_W} p \left\{ \hat{n}_{j+3} + \varepsilon_{jki} \, \theta_k \, \hat{n}_{i+3} \right\} ds$$

$$= \iint_{S} p \left\{ \hat{n}_{j+3} + \varepsilon_{jki} \, \theta_k \, \hat{n}_{i+3} \right\} ds + \iint_{S_1} p \left\{ \hat{n}_{j+3} + \varepsilon_{jki} \, \theta_k \, \hat{n}_{i+3} \right\} ds, \tag{103}$$

$$i, j, k = 1, 2, 3.$$

Note that (103) is essentially the same expression as (85) except that the index j is now replaced by $j + 3$. With the relationship established in (102), namely, $\hat{n}_{j+3}(Q) = n_{oj+3}(Q_o)$ where $Q \varepsilon S$ and $Q_o \varepsilon S_o$, the same analysis used in obtaining the final expression for X_j holds also for $\mathcal{M}_j$. For convenience let us write $\mathcal{M}_j = X_{j+3}$; then (103) will lead to an expression precisely the same as (99) except that X_j, n_{oj}, and n_{oi} are now replaced by X_{j+3}, n_{oj+3}, and n_{oi+3}. Thus from (103) we have

$$\left. \begin{aligned} X_{j+3} = \rho \iint_{S_o} \Big\{ &- g\,y\,[n_{oj+3} + \varepsilon_{jki}\,\theta_k\, n_{oi+3}] - g\, n_{oj+3}\,[y_1 + \theta_3 x - \theta_1 z] - \\ &- n_{oj+3} \left[\left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \varphi_1(\underline{x}; t) + \nabla \varphi_o(\underline{x}) \cdot \nabla \varphi_1 \right] + \end{aligned} \right\} \tag{104}$$

$$+ c\left[\left(n_{oj+3}+\varepsilon_{jki}\theta_k n_{oi+3}\right)\varphi_{ox} + n_{oj+3}\left(x_{1\ell}+\varepsilon_{\ell mn}\theta_m x_n\right)\varphi_{ox,\ell}\right]\Big\}ds$$

$$- \rho c\int_{\Gamma_o} n_{oj+3}(x,0,z)\,\varphi_{ox}(x,0,z)\left\{\frac{1}{g}\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)\varphi_1(x,0,z,t)+\right.$$

$$\left. + y_1 + \theta_3 x - \theta_1 z\right\}ds,$$

$$i,j,k = 1,2,3.$$

Note that from (100) we have

$$n_{oj+3}(x,y,z) = \varepsilon_{jki}\, x_k\, n_{oi}(x,y,z), \quad i,j,k=1,2,3. \tag{105}$$

Since $n_{oi}(x,y,z)=n_{oi}(x,0,z)$, $i=1,2,3$, follows from the wall-sidedness of the ship hull, and in the region S_{o1} we have $|y| \leq |\hat{\varphi}(x,z,t)|$, the difference between $n_{oj+3}(x,y,z)$ and $n_{oj+3}(x,0,z)$ is of order ε_S or ε_M. Thus the contour integral in (104) is the only contribution to the first-order expression for the moment component from the correction term, i.e., the integral over S_{o1} which is similar to (94). Furthermore, the wall-sidedness of the ship hull implies that $n_{o2}(x,0,z) = 0$ for $(x,y,z)\,\varepsilon\, S_{o1}$, so that $n_{o4}(x,0,z) = n_{o6}(x,0,z) = 0$. Thus the contour integral in (104) vanishes except for $j = 2$. Hence, it contributes to the yawing moment alone.

