A Semi-Analytical Formulation for the Elastoplastic Analysis of Imperfect Cylindrical Shells

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SUMMARY

A semi-analytical formulation is developed for the elastoplastic analysis of initially imperfect cylindrical shells under axial compression and lateral pressure. The formulation is based on a small-strain, moderate-rotation shell theory and a small-strain incremental constitutive theory. The basic shell equations and the partially inverted constitutive relations in total form are reduced to a set of coupled nonlinear algebraic/ordinary differential equations by means of a Fourier decomposition of the state variables, imperfections and loads in circumferential direction of the shell, and application of Galerkin's method. The governing nonlinear equations are solved with an incremental-iterative technique. The method of quasi-linearization is used to generate the governing equations of the iterative procedure which consistently takes into account both geometrical and material nonlinearities. Plasticity effects are described using a layered approach. The classical flow theory based on the von Mises yield surface, associative flow rule and the isotropic hardening law is used to describe the evolution of the plastic strains in the integration points. In every iteration a set of linear ordinary differential equations is solved numerically with a shooting method and a return mapping algorithm is used to integrate the constitutive equations locally. A number of elastic and elastoplastic buckling problems are solved for which results are known from literature. It is shown that the quadratic rate of convergence, characteristic for a Newton-type iteration procedure, is retained even for large load steps. A comparison between the present results and the results from literature shows a good agreement.
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NOTATIONS

In this report bold faced letters, e.g. A, denote matrices and underlined symbols, e.g. A̅, vectors. If the elements of a vector are given explicitly then these are placed between braces, e.g. \{ A_1, \ldots, A_n \}.

A  
A_i  
a  
\underline{a}_{i}^a, \underline{a}_{i}^b  
B_a, B_b  
c_{ij}  
\underline{D}  
D^a, D^b  
d^a, d^b  
d\lambda  
\underline{E}_{ij}  
E  
E^p  
E^p_{i}  
e^b  
H^{(i)}  
h^{(i)}_p, h^{(i)}_p  
h  
h_x  
k  
L  
L^\wedge  
L  
l  
l_c  
l_i  
M, M_x, M_y, M_{xy}  
m  
undeformed shell middle surface  
fundamental solution matrix for the \( i \)th shooting interval  
hardening parameter defined by (2.1.17)  
abscissae of the Gauss and Simpson integration rule, respectively.  
k/2 x k matrices used to define the boundary conditions (2.4.6)  
elastic constants used in the partially inverted constitutive equations  
elasticity matrix with in-plane stiffnesses  
stretching and bending stiffness matrix (defined by (2.1.11))  
nondimensional stretching and bending stiffness matrix (2.2.7)  
plastic multiplier  
matrices relating harmonic components of generalized plastic strains to harmonic components of basic variables  
Young's modulus  
harmonic components of the generalized plastic strains (see (2.4.5))  
tangent modulus  
equivalent plastic strain  
instantaneous moduli defined by (2.1.21)  
nondimensional instantaneous moduli (2.2.8), (2.2.9)  
shell thickness  
nondimensional reduced shear force defined by (2.3.2)  
dimension of the set of ordinary differential equations (2.4.1)  
in-plane elastoplastic moduli  
consistent elastoplastic moduli (4.4.13)  
shell length  
dimension of the set of algebraic equations (2.4.4)  
half-wavelength of the classical buckling mode (5.3.1)  
number of full waves of the \( i \)th Fourier term  
moment resultants  
dimension of \( E^p \) (\( m = 4 + 6*N \))
\( m, m_x, m_y, m_{xy} \quad \text{nondimensional moment resultants \( \sigma = M / ( \sigma_s h^2 \))'\)

\( N \quad \text{number of asymmetric terms used in the truncated Fourier series} \)

\( N, N_x, N_y, N_{xy} \quad \text{stress resultants} \)

\( n, n_x, n_y, n_{xy} \quad \text{number of full waves in circumferential direction of the imperfection \(2.4.1\)} \)

\( \mathbf{P} \quad \text{nondimensional stress resultants \( \sigma = N / ( \sigma_s h )\)} \)

\( \mathbf{p}, p_x, p_y, p_z \quad \text{matrix \(2.1.13\)} \text{used in the definition of the yield surface} \)

\( \mathbf{P} \quad \text{surface loads} \)

\( p_x, p_y, p_z \quad \text{nondimensional surface loads \( \sigma = PR / ( \sigma_s h )\)} \)

\( \mathbf{P}_{ext} \quad \text{dead external pressure} \)

\( R \quad \text{shell middle surface radius} \)

\( S \quad \text{vector with the v vectors} \mathbf{S}_i, \mathbf{S}_i = \left( S_{1i}, ..., S_{vi}, S_{wi} \right)^T \)

\( \mathbf{S}_i \quad \text{vector} \mathbf{Y} \text{ in shooting point} \mathbf{x}_i \)

\( s_{xy} \quad \text{nondimensional reduced shear force defined by \(2.3.2.\)} \)

\( U, V, W \quad \text{displacements in axial, circumferential and radial direction, respectively} \)

\( u, v, w \quad \text{nondimensional displacements \( u=U/h, v=V/h, w=W/h \)} \)

\( U_i, U_{i} \quad \text{displacement and nondimensional displacement vector \( u=U_i/h \)} \)

\( V_i, V_{i} \quad \text{particular solutions of \(4.2.4\)} \text{ for the} \ i^{th} \text{ shooting interval} \)

\( W_i, W_{i} \quad \text{initial geometric imperfection \( \mathbf{w} = \mathbf{W}/h \)} \)

\( w^g, w^s \quad \text{weights of Gauss and Simpson integration rule} \)

\( x, y, z \quad \text{axial, circumferential and radial coordinates, respectively} \)

\( \bar{x}, \bar{y}, \bar{z} \quad \text{nondimensional axial, circumferential and radial coordinate} \)

\( \bar{x}_i \quad \text{nondimensional axial coordinate of the} \ i^{th} \text{ shooting point} \)

\( \mathbf{Y} \quad \text{vector with harmonic components of the basic variables} \)

\( Z \quad \text{vector with harmonic components of the secondary variables} \)

\( \beta_1, \beta_2 \quad \text{vectors specifying the boundary conditions \(2.4.6\)} \)

\( \varepsilon, \varepsilon_x, \varepsilon_y, \gamma_{xy} \quad \text{membrane strains} \)

\( \varepsilon, \varepsilon_x, \varepsilon_y \quad \text{error norms for displacement and stress variables, respectively} \)

\( \eta, \eta_x, \eta_y, \eta_{xy} \quad \text{in-plane strains at a distance} \ z \text{ from the middle surface:} \ \eta = \varepsilon + z \kappa \)

\( \kappa, \kappa_x, \kappa_y, \kappa_{xy} \quad \text{bending strains} \)

\( \lambda \quad \text{load factor} \)

\( \mu \quad \text{path parameter} \)

\( \nu \quad \text{Poisson's ratio or number of shooting intervals} \)

\( \bar{\xi}_1, \bar{\xi}_2 \quad \text{axisymmetric and asymmetric imperfection amplitudes, see eqs. \(5.3.3\)} \)
\( \sigma_x, \sigma_y, \sigma_{xy}, \sigma_e, \sigma_r \)

- in-plane stresses
- equivalent stress
- reference stress
- initial yield stress

\( \sigma^* \)

- elastic trial stresses

\( \phi, \phi_x, \phi_y \)

- von Mises yield surface
- rotations around tangents to the shell middle surface
- initial rotations due to initial imperfections

Superscripts:

- \((\cdot)^e\)
  - elastic variable
- \((\cdot)^i\)
  - \(i\)-th Fourier component of \((\cdot)\)
- \((\cdot)^p\)
  - plastic variable
- \((\cdot)'\)
  - differentiation with respect to \(\tilde{x}\)
- \((\cdot)^*\)
  - differentiation with respect to \(\tilde{y}\)
- \((\cdot)^i(\cdot)\)
  - variables related to an intermediate, not converged, configuration
- \((\cdot)^n(\cdot)\)
  - variables related to a converged configuration
1. INTRODUCTION

Buckling of axially compressed cylindrical shells has been studied extensively in the past. This interest stems from the importance of the cylindrical shell as a construction element and from its very complex and interesting collapse behaviour. It is now well understood that in case of elastic shells this behaviour is strongly influenced by the effects of initial geometric imperfections and boundary conditions [1-3]. For thicker shells, as for example used in the off-shore industry, the effect of nonlinear material behaviour becomes important [4-6].

Nowadays, large general purpose finite element codes exist, like ABAQUS [7] or STAGS [8], with which one can simulate the buckling behaviour of cylindrical shells taking into account the above mentioned effects. However, these codes require a full two-dimensional discretization of the shell middle surface which generally leads to a large set of equations and long computing times, especially if accurate results are required. This makes that these codes are not very suited for a parametric study of the collapse behaviour of cylindrical shells. For a limited class of problems the so-called semi-analytical method [9-16] is an attractive alternative for a two-dimensional finite element or finite difference method. Basic concept in the semi-analytical method is that loads, imperfections and the response of the shell are in circumferential direction of the shell described by truncated Fourier series. If the loads, imperfections and response of the shell are such that they can be described with a small number of Fourier terms, this will lead to a much smaller set of equations than when a two-dimensional discretization of the shell middle surface is used. It is obvious however that this approach will pose restrictions on the class of problems that can be studied with it, but for a large number of problems very useful results can be obtained.

In this study we present a semi-analytical formulation for the elastoplastic buckling analysis of initially imperfect cylindrical shells that are loaded by axial compression and/or lateral pressure. Basic concept in the formulation is the reduction of the governing field equations to a set of coupled nonlinear algebraic/ordinary differential equations. This is achieved by expanding the stress resultants, displacements, imperfections and loads in circumferential direction of the shell into truncated Fourier series, and application of Galerkin's method. Basic variables in the differential equations are the quantities that can be prescribed at the edges of the shell, which makes it very easy to fullfill both the kinematic and the natural boundary conditions. The governing nonlinear equations are solved using an incremental-iterative technique. The method of quasi-linearization is used to generate the governing equations of the iterative procedure which consistently takes into account both geometrical and material nonlinearities. A layered approach is used to include the effects of nonlinear material behaviour. The classical flow theory of plasticity based on the von Mises yield surface, associative flow rule and isotropic hardening law is used to describe this behaviour locally in the integration points. The equations of the semi-analytical formulation are independent of the specific details of the plasticity theory used, which means that other models can easily be included. E.g. a model with
anisotropic hardening [17]. In every iteration a set of linearized differential equations is solved numerically with a shooting method and a return mapping algorithm [28,29] is used to integrate the constitutive equations locally. With the numerical model one can study accurately the combined effects of boundary conditions, initial imperfections and material nonlinearities. However, due to the use of Fourier series to describe the circumferential variation of the different variables only certain imperfections can be considered, otherwise the solution would become inefficient. In this study the imperfection consists therefore of two parts, an axisymmetric part and an asymmetric part with a certain number of waves in circumferential direction.

