Local buckling collapse of marine pipelines

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Local buckling collapse of marine pipelines

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Abstract

To keep up with the growing demand for oil and gas, the oil and gas industry ventures into deeper waters. This increases complexity of offshore projects. For a deep water pipeline project, South Stream, a pipeline test program has been developed. Part of this program is the investigation of the resistance of a pipeline against local buckling collapse due to external pressure. Response to external pressure is a decisive factor in design of marine pipelines, especially for deep water applications.

Local buckling of a pipeline is the buckling behaviour within the pipeline’s cross section. Other types of buckling behaviour such as upheaval and lateral buckling are not in the scope of this thesis. Buckling is defined as the state of a structure for which a relatively small increment in load leads to a relatively large increment in displacement. Generally this is reflected in a change in deformation shape and possibly a loss of stability.

A perturbation theory, first developed by Koiter [26], is applied to model this behaviour. The response of a structure that has initial geometric imperfections is obtained by using the response of an initially perfect structure, e.g. a straight beam, a perfect ring or a cylinder. From the principle of minimum potential energy an equilibrium state can be obtained. For a certain load, the bifurcation load, multiple equilibrium configurations are possible. The nature of this equilibrium state is investigated by expanding the load around this bifurcation load and expanding the displacement functions around their fundamental solutions, in terms of a bifurcation parameter. The value and sign of the post-bifurcation load coefficients determine the system’s initial post-bifurcation stability. Introduction of initial imperfections leads to modified post-bifurcation load coefficients. By using dimensionless identities, generality is enhanced. A linear-elastic material model is applied to obtain the bifurcation behaviour.

System collapse can occur in the elastic domain for unstable initial post-buckling behaviour or in the plastic domain due to material yielding. However, it is likely that collapse of a system with (small) initial geometric imperfections occurs due to an interaction of elastic and plastic buckling. Occurrence of buckling leads to relatively large displacements that induce material yielding. This can lead to loss of stiffness and introduces a possible collapse. Relatively thin walled rings and cylinders tend to collapse more in the elastic domain, while relatively thick walled rings and cylinders tend to collapse more in the plastic domain. This is due to the fact that thin walled structures require more deformation to induce yielding than thick walled structures.

When performing a collapse test, end caps are attached to the pipeline specimen. This is modelled by the introduction of boundary constraints. End caps are very stiff and are modelled as being rigid. Their influence on the
bifurcation and collapse behaviour of a cylinder is investigated. The constraints introduce boundary layer behaviour in the regions located close the end caps. It is found that the end cap constraints increase the buckling load of the cylinder with respect to an infinitely long cylinder (represented by a ring under plane strain condition). Besides, for relatively short cylinders, the buckling mode is altered. While a long cylinder prefers to collapse in an oval shape mode (described by 2 lobes), a short cylinder prefers to collapse in a buckling mode described by a higher number of lobes.

Because a collapse test is performed to estimate the collapse behaviour of a real life pipeline, it is required that the collapse shape that is observed in the test matches the oval collapse shape of a long real life pipeline. This results to a minimum required length of a tested pipeline specimen. A relation for the required length is obtained. An analytical method has been developed to determine the buckling load and mode of a constrained cylinder.

Finally, the analytically obtained results have been verified using finite element analysis (FEA) and experimental results obtained from literature.
Acknowledgements

The research reported in this thesis in mainly performed at INTECSEA Delft, the sponsor company for this work.

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Nomenclature

\( \ddot{a} \)  
Arbitrary nonlinear function \( a \) placeholder.

\( \dot{a} \)  
Arbitrary linear function \( a \) placeholder.

\( \tilde{a} \)  
Dimensionless identity. \( a \) is arbitrary placeholder.

\( \mathbf{a} \cdot \mathbf{b} \)  
Dot (inner) product of vectors \( \mathbf{a} \) and \( \mathbf{b} \).

\( a, b \)  
Derivative of arbitrary function \( a \) with respect to parameter \( b \).

\( \alpha \)  
Factor to account for end cap pressure.

\( \beta \)  
Factor to account for type of boundary constraints.

\( \gamma \)  
Ratio of radius and length \( r/L \).

\( \delta \)  
(Arbitrary) variation or Kronecker delta.

\( \epsilon \)  
Small parameter.

\( \varepsilon \)  
Material strain.

\( \varepsilon_{xx}, \varepsilon_{ss}, \varepsilon_{xs} \)  
Material strain components.

\( \zeta \)  
Bifurcation parameter. Implies the amount of buckling mode included in the solution.

\( \theta, \eta \)  
Dimensionless circumferential and axial coordinate.

\( \kappa \)  
Curvature.

\( \lambda \)  
Dimensionless load parameter. Defined differently in chapter 3 and chapter 4.

\( \lambda_1, \lambda_2 \)  
Initial post-buckling load perturbation coefficients.

\( \lambda_{col} \)  
Dimensionless collapse load.

\( \lambda_c \)  
Dimensionless (classical) bifurcation load.

\( \lambda_p \)  
Dimensionless purely plastic collapse load.

\( \nu \)  
Poisson’s ratio. In this thesis always equal to 0.3.

\( \xi \)  
Boundary layer coordinate at second boundary layer.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dot{\xi}$</td>
<td>Boundary layer coordinate at first boundary layer.</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Imperfection size parameter. For oval shaped imperfection equal to ovality.</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Radius.</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>Stress.</td>
</tr>
<tr>
<td>$\sigma_{ij}$</td>
<td>Cauchy stress tensor.</td>
</tr>
<tr>
<td>$\varphi$</td>
<td>Rotation of middle surface/line.</td>
</tr>
<tr>
<td>$\chi$</td>
<td>Ratio of wall thickness and diameter $t/2\rho$.</td>
</tr>
<tr>
<td>$\Pi$</td>
<td>Potential energy.</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>Additional potential energy due to initial imperfection.</td>
</tr>
<tr>
<td>$e$</td>
<td>Middle surface/line strain.</td>
</tr>
<tr>
<td>$f$</td>
<td>Arbitrary functional placeholder in chapter 2. Dimensionless stress potential function otherwise.</td>
</tr>
<tr>
<td>$f_y$</td>
<td>Yield strength.</td>
</tr>
<tr>
<td>$i, j, k$</td>
<td>Indices.</td>
</tr>
<tr>
<td>$m$</td>
<td>Axial half wave number.</td>
</tr>
<tr>
<td>$n$</td>
<td>Circumferential wave number.</td>
</tr>
<tr>
<td>$s_{ij}$</td>
<td>Deviatoric stress tensor.</td>
</tr>
<tr>
<td>$t$</td>
<td>Wall thickness.</td>
</tr>
<tr>
<td>$u, v, w$</td>
<td>$u$ denotes a generalised function in chapter 2. Axial, circumferential and radial deformation component otherwise.</td>
</tr>
<tr>
<td>$x, s, z$</td>
<td>Axial, circumferential and radial coordinate.</td>
</tr>
<tr>
<td>$D$</td>
<td>Fréchet derivative operator.</td>
</tr>
<tr>
<td>$E$</td>
<td>Young’s or elasticity modulus. In this thesis always equal to 210 GPa.</td>
</tr>
<tr>
<td>$F$</td>
<td>Stress potential function.</td>
</tr>
<tr>
<td>$I_1, I_2, I_3$</td>
<td>Invariants of the Cauchy stress tensor.</td>
</tr>
<tr>
<td>$J_1, J_2, J_3$</td>
<td>Invariants of the deviatoric stress tensor.</td>
</tr>
<tr>
<td>$M$</td>
<td>Bending moment or stress couple.</td>
</tr>
<tr>
<td>$M_{xx}, M_{ss}, M_{xs}$</td>
<td>Stress couples.</td>
</tr>
<tr>
<td>$N$</td>
<td>Normal force or stress resultant.</td>
</tr>
<tr>
<td>$N_{xx}, N_{ss}, N_{xs}$</td>
<td>Stress resultants.</td>
</tr>
</tbody>
</table>
\( P \)  
Pressure.

\( P_c \)  
Bifurcation pressure.

\( P_p \)  
Purely plastic collapse pressure.

\( U, W \)  
Strain and external energy.

\( U^* \)  
Strain energy density.

\( g \)  
Metric tensor.

\( i, j, k \)  
Orthonormal Cartesian base vectors.

\( k \)  
Curvature vector.

\( p, q \)  
Undeformed and deformed position vectors.

\( t, n, b \)  
Unit tangent, normal and binormal vectors.

\( \bar{u} \)  
Generalised imperfection function.

\( \hat{u} \)  
Placeholder for displacement function.

\( \hat{\hat{u}} \)  
Generalised imperfection shape function.

\( \hat{u} \)  
Generalised displacement function for an imperfect system.

\( u \)  
Generalised displacement function for a perfect system.
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Chapter 1

Introduction

Demand for oil and gas increases worldwide. Consequently prices for oil and gas products are rising. Now projects that proved to be too expensive in the past may become viable. The offshore oil and gas industry ventures into deeper waters. This introduces mayor challenges; the environment is highly pressurised and poorly accessible for installation, maintenance and repair activities.

When problems occur at such water depths, intervention has proven to be very difficult. Hence in deep water it is very important that a design complies with safety standards and its behaviour can be predicted accurately. INTECSEA, the sponsor company for this thesis work, is involved in the FEED-design (Front End Engineering and Design) of a deep water pipeline project, South Stream. On behalf of this project a material testing program is developed and executed. This material testing program comprehends a collapse test program.

To perform a collapse test, a pipeline joint is produced according to specifications and put into a pressurised chamber. External pressure is applied by pumping water into this chamber. The pipe joint is filled with water that is kept at atmospheric pressure during the execution of the test. At the end of the pipe joints, end caps are attached. These end caps are very stiff and have to ensure an air-tight seal of the pipe joint. If the influence of end cap pressure is to be removed it is possible to connect the end caps by a rigid rod. The end caps need to be connected to the cylinder walls in such a way that they do not transfer any axial forces. If the influence of end cap pressure needs to be taken into account this rigid rod can be removed and axial forces may be transferred from the end caps to the specimen wall. A typical setup of such a collapse test is given in fig. 1.1.

In fig. 1.2 an example is given of a collapsed pipeline specimen after the execution of the collapse test. To evaluate the buckle propagation, pressurising was continued after first occurrence of collapse in this test.

The tested pipeline joints are fabricated using the UOE process. A description of this fabrication process is given by Herynk et al. [17]. Small geometric imperfections are inherent to this process. Besides, cold expansion of the steel during fabrication (mainly because of the expansion step in the UOE process) leads to a reduction in compressive yield stress in hoop (circumferential) direction. This origins from the Bauschinger effect that for example is described by Hines et al. [18], Kyriakides et al. [28].

Goal of the test program is the determination of fabrication specifications
with respect to material properties and geometric imperfections. Further, the recovery of compressive strength due to heating of the pipe joints is investigated. The coating process during fabrication acts as a heat treatment of the joint. This treatment can lead to stress relaxation and a reduction of the Bauschinger effect. This is expected to result in a higher compressive strength in hoop direction and hence a higher fabrication factor that can be used in the criterion for collapse of an externally pressurised pipeline given in DNV-OS-FS101 \cite{9}.

This thesis focuses on the effect of length of the tested pipeline section in combination with applied boundary conditions. The boundary conditions represent the influence of the end caps on the collapse behaviour of a pipeline. Influence of dimensionless parameters, relating diameter, wall thickness and length of the test specimen, is investigated.

\textbf{Introduction}
1.1 Scope

The scope of this thesis comprehends certain aspects of the local buckling collapse behaviour of a pipeline section under external pressure. Local is referring to the buckling behaviour occurring in the pipeline’s cross section. Global buckling effects such as upheaval and lateral buckling are not considered. Buckling is characterised for a relatively small increment in load resulting to a relatively large increment in deformation. In general, before the occurrence of buckling, the response of a structure to a load is more or less linear. This response is the fundamental solution. After the occurrence of buckling, the response is characterised by a shape different from this fundamental shape.

The statement that buckling leads to structural failure, is not true by definition. However it may lead to, and is likely to initiate, structural failure. Depending on the properties of the structure, initial post-buckling behaviour is either stable or unstable. For stable initial post-buckling behaviour, deformations can only grow when the load is increased. For unstable initial post-buckling behaviour, deformations grow even though the load is not increased. In general, structural failure can be caused by a loss of stability. In terms of buckling behaviour this can be accomplished by unstable initial post-buckling behaviour or by plastic yielding leading to a reduction in stiffness and loss of stability. Deformations grow while the load is not increased necessarily. Plastic yielding due to occurring deformations is the reason that also stable initial post-buckling behaviour is likely to underlie collapse. It is possible that the stability of a structure is regained after first occurrence of initially unstable buckling. A new equilibrium configuration is found. This behaviour is named snap-through behaviour.

The pipeline section is modelled in two ways. First, the pipeline is modelled as a ring. This is the representation of a pipeline of infinite length. Second, the pipeline is modelled as a cylinder. This allows including length effects and boundary conditions at the end cap locations.

Collapse of a pipeline origins from an unstable response to a critical load. Although for relatively thick pipelines prediction of collapse loads is sensitive to applied material models, in this thesis only simple material models are considered. Material modelling is not the main interest. To obtain the collapse load it is necessary to introduce a definition of collapse. Various collapse load definitions are considered. A ring is analysed for occurrence of first yielding and for the full development of a plastic hinge. A cylinder is analysed for the occurrence of first yielding. Verification using finite element analysis is performed for occurrence of first yielding and for loss of stability under application of an elastic-perfectly plastic material model.

The elastic response of ring and cylinder representations of the pipeline joint is analysed. The adopted method is developed by Koiter [26] and described by Budiansky [5]. First, the response of a perfect structure is analysed. In this thesis, the adjective ‘perfect’ refers to the absence of initial geometric imperfections. The response of a perfect structure can be used to obtain the response of an imperfect structure. Here the adjective ‘imperfect’ refers to the presence of initial geometric imperfections in the model.

A framework of small strains and moderate rotations is adopted. Potential energy functionals are composed using thin shell theory and the Kirchhoff-Love Introduction
assumptions as given by for example Hoefakker [19], Ventsel and Krauthammer [47]. Equilibrium equations are obtained by minimising these potential energy functionals. The bifurcation load is defined as the load for which these equilibrium equations, for a perfect structure, lead to more than one possible solution. The first deformation solution is always the fundamental solution, while the other deformation solutions are the bifurcation solutions. The total solution is a combination of these solutions. The bifurcation behaviour can be observed from a kink (bifurcation) in the load-deformation diagram. Introduction of geometric imperfections tends to smooth the transition between the pre-bifurcation and post-bifurcation domain in this diagram.

For the ring model, the initial post-bifurcation behaviour is analysed, while for the cylinder model only the bifurcation point is analysed. It is assumed that the influence of an initial geometric imperfection on the buckling behaviour of a cylinder is similar to that of a ring. To include the influence of boundary conditions to the solutions of the cylinder model, boundary layer theory is applied, as for example used by Shen and Chen [37, 38, 39], Sun and Chen [44]. Besides, regular and singular perturbation theory, as for example given by Holmes [20], Verhulst [48], is applied to obtain estimate solutions for the boundary layer and regular domains. These solutions are combined to obtain the total solution. The bifurcation solution that accounts for boundary constraints is obtained by the application of a Galerkin method.

In the cylinder model, the end caps are assumed to behave rigid. To connect the end caps to the cylinder walls, two possible connection types are analysed: simply supported connections and clamped connections. Simply supported connections only constrain deformations, while clamped connections also constrain rotations. Analytical formulations are obtained and verified using finite element analysis later.

A pipeline specimen of certain length is used in a collapse test. To be able to predict behaviour of such a pipeline in practice, results from this test need to be extrapolated. To be able to make predictions for the real life behaviour of a pipeline, it has to be ensured that the collapse mode observed in the test corresponds to the collapse mode found in real life pipelines. Too short test specimens possibly lead to higher collapse modes. A minimum test specimen length is required to ensure an oval shaped collapse mode that is expected for a relatively long real life pipeline. End caps introduce additional stiffness to the system. It is expected that the buckling load obtained from tests is higher than the buckling load of real life pipeline. This overestimation of collapse pressure needs to be accounted for when results are translated back to real life collapse pressure values.

### 1.2 Objective

For this thesis several objectives have been formulated. The first objective is to obtain a clear understanding and give a mathematical description of the mechanism of local buckling collapse behaviour of pipelines under external (hydrostatic) pressure.

The second objective is to describe the influence of several parameters on the buckling load and buckling mode of a pipeline. The most important parameters are the diameter, wall thickness and length. The influence of end caps on the
buckling of a pipeline has to be investigated.

The final objective is to give a recommendation of the minimum joint length to use in a collapse test program for a pipeline of certain specifications.

1.3 Outline

Chapter 2 deals with general bifurcation and buckling theory. Koiter’s perturbation theory to obtain the (elastic) bifurcation and buckling behaviour of a structure is explained. In chapter 3 the bifurcation and buckling behaviour of a ring is analysed. Chapter 4 deals with the bifurcation and buckling behaviour of an externally pressurised cylinder constrained by end caps. In chapter 5 the results from chapters 3 and 4 are verified using finite element analyses. A quantitative and qualitative discussion of these results is reported in this chapter. Interpretation of the results, the conclusions from this study and the recommendations for further work are reported in chapter 6.
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Chapter 2

Buckling theory

2.1 Introduction

In this thesis the buckling and initial post-buckling behaviour of a pipeline model is analysed. The adopted method is developed by Koiter [26] and modified by Budiansky [5]. The initial-post buckling behaviour of a perfect structure can be used to determine the behaviour of a structure with initial geometric imperfections. The description of the adopted method in this chapter is largely based on Budiansky [5].

Only statics are taken into account. All time dependent (dynamic) terms are neglected. Hence the principle of minimum potential energy is introduced. An equilibrium state of a structure can be found by minimising the potential energy functional of this structure. By minimising this functional, equilibrium equations can be obtained. The method described by Koiter and Budiansky is based on perturbation theory. A solution to the minimisation of the potential energy functional or equilibrium equations is perturbed. This perturbation is expressed as a power series in terms of a small parameter $\zeta$, the bifurcation parameter. The first term in the expansion is the fundamental solution, the solution to the linearised problem. Now $\zeta$ describes the amount of bifurcation mode included in the solution. Also the load is expanded around its critical load value, the bifurcation load.

It is assumed that system, after application of the previous substitutions, needs to be satisfied for every order of $\zeta$ separately. This leads to an equilibrium system for each order of $\zeta$. The consequence is that besides the fundamental system, every system of equations is a system of linear differential equations.

Bifurcations from the fundamental solution path are the result of a singularity in the system. For a certain load, the bifurcation load, the system has multiple equilibrium solutions. This load is obtained from an Eigenvalue analysis. Analysis of the higher order load expansion terms gives the initial post-buckling behaviour of the system.

Initial imperfections are introduced by an additional potential energy functional. This functional is added to the potential energy functional of the perfect system. Consequently the additional potential functional should be equal to zero if there are no deformations or initial imperfections. The functions describing the initial imperfections are expanded around zero. Now the influence of initial
imperfections on the response of the structure can be analysed.

2.2 Preliminaries

2.2.1 The Fréchet derivative

Let \( f \) be a functional depending on the generalised function \( u \). This dependency is denoted by \( f[u] \). \( f \) is called Fréchet differentiable at the point \( u \) if it can be approximated by \( f[u + \epsilon u_1] \) and the following relation is satisfied [32]:

\[
\lim_{\epsilon \to 0} \left\| D_u f[u] u_1 \right\| = 0 \quad (2.1)
\]

Here the \( D_u \)-operator denotes the first order Fréchet derivative with respect to the function \( u \). For any fixed \( u_1 \) the following definition is obtained:

\[
D_u f[u] u_1 \equiv \lim_{\epsilon \to 0} \frac{f[u + \epsilon u_1] - f[u]}{\epsilon} \quad (2.2)
\]

This can be rewritten to:

\[
D_u f[u] u_1 = \frac{\partial}{\partial \epsilon} f[u + \epsilon u_1] \bigg|_{\epsilon=0} \quad (2.3)
\]

The generalised form for higher order Fréchet derivatives denotes [5]:

\[
D_{u...u} f[u] u_1 \ldots u_n = \frac{\partial^n}{\partial \epsilon_1 \ldots \partial \epsilon_n} f[u + \epsilon_1 u_i_1 + \ldots + \epsilon_n u_i_n] \bigg|_{\epsilon_1,\ldots,\epsilon_n=0} \quad (2.4)
\]

Where \( i_j \in \mathbb{N} \) and \( j = 1, 2, \ldots, n \).

Now let a functional \( f[u] \) denote a function depending on multiple functions e.g. \( v \), \( w \) and so on. Then \( u \) represents the set of functions \( \{v, w, \ldots\} \) and \( f[u] = f[v, w, \ldots] \). The Fréchet derivative of the functional \( f[u] \), denoted by \( D_{u...u} \), where the order of derivative is given by the amount of \( u \)-terms in the subscript of the operator, is defined by:

\[
D_{u...u} f[u] u_1 \ldots u_n = \frac{\partial^n}{\partial \epsilon_1 \ldots \partial \epsilon_n} f[u + \epsilon_1 u_i_1 + \ldots + \epsilon_n u_i_n] \bigg|_{\epsilon_1,\ldots,\epsilon_n=0} \quad (2.5)
\]

Where:

\[
\begin{align*}
\overset{*}{v} &= v + \epsilon_1 v_i_1 + \ldots + \epsilon_n v_i_n \\
\overset{*}{w} &= w + \epsilon_1 w_i_1 + \ldots + \epsilon_n w_i_n
\end{align*} \quad (2.6a/b)
\]

Note that the \( u_{i_j} \) represents a set \( \{v_{i_j}, w_{i_j}, \ldots\} \). To shorten the notation it is allowed to write \( D_{uu} f[u] u_1 u_1 \) as \( D_{uu} f[u] u_1^2 \) and so on.

Buckling theory
2.2.2 Expansions

The approximate value of a functional $f[u + \delta u]$ can be found using the power series expansion in terms of variations:

$$ f[u + \delta u] = f[u] + \delta f[u] + \delta^2 f[u] + \delta^3 f[u] + \ldots $$  \hfill (2.7)

It can also be approximated as a Taylor expansion terms of the Fréchet derivative of the functional:

$$ f[u + \delta u] = f[u] + D_u f[u] \delta u + \frac{1}{2!} D_{uu} f[u] (\delta u)^2 + \frac{1}{3!} D_{uuu} f[u] (\delta u)^3 + \ldots $$  \hfill (2.8)

The following is obtained for the generalised variation of a functional $f$:

$$ \delta^n f[u] = \frac{1}{n!} D_{u^n} f[u] (\delta u)^n $$  \hfill (2.9)

For a functional depending on multiple functions, a generalised Taylor expansion of $f[u + \delta u]$ is given by:

$$ f[u + \delta u] = f[u] + D_u f[u] \delta u + \frac{1}{2!} D_{uu} f[u] (\delta u)^2 + \frac{1}{3!} D_{uuu} f[u] (\delta u)^3 + \ldots $$  \hfill (2.10)

Let $f[u]$ describe a functional $f[v, w]$. Now $f[u]$ is expanded using a Taylor expansion at $u = u_0$. Introducing the notation $D_{u...u} f[u_0] = D_{u...u} f[u]|_{u=u_0}$ gives:

$$ f[u] = f[u_0] + D_u f[u_0] (u - u_0) + \frac{1}{2!} D_{uu} f[u_0] (u - u_0)^2 + \ldots $$  \hfill (2.11)

This expansion of $f[u]$ can also be expressed as a multivariate Taylor expansion of $f[v, w]$ at $v = v_0$ and $w = w_0$. If the expansion up to the second order derivative are given it will read:

$$ f[v, w] = f[v_0, w_0] + D_v f[v_0, w_0] (v - v_0) + D_w f[v_0, w_0] (w - w_0) + \frac{1}{2} D_{vv} f[v_0, w_0] (v - v_0)^2 + D_{vw} f[v_0, w_0] (v - v_0) (w - w_0) + \frac{1}{2} D_{ww} f[v_0, w_0] (w - w_0)^2 + \ldots $$  \hfill (2.12)

By comparing eqs. (2.11) and (2.12), by using the statement $f[u] = f[v, w]$, the following relations are obtained:

$$ D_u f[u_0] (u - u_0) = D_v f[v_0, w_0] (v - v_0) + D_w f[v_0, w_0] (w - w_0) $$  \hfill (2.13a)

$$ D_{uu} f[u_0] (u - u_0)^2 = D_{vv} f[v_0, w_0] (v - v_0)^2 + 2D_{vw} f[v_0, w_0] (v - v_0) (w - w_0) + D_{ww} f[v_0, w_0] (w - w_0)^2 $$  \hfill (2.13b)

This can be verified using eqs. (2.4) and (2.5) for arbitrary functionals. Hence the Taylor expansion given in eq. (2.11) is valid, also when $u$ represents a set of functions.

Analogous to eq. (2.7), the generalised variations of a functional $f[u]$ now reads:

$$ \delta^n f[u] = \frac{1}{n!} D_{u^n} f[u] (\delta u)^n $$  \hfill (2.14)

Buckling theory
2.2.3 Energy functionals

The potential energy of a perfect structure is denoted by functional $\Pi[u]$. Here $u$ is a generalised displacement field dependent on a load factor $\lambda$. Hence the potential energy is denoted by $\Pi[u(\lambda); \lambda]$. The potential energy of a structure is the sum of the strain energy due to the deformation of the structure and the applied external energy (work) due to the loads.

Because a static analysis is performed, the kinetic energy is neglected. Hence all time dependent terms drop out of the equations. The potential energy of a structure with initial geometric imperfections is given by:

$$\bar{\Pi}[u(\lambda); \bar{u}; \lambda] = \Pi[u(\lambda); \lambda] + \Psi[u(\lambda); \bar{u}; \lambda] \quad (2.15)$$

Here $\bar{\Pi}[u(\lambda); \bar{u}; \lambda]$ is the total potential energy of the structure with initial imperfections included. $\Pi[u(\lambda); \lambda]$ is the potential energy functional of the perfect structure. Hence $\Psi[u(\lambda); \bar{u}; \lambda]$ describes the additional potential energy of an imperfect structure with respect to the potential energy of a perfect structure. $\bar{u}$ describes the initial imperfection field of the structure.

The following relations should be valid for $\Psi[u(\lambda); \bar{u}; \lambda]$:

$$\Psi[u(\lambda); 0; \lambda] = 0 \quad (2.16a)$$
$$\Psi[0; \bar{u}; \lambda] = 0 \quad (2.16b)$$

If either the initial imperfection or the displacement is zero, the additional potential energy terms should be equal to zero.

2.3 Principle of minimum potential energy

Let $S$ be the action of a system on time interval $t \in [t_1, t_2]$. This action is defined by the following integral [46]:

$$S = \int_{t_1}^{t_2} \mathcal{L}[u(t); \dot{u}(t); t] dt \quad (2.17)$$

Where the Lagrangian $\mathcal{L}$ is defined by:

$$\mathcal{L}[u(t), \dot{u}(t), t] = K[\dot{u}(t), t] - \Pi[u(t), t] \quad (2.18)$$

The dot here denotes the first order derivative with respect to time $t$. $u$ denotes the generalised displacement and hence $\dot{u}$ represents the generalised velocity. $K$ is the kinetic energy, that depends on the velocity, and $\Pi$ is the potential energy functional, that depends on the displacement components of the system. The principle of stationary action or Hamilton’s principle [4] states that the dynamic system is in equilibrium if the action is stationary i.e. $\delta S = 0$. In general this will lead to minimising the action integral. Hamilton’s principle reads:

$$\delta S = \int_{t_1}^{t_2} \delta \mathcal{L}[u(t), \dot{u}(t), t] dt = 0 \quad (2.19)$$

The weak formulation of Hamilton’s principle is transformed to a stronger formulation by assuming that the equation above is fulfilled if its integrand is equal to zero. Hence the strong formulation of Hamilton’s principle reads:

$$\delta \mathcal{L}[u(t), \dot{u}(t), t] = \delta (K[\dot{u}(t), t] - \Pi[u(t), t]) = 0 \quad (2.20)$$

**Buckling theory**
In this thesis only statics are taken into account. Dynamic effects are
neglected. This allows for all time dependencies to be removed from the analysis.
This leaves the principle of minimum potential energy:

\[ \delta \Pi[u] = 0 \quad (2.21) \]

Note that in fact the obtained variational statement above represents
the principle of stationary potential energy. The obtained equilibrium configuration
can represent a minimum, a maximum or a saddle point in potential energy.
\[ \forall u_i \in u, \delta^2 \Pi > 0 \] indicates a stable equilibrium state and hence a minimum
in potential energy. The potential energy \( \Pi \) for a conservative system can be
described by:

\[ \Pi = U + W \quad (2.22) \]

Where \( U \) denotes the strain energy of the elastic body and \( W \) is the (negative)
energy of the applied external forces (work) onto the system.

### 2.4 Bifurcation buckling

The method described in this section is a short overview of Budiansky [5]. Let a
prime denote the Fréchet derivative with respect to the displacement field \( u \) i.e.
\[ \Pi'[u]u_1 \equiv D_u \Pi[u]u_1, \Pi''[u]u_1 \equiv D_{uu} \Pi[u]u_1 \] and so on.

Following the principle of minimum potential energy, the equilibrium state of
a system is obtained by setting the first variation of the potential energy to zero.
By using \( \delta \Pi = 0 \) and eq. (2.14), the following equilibrium equation is obtained:

\[ \Pi'[u(\lambda); \lambda]\delta u = 0 \quad (2.23) \]

The pre-buckling state of the system is described by \( u_0(\lambda) \). This is also
referred to as the fundamental solution. If the system is perturbed from this
state, the generalised displacement \( u \) can be determined using the small scalar
perturbation factor \( \zeta \), where it is important to note that \( \zeta \ll 1 \):

\[ u(\lambda) = u_0(\lambda) + \zeta u_1 + \zeta^2 u_2 + \zeta^3 u_3 + \ldots \quad (2.24) \]

Where

\[ \zeta = \langle u - u_0, u_1 \rangle = \langle \zeta u_1 + \zeta^2 u_2 + \zeta^3 u_3 + \ldots, u_1 \rangle \quad (2.25) \]

The \( \langle \bullet \rangle \) operator represents the bilinear inner product. If is assumed that
\( \langle u_1, u_1 \rangle = 1 \), it can be shown that \( \langle \zeta^2 u_2 + \zeta^3 u_3 + \ldots, u_1 \rangle = 0 \). This implies
that \( u_1 \) is orthogonal to \( u_2, u_3 \) and so on. Orthogonality of functions implies
independence of the functions.

Introduce the notation:

\[ \Pi_0^{(n)} \equiv \Pi^{(n)}[u(\lambda); \lambda]\bigg|_{u=u_0} \quad (2.26) \]

Where the superscripted \( (n) \)-term corresponds with the \( n \)th order Fréchet
derivative with respect to \( u \). The Taylor series expansion given in eq. (2.11) is
performed onto eq. (2.23). The following expansion is obtained:

\[ \Pi_0' \delta u + \Pi_0'' (u - u_0) \delta u + \frac{1}{2} \Pi_0''' (u - u_0)^2 \delta u + \frac{1}{6} \Pi_0'''' (u - u_0)^3 \delta u + \ldots = 0 \quad (2.27) \]

**Buckling theory**
The fundamental state \( u_0 \) can be derived from the equation:

\[ \Pi'_0 \delta u = 0 \]  

(2.28)

Application of the Euler-Lagrange equations on the system above leads to a system of differential equations. By solving this system, the fundamental solution is obtained.

It is allowed to set \( \Pi'_0 \delta u \) to zero because of eq. (2.23). Hence the term can be dropped in eq. (2.27). Dropping this term and substituting eq. (2.24) into eq. (2.27) leads to:

\[ \Pi'_0 (\zeta u_1 + \zeta^2 u_2 + \zeta^3 u_3 + \ldots) \delta u + \frac{1}{2} \Pi''_0 (\zeta u_1 + \zeta^2 u_2 + \ldots)^2 \delta u + \frac{1}{6} \Pi'''_0 (\zeta u_1 + \ldots)^3 \delta u + \ldots = 0 \]  

(2.29)

The load parameter \( \lambda \) is expanded at its critical state, the bifurcation load \( \lambda_c \), using the small perturbation parameter \( \zeta \):

\[ \lambda = \lambda_c + \zeta \lambda_1 + \zeta^2 \lambda_2 + \zeta^3 \lambda_3 + \ldots \]  

(2.30)

Introduce the identities:

\[ \Pi^{(n)} = \Pi^0 \big|_{\lambda=\lambda_c} \]  

(2.31a)

\[ \Pi^{(n)} = \frac{\partial \Pi^0}{\partial \lambda} \big|_{\lambda=\lambda_c} \]  

(2.31b)

\[ \Pi^{(n)} = \frac{\partial^2 \Pi^0}{\partial \lambda^2} \big|_{\lambda=\lambda_c} \]  

(2.31c)

Now perturbing the equilibrium equation given in eq. (2.27) around \( \lambda_c \) leads to:

\[ \left( \Pi'' + \{\lambda - \lambda_c\} \Pi''' + \frac{1}{2} \{\lambda - \lambda_c\}^2 \Pi'''' + \ldots \right) (\zeta u_1 + \zeta^2 u_2 + \zeta^3 u_3 + \ldots) \delta u 
+ \frac{1}{2} \left( \Pi^{'''} + \{\lambda - \lambda_c\} \Pi^{''''} + \ldots \right) (\zeta u_1 + \zeta^2 u_2 + \ldots)^2 \delta u 
+ \frac{1}{6} \left( \Pi^{'''''} + \ldots \right) (\zeta u_1 + \ldots)^3 \delta u + \ldots = 0 \]  

(2.32)

From eq. (2.30) can be obtained that:

\[ \lambda - \lambda_c = \zeta \lambda_1 + \zeta^2 \lambda_2 + \zeta^3 \lambda_3 + \ldots \]  

(2.33)

Substituting this relation into eq. (2.32) gives:

\[ \left( \Pi'' + \{\zeta \lambda_1 + \zeta^2 \lambda_2 + \ldots\} \Pi'''' + \frac{1}{2} \{\zeta \lambda_1 + \ldots\}^2 \Pi'''' + \ldots \right) (\zeta u_1 + \zeta^2 u_2 + \zeta^3 u_3 + \ldots) \delta u 
+ \frac{1}{2} \left( \Pi^{''''} + \{\zeta \lambda_1 + \ldots\} \Pi^{''''} + \ldots \right) (\zeta u_1 + \zeta^2 u_2 + \ldots)^2 \delta u 
+ \frac{1}{6} \left( \Pi^{'''''} + \ldots \right) (\zeta u_1 + \ldots)^3 \delta u + \ldots = 0 \]  

(2.34)

Rewriting this equation in terms of \( \zeta \) gives:

\[ \zeta \Pi'_0 u_1 \delta u + \zeta^2 \left( \Pi''_0 u_2 + \lambda_1 \Pi''_0 u_1 + \frac{1}{2} \Pi''''_0 u_1^2 \right) \delta u 
+ \zeta^3 \left( \Pi''_0 u_3 + \lambda_1 \Pi''_0 u_2 + \lambda_2 \Pi''_0 u_1 + \frac{1}{2} \lambda^2 \Pi''''_0 u_1 + \frac{1}{2} \Pi''''' u_1 \right) \delta u + O(\zeta^4) = 0 \]  

(2.35)

**Buckling theory**
In the equation above every single coefficient of a power of $\zeta$ must equal zero. The bifurcation equation is obtained by setting the coefficient of the first power of $\zeta$ to zero. The differential equation to determine the bifurcation load $\lambda_c$ and first order bifurcation shape $u_1$ is given by:

$$\Pi_c' u_1 \delta u = 0$$

(2.36)

The solutions $u_1$ to this system are only non-trivial if the load parameter $\lambda$ equals a critical bifurcation load parameter $\lambda_c$. A buckling shape $u_1$ is obtained by solving the system at the critical load. This solution will have an an arbitrary amplitude. It is chosen to assign a value to this amplitude based on letting the norm of one of the buckling shape functions be equal to one. If this function now is multiplied by a factor, then this factor will describe the amount of buckling included in the solution.

The analysis in this section results in the following solutions: $\lambda_c$, $u_0$ and $u_1$.

### 2.5 Initial post-buckling behaviour

The initial post buckling behaviour describes the behaviour of the structure just after occurrence of bifurcation buckling.

From eq. (2.35), besides the bifurcation equation, the following equations are obtained:

$$\Pi_c u_2 \delta u + \lambda_1 \Pi_c' u_1 \delta u + \frac{1}{2} \Pi_c'' u_1^2 \delta u = 0$$

(2.37a)

$$\Pi_c u_3 \delta u + \lambda_1 \Pi_c' u_2 \delta u + \lambda_2 \Pi_c'' u_1 \delta u + \frac{1}{2} \lambda_1^2 \Pi_c' u_1 \delta u
+ \Pi_c'' u_1 u_2 \delta u + \frac{1}{2} \lambda_1 \Pi_c'' u_1^2 \delta u + \frac{1}{6} \Pi_c''' u_1^3 \delta u = 0$$

(2.37b)

Now let $\delta u = u_1$ in eq. (2.37a). Because of the bifurcation equation given in eq. (2.36), the term $\Pi_c' u_1 u_2 = 0$ and drops out. Subsequently $\lambda_1$ can be obtained:

$$\lambda_1 = -\frac{\frac{1}{2} \Pi_c'' u_1^3}{\Pi_c' u_1^2}$$

(2.38)

With $\lambda_1$ being known, $u_2$ can be determined using eq. (2.37b). Using variational calculus and the Euler-Lagrange equations, a system of differential equations is obtained. The terms $\lambda_1 \Pi_c' u_1 \delta u$ and $\frac{1}{2} \Pi_c'' u_1^2 \delta u$ in eq. (2.37a) relate to constant terms in the system of equations.