Lastly let us substitute into the expressions (92) and (104) the various types of decomposition we made for φ_1, namely,

$$\left.\begin{aligned} \varphi_1(\underline{x};t) = \varphi_w(\underline{x};t) + \sum_{k=1}^{6}\int_0^t \ddot{\alpha}_k(\tau)\mathcal{K}_k^{(1)}(\underline{x};t-\tau)\,d\tau \\ + \sum_{k=1}^{6}\int_0^t \dot{\alpha}_k(\tau)\mathcal{K}_k^{(2)}(\underline{x};t-\tau)\,d\tau . \end{aligned}\right\} \tag{106}$$

Note that to be consistent with the use of the notation α for the displacements, we should also write $\theta_k = \alpha_{k+3}$, $k = 1,2,3,$ and $y_1 = \alpha_2$.

With the substitution for φ_1 from (106) and use of the boundary condition $n_{oj}(\underline{x}) = \mathcal{K}_{kn}^{(1)}(\underline{x};+0)$ for $\underline{X} \varepsilon S_o$ from (69a), the equations (99) and (104) lead to the same equations in the following form:

$$\left.\begin{aligned} X_j = X_{jo} - \sum_{k=1}^{6}\mu_{jk}\ddot{\alpha}_k(t) - \sum_{k=1}^{6} b_{jk}\dot{\alpha}_k(t) - \sum_{k=1}^{6} C_{jk}\alpha_k(t) - \\ -\sum_{k=1}^{6}\int_0^t \ddot{\alpha}_k(\tau)L_{jk}(t-\tau)\,d\tau - \sum_{k=1}^{6}\int_0^t \dot{\alpha}_k(\tau)M_{jk}(t-\tau)\,d\tau - X_{wj} , \end{aligned}\right\} \tag{107}$$

where $k, j = 1, 2, \ldots, 6,$

$$X_{jo} = \rho \iint_{S_o} \mathcal{K}_{jn}^{(1)}(\underline{x};+0)\left\{-gy + c\,\varphi_{ox}(\underline{x})\right\} dS , \tag{108a}$$

$$\mu_{jk} = \rho \iint_{S_o} \mathcal{K}_{jn}^{(1)}(\underline{x};+0)\,\mathcal{K}_k^{(1)}(\underline{x};+0)\,dS , \tag{108b}$$

$$b_{jk} = \rho \iint_{S_o} \mathcal{K}_{jn}^{(1)}(\underline{x};+0)\, \mathcal{K}_k^{(2)}(\underline{x};+0)\, dS, \tag{108c}$$

$$\left.\begin{aligned} L_{jk}(t) &= \rho \iint_{S_o} \mathcal{K}_{jn}^{(1)}(\underline{x};+0)\left\{\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)\mathcal{K}_k^{(1)}(\underline{x};t) + \nabla\varphi_o\cdot\nabla\mathcal{K}_k^{(1)}\right\} dS \\ &+ \frac{1}{g}\rho c \int_{\Gamma_o} \mathcal{K}_{jy}^{(1)}(x,0,z,+0)\,\varphi_{ox}(x,0,z)\left\{\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)\mathcal{K}_k^{(1)}(x,0,z,t)\right\} ds, \end{aligned}\right\} \tag{108d}$$

$$\left.\begin{aligned} M_{jk}(t) &= \rho \iint_{S_o} \mathcal{K}_{jn}^{(1)}(\underline{x};+0)\left\{\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)\mathcal{K}_k^{(2)}(\underline{x};t) + \nabla\varphi_o\cdot\nabla\mathcal{K}_k^{(2)}\right\} dS \\ &+ \frac{1}{g}\rho c \int_{\Gamma_o} \mathcal{K}_{jy}^{(1)}(x,0,z,+0)\,\varphi_{ox}(x,0,z)\left\{\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)\mathcal{K}_k^{(2)}(x,0,z,t)\right\} ds, \end{aligned}\right\} \tag{108e}$$