Semi-analytical formulations for the elastoplastic analysis of thin-walled shells of revolution, reminiscent of the one as presented in this report, have been given in the past by Wunderlich [11,12,13] and Esslinger [14,15,16]. Basic difference with our formulation is the manner in which the nonlinear effects are taken into account. In our formulation a path-independent integration procedure for the constitutive equations is used in conjunction with a consistently linearized set of equations. This ensures both a rapid convergence of the iteration procedure and that rather large load steps can be made. We have included a continuation procedure [38,39] that makes it possible to determine the solution up to and beyond the limit point.

Main objective of this report is to present the development of the semi-analytical model and to describe the numerical procedures used to solve it. The formulation will be checked by analyzing some elastic and elastoplastic buckling problems known from literature. In this report we do not intend to give a review of elastoplastic buckling of imperfect cylindrical shells nor do we give new results for this subject. These results will be published in the near future. The content on this report is as follows. In chapter 2 the semi-analytical model will be formulated. We focus our attention on the general lines; details are given in appendices. In chapter 3 linearized equations are consistently derived and in chapter 4 the iterative solution scheme and numerical procedures are discussed. In chapter 5, some numerical examples that can be compared with results known from literature, are analyzed. We conclude this report with chapter 6 where we discuss the results obtained and indicate some possible extensions and improvements of the present formulation.
2. THEORETICAL ANALYSIS

2.1. Basic shell equations and constitutive relations

In this section the basic equations from which the semi-analytical model will be derived are given. The shell equations are those according to a simplified form of the well known small-strain, moderate-rotations shell theory of Sanders [19] and Koiter [20]. The inelastic material behaviour is described by the classical small-strain $J_2$ flow theory based on the Von Mises yield surface, associative flow rule and the isotropic hardening law [22].

2.1.1. Kinematic relations

Let $h$, $l$ and $R$ denote the thickness, length and middle surface radius of the undeformed circular cylindrical shell, respectively. Material points on the undeformed shell middle surface are identified by surface coordinates $x,y$, where $x$ and $y$ measure the distance in the axial and circumferential direction, respectively. A radial coordinate $z$ measures the distance along the normal to the undeformed middle surface and is taken positive inwards. The components of the displacement vector of middle surface points are denoted as $U$, $V$ and $W$ in axial, circumferential and normal direction, respectively (Fig. 2.1). An initial geometric imperfection is specified as an initial stress-free radial displacement $ar{W}$, positive inwards. The membrane- and bending strains (changes of curvature) of the shell middle surface are approximated as follows:

\[
\begin{align*}
\epsilon &= \begin{bmatrix} \epsilon_x & \epsilon_y & \gamma_{xy} \end{bmatrix}^T \\
\kappa &= \begin{bmatrix} \kappa_x & \kappa_y & \kappa_{xy} \end{bmatrix}^T \\
\epsilon_x &= U_x + \frac{1}{2} \left( \phi_y + 2\phi \phi_y \right) \\
\epsilon_y &= V_y + \frac{1}{2} \left( \phi_x + 2\phi \phi x \right) \\
\gamma_{xy} &= \phi_y + \phi \phi_x + \phi \phi_x + \phi \phi_y \\
\kappa_x &= \phi_x, \kappa_y = \phi_y, \kappa_{xy} = \phi_{xy} \tag{2.1.1}
\end{align*}
\]

where a comma denotes partial differentiation with respect to the indicated coordinate. The rotations $\phi_x$ and $\phi_y$ of normals to the middle surface and the corresponding initial rotations $\bar{\phi}_x$ and $\bar{\phi}_y$ due to the geometric imperfection are defined as:

\[
\begin{align*}
\phi_x &= -\bar{W}_x \\
\phi_y &= -\left( \frac{1}{R} V + W_y \right) \\
\bar{\phi}_x &= -\bar{W}_x \\
\bar{\phi}_y &= -\bar{W}_y \tag{2.1.2}
\end{align*}
\]
These strain measures are a further simplification of the small-strain, moderate-rotation kinematic relations of Sanders, Koiter and are only valid under the assumptions of small strains, moderate rotations around tangents and small rotations around normals to the shell middle surface \([19,20]\). In contrast to the well known kinematic relations according to Donnell, that are only slightly simpler than the above relations, these relations can be used for the stability analysis of non-shallow shells \([21]\). The kinematic description of the shell is completed by approximating the in-plane Lagrangian strains at a distance \(z\) from the shell middle surface as follows:

\[
\mathbf{\eta} = \mathbf{\varepsilon} + z \mathbf{\chi} \\
\mathbf{\eta} = \{ \eta_x, \eta_y, \eta_{xy} \}^T
\]  

(2.1.3)

2.1.2. Equilibrium equations

Equilibrium of the shell can be formulated by means of the principle of virtual work. Let \(\mathbf{P} = \{ P_x, P_y, P_z \}^T\) be surface loads per unit undeformed middle surface, positive in positive coordinate direction. If \(A\) is the undeformed shell middle surface then the three-dimensional virtual work expression is approximated as follows:

\[
\int_A (\mathbf{N}^T \delta \mathbf{e} + \mathbf{M}^T \delta \mathbf{\chi}) dA = \int_A \mathbf{P}^T \delta \mathbf{U} dA + \text{VWE}
\]  

(2.1.4)

where VWE denotes the virtual work of the edge loads and \(\delta \mathbf{U} = \{ \delta U, \delta V, \delta W \}^T\) is a kinematically admissible variation of the displacement vector. The stress- and moment resultants in the above expression are defined as:

\[
\mathbf{N} = \int_h \sigma \, dz \\
\mathbf{N} = \{ N_x, N_y, N_{xy} \}^T
\]  

(2.1.5)

\[
\mathbf{M} = \int_h \sigma \, zdz \\
\mathbf{M} = \{ M_x, M_y, M_{xy} \}^T
\]  

(2.1.5)

where

\[
\sigma = \{ \sigma_x, \sigma_y, \tau_{xy} \}^T
\]  

(2.1.6)

is the vector of in-plane stresses. In Fig. 2.2 positive stress- and moment resultants acting on a shell element are shown. Note that an approximate state of plane stress is assumed, so that only stress resultants due to the in-plane stresses enter the virtual work expression. The actual values of the surface loads depend on the type of external or internal pressure. The simplest case is when we have a dead surface load, i.e. a load that does not change direction or magnitude during the deformation of the shell. In this case we have:

\[
\mathbf{P} = \{ 0, 0, p_{\text{ext}} \}^T
\]  

(2.1.7)
where \( P_{\text{ext}} \) is an external pressure per unit undeformed middle surface area. If we have an internal pressure then the value of \( P_z \) changes sign.

### 2.1.3. Constitutive equations

The basic mathematical formulation of the problem is completed by specifying the constitutive equations. First we derive relations between generalized stresses and generalized elastic strains. Here, generalized stresses are the stress- and moment resultants. Generalized strains are the membrane- and bending strains. The total, three-dimensional strains are assumed to remain small and are split into an elastic and a plastic part.

\[
\eta = \eta^e + \eta^p
\]  
(2.1.8)

If the elastic material behaviour is linear isotropic then we have the following elastic stress–strain relations

\[
\sigma = D (\eta - \eta^p)
\]  
(2.1.9)

where \( D \) is the usual plane stress elasticity matrix. If we assume that the shell wall has constant properties over the thickness then we can derive the following relations between generalized strains and generalized stresses from (2.1.3), (2.1.5) and (2.1.9):

\[
\begin{align*}
N &= D^s (\xi - \xi^p) \\
M &= D^b (\kappa - \kappa^p)
\end{align*}
\]  
\[
\begin{align*}
\xi^p &= (1/h) \int_h \eta^p \, dz \\
\kappa^p &= (12/h^3) \int_h \eta^p \, z \, dz
\end{align*}
\]  
(2.1.10)

where the stretching- and bending stiffnesses are:

\[
D^s = h \, D
\]

\[
D^b = h^3 / 12 \, D
\]  
(2.1.11)

The variables \( \xi^p \) and \( \kappa^p \) are denoted as plastic membrane- and bending strains, respectively. The difference between generalized strains and generalized plastic strains will be denoted as generalized elastic strains. The plastic membrane and bending strains depend on the plastic strain distribution \( \eta^p \) in the shell and are generally obtained by numerical integration of the appropriate expressions. We note here that equations (2.1.10) are independent of the details of the plasticity theory used, and are not necessarily restricted to time-independent inelastic deformations.
The plastic strains are generally governed by a set of rate equations which makes that the problem at hand is an initial boundary value problem that has to be solved step by step.

In this study we use the classical small-strain \( J_2 \) flow theory, based on the von Mises yield surface, associative flow rule and isotropic hardening law, to describe the evolution of the plastic strains. It is well known that if this theory is applied to bifurcation problems then the theoretically obtained buckling loads can be unrealistically high \([18,23]\). The use of a deformation theory of plasticity in these cases often results in bifurcation load predictions that are in good agreement with experimental data. However, the use of a deformation theory can only be justified for certain prebuckling states \([23]\), and it can not be applied generally. Because we are here dealing with a nonlinear collapse problem, in which we can expect that strongly nonproportional and even unloading occurs before the shell reaches its maximum load carrying capacity, the deformation theory is not an appropriate theory to use. It has been pointed out by Besseling \([18]\) that if a shell or plate has initial geometric imperfections it is in principle not correct to use an isotropic hardening rule to describe the hardening behaviour of the material. In this case a more realistic hardening law should be used. This extension will be realized in the near future in order to investigate the effect of a more correct description of the hardening behaviour on the collapse of shells which takes place at rather small strains.

An alternative to the classical flow theory could be a corner theory of plasticity, e.g. the one developed by Christoffersen and Hutchinson \([24]\). In this theory a corner develops at the stress point on the yield surface. This theory has successfully been used by Tvergaard \([25,26]\) in the study of buckling of axially compressed cylindrical shells. Tvergaard showed \([25]\) that in case of a long cylindrical shell when edge effects are neglected, a corner lowers the buckling stress and the average strain at which buckling occurs. However, the differences between results obtained with a corner theory and those obtained with the classical theory are only large for very thick shells. We do not use this model because of the numerical difficulties involved in following the development of the corners and because it has been pointed out by some researchers \([6,40]\) that when edge effects are taken into account, a good correlation between experimental results and theoretical results based on a flow theory with a smooth yield surface exists in case of thick shells.