Note that the homogeneous part of this equation ($\Pi_c' u_2 \delta u = 0$) with respect to $u_2$ is corresponds to the bifurcation equation given in eq. (2.36) with respect to $u_1$. In section 2.4 it is presumed that $u_2$ is orthogonal to $u_1$. Hence the part of the general solution corresponding to the solution to the homogeneous part of the differential equation has to drop out. Now the solution for $u_2$ is obtained.

If in eq. (2.37b) $u_1$ is substituted for $\delta u$, the term $\Pi_c' u_1 u_3 = 0$ and drops out due to eq. (2.36). Hence $\lambda_2$ can be obtained using:

$$\lambda_2 = -\frac{\frac{1}{2} \Pi_c'' u_1^3 + \Pi_c'' u_1^2 u_2 + \lambda_1 \left( \Pi_c' u_1 u_2 + \frac{1}{2} \Pi_c'' u_1^3 + \frac{1}{6} \lambda_1 \Pi_c''' u_1^3 \right)}{\Pi_c' u_1^2}$$

(2.39)

Systematic improvements can be made to the results by extending this analysis. In this analysis it is assumed that results are sufficiently correct for expansions up to order $\zeta^2$.
2.6 Initial imperfections

In practice, structures are never perfect. If the perfect structure is subject to an initial geometric imperfection, the analysis described previously can be modified to account for this effect. Assume that the imperfection does not cause any residual stresses in the structure and its shape can be described in a shape similar to the first order bifurcation shape $u_1$.

A modified energy functional that accounts for both the perfect behaviour and the imperfect behaviour of the structure is given in eq. (2.15). Now let $\tilde{u}$ describe the initial imperfection, such that $u = \xi \tilde{u}$, where the norm of (one of the components of) $\tilde{u}$ equals unity. Now $\xi$ is a measure of the imperfection amplitude. It can be considered analogous to the ovality used in the pipeline industry.

The variational equilibrium equation given in eq. (2.23) is modified for the adapted energy functional $\Pi$. This leads to the new equilibrium equation for the system with initial geometric imperfections:

$$\Pi'[u(\lambda); \lambda]\delta u + \Psi'[u(\lambda); \bar{u}; \lambda]\delta u = 0$$  \hspace{1cm} (2.40)

The Taylor expansion of this equation around the fundamental solution of the perfect system $u = u_0$ reads:

$$\Pi'_0\delta u + \Pi'_0(u - u_0)\delta u + \frac{1}{2}\Pi''_0(u - u_0)^2\delta u + \frac{1}{6}\Pi'''_0(u - u_0)^3\delta u$$

$$+ \ldots + \Psi'[u_0(\lambda); \bar{u}; \lambda]\delta u + \Psi''[u_0(\lambda); \bar{u}; \lambda](u - u_0)\delta u + \ldots = 0$$  \hspace{1cm} (2.41)

Now the Fréchet derivative with respect to the initial imperfection $\tilde{u}$ is introduced and denoted by an asterisk in superscript. This derivative is generated according to eq. (2.3). Let $\Psi[u_0(\lambda); 0; \lambda] = \Psi_0$. The following expansions are obtained:

$$\Psi'[u_0(\lambda); \bar{u}; \lambda] = \Psi'_0 + \Psi'^*_0\tilde{u} + \frac{1}{2}\Psi''_0\tilde{u}^2 + \ldots$$  \hspace{1cm} (2.42a)

$$\Psi''[u_0(\lambda); \bar{u}; \lambda] = \Psi''_0 + \Psi''_0\tilde{u} + \ldots$$  \hspace{1cm} (2.42b)

From eq. (2.16) follows that $\Psi'_0 = \Psi''_0 = \ldots = 0$. Hence these expansions can be rewritten:

$$\Psi'[u_0(\lambda); \bar{u}; \lambda] = \Psi'^*_0\tilde{u} + \frac{1}{2}\Psi''_0\tilde{u}^2 + \ldots$$  \hspace{1cm} (2.43a)

$$\Psi''[u_0(\lambda); \bar{u}; \lambda] = \Psi''_0\tilde{u} + \ldots$$  \hspace{1cm} (2.43b)

These relations are substituted into eq. (2.41) and the previously assumed relation $\tilde{u} = \xi \tilde{u}$ is used. $\Pi'_0\delta u = 0$ due to the equilibrium equation of the perfect system given in eq. (2.25). Solving this expression using the Euler-Lagrange equations, gives the fundamental solution. Now for the remaining terms the following relation is obtained:

$$\Pi''_0(u - u_0)\delta u + \frac{1}{2}\Pi''_0(u - u_0)^2\delta u + \frac{1}{6}\Pi'''_0(u - u_0)^3\delta u + \ldots$$

$$+ \xi \Psi'^*_0\tilde{u}\delta u + \frac{1}{2}\xi^2\Psi''_0\tilde{u}^2\delta u + \ldots + \xi\Psi''_0\tilde{u}(u - u_0)\delta u + \ldots = 0$$  \hspace{1cm} (2.44)

This equation is expanded at its bifurcation load $\lambda = \lambda_c$. Here $\Psi''_c = \Psi''_0|_{\lambda=\lambda_c}$ and so on, analogous to eq. (2.31). Further a dot above a functional represents Buckling theory.
the derivative with respect to \( \lambda \) such that \( \Pi''(u-u_0)\delta u + (\lambda - \lambda_c)\Pi''(u-u_0)\delta u + \frac{1}{2}(\lambda - \lambda_c)^2\Pi'''(u-u_0)\delta u + \ldots \)

\[
\begin{align*}
\Pi''(u-u_0)\delta u + (\lambda - \lambda_c)\Pi''(u-u_0)\delta u + \frac{1}{2}(\lambda - \lambda_c)^2\Pi'''(u-u_0)\delta u + \ldots \\
+ \frac{1}{2}\Pi''''(u-u_0)^2\delta u + \frac{1}{2}(\lambda - \lambda_c)\Pi''''(u-u_0)^2\delta u + \ldots \\
+ \frac{1}{2}\Pi''''(u-u_0)^3\delta u + \ldots + \xi\Psi'_{\varepsilon}\dot{u}\delta u + \xi(\lambda - \lambda_c)\Psi'_{\varepsilon}\ddot{u}\delta u + \ldots \\
+ \frac{1}{2}\xi^2\Psi''_{\varepsilon}\dot{u}^2\delta u + \ldots + \xi\Psi''_{\varepsilon}\dot{u}(u-u_0)\delta u + \ldots = 0
\end{align*}
\]

(2.45)

Now introduce the perturbation for \( u \) around the fundamental solution of the perfect structure \( u_0 \), analogous to eq. (2.24). It will be shown that this perturbation is valid. Further \( \lambda \) is perturbed around \( \lambda_c \) such that:

\[
\begin{align*}
u &= u_0 + \zeta_1 u_1 + \zeta^2 \tilde{u}_2 + \zeta^3 \tilde{u}_3 + \ldots \\
\lambda &= \lambda_c + \zeta_1 \tilde{\lambda}_1 + \zeta^2 \tilde{\lambda}_2 + \zeta^3 \tilde{\lambda}_3 + \ldots
\end{align*}
\]

(2.46a)

(2.46b)

This leads to the following identities:

\[
\begin{align*}
u - u_0 &= \zeta_1 u_1 + \zeta^2 \tilde{u}_2 + \zeta^3 \tilde{u}_3 + \ldots \\
\lambda - \lambda_c &= \zeta_1 \tilde{\lambda}_1 + \zeta^2 \tilde{\lambda}_2 + \zeta^3 \tilde{\lambda}_3 + \ldots
\end{align*}
\]

(2.47a)

(2.47b)

These identities are substituted into eq. (2.25). If now \( \zeta \) is related to \( \xi = \alpha \zeta^2 \), where \( \alpha \) can be set to a value such that this relation is valid. Now the following equilibrium equation is obtained:

\[
\begin{align*}
\zeta \Pi''(\zeta u_1 + \zeta^2 \tilde{u}_2 + \zeta^3 \tilde{u}_3 + \ldots)\delta u \\
+ (\zeta \tilde{\lambda}_1 + \zeta^2 \tilde{\lambda}_2 + \ldots)\Pi''(\zeta u_1 + \zeta^2 \tilde{u}_2 + \ldots)\delta u \\
+ \frac{1}{2}(\zeta \tilde{\lambda}_1 + \ldots)^2 \Pi''(\zeta u_1 + \ldots)\delta u + \ldots \\
+ \frac{1}{2}(\zeta \tilde{\lambda}_1 + \ldots)^2 \Pi'''(\zeta u_1 + \ldots)^2\delta u + \ldots \\
+ \frac{1}{2}(\zeta \tilde{\lambda}_1 + \ldots)^3 \Pi''''(\zeta u_1 + \ldots)^3\delta u + \ldots \\
+ \alpha \zeta^3 \Psi_{\varepsilon}^2\dot{u}\delta u + \alpha \zeta^4 (\zeta \tilde{\lambda}_1 + \ldots)\dot{\Psi}_{\varepsilon}^2\ddot{u}\delta u + \ldots \\
+ \alpha \zeta^6 \Psi_{\varepsilon}^3\dot{u}(\zeta u_1 + \ldots)\delta u + \ldots = 0
\end{align*}
\]

For symmetric buckling it has to follow that \( \lambda_1 = 0 \). If so, it is convenient to let \( \gamma = 3 \) [5]. Now the following equilibrium is obtained:

\[
\begin{align*}
\zeta \Pi''(u_1)\delta u + \zeta^2 (\Pi''\tilde{u}_2 + \tilde{\lambda}_1 \Pi'' u_1 + \frac{1}{2}\Pi''' u_2)\delta u \\
+ \zeta^3 \left( \Pi''_{\varepsilon} u_3 + \tilde{\lambda}_1 \Pi''_{\varepsilon} u_2 + \tilde{\lambda}_2 \Pi''_{\varepsilon} u_1 + \frac{1}{2}\tilde{\lambda}_1^2 \Pi''_{\varepsilon} u_1 \\
+ \Pi'''_{\varepsilon} u_1 \tilde{u}_2 + \frac{1}{2}\tilde{\lambda}_1 \Pi'''_{\varepsilon} u_1^2 + \frac{1}{6} \Pi''''_{\varepsilon} u_1^3 + \alpha \Psi_{\varepsilon}^2\dot{u} \right)\delta u + \mathcal{O}(\xi^4) = 0
\end{align*}
\]

(2.49)

Where all coefficients of \( \zeta \), \( \zeta^2 \), \ldots must vanish separately. Setting coefficient of \( \zeta \) to zero results in the bifurcation equation of the perfect system given in eq. (2.36). This also validates the fact that \( u_1 \) in the perfect system is equal to \( u_1 \) in the system with initial geometric imperfections. Setting the coefficient of \( \zeta^2 \) to zero and substituting \( \delta u = u_1 \) gives:

\[
\Pi''(u_1)\tilde{u}_2 + \tilde{\lambda}_1 \Pi''_{\varepsilon} u_1 + \frac{1}{2}\Pi''' u_2 = 0
\]

(2.50)

Buckling theory
The bifurcation equation results in
\[ \Pi''_c u_1 \tilde{u}_2 = \Pi''_c u_1 \delta u = 0. \]
The previous relation between \( \xi \) and \( \zeta \) removes \( \alpha \) by using \( \alpha = \xi \zeta^{-3} = \xi \zeta^{-3} \). Now \( \tilde{\lambda}_1 \) can be obtained from:
\[ \tilde{\lambda}_1 = -\frac{1}{2} \frac{\Pi'''_c u_1^3}{\Pi''_c u_1^2} = \lambda_1 \tag{2.51} \]

Now \( \tilde{u}_2 \) is obtained by solving the following equation using the Euler-Lagrange equations:
\[ \Pi''_c u_1 \delta u + \lambda_1 \Pi''_c u_1 \delta u + \frac{1}{2} \Pi'''_c u_1^2 \delta u = 0 \tag{2.52} \]

It follows that \( \tilde{u}_2 = u_2 \). In a similar fashion as described above, \( \tilde{\lambda}_2 \) can be obtained. The coefficient of \( \zeta^3 \) in eq. (2.52) is set to zero and the bifurcation equation is used to let \( \Pi''_c u_1 \tilde{u}_3 = \Pi''_c u_1 \delta u = 0 \). The substitution \( \delta u = u_1 \) is applied. The expression for \( \tilde{\lambda}_2 \) now reads:
\[ \tilde{\lambda}_2 = -\frac{1}{2} \frac{\Pi'''_c u_1^3}{\Pi''_c u_1^2} + \lambda_1 \left( \Pi''_c u_1 u_2 + \frac{1}{2} \Pi'''_c u_1^3 + \frac{1}{2} \lambda_1 \Pi''_c u_1^2 \right) - \frac{\xi \Psi'''_c \tilde{u} u_1}{\zeta^3 \Pi''_c u_1^2} \]
\[ = \lambda_2 - \frac{\xi \Psi'''_c \tilde{u} u_1}{\zeta^3 \Pi''_c u_1^2} \tag{2.53} \]

### 2.7 Bifurcation types

In fig. [2.11] for various types of initial post-buckling behaviour, relations are given for the bifurcation parameter \( \zeta \) and the dimensionless load factor \( \lambda \). The influence of initial imperfections with size \( \xi \) is (qualitatively) denoted by a dashed line for \( \xi > 0 \) and by a dash-dotted line for \( \xi < 0 \). Note that the branches located above the graph for the perfect structure can never be reached. Loading and deformation are assumed to start at the origin of the graph. The origin is located at the intersection point of the graphs of the perfect and imperfect systems. Here \( \zeta = \lambda = 0 \).

Figures [2.1c, 2.1d, and 2.1g] denote symmetric bifurcation behaviour and figs. [2.1a, 2.1b, 2.1e, and 2.1f] denote asymmetric bifurcation behaviour. Symmetric bifurcation implies that the sign of the imperfection does not influence initial post-buckling behaviour (apart from the direction). For asymmetric buckling the post-bifurcation behaviour varies for positive and negative imperfection sizes. In for example fig. [2.1a] a positive imperfection size leads to stable initial post-buckling behaviour while a negative imperfection size leads to unstable post-buckling behaviour.

In figs. [2.1g to 2.1k] snap-through behaviour is observed for the paths that first behave unstable and regain their stability for larger absolute values of \( \zeta \). For this type of behaviour the structure ‘snaps’ to another equilibrium configuration as soon as the graph reaches a local maximum (for load controlled behaviour).

Here stable initial post-buckling behaviour is characterised by an increase in \( \zeta \) requiring an increase in \( \lambda \). For unstable behaviour \( \zeta \) can grow even though the load parameter does not increase.

**Buckling theory**
Figure 2.1: Various types of initial post-bifurcation behaviour. Bifurcation point is at intersection of lines for $\xi = 0$. Buckling theory
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Chapter 3

Buckling of a ring

In this chapter, the behaviour of a ring subjected to external hydrostatic pressure will be analysed. Hydrostatic pressure is always directed perpendicular to the ring’s circumferential direction and co-rotates during deformation.

3.1 Assumptions

In order to derive the buckling behaviour of a perfectly circular ring, certain assumptions have to be made in modelling. The Kirchhoff-Love assumptions \[19, 47\] are adopted amongst others:

i  Straight sections initially normal to the middle line remain straight during deformation.
ii  Sections initially normal to the middle line remain normal to the middle line during deformation.
iii  Thickness of the ring does not change during deformation.
iv  Transverse normal stresses are small compared to other stress components and may be neglected.
v  The material model used is an isotropic linear elastic material model.
vi  The ring is modelled as a two dimensional model with one curvilinear coordinate axis describing the middle line.
vii  Middle line strains are small.
viii  Middle line rotations are moderately small.

The bifurcation analysis is performed for an isotropic elastic material model following Hooke’s law. Further, the influence of gravity and buoyancy forces in neglected. The structure is assumed to be homogeneous and the initial wall thickness is assumed to be constant.

3.2 Kinematics

Vector identities

Every material point on the middle line of an undeformed perfectly circular ring is described by the vector $p$. The ring is a plane object and hence its behaviour
is analysed in a two dimensional coordinate system. The unit vectors describing the (orthonormal) Cartesian coordinate system are $\hat{i}$ along the $x^1$-axis and $\hat{j}$ along the $x^2$-axis. The vector $\mathbf{p}$ is described in the Cartesian coordinate system by the relation $\mathbf{p} = x^1\hat{i} + x^2\hat{j}$. The model of the ring is given in fig. 3.1.

Now a two dimensional curvilinear coordinate system is introduced. One curvilinear axis describes the middle line of the perfect ring. This axis is also referred to as the circumferential or hoop axis. The other describes the normal component with respect to the former curvilinear axis. This axis is also referred to as the radial axis. The undeformed ring can by described fully by the circumferential coordinate $s \in [0, 2\pi\rho]$ and the radial coordinate $z \in [-\frac{1}{2}t, \frac{1}{2}t]$. Here $\rho$ denotes the radius and $t$ the wall thickness of the undeformed ring. The vector $\mathbf{p}$ can be described in the Cartesian coordinate system by:

$$\mathbf{p} = \rho \begin{bmatrix} \cos \left( \frac{s}{\rho} \right) \\ \sin \left( \frac{s}{\rho} \right) \end{bmatrix}$$

(3.1)

For every point along the curvilinear axis, a tangent unit vector $\mathbf{t}$ and a normal unit vector $\mathbf{n}$ (directed outward) are introduced. These unit vectors read:

$$\mathbf{t} = \frac{\mathbf{p}_s}{\|\mathbf{p}_s\|} = \begin{bmatrix} -\sin \left( \frac{s}{\rho} \right) \\ \cos \left( \frac{s}{\rho} \right) \end{bmatrix}$$

(3.2a)

$$\mathbf{n} = \frac{-\mathbf{t}_s}{\|\mathbf{t}_s\|} = \begin{bmatrix} \cos \left( \frac{s}{\rho} \right) \\ \sin \left( \frac{s}{\rho} \right) \end{bmatrix}$$

(3.2b)
The minus sign in the definition of the normal vector originates from the convention that the normal vector is directed outward with respect to the centre of curvature of the undeformed ring. If now a load is applied the system wants to deform. A material point on the middle line which initial location is described by the vector \( p \) translates according to the vector \( d = vt + wn \). Hence the new location of this material point is described by the sum of the initial position vector \( p \) and the deformation vector \( d \). This description of the location after deformation is denoted by the vector \( q \) such that:

\[
q = p + d = \begin{pmatrix}
(p + w) \cos \left( \frac{s}{\rho} \right) - v \sin \left( \frac{s}{\rho} \right) \\
(p + w) \sin \left( \frac{s}{\rho} \right) + v \cos \left( \frac{s}{\rho} \right)
\end{pmatrix}
\] (3.3)

The corresponding unit tangent vector of the deformed ring \( \hat{t} \) can be obtained using:

\[
\hat{t} = \frac{q_s}{\|q_s\|}
\] (3.4)

Curvature and rotation of the middle line

Now the curvature vector of the undeformed ring \( k \) and the curvature vector of the deformed ring \( \hat{k} \) are defined by:

\[
k = t_s = \frac{1}{\rho} \begin{pmatrix}
-\cos \left( \frac{s}{\rho} \right) \\
-\sin \left( \frac{s}{\rho} \right)
\end{pmatrix}
\] (3.5a)

\[
\hat{k} = \hat{t}_s
\] (3.5b)

The curvature parameter \( \kappa \) is defined as the difference between the norms of the curvature vectors in the deformed and the undeformed configuration:

\[
\kappa = \|\hat{k}\| - \|k\|
\] (3.6)

The post-deformation rotation \( \varphi \) of a material point on the middle line with respect to the initial configuration can be found by subtracting the direction angle of the vector \( \hat{t} \) from the direction angle of the vector \( t \). In the two dimensional Cartesian coordinate system this results to:

\[
\varphi = \arctan \left( \frac{\hat{t} \cdot \hat{j}}{\hat{t} \cdot \hat{i}} \right) - \arctan \left( \frac{t \cdot \hat{j}}{t \cdot \hat{i}} \right)
\] (3.7)

Strain of the middle line

The first fundamental form for the initial configuration of the ring in the \( s-z \)-coordinate system is given by:

\[
(dp)^2 = g_{ss} (ds)^2 + g_{sz} ds dz + g_{zs} dz ds + g_{zz} (dz)^2
\] (3.8)

Where \( dp \) denotes the infinitesimal arc length of the undeformed middle line. Now \( g \) denotes the metric tensor with respect to the undeformed middle line such that:

\[
g = \begin{bmatrix}
g_{ss} & g_{sz} \\
g_{zs} & g_{zz}
\end{bmatrix} = \begin{bmatrix}
p_s \cdot p_s & p_s \cdot p_z \\
p_z \cdot p_s & p_z \cdot p_z
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\] (3.9)
By combining eq. (3.8) and the metric given in eq. (3.9), the infinitesimal arc length of the undeformed middle line reads:

$$d\rho = ds$$ (3.10)

Now the first fundamental form of the deformed configuration of the ring is given by:

$$(dq)^2 = \bar{g}_{ss} (ds)^2 + \bar{g}_{sz} dz ds + \bar{g}_{zz} (dz)^2$$ (3.11)

Here $dq$ denotes the infinitesimal arc length of the deformed middle line. $\bar{g}$ denotes the metric tensor with respect to the deformed middle line. This metric reads:

$$\bar{g} = \begin{bmatrix} \bar{g}_{ss} & \bar{g}_{sz} \\ \bar{g}_{sz} & \bar{g}_{zz} \end{bmatrix} = \begin{bmatrix} q_s \cdot q_s & q_s \cdot q_z \\ q_z \cdot q_s & q_z \cdot q_z \end{bmatrix} = \begin{bmatrix} q_s \cdot q_s & 0 \\ 0 & 0 \end{bmatrix}$$ (3.12)

The definition of the norm of the vector $q_s$ is given by the square root of the inner product of this vector with itself, such that $\|q_s\| = \sqrt{q_s \cdot q_s}$. This definition, the relation given in eq. (3.11) and the metric given in eq. (3.12) are combined. Now it follows that:

$$dq = \|q_s\| ds$$ (3.13)

The strain in a material point on the middle line can be obtained from the infinitesimal arc length of the initial middle line $d\rho$ and the infinitesimal arc length of the deformed middle line $dq$ such that:

$$e = \frac{dq - d\rho}{d\rho} = \|q_s\| - 1$$ (3.14)

**Approximated kinematic relations**

If now the previously obtained kinematic relations for the middle line rotation $\varphi$, curvature $\kappa$ and strain $e$ are approximated using a multivariate Taylor series approximation with respect to the deformation functions and their derivatives, the following is obtained where h.o.t. stands for higher order terms:

$$\varphi = -w_s + \frac{1}{\rho} v + \text{h.o.t.}$$ (3.15a)

$$\kappa = -w_{ss} + \frac{1}{\rho} v_s + \text{h.o.t.}$$ (3.15b)

$$e = v_s + \frac{1}{\rho} w + \frac{1}{2} \left(w_s - \frac{1}{\rho} v\right)^2 + \text{h.o.t.}$$ (3.15c)

**Sanders’ shell equations**

In correspondence to the previously obtained kinematic relations, the kinematic relations from Sanders [36] with respect to the middle line of a ring, described by one curvilinear axis, read:

$$\varphi = -w_s + \frac{1}{\rho} v$$ (3.16a)

$$\kappa = \varphi_s$$ (3.16b)

$$e = v_s + \frac{1}{\rho} w + \frac{1}{2} \varphi^2$$ (3.16c)

**Buckling of a ring**
In agreement with the description above, \( v(s) \) describes the tangent displacement component and \( w(s) \) describes the normal displacement component. Further \( \varphi(s) \) describes the rotation, \( \kappa(s) \) the curvature and \( \epsilon(s) \) the strain of the middle line of the ring. The underlined term in eq. (3.16) will drop out for the Donnell-Mushtari-Vlasov (DMV) shell equations. Including these terms will give the Sanders shell equations. Rebel [34] states that application of the DMV equations is justified as long as the stresses arising from rotations are of comparable or smaller order than the stresses due to in plane stress resultants. At the occurrence of buckling the stress state changes from a pure plane stress regime to a more bending stress regime. Therefore it is assumed that the DMV equations should give a proper behaviour of the system close to first occurrence of buckling.

**Strain of a material point**

By using the previously stated assumptions, the occurring strain in a material point, \( \varepsilon(s, z) \), is obtained:

\[
\varepsilon = \epsilon + \kappa z
\]  

(3.17)

Where \( \epsilon = \epsilon(s) \) describes the middle surface strain and \( \kappa = \kappa(s) \) describes the curvature.

### 3.3 Constitutive relations

The fundamental constitutive equation used in this analysis is Hooke’s law. Hooke’s law relates occurring strains to occurring stresses. Its generalised form reads:

\[
\sigma_{ij} = c_{ijkl} \varepsilon_{kl}
\]  

(3.18)

Where Einstein’s summation convention needs to be applied. Here the tensor \( \sigma \) is the stress tensor, \( c \) is the stiffness tensor and \( \varepsilon \) is the strain tensor. For a ring it is assumed that it is free to strain in the out of plane (axial) direction (plane stress). Radial stresses and strains are neglected. This results in the out of plane stresses to be equal to zero. Hence Hooke’s law for a ring can be written as:

\[
\sigma = E\varepsilon
\]  

(3.19)

Where \( E \) denotes Young’s modulus, also called the modulus of elasticity. This modulus is a material property. For the steels used in this thesis it is always assumed that \( E = 210 \) GPa if not stated otherwise.

### 3.4 Potential energy

**Strain energy**

The strain energy of an infinitesimal volume is given by the strain energy density function. The generalised form of this function reads [35, 37]:

\[
U^* = \int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij}
\]  

(3.20)
For the ring this strain energy density function can be simplified to:

\[ U^* = \int_0^\sigma \varepsilon d\varepsilon \]  

(3.21)

Now using Hooke’s law as given in eq. (3.19), the previous relation can be rewritten to:

\[ U^* = \frac{1}{2} E \varepsilon^2 \]  

(3.22)

And by eq. (3.17) the strain energy density function reads:

\[ U^* = \frac{1}{2} E \varepsilon^2 + E \kappa z + \frac{1}{2} E \kappa^2 \varepsilon^2 \]  

(3.23)

The strain energy of the full system is obtained by integrating the strain energy density function over the volume of the structure. Because only two dimensions are analysed in the system, the strain energy per unit length is obtained. This is found by integrating \( U^* \) over the system area (domain) \( A = [-\frac{1}{2} l, \frac{1}{2} l] \times [0, 2\pi \rho] \):

\[ U = \iint_A U^* d\varepsilon d\sigma \]  

(3.24)

Substituting eq. (3.23) into the previous equation leads to the following equation for the strain energy per unit length for the system:

\[ U = \int_0^{2\pi \rho} \left( \frac{1}{2} M \kappa + \frac{1}{2} N \varepsilon \right) d\varepsilon \]  

(3.25)

Where \( M \) denotes the stress couple and \( N \) denotes the stress resultant. Note that the stress couple is directly related to the curvature and the stress resultant is directly related to the middle line strain. They are defined by:

\[ M = \frac{1}{12} E t^3 \kappa \]  

(3.26a)

\[ N = E t \varepsilon \]  

(3.26b)

**External energy**

If the system is loaded by an (radially directed) hydrostatic pressure, this means that the pressure is always directed perpendicular to the ring middle line or the cylinder middle surface. Hence the direction of the pressure depends on the deformed state of the structure. A dead pressure will be directed perpendicular to the ring middle line or the cylinder middle surface of the undeformed system. This direction will not change with respect to the deformation of the system.

The applied external energy per unit length due to hydrostatic external pressure can be obtained from the product of the enclosed volume change per unit length and the applied pressure. The enclosed volume change per unit length in a ring is described by Dyau and Kyriakides [12], Kyriakides and Corona [27] and reads:

\[ \int_0^{2\pi \rho} \left( w + \frac{1}{2} \left( \frac{1}{\rho} w^2 + w v_{,s} - v w_{,s} + \frac{1}{\rho} v^2 \right) \right) d\varepsilon \]  

(3.27)

Hence the applied energy per unit length \( W \) due to external hydrostatic pressure reads:

\[ W = \int_0^{2\pi \rho} P \left( w + \frac{1}{2} \left( \frac{1}{\rho} w^2 + w v_{,s} - v w_{,s} + \frac{1}{\rho} v^2 \right) \right) d\varepsilon \]  

(3.28)

**Buckling of a ring**
The external energy due to the hydrostatic pressure depends on both strain and rotation of the deformed structure. If dead pressure is assumed that does not depend on rotations and strains due to deformations, the following is obtained for the applied energy per unit length:

$$W_{\text{dead}} = \int_0^{2\pi \rho} P w ds \quad (3.29)$$

The result of this distinction in external energy is described by Kämmel [24]. The difference in results of bifurcation behaviour for hydrostatic and dead pressure is analysed later. The general analysis in this thesis is based on hydrostatic pressure, because this represents the real behaviour best.

**Total potential energy perfect system**

The total potential energy per unit length of the system reads:

$$\Pi = U + W \quad (3.30)$$

Hence:

$$\Pi = \int_0^{2\pi \rho} \left( \frac{1}{2} M \kappa + \frac{1}{2} N e + P \left\{ w + \frac{1}{2} \left( \frac{1}{\rho} w^2 + w w, s - v w, s + \frac{1}{\rho} v^2 \right) \right\} \right) ds \quad (3.31)$$

**Total potential energy system with initial imperfections**

Using the method as described for the perfect structure it is possible to obtain a potential energy functional for a ring with initial imperfections. It is assumed that the initial geometric imperfection does not cause any residual stress.

It may be assumed that the initial imperfection can be described in the framework of the perfect ring analysis using the two functions $\bar{v}(s)$ and $\bar{w}(s)$. The former function describes a displacement in the direction of the unit tangent vector $t$ and the latter describes a displacement in the direction of the normal unit vector $n$. Residual strains are only introduced by the additional displacement components $v(s)$ and $w(s)$. The total displacements with respect to the undeformed perfect ring are given by:

$$v_{\text{tot}} = \bar{v} + v \quad (3.32a)$$
$$w_{\text{tot}} = \bar{w} + w \quad (3.32b)$$

The strain is given by the difference of the total strain $\varepsilon_{\text{tot}}$ with respect to the undeformed perfect ring and the strain due to the initial imperfection $\varepsilon$:

$$\varepsilon = \varepsilon_{\text{tot}} - \varepsilon \quad (3.33)$$

Using eq. (3.17), the following identity is obtained for the initially imperfect structure:

$$\varepsilon = (\varepsilon_{\text{tot}} - \varepsilon) - (\kappa_{\text{tot}} - \bar{\kappa}) z \quad (3.34)$$

Where $\varepsilon_{\text{tot}}$ follows from substituting the values $v_{\text{tot}}$ and $w_{\text{tot}}$ for $v$ and $w$ into eq. (3.16). $\varepsilon$ is obtained by substituting $\bar{v}$ and $\bar{w}$ for $v$ and $w$ into the kinematic relations. Obtaining $\kappa_{\text{tot}}$ and $\bar{\kappa}$ is done in an analogous way.
Following the same analysis as for the perfect ring, the following is obtained for the strain energy per unit length for the ring with initial imperfections:

\[
\bar{U} = \int_0^{2\pi} \left( \frac{1}{2} E t^3 \{\kappa_{\text{tot}} - \bar{\kappa}\}^2 + \frac{1}{2} E t \{\epsilon_{\text{tot}} - \bar{\epsilon}\}^2 \right) ds
\]  

(3.35)

The external energy per unit length for the system with initial imperfections is found by the additional change in volume per unit length with respect to the additional displacements. Now this additional change is obtained by first substituting \( v = v_{\text{tot}} \) and \( w = w_{\text{tot}} \) into eq. (3.27) and subtracting this equation again, but now substituting \( v = \bar{v} \) and \( w = \bar{w} \).

The applied external energy per unit length now reads:

\[
\bar{W} = \int_0^{2\pi} P \left( \left\{ w_{\text{tot}} + \frac{1}{2} \left( \frac{1}{\rho} w_{\text{tot}}^2 + w_{\text{tot}} v_{\text{tot},s} - v_{\text{tot}} w_{\text{tot},s} + \frac{1}{\rho} v_{\text{tot}}^2 \right) \right\} \right) ds
\]

(3.36)

Now the total potential energy per unit length reads:

\[
\bar{\Pi} = \bar{U} + \bar{W}
\]  

(3.37)

It is convenient to introduce a functional \( \Psi \) that describes the difference in potential energy between the perfect and the imperfect system. This additional potential energy functional is defined by:

\[
\bar{\Pi} = \Pi + \Psi
\]  

(3.38)

### 3.5 Dimensionless variables

The following dimensionless variables are introduced:

\[
\chi = \frac{t}{2\rho} \quad \text{(3.39a)}
\]

\[
\lambda = \frac{4P\rho^3}{Et^3} \quad \text{(3.39b)}
\]

Here \( \chi \) describes the ratio of the wall thickness and the diameter of the ring. \( \lambda \) is the load parameter. The coordinate \( s \) is transformed into the dimensionless coordinate \( \theta = s\rho^{-1} \). Hence \( \theta \in [0, 2\pi] \). Also the following dimensionless identities are introduced:

\[
\tilde{v}(\theta) = v(\theta)\rho^{-1}
\]  

(3.40a)

\[
\tilde{w}(\theta) = w(\theta)\rho^{-1}
\]  

(3.40b)

\[
\tilde{\kappa} = \kappa \rho
\]  

(3.40c)

\[
\tilde{\Pi} = \frac{12\rho}{Et^3}\bar{\Pi}
\]  

(3.40d)

Here \( \tilde{v} \) and \( \tilde{w} \) are the dimensionless circumferential and radial deformation components. \( \tilde{\kappa} \) is the dimensionless curvature and \( \tilde{\Pi} \) is the dimensionless potential energy.
energy functional. Further the dimensionless stress resultant \( \tilde{N} \) and stress couple \( \tilde{M} \) is introduced:

\[
\tilde{M} = \tilde{\kappa} = \frac{12\rho}{Et^3} M \quad (3.41a)
\]

\[
\tilde{N} = e = \frac{1}{Et} N \quad (3.41b)
\]

From now on in this chapter the tildes are omitted for the dimensionless variable notations. This leads to the following dimensionless potential energy functional:

\[
\Pi = \int_{0}^{2\pi} \left( \frac{1}{2} M \kappa + \frac{3}{2} \chi^{-2} N e + 3\lambda \left\{ w + \frac{1}{2} (w^2 + wv_\theta - vw_\theta + v^2) \right\} \right) d\theta \quad (3.42)
\]

With:

\[
\varphi = -w_\theta + v \quad (3.43a)
\]

\[
M = \kappa = \varphi_\theta \quad (3.43b)
\]

\[
N = e = v_\theta + w + \frac{1}{2} \varphi^2 \quad (3.43c)
\]

To determine the dimensionless additional energy functional \( \Psi \), all remaining identities are multiplied by the same factor as their equivalent terms in the perfect system e.g. the term \( \bar{v} \) is multiplied by the same coefficient as the term \( v \) and so on. Now the following dimensionless value for the additional energy potential functional per unit length is determined:

\[
\Psi = \int_{0}^{2\pi} \left( \frac{1}{2} \chi^{-2} \{ 2\varphi \bar{\varphi} N + \varphi^2 \bar{\varphi}^2 \} + 3\lambda \left\{ \bar{w} \bar{w} + \frac{1}{2} (\bar{w} \bar{v}_\theta + \bar{v} \bar{w}_\theta - \bar{w} \bar{v}_\theta) + \bar{v} \bar{v} \right\} \right) d\theta \quad (3.44)
\]

Where besides eq. (3.43):

\[
\bar{\varphi} = -\bar{w}_\theta + \bar{v} \quad (3.45)
\]

### 3.6 Frèchet derivatives of potential energy functional

For the analysis performed in this section it is convenient to pre-calculate several Frèchet derivatives of the potential energy functional given in eq. (3.42).