$$\left.\begin{aligned} C'_{jk} &= \rho \iint_{S_o} \mathcal{K}_{jn}^{(1)}(\underline{x};+0)\left\{g\left[\underline{e}_2\cdot\underline{e}_k(\underline{x})\right] - c\left(\underline{e}_k\cdot\nabla\right)\varphi_{ox}\right\} dS \\ &+ \rho c \int_{\Gamma_o} \mathcal{K}_{jy}^{(1)}(x,0,z,+0)\left\{\left[\underline{e}_2\cdot\underline{e}_k(\underline{x})\right]\varphi_{ox}(x,0,z)\right\} ds, \end{aligned}\right\} \tag{108f}$$

where $$\sum_{k=1}^{6} \alpha_k \left[\underline{e}_2\cdot\underline{e}_k(\underline{x})\right] = y_1 + \theta_3 x - \theta_1 z,$$

$$\underline{e}_k = \begin{cases} \underline{e}_k\,, & k = 1,2,3, \\ \underline{e}_{k-3}\times\underline{X}\,, & k = 4,5,6, \end{cases}$$

$$C''_{jk} = \begin{pmatrix} 0 & 0 & 0 & 0 & -X_{30} & X_{20} \\ 0 & 0 & 0 & X_{30} & 0 & -X_{10} \\ 0 & 0 & 0 & -X_{20} & -X_{10} & 0 \\ 0 & 0 & 0 & 0 & -X_{60} & X_{50} \\ 0 & 0 & 0 & X_{60} & 0 & -X_{40} \\ 0 & 0 & 0 & -X_{50} & -X_{40} & 0 \end{pmatrix} , \tag{108g}$$

$$C_{jk} = C'_{jk} + C''_{jk} , \tag{108h}$$

$$X_{w_j} = \rho \iint_{S_0} \mathcal{K}^{(1)}_{jn}(\underline{x};+0)\left\{\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varphi_w(\underline{x};t) + \nabla\varphi_0 \cdot \nabla\varphi_w\right\} dS$$

$$+ \frac{1}{g}\rho c \int_{\Gamma_0} \mathcal{K}^{(1)}_{jy}(x,0,z,+0)\,\varphi_{0x}(x,0,z)\left\{\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\varphi_w(x,0,z,t)\right\} ds . \tag{108i}$$

It may be worth mentioning again that X_j given by (107) represents the force and moment (about $\hat{O}$, the origin of the ship coordinate system) acting on the instantaneous position of the ship. One can see from the definitions for the various quantities given in (108a) through (108i) that X_{j0} represents steady force and moment components, μ_{jk} is a constant depending only on ship geometry, b_{jk} and C_{jk} are constants depending upon the ship geometry and the constant average forward speed, and L_{jk} and M_{jk} are functions of time, geometry and the constant average forward speed. None of these quantities depends upon the unsteady oscillatory motion of the ship. (We note that b_{jk} , L_{jk} , M_{jk} , and C_{jk} do not have the same meanings here as in Ogilvie's paper.) X_j includes: a) the static buoyancy and wave resistance of the ship moving with a constant forward speed in its equilibrium position; b) the total hydrostatic and hydrodynamic force and moment acting on the ship due to its own motion; and c) the force and moment resulting from the action of the incident and diffracted waves upon the ship.

We remark in passing that the quantities C''_{jk} defined by (108g) arise only if the force and moment vectors are resolved along the steadily translating reference frame $Oxyz$. Thus, if we set $C''_{jk} \equiv 0$ for $j, k = 1, 2, \ldots, 6$, then the expression (107) will give the components of force and moment resolved along the ship coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$.

VII. The Equations of Motion.

Without the loss of generality, we shall suppose that the center of gravity of the ship is located at a point with the coordinates (0 , $\hat{y}_G$, 0) referred to the ship coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$. We shall also assume that the propeller thrust T is directed parallel to $\hat{O}\hat{x}$ along a line of action which is ℓ_T units below the center of gravity and lies in the $(\hat{x},\hat{y})$-plane.