In the remaining part of this section we will summarize the equations according to the \( J_2 \) flow theory. The equations will be given for the special case of a plane-stress state and will be cast in a form that is due to Simo and Taylor \([28]\). The Huber–von Mises yield criterion, the elastic stress-strain relations, the flow rule and the hardening law can be formulated as follows \([28]\):

\[
\phi = \frac{1}{2} \sigma^T P \sigma - \frac{1}{3} \sigma e^{\sigma} (\sigma P) s 0
\]
\[ \sigma = D (\eta - \eta^P) \]
\[ d\eta^P = d\lambda \ P \sigma \]
\[ de^P = d\lambda \left[ \frac{2}{3} \sigma^t P \sigma \right]^{1/2} \]  \hspace{1cm} (2.1.12)

where
\[ P = \frac{1}{3} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \]  \hspace{1cm} (2.1.13)

The equivalent stress \( \sigma_e \) is taken to be a function of the equivalent plastic strain \( e^P \) that is defined by (2.1.12). If the stress state is uni-axial then it can be shown that \( \sigma_e \) reduces to the axial stress and the equivalent plastic strain to the axial plastic strain. Thus the isotropic hardening behaviour of the material is determined by specifying an uniaxial stress-strain curve, the equivalent stress-equivalent plastic strain relations then follows from this curve. The so-called plastic multiplier \( d\lambda \) can be zero or positive depending on the following loading/unloading conditions:

\[ d\lambda = 0 \quad \text{if} \quad \phi = 0 \quad \text{and} \quad d\phi \leq 0 \quad \text{or} \quad \phi < 0 \]
\[ d\lambda > 0 \quad \text{if} \quad \phi = 0 \quad \text{and} \quad d\phi > 0 \]  \hspace{1cm} (2.1.14)

where
\[ \bar{\phi} = \frac{1}{2} \sigma^t P \sigma \]  \hspace{1cm} (2.1.15)

The actual value of the plastic multiplier is determined by imposing the consistency condition. This condition expresses that the stress state must remain on the yield surface during loading (if \( \phi = 0 \) then \( d\phi = 0 \)). Using standard arguments one can derive that:

\[ d\lambda = (\sigma^t P D \sigma + a)^{-1} \sigma^t P D \ d\eta \]  \hspace{1cm} (2.1.16)

where \( a \) is a hardening parameter given by:

\[ a = \frac{4}{9} \frac{d\sigma_e}{de^P} \kappa^2 \]  \hspace{1cm} (2.1.17)
The derivative $d\sigma_e / d\varepsilon^P$ follows from the uni-axial stress-strain curve. Using (2.1.12), (2.1.12), and (2.1.16) the following incremental equations for the plastic strains and the stresses are derived:

$$d\varepsilon^P = L_p \varepsilon = (\sigma^P P \sigma^P + a)^{-1} P \sigma^P \varepsilon^P d\varepsilon$$  \hspace{1cm} (2.1.18)

$$d\sigma = L \varepsilon = [ D - (\sigma^P P \sigma^P + a)^{-1} D \sigma^P \varepsilon^P ] d\varepsilon$$  \hspace{1cm} (2.1.19)

Relations between generalized stresses and generalized strains are obtained by substitution of (2.1.3) for the total strains into (2.1.19) and integration over the thickness of the shell. The incremental stress-strain relations in terms of generalized variables derived in this manner are usually written as:

$$dN = H^{(1)} d\varepsilon + H^{(2)} d\varepsilon$$

$$dM = H^{(2)} d\varepsilon + H^{(3)} d\varepsilon$$  \hspace{1cm} (2.1.20)

where the instantaneous moduli are defined as:

$$H^{(i)} = \int_h L z^{(i-1)} dz$$  \hspace{1cm} (2.1.21)

If the shell remains elastic then the integrals (2.1.21) can be evaluated ones and for all. Eqs. (2.1.20) reduce in that case to the incremental form of (2.1.10). If however the deformations are elastoplastic then the instantaneous moduli are obtained by numerical integration over the thickness of the shell.

2.2. Introduction of nondimensional stress– and displacement variables

The semi-analytical model will be formulated in terms of nondimensional variables. We will first introduce these variables and then we briefly summarize the basic equations in nondimensional form. Nondimensional variables are defined as follows:

$$n = N / (\sigma_e h)$$

$$m = M / (\sigma_e h^2)$$

$$e = R / h$$

$$n = \begin{pmatrix} n_x, n_y, n_{xy} \end{pmatrix}^T$$

$$m = \begin{pmatrix} m_x, m_y, m_{xy} \end{pmatrix}^T$$

$$e = \begin{pmatrix} e_x, e_y, e_{xy} \end{pmatrix}^T$$
\[ k = \begin{bmatrix} x \\ y \\ xy \end{bmatrix} \quad k = \begin{bmatrix} k_x & k_y & k_{xy} \end{bmatrix}^T \]

\[ u = \frac{U}{h} \quad u = \begin{bmatrix} u_x \quad v \quad w \end{bmatrix}^T \]

\[ p = PR/(\sigma_x h) \quad p = \begin{bmatrix} p_x \quad p_y \quad p_z \end{bmatrix}^T \] (2.2.1)

where \( \sigma_x \) denotes a reference stress that we take here equal to the initial yield stress of the material. It is noted here that \( e_{xy} \) is just a modified engineering strain \( \gamma_{xy} \) and not a tensor component as it is in continuum mechanics. Using the above definitions we can formulate the principle of virtual work as follows:

\[ \begin{align*}
2\pi L/R & \int \int \left( n_T \delta e + m_T \delta k \right) d\tilde{x} d\tilde{y} = 2\pi L/R \int \int p_T \delta u d\tilde{x} d\tilde{y} + vwe
\end{align*} \] (2.2.2)

where nondimensional coordinates \( \tilde{x} = x/R \) and \( \tilde{y} = y/R \) have been introduced. The strain–displacement relations can be expressed in the new nondimensional variables as follows:

\[ \begin{align*}
\varepsilon_x & = u_x + \frac{R}{2h} \phi_x (\phi_x + 2\phi_y) \\
\varepsilon_y & = v_y - w + \frac{R}{2h} \phi_y (\phi_y + 2\phi_x) \\
\varepsilon_{xy} & = u_y + v_x + \frac{R}{h} (\phi_x + \phi_y) \\
k & = \begin{bmatrix} k_x \\ k_y \end{bmatrix} \\
k & = \begin{bmatrix} k_x \\ k_y \end{bmatrix} \\
k & = \begin{bmatrix} k_x \\ k_y \end{bmatrix} + \begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix}
\end{align*} \] (2.2.3)

with

\[ \begin{align*}
\phi_x & = -\frac{h}{R} w \\
\phi_y & = -\frac{h}{R} (v + w) \quad (2.2.4)
\end{align*} \]

and where \( (\cdot) \) and \((\cdot)^*\) denote partial differentiation with respect to the nondimensional coordinates \( \tilde{x} \) and \( \tilde{y} \), respectively. The in-plane Lagrangian strains at a distance \( \tilde{z} \) from the shell middle surface are given as follows:

\[ \begin{align*}
\eta & = \frac{h}{R} (\varepsilon_x + \tilde{z} k_x) \\
\text{with} \quad \tilde{z} & = z/h
\end{align*} \] (2.2.5)
The elastic stress–strain relations in nondimensional form are:

$$\bar{n} = d^s (e^s - e^p)$$

$$\bar{e}^p = \frac{R}{h} \int_{-1/2}^{1/2} n^p \bar{z} \, dz$$

$$\bar{m} = d^b (k - k^p)$$

$$k^p = 12 \frac{R}{h} \int_{-1/2}^{1/2} n^p \bar{z}^2 \, dz$$

(2.2.6)

where the nondimensional stretching and bending stiffnesses are as follows:

$$d^s = h \frac{\hat{D}}{R}$$

$$d^b = \frac{1}{12} h \frac{\hat{D}}{R}$$

with

$$\hat{D} = \frac{1}{\sigma} D$$

(2.2.7)

The continuum constitutive equations (2.1.12)–(2.1.19) can be put into a nondimensional form by replacing all stresses or stress–like quantities (e.g. stiffnesses) by the same variables divided by the reference stress. We do not give these expressions here because they are rather trivial. The nondimensional form of (2.1.20) becomes:

$$d_n = h^{(1)} e_n + h^{(2)} d_k$$

$$d_m = h^{(2)} e_n + h^{(3)} d_k$$

$$h^{(i)} = \frac{h}{R} \int_{-1/2}^{1/2} \frac{1}{L/\sigma} \frac{\hat{D}^{(i-1)}}{\hat{D}} \, dz$$

(2.2.8)

In chapter 3, a derivation of the linearized semi-analytical model is given in which incremental relations between generalized plastic strains and generalized strains are used. These relations easily follow from (2.2.6), (2.1.18) and (2.2.5) and are written as follows:

$$d e^p = h^{(1)} e_p + h^{(2)} d_k$$

$$d k^p = 12 \left( h^{(2)} e_p + h^{(3)} d_k \right)$$

$$h^{(i)} = \frac{h}{R} \int_{-1/2}^{1/2} \frac{1}{L/\sigma} \frac{\hat{D}^{(i-1)}}{\hat{D}} \, dz$$

(2.2.9)
2.3. Kanonical set of equations in terms of nondimensional stress-- and displacement variables

In this section we formulate the equilibrium equations and the constitutive equations in total form, as a set of eight partial and three ordinary differential equations. These nonlinear equations are formulated in terms of basic and secondary variables. Basic variables are the generalized stresses and generalized displacements that can be prescribed at the edges of the shell. Secondary variables are three additional stress resultants.

Equilibrium equations and appropriate Kirchoff boundary conditions are derived from the principle of virtual work. In appendix A it is shown these equations can be formulated in terms of the nondimensional variables, as introduced in the previous paragraph, as follows:

\[
\begin{align*}
\dot{n}_{x} &= -s_{xy} \cdot \frac{h}{R} \cdot m_{xy} - p_{x} \\
\dot{s}_{xy} &= -n_{y} \cdot \frac{h}{R} \cdot m_{y} - \left[ \left( s_{xy} + \frac{h}{R} \cdot m_{xy} \right) \left( \phi_{x} + \frac{\ddot{w}}{R} \right) - \frac{h}{R} \cdot n \left( v + \frac{\ddot{w}}{R} \right) \right] - p_{y} \\
\dot{h}_{x} &= -n_{y} \cdot \frac{h}{R} \cdot m_{xy} + \left( s_{xy} + \frac{h}{R} \cdot m_{xy} \right) \left( \phi_{x} + \frac{\ddot{w}}{R} \right) - \frac{h}{R} \cdot n \left( v + \frac{\ddot{w}}{R} \right) - p_{z} \\
m_{x} &= -h_{x} - 2m_{xy} + \frac{R}{h} \cdot n \left( \phi_{x} + \frac{\ddot{w}}{R} \right) - \left( s_{xy} + \frac{h}{R} \cdot m_{xy} \right) \left( v + \frac{\ddot{w}}{R} \right)
\end{align*}
\]  

(2.3.1)

where \( s_{xy} \) and \( h_{x} \) are reduced shear forces defined by

\[
\begin{align*}
s_{xy} &= \frac{n_{xy}}{R} \cdot \frac{h}{R} \\
h_{x} &= \frac{h}{R} \cdot m_{x} + 2 \frac{h}{R} \cdot m_{xy} - n_{x} \left( \phi_{x} + \frac{\ddot{w}}{R} \right) - n_{xy} \left( \phi_{y} + \frac{\ddot{w}}{R} \right)
\end{align*}
\]  

(2.3.2)

The boundary conditions at the edges of the cylindrical shell are such that one can prescribe

\[
\begin{align*}
u \ or \ n_{x} : \ v \ or \ s_{xy} : \ w \ or \ h_{x} : \ \phi \ or \ m_{x}
\end{align*}
\]  

(2.3.3)