Here the \( u \)-term, as introduced in chapter 2, describes the two displacement functions \( v \) and \( w \). The \( n \)-th order Frèchet derivative of the potential energy functional \( \Pi \) in generalised form reads:

\[
\Pi^{(n)} \tilde{u}_1 \tilde{u}_2 \ldots \tilde{u}_n \quad (3.46)
\]

Now \( \Pi^{(n)} u_{i_1} u_{i_2} \ldots u_{i_n}, \) where \( i \in \mathbb{N} \), can be obtained by substituting \( \tilde{u}_1 = u_{i_1}, \tilde{u}_2 = u_{i_2} \) and so on in eq. (3.46). In this section the \( \bullet \) denotes a placeholder of a linear term of \( v \) and \( w \). The notation \( \cdot \) denotes a placeholder of both linear and nonlinear terms of \( v \) and \( w \). Using eq. (2.5), the following identities are obtained:

\[
\Pi' \tilde{u}_1 = \int_{0}^{2\pi} \left( M \tilde{M}_1 + 3\chi^{-2} N \tilde{N}_1 + 3\lambda \left\{ \bar{w}_1 + w \bar{w}_1 + v \bar{v}_1 + \frac{1}{2} (v_\theta \bar{w}_1 + w_\theta \bar{v}_1 - w_\theta \bar{v}_1) \right\} \right) d\theta \quad (3.47a)
\]

**Buckling of a ring**
\[ \Pi'' \hat{u}_1 \hat{u}_2 = \int_0^{2\pi} \left( \hat{M}_1 \hat{M}_2 + 3\chi^{-2} \left\{ \hat{N}_1 \hat{N}_2 + N \hat{\varphi}_1 \hat{\varphi}_2 \right\} + 3\lambda \left\{ \hat{w}_1 \hat{v}_2 + \hat{v}_1 \hat{w}_2 + \frac{1}{2} (\hat{w}_1 \hat{v}_{2,\theta} + \hat{v}_1 \hat{w}_{2,\theta} - \hat{w}_1 \hat{v}_{2,\theta} - \hat{v}_1 \hat{w}_{2,\theta}) \right\} \right) \, d\theta \]  
(3.47b)

\[ \Pi''' \hat{u}_1 \hat{u}_2 \hat{u}_3 = \int_0^{2\pi} \left( 3\chi^{-2} \left\{ \hat{N}_1 \hat{\varphi}_2 + \hat{\varphi}_1 \hat{N}_2 + \hat{\varphi}_1 \hat{\varphi}_2 \hat{N}_3 \right\} \right) \, d\theta \]  
(3.47c)

\[ \Pi'''' \hat{u}_1 \hat{u}_2 \hat{u}_3 \hat{u}_4 = \int_0^{2\pi} \left( 9\chi^{-2} \hat{\varphi}_1 \hat{\varphi}_2 \hat{\varphi}_3 \hat{\varphi}_4 \right) \, d\theta \]  
(3.47d)

Where:
\[ \hat{\varphi}_i = -\hat{w}_{i,\theta} + \hat{v}_i \]  
(3.48a)
\[ \hat{M}_i = \hat{\varphi}_{i,\theta} \]  
(3.48b)
\[ \hat{N}_i = \hat{v}_{i,\theta} + \hat{w}_i + \varphi \hat{\varphi}_i \]  
(3.48c)

### 3.7 Equilibrium equations

The equilibrium equations can be obtained from eq. (2.23). An expression for \( \Pi' \delta \mathbf{u} \) can be obtained by substituting \( \delta \mathbf{u} \) for \( \hat{\mathbf{u}} \) in eq. (3.47a). Now the following variational relation is obtained:

\[ \int_0^{2\pi} \left( M \delta M + 3\chi^{-2} N \delta N \right) \, d\theta = 0 \]  
(3.49)

Where the dimensionless constitutive relations given in eq. (3.43) are valid and:
\[ \delta \varphi = -\delta w_{,\theta} + \delta v \]  
(3.50a)
\[ \delta M = \delta \varphi_{,\theta} \]  
(3.50b)
\[ \delta N = \delta v_{,\theta} + \delta w + \varphi \delta \varphi \]  
(3.50c)

By using integration by parts, (3.49) can be rewritten to:

\[ \int_0^{2\pi} \left( \left\{ -3\chi^{-2} N_{,\theta} - M_{,\theta} + 3\chi^{-2} N \varphi + 3\lambda \varphi \right\} \delta w + \left\{ - M_{,\theta} + 3\chi^{-2} N + 3\chi^{-2} (N \varphi)_{,\theta} \right\} \delta \varphi \right) \, d\theta = 0 \]  
(3.51)

Now it is assumed that in order for the previous integral to be zero, the integrand has to be equal to zero. \( \delta v \) and \( \delta w \) are independent. Any continuous \( \delta v \) or \( \delta w \) is admissible. Therefore the coefficients of \( \delta v \) and \( \delta w \) have to match zero separately. Equalling the coefficients to zero leads to the following system of equilibrium equations:

\[ 3N_{,\theta} + \chi^2 \varphi_{,\theta} = 3\lambda \chi^2 \varphi \]  
(3.52a)
\[ \chi^2 \varphi_{,\theta} + 3N - 3(N \varphi)_{,\theta} = 3\lambda \chi^2 (1 + v_{,\theta} + w) \]  
(3.52b)

**Buckling of a ring**
Note that in eq. (3.52b) the coefficient of the pressure term \((1 + v, \theta + w) \approx 1 + e\). In a framework of small strain approximations this allows for the strain term to be neglected with respect to unity. This can simplify the equilibrium equations to:

\[
3N,\theta + \chi^2M,\theta - 3N\varphi = 3\lambda \chi^2 \varphi \tag{3.53a}
\]

\[
\chi^2M,\theta\theta - 3N - 3 (N\varphi),\theta = 3\lambda \chi^2 \tag{3.53b}
\]

Note that it also would have been possible to find stationary values of the potential energy by the application of the Euler-Lagrange equations on the potential energy functional. Doing this will give exactly the same result as obtained in this section.

### 3.8 Fundamental solution

The fundamental solution of the system can be obtained by analysing the equation \(\Pi_0'\delta u = 0\) as stated in eq. (2.28). By applying a method analogous to the method explained in section 3.7, a system of two equilibrium equations is obtained. Now the strain term is dropped again with respect to unity. The following system of differential equations is obtained:

\[
3N_0,\theta + \chi^2M_0,\theta - 3N_0\varphi_0 = 3\lambda \chi^2 \varphi_0 \tag{3.54a}
\]

\[
\chi^2M_0,\theta\theta - 3N_0 - 3 (N_0\varphi_0),\theta = 3\lambda \chi^2 \tag{3.54b}
\]

Where:

\[
\varphi_0 = -u_{0,\theta} + v_0 \tag{3.55a}
\]

\[
M_0 = \varphi_{0,\theta} \tag{3.55b}
\]

\[
N_0 = v_{0,\theta} + w_0 + \frac{1}{2} \varphi_0^2 \tag{3.55c}
\]

If now is assumed that \(v_0(\theta)\) and \(w_0(\theta)\) are constant with respect to \(\theta\), all derivative terms will drop out of the equilibrium equations. The following identity is obtained:

\[
w_0 + \frac{1}{2} \varphi_0^2 = -\lambda \chi^2 \tag{3.56}
\]

For the fundamental solution it may be assumed that no rigid body rotations occur. This allows for letting \(v_0 = 0\). Hence the fundamental solution reads:

\[
v_0(\theta) = 0 \tag{3.57a}
\]

\[
w_0(\theta) = -\lambda \chi^2 \tag{3.57b}
\]

### 3.9 Bifurcation solution

The bifurcation solution of the system is obtained by solving the generalised equation \(\Pi^2_1\delta u = 0\) as obtained in eq. (2.36). Now using eq. (3.47b) and performing the substitutions \(u_1\) for \(\tilde{u}_1\) and \(\delta u\) for \(\tilde{u}_2\). Set \(u \rightarrow u_0\) and let \(\lambda \rightarrow \lambda_c\). Putting the integral to zero leads to the following equation:

\[
\int_0^{2\pi} \left( \frac{M_1 \delta M + 3 \chi^{-1} N_1 \delta N}{3 \lambda_c} \left\{ - \varphi_1 \delta \varphi + w_1 \delta \varphi + v_1 \delta v + \frac{1}{2} \left( w_1 \delta v,\theta + v_1,\theta \delta w - v_1 \delta w,\theta - w_1,\theta \delta v \right) \right\} \right) \, d\theta = 0 \tag{3.58}
\]

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Where eq. (3.50) is valid except for the relation for $\delta N$. Here the quadratic term drops out, because the fundamental solution for the rotation $\varphi_0$ equals zero. Now, in the equation above is also valid:

\[
\delta N = \delta v, \theta + \delta w \quad (3.59a)
\]

\[
\varphi_1 = -w_{1, \theta} + v_1 \quad (3.59b)
\]

\[
M_1 = \varphi_{1, \theta} \quad (3.59c)
\]

\[
N_1 = v_{1, \theta} + w_1 \quad (3.59d)
\]

A similar method is applied as the method described in section 3.7! Applying integration by parts on the previous integral leads to the identity:

\[
\int_0^{2\pi} \left( \left\{-3\chi^2 N_{1, \theta} - M_{1, \theta} + 3\lambda c (\varphi_1 - \varphi_1) \right\} \delta v + \left\{-M_{1, \theta \theta} + 3\chi^2 N_1 + 3\lambda c (e_1 - \kappa_1) \right\} \delta w \right) \, d\theta = 0 \quad (3.60)
\]

Where:

\[
e_1 = N_1 \quad (3.61a)
\]

\[
\kappa_1 = \varphi_{1, \theta} \quad (3.61b)
\]

Setting the coefficients of $\delta v$ and $\delta w$ to zero separately, results in the following system of linear ordinary differential equations:

\[
3N_{1, \theta} + \chi^2 M_{1, \theta} = 3\lambda c \chi^2 (\varphi_1 - \varphi_1) \quad (3.62a)
\]

\[
\chi^2 M_{1, \theta \theta} - 3N_1 = 3\lambda c \chi^2 (e_1 - \kappa_1) \quad (3.62b)
\]

The bifurcation buckling load parameter $\lambda c$ can be determined by performing an Eigenvalue analysis on the system above. After substituting this solution back into the system of equations, a relation between the first order buckling modes $v_1(\theta)$ and $w_1(\theta)$ can be obtained. Note that by this obtained relation it is only possible to obtain a first order buckling shape, not an exact configuration.

Now two functions are introduced to describe the first order displacement components $v_1$ and $w_1$ (buckling shape). In appendix A it is shown that it is allowed to assume such a function shape. Note that by eqs. (A.24) and (A.35), the obtained solution allow for an arbitrary rotation of the buckling shape. For a perfect ring this orientation does not have any influence on the buckling load values and therefore is neglected.

The following first order buckling modes are assumed:

\[
v_1(\theta) = \hat{V}_1 \sin(n\theta) \quad (3.63a)
\]

\[
w_1(\theta) = \hat{W}_1 \cos(n\theta) \quad (3.63b)
\]

Where $n \in \{2, 3, \ldots\}$ denotes the circumferential wave number. Now substitute these shapes into the bifurcation equations given in eq. (3.62). The system will be solved for the case of the Sanders shell equations, where the underlined terms are included, and the DMV shell equations, where the underlined terms are omitted.

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3.9.1 DMV shell equations

After substitution of the trial functions, given in eq. (3.63), in the bifurcation equations given in eq. (3.62) the following system is obtained:

\[
\begin{bmatrix}
-3n^2 & -3n (\lambda_c \chi^2 + 1) \\
-3n (\lambda_c \chi^2 + 1) & 3 (n^2 - 1) \lambda_c \chi^2 - (n^4 \chi^2 + 3)
\end{bmatrix}
\begin{bmatrix}
\hat{V}_1 \\
\hat{W}_1
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] (3.64)

To obtain the buckling load parameter from the system above, the matrix must be singular. This can be obtained by setting its determinant to zero. The roots of this characteristic equation are:

\[
\lambda_{c,DMV} = -\frac{1}{2} \left( n^2 + 1 \pm \sqrt{(n^2 + 1)^2 + \frac{4}{3} \chi^2 n^4} \right) \chi^{-2}
\] (3.65)

The only interesting root, leading to a positive value for \( \lambda_c \), is:

\[
\lambda_{c,DMV} = -\frac{1}{2} \left( n^2 + 1 - \sqrt{(n^2 + 1)^2 + \frac{4}{3} \chi^2 n^4} \right) \chi^{-2}
\] (3.66)

Because \( \chi^2 \ll 1 \), it is allowed to approximate \( \lambda_c \) at \( \chi = 0 \) using the Taylor series expansion:

\[
\lambda_{c,DMV} = \frac{n^4}{3 (n^2 + 1)} + \mathcal{O}(\chi^2)
\] (3.67)

It is interesting to note that if in eq. (3.62) \( c_1 \) is neglected with respect to \( \kappa_1 \), the first term of the approximation of \( \lambda_c \) eq. (3.67) is obtained as a solution. This can be explained due to the fact that a framework of small strains and moderately small rotations is adopted. This allows for dropping the strain term with respect to the curvature term.

3.9.2 Sanders shell equations

Now the same analysis as explained in the previous section is performed for the Sanders shell equations. The following system is obtained:

\[
\begin{bmatrix}
-n^2 (\chi^2 + 3) & -n (n^2 \chi^2 + 3) \\
-n (n^2 \chi^2 + 3) & 3 (n^2 - 1) \lambda_c \chi^2 - (n^4 \chi^2 + 3)
\end{bmatrix}
\begin{bmatrix}
\hat{V}_1 \\
\hat{W}_1
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\] (3.68)

Setting the determinant of the matrix above to zero and solving this characteristic equation for \( \lambda_c \) leads to:

\[
\lambda_{c,Sanders} = \frac{n^2 - 1}{3 + \chi^2}
\] (3.69)

In the chosen framework it is assumed that \( \chi^2 \ll 1 \). This allows for this small term to be dropped with respect to unity. By taking the Taylor series approximation of eq. (3.69) at \( \chi = 0 \), the classical bifurcation buckling load plus some higher order terms are obtained and read:

\[
\lambda_{c,Sanders} = \frac{1}{3} \left( n^2 - 1 \right) + \mathcal{O}(\chi^2)
\] (3.70)

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3.9.3 Comparison bifurcation loads

In fig. 3.2, the first non-zero terms of the series approximations of $\lambda_{c;DMV}$ and $\lambda_{c;Sanders}$ are compared. It is verified that the values match closely. Most significant difference in values is found for $n = 2$. The ratio between the DMV approximation and the Sanders approximation is equal to $\frac{n^4}{n^4-1}$. It follows that the DMV approximation always gives a slightly higher value for bifurcation pressure. For $n = 2$ this value is about 6.67% higher than the Sanders value.

![Figure 3.2: Critical dimensionless pressures for DMV and Sanders kinematics](image)

The values based on the DMV equations that are derived, show useful to verify the buckling analysis of a perfect cylinder in chapter 4. The rest of the bifurcation buckling analysis in this chapter will be performed using the Sanders kinematic equations and solutions.

3.9.4 First order buckling shape

Now the bifurcation shape is analysed. This is done by feeding the critical value for the load parameter given in eq. (3.69) back into the system given in eq. (3.68). The system being singular is prerequisite to find the critical load parameter. Hence substituting this parameter back into the system makes it singular. This implies that the matrix should have a non-empty nullspace (besides the null vector). This nullspace contains a vector describing the ratio of the factors $\hat{V}_1$ and $\hat{W}_1$.

Rewriting the matrix of the system to reduced row echelon form after substitution of the critical buckling load parameter leads to a matrix $K$:

$$
K = \begin{bmatrix}
1 & n^2\chi^2+3 \\
0 & n(\chi^2+3)
\end{bmatrix}
$$

(3.71)

The nullspace of this matrix is given by:

$$
\text{Null}(K) = \left\{ \begin{bmatrix}
-\frac{n^2\chi^2+3}{n(\chi^2+3)} \\
1
\end{bmatrix} \right\}
$$

(3.72)

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This leads to $\hat{V}_1 = -\frac{n^2 \chi^2 + 3}{n(\chi^2 + 3)}$ and $\hat{W}_1 = 1$. It is allowed to put a shared multiplication factor in front of both parameters. Setting $\hat{W}_1 = 1$ is in correspondence with Budiansky [5], because he states that a suitable value for the norm of the first order generalised displacement $u_1$ is unity. This allows, in this case, the factor $\zeta$ to be a good representation of the amount of buckling shape that is included into the system. Generally it is possible to satisfy this constraint for only one displacement component. In this analysis this is done for $w_1$.

The buckling shape that is obtained reads:

$$v_1(\theta) = -\frac{n^2 \chi^2 + 3}{n(\chi^2 + 3)} \sin(n\theta) \quad (3.73a)$$

$$w_1(\theta) = \cos(n\theta) \quad (3.73b)$$

Now assume that the norm $\|f\|$ of a function $f(u)$ on domain $\Omega \in [0, 2\pi]$ can be obtained by the following relation:

$$\|f\| = \frac{1}{\pi} \int_{\Omega} f(u)^2 \delta u \quad (3.74)$$

This definition for the norm of a function will satisfy $\|w_1\| = 1$. Note that in general $\|v_1\| \neq 1$.

### 3.9.5 Bifurcation load for dead pressure

The analysis can also be performed for dead pressure instead of hydrostatic pressure. Therefore the external energy functional given in eq. (3.28) is substituted for eq. (3.29). Sanders shell equations will give the same result as the DMV shell equations. If an analysis is performed analogous to the previous analysis for the hydrostatic pressure case, the bifurcation load for the system loaded by dead pressure can be determined.

The fundamental solution will match the fundamental solution given in eq. (3.57). This makes sense because no rotations occur for the fundamental solution. Further the first order strains in the hydrostatic pressure case were neglected with respect to unity. Hence the result for the dead pressure case is as expected.

Now the bifurcation equations are obtained for the dead pressure case. Performing the Eigenvalue analysis on this set of equations using the ansatzes introduced in eq. (3.63) leads to the following bifurcation load parameter:

$$\lambda_{c,\text{dead}} = \frac{1}{4} n^2 \quad (3.75)$$

If this is compared to the first term of eq. (3.70) it can be seen that the difference in bifurcation load between the hydrostatic pressure case and the dead pressure case is significant for relatively small values for $n$.

The obtained results for $\lambda_{c,\text{Sanders}}$ in eq. (3.70) and $\lambda_{c,\text{dead}}$ in eq. (3.75) correspond to the first terms of the expansions obtained in Kämmer [24].

The buckling shape of the dead pressure load can be obtained by using the nullspace of the matrix used in the Eigenvalue analysis where the bifurcation load parameter $\lambda_{c,\text{dead}}$ is substituted for the load parameter $\lambda$. This leads to the first order buckling shapes of:

$$v_1(\theta) = -\frac{1}{n} \sin(n\theta) \quad (3.76a)$$

$$w_1(\theta) = \cos(n\theta) \quad (3.76b)$$

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3.10 Initial post-buckling

To be able to predict the behaviour of a system right after occurrence of bifurcation buckling, it will suffice to analyse the nature of the equilibrium around its bifurcation point. The procedure as described in chapter 2 is followed to obtain the initial post-buckling behaviour of a perfectly circular ring.

3.10.1 Post-buckling load parameter $\lambda_1$

To obtain a value for $\lambda_1$, eq. (2.38) is used. First an expression for $\Pi''''u_3^1$ is obtained. This can be done by making use of eq. (3.47c) and applying eqs. (2.26) and (2.31a). Now the following is obtained:

$$\Pi''''u_3^1 = \int_0^{2\pi} \left( 9\chi^{-2}N_1\varphi_1^2 \right) \, d\theta \quad (3.77)$$

Where:

$$\varphi_1 = -w_{1,\theta} + v_1$$  \hspace{1cm} (3.78a)

$$N_1 = v_{1,\theta} + w_1$$  \hspace{1cm} (3.78b)

Substituting he first order buckling deformation functions given in eq. (3.73) gives:

$$\Pi''''u_3^1 = 0 \quad (3.79)$$

Now $\Pi''u_1^1$ is obtained by making use of eq. (3.47b) and applying eqs. (2.26) and (2.31b). This leads to:

$$\Pi''u_1^1 = \int_0^{2\pi} 3 \left( -\varphi_1^2 + \varphi_1^2 + w_{1,\theta} - v_{1,v_{1,\theta}} + v_1^2 \right) \, d\theta \quad (3.80)$$

Substituting the buckling solution given in eq. (3.73) gives:

$$\Pi''u_1^1 = -3\pi \left( n^2 - 1 \right) \quad (3.81)$$

Now $\lambda_1$ is obtained by using eq. (2.38) and the previously obtained values:

$$\lambda_1 = 0 \quad (3.82)$$

Note that for symmetric buckling it is expected that $\lambda_1 = 0$. Hence the result meets the expectations.

3.10.2 Second order buckling shape

The second order buckling shape $u_2$ is described by the displacement functions $v_2(\theta)$ and $w_2(\theta)$. A system of linear ordinary differential equations can be obtained from eq. (2.37a). By using eq. (3.82) this reduces to:

$$\Pi''u_2\delta u + \frac{1}{2}\Pi''''u_2^2\delta u = 0 \quad (3.83)$$

Now it is clear that the first term leads to the bifurcation equation given in eq. (2.36). The only difference is that the terms $v_1$ and $w_1$ have to be substituted for $v_2$ and $w_2$ respectively. The second term in eq. (3.83) in fact adds a term to
each equation in the new bifurcation equation. By making use of eq. \([3.47c]\) the following identity is found:

\[
\Pi_c'' u_1^2 \delta u = \int_0^{2\pi} \left( 6 \chi^{-2} \varphi_1 \{ N_1 - \varphi_1, \theta \} \delta v + 3 \chi^{-2} \left\{ 2 (N_1 \varphi_1, \theta) + \varphi_1^2 \right\} \delta w \right) d\theta
\]  

(3.84)

Where:

\[
\varphi_1 = -w_1, \theta + v_1 \quad (3.85a)
\]

\[
N_1 = v_1, \theta + w_1 \quad (3.85b)
\]

Substituting the bifurcation solution given in eq. (3.73) in eq. (3.84) leads to:

\[
\Pi_c'' u_1^2 \delta u = \int_0^{2\pi} \left( - \frac{9 \lambda^2}{n \chi^2} \left( 3 + \chi^2 \right) \sin (2n \theta) \delta v 
- \frac{9 \lambda^2}{2n^2 \chi^2} \left\{ \left( 4 \chi^2 n^2 + 3 \right) \cos (2n \theta) - 3 \right\} \delta w \right) d\theta
\]  

(3.86)

Multiplying this with a factor \(\frac{1}{2}\) and summing it with eq. (3.60) where \(v_1\) and \(w_1\) are substituted by \(v_2\) and \(w_2\) gives description of eq. (3.83). If now each coefficient of \(\delta v\) and \(\delta w\) is set to zero separately, the following system of equations is obtained:

\[
3 N_{2, \theta} + \chi^2 M_{2, \theta} = - \frac{9 \lambda^2}{2n} \left( 3 + \chi^2 \right) \sin (2n \theta) \quad (3.87a)
\]

\[
\chi^2 M_{2, \theta \theta} - 3 N_2 - 3 \lambda c \chi^2 (e_2 - \kappa_2) = - \frac{9 \lambda^2}{4n^2} \left\{ \left( 4 \chi^2 n^2 + 3 \right) \cos (2n \theta) - 3 \right\} \quad (3.87b)
\]

Where:

\[
\varphi_2 = -w_{2, \theta} + v_2 \quad (3.88a)
\]

\[
M_2 = \varphi_{2, \theta} \quad (3.88b)
\]

\[
N_2 = v_{2, \theta} + w_2 \quad (3.88c)
\]

Further \(e_2 = N_2\) and \(\kappa_2 = \varphi_{2, \theta}\). Solving the homogeneous part of the system, dropping the right-hand side of eq. (3.87), would lead to the first order bifurcation solution. However in section \(2.4\) it was stated that the first order buckling solution needs to be independent of the higher order solutions. Therefore the solution to the homogeneous part of the differential equations has to be neglected.

To obtain a solution to the inhomogeneous system of differential equations, a solution is sought that is independent to the first order bifurcation solution. A solution is for \(v_2\) and \(w_2\) is found if the following functions are assumed:

\[
v_2(\theta) = \hat{V}_2 \sin (2n \theta) \quad (3.89a)
\]

\[
w_2(\theta) = \hat{W}_2, + \hat{W}_2 \cos (2n \theta) \quad (3.89b)
\]

By substituting these values into eq. (3.87) gives two equations. The first is a coefficient of \(\sin (2n \theta)\) set to zero and the second one is the sum of a coefficient of \(\cos (2n \theta)\) and a constant set to zero. Setting both coefficients and the constant term to zero separately gives three equations. These can be solved for the three

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unknowns $\hat{V}_2$, $\hat{W}_2$, and $\hat{W}_2$. Now the following solutions for the second order bifurcation solutions $v_2$ and $w_2$ are obtained:

$$v_2(\theta) = \frac{9\lambda^2}{8n^4} \sin(2n\theta) \quad (3.90a)$$

$$w_2(\theta) = -\frac{9(3 + \chi^2) \lambda^2}{4n^2(n^2\chi^2 + 3)} \quad (3.90b)$$

### 3.10.3 Post-buckling load parameter $\lambda_2$

To derive a value for $\lambda_2$ eq. (2.39) is used. Because previously it is obtained that $\lambda_1 = 0$, the terms in the equation for $\lambda_2$ related to $\lambda_1$ will drop out. Hence the equation reduces to:

$$\lambda_2 = -\frac{1}{3} \Pi'''_c \mathbf{u}_1^4 + \Pi''_c \mathbf{u}_2^2 \quad (3.91)$$

First $\Pi'''_c \mathbf{u}_1^4$ is analysed. The substitutions $\hat{u}_1 = \mathbf{u}_1$, $\hat{u}_2 = \mathbf{u}_1$, $\hat{u}_3 = \mathbf{u}_1$ and $\hat{u}_4 = \mathbf{u}_1$ are performed in eq. (3.47d). Now setting $\mathbf{u} \rightarrow \mathbf{u}_0$ and $\lambda \rightarrow \lambda_c$ leads to the following identity:

$$\Pi'''_c \mathbf{u}_1^4 = \frac{27\pi}{4\lambda^2} \left( \frac{3\lambda}{n} \right)^4 \quad (3.92)$$

With:

$$\varphi_1 = -w_{1,\theta} + v_1 \quad (3.93)$$

Introducing the values for $v_1(\theta)$ and $w_1(\theta)$ in the identity above leads to:

$$\Pi'''_c \mathbf{u}_2^2 \mathbf{u}_2 = \frac{27\pi}{4\lambda^2} \left( \frac{3\lambda}{n} \right)^4 \quad (3.94)$$

Now the term $\Pi''_c \mathbf{u}_2^2 \mathbf{u}_2$ is analysed. To do so eq. (3.47c) is used. First the substitutions $\hat{u}_1 = \mathbf{u}_1$, $\hat{u}_2 = \mathbf{u}_1$ and $\hat{u}_3 = \mathbf{u}_2$ are performed. Then let $\mathbf{u} \rightarrow \mathbf{u}_0$ and $\lambda \rightarrow \lambda_c$. The following is obtained:

$$\Pi''_c \mathbf{u}_2^2 \mathbf{u}_2 = \frac{27\pi}{4\lambda^2} \left( \frac{3\lambda}{n} \right)^4 \quad (3.95)$$

With:

$$\varphi_1 = -w_{1,\theta} + v_1 \quad (3.96a)$$

$$N_1 = v_{1,\theta} + w_1 \quad (3.96b)$$

$$\varphi_2 = -w_{2,\theta} + v_2 \quad (3.96c)$$

$$N_2 = v_{2,\theta} + w_2 \quad (3.96d)$$

The obtained values for $v_1(\theta)$, $w_1(\theta)$, $v_2(\theta)$ and $w_2(\theta)$ are introduced in this relation. Hence the following is determined:

$$\Pi''_c \mathbf{u}_2^2 \mathbf{u}_2 = -\frac{1}{2} \left( \frac{(3 + \chi^2)(n^2\chi^2 + 9)}{8(n^2\chi^2 + 3)\chi^2} \right) \left( \frac{3\lambda}{n} \right)^4 \quad (3.97)$$

Because $\Pi''_c \mathbf{u}_2^2$ is obtained in eq. (3.81), $\lambda_2$ can be obtained from eq. (3.91). Hence the expression for the post-buckling load parameter $\lambda_2$ reads:

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\[ \lambda_2 = \frac{27 (6 \{n^2 - 1\} - \{n^2x^2 + 3\}) (n^2 - 1)^2}{8n^4 (x^2 + 3)^3 (n^2x^2 + 3)} \lambda_c \] (3.98)

### 3.11 Initial imperfections

Because the previously obtained \( \lambda_1 = 0 \), the analysis for an ring with initial imperfections is straightforward as described in section \( \text{[277]} \). Compared to the perfect ring analysis, the only parameter that needs to be adjusted is \( \lambda_2 \). The adjusted second order post-buckling load parameter is denoted by: \( \tilde{\lambda}_2 \) and can be obtained using eq. (2.53). This equation reads:

\[ \tilde{\lambda}_2 = \lambda_2 - \frac{\xi \Psi_0 \hat{u}_1}{\zeta \Omega \hat{u}_1^*} \] (3.99)

It has to be assumed that the initial imperfection of the ring can be described in the shape of the first order buckling mode given in eq. (3.73). Now a factor \( \xi \) is introduced that denotes the imperfection of the ring. This scalar imperfection parameter is also referred to as the ovality. The ovality is defined as:

\[ \xi = \frac{\rho_{\text{max}} - \rho_{\text{min}}}{\rho_{\text{max}} + \rho_{\text{min}}} \] (3.100)

This ovality will be equal to the amplitude of the dimensionless imperfection function \( \bar{w}(\theta) \). Dimensions in the imperfection function are removed by dividing by the mean radius. If the shape of the imperfection is described by \( \hat{v}(\theta) = v_1(\theta) \) and \( \hat{w}(\theta) = w_1(\theta) \). The following relations are valid:

\[ \bar{v} = \xi \hat{v} \] (3.101a)
\[ \bar{w} = \xi \hat{w} \] (3.101b)

An energy functional \( \Psi[v(\lambda), w(\lambda); \hat{v}, \hat{w}; \lambda] \) containing the additional potential energy of the imperfect system with respect to the perfect system is derived in eq. (3.44). Now \( \Psi_0 \hat{u}_1 \) can be obtained and reads:

\[ \Psi_0 \hat{u}_1 = \int_0^{2\pi} 3\lambda_c \left( -\varphi_1 \hat{\varphi} + w_1 \hat{w} + \frac{1}{2} \left\{ w_1 \hat{v}_{1,\theta} + \hat{w}v_1,\theta - v_1 \hat{w}_{1,\theta} - \hat{v}w_1,\theta \right\} + v_1 \hat{v} \right) d\theta \] (3.102)

Where:

\[ \varphi_1 = -w_{1,\theta} + v_1 \] (3.103a)
\[ \hat{\varphi} = -\hat{w}_{1,\theta} + \hat{v} \] (3.103b)

Substituting eq. (3.73) and assuming the previously stated \( \hat{v} = v_1 \) and \( \hat{w} = w_1 \) gives:

\[ \Psi_0 \hat{u}_1 = -3\pi (n^2 - 1) \lambda_c \] (3.104)

Now \( \tilde{\lambda}_2 \) is obtained by using eq. (3.99). Substituting eq. (3.81) leads to:

\[ \tilde{\lambda}_2 = \lambda_2 - \frac{\xi}{\zeta} \lambda_c \] (3.105)

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3.12 Results

If the load factor is written as a function of the bifurcation factor $\zeta$, the following relation can be obtained:

$$\frac{\lambda}{\lambda_c} = 1 + \frac{27}{8n^4(\chi^2 + 3)^4} \left( \frac{n^2 - 1}{n^2 \chi^2 + 3} \right)^2 \zeta^2 - \frac{\xi}{\zeta} \quad (3.106)$$

In fig. 3.3 this relation is plotted for $\chi = \frac{1}{21}$ and $n = 2$. Note that this relation does not depend heavily on the choice for $\chi$, as long as $\chi^2 \ll 1$. For $n = 2$ and $\chi^2 \ll 1$ the following approximation is valid:

$$\frac{\lambda}{\lambda_c} \approx 1 + \frac{45}{128} \zeta^2 - \frac{\xi}{\zeta} \quad (3.107)$$

Figure 3.3: Imperfection sensitivity of a ring with $\chi = \frac{1}{21}$ and $n = 2$

It is important to note that the equations in this section are valid for loads relatively close to the bifurcation load of the perfect system.

The dimensionless displacement functions can be combined to:

$$w(\theta, \lambda) = -\lambda \chi^2 + \zeta \cos(n\theta) - \zeta^2 \frac{9}{4n^3(3 + n^2 \chi^2)} \lambda_c^2$$

$$v(\theta, \lambda) = -\zeta \frac{3 + n^2 \chi^2}{n(3 + \chi^2)} \sin(n\theta) + \zeta^2 \frac{9 \lambda_c^2}{8n^3} \sin(2n\theta) \quad (3.108b)$$

A qualitative description of the bifurcation shapes of a perfect ring for various values of $n$ is given in fig. 3.4.

3.13 Influence initial imperfection for small load

In the previous section, the influence of initial imperfections on the structure under a load close to the bifurcation load is analysed. For small loads much...
Figure 3.4: Initial and bifurcation shapes of a perfect ring for various values of $n$.

smaller than the bifurcation load the results of this analysis are not reliable. To obtain the structural behaviour for small loads, the total potential energy functional per unit length $\Pi$ of the structure with initial imperfections is used.

Equilibrium in the structure will occur when the potential energy functional is stationary. Hence the first variation of this energy functional must be equal to zero. This can be satisfied by satisfying a set of equilibrium equations that can be obtained from applying the Euler-Lagrange equations with respect to $v$ and $w$ onto $\Pi$. Now the following set of equilibrium equations is obtained:

$$3N,\theta + \chi^2 M,\theta - 3 (N + \varphi \bar{\varphi}) \varphi + 3 (\varphi \bar{\varphi})_{,\theta} = 3\lambda \chi^2 (\varphi + \bar{\varphi})$$

$$\chi^2 M,\theta\theta - 3N - 3 \{N + \varphi \bar{\varphi} \} \varphi + 3 \varphi \bar{\varphi} = 3\lambda \chi^2 (1 + v,\theta + w + \bar{v},\theta + \bar{w})$$

Where:

$$\varphi = -w,\theta + v$$

$$M = \varphi,\theta$$

$$N = v,\theta + w + \frac{1}{2} \varphi^2$$

$$\bar{\varphi} = -\bar{w},\theta + \bar{v}$$

In eq. (3.109b), the coefficient of $3\lambda \chi^2$ at the right-hand side of the equation can be simplified in the framework of small middle surface strains and small initial imperfections. It is allowed to drop the terms $v,\theta + w + \bar{v},\theta + \bar{w}$ with respect to unity, because they their summation with unity approximates $1 + e + \bar{e} \approx (1 + e) (1 + \bar{e}) \approx 1$. Hence the system of equations is simplified to:

$$3N,\theta + \chi^2 M,\theta - 3 (N + \varphi \bar{\varphi}) \varphi + 3 (\varphi \bar{\varphi})_{,\theta} = 3\lambda \chi^2 (\varphi + \bar{\varphi})$$

$$\chi^2 M,\theta\theta - 3N - 3 \{N + \varphi \bar{\varphi} \} \varphi + 3 \varphi \bar{\varphi} = 3\lambda \chi^2$$

Now the description for the imperfection is introduced again as the product of the ovality and a shape function that is described by the first order bifurcation mode shape given in eq. (3.73) such that eq. (3.101) is valid. The following expansion of the displacement components $v$ and $w$ in terms of the small ovality term $\|\xi\| \ll 1$ is introduced:

$$v(\theta) = v_0 + \xi v_1$$

$$w(\theta) = w_0 + \xi w_1$$
The fundamental solutions $v_0$ and $w_0$ are obtained by substituting these expansions and the relations $\bar{v} = \xi \hat{v}$ and $\bar{w} = \xi \hat{w}$ into the simplified equilibrium equations given in eq. (3.111). Here $\hat{v}$ and $\hat{w}$ describe the imperfection shape functions. Setting the small ovality parameter $\xi$ to zero and solving for constant fundamental solutions with respect to dimensionless coordinate $\theta$ leads to same identity as obtained in eq. (3.56). By constraining rigid ring rotations or displacements it is allowed to let $v_0 = 0$ and hence exactly the same fundamental solutions are obtained as in eq. (3.57):

$$v_0(\theta) = 0 \quad (3.113a)$$

$$w_0(\theta) = -\lambda \chi^2 \quad (3.113b)$$

After obtaining this fundamental solution, the non-simplified equilibrium equations given in eq. (3.109) are used to determine $v_1$ and $w_1$. The expansions given in eq. (3.112) are substituted into the system of equilibrium equations. Now take for the two equations the coefficient terms of the first power of $\xi$. Note that in the analysis previously the same type of analysis is performed on which only for the determination of the fundamental solution the equilibrium equations are simplified. In fact this will give a residual term in the non-simplified equilibrium equation, but it can be verified that this term is insignificant to the solution of the system. The following equilibrium equations are obtained for $v_1$ and $w_1$:

$$3N_{1,\theta} + \chi^2 M_{1,\theta} = 0 \quad (3.114a)$$

$$\chi^2 M_{1,\theta\theta} - 3N_{1} = 3\lambda \chi^2 (c_1 + \hat{e} - \kappa_1 - \hat{\kappa}) \quad (3.114b)$$

Where:

$$M_1 = -w_{1,\theta\theta} + v_{1,\theta} \quad (3.115a)$$

$$N_1 = v_{1,\theta} + w_1 \quad (3.115b)$$

$$c_1 = N_1 \quad (3.115c)$$

$$\kappa_1 = M_1 \quad (3.115d)$$

$$\hat{e} = \hat{v}_{,\theta} + \hat{w} \quad (3.115e)$$

$$\hat{\kappa} = -\hat{w}_{,\theta\theta} + \hat{v}_{,\theta} \quad (3.115f)$$

Now, as in eq. (3.63), the following functions are assumed for $v_1$ and $w_1$:

$$v_1(\theta) = \hat{V}_1 \sin(n\theta) \quad (3.116a)$$

$$w_1(\theta) = \hat{W}_1 \cos(n\theta) \quad (3.116b)$$

Substituting these relations into eq. (3.114) will lead to two equations. By solving these equations, the following equations are obtained for $v_1$ and $w_1$:

$$v_1(\theta) = -\frac{\lambda}{\lambda_c - \lambda} \left( \frac{n^2 \chi^2 + 3}{n (\chi^2 + 3)} \right) \sin(n\theta) \quad (3.117a)$$

$$w_1(\theta) = \frac{\lambda}{\lambda_c - \lambda} \cos(n\theta) \quad (3.117b)$$

To be able to relate it to the method described previously the amount of first order buckling mode included is compared. Hence $\zeta v_1(\theta)$ and $\zeta w_1$ with $v_1$ and $w_1$.
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$w_1$ obtained in eq. (3.73) is related to $\xi v_1$ and $\xi w_1$ with $v_1$ and $w_1$ obtained in eq. (3.117). If it is assumed that the two methods describe the same function for either $v$ or $w$ the following relation is obtained between $\zeta$ and $\xi$:

$$\zeta = \frac{\lambda \xi}{\lambda_c - \lambda}$$ (3.118)

From this the relation is obtained:

$$\frac{\lambda}{\lambda_c} = 1 - \frac{\xi}{\xi + \xi}$$ (3.119)

In fig. 3.5 the bifurcation response of a ring is given for various wave numbers $n$ describing the initial geometric imperfection.