In writing down the equations of motion for the ship the following result of Euler in rigid-body dynamics may be used: The motion of a rigid body can be determined by treating separately the motion of the center of gravity and the rotational motion about the center of gravity. Moreover, the motion of the center of gravity can be determined as the motion of a particle of the total mass m of the body subjected to the total force F applied to the body, and the rotational motion of the body about its center of gravity can be determined as if this point were fixed and the body were subject to the moment of the applied force about this point. If we use the steadily translating coordinate system $Oxyz$ as our inertial reference frame, then the motion of the center of gravity is given by

$$\left.\begin{aligned} m\ddot{x}_1 &= F_1 \\ m\ddot{y}_1 &= F_2 \\ m\ddot{z}_1 &= F_3 \end{aligned}\right\} \qquad (109)$$

where $\underline{F} = F_1 \underline{e}_1 + F_2 \underline{e}_2 + F_3 \underline{e}_3$ is the total external force acting on the ship and $\ddot{x}_1$, $\ddot{y}_1$, and $\ddot{z}_1$ are the three components of the rectilinear acceleration of the center of gravity in the Ox-, Oy-, and Oz-directions, respectively. In our present generalized notation we may write these quantities as $\ddot{x}_1 = \ddot{\alpha}_1$, $\ddot{y}_1 = \ddot{\alpha}_2$, and $\ddot{z}_1 = \ddot{\alpha}_3$. The motion of the ship about the center of gravity is, then determined by Euler's equations of motion:

$$\left.\begin{aligned}
&I_1 \dot{\omega}_1 - I_{12}\dot{\omega}_2 - I_{13}\dot{\omega}_3 + \omega_2\left[-I_{13}\omega_1 - I_{23}\omega_2 + I_3\omega_3\right] - \\
&\qquad - \omega_3\left[-I_{12}\omega_1 + I_2\omega_2 - I_{23}\omega_3\right] = \hat{\mathfrak{M}}_{G1}, \\
&-I_{12}\dot{\omega}_1 + I_2\dot{\omega}_2 - I_{23}\dot{\omega}_3 + \omega_3\left[I_1\omega_1 - I_{12}\omega_2 - I_{13}\omega_3\right] - \\
&\qquad - \omega_1\left[-I_{13}\omega_1 - I_{23}\omega_2 + I_3\omega_3\right] = \hat{\mathfrak{M}}_{G2}, \\
&-I_{13}\dot{\omega}_1 - I_{23}\dot{\omega}_2 + I_3\dot{\omega}_3 + \omega_1\left[-I_{12}\omega_1 + I_2\omega_2 - I_{23}\omega_3\right] - \\
&\qquad - \omega_2\left[I_1\omega_1 - I_{12}\omega_2 - I_{13}\omega_3\right] = \hat{\mathfrak{M}}_{G3},
\end{aligned}\right\} \tag{110}$$

where

$$\underline{\mathfrak{M}}_G = \hat{\mathfrak{M}}_{G1}\hat{\underline{e}}_1 + \hat{\mathfrak{M}}_{G2}\hat{\underline{e}}_2 + \hat{\mathfrak{M}}_{G3}\hat{\underline{e}}_3 \tag{111}$$

is the moment of the total external force $\underline{F}$ about the center of gravity and

$$\underline{\omega} = \omega_1\hat{\underline{e}}_1 + \omega_2\hat{\underline{e}}_2 + \omega_3\hat{\underline{e}}_3 \tag{112}$$

is the angular velocity of the ship as a rotating rigid body. In order to define the quantities I_{ij} , let us introduce the function $\rho_s(\hat{x},\hat{y},\hat{z})$, the density of the material of the ship. Then evidently