The remaining equations are derived from the partially inverted constitutive equations in total form (2.2.6) and the kinematic relations. If the strains are eliminated from the partially inverted constitutive equations with the strain-displacement relations we get a set of differential equations formulated in terms of generalized stresses and displacements. The derivation of these equations is given in appendix B.
Eqs. (2.3.1) are completed by the following four partial differential equations for the generalized displacement variables:

\[
\begin{align*}
\ddot{u} &= c_{11} \dddot{n} + c_{14} \left[ \dddot{v} - \dddot{w} + \frac{h}{2R} (\dddot{v} + \dddot{w}) (\dddot{v} + \dddot{w} + 2\dddot{w}) \right] + (e^p_x - c_{14} e^p_y) \\
&\quad = \frac{R}{2h} \dddot{\phi} \left( \dddot{\phi} + 2\dddot{\phi} \right) \\
\ddot{v} &= c_{22} \dddot{n} + c_{26} \left[ \dddot{2\phi} + \frac{R}{h} \dddot{u} - (\dddot{\phi} + \dddot{\phi}) (\dddot{v} + \dddot{w}) - \dddot{\phi} \dddot{w} \right] + (e^p_y - c_{22} (k^p_x + h e^p_y)) \\
&\quad = -\frac{R}{h} \dddot{\phi} \left( \dddot{\phi} + \dddot{\phi} - \dddot{\phi} \right) \\
\ddot{w} &= -\frac{R}{h} \dddot{\phi} \\
\dddot{\phi} &= c_{33} \dddot{m} - \frac{h}{R} c_{35} (\dddot{v} + \dddot{w}) + (\kappa^p_x - c_{35} \kappa^p_y) \\
\end{align*}
\]  

where \( c_{ij} \) are constants that depend on the elastic material parameters. The following three differential equations can be used to eliminate the secondary variables \( n_y, m_y \) and \( m_{xy} \) from (2.3.1) and (2.3.4)

\[
\begin{align*}
n_y &= c_{41} \dddot{n} + c_{44} \left[ \dddot{v} - \dddot{w} + \frac{h}{2R} (\dddot{v} + \dddot{w}) (\dddot{v} + \dddot{w} + 2\dddot{w}) \right] - c_{44} e^p_y \\
m_y &= c_{53} \dddot{m} - \frac{h}{R} c_{55} (\dddot{v} + \dddot{w}) - c_{55} \kappa^p_y \\
m_{xy} &= c_{62} \dddot{n} + c_{66} \left[ \dddot{2\phi} + \frac{h}{R} \dddot{u} - (\dddot{\phi} + \dddot{\phi}) (\dddot{v} + \dddot{w}) - \dddot{\phi} \dddot{w} \right] - c_{66} (\kappa^p_y + h e^p_y) \\
\end{align*}
\]

(2.3.5)

The equations that follow from the partially inverted constitutive equations depend not only on the basic and secondary variables, but also on the generalized plastic strains. These latter variables are considered to be functions of the displacements; i.e., if we know the basic variables and the gradients of these variables then we can calculate the strains and integrate the continuum constitutive equations. This yields new values for the plastic strains. Plastic membrane and bending strains are then obtained by numerical integration over the thickness of the shell.

An advantage of the formulation given here, is that the variables that can be prescribed at the edges of the shell enter the governing equations directly as independent variables. This makes it very easy to satisfy both the kinematical and the natural boundary conditions, both for elastic and elastoplastic shells.
2.4 Reduction of the governing equations to a set of coupled algebraic/differential equations.

Essential element in the semi-analytical formulation is the use of truncated Fourier series to describe the circumferential variation of basic and secondary variables, loads and imperfections. It is obvious that this approach will only be efficient if the circumferential variation of the different variables can be described by a small number of Fourier terms. We therefore limit ourselves in the first place to the analysis of shells with an idealized imperfection of the following form:

\[
\bar{w}(\bar{x},\bar{y}) = \bar{w}^s(\bar{x}) + \bar{w}^n(\bar{x})\cos(n\bar{y})
\]  

(2.4.1)

where \(\bar{w}^s\) and \(\bar{w}^n\) are the amplitudes of an axisymmetric and an asymmetric imperfection, respectively; \(n\) is the number of full waves of the asymmetric imperfection in circumferential direction of the shell. We note here that the present formulation actually allows that more asymmetric terms are included in (2.4.1), but it is questionable whether the response of the shell in that case can efficiency be described by Fourier series. Thus, in this study we will concentrate on an imperfection as specified above. We now assume that the response of the shell can efficiently be described by the following truncated Fourier series for the basic variables:

\[
\begin{align*}
\bar{u}(\bar{x},\bar{y}) &= u^s(\bar{x}) + \Sigma_{i=1}^{N} u^i(\bar{x})\cos(l_i\bar{y}) \\
\bar{n}_x(\bar{x},\bar{y}) &= n^s_x(\bar{x}) + \Sigma_{i=1}^{N} n^i_x(\bar{x})\cos(l_i\bar{y}) \\
\bar{v}(\bar{x},\bar{y}) &= \Sigma_{i=1}^{N} v^i(\bar{x})\sin(l_i\bar{y}) \\
\bar{s}_{xy}(\bar{x},\bar{y}) &= \Sigma_{i=1}^{N} s_{xy}^i(\bar{x})\sin(l_i\bar{y}) \\
\bar{w}(\bar{x},\bar{y}) &= w^s(\bar{x}) + \Sigma_{i=1}^{N} w^i(\bar{x})\cos(l_i\bar{y}) \\
\bar{h}_x(\bar{x},\bar{y}) &= h^s_x(\bar{x}) + \Sigma_{i=1}^{N} h^i_x(\bar{x})\cos(l_i\bar{y}) \\
\bar{\phi}_x(\bar{x},\bar{y}) &= \phi^s_x(\bar{x}) + \Sigma_{i=1}^{N} \phi^i_x(\bar{x})\cos(l_i\bar{y}) \\
\bar{m}_x(\bar{x},\bar{y}) &= m^s_x(\bar{x}) + \Sigma_{i=1}^{N} m^i_x(\bar{x})\cos(l_i\bar{y}) \\
\bar{n}_y(\bar{x},\bar{y}) &= n^s_y(\bar{x}) + \Sigma_{i=1}^{N} n^i_y(\bar{x})\cos(l_i\bar{y})
\end{align*}
\]  

(2.4.2)

where \(n\) is the number of asymmetric Fourier terms and \(l_i\) are integers denoting the number of full waves in circumferential direction. Note that it is assumed that the deformations remain symmetric with respect to the plane \(\bar{y} = 0\). This means that shells loaded by torsion cannot be analyzed by solutions of the form assumed here. The secondary variables are approximated by:

\[
\bar{n}_y(\bar{x},\bar{y}) = n^s_y(\bar{x}) + \Sigma_{i=1}^{N} n^i_y(\bar{x})\cos(l_i\bar{y})
\]
\[
\begin{align*}
    m_y(x, y) &= m_y^0(x) + \sum_{i=1}^{N} m_y^i(x) \cos(l_i y) \\
    m_{x y}(x, y) &= \sum_{i=1}^{N} m_{x y}^i(x) \sin(l_i y)
\end{align*}
\]  
(2.4.3)

Here, N and \( l_i \) must be chosen such that the response is described accurately. In practice this will mean that the number \( l_i \) will be a multiple of the number of full waves in circumferential direction n. For example if \( N = 3 \) then \( l_i = n, 2n, 3n \).

When the truncated series are substituted into the governing equations then these will not be satisfied. In order to eliminate the \( y \)-dependence Galerkin's method is used. Substitution of the Fourier series into the governing equations and application of Galerkin's method yields a set of first-order nonlinear differential equations and a set of nonlinear algebraic equations. These two sets are coupled. The resulting expressions are lengthy and are therefore given in appendix C. The resulting equations can be written as follows:

\[
\begin{align*}
    \dot{Y} &= F(x, y, Z, E^p; \lambda) \\
    Z &= G(x, y, E^p) \\
    Y &\in \mathbb{R}^k : k = 6 + 8N \\
    Z &\in \mathbb{R}^l : l = 2 + 3N
\end{align*}
\]  
(2.4.4)

where \( Y \) and \( Z \) are vectors that contain all the Fourier terms of the basic and secondary variables, respectively. The vector \( E^p \) contains the harmonic components of the plastic membrane and bending strains and depends therefore on the plastic strain distribution throughout the entire shell. The elements of \( E^p \) are defined in appendix C. Thus,

\[
\begin{align*}
    Y &= \left\{ ..., \frac{i}{n}, \frac{i}{x}, \frac{i}{xy}, \frac{i}{x}, \frac{i}{m}, \frac{i}{u}, \frac{i}{v}, \frac{i}{w}, \frac{i}{\phi}, ..., \right\}^T \\
    Z &= \left\{ ..., \frac{i}{y}, \frac{i}{m}, \frac{i}{m}, \frac{i}{xy}, ..., \right\}^T \\
    E^p &= \left\{ ..., \frac{e_p}{x}, \frac{e_p}{y}, \frac{e_p}{xy}, \frac{e_p}{x}, \frac{e_p}{x}, \frac{e_p}{y}, \frac{e_p}{xy}, \frac{e_p}{xy}, \frac{e_p}{xy}, ..., \right\}^T
\end{align*}
\]  
(2.4.5)

where the notation \( \{ ..., \frac{i}{j}, ..., \} \) is used to denote that the vector contains all Fourier terms of the variable \( j \). In the above equation \( \lambda \) denotes a load parameter that determines the loading intensity. The boundary conditions can be written as:
\begin{align*}
\mathbf{B}_a \mathcal{Y}(\bar{x} = 0) &= \lambda \mathbf{\beta}_1 \\
\mathbf{B}_b \mathcal{Y}(\bar{x} = L/R) &= \lambda \mathbf{\beta}_2 
\end{align*}
\tag{2.4.6}

where \( \mathbf{B}_a \) and \( \mathbf{B}_b \) are \( k/2 \times k \) matrices and \( \mathbf{\beta}_1, \mathbf{\beta}_2 \) are \( k/2 \) vectors. If the entire shell remains elastic, i.e., no plastic deformations occur, then the secondary variables can be eliminated directly from (2.4.4). What remains is a set of nonlinear ordinary differential equations formulated entirely in terms of basic variables. If plastic deformations occur then \( \mathcal{Z} \) can still be eliminated but then the resulting set of differential equation depends on \( \mathcal{E}_p \). In its turn, \( \mathcal{E}_p \) depends on the entire history of the basic variables and the gradients of these variables (more precisely, the displacement variables that are part of \( \mathcal{Y} \)).
3. LINEARIZATION OF THE SET OF ALGEBRAIC/DIFFERENTIAL EQUATIONS

The governing nonlinear algebraic/differential equations as derived in the previous chapter must be solved step by step because no explicit expression for the plastic strains can be given. The plastic strains are obtained by numerical integration of the constitutive equations that are given in rate form. For every load step an iteration procedure is used in order to satisfy the nonlinear equations with a certain accuracy. Here we use the Newton-Raphson method as iteration procedure (it is noted here that this method is known as quasi-linearization when applied to nonlinear ordinary differential equations [30]). In this section the set of nonlinear algebraic/differential equations is linearized and it is shown how we can reduce the problem to the solution of a set of linear ordinary differential equations that is formulated in terms of increments of the basic variables.

3.1. Linearized equations for elastic cross sections

An elastic cross section is characterized by the fact that the material behaviour is elastic in all points of the cross section, whereby the term cross section is used to denote a slice of the shell with constant axial coordinate. This means that in every point the material has not yielded yet or is unloading.