![Initial and bifurcation shapes of a ring with an initial imperfection described by wave number $n$.](image)

3.14 Matching results

To give a complete description of the relation of $\lambda$ and $\zeta$ it is necessary to match the solution for small a load parameter given in eq. (3.119) and the solution for load parameters within close range of the bifurcation load given in eq. (3.106). In this section the inner solution will refer to the solution for a small load and the outer solution will refer to the solution for loads close to the bifurcation load.

It must be assumed that the ovality is small such that $\|\xi\| \ll 1$. The matching of the two functions is analogous to matching functions in a boundary layer as described by Howison [22]. Here the inner solution will describe the behaviour of the system at the boundary and the outer solution will describe its behaviour at further distance of this layer. Now assume that the inner solution is valid for $\zeta = O(\xi)$ and the outer solution is valid for $\zeta \gg O(\xi)$.

**Outer solution**

The outer solution is given by eq. (3.106). Because it is stated that this solution is valid if $\zeta \gg O(\xi)$, $\xi$ may be dropped with respect to $\zeta$. Hence the following approximation is valid for the outer solution:

$$\frac{\lambda}{\lambda_c} = 1 + \frac{27}{8n^4 (\chi^2 + 3)^3 (n^2 \chi^2 + 3)} \left( \frac{n^2 \chi^2 + 3}{n^2 \chi^2 + 3} \right)^2 (n^2 - 1)^2 \zeta^2$$ (3.120)

The limit of this function for $\zeta \to 0$ is equal to one.
Inner solution

The inner solution is given by eq. (3.119). Introducing the substituting \( \zeta = X\xi \) leads to the inner solution:

\[
\frac{\lambda}{\lambda_c} = \frac{X}{1+X} \tag{3.121}
\]

The limit of this function for \( X \to \infty \) is equal to one. So it is verified that the inner solution and the outer solution have a shared limit equal to unity.

Matching

To obtain a matched approximate function for \( \lambda/\lambda_c \) that is valid for both small and larger values of \( \lambda \), the inner and outer solutions are matched. This is done by taking the sum of the inner and outer solutions, eqs. (3.120) and (3.121), and subtracting their shared limit. After substituting \( X = \zeta/\xi \) back into the equation, the following matched function is obtained:

\[
\frac{\lambda}{\lambda_c} = 1 + \frac{27 (6 \{n^2 - 1\} - \{n^2\chi^2 + 3\}) (n^2 - 1)^2}{8n^4 (\chi^2 + 3)^3 (n^2\chi^2 + 3)} \zeta^2 - \frac{\zeta}{\zeta + \xi} \tag{3.122}
\]

In fig. 3.6 a result of the matching process is displayed.

![Figure 3.6: Matching inner and outer solution for \( \chi = 1/21 \), \( n = 2 \) and \( \xi = 0.02 \)](image)

3.15 Relation results to literature

In this section parameters are introduced again that contain dimensions. Therefore a tilde denotes that a parameter is dimensionless.

Buckling of a ring
3.15.1 Timoshenko formulation

To obtain the Timoshenko equation for collapse of a ring, it is assumed that collapse occurs at first yielding of the material. First yielding occurs when the outer fibre stress reaches the yield strength of the material \( f_y \). The occurring strain can be obtained using eq. (3.17). Now the occurring stress can be obtained from this strain and eq. (3.19). To be able to use the dimensionless form \( \tilde{\kappa} \), the following identity is used: \( \kappa = \tilde{\kappa} \rho^{-1} \). For the occurring stress the following is obtained:

\[
\sigma = E\varepsilon + \frac{Ez}{\rho} \tilde{\kappa} \tag{3.123}
\]

The dimensionless kinematic relations given in eq. (3.43) are substituted into this relation. For the displacement functions \( v \) and \( w \), Timoshenko and Gere use the ones obtained from the analysis for small \( \lambda \) as performed in section 3.13:

\[
v(\theta) = -\frac{\lambda\xi}{\lambda_c - \lambda} \left( \frac{n^2\chi^2 + 3}{n(\chi^2 + 3)} \right) \sin(n\theta) \tag{3.124a}
\]

\[
w(\theta) = \frac{\lambda\xi}{\lambda_c - \lambda} \cos(n\theta) - \lambda\chi^2 \tag{3.124b}
\]

Using these displacement functions, it follows that the maximum absolute stress value occurs for \( z = \frac{t}{2} \) and \( \theta = \frac{1}{2} \pi \). Because collapse is assumed to occur at first yielding of the material, the collapse criterion is the extreme fibre stress to reach the (tensile or compressive) yield stress. If it is assumed that both the compressive and tensile yield stress have the same absolute value, first yielding will be reached in the compressive region. Hence the maximum stress is put equal to the compressive (negative sign) yield strength of the material. Rewriting the equation to dimensional parameters will lead to the following criterion for occurrence of collapse of a ring:

\[
P^2 - \left( \frac{f_y t}{\rho} + \left\{ 1 - \xi + 6\xi \frac{P_c}{t} \right\} P_c \right) P + \frac{f_y t}{\rho} P_c = 0 \tag{3.125}
\]

Where:

\[
P_c = \frac{E(n^2 - 1)}{12 + (\xi/\rho)^2} \left( \frac{t}{\rho} \right)^3 \tag{3.126}
\]

For \( \|\xi\| \ll 1 \) and \( (t/\rho)^2 \ll 1 \) it is allowed to simplify the collapse criterion to the Timoshenko collapse criterion. When this second order polynomial is solved for \( P \), the collapse load is obtained:

\[
P^2 - \left( \frac{f_y t}{\rho} + \left\{ 1 + 6\xi \frac{P_c}{t} \right\} P_c \right) P + \frac{f_y t}{\rho} P_c = 0 \tag{3.127}
\]

Where \( P_c \) is the classical buckling load for an elastic ring defined by:

\[
P_c = \frac{E(n^2 - 1)}{12} \left( \frac{t}{\rho} \right)^3 \tag{3.128}
\]

For an infinitely long cylinder it can be assumed that no strains occur in the out of plane direction of the ring. From Hooke’s law follows that the modulus of elasticity is adjusted by a factor \( (1 - \nu^2) \). In other words, a plane stress Buckling of a ring.
criterion is modified to a plane strain criterion. The classical buckling load for an infinitely long cylinder is given by:

\[ P_c = \frac{E}{12(1-\nu^2)} \left( \frac{t}{\rho} \right)^3 \]  

(3.129)

This relation is used in the Timoshenko collapse criterion given in eq. (3.127) to obtain the collapse criterion at first yielding for a long cylinder. It is convenient to rewrite this criterion into dimensionless form. First the dimensionless yield strength is introduced:

\[ \tilde{f}_y = \frac{f_y}{E} \]  

(3.130)

By making use of this identity, eq. (3.127) can be rewritten to the dimensionless Timoshenko criterion:

\[ \lambda^2 = \left( \tilde{f}_y + \left(1 + \frac{3\kappa}{\chi} \right) \lambda_c \right) \lambda + \tilde{f}_y \lambda_c = 0 \]  

(3.131)

Solving this polynomial for \( \lambda \) results to the dimensionless collapse load. Here \( \lambda_c \) is the dimensionless bifurcation load for a ring or a cylinder such that:

\[ \lambda_c = \begin{cases} \frac{1}{3} (n^2 - 1), & \text{for a ring} \\ \frac{n^2 - 1}{3(1-\nu^2)}, & \text{for a cylinder} \end{cases} \]  

(3.132a, 3.132b)

### 3.15.2 DNV formulation

The DNV equation is based on the same idea as the Timoshenko equation. Only now collapse is assumed after the full development of a plastic hinge. This plastic hinge is based on an elastic-perfectly plastic material model. The plastic hinge is fully developed as soon as the stress reaches the yield strength in the full cross section of the ring wall. The modulus of elasticity is assumed to be constant until the criterion is satisfied. For the full development of the plastic hinge the following criterion has to be satisfied:

\[ \frac{\|M\|}{M_{pl}} + \left( \frac{N}{N_{pl}} \right)^2 = 1 \]  

(3.133)

Where:

\[ M_{pl} = \frac{1}{4} f_y t^2 \]  

(3.134a)

\[ N_{pl} = f_y t \]  

(3.134b)

Collapse of the system is assumed for the load at which the criterion is satisfied at any location. Therefore the criterion is analysed for all coordinates \( \theta \in [0, 2\pi] \). \( M \) and \( N \) can be obtained from eq. (3.41) and the dimensionless kinematic relations given in eq. (3.43). The displacement relations for small \( \lambda \ll \lambda_c \) given in eq. (3.124) are used, as in the Timoshenko criterion, to obtain the DNV collapse criterion.
It can be verified that critical results are obtained for $\theta = 0$. The following identities are obtained at this location:

\[ M(0) = \frac{PP_c \xi \rho^2}{P_c - P} \quad (3.135a) \]
\[ N(0) = -\frac{(1 + \xi) P_c - P}{P_c - P} P \rho \quad (3.135b) \]

Substituting these identities into the plastic hinge criterion leads to a new collapse criterion. Further introduce a purely plastic yield pressure $P_p = \frac{t}{\rho} f_y$, which describes the pressure to induce yielding in a ring under compression only. The criterion becomes:

\[ (P - P_c) (P^2 - P_p^2) = 4 P P_c \xi \left( \frac{4 P_p}{t} + \frac{1}{2} P \left( 1 + \frac{\xi}{2 (1 - P/P_c)} \right) \right) \quad (3.136) \]

It should always be the case that $P \leq P_p$ and $P < P_c$. Further for practical cases it is assumed that the ring is thin to moderately thin ($\|\frac{t}{\rho}\| \ll 1$) and $\|\xi\| \ll 1$. Now the criterion can be approximated by:

\[ (P - P_c) (P^2 - P_p^2) = 4 P P_c P_p \frac{\xi}{t} \quad (3.137) \]

This criterion matches the buckling collapse criterion of DNV \cite{9} for a pipeline if the following identities are used:

\[ P_p = \frac{t}{\rho} f_y \alpha_{fab} \quad (3.138a) \]
\[ P_c = \frac{E (n^2 - 1)}{12 (1 - \nu^2)} \left( \frac{t}{\rho} \right)^3 \quad (3.138b) \]

With the classical elastic collapse pressure for an infinitely long cylinder $P_c$ is obtained from eq. (3.129). Further $\alpha_{fab}$ is a fabrication factor that accounts for the reduction in compressive yield strength of the material due to the Bauschinger effect. The Bauschinger effect occurs due to cold bending of the plates during the fabrication process of the pipeline joints. To match with DNV regulations it is necessary to let $n = 2$.

For this criterion it is also convenient to rewrite it to dimensionless form. In order to do so the dimensionless plastic collapse load is introduced and reads:

\[ \lambda_p = \alpha_{fab} \tilde{f}_y \quad (3.139) \]

Equation (3.137) can now be rewritten to its dimensionless form:

\[ (\lambda - \lambda_c) (\lambda^2 - \lambda_p^2) = 2 \lambda \lambda_c \lambda_p \frac{\xi}{\chi} \quad (3.140) \]

Where $\lambda_c$ is given by eq. (3.132).
Chapter 4

Buckling of a cylinder

In the previous chapter the bifurcation and buckling behaviour of a ring is analysed. A ring is the two dimensional model of a pipeline joint. A cylinder is the three dimensional model of a pipeline joint. Therefore a new curvilinear coordinate axis is introduced, the axial axis. The axial coordinate is denoted by $x$ and its dimensionless equivalent is denoted by $\eta$.

In this chapter Koiter’s perturbation theory is used to obtain the bifurcation behaviour of the system. This theory is combined with boundary layer theory. A cylinder is found to have strong boundary layer behaviour when subject to external pressure and certain boundary conditions i.e. clamped or simply supported edges. It is assumed that between the boundary layers there is a region for which the effect of the boundary conditions is insignificant, the regular domain.

Assumptions as given in section 3.1 are adopted for the cylinder model. Only for the cylinder model, an additional curvilinear axis is added; the axial axis.

4.1 Kinematics

Kinematic relations that are adopted for the cylinder analysis are similar to ones adopted for the ring analysis. Every material point of the perfect undeformed cylinder can be described by a vector $p(x^1, x^2, x^3) = x^1 \mathbf{i} + x^2 \mathbf{j} + x^3 \mathbf{k}$. The unit vectors in the Cartesian coordinate system are denoted by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ respectively.

A curvilinear coordinate system is introduced. It is constructed using a circumferential coordinate $s \in [0, 2\pi\rho]$, an axial coordinate $x \in [0, L]$ and a radial coordinate $z \in [-\frac{1}{2}t, \frac{1}{2}t]$. Here $\rho$ denotes the mean radius of the cylinder, $L$ is the cylinder length and $t$ is its wall thickness. For every admissible location on the middle surface of the cylinder ($z = 0$), a local coordinate system is introduced. This local coordinate system is described by a vector tangent to the circumferential direction, $\mathbf{t}$, a normal vector perpendicular to the middle surface $\mathbf{n}$ and a binormal vector $\mathbf{b}$, which in the chosen coordinate system is tangent to the axial coordinate axis. The displacement of a material point on the middle surface is denoted by the vector $d = u \mathbf{b} + v \mathbf{t} + w \mathbf{n}$. The location of a material point on the deformed middle surface now can be described by the vector $q = p + d$. These identities are illustrated in fig. 4.1.

Kinematic relations as described by Sanders [36] are introduced. Rotations
about the shell normal are neglected. Underlined terms are dropped to obtain the nonlinear Donnell-Mushtari-Vlasov (DMV) equations. These equations prove to be useful for solving the problem for a cylinder analytically.

Rotations of the middle surface are given by:

\[ \varphi_x = -w_x \]  
\[ \varphi_s = -w_s + \frac{1}{\rho} v \]

Curvatures are given by:

\[ \kappa_{xx} = \varphi_{x,x} \]  
\[ \kappa_{ss} = \varphi_{s,s} \]  
\[ \kappa_{xs} = \frac{1}{2} (\varphi_{s,x} + \varphi_{x,s}) \]

Middle surface strains are given by:

\[ e_{xx} = u_x + \frac{1}{2} \varphi_x^2 \]  
\[ e_{ss} = v_s + \frac{1}{\rho} w + \frac{1}{2} \varphi_s^2 \]  
\[ e_{ss} = \frac{1}{2} (v_{s,x} + u_{s,s} + \varphi_x \varphi_s) \]
Due to the assumptions stated in section 3.1, a relation between the middle surface strain, curvature and the strain at an arbitrary material point of the cylinder can be determined using the following relation:

\[ \varepsilon_{ij} = \varepsilon_{ij} + \kappa_{ij}z \]  

(4.4)

The indices \( i, j \in \{x, s\} \). Further \( \varepsilon_{ij} = \varepsilon_{ji} \).

### 4.2 Constitutive relations

Constitutive relations are used to relate material strains to occurring stresses. Using these relations, the strain energy of the system can be determined. Unless stated otherwise, the cylinder is assumed to behave perfectly elastic. The well-known constitutive description for an elastic material model is Hooke’s law. Its generalised description is given in eq. (3.18). For a shell, a two dimensional object, this constitutive law is written as:

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{ss} \\
\sigma_{xs}
\end{bmatrix} =
\frac{E}{1-\nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1-\nu
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{ss} \\
\varepsilon_{xs}
\end{bmatrix} +
\frac{Ez}{1-\nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1-\nu
\end{bmatrix}
\begin{bmatrix}
\kappa_{xx} \\
\kappa_{ss} \\
\kappa_{xs}
\end{bmatrix}
\]  

(4.5)

This identity is combined with the kinematic relation given in eq. (4.4). The following relation is obtained for occurring stresses, middle surface strains and curvatures:

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{ss} \\
\sigma_{xs}
\end{bmatrix} =
\frac{E}{1-\nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1-\nu
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{xx} \\
\varepsilon_{ss} \\
\varepsilon_{xs}
\end{bmatrix} +
\frac{Ez}{1-\nu^2}
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & 1-\nu
\end{bmatrix}
\begin{bmatrix}
\kappa_{xx} \\
\kappa_{ss} \\
\kappa_{xs}
\end{bmatrix}
\]  

(4.6)

### 4.3 Potential energy

To obtain the system’s equilibrium configuration, its potential energy has to be minimised. This implies minimising an integral and can be accomplished by letting its first variation be equal to zero. Now the Euler-Lagrange equations are applied to obtain a system of equilibrium equations. First a relation for the potential energy functional has to be found. It consists of the sum of the strain energy and the sum of all externally applied energy (work).

#### 4.3.1 Strain energy

The strain energy of an infinitesimal volume is given in eq. (3.20). By combining this and eq. (4.4), the following is obtained for the strain energy density function:

\[
U^* = \frac{1}{2} \sigma_{xx} (e_{xx} + \kappa_{xx}z) + \frac{1}{2} \sigma_{ss} (e_{ss} + \kappa_{ss}z) + \sigma_{xs} (e_{xs} + \kappa_{xs}z)
\]  

(4.7)

The strain energy of an infinitesimal area of shell is can be determined by integrating the strain energy density function over the cylinder through-thickness coordinate \( z \in [-\frac{1}{2}t, \frac{1}{2}t] \) and by using eq. (4.6),

\[
\int_{-\frac{1}{2}t}^{\frac{1}{2}t} U^* dz = \frac{1}{2} N_{ij} \varepsilon_{ij} + \frac{1}{2} M_{ij} \kappa_{ij}
\]  

(4.8)

Buckling of a cylinder
Here Einstein's summation convention is applied for \( i, j \in \{x, s\} \). The stress couple \( M_{ij} \) denotes the bending moment per unit length in \( ij\)-direction. The stress resultant \( N_{ij} \) denotes the axial force per unit length in \( ij\)-direction. Using earlier assumptions these identities can be used to reversely calculate the occurring stresses. The stress couples are given by:

\[
M_{xx} = \frac{E_t^3}{12(1-\nu^2)} (\kappa_{xx} + \nu \kappa_{ss}) \tag{4.9a}
\]

\[
M_{ss} = \frac{E_t^3}{12(1-\nu^2)} (\kappa_{ss} + \nu \kappa_{xx}) \tag{4.9b}
\]

\[
M_{xs} = \frac{E_t^3}{12(1+\nu)} \kappa_{xs} \tag{4.9c}
\]

Here \( \frac{E_t^3}{12(1-\nu^2)} \) represents the flexural rigidity of the shell. The stress resultants \( N_{ij} \) are given by:

\[
N_{xx} = \frac{E_t}{1-\nu^2} (e_{xx} + \nu e_{ss}) \tag{4.10a}
\]

\[
N_{ss} = \frac{E_t}{1-\nu^2} (e_{ss} + \nu e_{xx}) \tag{4.10b}
\]

\[
N_{xs} = \frac{E_t}{1+\nu} e_{xs} \tag{4.10c}
\]

Where \( \frac{E_t}{1-\nu^2} \) is the axial rigidity of the shell. The shell domain is described by \( \Omega = [0,L] \times [0,2\pi\rho] \), where the axial coordinate is \( x \in [0,L] \) and the circumferential coordinate is \( s \in [0,2\pi\rho] \). The strain energy of the system can be obtained by integrating the strain energy per unit area as given in eq. (4.8) over the shell domain \( \Omega \):

\[
U = \int_{\Omega} \frac{1}{2} (N_{ij} e_{ij} + M_{ij} \kappa_{ij}) \, d\Omega \tag{4.11}
\]

4.3.2 External energy

The external energy due to external pressure on the shell is adopted from Dyau and Kyriakides [12]. It is dependent on the change of volume enclosed by the cylinder shell and the applied pressure. The corresponding energy integral functional reads:

\[
W_P = \int_{\Omega} P \left( w + \frac{1}{2} \left( wu_x - w_x u + \frac{1}{\rho} w^2 + vw_s - vw_s + \frac{1}{\rho} v^2 \right) \right) \, d\Omega \tag{4.12}
\]

If end caps are attached to the cylinder, they will be subject to the same hydrostatic pressure as the external pressure on the cylinder \( P \). It is assumed that the end caps do not rotate under application of load. Hence the end cap pressure can be treated as a dead pressure: the pressure direction is constant and independent of deformations.

The general formulation for the external work of a force \( F \) is defined by \( F \cdot u \), where \( u \) is the displacement of a point at which \( F \) is applied and the dot denotes the inner product of the two vectors. The value of the stress resultant \( N_{xx} \) at the boundaries, the locations of the end caps, can be obtained from the pressure

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and the ratio of the area and perimeter of an end cap: $\frac{\pi r^2}{2\pi r} = \frac{1}{2} P$. The energy corresponding to the work done by this end cap pressure reads:

$$W_E = \int_0^{2\pi r} \frac{1}{2} P \rho \{ u(L) - u(0) \} \, ds $$

This can be rewritten such that the expression for the external energy due to the end cap pressure on the shell edge reads:

$$W_E = \int_0^L \int_0^{2\pi r} \frac{1}{2} P \rho u_x \, dx \, ds$$

(4.13)

This can be rewritten such that the expression for the external energy due to the end cap pressure on the shell edge reads:

$$W_E = \int_0^L \frac{1}{2} P \rho u_x \, ds$$

(4.14)

Note that this term drops out in the equilibrium equations and needs to be introduced in the analysis using a boundary condition. The potential energy of the system can be obtained by combining the previously obtained relations such that:

$$\Pi = U + W_P + \alpha W_E$$

(4.15)

The factor $\alpha \in \{0, 1\}$ denotes whether to include the influence of the end-cap pressure in the analysis.

### 4.4 System of equations

#### 4.4.1 Equilibrium equation

Equilibrium equations follow from minimising the potential energy functional given in eq. (4.15). As obtained in eq. (2.23), this can be accomplished by setting its first variation to zero such that:

$$\delta \Pi = \Pi' \delta u = 0$$

(4.16)

Making use of the Euler-Lagrange equations results in a system of three equilibrium equations, one for each displacement component $u$, $v$ and $w$:

\begin{align*}
N_{xx,s} + N_{sx,s} &= P \varphi_x \\
N_{ss,s} + N_{sx,x} + \frac{1}{\rho} (M_{ss,s} + M_{sx,s} - \varphi_s N_{ss} - \varphi_x N_{sx}) &= P \varphi_s \\
M_{xx,ss} + 2M_{xs,xs} + 2M_{ss,ss} - \frac{1}{\rho} N_{ss} - (N_{xx} \varphi_x + N_{xx} \varphi_x)_s + (N_{sx} \varphi_s + N_{sx} \varphi_s)_x &= \frac{1}{P} (1 + u_x + v_s + \frac{1}{\rho} w) 
\end{align*}

(4.17)

The underlined terms drop out for the DMV kinematic equations. In further analysis the DMV approximations are used. A potential function $F$ is introduced. This function is equivalent to the Airy-stress function, but is modified to account for the rotation of the pressure during deformation. This potential function is defined by:

\begin{align*}
F_{s,x} &\equiv N_{ss} + Pw \\
F_{s,s} &\equiv N_{xx} + Pw
\end{align*}

(4.18)

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Choosing a potential stress function in this form satisfies the first two equilibrium equations. However, two unknown functions, $F$ and $w$, remain. To be able to solve this system, an additional equation is needed. This equation is called the compatibility condition.

### 4.4.2 Compatibility condition

To obtain the compatibility condition, the following identity is evaluated:

$$e_{xx,ss} + e_{ss,xx} - 2e_{xs,xs} = 1 + \nu \frac{E_t}{\rho} N_{xs}$$  \hspace{1cm} (4.19)

Rewriting eq. (4.10) gives:

$$e_{xx} = \frac{1}{E_t} (N_{xx} - \nu N_{ss}) $$  \hspace{1cm} (4.20a)

$$e_{ss} = \frac{1}{E_t} (N_{ss} - \nu N_{xx}) $$  \hspace{1cm} (4.20b)

$$e_{xs} = 1 + \nu \frac{E_t}{\rho} N_{xs} $$  \hspace{1cm} (4.20c)

After combining eqs. (4.18) to (4.20), the following relation can be obtained:

$$e_{xx,ss} + e_{ss,xx} - 2e_{xs,xs} = 1 + \nu \frac{E_t}{\rho} w_{xx} - L_{11} [w, w] $$  \hspace{1cm} (4.23)

Here $\nabla^4$ denotes the biharmonic operator and $\nabla^2$ is the Laplace operator such that:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2} $$  \hspace{1cm} (4.22a)

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^2}{\partial x^2 \partial s^2} + \frac{\partial^4}{\partial s^4} $$  \hspace{1cm} (4.22b)

The following relation is determined using the DMV kinematic relations given in section 4.1:

$$e_{xx,ss} + e_{ss,xx} - 2e_{xs,xs} = \frac{1}{\rho} w_{xx} - L_{11} [w, w] $$  \hspace{1cm} (4.23)

Operator $L_{11}$ denotes a bilinear operator such that:

$$L_{11} = \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial s^2} - 2 \frac{\partial^2}{\partial x \partial s} \frac{\partial^2}{\partial x \partial s} + \frac{\partial^2}{\partial s^2} \frac{\partial^2}{\partial x^2} $$  \hspace{1cm} (4.24)

Substituting eq. (4.23) into eq. (4.21) results in the compatibility equation:

$$\frac{1}{E_t} \nabla^4 F - \frac{1}{\rho} w_{xx} + \frac{1}{2} L_{11} [w, w] - P \frac{1 - \nu}{E_t} \nabla^2 w = 0 $$  \hspace{1cm} (4.25)

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4.4.3 System of equilibrium equations

Now that the compatibility equation is obtained, it is convenient to rewrite the equilibrium equation in terms of $F$ and $w$. This is done by substituting eq. (4.18) into eq. (4.17c) and leads to the following equilibrium equation:

$$
\frac{E t^3}{12 (1 - \nu^2)} \nabla^4 w + \frac{1}{\rho} F_{xx} - L_{11} [F, w] + P \left\{ 1 - w \left( \frac{1}{\rho} - \nabla^2 w \right) \right\} + P \left\{ u_x + v_s + \frac{1}{\rho} w + \varphi_x^2 + \varphi_s^2 \right\} = 0
$$

(4.26)

Using eq. (4.3), the following identity can be obtained:

$$
u_x + v_s + \frac{1}{\rho} w + \varphi_x^2 + \varphi_s^2 = \varepsilon_{xx} + \varepsilon_{ss} + \frac{1}{2} \varphi_x^2 + \frac{1}{2} \varphi_s^2
$$

(4.27)

Because a framework of small strains and moderate rotations is adopted, linear strain terms and quadratic rotation terms can be neglected with respect to unity. Hence it is allowed to simplify eq. (4.26) to the equilibrium equation:

$$
\frac{E t^3}{12 (1 - \nu^2)} \nabla^4 w + \frac{1}{\rho} F_{xx} - L_{11} [F, w] + P \left\{ 1 - w \left( \frac{1}{\rho} - \nabla^2 w \right) \right\} = 0
$$

(4.28)

Combining this equilibrium equation and the previously obtained compatibility condition gives the following system of equations that has to be satisfied for a cylinder under external hydrostatic pressure:

$$
\frac{E t^3}{12 (1 - \nu^2)} \nabla^4 F - \frac{1}{\rho} w_{xx} - \frac{1}{2} L_{11} [w, w] - P \frac{1 - \nu}{E t} \nabla^2 w = 0
$$

(4.29a)

$$
\frac{1}{E t} \nabla^4 w - \frac{1}{\rho} F_{xx} - \frac{1}{2} L_{11} [F, w] - P \frac{1 - \nu}{E t} \nabla^2 w = 0
$$

(4.29b)

4.5 Boundary conditions

The system can only be solved if certain boundary conditions are applied. The behaviour of the end caps attached to the cylinder boundaries is modelled through boundary conditions. One of both end caps is allowed to move in axial direction rigidly. The other one is fully constrained. The end caps are assumed to behave completely rigid. Hence the radial deformation $w$ is constrained at the end caps, such that $\forall x \in \{0, L\} : w = 0$.

Two possibilities for the connection between the ends caps and the cylinder walls are investigated:

i Simply supported - Rotations are allowed, but the value of the stress couple $M_{xx}$ should be zero. This results to $\forall x \in \{0, L\} : w_{xx} = 0$.

ii Clamped - Rotations are constrained at the boundaries. Hence a stress couple with respect to the axial coordinate can exist at the boundaries, such that $\forall x \in \{0, L\} : w_x = 0$.

Regardless of the support conditions described above, the end cap pressure is introduced by the equilibrium of resultant forces in axial direction in the cylinder wall. Rotational rigid body constraints are assumed to be absent. Hence there

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should not be any resultant shear force at the end cap locations. Resultant forces can be obtained by integrating the stress resultants over the circumference of the cylinder. Now axial and shear force equilibria at the boundaries are given by:

\[ \forall x \in \{0, L\} : \begin{cases} \int_{0}^{2\pi\rho} N_{xx} ds + \alpha P \pi \rho^2 = 0, \\ \int_{0}^{2\pi\rho} N_{xx} ds = 0 \end{cases} \] (4.30a) (4.30b)

Using eq. (4.18), this condition can be rewritten to:

\[ \forall x \in \{0, L\} : \begin{cases} \int_{0}^{2\pi\rho} F_{ss, xx} ds + P \left( \alpha \pi \rho^2 - \int_{0}^{2\pi\rho} w ds \right) = 0, \\ \int_{0}^{2\pi\rho} F_{ss} ds = 0 \end{cases} \] (4.31a) (4.31b)

Continuity of the stress resultant in axial direction is ensured by:

\[ \forall x \in \{0, L\} : N_{xx} |_{s=2\pi\rho} - N_{xx} |_{s=0} = 0 \] (4.32)

This is equivalent to:

\[ \forall x \in [0, L] : \int_{0}^{2\pi\rho} N_{xx, ss} ds = 0 \] (4.33)

Substitution of eq. (4.18) and rewriting to integral form leads to:

\[ \forall x \in [0, L] : \int_{0}^{2\pi\rho} (F_{ss, ss} - P w, ss) ds = 0 \] (4.34)

The final constraint is the closed (or periodic) condition as given by Shen and Chen [37, 38, 39] and Sun and Chen [44]:

\[ \forall x \in [0, L] : \int_{0}^{2\pi\rho} v, ss ds = 0 \] (4.35)

This condition should be satisfied on all admissible \( x \in [0, L] \). By making use of eqs. (4.3), (4.18) and (4.20), the closed condition can be written to:

\[ \forall x \in [0, L] : \int_{0}^{2\pi\rho} \left\{ \frac{1}{Et} (F_{ss, xx} - \nu F_{ss, ss}) - w \left( \frac{1}{\rho} + \frac{P (1 - \nu)}{Et} \right) - \frac{1}{2} w, ss \right\} ds = 0 \] (4.36)

The previous results in the following set of constraints for the cylinder problem:

\[ \forall x \in \{0, L\} : \begin{cases} w = 0, \\ w, x = 0 \text{ or } w, xx = 0, \\ \int_{0}^{2\pi\rho} F_{ss, ss} ds + P \left( \alpha \pi \rho^2 - \int_{0}^{2\pi\rho} w ds \right) = 0, \\ \int_{0}^{2\pi\rho} F_{ss} ds = 0 \end{cases} \] (4.37a) (4.37b) (4.37c) (4.37d)

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∀x ∈ [0, L] :
\[
\begin{align*}
&\int_0^{2\pi\rho} (F_{,ssss} - Pw_{,s}) \, ds = 0, \\
&\int_0^{2\pi\rho} \left\{ \frac{1}{Et} (F_{,xx} - \nu F_{,ss}) - w \left( \frac{1}{\rho} + \frac{P (1 - \nu)}{Et} \right) - \frac{1}{2} w_{,s}^2 \right\} \, ds = 0
\end{align*}
\]
(4.37e)
(4.37f)

Here eq. (4.37b) is the boundary condition that varies for a clamped and a simply supported cylinder.

### 4.6 Nondimensionalisation of system

To be able to generalise the analysis’ results, it is convenient to rewrite the system to dimensionless form. By choosing dimensionless variables in a way such that the highest order derivative terms in either equilibrium equation are multiplied by a small factor the system’s boundary layers can be analysed.

The following dimensionless variables are introduced:

\[
\eta = \frac{x}{L}, \theta = \frac{s}{\rho}, \tilde{w} = \frac{w}{\rho}, \tilde{f} = \frac{F}{Et\rho^2}, \lambda = \frac{4P\rho^3 (1 - \nu^2)}{Et^3}, \gamma = \frac{\rho}{L}, \chi = \frac{t}{2\rho}
\]

(4.38)

Here \( \eta \in [0, 1] \) and \( \theta \in [0, 2\pi] \) describe the dimensionless domain of the cylinder. Using these dimensionless identities, eq. (4.29) can be rewritten to its dimensionless equivalent. The parameter small parameter \( \chi \) in front of the highest order derivative term in eq. (4.39) indicates that the system contains boundary layers. Until further notice dimensionless parameters will be used.

From now on the tildes on top of the dimensionless variables are omitted. The dimensionless system reads:

\[
\begin{align*}
\chi^2 \nabla^4 w + 3\gamma^2 \left( 1 - \nu^2 \right) \{ f_{,\eta\eta} - L_{11} [w, f] \} \\
+ 3\lambda \chi^2 \{ 1 - w (1 - \nabla^2 w) \} &= 0 \quad (4.39a) \\
\nabla^4 f - \gamma^2 \{ w_{,\eta} - \frac{1}{2} L_{11} [w, w] \} - \frac{\lambda \chi^2}{1 + \nu} \nabla^2 w &= 0 \quad (4.39b)
\end{align*}
\]

Where \( \nabla^2 \) denotes the quadratic dimensionless Laplace operator, \( \nabla^4 \) denotes the quadratic dimensionless biharmonic operator and \( L_{11} \) denotes a dimensionless bilinear operator such that:

\[
\begin{align*}
\nabla^2 &= \gamma^2 \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \theta^2} \\
\nabla^4 &= \gamma^4 \frac{\partial^4}{\partial \eta^4} + 2\gamma^2 \frac{\partial^4}{\partial \eta^2 \partial \theta^2} + \frac{\partial^2}{\partial \theta^2} \\
L_{11} &= \frac{\partial^2}{\partial \eta^2} \frac{\partial^2}{\partial \theta^2} - 2 \frac{\partial^2}{\partial \eta \partial \theta} \frac{\partial^2}{\partial \eta \partial \theta} + \frac{\partial^2}{\partial \theta^2} \frac{\partial^2}{\partial \eta^2}
\end{align*}
\]

(4.40a)
(4.40b)
(4.40c)

This system is constrained by dimensionless boundary conditions. These boundary conditions can be obtained by introducing the dimensionless variables

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in eq. (4.37). The following dimensionless boundary conditions need to be satisfied for the cylinder problem:

\[
\forall \eta \in \{0, 1\} : \begin{cases}
w = 0, \\
w, \eta = 0 \text{ or } w, \eta\eta = 0, \\
(1 - \nu^2) \int_0^{2\pi} f_{,\theta\theta} d\theta + \lambda \chi^2 \left( \alpha \pi - \int_0^{2\pi} w d\theta \right) = 0, \\
\int_0^{2\pi} f_{,\theta\theta} d\theta = 0,
\end{cases}
\] (4.41a)

\[
\forall \eta \in [0, 1] : \begin{cases}
\int_0^{2\pi} \left( f_{,\theta\theta} - \frac{\lambda \chi^2}{1 - \nu^2} w_{,\theta} \right) d\theta = 0, \\
\int_0^{2\pi} \left( \gamma^2 f_{,\eta\eta} - \nu f_{,\theta\theta} - w \left( 1 + \frac{\lambda \chi^2}{1 + \nu} \right) - \frac{1}{2} w_{,\theta}^2 \right) d\theta = 0
\end{cases}
\] (4.41b)

### 4.7 Solution expansions and boundary layer

#### 4.7.1 Expansions

In the Koiter [26] method the solutions are expanded as a power series about its fundamental (pre-buckling) solution. The expansion is done in terms of a parameter \( \zeta \). The value of this parameter describes the amount of first order buckling mode that is included in the solution. Further the dimensionless load parameter is expanded in a similar way. The series expansions read:

\[
w = w_0(\lambda) + \zeta w_1 + \zeta^2 w_2 + \ldots \]
(4.42a)

\[
f = f_0(\lambda) + \zeta f_1 + \zeta^2 f_2 + \ldots \]
(4.42b)

\[
\lambda = \lambda_c + \zeta \lambda_1 + \zeta^2 \lambda_2 + \ldots
\] (4.42c)

Alternatively the load parameter can be expanded in terms of the initial post-buckling coefficients \( a \) and \( b \), such that
\[
\lambda - \lambda_c = \zeta \lambda_1 + \zeta^2 \lambda_2 + \ldots
\]

This expansion is related to eq. (4.42c) by the relations \( \lambda_1 = a \lambda_c \) and \( \lambda_2 = b \lambda_c \). For symmetric buckling, where \( a = 0 \), the sign of buckling coefficient \( b \) will determine the initial post-buckling stability of the system. In this thesis, the expansion as given in eq. (4.42c) is used.