$$m = \iiint_V \rho_s(\hat{x},\hat{y},\hat{z})\, dV$$

and the I_{ij}'s are given by

$$\left.\begin{aligned}
I_1 &= \iiint_V \rho_s\left\{(\hat{y}-\hat{y}_G)^2+\hat{z}^2\right\} dV , & I_2 &= \iiint_V \rho_s\left\{\hat{z}^2+\hat{x}^2\right\} dV , \\
I_3 &= \iiint_V \rho_s\left\{\hat{x}^2+(\hat{y}-\hat{y}_G)^2\right\} dV , & I_{12} &= \iiint_V \rho_s\, \hat{x}(\hat{y}-\hat{y}_G)\, dV , \\
I_{23} &= \iiint_V \rho_s(\hat{y}-\hat{y}_G)\hat{z}\, dV , & I_{31} &= \iiint_V \rho_s\, \hat{z}\hat{x}\, dV ,
\end{aligned}\right\} \tag{113}$$

where V is the total volume of the ship, including the part above the waterplane.

Let us now show that the quantities ω_i may be identified, to the first order of approximation, with the quantities $\dot{\theta}_i$ which we have introduced in the previous chapters. It is a well-known result in the kinematics of a rotating reference frame that

$$d\hat{\underline{e}}_i/dt = \varepsilon_{ijk}\,\omega_k\,\hat{\underline{e}}_j \quad , \tag{114}$$

for an arbitrary rotational motion of the coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$. On the other hand, for small angular displacements of the coordinate system $\hat{O}\hat{x}\hat{y}\hat{z}$ we have the relationships

$$\hat{\underline{e}}_i = \underline{e}_i + \varepsilon_{ijk}\theta_k \underline{e}_j + O(\varepsilon_M^2)$$

and

$$\underline{e}_j = \hat{\underline{e}}_j + \varepsilon_{jki}\theta_k \hat{\underline{e}}_i + O(\varepsilon_M^2)$$

which can be derived easily from the transformation (4). Thus we have

$$d\hat{\underline{e}}_i/dt = \varepsilon_{ijk}\dot{\theta}_k \underline{e}_j + O(\varepsilon_M^2) = \varepsilon_{ijk}\dot{\theta}_k[\hat{\underline{e}}_j + \varepsilon_{jmn}\theta_m \hat{\underline{e}}_n] + O(\varepsilon_M^2)$$

$$= \varepsilon_{ijk}\dot{\theta}_k \hat{\underline{e}}_j + O(\varepsilon_M^2), \tag{115}$$

where we have assumed that the $\dot{\theta}_k$'s are also of the order ε_M . Hence from (114) and (115) we have

$$\omega_k = \dot{\theta}_k + O(\varepsilon_M^2) \equiv \dot{\alpha}_{k+3} + O(\varepsilon_M^2), \quad k = 1, 2, 3. \tag{116}$$

Let us now write the equations of motion for the center of gravity of the ship according to (109). To obtain the total external force acting on the ship, we must add to X_j the other forces acting on the ship, for example, the gravity force, propeller thrust, forces due to wind gusts and artificial restraints, etc. Thus the equations for the ship corresponding to (109) may be written as

$$m\ddot{\alpha}_j(t) = X_j(t) + T_j(t) - mg\,\delta_{2j} + G_j(t), \quad j = 1, 2, 3, \tag{117}$$

where T_j represents the three components of the propeller thrust resolved along the Ox-, Oy-, and Oz-directions, respectively, and by our assumption about the line of action of the propeller thrust we have

$$T_j = \underline{T} \cdot \underline{e}_j = (T\hat{\underline{e}}_1) \cdot \underline{e}_j = T(\delta_{1j} + \varepsilon_{1jk}\theta_k), \qquad j = 1, 2, 3, \tag{118}$$