Let a superscript i denote an approximate solution for a certain value of the load parameter λ and let \( \Delta(\cdot) \) denote a correction to this solution, thus:

\[
i_{Y} = Y + \Delta Y ; \quad i_{Z} = Z + \Delta Z
\]

Substitution of these expression into the governing nonlinear equations, and neglecting higher-order terms yields:

\[
\Delta Y' = \left( I - Y' \right) + I_{Y} \Delta Y + I_{Z} \Delta Z = R_{Y} + I_{Y} \Delta Y + I_{Z} \Delta Z
\]

\[
\Delta Z = \left( I - Z \right) + I_{Y} \Delta Y = R_{Z} + I_{Y} \Delta Y
\]

where

\[
F_{Y} = \left[ \frac{\partial F}{\partial Y} \right] \quad F_{Z} = \left[ \frac{\partial F}{\partial Z} \right] \quad G_{Y} = \left[ \frac{\partial G}{\partial Y} \right]
\]

\[
F_{Y} : R^{k} \times R^{k} \quad F_{Z} : R^{k} \times R^{l} \quad G_{Y} : R^{l} \times R^{k}
\]

and the residuals \( R_{Y} \) and \( R_{Z} \) are elements of \( R^{k} \) and \( R^{l} \). Recall that \( k \) and \( l \) are the dimensions of the set of algebraic equations and ordinary differential equations, respectively. The elements of the Jacobians can explicitly be expressed in
terms of $Y$ and $Z$ and are easily derived from the governing equations as given in appendix C. The second set of equations from (3.1.2) can be used to eliminate $\Delta Z$ from the linearized ordinary differential equations, this yields

$$
\Delta Y' = \left( R_f + F_{r} \right) + \left( F_y + F_{g} \right) \Delta Y = R + A \Delta Y
$$

(3.1.4)

where we have dropped the superscript $i$ because it is obvious that the jacobians and residuals are calculated in the last obtained configuration. This set of equations is completed by specifying the appropriate boundary conditions. Substitution of (3.1.1) into (2.4.6) yields:

$$
B_a \Delta Y(\tilde{\alpha} = 0) = \lambda \beta_1 - B_a Y(\tilde{\alpha} = 0)
$$

$$
B_b \Delta Y(\tilde{\alpha} = L/R) = \lambda \beta_2 - B_b Y(\tilde{\alpha} = L/R)
$$

(3.1.5)

If all cross sections are elastic then the linearized problem is completely specified by the linearized ordinary differential equations (3.1.4) and the boundary conditions (3.1.5). This linear two-point boundary value problem can be solved for $\Delta Y$ with standard numerical techniques. The increments of the secondary variables $\Delta Z$ follow from the algebraic equations (3.1.2). However, if elastoplastic deformations occur in a cross section then the vector $E_P$ is not constant anymore. Separate equations that take into account the effect of material nonlinearities must be given for such a cross section.

3.2. Linearized equations for elastoplastic cross sections

Two alternative derivations of the linearized equations for elastoplastic cross sections will be given. The first derivation uses the incremental form of the constitutive equations as given by (2.2.9). Since it is rather unusual to derive linearized equations, that describe the global behaviour of the shell, with these equations we also give an alternative derivation based on (2.2.8). However, in the latter approach we can not derive the linearized equations directly from the equations as given in appendix C.

Method 1:

We first note that only the equations that follow from the partially inverted constitutive equations in total form depend on $E_P$. These equations can be formulated as follows:

$$
\bar{Y}_r' = F_r(\tilde{\alpha}, \bar{Y}_r, E_P)
$$

$$
\bar{Z} = G(\tilde{\alpha}, \bar{Y}_r, E_P)
$$

(3.2.1)
where

\[ Y_r = \{ ..., u, v, \phi_x, ... \}^T \]  \hspace{1cm} (3.2.2)

Here, \( Y_r \) is a subvector of \( Y \) with dimension \( l \). Note that \( Y_r \) has the same dimension as \( Z_r \), so \( F \) does not depend on the vector of secondary variables \( Z_r \) and that the remaining equations of (2.4.4), i.e. those equations not contained in (3.2.1), remain unchanged. Applying the method of quasi-linearization yields:

\[ \Delta Y_r = R_{rf} + F_{ry} \Delta Y + F_{re} \Delta E^P \]
\[ \Delta Z_r = R_g + G_y \Delta Y + G_c \Delta E^P \]  \hspace{1cm} (3.2.3)

where

\[ F_{re} = \left[ \frac{\partial F}{\partial E^P} \right] \]
\[ G_c = \left[ \frac{\partial G}{\partial E^P} \right] \]

\[ F_{re} : R^l \times R^m \]
\[ G_c : R^l \times R^m \]  \hspace{1cm} (3.2.4)

and \( m = 4 + 6^* N \). The matrix \( F_{re} \) is an \( l \times k \) submatrix of \( F \) and \( R_{rf} \) is an \( l \) dimensional subvector of \( R_f \). Relations between increments of \( E^P \) and increments of the basic variables \( \Delta Y \) (more specifically, the displacement variables that are elements of \( Y \)) can be derived with the constitutive relations (2.2.10). In appendix D it is shown that these relations can be written as follows:

\[ \Delta E^P = E_{11} \Delta Y_r + E_{12} \Delta Y \]  \hspace{1cm} (3.2.5)

where \( E_{11} \) and \( E_{12} \) are \( m \times l \) and \( m \times k \) matrices, respectively. These matrices depend on \( h_p^{(1)} \) and are obtained by numerical integration of the appropriate expressions along the circumference of the shell. Substitution of (3.2.5) into the first equation of (3.2.3) and inversion of the resulting expression yields:

\[ \Delta Y_r = (I - F_{re} E_{11})^{-1} \left[ R_{rf} + (F_{ry} + F_{re} E_{12}) \Delta Y \right] \]
\[ = R_{rf} + F_{ry} \Delta Y \]  \hspace{1cm} (3.2.6)
Substitution of (3.2.5) into (3.2.3) and elimination \( \Delta Y \) with (3.2.6) results in the following expression for the increments of secondary variables:

\[
\Delta Z = \left( \frac{R}{g} + G \frac{E}{e_{11}} \tilde{R} \right) + \left( G \frac{E}{e_{12}} + G \frac{E}{e_{11}} \tilde{F} \right) \Delta Y
\]

\[
= \tilde{R} + \tilde{G} \Delta Y
\]

(3.2.7)

If we now add the remaining linearized equations from (2.4.1), i.e. those not contained in (3.2.3), to (3.2.6) we get a set of equations that can be written as follows:

\[
\Delta Y' = \tilde{R} + \tilde{F} \Delta Y + \tilde{F} \Delta Z
\]

(3.2.8)

As for elastic cross sections, (3.2.7) can be used to eliminate \( \Delta Z \) from (3.2.8):

\[
\Delta Y' = \left( \frac{\tilde{R}}{I} + \frac{\tilde{F}}{z-g} \right) + \left( \frac{\tilde{F}}{y} + \frac{\tilde{G}}{z} \right) \Delta Y = \tilde{R} + \tilde{A} \Delta Y
\]

(3.2.9)

This is the required set of linear ordinary differential equations for the elastoplastic cross sections that together with the corresponding set for elastic cross sections (3.1.4) and the boundary conditions (3.1.5) form a linear two-point boundary value problem. It is worth mentioning here that, in order to get equations into the form (3.2.9), we have to invert the \( 1 \times 1 \) matrix \( I - \frac{F}{z} \frac{E}{e_{11}} \). This makes that the numerical evaluation of \( \tilde{R} \) and \( \tilde{A} \) is much more expensive than the corresponding elastic quantities \( R \) and \( A \).

**Method 2:**

In this section we give an alternative derivation of the linearized set of equations for elastoplastic cross sections. This derivation will be based on the constitutive relations (2.2.8) and (2.2.6). With these equations the incremental relations for the generalized stresses can be written as:

\[
\Delta \bar{n} = h^{(1)} \Delta \bar{e} + h^{(2)} \Delta \bar{k} + \left( d^{(1)} \bar{e} - \bar{n} \right)
\]

\[
\Delta \bar{m} = h^{(2)} \Delta \bar{e} + h^{(3)} \Delta \bar{k} + \left( d^{(2)} \bar{e} - \bar{m} \right)
\]

(3.2.10)

where

\[
\bar{e} = e - e^P
\]

\[
\bar{k} = k - k^P
\]

(3.2.11)
The terms between brackets in (3.2.10) are generally not zero because stress- and moment resultants are taken as independent variables, i.e. in numerical computations (2.2.6) is not used to compute the stress- and moment resultants. Introduction of the reduced shear force \( s_{xy} \), substitution of the Fourier expansions into (3.2.10) and application of Galerkin’s method gives a set of equations that after some reordering can be written as

\[
\Delta Y = A_{11} \Delta Y_{-r} + A_{12} \Delta Y_{-s} + R_{-s}
\]

\[
\Delta Z = A_{21} \Delta Y_{-r} + A_{22} \Delta Y_{-z} + R_{-z}
\]  

(3.2.12)

where \( Y \) is given by (3.2.2), \( A_{11} \) and \( A_{21} \) are 1 x 1 and \( A_{12} \), \( A_{22} \) are 1 x k matrices. \( R_{-s}, R_{-z} \) and

\[
Y_{-s} = (..., n_i, s_i, m_i, ...)^T
\]

\[
Z_{-z} = (..., n_i, m_i, m_i, ...)^T
\]

(3.2.13)

are m-vectors. We do not give the detailed derivation of (3.2.12) here because the derivation itself is straightforward and because the resulting expressions are very lengthy. The matrices \( A_{ij} \) depend on the instantaneous moduli \( h^{(i)} \) and are obtained by numerical integration of the appropriate expressions along the circumference of the shell. Equations (3.2.12,) can be written as follows:

\[
\Delta Y_{-r} = A_{11}^{-1} (I_s - A_{12}) \Delta Y_{-s} + A_{11}^{-1} R_{-s}
\]

(3.2.14)

where \( I_s \) is a 1 x k matrix such that

\[
Y_{-s} = I_s Y
\]

(3.2.15)

Elimination of \( \Delta Y_{-r} \) from (3.2.12,) yields:

\[
\Delta Z = [A_{22} + A_{21} A_{11}^{-1} (I_s - A_{12})] \Delta Y_{-s} + (R_{-z} + A_{21} A_{11}^{-1} R_{-s})
\]

(3.2.16)

The equations (3.2.14) and (3.2.16) are equivalent to (3.2.6) and (3.2.7). Again, an 1 x 1 matrix must be inverted in order to formulate the equations as a set of first order ordinary differential equations formulated in terms of increments of the basic variables.
**Remark 1:** If the residuals $R_s$ and $R_z$ are zero then the following equations have been satisfied:

\[ d^s (e^s) - n^i = 0 \]
\[ d^b (e^b) - m^i = 0 \]  

(3.2.17)

where $(\cdot)^i$ denotes the $i^{th}$ Fourier component of $(\cdot)$. It can be shown that these equations are equivalent to the equations that are given in appendix C (that is, the equations following from the partially inverted constitutive equations in total form).

**Remark 2:** The linearized equations in this paragraph have been derived with the incremental constitutive equations as given in paragraph 2.2. These incremental equations are based on the so-called continuum elastoplastic moduli. It is now well understood that, if a Newton type iteration procedure is used, the use of these moduli in conjunction with a path independent integration method for the continuum constitutive equations will result in a loss of the quadratic convergence rate [27-29]. This rate of convergence is retained if instead of the continuum the so-called consistent elastoplastic moduli are used. These moduli are derived by differentiation of the stress integration algorithm, this is discussed in paragraph 4.4. Thus in order to retain the convergence properties of Newton's method the continuum elastoplastic moduli must be replaced by the consistent moduli, nothing else changes.