Now \( w_0(\lambda) \) and \( f_0(\lambda) \) are expanded as a Taylor series about their fundamental solution for the critical buckling load \( w_0(\lambda_c) \) and \( f_0(\lambda_c) \). These series expansions read:

\[
w_0(\lambda) = w_0(\lambda_c) + (\lambda - \lambda_c) w_{0,\lambda}(\lambda_c) + \frac{1}{2} (\lambda - \lambda_c)^2 w_{0,\lambda\lambda}(\lambda_c) + \ldots
\] (4.43a)

\[
f_0(\lambda) = f_0(\lambda_c) + (\lambda - \lambda_c) f_{0,\lambda}(\lambda_c) + \frac{1}{2} (\lambda - \lambda_c)^2 f_{0,\lambda\lambda}(\lambda_c) + \ldots
\] (4.43b)

Using eq. (4.42c) such that \( \lambda - \lambda_c = \zeta \lambda_1 + \zeta^2 \lambda_2 + \ldots \), the Taylor series expansions can be rewritten:

\[
w_0(\lambda) = w_0(\lambda_c) + \zeta \lambda_1 w_{0,\lambda}(\lambda_c) + \zeta^2 \left\{ \lambda_2 w_{0,\lambda}(\lambda_c) + \frac{1}{2} \lambda_1^2 w_{0,\lambda\lambda}(\lambda_c) \right\} + \ldots
\] (4.44a)
\[ f_0(\lambda) = f_0(\lambda_c) + \zeta_1 f_{0,\lambda}(\lambda_c) + \zeta_2 \{ \lambda_2 f_{0,\lambda,\lambda}(\lambda_c) + \frac{1}{2} \lambda_1^2 f_{0,\lambda,\lambda,\lambda}(\lambda_c) \} + \ldots \]  

(4.44b)

Substituting this in the power series expansions of \( w \) and \( f \) in eq. (4.42) results to the following expansions:

\[
\begin{align*}
w &= w_0(\lambda_c) + \zeta \{ \lambda_1 w_{0,\lambda}(\lambda_c) + w_1 \} \\
&\quad + \zeta^2 \{ \lambda_2 w_{0,\lambda,\lambda}(\lambda_c) + \frac{1}{2} \lambda_1^2 w_{0,\lambda,\lambda,\lambda}(\lambda_c) + w_2 \} + \ldots \\
\tag{4.45a}
\end{align*}
\]

\[
\begin{align*}
f &= f_0(\lambda_c) + \zeta \{ \lambda_1 f_{0,\lambda}(\lambda_c) + f_1 \} \\
&\quad + \zeta^2 \{ \lambda_2 f_{0,\lambda,\lambda}(\lambda_c) + \frac{1}{2} \lambda_1^2 f_{0,\lambda,\lambda,\lambda}(\lambda_c) + f_2 \} + \ldots \\
\tag{4.45b}
\end{align*}
\]

It is interesting to note that for symmetric buckling it is expected that \( \lambda_1 = 0 \) and hence the previous equations will be reduced to:

\[
\begin{align*}
w &= w_0(\lambda_c) + \zeta w_1 + \zeta^2 \{ \lambda_2 w_{0,\lambda,\lambda}(\lambda_c) + w_2 \} + \ldots \\
\tag{4.46a}
\end{align*}
\]

\[
\begin{align*}
f &= f_0(\lambda_c) + \zeta f_1 + \zeta^2 \{ \lambda_2 f_{0,\lambda,\lambda}(\lambda_c) + f_2 \} + \ldots \\
\tag{4.46b}
\end{align*}
\]

### 4.7.2 Boundary layers

Boundary layers (b.l.) are characterised by a rapidly varying response within a small subdomain. In differential equations, an indication of the occurrence of a boundary layer is a small factor or coefficient multiplying the highest order derivative term.

Now assume that the various expansion orders as introduced in eq. (4.45) can be characterised by:

\[
w_j = \begin{cases} 
w_j^{[i]} + w_j^{[r]}, & \text{near } \eta = 0 \\
w_j^{[r]}, & \text{away from b.l.} \\
w_j^{[i]} + w_j^{[I]}, & \text{near } \eta = 1 
\end{cases} 
\tag{4.47a}
\]

\[
f_j = \begin{cases} 
f_j^{[i]} + f_j^{[r]}, & \text{near } \eta = 0 \\
f_j^{[r]}, & \text{away from b.l.} \\
f_j^{[i]} + f_j^{[I]}, & \text{near } \eta = 1 
\end{cases} 
\tag{4.47b}
\]

The \([r]\)-superscript denotes the regular solution or outer solution. The \([i]\)-superscript denotes boundary layer solution or inner solution near \( \eta = 0 \) with respect to the regular solution. The \([I]\)-superscript denotes the boundary layer solution near \( \eta = 1 \) with respect to the regular solution. Equation (4.47) is valid for all admissible \( j \), such that the total solution for the \( j \)th order expansion can be estimated by:

\[
w_j \approx w_j^{[i]} + w_j^{[r]} + w_j^{[I]} 
\tag{4.48a}
\]

\[
f_j \approx f_j^{[i]} + f_j^{[r]} + f_j^{[I]} 
\tag{4.48b}
\]

Note that eqs. (4.47) and (4.48) cannot be valid simultaneously if either boundary layer solution is not equal to zero. If the boundary layer solutions values outside of their boundary layer domains is very small it is possible that eqs. (4.47) and (4.48) are almost satisfied. Hence the approximate relation in eq. (4.48).

\section*{Buckling of a cylinder}
For separate analyses for the boundary layer and regular domains the relations given in eq. (4.47) are used. For a graphical representation, see fig. 4.2. The total solution of the system can be obtained by the approximation given in eq. (4.48). This results to a continuous solution that does not match the boundary constraints exactly because of some residual of the boundary layer solution on the other side of the cylinder.

Figure 4.2: Inner (boundary layer) and outer (regular) solutions

To perform the boundary layer analysis, the following boundary layer coordinates are introduced:

\[ \hat{\xi} = \frac{\eta}{\sqrt{\chi}} \quad \text{and} \quad \bar{\xi} = \frac{1 - \eta}{\sqrt{\chi}} \quad (4.49) \]

These coordinates rescale the axial coordinate near the boundaries. \( \hat{\xi} \) denotes the boundary layer coordinate near \( \eta = 0 \) and \( \bar{\xi} \) denotes the boundary layer coordinate near \( \eta = 1 \). Dependencies on these boundary layer coordinates are added to the solution functions. Hence \( w(\eta, \theta) \rightarrow w(\eta, \theta, \hat{\xi}, \bar{\xi}) \) and \( f(\eta, \theta) \rightarrow f(\eta, \theta, \hat{\xi}, \bar{\xi}) \). Note that \( w(\eta, \theta) \) and \( f(\eta, \theta) \) can be obtained reversely by using eq. (4.49).

The dependencies for every \( j \)th order bifurcation solution as given in eq. (4.48) generally read:

\[
\begin{align*}
{w_j}(\eta, \theta, \hat{\xi}, \bar{\xi}) &\approx {w_j}^i(\theta, \hat{\xi}) + {w_j}^{[i]}(\eta, \theta) + {w_j}^{[I]}(\theta, \bar{\xi}) \\
{f_j}(\eta, \theta, \hat{\xi}, \bar{\xi}) &\approx {f_j}^i(\theta, \hat{\xi}) + {f_j}^{[i]}(\eta, \theta) + {f_j}^{[I]}(\theta, \bar{\xi})
\end{align*}
\] (4.50a,b)

The boundary layer solutions are constrained by boundary conditions at infinity. These boundary conditions ensure that secular terms are removed such that:

\[
\begin{align*}
\lim_{\xi \to \infty} {w_j}^i &= 0 \quad (4.51a) \\
\lim_{\xi \to \infty} {f_j}^i &= 0 \quad (4.51b) \\
\lim_{\xi \to \infty} {w_j}^{[i]} &= 0 \quad (4.51c) \\
\lim_{\xi \to \infty} {f_j}^{[i]} &= 0 \quad (4.51d)
\end{align*}
\]

Because of the boundary layer coordinates’ dependencies \( \hat{\xi}(\eta) \) and \( \bar{\xi}(\eta) \) on the axial coordinate, the differential operators with respect to \( \eta \) need adjustment.

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These adjustments can be obtained by application of the chain rule. The following adjustments are applied:

\[
\begin{align*}
\frac{\partial}{\partial \eta} & \rightarrow \frac{\partial}{\partial \eta} + \frac{\partial \xi}{\partial \eta} \frac{\partial}{\partial \xi} + \frac{\partial \xi}{\partial \eta} \frac{\partial}{\partial \xi} \\
\frac{\partial^2}{\partial \eta^2} & \rightarrow \left( \frac{\partial}{\partial \eta} + \frac{\partial \xi}{\partial \eta} \frac{\partial}{\partial \xi} + \frac{\partial \xi}{\partial \eta} \frac{\partial}{\partial \xi} \right)^2 \\
\frac{\partial^4}{\partial \eta^4} & \rightarrow \left( \frac{\partial}{\partial \eta} + \frac{\partial \xi}{\partial \eta} \frac{\partial}{\partial \xi} + \frac{\partial \xi}{\partial \eta} \frac{\partial}{\partial \xi} \right)^4
\end{align*}
\] (4.52a, b, c)

From eq. (4.49) follows:

\[
\frac{\partial \xi}{\partial \eta} = \frac{1}{\sqrt{\chi}} \frac{\partial \xi}{\partial \eta} = -\frac{1}{\sqrt{\chi}}
\] (4.53)

Hence eq. (4.52) can be written to:

\[
\begin{align*}
\frac{\partial}{\partial \eta} & \rightarrow \frac{\partial}{\partial \eta} + \frac{1}{\sqrt{\chi}} \frac{\partial}{\partial \xi} - \frac{1}{\sqrt{\chi}} \frac{\partial}{\partial \xi} \\
\frac{\partial^2}{\partial \eta^2} & \rightarrow \left( \frac{\partial}{\partial \eta} + \frac{1}{\sqrt{\chi}} \frac{\partial}{\partial \xi} - \frac{1}{\sqrt{\chi}} \frac{\partial}{\partial \xi} \right)^2 \\
\frac{\partial^4}{\partial \eta^4} & \rightarrow \left( \frac{\partial}{\partial \eta} + \frac{1}{\sqrt{\chi}} \frac{\partial}{\partial \xi} - \frac{1}{\sqrt{\chi}} \frac{\partial}{\partial \xi} \right)^4
\end{align*}
\] (4.54a, b, c)

These modified operators are reflected in all operators containing derivative terms with respect to axial coordinate \( \eta \).

## 4.8 Fundamental solutions

The fundamental solutions are the pre-buckling or bifurcation solutions. They depend on the load parameter \( \lambda \). For the cylinder problem, it is assumed that the fundamental solution describes an axisymmetric shape. Hence all dependencies of \( w_0 \) on coordinate \( \theta \) are removed from the equations. Further it is assumed that the stress resultants in axial and circumferential directions are only dependent on the axial coordinate the i.e. \( \theta \)-dependency is removed. Equation (4.45) is substituted into eq. (4.39). After setting \( \zeta \) to zero, the following system of equations needs to be satisfied for the fundamental solutions \( w_0(\eta) \) and \( f_0(\eta, \theta) \):

\[
\chi^2 \gamma^4 w_{0,\eta\eta\eta\eta} + 3\gamma^2 (1 - \nu^2) \{ f_{0,\eta\eta} - w_{0,\eta\eta} f_{0,\theta} \} + 3\lambda \chi^2 \{ 1 - w_0 (1 - \gamma^2 w_{0,\eta\eta}) \} = 0
\] (4.55a)

\[
\nabla^4 f_0 - \gamma^2 \left( 1 + \frac{\lambda \chi^2}{1 + \nu} \right) w_{0,\eta\eta} = 0
\] (4.55b)

Additionally the following boundary conditions should be satisfied:

\[
\forall \eta \in \{ 0, 1 \}:
\]

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\[
\begin{align*}
\left\{ \begin{array}{l}
w_0 = 0, \\
w_{0, \eta} = 0 \text{ or } w_{0,\eta\eta} = 0, \\
(1 - \nu^2) \int_0^{2\pi} f_{0, \theta\theta} d\theta + \lambda \pi \chi^2 (\alpha - 2w_0) = 0, \\
\int_0^{2\pi} f_{0, \theta\theta} d\theta = 0
\end{array} \right. \\
\forall \eta \in [0, 1]:
\end{align*}
\]

Where either the first order derivative \( w_{0, \eta} = 0 \) or the second order derivative \( w_{0,\eta\eta} = 0 \) for clamped or simply supported boundaries respectively. By using eq. (4.50), the fundamental solutions can be written as:

\[
\begin{align*}
w_0(\eta, \xi, \hat{\xi}, \lambda) &= w_{0}^{[i]}(\hat{\xi}, \lambda) + w_{0}^{[r]}(\eta, \lambda) + w_{0}^{[I]}(\hat{\xi}, \lambda) \\
f_0(\eta, \theta, \xi, \hat{\xi}, \lambda) &= f_{0}^{[i]}(\theta, \xi, \lambda) + f_{0}^{[r]}(\eta, \theta, \lambda) + f_{0}^{[I]}(\theta, \hat{\xi}, \lambda)
\end{align*}
\]

Note that, as mentioned earlier, by the introduction of the boundary layer coordinates, the differential operators need to be modified using eq. (4.54).

### 4.8.1 Regular fundamental solution

The regular fundamental solution can be obtained by application of regular perturbation theory. The regular solution is expanded in terms of the small parameter \( \chi \). Equation (4.57) is substituted in eq. (4.55). As given in eq. (4.47), for the subdomain beyond the boundary layers, only regular terms are taken into account. This newly obtained system is referred to as the regular fundamental system.

Introduce the following expansions:

\[
\begin{align*}
w_{0}^{[r]}(\eta, \lambda) &= \sum_{j=2}^{\infty} \chi^j w_{0,j}^{[r]}(\eta, \lambda) = \chi^2 w_{0,2}^{[r]}(\eta, \lambda) + \chi^3 w_{0,3}^{[r]}(\eta, \lambda) + \ldots \\
f_{0}^{[r]}(\eta, \theta, \lambda) &= \sum_{j=2}^{\infty} \chi^j f_{0,j}^{[r]}(\eta, \theta, \lambda) = \chi^2 f_{0,2}^{[r]}(\eta, \theta, \lambda) + \chi^3 f_{0,3}^{[r]}(\eta, \theta, \lambda) + \ldots
\end{align*}
\]

These expansions are substituted in the regular fundamental system and its corresponding boundary conditions. Every order of \( \chi \) is treated separately. The lowest order of \( \chi \) in the system results to:

\[
\begin{align*}
\gamma^2 (1 - \nu^2) f_{0,2,\eta\eta}^{[r]} + \lambda &= 0 \\
\nabla^4 f_{0,2}^{[r]} - \gamma^2 w_{0,2,\eta\eta}^{[r]} &= 0
\end{align*}
\]

And the constraints:

\[
\forall \eta \in \{0, 1\} : \text{Buckling of a cylinder}
\]
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\[
\begin{align*}
&\left\{ (1 - \nu^2) \int_0^{2\pi} f_{0:2,\theta\theta}^{[r]} d\theta + \alpha \lambda \pi = 0, \right. \\
&\left. \int_0^{2\pi} f_{0:2,\theta\theta}^{[r]} d\theta = 0 \right\}, \quad (4.60a) \\
&\forall \eta \in [0, 1] : \left\{ \int_0^{2\pi} f_{0:2,\theta\theta}^{[r]} d\theta = 0, \right. \\
&\left. \int_0^{2\pi} \left( \gamma^2 f_{0:2,\eta\eta}^{[r]} - \nu f_{0:2,\theta\theta}^{[r]} \right) d\theta - 2\pi w_{0:2}^{[r]} = 0 \right\}. \quad (4.60c)
\end{align*}
\]

Solving eq. (4.59a) by direct integration, leads to:

\[
f_{0:2}^{[r]}(\eta, \theta, \lambda) = \frac{-\lambda \eta^2}{2\gamma^2 (1 - \nu^2)} + f_{0:2:1}^{[r]}(\theta, \lambda) \eta + f_{0:2:2}^{[r]}(\theta, \lambda) \quad (4.61)
\]

This newly obtained relation is substituted in eq. (4.60d). It follows that \( w_{0:2}^{[r]}(\eta, \lambda) \) should be expressed as a polynomial of first degree:

\[
w_{0:2}^{[r]}(\eta, \lambda) = w_{0:2:1}^{[r]}(\lambda) \eta + w_{0:2:2}^{[r]}(\lambda) \quad (4.62)
\]

Now the coefficients of all possible powers of \( \eta \) in eq. (4.60d) are put to zero separately. This leads to the relations:

\[
\begin{align*}
\int_0^{2\pi} f_{0:2:1,\theta\theta}^{[r]} d\theta &= -\frac{2\pi}{\nu} w_{0:2:1}^{[r]} \quad (4.63a) \\
\int_0^{2\pi} f_{0:2:2,\theta\theta}^{[r]} d\theta &= -\frac{2\pi}{\nu} \left( w_{0:2:2}^{[r]} + \frac{\lambda}{1 - \nu^2} \right) \quad (4.63b)
\end{align*}
\]

Introducing the previous substitutions in eq. (4.60a) and solving at each boundary for \( w_{0:2:1}^{[r]} \) and \( w_{0:2:2}^{[r]} \) leads to:

\[
w_{0:2}^{[r]}(\eta, \lambda) = -\frac{1 - \frac{1}{2} \alpha \nu}{1 - \nu^2} \lambda \quad (4.64)
\]

Equation (4.61) is substituted into eq. (4.59b) and solved for each power of \( \eta \). Solution parts that do not contribute to the second order derivatives of the dimensionless potential stress function are neglected. They do not have physical meaning:

\[
\begin{align*}
f_{0:2:1}^{[r]}(\theta, \lambda) &= \frac{1}{2} f_{0:2:1:1}^{[r]}(\lambda) \theta^3 + \frac{1}{2} f_{0:2:1:2}^{[r]}(\lambda) \theta^2 + f_{0:2:1:3}^{[r]}(\lambda) \theta \\
f_{0:2:2}^{[r]}(\theta, \lambda) &= \frac{1}{2} f_{0:2:2:1}^{[r]}(\lambda) \theta^3 + \frac{1}{2} f_{0:2:2:2}^{[r]}(\lambda) \theta^2 
\end{align*}
\]

The previous relations are substituted in eqs. (4.60b), (4.60c) and (4.63). These can be solved for the unknown functions with respect to \( \lambda \) given in the previous equation such that:

\[
f_{0:2}^{[r]}(\eta, \theta, \lambda) = \frac{-\lambda}{2(1 - \nu^2)} \left( \frac{\eta^2}{\gamma^2} + \frac{\alpha \theta^2}{2} \right) \quad (4.66)
\]

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A similar analysis is performed for higher order expansions in eq. (4.58). This results to the following regular fundamental solutions:

\[
\begin{align*}
    w_0^{|r|}(\eta, \lambda) &= -\left(1 - \frac{1}{4}\alpha\nu\right) \frac{\lambda\chi^2}{1 - \nu^2} \\
    f_0^{|r|}(\eta, \theta, \lambda) &= -\frac{\lambda\chi^2}{1 - \nu^2} \frac{\eta^2}{2\gamma^2} - \left(\frac{\lambda\chi^2}{2(1 - \nu^2)}\right) \left(\frac{\eta^2}{2\gamma^2} + \frac{\theta^2}{2}\right) (1 - \frac{1}{4}\alpha\nu)
\end{align*}
\]  

(4.67a)  

(4.67b)

These solutions satisfy the regular fundamental system and the corresponding constraints. These fundamental solutions represent a membrane stress state. The influence of the support conditions on the fundamental solutions, given in eqs. (4.56a) and (4.56b), is analysed by the implementation of a boundary layer method.

4.8.2 Boundary layer fundamental solution

At the boundaries the functions \(w_0\) and \(f_0\) are approximated using eq. (4.47). Using eq. (4.57), the following can be obtained for the solutions of the cylinder problem in the boundary layer subdomains:

\[
\begin{align*}
    w_0(\eta, \hat{\xi}, \hat{\xi}, \lambda) &= \begin{cases} 
    w_0^{|i|}(\hat{\xi}, \lambda) + w_0^{|r|}(\eta, \lambda), & \text{near } \eta = 0 \\
    w_0^{|i|}(\eta, \lambda) + w_0^{|I|}(\hat{\xi}, \lambda), & \text{near } \eta = 1 
    \end{cases} \\
    f_0(\eta, \theta, \hat{\xi}, \hat{\xi}, \lambda) &= \begin{cases} 
    f_0^{|i|}(\theta, \hat{\xi}, \lambda) + f_0^{|r|}(\eta, \theta, \lambda), & \text{near } \eta = 0 \\
    f_0^{|i|}(\eta, \theta, \lambda) + f_0^{|I|}(\hat{\xi}, \lambda), & \text{near } \eta = 1 
    \end{cases}
\end{align*}
\]  

(4.68a)  

(4.68b)

First the boundary near \(\eta = 0\) is analysed. This boundary layer is referred to as boundary layer 1 from now on. The boundary layer near \(\eta = 1\) is denoted as boundary layer 2.

**Boundary layer 1**

The part of eq. (4.58) that is valid on boundary layer 1 is substituted into eq. (4.55). The regular equilibrium equations are already satisfied in the previous section. Hence the contribution of these equations can be removed from the boundary layer system. Due to nonlinearity of the system some coupling between inner and outer solutions exists. Taking advantage of eq. (4.67), the equilibrium equations for the boundary layer near \(\eta = 0\) read:

\[
\begin{align*}
    \chi^2\nabla^4 f_0^{|i|} - \gamma^2 \left(\chi + \frac{\lambda\chi^3}{1 + \nu}\right) w_0^{|i|}_0 & = 0 \\
    -3\lambda \left\{ \chi^3 w_0^{|i|} - \gamma^2 \chi^2 (w_0^{|i|} + w_0^{|r|}) w_0^{|i|}_0 \right\} - 3\lambda \left\{ \chi^3 w_0^{|i|} - \gamma^2 \chi^2 (w_0^{|i|} + w_0^{|r|}) w_0^{|i|}_0 \right\} & = 0 \\
    \chi^2\nabla^4 f_0^{|i|} - \gamma^2 \left(\chi + \frac{\lambda\chi^3}{1 + \nu}\right) w_0^{|i|}_0 & = 0
\end{align*}
\]  

(4.69a)  

(4.69b)

This is a system of nonlinear partial differential equations. It is constrained by the following relations where, by eq. (4.49), for \(\eta = 0 : \xi = 0\).

for \(\eta = 0 : \)

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\[
\begin{align*}
\begin{cases}
    w^{[i]}_0 + w^{[r]}_0 = 0, \\
    w^{[i]}_{0,\xi} + \sqrt{\chi}w^{[i]}_{0,\eta} = 0 \text{ or } w^{[i]}_{0,\xi\xi} + \chi w^{[r]}_{0,\eta\eta} = 0, \\
(1 - \nu^2) \int_0^{2\pi} f^{[i]}_{0,\theta\theta} d\theta - 2\pi \lambda \chi w^{[i]}_0 = 0, \\
\int_0^{2\pi} f^{[i]}_{0,\theta} d\theta = 0
\end{cases}
\end{align*}
\]

∀\(\eta \in [0, 1] : \)

\[
\begin{align*}
\begin{cases}
    \int_0^{2\pi} f^{[i]}_{0,\theta\theta\theta} d\theta = 0, \\
\int_0^{2\pi} \left( \gamma^2 f^{[i]}_{0,\xi\xi} - \nu \chi f^{[i]}_{0,\theta\theta} \right) d\theta - 2\pi \lambda \chi w^{[i]}_0 \left( \chi + \frac{\lambda \chi^3}{1 + \nu} \right) = 0
\end{cases}
\end{align*}
\]

The approach to solve this system is to introduce a power series expansion for \(w^{[i]}_0\) and \(f^{[i]}_0\). The expanded solutions are substituted in eq. (4.69). The system of differential equations needs to be satisfied for each order of the small parameter \(\chi\) separately. This method is called singular perturbation method. The difference of this method with regular perturbation method (as applied for the regular fundamental solution) is the fact that now the axial coordinate is rescaled. This results in terms to be included in the analysis that would drop out in the regular analysis. Now the fundamental boundary layer solution is expanded as:

\[
w^{[i]}_0(\hat{\xi}, \lambda) = \sum_{j=2}^{\infty} \chi^j w^{[i]}_{0,j}(\hat{\xi}, \lambda) = \chi^2 w^{[i]}_{0,2}(\hat{\xi}, \lambda) + \chi^3 w^{[i]}_{0,3}(\hat{\xi}, \lambda) + \ldots
\]

\[
f^{[i]}_0(\theta, \hat{\xi}, \lambda) = \sum_{j=3}^{\infty} \chi^j f^{[i]}_{0,j}(\theta, \hat{\xi}, \lambda) = \chi^3 f^{[i]}_{0,3}(\theta, \hat{\xi}, \lambda) + \chi^4 f^{[i]}_{0,4}(\theta, \hat{\xi}, \lambda) + \ldots
\]

Lower order expansions than the ones included above will result in trivial solutions. Hence they are disregarded. The expansions are substituted in eq. (4.69). Every order of \(\chi\) can be analysed separately. The lowest order equations read:

\[
\begin{align*}
\gamma^2 w^{[i]}_{0,2,\xi\xi\xi\xi} + 3(1 - \nu^2) f^{[i]}_{0,3,\xi\xi} &= 0 \\
w^{[i]}_{0,2,\xi\xi} - \gamma^2 f^{[i]}_{0,3,\xi\xi\xi\xi} &= 0
\end{align*}
\]

And the corresponding constraints are:

for \(\eta = 0 : \)

\[
\begin{align*}
\begin{cases}
    w^{[i]}_{0,2} + w^{[r]}_{0,2} = 0, \\
    w^{[i]}_{0,2,\xi} = 0 \text{ or } w^{[i]}_{0,2,\xi\xi} = 0, \\
\int_0^{2\pi} f^{[i]}_{0,3,\theta\theta} d\theta = 0, \\
\int_0^{2\pi} f^{[i]}_{0,3,\theta\xi} d\theta = 0
\end{cases}
\end{align*}
\]

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∀ η ∈ [0, 1] :
\[
\begin{align*}
\int_0^{2\pi} f_{0,3.00a}^{[i]} d\theta &= 0, \\
\int_0^{2\pi} \gamma^2 f_{0,3.\xi,\xi}^{[i]} d\theta - 2\pi u_{0,2}^{[i]} &= 0
\end{align*}
\]
(4.73e)
(4.73f)

Combining eq. (4.72) and neglecting integration constants leads to:
\[
w_{0,2.\dot{\xi},\dot{\xi}}^{[i]} + \frac{3(1-\nu^2)}{\gamma} u_{0,2.\dot{\xi}}^{[i]} = 0
\]
(4.74)

Equation (4.74) is solved for \( w_{0,1}^{[i]}(\dot{\xi},\lambda) \). This differential has four complex Eigenvalues and a repeated Eigenvalue valued zero. The following is obtained for the general solution:
\[
w_{0,2}^{[i]}(\dot{\xi},\lambda) = \left\{\begin{array}{l}
w_{0,2,1}^{[i]}(\lambda) \cos\left(\psi_{\dot{\xi}}\right) + w_{0,2,2}^{[i]}(\lambda) \sin\left(\psi_{\dot{\xi}}\right) \\
+ \left\{w_{0,2,3}^{[i]}(\lambda) \cos\left(\psi_{\dot{\xi}}\right) + w_{0,2,4}^{[i]}(\lambda) \sin\left(\psi_{\dot{\xi}}\right)\right\} \exp\left(-\psi_{\dot{\xi}}\right)
\end{array}\right.
\]
(4.75)

Where:
\[
\psi = \frac{1}{\gamma} \sqrt{\frac{3}{4}(1-\nu^2)}
\]
(4.76)

For the boundary layer solution, secular terms (i.e. terms growing without bounds with respect to the axial coordinate) are neglected. This results to:
\[
w_{0,2}^{[i]}(\dot{\xi},\lambda) = \left\{w_{0,2,1}^{[i]}(\lambda) \cos\left(\psi_{\dot{\xi}}\right) + w_{0,2,2}^{[i]}(\lambda) \sin\left(\psi_{\dot{\xi}}\right)\right\} \exp\left(-\psi_{\dot{\xi}}\right) + w_{0,2,6}^{[i]}(\lambda)
\]
(4.77)

Substituting this solution into eq. (4.72a) and integrating twice with respect to \( \dot{\xi} \) leads to:
\[
f_{0,2}^{[i]}(\theta,\dot{\xi},\lambda) = \frac{1}{\sqrt{3(1-\nu^2)}} \left\{w_{0,2,2}^{[i]}(\lambda) \cos\left(\psi_{\dot{\xi}}\right) - w_{0,2,1}^{[i]}(\lambda) \sin\left(\psi_{\dot{\xi}}\right)\right\} \exp\left(-\psi_{\dot{\xi}}\right)
\]
(4.78)

Using eq. (4.73), the following is obtained for the unknown function:
\[
\begin{align*}
w_{0,2,1}^{[i]}(\lambda) &= \frac{1 - \frac{1}{2}\alpha\nu}{1 - \nu^2} \\
w_{0,2,2}^{[i]}(\lambda) &= \frac{1 - \frac{1}{2}\alpha\nu}{1 - \nu^2} \beta \lambda \\
w_{0,2,6}^{[i]}(\lambda) &= 0
\end{align*}
\]
(4.79a)
(4.79b)
(4.79c)

Where \( \beta \in \{0,1\} \) such that \( \beta = 0 \) for simply supported boundaries and \( \beta = 1 \) for clamped boundaries. From the previously obtained functions and the assumption that for the fundamental solution, the stress resultants in axial and

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circumferential direction are independent of the circumferential coordinate \( \theta \), it is assumed:

\[
\frac{\partial^2}{\partial \xi^2} f_{0:3}^{[i]}(\theta, \xi, \lambda) = \hat{g}(\xi, \lambda) \\
\frac{\partial^2}{\partial \theta^2} f_{0:3}^{[i]}(\theta, \xi, \lambda) = \check{g}(\xi, \lambda)
\]

(4.80a)  
(4.80b)

Where \( \hat{g}(\xi, \lambda) \) and \( \check{g}(\xi, \lambda) \) are arbitrary functions introduced to denote dependencies. Equation (4.80a) is satisfied by eq. (4.78). Introducing eq. (4.78) gives into eq. (4.80b) leads to:

\[
f_{0:3:1}^{[i]}(\lambda) = \frac{1}{2} f_{0:3:1:1}^{[i]}(\lambda) \theta^2 + f_{0:3:1:2}^{[i]}(\lambda) \theta + f_{0:3:1:3}^{[i]}(\lambda)
\]

(4.81a)  
\[
f_{0:3:2}^{[i]}(\lambda) = \frac{1}{2} f_{0:3:2:1}^{[i]}(\lambda) \theta^2 + f_{0:3:2:2}^{[i]}(\lambda) \theta + f_{0:3:2:3}^{[i]}(\lambda)
\]

(4.81b)

Because eq. (4.78) is a potential function, only derivative terms with respect to the axial and circumferential coordinates from order two have physical meaning. Hence lower order derivatives can be disregarded. Further the term \( f_{0:3:1:1}^{[i]}(\lambda) \xi \theta^2 \) proves to be a secular term and hence is removed:

\[
f_{0:3}^{[i]}(\theta, \xi, \lambda) = \frac{1}{\sqrt{3(1-\nu^2)}} \left\{ w_{0:2:2}^{[i]}(\lambda) \cos (\psi \xi) - w_{0:2:1}^{[i]}(\lambda) \sin (\psi \xi) \right\} \exp (-\psi \xi)
\]

(4.82)  
\[
f_{0:3}^{[i]}(\theta, \xi, \lambda) = \frac{1}{\sqrt{3(1-\nu^2)}} \left\{ w_{0:2:2}^{[i]}(\lambda) \cos (\psi \xi) - w_{0:2:1}^{[i]}(\lambda) \sin (\psi \xi) \right\} \exp (-\psi \xi)
\]

(4.83a)  
\[
f_{0:3}^{[i]}(\theta, \xi, \lambda) = \frac{1}{\sqrt{3(1-\nu^2)}} \left\{ w_{0:2:2}^{[i]}(\lambda) \cos (\psi \xi) - w_{0:2:1}^{[i]}(\lambda) \sin (\psi \xi) \right\} \exp (-\psi \xi)
\]

(4.83b)

The boundary conditions given in eq. (4.73) are only satisfied when \( f_{0:3:1:2}^{[i]}(\lambda) = 0 \) and \( f_{0:3:1:3}^{[i]}(\lambda) = 0 \). This results to the boundary layer solutions:

\[
w_{0:2}^{[i]}(\xi, \lambda) = \frac{\lambda (1 - \frac{1}{2} \alpha \nu)}{1 - \nu^2} \left\{ \cos (\psi \xi) + \beta \sin (\psi \xi) \right\} \exp (-\psi \xi)
\]

(4.83a)  
\[
f_{0:3}^{[i]}(\theta, \xi, \lambda) = \frac{\lambda (1 - \frac{1}{2} \alpha \nu)}{\sqrt{3(1-\nu^2)^2}} \left\{ \beta \cos (\psi \xi) - \sin (\psi \xi) \right\} \exp (-\psi \xi) + O(\chi^3)
\]

(4.83b)

Substituting this relation in eq. (4.77) leads to:

\[
w_{0:2}^{[i]}(\xi, \lambda) = \frac{\lambda \chi^2 (1 - \frac{1}{2} \alpha \nu)}{1 - \nu^2} \left\{ \cos (\psi \xi) + \beta \sin (\psi \xi) \right\} \exp (-\psi \xi) + O(\chi^3)
\]

(4.84a)  
\[
f_{0}^{[i]}(\theta, \xi, \lambda) = \frac{\lambda \chi^3 (1 - \frac{1}{2} \alpha \nu)}{\sqrt{3(1-\nu^2)^2}} \left\{ \beta \cos (\psi \xi) - \sin (\psi \xi) \right\} \exp (-\psi \xi) + O(\chi^4)
\]

(4.84b)

**Boundary layer 2**

Similar to the analysis performed for boundary layer 1, an analysis is performed for boundary layer 2. The part of eq. (4.68) valid on boundary layer 2 is
substituted in eq. (4.55). The system that has to be satisfied for the boundary layer near \( \eta = 1 \) reads:

\[
\chi^4 w_0^{[\xi]} + 3\gamma^2 (1 - \nu^2) \left\{ \frac{j^{[\eta]}_{0,0} - \left( j^{[\eta]}_{0,0} + j^{[\eta]}_{0,\theta} \right) w_0^{[\xi]}}{\chi} \right\} - 3\lambda \left\{ \chi^3 w_0^{[\xi]} - \gamma^2 \chi^2 \left( w_0^{[\xi]} + w_0^{[\xi]} \right) \right\} = 0
\]

\[
\lambda^2 \nabla^6\sigma_0^{[\xi]} - \gamma^2 \left( \chi + \frac{\lambda^3}{1 + \nu} \right) w_0^{[\xi]} = 0
\]