$G_j(t)$ represents all other external forces besides the force $X_j(t)$, the propeller thrust T_j, and the gravity force $-mg\,\delta_{2j}$. The expression (117) can be further put into the following form if (107) is substituted for $X_j(t)$:

$$\left.\begin{aligned} &\sum_{k=1}^{6} (m_{jk} + \mu_{jk})\,\ddot{\alpha}_k(t) + \sum_{k=1}^{6} b_{jk}\dot{\alpha}_k(t) + \sum_{k=1}^{6} C_{jk}\alpha_k(t) \\ &\quad + \sum_{k=1}^{6} \int_0^t \ddot{\alpha}_k(\tau) L_{jk}(t-\tau)\,d\tau + \sum_{k=1}^{6} \int_0^t \dot{\alpha}_k(\tau) M_{jk}(t-\tau)\,d\tau \\ &= X_{jo} + T_j(t) - mg\,\delta_{2j} + G_j(t) - X_{wj}(t), \\ &\qquad\qquad j = 1, 2, 3, \end{aligned}\right\} \tag{119}$$

where

$$m_{jk} = \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \end{pmatrix}, \qquad T_j = \begin{pmatrix} T \\ T\alpha_6 \\ -T\alpha_5 \end{pmatrix}, \tag{120}$$

$$j = 1, 2, 3; \quad k = 1, 2, \cdots, 6.$$

We shall next write Euler's equations for the rotational motion of the ship about its center of gravity according to (110). Let $\underline{r}_G$ be the vector directed from the center of gravity of the ship to a point Q on the wetted ship hull. Then the moment about the center of gravity of the ship due to the pressure distribution around the wetted ship hull is given by

$$\iint_{S_W} p\, \underline{r}_G \times \underline{n}\, ds .$$

But by the assumption about the location of the center of gravity we have

$$\underline{r}_G = \hat{\underline{X}}' - \hat{y}_G \hat{\underline{e}}_2 , \tag{121}$$

where, as before, $\hat{\underline{X}}'$ is the position vector of the point on a ship hull. Hence

$$\begin{aligned} \iint_{S_W} p\, \underline{r}_G \times \underline{n}\, ds &= \iint_{S_W} p\, \hat{\underline{X}}' \times \underline{n}\, ds - y_G \iint_{S_W} p\, \hat{\underline{e}}_2 \times \underline{n}\, ds \\ &= \left\{ \hat{X}_{j+3} + \varepsilon_{jm2}\, \hat{y}_G\, \hat{X}_m \right\} \hat{\underline{e}}_j , \qquad j, m = 1, 2, 3, \end{aligned} \tag{122}$$

where the repeated indices imply summation and

$$\hat{X}_{j+3} = \hat{\underline{e}}_j \cdot \iint_{S_W} p\, \hat{\underline{X}}' \times \underline{n}\, ds , \quad j = 1, 2, 3 ,$$

$$\hat{X}_m = \hat{\underline{e}}_m \cdot \iint_{S_W} p\, \underline{n}\, ds , \qquad m = 1, 2, 3 .$$

Clearly, $\hat{X}_{j+3}$ and $\hat{X}_j$ are, respectively, the components of the moment (about $\hat{O}$) and the force due to the pressure distribution when those two vectors are resolved along the ship coordinate system. As it was remarked at the end of the last chapter that the $\hat{X}_{j+3}$'s and the $\hat{X}_j$'s may still be given by the expression (107) if we put $C''_{jk} \equiv 0$, $j, k = 1, 2, \ldots, 6$ in (108g). This is, in fact, equivalent to replacing C_{jk} by C'_{jk} in the expression (107). Let us denote by $T_{j+3}(t)$ the three components of the moment about the center of gravity due to the propeller thrust $\underline{T}$ and by $G_{j+3}(t)$ that due to all the other external forces besides the pressure distribution and the thrust. Then, by neglecting higher-order terms, we may write the equations corresponding to (110) in the following form:

$$\sum_{k=4}^{6} m_{j+3,k}\, \ddot{\alpha}_k(t) = \hat{X}_{j+3}(t) + \hat{y}_G \sum_{m=1}^{3} \varepsilon_{jm2}\, \hat{X}_m(t) + T_{j+3}(t) + G_{j+3}(t), \quad j = 1, 2, 3.$$