**Remark 3:** All Fourier terms are coupled in the linearized equations because of the geometrical and material non-linearities. Other authors (e.g., [10-13]) consider the nonlinear terms as pseudo-loads in order to get set of equations that are decoupled in all Fourier components, as is the case in linear analysis. Although the numerical solution of these decoupled equations is much cheaper then the solution of the full set of coupled equations, much more iterations are needed to obtain a converged solution. Because the constitutive equations have to be integrated for every iteration this can be very expensive. In our formulation a rapid convergence is ensured. It will also be shown that rather large load steps can be taken.
4. NUMERICAL SOLUTION PROCEDURES

In the previous chapter it has been pointed out that the set of governing nonlinear equations are solved iteratively with a Newton type procedure (method of quasi-linearization). A set of linearized equations has therefore been derived that takes into account both geometrical and material nonlinearities. In this chapter we will discuss in some detail the different steps involved in the solution procedure. Special attention will be given to the solution of the linear two-point boundary value problem, the integration of the constitutive law and the numerical evaluation of the elements of generalized plastic strains, stiffnesses, etc.


Let a converged solutions of the nonlinear two-point boundary value problem be identified by a left superscript \(n\), \(n+1\), etc. Intermediate solutions, i.e., solutions during the iteration process that do not satisfy the governing nonlinear equations, are denoted by a left superscript \(i\), \(i+1\), etc. If we assume that a converged solution for \(\lambda = \eta_n\) is known, and that the load parameter is increased by \(\Delta \lambda\), then the solution scheme consists of the following steps:

STEP 1: Calculate the elements of \(i^1A\) and \(i^1R\) for elastic cross sections and the elements of \(i^1E_p\) and \(i^1E_{ij}\) for elastoplastic cross sections. This step includes the numerical evaluation of the elements of \(i^1E_{p}p\) and \(i^1E_{ij}r\) or \(i^1A_{ij}\).

STEP 2: Solve the linear two-point boundary value problem for \(\Delta Y\). This should not only give the values of \(\Delta Y\) for a certain axial grid, but also the values of \(\Delta Y'\). These derivatives are needed when the strains are calculated.

STEP 3: Calculate \(\Delta Z\) with (2.3.4) and update \(Y, Y'\) and \(Z\).

STEP 4: Calculate the total strains \(i^1+1\eta\) in all integration points with (2.4.2), (2.2.3) and (2.2.4).

STEP 5: Integrate the constitutive equations (2.1.12) for a strain increment \(\Delta \eta = i^1+1\eta - i\eta\). Note that the constitutive equations are integrated from the converged state to the last intermediate state.

STEP 6: Check the convergence of the iteration procedure. We use the following convergence criteria

\[
\varepsilon_d = i^1+1S_d - iS_d \parallel i^1+1S_d - iS_d \parallel < \varepsilon_d
\]

\[
\varepsilon_s = i^1+1S_s - iS_s \parallel i^1+1S_s - iS_s \parallel < \varepsilon_s
\]

where \(\parallel\) is the Euclidean norm of a vector. The vectors \(S_d\) and \(S_s\) contain the generalized displacement and generalized stress variables in selected mesh points (see also the next paragraph). If the error norms are satisfied then the plastic strains are updated and the load intensity parameter is increased. If the solution did not converge yet then go to step 1.

Note that the strain increment is calculated with respect to the converged state and that plastic strains are updated only after a converged solution has been obtained. This procedure ensures that all not converged intermediate states are "forgotten", thus the integration procedure for the constitutive equations is path-independent.
It is well known that Newton type iteration procedures fail to converge when limit points or bifurcation points are reached [30]. The solution to this problem is to treat the load intensity parameter as an unknown and to introduce an additional equation such that the problem remains well posed [38,39]. We will include such a continuation procedure in our formulation in order to be able to determine the solution up to and beyond the limit point. Using $\lambda$ as an unknown and linearizing the governing algebraic/differential equations with respect to $\lambda$ gives the following equations:

$$\Delta Y' = A\Delta Y + R + P\Delta \lambda$$

(4.1.1)

$$B_a \Delta Y(\vec{x}=0) = \Delta \lambda \frac{\partial}{\partial x} Y(\vec{x}=0)$$

$$B_b \Delta Y(\vec{x}=L/R) = \Delta \lambda \frac{\partial}{\partial x} Y(\vec{x}=L/R)$$

where

$$P = \partial F/\partial \lambda$$

(4.1.2)

These linearized equations have to be completed by a constraint equation that will be defined in the next paragraph after the differential equations have been discretized.

4.2. Solution of the constrained linear two-point boundary value problem

The linearized two-point boundary value problem is solved with a linear shooting method [30]. This method is chosen because it yields accurate results and because it has successfully been applied in the past to a wide variety of cylindrical shell buckling and vibration problems, e.g. [33,33].

It is well known that numerical instabilities cause that "single shooting" can only be applied to very short shells. If longer shell are to be analyzed one must use the so-called multiple or parallel shooting method. Let the integration interval be divided into $v$ subintervals and let $\vec{x}_i$ be the axial coordinates of the shooting points. Thus,

$$\vec{x}_1 = 0 < \ldots < \vec{x}_i < \ldots < \vec{x}_{v+1} = L/R$$

(4.2.1)

The solution of the linear BVP, within every subinterval, can be written as follows [30]:

$$\Delta Y(\vec{x}) = U_i(\vec{x})\Delta S_i + V_i(\vec{x}) + W_i(\vec{x})\Delta \lambda$$

$$\vec{x}_i < \vec{x} < \vec{x}_{i+1}$$

(4.2.2)
where $U_1(x)$ is the $k \times k$ fundamental solution matrix over the interval $[x_i, x_{i+1}]$ and is determined by the following homogeneous initial value problem:

$$U'_1 = AU_1 \quad \quad U_1(x_i) = I \quad (4.2.3)$$

where $I$ is an $k \times k$ identity matrix. $V_1$ and $W_1$ are elements of $R^k$ and are the solutions of the following inhomogeneous initial value problems:

$$V'_1 = AV_1 + R \quad \quad V_1(x_i) = 0$$
$$W'_1 = AW_1 + P \quad \quad W_1(x_i) = 0 \quad (4.2.4)$$

The vectors $\Delta S_i$ are equal to the values of the solution vector $\Delta Y$ in the shooting points $x_i$, which can easily be verified with the above equations. These vectors can be determined by imposing the continuity of the $v$ solutions $\Delta Y_1$ at the interval boundaries. These continuity conditions can be written as:

$$U_1(x_{i+1}) \Delta S_{i+1} = I \Delta S_i + \Delta \lambda \Delta W_1(x_{i+1}) = -V_1(x_{i+1}) \quad 1 \leq i \leq v - 1 \quad (4.2.5)$$

The boundary conditions at the edges of the shell become:

$$B_a \Delta S_{v+1} - B_{v} \Delta \lambda - B - Y(x = 0)$$
$$B_{v} U_1(x_{v+1}) \Delta S_{v+1} + [B_{v} W_1(x_{v+1}) - B_{v} \Delta Y_1(x_{v+1}) - B_{v} V_1(x_{v+1})] \Delta \lambda = 0 \quad (4.2.6)$$

Because $\lambda$ is treated as an unknown we must complete the linear equations (4.2.5) and (4.2.6) with an additional constraint equation. From the many possibilities we have chosen the linear constraint equation that is formulated as follows:

$$S^T \cdot \Delta S + \Delta \lambda = \alpha \Delta \mu \quad (4.2.7)$$

where

$$S = \{ S_1^T, \ldots, S_{i-1}^T, \ldots, S_v^T \}^T$$
\[
\dot{S} = \frac{\partial S}{\partial \mu}
\]

\[
\lambda = \frac{\partial \lambda}{\partial \mu}
\]  

(4.2.8)

In these equations \( S \) and \( \lambda \) are considered to be a function of the path parameter \( \mu \). The parameter \( \alpha \) equals 1 in the first iteration (prediction) and 0 in the following (correction). In this study we approximate the path derivatives as follows

\[
\dot{S} = \left( \frac{nS}{n} - \frac{n-1}{n} S \right) \lambda \left( \frac{nS}{n} - \frac{n-1}{n} S \right) \]

\[
\dot{\lambda} = \left( \frac{n\lambda}{n} - \frac{n-1}{n} \lambda \right) \lambda \left( \frac{n\lambda}{n} - \frac{n-1}{n} \lambda \right) 
\]  

(4.2.9)

where the superscript \( n \) and \( n-1 \) denote the last two obtained converged solutions and

\[
\tilde{S} = \begin{bmatrix} \tilde{S}^T, \lambda \end{bmatrix}^T
\]

(4.2.10)

The first increment in the computational procedure is calculated with \( \lambda \) prescribed. The vector \( S \) contains variables with different physical meaning namely displacement, rotations and stress and moment resultants. It may therefore be necessary to apply some sort of scaling [34]. In that case (4.2.7) can be replaced by

\[
\dot{S}^T V \Delta S + \beta \lambda \Delta \lambda = \alpha \Delta \mu
\]  

(4.2.11)

where \( V \) is a diagonal matrix \( V = \text{diag} \left( V_1, V_2, ..., V_{n\times n} \right) \) and \( V \) and \( \beta \) are scaling factors. The linear equations (4.2.5) – (4.2.7) have a special structure that is shown in Fig.4.1. We note here that the band matrix following from the shooting procedure is not symmetric. The symmetry of the entire set is also destroyed by the constraint equation. However, special equation solvers for these type of equations do exist [31]. We do not use these because the time needed to solve the linear set of equations is here a small fraction of the total computational cost. We solve the linear set with routines from the Linpack package for general full matrices.

The initial value problems (4.2.3) and (4.2.4) are integrated with the classical fourth order Runge-Kutta method. In contrast to elastic problems, we cannot use routines in which the step-size is automatically adjusted so that some sort of local error criterion is satisfied. This is of course due to the fact that one must follow the stress and strain history in the integration points which therefore must be fixed. An additional difficulty is that the functions must be evaluated in points between grid points, that is when \( x_i \) and \( x_{i+1} \) are two adjacent grid points then functions have to
be evaluated in \( x_i + 1/2h_i \), where \( h_i \) is the step-size. If the shell is elastic this is no problem, if the shell is elastoplastic however then we must know the plastic strains and the elastoplastic stiffnesses in these points. This implies that after we have determined the solution in the grid points with the shooting method we also must determine the solution in the intermediate points. The solution in intermediate points is determined with the following interpolation formula

\[
Y(\tilde{x}) = Y(\tilde{x}_j)(1 - 3\xi^2 + 2\xi^3) + Y(\tilde{x}_{j+1})(3\xi^2 - 2\xi^3) \\
+ Y'_{-}(\tilde{x}_j)(\xi - 2\xi^2 + \xi^3) + Y'_{+}(\tilde{x}_{j+1})(-\xi^2 + \xi^3)
\]  

(4.2.12)

where

\[
\xi = (\tilde{x} - \tilde{x}_j)/(\tilde{x}_{j+1} - \tilde{x}_j)
\]  

(4.2.13)

We note here that \( \tilde{x}_j \) denotes an arbitrary grid point in a shooting interval, not a shooting point. The value of \( \Delta Y'_{-} \) is found from the differential equations (3.1.4),(3.2.6) and \( \Delta Z \) is found from (3.1.2) or (3.2.3).