This system is equivalent to eq. (4.69). It is constrained by constraints equivalent to eq. (4.73). Only the boundary location it moved to \( \eta = 1 \) and similarly by eq. (4.49), \( \xi = 0 \). The regular fundamental solution contributions in the constraints are independent of coordinate \( \eta \). Hence the fundamental boundary layer solutions for boundary layer 2 are equivalent to the boundary layer solutions for boundary layer 1 such that:

\[
u_0^{[\xi]} (\xi, \lambda) = \frac{\lambda^2 \left( 1 - \frac{1}{2} \alpha \nu \right)}{1 - \nu^2} \left\{ \cos \left( \frac{\psi_0}{\xi} \right) + \beta \sin \left( \frac{\psi_0}{\xi} \right) \right\} \exp \left( -\frac{\psi_0}{\xi} \right) + O(\chi^3)
\]

\[

j_0^{[\xi]} (\theta, \xi, \lambda) = \frac{\lambda^3 \left( 1 - \frac{1}{2} \alpha \nu \right)}{\sqrt{3} (1 - \nu^2)^2} \left\{ \beta \cos \left( \frac{\psi_0}{\xi} \right) - \sin \left( \frac{\psi_0}{\xi} \right) \right\} \exp \left( -\frac{\psi_0}{\xi} \right) + O(\chi^4)
\]

### 4.9 Bifurcation solutions

Now that the fundamental solutions are obtained, the first order perturbation of the system is analysed. As for the fundamental system, the expansions around the critical fundamental solutions, eq. (4.45), are introduced into the equilibrium system and constraints given in eqs. (4.39) and (4.41). The equations and constraints of order \( O(\chi) \) are evaluated as being the bifurcation system. This leads to the following bifurcation equations:

\[
\chi^2 \nabla^4 w_1 + 3\gamma^2 (1 - \nu^2) \left\{ f_1, \eta - w_{0c, \eta} f_{0c, \theta} - L_{11} \left[ w_1, f_{0c} \right] \right\} + 3\lambda_1 \lambda_2^2 \left\{ w_1 \left( \chi^2 w_{0c, \eta} - 1 \right) + w_{0c} \nabla^2 w_1 \right\} + \lambda_1 \lambda_2^2 \left\{ \chi^2 w_{0c, \eta} - 1 \right\} \Delta + 3\lambda_1 \lambda_2 \left\{ \chi^2 w_{0c, \eta} - 1 \right\} \Delta + 3\lambda_1 \chi^2 \left\{ 1 + w_{0c} \right\} \left( \chi^2 w_{0c, \eta} - 1 \right) = 0
\]

\[
\nabla^4 f_1 - \gamma^2 \left\{ w_1, \eta - w_{0c, \eta} w_{1, \theta} \right\} - \frac{\lambda^2}{1 + \nu} \nabla^2 w_1
\]

\[
+ \lambda_1 \nabla^4 f_{0c, \lambda} - \lambda_2 \chi^2 w_{0c, \eta} \eta - \frac{\lambda^2}{1 + \nu} \left\{ w_{0c, \eta} + \lambda_2 w_{0c, \lambda} \right\} = 0
\]

And the corresponding constraints read:

\[
\forall \eta \in \{0, 1\} : 
\]

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wave number of the bifurcation mode in axial direction and

The regular bifurcation solutions can be obtained by neglecting the boundary

of the bifurcation mode in circumferential direction.

and hence eqs. (4.88a) and (4.88b) can be neglected. This leads to:

\[
\begin{align*}
\{ & w_1 + \lambda_1 w_{0c,\lambda} = 0, \\
& w_{1,\eta} + \lambda_1 w_{0c,\lambda,\eta} = 0 \text{ or } w_{1,\eta\eta} + \lambda_1 w_{0c,\lambda,\eta\eta} = 0, \\
& \int_0^{2\pi} \left( f_{1,\theta\theta} - \frac{\lambda_c^2}{1 - \nu^2} w_1 \right) d\theta + \lambda_1 \int_0^{2\pi} f_{0c,\lambda,\theta\theta} d\theta \\
& + \frac{\lambda_1 \pi \lambda_c^2}{1 - \nu^2} \left( \alpha - 2w_{0c} - 2\lambda_c w_{0c,\lambda} \right) = 0, \\
& \int_0^{2\pi} f_{1,\eta\theta} d\theta + \lambda_1 \int_0^{2\pi} f_{0c,\lambda,\eta\theta} d\theta = 0 \\
\}
\end{align*}
\] (4.88d)

\[
\forall \eta \in [0, 1]:
\begin{align*}
\int_0^{2\pi} \left( f_{1,\theta\theta\theta} - \frac{\lambda_c \chi^2}{1 - \nu^2} w_{1,\theta} \right) d\theta + \lambda_1 \int_0^{2\pi} f_{0c,\lambda,\theta\theta\theta} d\theta &= 0, \\
\int_0^{2\pi} \left( \gamma^2 f_{1,\eta\eta} - \nu f_{1,\theta\theta} - w_1 \left( 1 + \frac{\lambda_c \chi^2}{1 + \nu} \right) \right) d\theta \\
& + \lambda_1 \int_0^{2\pi} \left( \gamma^2 f_{0c,\lambda,\eta\eta} - \nu f_{0c,\lambda,\theta\theta} \right) d\theta \\
& - 2\pi \lambda_1 \left( w_{0c,\lambda} \left( 1 + \frac{\lambda_c \chi^2}{1 + \nu} \right) + \frac{\chi^2}{1 + \nu} w_{0c} \right) = 0
\end{align*}
\] (4.88f)

In the equations above, the c-subscript implies that the subscripted function

is evaluated for \( \lambda = \lambda_c \). For example \( w_{0c} = w_0(\eta, \lambda) |_{\lambda=\lambda_c} = w_0(\eta, \lambda_c), w_{0,\lambda} = \frac{\partial}{\partial \lambda} w_0(\eta, \lambda) |_{\lambda=\lambda_c}, \) etcetera.

\section{4.9.1 Regular bifurcation solution}

The regular bifurcation solutions can be obtained by neglecting the boundary

layer solutions in both the fundamental part and the bifurcation part of the

solution. The previously obtained eq. (4.87) is substituted for the fundamental

solution in eqs. (4.87) and (4.88). Only regular solutions are taken into account

and hence eqs. (4.88a) and (4.88b) can be neglected. This leads to:

\[
\chi^2 \nabla^4 w_1^{[r]} + 3\gamma^2 \left( 1 - \nu^2 \right) f_{1,\eta\eta}^{[r]} \\
+ 3\lambda_c \chi^2 \left( \frac{1}{2} \alpha \gamma^2 w_1^{[r]} + w_1^{[r]} - w_1^{[r]} \right) = 0
\] (4.89a)

\[
\nabla^4 f_1^{[r]} - \gamma^2 w_1^{[r]} - \frac{\lambda_c \chi^2}{1 + \nu} \nabla^2 w_1^{[r]} = 0
\] (4.89b)

Now the following ansatzes are introduced for the regular bifurcation solutions:

\[
w_1^{[r]}(\eta, \theta) = w_1^{[r]}(n, m) \sin (m \pi \eta) \cos (n \theta)
\] (4.90a)

\[
f_1^{[r]}(\eta, \theta) = f_1^{[r]}(n, m) \sin (m \pi \eta) \cos (n \theta)
\] (4.90b)

Where \( m \in \{ \tilde{m} \in \mathbb{N} | \tilde{m} \geq 1 \} \) and \( n \in \{ \tilde{n} \in \mathbb{N} | \tilde{n} \geq 2 \} \) such that \( m \) is the half

wave number of the bifurcation mode in axial direction and \( n \) is the wave number

of the bifurcation mode in circumferential direction.

\section*{Buckling of a cylinder}
The ansatzes comply with the constraints for the regular part of the bifurcation solution. Substitution of the ansatzes into eq. (4.89) results to the following linear system:

\[
\begin{bmatrix}
-\frac{\lambda_c}{1-\nu} \left( n^2 + \frac{1}{2} \alpha \bar{m}^2 + 1 \right) + \frac{\left( n^2 + \bar{m}^2 \right)^2}{3(1-\nu)} & -\bar{m}^2 \\
\lambda_c \left( n^2 + \bar{m}^2 \right) + \frac{1+\nu}{\chi^2} \bar{m}^2 & \frac{-\bar{m}^2}{n^2 + \bar{m}^2}
\end{bmatrix}
\begin{bmatrix}
\frac{\chi^2 w_{11}^{[1]}(n,m)}{1+\nu} \\
\left( n^2 + \bar{m}^2 \right) f_{11}^{[1]}(n,m)
\end{bmatrix} = 0
\]

(4.91)

Here \( \bar{m} = \pi \gamma m \). The solutions of this system can only be non-trivial if the system is singular. Singularity of the system follows from the determinant of the matrix being equal to zero. Hence:

\[
\begin{vmatrix}
-\frac{\lambda_c}{1-\nu} \left( n^2 + \frac{1}{2} \alpha \bar{m}^2 + 1 \right) + \frac{\left( n^2 + \bar{m}^2 \right)^2}{3(1-\nu)} & -\bar{m}^2 \\
\lambda_c \left( n^2 + \bar{m}^2 \right) + \frac{1+\nu}{\chi^2} \bar{m}^2 & \frac{-\bar{m}^2}{n^2 + \bar{m}^2}
\end{vmatrix} = 0
\]

(4.92)

This leads to the critical value for the load parameter with respect to the regular solution only, denoted by \( \lambda_c^{[1]}(n,m) \). This will be adapted later, to account for the boundary conditions and boundary layer solutions. Equation (4.92) is solved for \( \lambda_c \). It is obtained that:

\[
\lambda_c^{[1]}(n,\bar{m}) = \frac{1}{3} \left( n^2 + \bar{m}^2 \right)^2 + \frac{\left( 1-\nu^2 \right) \bar{m}^4}{\chi^2 \left( n^2 + \bar{m}^2 \right)^2} - \frac{1}{n^2 + \frac{1}{2} \alpha \bar{m}^2 + \frac{n^2 + \nu \bar{m}^2}{n^2 + \bar{m}^2}}
\]

(4.93)

Where \( \bar{m} = \pi \gamma m \) such that:

\[
\lambda_c^{[1]}(n,m) = \frac{1}{3} \left( n^2 + \pi^2 \gamma^2 m^2 \right)^2 + \frac{\left( 1-\nu^2 \right) \pi^4 \gamma^4 m^4}{\chi^2 \left( n^2 + \pi^2 \gamma^2 m^2 \right)^2} - \frac{1}{n^2 + \frac{1}{2} \alpha \pi^2 \gamma^2 m^2 + \frac{n^2 + \nu \pi^2 \gamma^2 m^2}{n^2 + \pi^2 \gamma^2 m^2}}
\]

(4.94)

If only the regular part of the solution is analysed to obtain the bifurcation behaviour of the system, a relation between the undetermined coefficients \( w_{11}^{[1]}(n,m) \) and \( f_{11}^{[1]}(n,m) \) can be obtained by determining the nullspace of the matrix given in eq. (4.91) after substitution of the critical load parameter given above. This results in an Eigenvector. Assuming that the amount of bifurcation mode included is given by the parameter \( \zeta \), \( w_{11}^{[1]}(n,m) \) can be set to one. This results to:

\[
w_{11}^{[1]}(n,\bar{m}) = 1
\]

(4.95a)

\[
f_{11}^{[1]}(n,\bar{m}) = -\frac{\left( n^2 + \bar{m}^2 \right)^2}{3(1+\nu)} + \frac{\bar{m}^2 \left( n^2 + \frac{1}{2} \alpha \bar{m}^2 + 1 \right)}{\left( n^2 + \bar{m}^2 \right)^2}
\]

(4.95b)

\[
\]

Where again \( \bar{m} = \pi \gamma m \), such that:

\[
w_{11}^{[1]}(n,m) = 1
\]

(4.96a)

**Buckling of a cylinder**
\[ f_{1:1}^{[r]}(n, m) = \frac{(n^2 + \pi^2 \gamma^2 m^2) \chi^2 + \pi^2 \gamma^2 m^2 (n^2 + \frac{1}{2} \alpha \pi^2 \gamma^2 m^2 + 1)}{3 (1 + \nu) \left( n^2 + \pi^2 \gamma^2 m^2 \right)^2} \] (4.96b)

In further analysis, where the influence of the boundary conditions on the bifurcation load is analysed the values of these coefficient are assumed to be undetermined yet again such that:

\[ w_{1:1}^{[r]}(n, m) = \hat{W}(n, m) \] (4.97a)
\[ f_{1:1}^{[r]}(n, m) = \hat{F}(n, m) \] (4.97b)

4.9.2 Boundary layer bifurcation solution

The bifurcation solution has to satisfy some constraints at the boundaries. The regular solution does not satisfy all of these constraints. To be able to satisfy these constraints, the bifurcation equations need to be satisfied for a boundary layer solution. As done for the fundamental boundary layer solutions, singular perturbation theory is applied to obtain the bifurcation boundary layer solutions.

The boundary layer bifurcation equations are obtained by substituting the corresponding boundary layer solutions described by eqs. (4.47) and (4.50) into eq. (4.87) and the constraints in eq. (4.88). Hereafter the regular bifurcation equations as obtained in eq. (4.89) and the regular parts of the constraints are subtracted from this system. This is allowed because the regular bifurcation equations are satisfied separately.

**Boundary layer 1**

Now the following expansions are introduced:

\[ w_1^{[i]}(\theta, \hat{\xi}) = \cos(n \theta) \sum_{j=0}^{\infty} \chi^j w_1^{[i]}(\hat{\xi}) = \cos(n \theta) w_1^{[i]}(\hat{\xi}) + \ldots \] (4.98a)
\[ f_1^{[i]}(\theta, \hat{\xi}) = \cos(n \theta) \sum_{j=1}^{\infty} \chi^j f_1^{[i]}(\hat{\xi}) = \cos(n \theta) \chi f_1^{[i]}(\hat{\xi}) + \ldots \] (4.98b)

Using these expansions and neglecting all higher orders of \( \chi \) leads to the system to be satisfied for the boundary layer bifurcation solutions:

\[ \gamma^2 w_1^{[i]}_{1:0, \hat{\xi} \hat{\xi} \hat{\xi} \hat{\xi}} + 3 \left( 1 - \nu^2 \right) f_1^{[i]}_{1:1, \hat{\xi} \hat{\xi}} = 0 \] (4.99a)
\[ w_1^{[i]}_{1:0, \hat{\xi} \hat{\xi}} - \gamma^2 f_1^{[i]}_{1:1, \hat{\xi} \hat{\xi} \hat{\xi} \hat{\xi}} = 0 \] (4.99b)

This equation is equivalent to eq. (4.72). Hence it is assumed that the solution of this equation is described by:

\[ w_{1:0}^{[i]}(\hat{\xi}) = \left\{ w_{1:0}^{[i]}(\psi \hat{\xi}) + w_{1:0}^{[i]}(\psi \hat{\xi}) \sin(\psi \hat{\xi}) \right\} \exp(-\psi \hat{\xi}) \] (4.100a)
\[ f_{1:1}^{[i]}(\hat{\xi}) = \frac{1}{\sqrt{3 (1 - \nu^2)}} \left\{ w_{1:0}^{[i]}(\psi \hat{\xi}) - w_{1:0}^{[i]}(\psi \hat{\xi}) \sin(\psi \hat{\xi}) \right\} \exp(-\psi \hat{\xi}) \] (4.100b)

Bounding of a cylinder
Now all higher order approximations of \( w^{[i]}_1 \) and \( f^{[i]}_1 \) are neglected. Using the constraints from eq. (4.88), the following is obtained for the undetermined constants:

\[
\begin{align*}
    w^{[i]}_{1:0:1} &= 0 \\
    w^{[i]}_{1:0:2} &= -\frac{\beta m \pi \gamma \sqrt{\chi}}{\left\{ \frac{3}{4} \left( 1 - \nu^2 \right) \right\}^{\frac{1}{4}}} \hat{W}(n, m)
\end{align*}
\]

(4.101a) \hspace{1cm} (4.101b)

And further it is necessary that:

\[
    \lambda_1 = 0
\]

(4.102)

By combining the previously obtained relations, the following is obtained for the boundary layer bifurcation solution for the boundary layer near \( \eta = 0 \):

\[
\begin{align*}
    w^{[i]}_1(\theta, \hat{\xi}) &= -\hat{W}(n, m) \frac{\beta m \pi \gamma \sqrt{\chi}}{\left\{ \frac{3}{4} \left( 1 - \nu^2 \right) \right\}^{\frac{1}{4}}} \sin \left( \psi \hat{\xi} \right) \cos \left( n\theta \right) \exp \left( -\psi \hat{\xi} \right) \\
    f^{[i]}_1(\theta, \hat{\xi}) &= -\hat{W}(n, m) \frac{4 \beta m \pi \gamma \chi}{\left\{ 12 \left( 1 - \nu^2 \right) \right\}^{\frac{1}{4}}} \cos \left( \psi \hat{\xi} \right) \cos \left( n\theta \right) \exp \left( -\psi \hat{\xi} \right)
\end{align*}
\]

(4.103a) \hspace{1cm} (4.103b)

**Boundary layer 2**

The boundary layer bifurcation solution for the boundary layer near \( \eta = 1 \) is obtained using the same method as the solution for the other boundary layer. This results in almost the same result. Due to eq. (4.88b) the result varies slightly from eq. (4.103). The result reads:

\[
\begin{align*}
    w^{[i]}_1(\theta, \hat{\xi}) &= \hat{W}(n, m) \frac{(-1)^m \beta m \pi \gamma \sqrt{\chi}}{\left\{ \frac{3}{4} \left( 1 - \nu^2 \right) \right\}^{\frac{1}{4}}} \sin \left( \psi \hat{\xi} \right) \cos \left( n\theta \right) \exp \left( -\psi \hat{\xi} \right) \\
    f^{[i]}_1(\theta, \hat{\xi}) &= \hat{W}(n, m) \frac{4 (-1)^m \beta m \pi \gamma \chi}{\left\{ 12 \left( 1 - \nu^2 \right) \right\}^{\frac{1}{4}}} \cos \left( \psi \hat{\xi} \right) \cos \left( n\theta \right) \exp \left( -\psi \hat{\xi} \right)
\end{align*}
\]

(4.104a) \hspace{1cm} (4.104b)

**4.10 Bifurcation load**

Now that the bifurcation shapes are obtained in terms of two undetermined coefficients \( w^{[i]}_{1:1} (n, m) \) and \( f^{[i]}_{1:1} (n, m) \), it is possible to apply a Galerkin procedure to obtain the bifurcation load that accounts for the boundary conditions of the system.

First the regular bifurcation equations given in eq. (4.89) are written in operator form such that:

\[
\begin{align*}
    L_{bif:1}[w, f] &= \chi^2 \nabla^4 w + 3 \gamma^2 \left( 1 - \nu^2 \right) f,_{\eta\eta} + 3 \lambda c \chi^2 \left( \frac{1}{2} \alpha \gamma^2 w,_{\eta\eta} + w,_{\theta\theta} - w \right) \\
    L_{bif:2}[w, f] &= \nabla^4 f - \gamma^2 w,_{\eta\eta} = \frac{\lambda c \chi^2}{1 + \nu} \nabla^2 w
\end{align*}
\]

(4.105a) \hspace{1cm} (4.105b)
As done by Sun and Chen [44], it is assumed that the regular bifurcation equations need to be satisfied for the total bifurcation solutions. The adopted weighting functions are the shapes of the regular bifurcation solutions, eq. (4.90), such that:

\[
\int_0^{2\pi} \int_0^L w_{1,1} [w_1, f_1] \cos (n\theta) \sin (m\pi\eta) \, d\eta \, d\theta = 0 \quad (4.106a)
\]

\[
\int_0^{2\pi} \int_0^L w_{1,2} [w_1, f_1] \cos (n\theta) \sin (m\pi\eta) \, d\eta \, d\theta = 0 \quad (4.106b)
\]

It makes sense that a better approximation is obtained when (instead of only the regular equation) the total bifurcation equation is satisfied with respect to the total bifurcation solution. In order to obtain a workable analytical solution only the regular parts of the bifurcation equations are used (combined with the total bifurcation solution). This neglects the influence of the fundamental boundary layer solutions. After evaluation of the integrals, the system can be written as:

\[
[M] [\hat{W}(n,m)] = 0 \quad (4.107)
\]

This system includes a small exponential term that can be neglected. As long as \(\gamma\) is not too large this is allowed. The system above can only have non-trivial solutions if matrix \([M]\) is singular i.e. its determinant is zero valued. From this equation the bifurcation load is obtained that accounts for the boundary constraints. The bifurcation load for a perfect cylinder reads:

\[
\lambda_c(n,m) = \lambda^{[r]}_c(n,m) + \frac{1}{3} \gamma \beta \bar{m}^4 \left\{ \frac{3 (1 - \nu^2) + \chi^2 \bar{m}^4}{\frac{2n^2 + \bar{m}^2}{n^2 + \bar{m}^2}} \right\} \left( \frac{\sqrt{2} \chi \left\{ 3 (1 - \nu^2) + \chi^2 \bar{m}^4 \right\}}{8 \left\{ 3 (1 - \nu^2) \right\}^{\frac{1}{4}}} - \gamma \beta \chi \bar{m}^2 \right\}
\]

\[
\lambda_c(n,m) = \lambda^{[r]}_c(n,m) = \frac{n^4}{3 (n^2 + 1)} \quad (4.108)
\]

Note that also here \(\bar{m} = \pi \gamma m\). Bifurcation will occur for the lowest possible value of \(\lambda_c(n,m)\) for all admissible values of \(n\) and \(m\). The relation for \(\lambda^{[r]}_c(n,m)\) is given in eq. (4.94). It is interesting to note that for a very long cylinder the critical load parameters approach the critical load for a ring (under plane strain condition) based on the DMV kinematics given in eq. (3.67).

\[
\lim_{m \to 0} \lambda_c(n,m) = \lim_{m \to 0} \lambda^{[r]}_c(n,m) = \frac{n^4}{3 (n^2 + 1)} \quad (4.109)
\]

Keep in mind that the definitions for the dimensionless load factor for a ring in eq. (3.30) and for a cylinder in eq. (4.38) are slightly different. This difference results from plane stress that is assumed for a ring and plane strain that is assumed for an infinitely long cylinder.

As mentioned earlier, the bifurcation load and the corresponding bifurcation mode of a cylinder is determined by eq. (4.108). The bifurcation load is the minimum value for \(\lambda_c(n,m)\) for all admissible \(n\) and \(m\). The corresponding values of \(n\) and \(m\) describe the bifurcation shape.

**Buckling of a cylinder**
In figs. 4.3 to 4.5 a qualitative description of the cylinder bifurcation load with respect to its bifurcation mode is given. A dark background colour denotes a high bifurcation load and a light background colour denotes a low bifurcation load. The minimum admissible bifurcation load is denoted by a red dot. This describes the likely bifurcation mode of an elastic cylinder.

It appears that higher values of $\gamma$ result to higher values of $n$ describing the bifurcation mode. Further clamped boundary conditions appear to lead to a higher bifurcation mode than simply supported boundary conditions although this influence is only significant for relatively short cylinders. Further do thinner cylinders tend to bifurcate in higher modes than thicker cylinders do.

These conclusions are supported by fig. 4.6. In these figures the change in bifurcation mode is reflected by a kink in the graph. For small values of $\gamma$ the bifurcation mode is represented by $n = 2$. For shorter cylinder sections the bifurcation shapes are described by higher values of $n$. Further it appears that the end cap pressure tends to decrease the bifurcation load value, especially for relatively short cylinders.

4.11 Comparison analytical bifurcation load with literature

Previous work has been done to describe the bifurcation behaviour of a finite length cylinder. Several relations for the critical bifurcation load of a simply supported cylinder are compared to the relation obtained in eq. (4.108).

First, the relation obtained by Sturm [43] is evaluated. This relation is similar to the relations obtained by von Mises [50, 51]. Second, the relation is evaluated that is obtained by Batdorf [3]. A similar relation is also reported later by NASA-SP-8007 [31] and Shen and Chen [37].

For simply supported cylinder edges ($\beta = 0$) various relations for the bifurcation load are compared. In figs. 4.7 and 4.8 results are given for a relatively thick walled and a relatively thin walled cylinder respectively.

From these figures follows that the relation obtained by Batdorf [3] gives slightly higher critical values for the bifurcation pressure compared to the other two relations. Especially for relatively short cylinders (small values of $\gamma$) this relation is expected to over-predict the bifurcation load significantly. This can be explained by the fact that Batdorf [3] uses Donnell’s equilibrium equations as first obtained by Donnell [10] and does not account for the rotation of the load vector during deformation. This leads to an over-prediction of the critical bifurcation pressure that is significant, specifically, for small values of $\gamma$.

The analytical solution for the bifurcation load of a simply supported cylinder obtained by Sturm [43] is found to be more or less in the range between the Batdorf relation and the relation obtained in this thesis. However Sturm [43] reports that the difference between the analytical bifurcation solutions for simply supported and clamped cylinders is found to be up to 50%, even for relatively small values of $\gamma$. The solution for a clamped cylinder is found to be considerably higher than the results obtained through experiments.

For the relation for the bifurcation pressure of a cylinder obtained in this thesis, the difference in bifurcation pressure for simply supported and clamped cylinders is only significant for relatively large values of $\gamma$. This is in correspondence with
Figure 4.3: Contour plots for $\lambda_c(n, m)$ with $\chi = 1/20$ and $\alpha = 1$
Figure 4.4: Contour plots for $\lambda_c(n,m)$ with $\chi = 1/4\alpha$ and $\alpha = 1$
Figure 4.5: Contour plots for $\lambda_c(n, m)$ with $\chi = 1/100$ and $\alpha = 1$
Figure 4.6: Critical bifurcation loads for a cylinder
Figure 4.7: Comparison critical bifurcation loads of an externally pressurised cylinder for $\chi = 1/20$ and $\beta = 0$
Figure 4.8: Comparison critical bifurcation loads of an externally pressurised cylinder for $\chi = 1/100$ and $\beta = 0$
expectations that boundary constraints have more influence on the bifurcation behaviour of shorter cylinders (higher values of $\gamma$) than on the bifurcation behaviour of longer cylinders.

4.12 Effect $\chi$ and $\gamma$ on first change in bifurcation mode

Goal of a collapse test is to estimate the real life collapse capacity of a pipeline. The collapse load obtained through such a test only has practical meaning if the collapse mode of the tested section matches the collapse mode that would occur for a real life pipeline.

Because a pipeline has a very long unconstrained length, it would have a very low $\gamma$ value. As found through the analysis of a ring and in fig. 4.6, the collapse mode corresponding to such a cylinder would be described by the circumferential wave number $n = 2$.

To ensure an elastic bifurcation mode in this shape, a maximum value of $\gamma$ (a minimum length) is required. The corresponding value of $\gamma$ can be obtained by using eq. (4.108). This function for the bifurcation load is evaluated for $n = 2$ and $n = 3$. The value of $\gamma$ for which these function intersect is the maximum value of $\gamma$ for which the collapse mode would be governed by $n = 2$. In for example figs. 4.3 to 4.5 it is found that the bifurcation mode of a cylinder under hydrostatic pressure is always described by the axial half wave number $m = 1$.

The point of intersection of the graphs is obtained by a numerical method. The results are given in fig. 4.9. It follows that a higher value of $\chi$ leads to a higher value of $\gamma$ for which the oval bifurcation shape is maintained.

Thus, to ensure an oval bifurcation shape, a thick walled test specimen can be of shorter length than a thin walled test specimen.

Buckling of a cylinder
4.13 Bifurcation amplitudes

The relation between the coefficients $\hat{W}(n,m)$ and $\hat{F}(n,m)$ can be obtained by substituting the newly obtained relation for $\lambda_c$ back into eq. (4.107). The vector contained in the nullspace of this matrix not being the null vector, the Eigenvector, describes the relation between the coefficients. By letting $\hat{W}(n,m) = 1$, the bifurcation parameter $\zeta$ describes the amount of bifurcation mode included in the solution. From analysis of the Eigenvector now follows:

$$\hat{F}(n,\bar{m}) = f^{[1]}_{1,1}(n,\bar{m})$$

$$= \left\{ \begin{array}{l}
\chi^2 \left\{ \frac{2(n^2 + \bar{m}^2)^2 - n^4}{1 + \nu} + \frac{n^2 \bar{m}^2 (\nu + n^2 + \frac{1}{2} \alpha \bar{m}^2)}{1 - \nu^2} \right\} - 6 \left\{ n^2 + 1 + \frac{1}{2} \alpha \bar{m}^2 \right\} \\
- \frac{6 \gamma \beta \chi^2 \sqrt{2} \bar{m}^2}{\{3(1 - \nu^2) + \chi^2 \bar{m}^4\} \{3(1 - \nu^2)\}^{\frac{3}{2}}} \left\{ \begin{array}{l}
4 \chi^2 (1 - \nu) \left\{ \left( n^2 + \bar{m}^2 \right)^2 - \frac{1}{2} n^4 \right\} \\
+ 2 \chi^2 n^2 \bar{m}^2 \left( \nu + n^2 + \frac{1}{2} \alpha \bar{m}^2 \right) \\
- 12 (1 - \nu^2) \left( 1 + n^2 + \frac{1}{2} \alpha \bar{m}^2 \right) \end{array} \right\} \\
+ 3(n^2 + \bar{m}^2)^2 \left( n^2 + \frac{1}{2} \alpha \bar{m}^2 + \frac{n^2 + \nu \bar{m}^2}{n^2 + \bar{m}^2} \right) \left( \frac{\sqrt{2}}{8 \gamma \beta \chi^2 \bar{m}^2} \left\{ 3 \left( 1 - \nu^2 \right) + \bar{m}^4 \right\}^{\frac{3}{2}} - 1 \right) \end{array} \right\}$$

(4.110)

Where $f^{[1]}_{1,1}(n,\bar{m})$ is given in eq. (4.95) and $\bar{m} = \pi \gamma m$.

4.14 Initial imperfections

A real life pipeline can never be free of initial geometric imperfections. These initial imperfections need to be accounted for. It is assumed that the influence of initial geometric imperfections on an externally pressurised ring, in the shape of the first order bifurcation mode, is similar to the influence of initial geometric imperfections on an externally pressurised cylinder. The second order bifurcation terms of order $O(\zeta^2)$ are neglected and hence the initial post-bifurcation behaviour is assumed to be neutral.

The relation for a pressurised ring given in eq. (3.122) is adjusted for a cylinder such that:

$$\frac{\lambda}{\lambda_c} = 1 - \frac{\xi}{\zeta + \xi} = \frac{\zeta}{\zeta + \xi}$$

(4.111)

Where $\xi$ is obtained using eq. (3.100). It describes the amount of initial imperfection in the system according to:

$$\bar{w}(\eta, \theta) = \xi \cos (n \theta) \sin (\pi m \eta)$$

(4.112)

This denotes the amount of imperfection in radial direction included in the system. The shape of this function corresponds to the first order bifurcation solution of the radial deformation component of a constrained cylinder. By rewriting eq. (4.111), an expression for the bifurcation parameter $\zeta$ is found:

$$\zeta = \frac{\lambda \xi}{\lambda_c - \lambda}$$

(4.113)

Buckling of a cylinder
4.15 Combined elastic and plastic buckling

The collapse pressure of a cylinder can be obtained by making use of the previously obtained dimensionless solutions. First these dimensionless solutions are transformed into relations for the stress resultants \( N_{xx}, N_{ss} \) and \( N_{xs} \) and stress couples \( M_{xx}, M_{xx} \) and \( M_{xx} \).

From these relations and making use of the assumptions stated earlier, the Cauchy stress tensor can be obtained for each material point of the cylinder.

The Von Mises yield criterion is adopted. Besides, it is assumed that buckling occurs at first yielding. Now it is possible to obtain a load that corresponds to the occurrence of first yielding in the cylinder, the collapse load.

4.15.1 Von Mises yield criterion

The adopted yield criterion is the von Mises yield criterion. The \( x \)-axis and \( s \)-axis are assumed to be orthogonal. The Cauchy stress tensor can be written as:

\[
\sigma_{ij} = \begin{bmatrix}
\sigma_{xx} & \sigma_{xs} & \sigma_{xz} \\
\sigma_{sx} & \sigma_{ss} & \sigma_{sz} \\
\sigma_{zx} & \sigma_{sz} & \sigma_{zz}
\end{bmatrix}
\]  

(4.114)

The invariants of the Cauchy stress tensor are denoted by \( I \). For the von Mises yield criterion it is assumed that the hydrostatic part of the stress tensor does not affect yielding. The mean hydrostatic stress tensor is obtained by averaging the coefficients at the main diagonal and multiplying this with the identity tensor. This tensor can be written as \( \frac{1}{3} \sigma_{kk} \delta_{ij} \). Here \( \delta_{ij} \) denotes the identity tensor or the Kronecker delta. Note that the first invariant of the Cauchy stress tensor is given by \( I_1 = \sigma_{kk} \). Einstein’s summation convention is applied with respect to repeated indices.

The deviatoric stress tensor is defined as the difference between the Cauchy stress tensor and the mean hydrostatic stress tensor such that:

\[
s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij}
\]  

(4.115)

Or in matrix notation:

\[
s_{ij} = \begin{bmatrix}
s_{xx} & s_{xs} & s_{xz} \\
s_{sx} & s_{ss} & s_{sz} \\
s_{zx} & s_{sz} & s_{zz}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{2}{3} \sigma_{xx} - \frac{1}{3} \sigma_{ss} & \frac{1}{3} \sigma_{ss} & \frac{1}{3} \sigma_{xx} \\
\frac{1}{3} \sigma_{ss} & \frac{2}{3} \sigma_{xx} - \frac{1}{3} \sigma_{ss} & \frac{1}{3} \sigma_{ss} \\
\frac{1}{3} \sigma_{ss} & \frac{1}{3} \sigma_{ss} & \frac{2}{3} \sigma_{xx} - \frac{1}{3} \sigma_{ss}
\end{bmatrix}
\]  

(4.116)

The invariant of the deviatoric stress tensor are denoted by \( J \). They result from the characteristic equation of the Eigenvalue analysis of this tensor. This relates the square root of the \( J_2 = \frac{1}{2} s_{ij} s_{ij} \) invariant to the uniaxial yield strength \( f_y \) of the material. First yielding occurs for \( \frac{J_2}{f_y} \):

\[
\sqrt{J_2} = k
\]  

(4.117)

Where \( k \) denotes the yield stress in pure shear \( [29] \). Solving this equation for pure uniaxial tension or pure uniaxial compression it is obtained that first
yielding occurs for:
\[ \sqrt{3J_2} = f_y \]  (4.118)

Where \( f_y \) denotes the absolute value of the uniaxial yield strength of the material. The absolute value of the compressive yield strength is assumed to be equal to absolute value of the tensile yield strength. Yielding is assumed to occur when the equivalent von Mises stress \( \sigma_{VM} = \sqrt{3J_2} \) exceeds the material yield strength \( f_y \).

By rewriting the previous equation the following criterion is obtained for material yielding:
\[
\frac{1}{2} \left\{ (\sigma_{xx} - \sigma_{ss})^2 + (\sigma_{xx} - \sigma_{zz})^2 + (\sigma_{ss} - \sigma_{zz})^2 \right\} + 3 \left( \sigma_{xx}^2 + \sigma_{ss}^2 + \sigma_{zz}^2 \right) \leq f_y^2
\]  (4.119)

For a cylinder it is assumed that all stresses with respect to radial coordinate \( z \) disappear. Hence the previous relation can be simplified to the two dimensional version of the von Mises yield criterion:
\[
\sigma_{VM} = \sqrt{\sigma_{xx}^2 + \sigma_{ss}^2 - \sigma_{xx}\sigma_{ss} + 3\sigma_{xs}^2} \leq f_y
\]  (4.120)

### 4.15.2 Occurring stresses

The previously obtained dimensionless solutions can be transformed in their dimensioned equivalents by making use of eq. (4.38). Further eq. (4.113) is adopted and the fundamental solutions are evaluated for \( \lambda \), not the critical value.

It is important to note that the initial geometric imperfection, quantified by \( \xi \), has a shape described by the axial half wave number \( m \) and the circumferential wave number \( n \). Using the von Mises yield criterion given in eq. (4.120) the occurrence of buckling is investigated.

Because the second order post-buckling coefficient \( \lambda_2 \) is neglected, it is assumed that at onset of elastic bifurcation, displacements and stresses will go to infinity. In this framework the lowest possible elastic bifurcation load always leads to buckling and a corresponding buckling mode. However for an imperfect cylinder it is possible that that it collapses in the mode corresponding to its imperfection mode.