Or, with the use of (107) this last equation may be further written in a form parallel to (119):

$$\left.\begin{aligned}
&\sum_{k=1}^{6}\left(m_{j+3,k}+\mu_{j+3,k}+\hat{y}_G\sum_{m=1}^{3}\varepsilon_{jm2}\mu_{mk}\right)\ddot{\alpha}_k(t)\\
&+\sum_{k=1}^{6}\left(b_{j+3,k}+\hat{y}_G\sum_{m=1}^{3}\varepsilon_{jm2}b_{mk}\right)\dot{\alpha}_k(t)+\sum_{k=1}^{6}\left(C'_{j+3,k}+\hat{y}_G\sum_{m=1}^{3}\varepsilon_{jm2}C'_{mk}\right)\alpha_k(t)\\
&+\sum_{k=1}^{6}\int_0^t\ddot{\alpha}_k(\tau)\left\{L_{j+3,k}(t-\tau)+\hat{y}_G\sum_{m=1}^{3}\varepsilon_{jm2}L_{mk}(t-\tau)\right\}d\tau\\
&+\sum_{k=1}^{6}\int_0^t\dot{\alpha}_k(\tau)\left\{M_{j+3,k}(t-\tau)+\hat{y}_G\sum_{m=1}^{3}\varepsilon_{jm2}M_{mk}(t-\tau)\right\}d\tau\\
&=\left(X_{j+3,0}+\hat{y}_G\sum_{m=1}^{3}\varepsilon_{jm2}X_{m0}\right)+T_{j+3}(t)+G_{j+3}(t)\\
&\quad-\left(X_{w\,j+3}+\hat{y}_G\sum_{m=1}^{3}\varepsilon_{jm2}X_{wm}\right),
\end{aligned}\right\} \tag{123}$$

where

$$m_{j+3,k}=\begin{pmatrix}0&0&0&I_1&-I_{12}&-I_{13}\\0&0&0&-I_{12}&I_2&-I_{23}\\0&0&0&-I_{13}&-I_{23}&I_3\end{pmatrix},\qquad \begin{aligned}j&=1,2,3,\\k&=1,2,\cdots,6,\end{aligned}$$

and

$$T_{j+3}=\left(0,\ 0,\ T\cdot l_T\right),\qquad j=1,2,3.$$

Note that (119) and (123) together form a set of six integro-differential equations for $\alpha_k(t)$, $k = 1, 2, \ldots, 6$. Such equations are generally solved by method of the Laplace transforms. Thus the information about the position and velocity of both ship and fluid at the initial instant $t=0$, i.e., $\alpha_k(0)$, $\dot{\alpha}_k(0)$, $\gamma(x,\zeta,0)$, and $\gamma_t(x,\zeta,0)$, together with the integral equations for $\varphi_W(\underline{x};t)$, $\mathcal{K}_k^{(1)}(\underline{x};t)$, and $\mathcal{K}_k^{(2)}(\underline{x};t)$, and the above set of integro-differential equations uniquely determine the behavior of both ship and fluid at later instants of time, provided one grants uniqueness of solution of the integral equations.

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The motion of a freely floating or submerged body, which is moving with a constant average forward speed and oscillating arbitrarily in any of the six degrees of freedom, is formulated as an initial-value problem. The seaway is assumed to be arbitrary. The body is assumed to be 'smoth', but no symmetry of the body is required. The fundamental assumption is that both the free-surface disturbance due to forward motion of the body and the oscillations are small enough so that the problem may be linearized.

By an approach similar to that of Wehausen (1965), it is shown how the present treatment of the problem leads also to Ogilvie's (1965) modified results of Cummins' (1962) decomposition of the velocity potential for the case of an oscillating body with a constant average forward speed. The linearized equations of motion of the body are then derived as a set of six integro-differential equations. Existence and uniqueness theorems are not established either for the boundary-value problem or for the integral equation which is constructed.

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