4.3. Numerical integration in radial and circumferential direction

In step 1 of the solution scheme, the elements of \( R \) and \( A \) are calculated. If the cross section is elastoplastic this requires the numerical evaluation of integrals in radial and circumferential direction of the shell. As an example, let us consider the calculation of the harmonic components of generalized plastic strains, e.g. the element \( (e^p_{k}) \) of \( E^p \) that is defined by:

\[
(e^p_{k}) = \frac{1}{\pi} \int_0^{2\pi} e^p \cos(l\tilde{y}) \, d\tilde{y}
\]  

(4.3.1)

With the definition of \( e^p_x \) follows:

\[
(e^p_{k}) = \frac{1}{R} \int_0^{2\pi} 1/2 \int_0^{1/2} e^p \cos(l\tilde{y}) \, dz \, d\tilde{y}
\]  

(4.3.2)

In section 3 it was argued that the integers \( l \) are multiples of the circumferential wavenumber of the asymmetric initial imperfection (in the case that we specify only one asymmetric imperfection). If we denote this wavenumber as \( n \) and we consider the assumed symmetry of the deformation pattern with respect to \( \tilde{y} = 0 \) then it can easily be seen
that we can limit the numerical integration in circumferential direction to the interval \([0, \pi/n]\). The integral (4.3.2) can then be approximated as follows

\[
\left(\mathbf{e}_p^c\right)(\chi) = \frac{R}{h} \sum_{i=1}^{N_g} \sum_{j=1}^{N_s} w_i^g w_j^s \eta^c(\chi, \tilde{a}_i, \tilde{b}_j) \cos(l \tilde{a}_i)
\]

(4.3.3)

where \(w_i^g\) and \(a_i\) are the weights and abscissae according to the Gauss integration rule [30], and \(N_g\) denotes the number of Gauss points. Likewise, \(w_j^s\) and \(b_j\) are the weights and the abscissae of Simpson’s rule and \(N_s\) is the number of integration points in radial direction. The number of integration points depends on the required accuracy with which (4.3.2) has to be approximated. In thickness direction 7 integration points are generally sufficient. In circumferential direction the number of points will depend on the number of Fourier terms included in the analysis. If the highest wavenumber in the Fourier expansion is for example 5n then we divide the integration interval into 10 subintervals. In every subinterval a three-point Gaussian quadrature rule is applied.

4.4. Integration of the constitutive law

An important step in every elastoplastic calculation is the integration of the constitutive equations, in all integration points, for a finite increment of the strains. Radial return or, more generally, return mapping algorithms have become very popular recently. Advantage of these algorithms is that they are efficient and reasonably accurate, even for relatively large strain increments. Besides this, the implicit algorithms are unconditionally stable and can consistently be linearized. This means that when they are used in conjunction with the Newton-Raphson method, the quadratic rate of convergence is kept when a path independent iteration procedure is used. In this study we have used the return mapping algorithm from Simo and Taylor [28] and Ramm and Matzenmiller [29]. Because this algorithm is well documented in the literature we just give a sketch of it and only summarize the most important formulae. For details the reader is referred to the above mentioned references.

The algorithm basically consists of two steps. In the first or predictor step an elastic trial stress state is calculated by assuming that the entire strain increment from the last converged configuration to the current intermediate configuration is elastic. This can be expressed as follows:

\[
\mathbf{\sigma}^* = D \left( \mathbf{n} - \mathbf{n}^p \right)
\]

(4.4.1)

where \(\mathbf{\sigma}^*\) denotes the trial stress state, \(\mathbf{n}\) the current configuration and \(\mathbf{n}^p\) the last obtained converged configuration. If
\[
\frac{1}{2} \sigma^* P \sigma - \frac{1}{3} \sigma^2 c_c (e_c) < 0
\]  

(4.4.2)

then the trial stress state lies inside the yield surface and in that case equals the final stress state \( i_\sigma \). If the trial stress state lies outside the yield surface then the stress state is brought back to a properly updated new yield surface by means of what is called plastic relaxation (this is the second or corrector step). The change in plastic strains in this relaxation phase is governed by the flow rule. The algorithm that we consider here is fully implicit which means that the flow rule is satisfied at the end of the strain increment. Thus we have:

\[
\Delta i_{\eta_p} = \Delta \lambda P \sigma
\]  

(4.4.3)

where the true stresses \( i_\sigma \) are given by:

\[
i_\sigma = D [i_{\eta} - \eta_p - \Delta \eta_p] = i_\sigma^* - D \Delta \eta_p
\]  

(4.4.4)

The reason why the last phase is called a plastic relaxation phase is obvious when one looks at expression (4.4.4).

From (4.4.3) and (4.4.4) the following expression for the stresses is derived:

\[
i_\sigma = [I - \Delta \lambda D P]^{-1} i_\sigma^*
\]  

(4.4.5)

From (2.1.13) it follows that the current value of the equivalent plastic strain is given by:

\[
i_{\text{ep}} = n_c \epsilon + \Delta \lambda \left( \frac{2}{3} \sigma P \sigma \right)^{1/2}
\]  

(4.4.6)

The plastic multiplier \( \Delta \lambda \) is derived from the condition that the final stress state must be on the yield surface. Substitution of (4.4.5) and (4.4.6) into the yield surface (2.1.11) gives a nonlinear equation in terms of \( \Delta \lambda \):

\[
\phi = \phi(\Delta \lambda) = 0
\]  

(4.4.7)

The value of the plastic multiplier is determined numerically with the Newton–Raphson method in order to ensure a proper rate of convergence. In the paper of Ramm and Matzenmiller an explicit expression of the yield surface in terms of trial stress and the plastic multiplier is given.
If a path–independent iteration procedure is used then the proper elastoplastic moduli are obtained by differentiating the integration algorithm. The derivation of these moduli goes as follows. The total stain in the current configuration can be decomposed as:

\[
\dot{\varepsilon} = \varepsilon + \Delta \dot{\varepsilon} + \Delta \varepsilon = \varepsilon + (D)^{-1} (\sigma - \sigma) + \Delta \lambda \dot{P} \sigma
\]

(4.4.8)

and can be differentiated

\[
d\dot{\varepsilon} = [(D)^{-1} + \Delta \lambda \dot{P}] d\dot{\varepsilon} + d\Delta \lambda \dot{P} \sigma
\]

(4.4.9)

Note that the derivatives of variables in the converged configuration vanish because this configuration is fixed. The derivative of the plastic multiplier follows from the derivative of the yield surface.

\[
d\phi = \sigma \dot{P} \sigma - \frac{2}{3} \sigma \varepsilon^{P} \sigma \varepsilon^{P} + d\dot{P} \varepsilon = 0
\]

(4.4.10)

The derivative of \(\dot{\varepsilon}^{P}\) follows from (4.4.6). Substituting this derivative into (4.4.10) and reordering the resulting expression gives:

\[
d\Delta \lambda = \frac{1}{A} \dot{\varepsilon} \sigma \dot{P} \sigma
\]

(4.4.11)

Finally, substituting (4.4.11) into (4.4.9) and inverting the resulting equation gives:

\[
d\dot{\varepsilon} = L d\dot{\varepsilon}
\]

(4.4.12)

with

\[
L = [H^{-1} - (A + \sigma \dot{P} \sigma \dot{P})^{-1} (\sigma \dot{P})^{-1} (\sigma \dot{P})^{-1}]
\]

(4.4.13)

and

\[
H = (D)^{-1} + \Delta \lambda \dot{P}
\]

(4.4.14)

We note here that if \(\Delta \lambda\) equals zero then the consistent elastoplastic moduli \(\dot{L}\) reduce to the continuum moduli \(L\) as given in chapter 4.2.
5. NUMERICAL EXAMPLES

A Fortran77 program (EPAC: ElastoPlastic Analysis of Cylindrical shells), based on the theory as presented in the previous chapters has been developed. The program runs on Sun workstations and Convex minisupercomputers. In order to validate the coding a number of elastic and elasto-plastic buckling problems known from literature have been analyzed.

5.1. Elastic bifurcation buckling of an axially compressed cylinder

Elastic bifurcation buckling of axially compressed cylindrical shells is a well studied problem and is therefore taken as a first test case. Although the present formulation is not especially developed for this type of problems, bifurcation points on a nonlinear prebuckling path can be found as follows. The asymmetric response of the shell is described by taking N=1 in the Fourier expansion and the wavenumber of the asymmetric term corresponds to the wavenumber of the required buckling mode. If a perfect shell is analyzed then the shooting matrix, i.e. the coefficient matrix of the linear set of equations as depicted in fig. 4.1, becomes singular in the bifurcation point, because two incremental solutions are possible. One solution along the fundamental path and one along the postbuckling path. If the solution is determined step by step along the prebuckling path, the determinant of the shooting matrix will change sign when the single bifurcation point is passed. Although the actual solution in the bifurcation point cannot be determined with the iterative scheme as described in this report, no difficulties are encountered in practice when bifurcation points are passed in the above described manner. Thus monitoring the value of the determinant of the shooting matrix will tell us approximately for what value of the load parameter a bifurcation point occurs. This method closely resembles the so-called determinant plotting method [36]. We note here that the above described method is inefficient and only serves as a check of the present formulation.

We have analyzed an elastic shell with R/h=100 and L/R=3.2 for eight sets of boundary conditions with the determinant plotting method. The same shell has also been analyzed by Yamaki [34] and Rotter&Teng [35]. Yamaki obtained results for Donnell's theory [8] and the modified Flugge theory [34] using double Fourier series and Galerkin's method. The results of Rotter&Teng are obtained with a finite element method applied to a shell theory that reduces to the modified Flugge theory in case of cylindrical shells. In table 1 we have given the boundary conditions using the same notation as Yamaki. In table 2, the results of our analysis are compared with the results of Yamaki and Rotter&Teng. In the mode column of this table the number of full waves of the buckling mode is given and it is indicated whether the buckling mode is symmetric (S) or anti-symmetric (A) with respect to x/L = 0.5. As can be seen the agreement between the different results is good.
5.2. Nonlinear elastic collapse of a cylindrical shell with a two-mode imperfection

As a second testcase we consider the nonlinear elastic collapse of a clamped cylindrical shell with one axisymmetric and one asymmetric imperfection. This example is taken from [3]. The following three two-mode imperfections are considered:

1) \( \bar{w} = 0.5 \cos(2\pi x/L) + 0.05 \cos(\pi x/L) \cos(13y/R) \)

2) \( \bar{w} = -0.5 \cos(2\pi x/L) + 0.05 \sin(\pi x/L) \cos(13y/R) \)

3) \( \bar{w} = 0.5 \cos(2\pi x/L) + 0.05 \sin(\pi x/L) \cos(13y/R) \) \hspace{1cm} (5.2.1)

The geometry of the shell is as follows: \( R/h = 1000 \) and \( L/R = 1 \), and the boundary conditions are

\[
\begin{align*}
\bar{n}_x &= \bar{n}_x & \bar{v} = w = \phi_x &= 0 & \tilde{x} = 0, \ L/R
\end{align*}
\]  \hspace{1cm} (5.2.2)

In table 3 the calculated limit loads are given. These loads are normalized by the classical buckling load of a perfect elastic shell (\( \bar{E}/(R, c=3(1-\nu^2)^{1/2} \)) and have been obtained with \( N=1 \) or \( N=2 \) (wavenumbers 13 and 26) in the Fourier series. In case 2 and 3, the response of the shell is symmetric with respect to the plane \( x = 0.5L \) because of the symmetry of the geometry, loads and imperfections. Thus, only a half shell needed to be analyzed. This was done by prescribing symmetry conditions \( (u = s, h = \phi_x) = 0 \) at \( x/L = 0.5 \). If the solution was determined over the entire shell 8 shooting intervals were chosen, otherwise 4. In every shooting interval 10 grid points were used. The results according to [3] have also been given in table 3. The agreement with our results is especially good when \( N=1 \). The reason for this is that in [3] solutions are obtained with the so-called extended analysis in which the circumferential variation of the radial displacement is also described by one Fourier term.