Using eqs. (4.6), (4.9) and (4.10) the following relations are obtained for the Cauchy stress components:
\[
\sigma_{xx} = \frac{N_{xx}}{t} + \frac{12zM_{xx}}{t^3} \]  (4.121a)
\[
\sigma_{ss} = \frac{N_{ss}}{t} + \frac{12zM_{ss}}{t^3} \]  (4.121b)
\[
\sigma_{xs} = \frac{N_{xs}}{t} + \frac{12zM_{xs}}{t^3} \]  (4.121c)

Where \( z \) is the radial coordinate that equals zero at the cylinder middle surface. The stresses are extreme at either in- or outside of the cylinder wall. The outside of the cylinder wall is defined for \( z = \frac{4}{3}t \) and the inside is defined for \( z = -\frac{1}{2}t \). Combining this with the previous relations, the extreme fibre stresses can be found using:
\[
\sigma_{xx}^{[in]} = \frac{N_{xx}}{t} - \frac{6M_{xx}}{t^2} \]  (4.122a)

**Buckling of a cylinder**
The von Mises stresses at the cylinder inside and outside can be obtained by substituting these relations in eq. (4.120). A plot of typical occurring von Mises stresses in a cylinder just before occurrence of collapse is given in figs. 4.10 (simply supported end caps) and in figs. 4.11 (clamped end caps).

It follows that for a cylinder with clamped end caps high stress concentrations occur in the boundary layers. These however are not most likely not critical for collapse behaviour, because yielding in the boundary layer domains probably leads to a stress redistribution through the boundary layer domains. The stiffness gained from the rigid end caps ensures this does not lead to collapse. High von Mises stresses in the regular domain can lead to collapse and are expected to be governing in predicting collapse behaviour of a cylinder.

4.15.3 Purely plastic collapse resistance

An upper value of the collapse pressure of a cylinder should be given by the maximum plastic capacity of a pipeline. The maximum plastic capacity is given when deformation only occurs in the fundamental shape. This implies an absence of bending influences. For this analysis it is assumed that the cylinder does not ‘feel’ its boundaries. The stress in circumferential direction can be obtained from a simple force equilibrium in the cross section of the cylinder (thus a ring) giving:

\[ \sigma_{ss} = \frac{P \rho}{t} \]  

(4.123)

Now the plastic collapse capacity of a cylinder is investigated for an infinitely long cylinder where the occurrence of axial strain is assumed to be zero (plane strain). Using Hooke’s law this results to an axial stress component of:

\[ \sigma_{xx} = \frac{\nu P \rho}{t} \]  

(4.124)

When either end cap is allowed to move in axial direction the stress in axial direction is given by:

\[ \sigma_{xx} = \frac{P \rho \alpha}{2t} \]  

(4.125)

Only the regular solution domain is taken into account to obtain the plastic collapse resistance. Shear stresses are assumed to be absent. Collapse is defined when the equivalent von Mises stress in eq. (4.120) exceeds the uniaxial yield strength \( f_y \). This leads to the following plastic collapse loads \( P_p \):
Figure 4.10: Typical distribution of Mises stress [MPa] for a cylinder at collapse pressure $P = 8.70$ MPa. Here $\xi = 0.5\%$, $L = 3\,$m, $\rho = 0.4\,$m, $t = 20\,$mm, $f_y = 450$ MPa, $n = 2$, $m = 1$, $\alpha = 1$ and $\beta = 0$.

**Buckling of a cylinder**
Figure 4.11: Typical distribution von Mises stress [MPa] for a cylinder at collapse pressure $P = 8.73$ MPa. Here $\xi = 0.5\%$, $L = 3$ m, $\rho = 0.4$ m, $t = 20$ mm, $f_y = 450$ MPa, $n = 2$, $m = 1$, $\alpha = 1$ and $\beta = 1$.
\[ P_p = \frac{f_y t}{\mu \sqrt{1 - \frac{1}{2} \alpha + \frac{1}{4} \alpha^2}} \quad \text{, otherwise} \quad (4.126b) \]

Using the dimensionless identities given in eq. (4.38) and the dimensionless uniaxial yield strength given in eq. (3.130), these solutions for plastic collapse load can be written to their dimensionless equivalents \( \lambda_p \):

\[
\lambda_p = \frac{f_y}{\sqrt{1 - \frac{1}{2} \alpha + \frac{1}{4} \alpha^2}} \quad \text{, for infinitely long cylinder} \quad (4.127a)
\]

\[
\lambda_p = \frac{f_y}{\sqrt{1 - \frac{1}{2} \alpha + \frac{1}{4} \alpha^2}} \quad \text{, otherwise} \quad (4.127b)
\]

Dimensionless purely plastic collapse loads for various steel grades (given in table 5.1) are displayed in fig. 4.12.

![Graph showing dimensionless plastic collapse loads for various steel grades](image)

Figure 4.12: Dimensionless plastic collapse loads according to eq. (4.127) for various steel grades.

From this figure follows that for thin walled cylinders (small \( \chi \)-values), plastic collapse loads are relatively high compared to thick walled cylinders. This effect partly explains collapse of thin walled pipelines to occur more in the elastic regime, while collapse of thick walled pipelines is governed by their plastic behaviour.

### 4.15.4 Combined elastic and plastic collapse

To predict the collapse load of a cylinder it is necessary to define the occurrence of collapse. In this analytical cylinder model, collapse is defined to occur at first material yielding in the regular domain. Hence high von Mises stresses in the boundary layer domains are assumed not to lead to collapse. First yielding is assumed to occur for the first load for which the von Mises stress equals the material uniaxial yield strength.

**Buckling of a cylinder**
A formulation for the occurring equivalent stresses is obtained earlier. A typical equivalent stress distribution for a simply supported cylinder is given in fig. 4.10, and a typical distribution for a clamped cylinder is given in fig. 4.11. The maximum value of the occurring von Mises stress through the cylinder wall is obtained using an iterative procedure. For several grid points in the cylinder domain the occurring von Mises stresses are obtained. The grid is refined around the location that possesses the highest occurring stress. This is done several times to obtain the location and value of the maximum von Mises stress. Now the von Mises criterion is checked. If yielding does not occur, the applied pressure is increased. If it does occur, the applied pressure is decreased. This procedure is repeated until the von Mises criterion for first yielding is satisfied up to some threshold range. The corresponding load is reported as the collapse load belonging to the initial imperfection shape.

After obtaining the collapse load corresponding to the initial imperfection shape and size, the elastic collapse load and mode shape is analysed. The cylinder is assumed to collapse at the lowest value of these two loads. The occurring collapse mode shape is the mode corresponding to this load.

When the elastic collapse mode shape is equal to the mode shape of the initial imperfection the structure will always collapse at a load lower than its elastic collapse load.

In chapter 5 results obtained using this method are compared to results obtained using finite element analysis. Some sample values for cylinders with $\chi = \frac{1}{20}$ and $\chi = \frac{1}{100}$ are given in tables 4.1 and 4.2. Collapse is investigated for some of the steel grades given in table 5.1. Initial imperfection shapes are taken close to the preferred elastic bifurcation shape.

The results given in these tables suggest that although end cap pressure leads to a lower bifurcation load it can lead to a higher collapse load. This is due to the fact that the axial compressive stress resulting from this load reduces the von Mises stress due to pressure on the cylinder. It follows that this effect appears for relatively thick walled cylinder (high values of $\chi$). For relatively thin walled cylinders (low values of $\chi$), the reduction in bifurcation load is more significant than the gain in collapse load due to end cap pressure.

It is generally true that clamped boundaries lead to higher collapse loads than simply supported boundaries when other parameters are kept constant. Thick walled cylinders tend to collapse relatively close to their plastic collapse load, while thin walled cylinders tend to collapse more into the elastic deformation regime. As found in fig. 3.3, the loading path is first governed by a linear response and later by a post-buckling response. If plasticity starts to develop in the pre-buckling (linear) part, for small values of $\zeta$, the structure’s deformation shape at occurrence of collapse is mainly described by the shape of the initial imperfection. If plasticity starts to develop after occurrence of buckling, for larger values of $\zeta$, the structure’s deformation shape at occurrence of collapse will mainly be governed by its bifurcation shape. In practice, the former is seen as plastic collapse behaviour and the latter as elastic collapse behaviour.
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Table 4.1: Collapse load at first yielding for $\chi = \frac{1}{20}$ and $\xi = 0.01$
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<td>0.824</td>
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Table 4.2: Collapse load at first yielding for \( \chi = \frac{1}{100} \) and \( \xi = 0.01 \)
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Chapter 5

Verification and discussion results

5.1 Introduction

In this chapter, previously obtained analytical results for the local buckling collapse behaviour of a pipeline, are compared to results from finite element analysis (FEA) and published experimental work.

To perform the FEA use is made of the commercial software code ANSYS 14.0. To be able to perform this analysis for a vast combination of input parameters APDL-scripts (ANSYS parametric design language) are developed. Examples of applied scripts are given in appendix C. These will be explained later.

In FEA dimensions have to be introduced again. Reporting of the results in this chapter is done in dimensionless identities however. Therefore the previously introduced dimensionless identities are used to rewrite the results to their dimensionless equivalents.

Two types of finite element models (FEM) are developed. A ring model is developed to verify the results obtained in chapter 3 and a cylinder model is developed to verify the results obtained in chapter 4. In FEA it can be difficult or even impossible to obtain the bifurcation behaviour of a perfect structure. If, for a perfect structure, no arbitrary perturbations from the fundamental solution path occur, this fundamental solution is likely to be the obtained equilibrium state, even for loads beyond the bifurcation load (if numerical irregularities are insignificant). To ensure that bifurcation behaviour is found, a small initial geometric imperfection is introduced into the system. The response obtained from this imperfect system can be extrapolated to obtain the values corresponding to the bifurcation behaviour of the perfect structure.

A large displacement analysis is performed (the geometric stiffness matrix is recalculated after each load step) and depending on the analysis, the Newton-Raphson or the arc-length solver is used. Here the Newton-Raphson solver uses a load controlled algorithm, while the arc-length method uses a displacement controlled algorithm. This results to the fact that the arc-length solver is able to capture unstable post-buckling behaviour, while the Newton-Raphson solver is only capable to capture stable branches of the post-buckling behaviour solution.

Properties and information of the used elements and the performed FEA are
5.1.1 Ring model

To model the ring representation of a pipeline, a system is developed using BEAM188 elements. These elements in fact are three dimensional elements. Because only a two dimensional representation is required, the elements are constrained in a fixed plane. The used beam elements follow Timoshenko beam theory. In the analytical analysis Euler-Bernoulli beam theory is applied. The difference is that Timoshenko theory accounts for shear to occur within the beam cross section, while Euler-Bernoulli theory neglects this. As long as the ring can be considered as relatively thin it can be assumed that the difference between these theories is insignificant. For thicker rings the application of Timoshenko beam theory is assumed to lead to slightly lower values for the bifurcation load compared to Euler-Bernoulli beam theory. This because including this shear effectively reduces the beam stiffness.

The ring wall cross section is assumed to behave rigidly. This means that the wall thickness does not change under straining in the ring and hence the wall thickness is assumed to be constant during deformation. Hydrostatic pressure is applied perpendicular to axial direction of the beam elements, directed towards the ring centre. It is assumed that this direction stays perpendicular during deformation. In fig. 5.1 the deformed and undeformed configuration of the finite element representation is given of a pressurised ring. Different elements are displayed in various colours.

Initial geometric imperfections are introduced either by moving all ring nodes by a small random initial translation of the order of $0.1\%$ of the ring mean radius $\rho$, or by introducing a small imperfection in a shape close to the first order bifurcation shape of the ring.
The ring is constrained by the introduction of multi point constraints (MPC). This ensures that rigid body translations and rotations are prevented. All nodes that are part of the ring are connected to a pilot node by a force-distributed constraint. Influence of mesh and element sizes is investigated, but not reported in this thesis. It is tried to use as few as possible elements, while the results are within close range of the structure modelled by significantly more elements. Further the buckling mode found from the analysis always has to be compared to the imperfection mode introduced before application of load.

A typical APDL script for the analysis of a pressurised ring is given in appendix C.1.

5.1.2 Cylinder model

To verify the relations obtained in chapter 4, it is necessary to include an additional spatial dimension in the FEA. This is done by making use shell elements to describe a cylinder. The element types that are used are SHELL181 elements. Contrary to the beam elements used for the ring model, these elements account for change in wall thickness during deformation. It is assumed that this influence is small.

The SHELL181 element is based on Mindlin-Reissner shell theory. This theory takes into account transverse shear (deformations). The difference of this theory and the applied theory in chapter 4 based on the Kirchhoff-Love assumptions, is analogous to the difference due to Timoshenko and Euler-Bernoulli beam theory. For relatively thin cylinders the effect of this difference is assumed to be insignificant and for relatively thick cylinders the effect is assumed to be small, but a slightly lower bifurcation load is expected due to including transverse shear in the FEA.

Hydrostatic pressure is applied to the cylinder walls. This pressure is directed perpendicular to the shell middle surface directed to the cylinder inside. The pressure stays perpendicular to the shell middle surface during deformation.

In fig. 5.2 the deformed configuration of a typical finite element model (FEM) of a cylinder with end caps is displayed. The various elements can be distinguished by the variation in colour.

The end caps are modelled by rigid surface MPC’s. The radial displacement component is constrained and either simply supported or clamped cylinder wall-to-end cap connections are introduced by setting the corresponding options for the MPC elements. The nodes connected by the MPC elements are constrained in a rigid plane. These planes are connected to a pilot node at each end. One of the pilot nodes is constrained in all directions, while the other is only allowed to move in axial direction. Hence no resulting axial force shall be transferred to the pilot nodes (when end cap pressure is lacking).

Initial imperfections are introduced in a shape close to the first order regular bifurcation mode. Also for a cylinder, it is verified that the used mesh is fine enough such that further refinement of the mesh does not lead to a significant improvement of the estimate solution. It is ensured that bifurcation occurs in the mode that can be expected by the introduction of the imperfection mode. To obtain the lowest possible bifurcation load it is necessary to analyse various initial imperfection mode shapes.

A typical APDL script for the buckling analysis of a constrained cylinder is given in appendix C.2.
5.1.3 Bifurcation parameter

It is convenient to report the results of FEA in dimensionless form. An important parameter in the analytical results is the bifurcation parameter $\zeta$. The value of this parameter can be obtained from the maximum radial deformation value.

The dimensionless radial deformation solution for a ring is given by eq. (3.108). The maximum absolute value for $w$ is found by substituting $\cos(n\theta) = -1$ such that:

$$w_{\text{max}} = -\lambda \chi^2 - \zeta^2 - \frac{9 (3 + \chi^2) \lambda^2}{4n^2 (3 + n^2 \chi^2)}$$

(5.1)

For a cylinder the same relation can be used, but it has to be noted that $\lambda$ is defined slightly different for a ring and a cylinder. The value of the bifurcation parameter can be obtained by solving the identity above for $\zeta$ using the assumption that $\zeta$ is positive. Note that $w_{\text{max}}$ is negative when directed inwards and positive when directed outwards.

5.2 Perfect ring initial post-bifurcation

To obtain the initial post-bifurcation behaviour of a perfect ring using FEA is necessary to introduce a small imperfection. For this analysis, a random disturbance of the node location of the order of 0.1% of the mean ring radius is introduced. The smaller the imperfection size, the more the loading path should resemble to the loading path of a perfect ring and the clearer the kink in the graph should be observed.

In figs. 5.3 and 5.4 the analytical result of the initial post-bifurcation behaviour of a ring is compared to the initial post-bifurcation behaviour obtained from FEA for various typical values of $\chi$. The results of several FEA runs have been given for which each run has a different random imperfection.
Figure 5.3: Amount of buckling mode with respect to load for perfect ring

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Figure 5.4: Amount of buckling mode with respect to load for perfect ring
The influence of initial imperfections in the FEA results is observed by smoothing of the expected kink at $\zeta = 0$ in the graph. The expected kink is the bifurcation point. All buckling modes that are observed in this analysis are described by the circumferential wave number $n = 2$. Further, it appears that the analytical results slightly under-predict the initial post-bifurcation stability for a ring. The FEA results to higher loads than the analytical analysis, to obtain the same amount of bifurcation mode. The used values of $\lambda_c$ are the analytical values and for $n = 2$ it is valid that $\lambda_c = 1$. If the FEA results given in figs. 5.3 and 5.4 are extrapolated to remove the influence of initial imperfections, it can be concluded that the FEA leads to slightly higher bifurcation loads than the analytical analysis. Still, the results are consistent for various values of $\chi$. It appears that the influence of $\chi$ is not significant. This corresponds to the approximation that is obtained in eq. 3.107 that is valid for small values of $\chi^2$ with respect to unity.

5.3 Imperfect ring initial post-bifurcation

Now the influence of the imperfection size $\xi$ on the initial post-bifurcation behaviour of a ring is investigated by FEA. Because in the previous section is verified that a perfect ring prefers to buckle in an oval shape in this section the influence of an oval shaped imperfection is investigated. In fig. 5.5 the bifurcation parameter is plotted against the dimensionless load factor divided by its analytical bifurcation load. This is done for various imperfection sizes and values of $\chi$.

From this figure appears that the analytical formulation is a close match to the FEA results. For larger values of the bifurcation parameter $\zeta$, the analytical and FEA results start to deviate slightly. This is in correspondence with the deviation between analytical and FEA results that is found for a perfect cylinder in figs. 5.3 and 5.4.

In fig. 5.5, the dimensionless solutions for various values of $\chi$ are very similar.

5.4 Collapse load of a ring

The API steel grades given in table 5.1 are used in the analysis of a ring. The yield strength values used in calculations are the values in MPa.

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<td>X70</td>
<td>70,000</td>
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</table>

Table 5.1: API steel grades

Note that the API specification 5L uses slightly different values for the yield strength of X60 and X70 of 415 MPa and 485 MPa respectively.
Figure 5.5: Amount of buckling mode with respect to load for ring with initial imperfections. Lines are analytical results and points are FEA results.

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The collapse load of a ring is analysed for two possible definitions. The collapse definitions are given in section 3.15. The first definition assumes collapse to occur at first material yielding and the second assumes it to occur at the full development of a plastic hinge. The former is also referred to as the Timoshenko criterion and the latter is referred to as the DNV criterion.

To obtain FEA results for the collapse at first yield, an elastic material is used and the response of the model is analysed. The load that corresponds to first material yielding is calculated and reported as the collapse load. To perform FEA verification of the occurrence of collapse at the full development of a plastic hinge, an elastic-perfectly plastic material model is introduced. Now the collapse load is defined as the maximum load for which the system is still stable. It is likely that the system collapses before the full development of a plastic hinge.

The collapse loads resulting from the criteria given in eqs. (3.131) and (3.140), combined with the FEA results, are given in figs. 5.6 and 5.7. Analytical results are plotted as lines and FEA results are plotted as points. Imperfection shapes in this analysis are characterised by the circumferential wave number $n = 2$.

It follows that rings with smaller $\chi$-values collapse closer to their elastic bifurcation load. Further it appears that rings with relatively small imperfection sizes collapse closer to their elastic bifurcation load than rings with larger imperfections. Besides, a higher yield strength of the applied material leads to higher collapse loads, closer to the elastic bifurcation loads.

In industry practice one tends to say that thick walled rings and cylinders have a plastic collapse behaviour and thin walled cylinders have an elastic collapse behaviour while in fact a single mechanism comes into play. Thick walled cylinders (high values of $\chi$) allow less deformation before plasticity sets in than thin walled cylinders (low values of $\chi$). Hence they will collapse in the deformation domain that is governed by the fundamental solution and the imperfection shape. Thin walled cylinders will collapse in the deformation domain that is governed by the bifurcation shape.

An elastic-perfectly plastic ring is expected not to be able to develop a full plastic hinge before loss of stability occurs. Besides, it is also not expected to lose stability at first yielding. Hence it is expected that FEA results for the elastic-perfectly plastic material model should be between the results for the criterion of collapse at first yield and the criterion of collapse at full development of a plastic hinge. In figs. 5.6 and 5.7 this expectation is confirmed.

### 5.5 Bifurcation load of a cylinder

#### 5.5.1 Comparison FEA

The analytical solutions for the bifurcation load of a cylinder, as given in fig. 4.6, are validated using FEA. The results are plotted in fig. 5.8. To obtain the critical bifurcation load of a cylinder, a small initial imperfection is introduced in a shape described by the circumferential wave number $n$ and the axial half-wave number $m = 1$. The cylinder is likely to buckle in the shape of the initial imperfection. Therefore imperfections with several circumferential wave numbers $n$ are analysed. It is verified that the cylinder buckles in the shape of the imperfection. The bifurcation load is obtained by extrapolating the $\zeta - \lambda$ load graph of the imperfect cylinder to the perfect cylinder case. The shape leading
Figure 5.6: Influence initial imperfection size and steel grade on collapse load of a ring. Lines are analytical results and points are FEA results.

Verification and discussion results
Figure 5.7: Influence initial imperfection size and steel grade on collapse load of a ring. Lines are analytical results and points are FEA results.
to the lowest possible bifurcation load is reported as the bifurcation shape and the corresponding load as the critical bifurcation load.

It appears that higher values for the bifurcation loads are found for FEA results compared to analytically obtained results. This could be explained by the fact that in the Galerkin procedure that is used to obtain the bifurcation load in section 4.10, the boundary layer solution part of the fundamental solution is neglected. So in fact more stiffness should be inherited from the boundaries than is taken into account in the analytical formulation in eq. (4.108).

Besides, according to FEA the cylinder bifurcates in a higher mode than is obtained through the analytical formulation. This could also contribute to the reduced stiffness of the analytical model. One could say that the length of the cylinder analysed in the analytical model in fact is a little larger than expected through the value of $\gamma$.

The trend of the analytical results and the FEA results corresponds. The influence of the factors $\alpha$ and $\beta$ also corresponds.

It must be noted that although long cylinders have a stable initial post-bifurcation load, from FEA appears that shorter cylinder develop a snap-through bifurcation behaviour. First, stability is lost, but after increase of deformations stability is regained. The structure ‘snaps’ to another equilibrium state. When plasticity is included it is very unlikely that the snap-through behaviour is observed. Larger deformations lead to plasticity that probably leads to instability before the snap-through behaviour is observed.

### 5.5.2 Comparison experimental results

For a collapse test, the influence of pipeline properties on the collapse load of a thin walled pipeline section has been experimentally investigated by Fairbairn [13], Windenburg and Trilling [52], Sturm [43]. In fig. 5.9 the results of these experiments are compared to the critical bifurcation load obtained by eq. (4.108).

It is assumed that, for small initial imperfections, collapse occurs close to the critical bifurcation load. Thin walled cylinders (small value of $\chi$) are observed to collapse in the elastic domain. Hence the collapse load of a thin pipeline section is expected to be a good measure of the corresponding bifurcation load. This is valid as long as the purely plastic collapse load is much higher than the bifurcation load.

From this figure follows that especially for relatively small critical loads, the experimental and analytical results are close. For higher bifurcation loads it can be expected that plastic yielding starts to come into play and hence a reduction of the collapse load should be observed. Further, it appears that for thinner cylinders (small values of $\chi$), the analytical expression gives a better description for the collapse load than it does for thicker cylinders (higher values of $\chi$).

### 5.6 Collapse load of a cylinder

The analytical collapse load of a cylinder is obtained using an iterative procedure described in section 2.15.3. The load for which first yielding occurs in a cylinder is obtained through FEA by using an elastic model.

The response of this model is analysed. The load for which first yielding occurs in the regular domain, is reported as the collapse load. Results are given
Figure 5.8: Bifurcation loads of a constrained cylinder. Lines are analytical solution and points are FEA solutions. The circumferential wave number of the bifurcation shape $n$ can be distinguished by colour of the point, such that $n \in \{2, 3, 4, 5, 6, 7, \ldots\}$.
Fairbairn [13] ($\alpha = 1$)
Windenburg and Trilling [52] ($\alpha = 1$)
Sturm [43] ($\alpha = 0$)
Sturm [44] ($\alpha = 1$)

Figure 5.9: Comparison analytical bifurcation loads ($\beta = 0$) to experimental results

Verification and discussion results
in tables 5.2 and 5.3.

<table>
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<th>β</th>
<th>γ</th>
<th>n</th>
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Table 5.2: Collapse load at first yielding for a cylinder with \( \chi = 1/20 \) and \( \xi = 0.01 \). Comparison analytical and FEA results

For all investigated cases the analytical results underestimate the collapse load with respect to the FEA verifications. For cylinders with very small values of \( \gamma \) the analytical results are a close match to the FEA results. For the cylinder with \( \chi = 1/100 \) its collapse behaviour is expected to be governed by its plastic collapse load. It appears that the relative difference is the analytical results and the FEA results are governed by the underestimation of the bifurcation load as given in fig. 5.8. For the relatively thick walled cylinder, with \( \chi = 1/20 \), it is expected that its collapse behaviour is governed by its plastic collapse load given in eq. (4.127). The collapse loads are much smaller than the purely elastic collapse loads and of order of the plastic collapse load. Results generally are of correct order but there is a significant difference between analytical and FEA results for some combinations of parameters.

### 5.7 First change in bifurcation mode

In section 4.12 a relation is obtained for \( \chi \) and the maximum value of \( \gamma \) for which the oval shaped bifurcation mode occurs. The corresponding value of \( \gamma \) is obtained by determining the value of \( \gamma \) for which the relations of \( \gamma \) and \( \lambda_c \) for \( n = 2 \) and \( n = 3 \) intersect.

These results are verified using FEA. A small imperfection in the shape characterised by either \( n = 2 \) or \( n = 3 \) is introduced. The bifurcation load is obtained for several values of \( \gamma \). The value of \( \gamma \) where these curves intersect is reported. This results to fig. 5.10.

From this figure follows that the analytical results over-predict the FEA results. It appears that, in a collapse test to investigate collapse resistance of a real life pipeline, the maximum allowable value for \( \gamma \) should be lower than the analytical results indicate. Hence the length of the tested specimen should be more than the analytical results suggest.
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Table 5.3: Collapse load at first yielding for a cylinder with $\chi = 1/100$ and $\xi = 0.01$. Comparison analytical and FEA results

However, fig. 5.10 indicates that the trend found in analytical results is similar to the trend found in FEA results. If the analytical results are scaled by a factor 0.63 it proves to be a very close match to the FEA. The rescaled results are also given in fig. 5.10. Note that this scaling factor does not have physical meaning and is found by trial and error.

**Verification and discussion results**
Figure 5.10: Critical $\gamma$ for mode switch. Comparison FEA and analytical results for $\alpha = 0$ and $\beta = 0$. For a combination of $\gamma$ and $\chi$ located below the curve, a buckling shape characterised by $n = 2$ is expected. When located above the curve, a higher buckling mode is expected.
Chapter 6

Conclusion and recommendations

6.1 Conclusions of this research

In this thesis an overview has been given of the local buckling collapse behaviour of a pipeline. First, the collapse behaviour of a ring has been analysed and later, the collapse behaviour of a cylinder is investigated. For a ring the initial post-buckling behaviour is analysed. This is found to be stable. FEA shows that this behaviour is even more stable than analytically obtained (higher value of $\lambda_2$).

It is found that the collapse behaviour of a pipeline is highly dependent on the value of $\chi$, the ratio between wall thickness and diameter. Relatively thick walled pipelines tend to collapse in the plastic domain while relatively thin walled pipelines tend to collapse in the elastic domain. The plastic domain is defined as the domain for which buckling is not yet clearly observed at the onset of collapse. The fundamental solution governs the equilibrium solution at the onset of collapse. The elastic domain is defined as the domain for which the occurrence of buckling can be observed at the onset of collapse. Now the bifurcation solution governs the equilibrium solution at the onset of collapse.

This difference can be explained by the fact that for thin pipelines the purely plastic collapse load $\lambda_p$ is much higher than the critical (elastic) load $\lambda_c$. For a cylinder, the initial post-bifurcation behaviour is not investigated, but it is assumed to be neutral. The collapse load can never by higher than both $\lambda_p$ and $\lambda_c$. This results to collapse in the elastic domain. For thick walled pipelines, it is likely that the critical load is much higher than the purely plastic collapse load. This results to collapse in the plastic domain.

A cylinder is analysed for various types of boundary constraints. The application of end cap pressure has been analysed through a factor $\alpha$. For $\alpha = 0$, end cap pressure is neglected and for $\alpha = 1$ end cap pressure is taken into account. Further, two types of boundary constraints are analysed such that $\beta = 0$ for a simply supported connection of the end caps to the cylinder walls and $\beta = 1$ for a clamped connection.

It is found that including the end cap pressure leads to a reduction in bifurcation load and a possible reduction of the order of the bifurcation mode.
shape. This is the case for relatively short cylinders in particular. For long cylinders, the influence of $\alpha$ on the bifurcation load is insignificant. However $\alpha$ proves to influence purely plastic buckling. Due to the introduction of an axial compressive stress component, the equivalent von Mises stress in the pressurised cylinder is reduced. In this thesis, collapse of a cylinder is defined using the von Mises failure criterion. Hence the introduction of an axial stress component (due to hydrostatic pressure) leads to an increase in purely plastic collapse load. This is also valid for long cylinders. End cap pressure reduces the critical load for thin walled cylinders that tend to collapse in the elastic domain. For thick walled cylinders, whose collapse is governed by the purely plastic collapse load, end cap pressure tends to increase collapse load.

A clamped cylinder has a slightly higher collapse load than a simply supported cylinder. This is explained by the stiffness of the system being higher for a clamped cylinder than for a simply supported cylinder. Clamped cylinders tend to collapse in a higher (or equal) collapse mode shape than simply supported cylinders. This is also caused by additional stiffness gained from clamped edges compared to simply supported edges. The influence of these boundary conditions is only significant for relatively short cylinders. For long cylinders this can be neglected.

The effect of cylinder length is modelled by introduction of dimensionless parameter $\gamma$. Short cylinders will have higher collapse loads bounded by their purely plastic collapse load. The collapse mode shape of short cylinders is generally described by higher circumferential wave numbers than the collapse mode of longer cylinders. The effect of the length of a cylinder on the collapse mode is investigated. A critical value is determined for $\gamma$ when the bifurcation mode shape switches its circumferential wave number from $n = 2$ to $n = 3$. The corresponding value of $\gamma$ is of interest because this value corresponds to the minimum required length of a pipeline section in a collapse test. A hybrid relation has been obtained by combining the analytical relation for this value of $\gamma$ and results obtained through FEA.

Analytically obtained values for bifurcation load $\lambda_c$ prove to be lower than values obtained through FEA. This can be explained by the fact that the boundary layer part of the fundamental solution is neglected when obtaining an analytical value for $\lambda_c$. Hence some stiffness resulting from the fundamental solution is lost. This results to a slightly lower value for the bifurcation load required to bifurcate the system. This deviation in $\lambda_c$ is found to reflect in the collapse load analysis. Collapse loads for cylinders obtained by analytical analysis are lower compared to collapse loads obtained through FEA.

### 6.2 Recommendations for further work

In this thesis the influence of material properties is largely neglected. Only basic material models are taken into account such as an elastic and an elastic-perfectly plastic material model. Especially for thick walled pipelines it is expected that the material modelling has a significant influence on the bifurcation behaviour. Because thin walled cylinders tend to collapse in the elastic domain, this significance is expected to be much less.

The structure’s material properties are assumed to be homogeneous and isotropic. The influence of inhomogeneity and anisotropy is neglected. For
pipelines manufactured by the UOE process, a significant reduction of compressive yield strength in hoop direction occurs due to the Bauschinger effect. This results to material properties to become anisotropic. This can have a significant influence on the buckling behaviour of a (relatively thick) pipeline.

In this thesis, the wall thickness of a pipeline is assumed to be constant. Introducing wall thickness variations leads to local weak spots that could induce the occurrence of buckling earlier than expected. The influence of such weak spots could be investigated.

From FEA it is found that constrained elastic cylinders develop snap-through behaviour after occurrence of buckling. This behaviour is only found for shorter cylinders. Long cylinders tend to have a stable initial post-buckling behaviour. The initial post-buckling behaviour of cylinders could be investigated. This can lead to a threshold value for the length at which the post-buckling behaviour turns from stable to snap-through.

Shell theory is used to model the pipeline. Therefore the Kirchhoff assumptions are introduced. For thick walled pipelines in particular, transverse shear stresses are expected to have a significant influence on the buckling behaviour of a pipeline. Further the plane normal to the shell middle surface is not expected to stay straight during deformations (due to shear). The influence of these assumptions can be investigated.

In this thesis only the influence of external pressure on collapse behaviour of a pipeline is investigated. The critical scenario for a deep water pipeline with respect to collapse, is likely to be the sag bend. Here bending strains are introduced in the pipeline combined with high external pressure. It is important to account for bending (strain) introduced in the system when developing a design formulation.

The end caps attached to the pipeline specimen’s ends are modelled to be fully rigid. The influence of adding some flexibility to these end caps can be investigated.
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Appendix A

Profound analysis ring bifurcation

In this appendix the derivation of the solution of the bifurcation equation of a ring is given. The bifurcation equations are given in eq. (3.62) and read:

\[
\begin{align*}
3N_{1,\theta} + \chi^2 M_{1,\theta} &= 3\lambda_c \chi^2 (\varphi_1 - \varphi_1) \\
\chi^2 M_{1,\theta\theta} - 3N_1 &= 3\lambda_c \chi^2 (e_1 - \kappa_1)
\end{align*}
\]

(A.1a)  
(A.1b)

These are the bifurcation equations based on Sanders shell equations. The underlined terms are dropped for the DMV shell relations.

\[
\begin{align*}
\varphi_1 &= -w_{1,\theta} + v_1 \\
M_1 &= \kappa_1 = \varphi_{1,\theta} \\
N_1 &= e_1 = v_{1,\theta} + w
\end{align*}
\]

(A.2a)  
(A.2b)  
(A.2c)

A.1 Sanders shell

A.1.1 General solution system

Substituting eq. (A.2) into eq. (A.1) and rewriting the system leads to the following equilibrium equations:

\[
\begin{align*}
v_{1,\theta\theta} + (1 + \lambda_c \chi^2) w_{1,\theta\theta} + v_{1,\theta} + (1 + \lambda_c \chi^2) w_1 &= 0 \\
\chi^2 w_{1,\theta\theta} - (3 + \chi^2) v_{1,\theta\theta} - 3w_{1,\theta} &= 0
\end{align*}
\]

(A.3a)  
(A.3b)

General solutions to this system of differential equations can be obtained by substituting the following in the system above:

\[
\begin{align*}
v_1 &= \hat{V}_1 \exp (p\theta) \\
w_1 &= \hat{W}_1 \exp (p\theta)
\end{align*}
\]

(A.4a)  
(A.4b)

This leads to the linear system:

\[
\begin{bmatrix}
(1 + p^2) & (1 + p^2) (1 + \lambda_c \chi^2) \\
-(3 + \chi^2) p^2 & (p^2 \chi^2 - 3) p
\end{bmatrix}
\begin{bmatrix}
\hat{V}_1 \\
\hat{W}_1
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(A.5)
This system has non-trivial solutions if and only if its determinant is equal to zero. This leads to the characteristic polynomial of sixth degree for \( p \):

\[
p^6 + \left\{ 2 + \lambda_c (3 + \chi^2) \right\} p^4 + \left\{ 1 + \lambda_c (3 + \chi^2) \right\} p^2 = 0 \quad (A.6)
\]

This characteristic equation has six solutions such that:

\[
p^2 \in \{-1 - \lambda_c (3 + \chi^2), -1, 0\} \quad (A.7)
\]

Hence:

\[
p \in \{i\sqrt{\Lambda}, -i\sqrt{\Lambda}, i, -i, 0, 0\} \quad (A.8)
\]

Where:

\[
\Lambda = 1 + \lambda_c (3 + \chi^2) \quad (A.9)
\]

Now for each Eigenvalue \( p \), the corresponding Eigenvector \( \mu \) is determined. This is done by substituting the Eigenvalue into the matrix of eq. (A.5). The nullspace of the new matrix is obtained. It appears that the nullity of all nullspaces is one. Hence the corresponding Eigenvector is equal to the vector contained in the nullspace. The Eigenvalue \( p = 0 \) is a repeated Eigenvalue. Because the nullity of the corresponding Eigenspace is equal to one, it is necessary to obtain a second independent Eigenvector corresponding with the repeated Eigenvalue. This can be done by performing the following substitution in eq. (A.3):

\[
\begin{bmatrix} v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta + \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \exp (p\theta) \quad (A.10)
\]

Where \( p = 0 \) and \( \begin{bmatrix} 1, 0 \end{bmatrix}^\top \theta + \begin{bmatrix} \rho_1, \rho_2 \end{bmatrix}^\top \) represents the additional independent Eigenvector. After the substitution and solving for \( \rho_1 \) and \( \rho_2 \), it is obtained that \( \rho_1 \) is arbitrary and \( \rho_2 = -\left(1 + \frac{\lambda_c}{3\chi}\right)^{-1} \). It is chosen that \( \rho_1 = 0 \). Now the following Eigenvectors \( \mu \) corresponding to the Eigenvalues given in eq. (A.8) are obtained:

\[
\mu \in \left\{ \begin{bmatrix} 1 + \lambda_c \chi^2 \\ -i\sqrt{\Lambda} \end{bmatrix}, \begin{bmatrix} 1 + \lambda_c \chi^2 \\ i\sqrt{\Lambda} \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} (1 + \lambda_c \chi^2) \theta \\ -1 \end{bmatrix} \right\} \quad (A.11)
\]

Now the general solution can be obtained by the superposition of all possible solutions:

\[
\begin{bmatrix} v_1 \\ w_1 \end{bmatrix} = \sum_{i=1}^{6} C_i \mu_i \exp (p_i\theta) \quad (A.12)
\]

Using Euler’s formula the general solution can be rewritten to:

\[
v_1(\theta) = C_1 \left( 1 + \lambda_c \chi^2 \right) \cos \left( \sqrt{\Lambda} \theta \right) + C_2 \left( 1 + \lambda_c \chi^2 \right) \sin \left( \sqrt{\Lambda} \theta \right) + C_3 \cos (\theta) + C_4 \sin (\theta) + C_5 \left( 1 + \lambda_c \chi^2 \right) \theta + C_6
\]

\[
w_1(\theta) = C_1 \sqrt{\Lambda} \sin \left( \sqrt{\Lambda} \theta \right) - C_2 \sqrt{\Lambda} \cos \left( \sqrt{\Lambda} \theta \right) + C_3 \sin (\theta) - C_4 \cos (\theta) - C_5
\]
A.1.2 Boundary conditions

Now that the general solutions are obtained, they can be used to obtain the solution that satisfies the boundary conditions. The boundary conditions used in this analysis constrain rigid body displacements and rotations and besides continuity needs to be ensured on $\theta = 0$ and $\theta = 2\pi$. The continuity conditions follow from the following three boundary conditions:

$$
v(0) = v(2\pi) \quad (A.14a)
$$

$$
w(0) = w(2\pi) \quad (A.14b)
$$

$$
w,\theta(0) = w,\theta(2\pi) \quad (A.14c)
$$

Rigid body constraints give two equations to fix the centre of gravity and one equation to constrain the rigid body rotation. Because a homogeneous material is assumed the centre of gravity coincides with the centre of area. Conventions from figure 3.1 are used. The centre of are will be constrained along the $x_1$ and the $x_2$-axis. The location of the centre of area along an axis can be obtained by dividing the first moment of area ($S_{x_1}$ or $S_{x_2}$) along that axis by the total area. Because the location of the centre of area is set to $x_1 = 0$ and $x_2 = 0$ is is sufficient to let the first moment of area in each direction be equal to zero:

$$
S_{x_1} = \int_A \{(p + d) \cdot \hat{i}\} \, dA = 0 \quad (A.15a)
$$

$$
S_{x_2} = \int_A \{(p + d) \cdot \hat{j}\} \, dA = 0 \quad (A.15b)
$$

Where $A = [-\frac{1}{2} t, \frac{1}{2} t] \times [0, 2\pi]$ and $dA = \rho \, d\theta$. $\hat{i}$ and $\hat{j}$ are the unit vectors in $x_1$ and $x_2$ direction respectively and the dot denotes the inner product of two vectors. It can be obtained that:

$$
p + d \rho = (1 + w) \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} + v \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \quad (A.16)
$$

Here $v$ and $w$ represent the dimensionless displacement functions $v(\theta)$ and $w(\theta)$. Making the first moments of area dimensionless by dividing by $\rho^2 t$ leads to the following two boundary conditions that constrain the centre of area:

$$
\int_0^{2\pi} \{(1 + w) \cos(\theta) - v \sin(\theta)\} \, d\theta = 0 \quad (A.17a)
$$

$$
\int_0^{2\pi} \{(1 + w) \sin(\theta) + v \cos(\theta)\} \, d\theta = 0 \quad (A.17b)
$$

Only one more boundary condition is needed. This is the condition to constrain the rigid body rotation. The only vector component that represents a rotation is $v \hat{t}$. No rigid body rotation occurs if:

$$
\int_0^{2\pi} v \, d\theta = 0 \quad (A.18)
$$

Reintroduction of the following expansions allows for rewriting the boundary conditions for each order of expansion:

$$
v = v_0(\lambda_c) + \zeta v_1 + \zeta^2 v_2 + \ldots \quad (A.19a)
$$

Profound analysis ring bifurcation
\[ w = w_0(\lambda_c) + \zeta w_1 + \zeta^2 w_2 + \ldots \] (A.19b)

In these expansions \( v_0(\lambda_c) \) and \( w_0(\lambda_c) \) are the fundamental solutions as obtained in eq. (3.57) where \( \lambda \rightarrow \lambda_c \). Furthermore \( v_1 \) and \( w_1 \) are the general solutions as obtained in eq. (A.13). Now the boundary conditions corresponding to the fundamental solutions can be obtained by letting \( \zeta = 0 \). It follows that the fundamental solution complies with the boundary conditions.

The boundary conditions constraining the first order solutions can be obtained by dividing the boundary conditions by \( \zeta \) and thereupon letting \( \zeta = 0 \). This is equivalent to obtaining the coefficient of the first power of \( \zeta \) in the boundary conditions. Now the following boundary conditions are obtained:

\[
\begin{align*}
    v_1(0) - v_1(2\pi) &= 0 \quad \text{(A.20a)} \\
    w_1(0) - w_1(2\pi) &= 0 \quad \text{(A.20b)} \\
    w_{1,\theta}(0) - w_{1,\theta}(2\pi) &= 0 \quad \text{(A.20c)} \\
    \int_0^{2\pi} \{ w_1(\theta) \cos(\theta) - v_1(\theta) \sin(\theta) \} \, d\theta &= 0 \quad \text{(A.20d)} \\
    \int_0^{2\pi} \{ w_1(\theta) \sin(\theta) + v_1(\theta) \cos(\theta) \} \, d\theta &= 0 \quad \text{(A.20e)} \\
    \int_0^{2\pi} v_1 \, d\theta &= 0 \quad \text{(A.20f)}
\end{align*}
\]

A.1.3 Solution of system

By substituting the general solution given in eq. (A.13) into the boundary condition it is possible to write the boundary conditions as a system of a six-by-six coefficient matrix multiplied by a vector containing all \( C \)'s being equal to the null vector. This homogeneous system has only non-trivial solutions if and only if the determinant of the coefficient matrix is equal to zero. Solving this characteristic equation for \( \lambda_c \) under the assumption that \( \lambda_c \) is positive leads to the following buckling load parameter:

\[ \lambda_{c; Sanders} = \frac{n^2 - 1}{3 + \chi^2} \] (A.21)

This obtained value exactly corresponds to eq. (3.69). Now the system can be solved for the unknown coefficients if this newly obtained value for \( \lambda_c \) is substituted back into the system. This results in only \( C_1 \) and \( C_2 \) being non-zero. Besides, it is obtained that they have an arbitrary value. The general solution that complies with the boundary conditions now reads:

\[
\begin{align*}
    v_1(\theta) &= C_1 \frac{3+n^2+2}{3+\chi^2} \cos(n\theta) + C_2 \frac{3+n^2-2}{3+\chi^2} \sin(n\theta) \quad \text{(A.22a)} \\
    w_1(\theta) &= C_1 n \sin(n\theta) - C_2 n \cos(n\theta) \quad \text{(A.22b)}
\end{align*}
\]

These expressions can be rewritten based on the following trigonometric identities:

\[
\begin{align*}
    A \cos(n\theta - \psi) &= A \cos(\psi) \cos(n\theta) + A \sin(\psi) \sin(n\theta) \quad \text{(A.23a)} \\
    A \sin(n\theta - \psi) &= A \cos(\psi) \sin(n\theta) - A \sin(\psi) \cos(n\theta) \quad \text{(A.23b)}
\end{align*}
\]
The new expressions for the first order solutions now read:

\[ v_1(\theta) = -C \frac{3 + n^2 \chi^2}{n(3 + \chi^2)} \sin (n\theta - \psi) \]  
\[ w_1(\theta) = C \cos (n\theta - \psi) \]  

(A.24a)  
(A.24b)

Where \( C \) is an arbitrary amplitude parameter and \( \psi \) is an arbitrary phase shift. Because of the symmetry of the structure the orientation does not influence the buckling load parameter and hence a fixed value for \( \psi \) can be chosen for the bifurcation analysis. It must be kept in mind that although a fixed orientation is found in chapters 3 and 4 due to the choice of an ansatz, the orientation should be arbitrary. The amplitude parameter is arbitrary, because in the bifurcation analysis only a shape is found.

If now is chosen that \( \psi = 0 \) and \( C = 1 \), exactly the solution as obtained in eq. (3.73) is found. For small values of \( \chi \) the following approximations are valid:

\[ \lambda_c = \frac{1}{3} (n^2 - 1) \]  
\[ v_1(\theta) = -\frac{1}{n} \sin (n\theta) \]  
\[ w_1(\theta) = \cos (n\theta) \]  

(A.25a)  
(A.25b)  
(A.25c)

A.2 Donnell-Mushtari-Vlasov shell

Substituting eq. (A.2) into eq. (A.1), dropping the underlined terms and rewriting leads to the following bifurcation equations based on the DMV shell equations:

\[ v_{1,\theta\theta} + (1 + \lambda_c \chi^2) w_{1,\theta} = 0 \]  
\[ \frac{1}{4} \chi^2 w_{1,\theta\theta\theta\theta} + \lambda_c \chi^2 w_{1,\theta\theta} + (1 + \lambda_c \chi^2) v_{1,\theta} + (1 + \lambda_c \chi^2) w_{1} = 0 \]  

(A.26a)  
(A.26b)

To solve these equations, a similar method is applied as in the previous section. In the system above, the identities given in eq. (A.4) are substituted. This leads to the system:

\[
\begin{bmatrix}
 p^2 \\
 (1 + \lambda_c \chi^2) p \\
 (1 + \lambda_c \chi^2) p \\
 (1 + \lambda_c \chi^2) p
\end{bmatrix}
\begin{bmatrix}
 \hat{V}_1 \\
 \hat{V}_1 \\
 \hat{W}_1 \\
 \hat{W}_1
\end{bmatrix}
= 
\begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0
\end{bmatrix}
\]  

(A.27)

This homogeneous system can only have non-trivial solutions if and only if the determinant of the matrix is equal to zero. This leads to a characteristic polynomial of sixth order for \( p \):

\[ \frac{1}{4} \chi^2 p^6 + \lambda_c \chi^2 p^4 - \lambda_c \chi^2 (1 + \lambda_c \chi^2) p^2 = 0 \]  

(A.28)

The roots of this polynomial are:

\[ p \in \left\{ \sqrt{-\Lambda_1 + \Lambda_2}, -\sqrt{-\Lambda_1 + \Lambda_2}, i \sqrt{\Lambda_1 + \Lambda_2}, -i \sqrt{\Lambda_1 + \Lambda_2}, 0, 0 \right\} \]  

(A.29)

Where:

\[ \Lambda_1 = \frac{3}{2} \lambda_c \]  

(A.30a)

Profound analysis ring bifurcation
\[ \Lambda_2 = \sqrt{3\lambda_c + \lambda_c^2 \left( \frac{n^2}{4} + 3\chi^2 \right)} \] 

(A.30b)

It follows that \( \Lambda_2 > \Lambda_1 \). Hence all square roots in eq. (A.29) are real valued. The Eigenvectors corresponding to the Eigenvalues can be obtained by substituting the Eigenvalues into the matrix in eq. (A.27) and obtaining the nullspace of the matrix. The Eigenvector is the vector contained in this nullspace with exception of the null vector. The null vector is irrelevant. Note that there is a shared Eigenvalue \( p = 0 \). Hence for one of the two occurrences of this Eigenvalue the corresponding Eigenvector is adjusted corresponding the method explained in the previous section. This leads to the following set of Eigenvectors corresponding to the Eigenvalues given in eq. (A.29).

\[
\mu \in \begin{cases} 
1 + \frac{1 + \lambda_c \chi^2}{\sqrt{-\Lambda_1 + \Lambda_2}} & 1 + \frac{1 + \lambda_c \chi^2}{\sqrt{-\Lambda_1 + \Lambda_2}} \\
1 + \frac{1 + \lambda_c \chi^2}{-\sqrt{\Lambda_1 + \Lambda_2}} & 1 + \frac{1 + \lambda_c \chi^2}{\sqrt{\Lambda_1 + \Lambda_2}} \\
1 + \frac{1 + \lambda_c \chi^2}{\sqrt{-\Lambda_1 + \Lambda_2}} & 1 + \frac{1 + \lambda_c \chi^2}{\sqrt{\Lambda_1 + \Lambda_2}} \\
1 + \frac{1 + \lambda_c \chi^2}{\sqrt{\Lambda_1 + \Lambda_2}} & 1 + \frac{1 + \lambda_c \chi^2}{\sqrt{\Lambda_1 + \Lambda_2}} \\
1 & 0 \\
\theta & -1
\end{cases}
\] 

(A.31)

The general solution for \( v_1(\theta) \) and \( w_1(\theta) \) can be composed using eq. (A.12). Using Euler’s formula and rewriting for the undetermined coefficients the following general solution reads:

\[
v_1(\theta) = \begin{cases} 
C_1 \cosh \left( \sqrt{-\Lambda_1 + \Lambda_2} \theta \right) - C_2 \sinh \left( \sqrt{-\Lambda_1 + \Lambda_2} \theta \right) \\
+ C_3 \cos \left( \sqrt{\Lambda_1 + \Lambda_2} \theta \right) + C_4 \sin \left( \sqrt{\Lambda_1 + \Lambda_2} \theta \right) \\
+ C_5 \theta + C_6
\end{cases}
\]

(A.32a)

\[
w_1(\theta) = \sqrt{-\Lambda_1 + \Lambda_2} \left\{ -C_1 \sinh \left( \sqrt{-\Lambda_1 + \Lambda_2} \theta \right) + C_2 \cosh \left( \sqrt{-\Lambda_1 + \Lambda_2} \theta \right) \\
+ \sqrt{\Lambda_1 + \Lambda_2} \left\{ C_3 \sin \left( \sqrt{\Lambda_1 + \Lambda_2} \theta \right) - C_4 \cos \left( \sqrt{\Lambda_1 + \Lambda_2} \theta \right) \right\} - C_5
\]

(A.32b)

Now the boundary conditions given in eq. (A.20) are introduced again. The general solution above is substituted into these boundary conditions. The system of boundary conditions is rewritten to a homogeneous linear system with respect to the indeterminate coefficients. This system can only have non-trivial solutions if its determinant is equal to zero. Setting this determinant to zero leads to a characteristic equation. Obtaining the roots of this characteristic equation results in several possible solutions for \( \lambda_c \). The only root of interest, having a positive value is given by:

\[
\lambda_c; \text{DMV} = -\frac{1}{\sqrt{2}} \left( n^2 + 1 - \sqrt{(n^2 + 1)^2 + \frac{4\chi^2n^4}{4}} \right) \chi^{-2}
\]

(A.33)

This solution exactly matches eq. (3.65). This solution is substituted back into the general solution. The boundary condition equations now can be solved for the six indeterminate coefficients. This results in all coefficients but \( C_3 \) and \( C_4 \) to be zero. The remaining two coefficients are free. Hence the general solution complying with the boundary conditions reads:

\[
v_1(\theta) = \frac{1}{2} \left\{ \sqrt{(n^2 + 1)^2 + \frac{4\chi^2n^4}{4}} \right\} \left[ C_1 \cos (n\theta) + C_2 \sin (n\theta) \right]
\]

(A.34a)

Profound analysis ring bifurcation
\[ w_1(\theta) = C_1 n \sin(n\theta) - C_2 n \cos(n\theta) \quad (A.34b) \]

By using eq. (A.23), it is possible to rewrite the general solution to:

\[ v_1(\theta) = -C \frac{1}{2n} \left\{ \sqrt{(n^2 + 1)^2} + \frac{4}{3} \chi^2 n^4 - (n^2 - 1) \right\} \sin(n\theta - \psi) \quad (A.35a) \]
\[ w_1(\theta) = C \cos(n\theta - \psi) \quad (A.35b) \]

Here \( C \) is a free amplitude parameter and \( \psi \) is a free phase parameter. As in the previous section a general solution can be described by setting \( C = 1 \) and \( \psi = 0 \):

\[ v_1(\theta) = -\frac{1}{2n} \left\{ \sqrt{(n^2 + 1)^2} + \frac{4}{3} \chi^2 n^4 - (n^2 - 1) \right\} \sin(n\theta) \quad (A.36a) \]
\[ w_1(\theta) = \cos(n\theta) \quad (A.36b) \]

This corresponds with the ansatzes introduced in eq. (3.63). For small values of \( \chi \) the following approximations are valid:

\[ \lambda_c = \frac{n^4}{3(n^2 + 1)} \quad (A.37a) \]
\[ v_1(\theta) = -\frac{1}{n} \sin(n\theta) \quad (A.37b) \]
\[ w_1(\theta) = \cos(n\theta) \quad (A.37c) \]
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Appendix B

FEA results ring collapse

In this appendix results of the FEA for a ring are given for two different collapse criteria. The results denoted in the tables are used in figs. 5.6 and 5.7. Note that for all analyses performed in this appendix an oval initial imperfection shape is applied and consequently an oval shaped collapse mode is observed.

In tables B.1 to B.4 results are given for the collapse criterion that assumes collapse to occur for first yielding of the ring. This is also known as the Timoshenko criterion. In tables B.5 to B.8 results are given of FEA of a ring using an elastic-perfectly plastic material model. These results are used to verify the collapse criterion based on the full development of a plastic hinge (DNV criterion).

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( \lambda_{col} ) for steel grade</th>
<th>X42</th>
<th>X52</th>
<th>X60</th>
<th>X65</th>
<th>X70</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.996</td>
<td>1.001</td>
<td>1.003</td>
<td>1.006</td>
<td>1.008</td>
<td></td>
</tr>
<tr>
<td>0.002</td>
<td>0.972</td>
<td>0.984</td>
<td>0.989</td>
<td>0.991</td>
<td>0.994</td>
<td></td>
</tr>
<tr>
<td>0.005</td>
<td>0.910</td>
<td>0.931</td>
<td>0.943</td>
<td>0.950</td>
<td>0.955</td>
<td></td>
</tr>
<tr>
<td>0.010</td>
<td>0.821</td>
<td>0.857</td>
<td>0.876</td>
<td>0.888</td>
<td>0.898</td>
<td></td>
</tr>
<tr>
<td>0.020</td>
<td>0.682</td>
<td>0.734</td>
<td>0.763</td>
<td>0.782</td>
<td>0.797</td>
<td></td>
</tr>
<tr>
<td>0.050</td>
<td>0.451</td>
<td>0.509</td>
<td>0.545</td>
<td>0.571</td>
<td>0.590</td>
<td></td>
</tr>
<tr>
<td>0.100</td>
<td>0.288</td>
<td>0.338</td>
<td>0.370</td>
<td>0.394</td>
<td>0.410</td>
<td></td>
</tr>
<tr>
<td>0.200</td>
<td>0.168</td>
<td>0.204</td>
<td>0.226</td>
<td>0.245</td>
<td>0.257</td>
<td></td>
</tr>
</tbody>
</table>

Table B.1: Collapse load at first yielding for \( \chi = 1/100 \)
### Table B.2: Collapse load at first yielding for \( \chi = \frac{1}{50} \)

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( \lambda_{col} ) for steel grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.958 0.972 0.979 0.984 0.986</td>
</tr>
<tr>
<td>0.002</td>
<td>0.907 0.931 0.943 0.950 0.955</td>
</tr>
<tr>
<td>0.005</td>
<td>0.782 0.828 0.852 0.866 0.876</td>
</tr>
<tr>
<td>0.010</td>
<td>0.641 0.701 0.732 0.756 0.770</td>
</tr>
<tr>
<td>0.020</td>
<td>0.475 0.538 0.574 0.600 0.619</td>
</tr>
<tr>
<td>0.050</td>
<td>0.271 0.319 0.350 0.374 0.391</td>
</tr>
<tr>
<td>0.100</td>
<td>0.158 0.192 0.214 0.230 0.242</td>
</tr>
<tr>
<td>0.200</td>
<td>0.089 0.108 0.122 0.132 0.142</td>
</tr>
</tbody>
</table>

### Table B.3: Collapse load at first yielding for \( \chi = \frac{1}{30} \)

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( \lambda_{col} ) for steel grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.828 0.888 0.914 0.929 0.938</td>
</tr>
<tr>
<td>0.002</td>
<td>0.736 0.808 0.842 0.864 0.877</td>
</tr>
<tr>
<td>0.005</td>
<td>0.578 0.653 0.696 0.725 0.742</td>
</tr>
<tr>
<td>0.010</td>
<td>0.444 0.511 0.552 0.583 0.602</td>
</tr>
<tr>
<td>0.020</td>
<td>0.312 0.367 0.401 0.427 0.446</td>
</tr>
<tr>
<td>0.050</td>
<td>0.168 0.204 0.228 0.245 0.257</td>
</tr>
<tr>
<td>0.100</td>
<td>0.098 0.118 0.134 0.146 0.154</td>
</tr>
<tr>
<td>0.200</td>
<td>0.053 0.065 0.074 0.082 0.086</td>
</tr>
</tbody>
</table>

### Table B.4: Collapse load at first yielding for \( \chi = \frac{1}{20} \)

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( \lambda_{col} ) for steel grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.490 0.593 0.660 0.708 0.739</td>
</tr>
<tr>
<td>0.002</td>
<td>0.444 0.535 0.593 0.636 0.665</td>
</tr>
<tr>
<td>0.005</td>
<td>0.360 0.430 0.478 0.511 0.535</td>
</tr>
<tr>
<td>0.010</td>
<td>0.281 0.336 0.374 0.401 0.422</td>
</tr>
<tr>
<td>0.020</td>
<td>0.199 0.240 0.269 0.290 0.305</td>
</tr>
<tr>
<td>0.050</td>
<td>0.110 0.134 0.151 0.163 0.173</td>
</tr>
<tr>
<td>0.100</td>
<td>0.064 0.079 0.089 0.097 0.103</td>
</tr>
<tr>
<td>0.200</td>
<td>0.035 0.043 0.049 0.054 0.058</td>
</tr>
</tbody>
</table>

**FEA results ring collapse**
<table>
<thead>
<tr>
<th>ξ</th>
<th>X42</th>
<th>X52</th>
<th>X60</th>
<th>X65</th>
<th>X70</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>1.002</td>
<td>1.008</td>
<td>1.013</td>
<td>1.013</td>
<td>1.015</td>
</tr>
<tr>
<td>0.002</td>
<td>0.990</td>
<td>0.993</td>
<td>0.998</td>
<td>1.003</td>
<td>1.004</td>
</tr>
<tr>
<td>0.005</td>
<td>0.934</td>
<td>0.960</td>
<td>0.962</td>
<td>0.969</td>
<td>0.974</td>
</tr>
<tr>
<td>0.010</td>
<td>0.859</td>
<td>0.890</td>
<td>0.907</td>
<td>0.918</td>
<td>0.925</td>
</tr>
<tr>
<td>0.020</td>
<td>0.741</td>
<td>0.785</td>
<td>0.811</td>
<td>0.829</td>
<td>0.840</td>
</tr>
<tr>
<td>0.050</td>
<td>0.524</td>
<td>0.583</td>
<td>0.618</td>
<td>0.643</td>
<td>0.659</td>
</tr>
<tr>
<td>0.100</td>
<td>0.362</td>
<td>0.411</td>
<td>0.442</td>
<td>0.465</td>
<td>0.484</td>
</tr>
<tr>
<td>0.200</td>
<td>0.233</td>
<td>0.274</td>
<td>0.300</td>
<td>0.320</td>
<td>0.335</td>
</tr>
</tbody>
</table>

Table B.5: Collapse load for elastic-perfectly plastic material where $\chi = \frac{1}{100}$

<table>
<thead>
<tr>
<th>ξ</th>
<th>X42</th>
<th>X52</th>
<th>X60</th>
<th>X65</th>
<th>X70</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.971</td>
<td>0.983</td>
<td>0.988</td>
<td>0.992</td>
<td>0.994</td>
</tr>
<tr>
<td>0.002</td>
<td>0.929</td>
<td>0.950</td>
<td>0.960</td>
<td>0.966</td>
<td>0.970</td>
</tr>
<tr>
<td>0.005</td>
<td>0.824</td>
<td>0.863</td>
<td>0.884</td>
<td>0.897</td>
<td>0.905</td>
</tr>
<tr>
<td>0.010</td>
<td>0.706</td>
<td>0.754</td>
<td>0.781</td>
<td>0.800</td>
<td>0.813</td>
</tr>
<tr>
<td>0.020</td>
<td>0.565</td>
<td>0.620</td>
<td>0.652</td>
<td>0.675</td>
<td>0.690</td>
</tr>
<tr>
<td>0.050</td>
<td>0.353</td>
<td>0.402</td>
<td>0.432</td>
<td>0.456</td>
<td>0.473</td>
</tr>
<tr>
<td>0.100</td>
<td>0.227</td>
<td>0.265</td>
<td>0.289</td>
<td>0.308</td>
<td>0.321</td>
</tr>
<tr>
<td>0.200</td>
<td>0.135</td>
<td>0.162</td>
<td>0.178</td>
<td>0.192</td>
<td>0.202</td>
</tr>
</tbody>
</table>

Table B.6: Collapse load for elastic-perfectly plastic material where $\chi = \frac{1}{50}$

<table>
<thead>
<tr>
<th>ξ</th>
<th>X42</th>
<th>X52</th>
<th>X60</th>
<th>X65</th>
<th>X70</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>0.854</td>
<td>0.911</td>
<td>0.934</td>
<td>0.946</td>
<td>0.954</td>
</tr>
<tr>
<td>0.002</td>
<td>0.771</td>
<td>0.839</td>
<td>0.871</td>
<td>0.890</td>
<td>0.902</td>
</tr>
<tr>
<td>0.005</td>
<td>0.638</td>
<td>0.707</td>
<td>0.744</td>
<td>0.768</td>
<td>0.784</td>
</tr>
<tr>
<td>0.010</td>
<td>0.521</td>
<td>0.586</td>
<td>0.624</td>
<td>0.651</td>
<td>0.668</td>
</tr>
<tr>
<td>0.020</td>
<td>0.405</td>
<td>0.459</td>
<td>0.492</td>
<td>0.517</td>
<td>0.534</td>
</tr>
<tr>
<td>0.050</td>
<td>0.247</td>
<td>0.287</td>
<td>0.314</td>
<td>0.334</td>
<td>0.348</td>
</tr>
<tr>
<td>0.100</td>
<td>0.153</td>
<td>0.182</td>
<td>0.200</td>
<td>0.214</td>
<td>0.225</td>
</tr>
<tr>
<td>0.200</td>
<td>0.087</td>
<td>0.105</td>
<td>0.117</td>
<td>0.127</td>
<td>0.134</td>
</tr>
</tbody>
</table>

Table B.7: Collapse load for elastic-perfectly plastic material where $\chi = \frac{1}{30}$
Table B.8: Collapse load for elastic-perfectly plastic material where $\chi = \frac{1}{20}$

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\lambda_{col}$ for steel grade</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>X42</td>
</tr>
<tr>
<td>0.001</td>
<td>0.509</td>
</tr>
<tr>
<td>0.002</td>
<td>0.475</td>
</tr>
<tr>
<td>0.005</td>
<td>0.410</td>
</tr>
<tr>
<td>0.010</td>
<td>0.344</td>
</tr>
<tr>
<td>0.020</td>
<td>0.273</td>
</tr>
<tr>
<td>0.050</td>
<td>0.172</td>
</tr>
<tr>
<td>0.100</td>
<td>0.106</td>
</tr>
<tr>
<td>0.200</td>
<td>0.060</td>
</tr>
</tbody>
</table>

FEA results ring collapse
Appendix C

APDL scripts

In this appendix two typical scripts are given. These are used to analyse the bifurcation, buckling and collapse behaviour of a ring and a cylinder respectively. They can be loaded into the finite element software ANSYS.

C.1 Ring model

The script below denotes a typical APDL script to analyse the buckling behaviour of a ring under external pressure.

```apdl
!------------------------------------------------!
! Ansys script to determine ring buckling          
!------------------------------------------------!

!------------------------------------------------!
finish
/clear
/title, Ring bifurcation buckling and collapse
/prep7
/units,si

!--- Constants ---!
*afun, rad
pi = acos(-1) ! Set pi
*afun, deg

seltol,1e-7 ! Selection tolerance

!--- Input parameters ---!

Dt = 30 ! = D/t = 1/chi
D = 0.8 ! = 2*rho
E = 210e9 ! Young's modulus
nu = 0.3 ! Poisson ratio
nrElm = 40 ! Number of elements to define ring
xi = 0.002 ! Imperfection amplitude (rho_max-rho_min)/(rho_max+rho_min)

n = 2 ! Circumferential imperfection wave number
lambda = 1.2 ! Dimensionless load parameter
RandomOffset = 0 ! Random disturbance of the node
```
YieldStress = 290e6 ! Yield stress | X70 ~ 480MPa | X65 ~ 450 MPa | X60 ~ 410MPa | X52 ~ 360MPa | X42 ~ 290MPa

SubSteps = 1000 ! Number of substeps

chi = 1/Dt
rho = D/2
t = 2*rho*chi
I = t**3/12
P = lambda*E/4*(t/rho)**3

!--- Element properties ---!
et,1, beam188
et,2, target169 ! Force based constraints
et,3, conta175 ! Pilot node element

! keyopt, element#, keyopt#, keyopt value
keyopt,1,2,1
keyopt,2,2,1 ! Boundary conditions specified by user
keyopt,2,4,111 ! Constraints in all directions will be passed from pilot node
keyopt,3,2,2 ! MPC Contact algorithm
keyopt,3,4,1 ! Force-distributed constraint
keyopt,3,12,5 ! Always bonded behaviour of contact surface

r,2

!--- Sections ---!
sectype,1, beam, rect
secdata, t, 1 ! Beam with height t and unit breadth

!--- Materials ---!
mptemp, _Dt1001, 298
mpdata, ex, 1,, E
mpdata, prxy, 1,, nu

*if,1, eq, 0, then ! Only run for elastic perfectly-plastic analysis
  tb, bkin, 1, 1, 2, 0 ! Kinematic hardening
tbtemp, 298
tbdata, 1, YieldStress, 0 ! Yield stress | Tangent modulus
*endif

dTheta = 360/nrElm

!--- Nodes ---!
n, 1, 0
*do, i, 1, nrElm
  Theta = (i-1)*dTheta
  Wi = x*i* rho*cos(n*Theta)
  Wi = -x*i* rho/n*sin(n*Theta)
  X = rho*cos(Theta) + Wi*cos(Theta) + Vi*sin(Theta) + RAND
      (-1,1)*RandomOffset*rho
  Y = rho*sin(Theta) + Wi*sin(Theta) + Vi*cos(Theta) + RAND
      (-1,1)*RandomOffset*rho
  n,, X, Y
*endo

nsel,s,, 2, nrElm + 1
csys, 1

--- APDL scripts ---
!--- Elements ---!
real,1
type,1
*do,i,2,nrElm+1
  *if.i,eq,nrElm+1,then
    j = 2
  *else
    j = i + 1
  *endif
  e,i,j
*enddo

!--- MPC ---!
real,2
type,3
allsel
esurf
type,2
tshap,pilo
e,1

!--- Boundary conditions ---!
csys,0
d,1,ux,0
d,1,uy,0
d,1,rotz,0
d,all,uz,0
d,all,rotx,0
d,all,roty,0

!--- Loads and solver settings ---!
/solu
antype,0
nlgeom,1
outres,all,all
solcontrol,on

! Either Newton-Raphson or arc-length solver
nropt,full
!arclen,on,25,1/1000 ! Default: 25,1/1000

esel,s,etype,,1
sfbeam,all,2,press,-P
allsel
time,1
solve
finish
C.2 Cylinder model

The script below denotes a typical APDL script to analyse the buckling behaviour of a cylinder under external pressure constrained by end caps.

```apdl
!------------------------------------------------!
! Ansys script to determine cylinder buckling 
!------------------------------------------------!
finish
/clear
/title, Cylinder bifurcation buckling and collapse
/prep7
/units, si

!--- Constants ---!
*afun, rad
pi = acos(-1) ! set pi
*afun, deg
seltol, 1e-7 ! selection tolerance

!--- Input parameters ---!
!------------------------------------------------!
Dt = 100 ! = D/t = 1/chi
LD = 20 ! = L/D = L/2/rho = 1/2/gamma => gamma = 1/2/LD
D = 0.8 ! = 2*rho
E = 210e9 ! Young’s modulus
nu = 0.3 ! Poisson ratio
nrElm = 100 ! number of elements through the circumference
xi = 0.01 ! imperfection amplitude (rho_max-rho_min)/(rho_max+rho_min)
n = 2 ! circumferential imperfection wave number
m = 1 ! axial half wave number imperfection
lambda = 1.5 ! dimensionless load parameter
RandomOffset = 0e-3 ! random disturbance of the node
AspectRatio = 10 ! element size in axial direction divided by element size in circumferential direction
YieldStress = 480e6 ! yield stress | X70 ~ 480 MPa | X65 ~ 450 MPa | X60 ~ 410 MPa | X52 ~ 360 MPa | X42 ~ 290 MPa
SubSteps = 10 ! number of substeps
alpha = 1 ! include end cap pressure for alpha=1 and not include end cap pressure for alpha = 0
beta = 0 ! for beta = 0 simply supported and for beta = 1 clamped

!------------------------------------------------!
chi = 1/Dt
rho = D/2
L = D*LD
t = 2*rho*chi
P = lambda*E/4*(t/rho)**3/(1-nu**2)
AxialElementCnt = nint(nrElm*L/2/pi/rho/AspectRatio)
Fec = P*pi*rho**2*alpha

!--- Element properties ---!
et, 1, shell181
```

APDL scripts
et,2,target170 ! 3d target element for rigid mpc
et,3,conta175 ! Pilot node

!--- Keyoptions ---!
keyopt,2,2,1 ! Boundary conditions specified by user

*if,beta,eq,0,then
  keyopt,2,4,000111 ! Constraints in all directions will be passed from pilot node except for rotations
*elseif,beta,eq,1,then
  keyopt,2,4,111111 ! Constraints in all directions will be passed from pilot node
*else
  /eof
*endif

keyopt,3,2,2 ! MPC Contact algorithm
keyopt,3,4,0 ! Rigid constraint ==> 0 | Force-distributed constraint ==> 1
keyopt,3,12,5 ! Always bonded behaviour of contact surface

!--- Sections ---!
sectype,, shell
secdata,,t ! Shell thickness
r,2 ! Real constant set for z=0
r,3 ! Real constant set for z=L

!--- Materials ---!
mptemp,1,298
mpdata,ex,1,,E
mpdata,prxy,1,, nu

*if,1,eq,0,then ! Only run for elastic perfectly-plastic analysis
  tb,bkin,1,1,2,0 ! Kinematic hardening
tbtemp,298
tbdata,1,YieldStress,0 ! Yield stress | Tangent modulus
*endif

dTheta = 360/nrElm

!--- Nodes ---!
n,1,0,0,0
n,2,0,0,L

*do,j,1,AxialElementCnt+1
  *do,i,2,nrElm+1
    Z = (j-1)*L/AxialElementCnt
    Theta = (i-1)*dTheta
    Wi = xi*rho*cos(n*Theta)*sin(pi*m*Z/L*360/2/pi)
    X = (rho+Wi)*cos(Theta)
    Y = (rho+Wi)*sin(Theta)
    XMod = X + RAND(-1,1)*RandomOffset*rho*sin(pi*Z/L*360/2/pi)
    YMod = Y + RAND(-1,1)*RandomOffset*rho*sin(pi*Z/L*360/2/pi)
    Zmod = Z + RAND(-1,1)*RandomOffset*rho*sin(pi*Z/L*360/2/pi)
    *if,Z,eq,L,then
      Xmod = rho*cos(Theta)
      Ymod = rho*sin(Theta)
      Zmod = Z
    *endif
  *endif
*do,i,1
  *do,j,1

APDL SCRIPTS
*endif
endif
nsel,s,,all
nsel,u,,,1,2
csys,1
nrotat,,all
csys,0
allsel

!--- Elements ---!
do,i,1,AxialElementCnt
  ElmAdd = (i-1)*nrElm
do,j,3,nrElm+2
  *if,j,eq,nrElm+2,then
    e,j+ElmAdd,j+1-nrElm+ElmAdd,j+1+ElmAdd,j+nrElm+ElmAdd
  *else,then
    e,j+ElmAdd,j+1*ElmAdd,j+1*nrElm+ElmAdd,j+nrElm+ElmAdd
  *endif
endo
do
endo

!--- MPC for z=0 ---!
real,2
type,3
nsel,s,loc,z,0
esurf
allsel
type,2
tshap,pilo
e,1

!--- MPC for z=L ---!
real,3
type,3
nsel,s,loc,z,L
esurf
allsel
type,2
tshap,pilo
e,2

!--- Boundary conditions ---!
csys,0
d,1,ux,0
d,1,uy,0
d,1,uz,0
d,1,rotx,0
d,1,roty,0
d,1,rotz,0
d,2,ux,0
d,2,uy,0
d,2,rotx,0
d,2,roty,0
d,2,rotz,0

APDL scripts
!--- Loads and solver settings ---!

/solu
antype,0
nlgeom,1
outres,all,all
solcontrol,on

! Either Newton-Raphson or arc-length solver
nropt,full
!arclen,on,25,1/1000 ! Default: 25,1/1000
nsubst,SubSteps,1e15,SubSteps/4

esel,s,type,,1
sfe,all,2,pres,0,P ! Wall pressure
allsel

*if,Fec,ne,0,then ! End cap pressure
  f,1,fz,Fec
  f,2,fz,-Fec
*endif

time,1

solve

finish
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