In fig. 5.1 we have shown the load versus end-shortening curves for case 2 with \( N=1 \). As can be seen, at the limit point of the shell the solution "snaps back". In order to pass the limit point it was necessary to use the continuation method as outlined in chapter 4.2, no scaling was applied. Generally four iterations were sufficient to satisfy the convergence criteria \( (\epsilon = \epsilon_d = 10^{-5}) \). The load versus end-shortening curves for case 1 and 3 are similar. For the cases 1 and 3 a noticeable difference is found between the maximum loads determined with \( N=1 \) and \( N=2 \). When \( N=2 \) convergence difficulties were encountered in the neighbourhood of the limit point. In case 3 the sign of the shooting determinant changed. Some additional computer runs showed that the perfect shell has a lowest non-dimensional buckling load of 0.910 for a mode with 26 full waves around the circumference. The most plausible explanation for the above mentioned convergence problems is that the shell bifurcates from a mode dominated by 13 full waves in circumference to a mode with 26 waves. Because the program is not especially developed for this type
of bifurcation problems this could not be further investigated. In figures 5.2-5.7 the calculated radial displacements at the limit point are shown.

5.3. Elastoplastic buckling of a long imperfect cylinder

This case concerns the buckling of a long cylindrical shell with one axisymmetric and one asymmetric imperfection. The effects of boundary conditions are completely neglected in this example. The axisymmetric imperfection is taken to be affine to the axisymmetric buckling mode of the perfect shell, this mode corresponds to the lowest buckling load (tangent modulus load) of the perfect plastic shell. The normalized buckling load \( n_{\text{tan}} \) and the normalized half wavelength \( l_{c} \) of the buckling mode are given by [37]:

\[
\begin{align*}
  n_{\text{tan}} &= N_{\text{tan}}/(\sigma_{y} h) = 2\frac{E}{\sigma_{y} R} \left[ 3 \left( \frac{5 - 4v}{2} \right)^{1/2} - \left( 1 - 4v \right) \right]^{-1/2} \\
  l_{c} &= \frac{L_{c}}{R} = \frac{h}{2\pi \left[ \frac{h}{R} \left( \mu + 3 \right) \right]^{1/2}} \left[ \frac{3 \left( \frac{5 - 4v}{2} \right)}{\left( 1 - 4v \right)^{2}} \right]^{-1/4}
\end{align*}
\]  

(5.3.1)

where \( \mu = E/\epsilon_{t} \), \( \epsilon_{t} = d\sigma/d\varepsilon \) is the tangent modulus of the stress–strain curve, and \( v \) is Poisson's ratio. This problem has been studied in the past by Tvergaard [25] who obtained numerical results for a Sanders-Koiter type shell theory with a finite-element method in combination with a Fourier decomposition of the displacements in circumferential direction of the shell. Tvergaard used the corner theory of Cristoffersen and Hutchinson [24] to describe the material behaviour. However, some results are also given for the \( J_{2} \) flow theory (this is a limiting case in the corner theory).

The uni-axial material behaviour is specified as follows:

\[
\sigma = \sigma_{y} \quad \varepsilon = \frac{\sigma_{y}}{E} \left[ \frac{1 - \left( \frac{\varepsilon}{\sigma_{y}} \right)^{n} - 1}{\frac{1}{n} + 1} \right] \quad \sigma > \sigma_{y}
\]

(5.3.2)

where \( n \) is a hardening parameter and \( \sigma_{y} \) is the initial yield stress. The data for the shell that we consider here is as follows:

**Geometry:**

\[
R/h = 100 \quad ; \quad L/R = 2 \times \frac{L_{c}}{R} = 0.3211
\]

**Material parameters:**

\[
\sigma_{y}/E = 0.0025 \quad ; \quad n = 10 \quad ; \quad v = 0.3
\]
Boundary conditions:

\[ \begin{align*}
&\text{at } x/L = 0: \quad s_{xx} = h = \phi_x = 0 \quad u = \bar{u} \\
&\text{at } x/L = 1: \quad s_{xx} = h = \phi_x = 0 \quad (\text{symmetry})
\end{align*} \]

Geometric imperfection:

\[ w = \bar{\xi}_1 \cos(2\pi x/L) + \bar{\xi}_2 \cos(\pi x/L) \cos(n y/R) \]  \hspace{1cm} (5.3.3)

Numerical results are obtained by shooting over 4 intervals. In every interval 10 grid points are used. Over the thickness of the shell 7 integration points are used, a higher number of integration points across the thickness hardly influences the results. The number of integration points and the integration interval along the circumference of the shell are chosen according to the rule as discussed in paragraph 4.3. The program EPAC requires a piece-wise linear stress-strain curve as input. The curve (5.3.2) is therefore approximated by a number of straight segments. The points that define this curve are given in table 4 and the approximated curve is shown in fig. 5.8.

It is known from the work of Tvergaard [25] that this shell first bifurcates into an axisymmetric mode given by (5.3.1), then attains a limit load that is slightly higher than the bifurcation load and then eventually bifurcates into an asymmetric mode with an axial wavelength twice as long as the axisymmetric mode and with 9 full waves along the circumference. To simulate this behaviour we have analyzed a shell with a very small axisymmetric imperfection (\( \bar{\xi}_1 = 10^{-4} \)) and a very small asymmetric imperfection (\( \bar{\xi}_2 = 10^{-4} \)). The results of this analysis are shown in fig. 5.9-5.12. In fig. 5.9 the axial load \( n_x^* \) per unit length versus average axial strain \( u_x/L \) curve is given. We have found a first bifurcation point for \( n_x^* = 1.1508 \). The exact value of \( n_x^* \) according to (5.3.1) and the stress-strain data in table 4 is \( n_x^* = 1.1493 \). Both the average axial strain at the bifurcation points and the limit point as well as the value of the axial load at these points are in very good agreement with the results found by Tvergaard [25], see table 5. In fig. 5.10 the radial displacement at the first bifurcation point, the limit point and the secondary bifurcation point are shown. Even in the post-buckling range the radial displacement is more or less affine to the sinusoidal buckling mode. However, this is not the case for the circumferential stress resultant as is shown in fig. 5.11. As can be seen, at the limit point \( n_y \) is sinusoidal of shape, at the secondary bifurcation point however, higher harmonics become important. The reason for the secondary bifurcation is clear from this picture. At the places where the radial displacement points inwards (positive \( w \)), large circumferential compressive forces occur which eventually leads to the secondary bifurcation. In fig. 5.12 we have shown the response of the shell just after the secondary bifurcation.

In figure 5.13 we have shown the calculated response of a shell with different axisymmetric imperfections, namely \( \bar{\xi}_1 = 0.02, 0.1 \) and 0.4. The points where the shell bifurcated into an asymmetric mode have been indicated in the figure. For the two smaller imperfections the buckling mode has 9 full waves in circumferential direction of the
shell whereas for the largest imperfection the buckling mode has 8 full waves. The points have been determined accurately with the determinant plotting method first. After this, the analysis was repeated by analyzing a shell with a very small asymmetric imperfection ($\bar{\xi}_2 = 10^{-4}$) yielding the postbuckling curves as shown in figure 5.13. In table 5 a comparison is made with the Tvergaard's results [25], generally the agreement is good.

In fig. 5.14 results are given for the case that $\bar{\xi}_1 = 0.1$ and $\bar{\xi}_2 = 0$, -0.01, -0.1 and -0.4. The asymmetric imperfection has 9 full waves in circumferential direction and the response of the shell is described with $N=2$ (wavenumbers 9 and 18). The reason why $N = 2$ is chosen will be motivated later. In all these cases the axial end-displacement $u_L$ of the shell is prescribed with $\Delta u_L = 0.004$. In table 5 we compare our results with those of Tvergaard [25] (not for $\bar{\xi}_2 = -0.4$). Generally the agreement is good, we should note however that Tvergaards results for the two-mode imperfection are obtained with the corner theory instead of the $J_2$ flow theory. In fig. 5.15 the effect of the value of the prescribed end-displacement on the accuracy of the results is investigated. Results are given for $\Delta u_L = 0.004$ and 0.032. In fig. 5.17 and 5.18 we compare the radial displacement and the circumferential stress resultants for these cases. From these figures follows that the accuracy of the solution remains unchanged even if rather large load steps are taken. However, we should remind that the prescribed average axial strain is still rather small ($-0.1\%$ for $\Delta u_L = 0.032$). In table 6 we have compared the convergence rates of the iteration procedure for the cases that we use consistent or continuum elastoplastic moduli. The advantage of the consistent formulation is clear from this table. In fig. 5.16 results are given for $\bar{\xi}_1 = 0.1$ and $\bar{\xi}_2 = -0.01$ for $N = 1, 2, 3$ and 4. From this figure follows that with $N=1$ the response of the shell is not described accurately after the limit load has been obtained. However, there is hardly any difference in the value of the limit load. The cases with $N = 2, 3$ and 4 hardly differ. Thus for this kind of analysis two asymmetric terms in Fourier expansion is enough to predict the load versus end-shortening behaviour of these shells.

5.4. Elastoplastic bifurcation buckling of a clamped cylinder

The last example concerns the elastoplastic buckling of a perfect shell that is clamped at one edge. This problem has been studied by Tvergaard [26] who showed that bifurcation into an asymmetric mode can occur after the load maximum and that localization of the deformations at the edges of the shell delays the occurence of the secondary bifurcation point. Tvergaard has only given results obtained with a corner theory. The shell that we analyze is the same as the one in the previous paragraph only now with $L=5L_c, 7.5L_c, 10L_c$ and $15L_c$ and with different boundary conditions. These boundary conditions are specified as:

\[
\begin{align*}
\text{at } x/L = 0: & \quad v = w = \phi_x = 0 \quad u = \bar{u} \\
\text{at } x/L = 1: & \quad s_{xy} = h_x = u = \phi_x = 0 \quad (\text{symmetry})
\end{align*}
\]
For the case that $L=10\ell_c$ we have shown the results of the analysis in figures 5.19-5.22. Shooting over eight intervals was applied and in every interval 5 grid points were used. Across the thickness 7 integration points were used. In fig. 5.19 the calculated normalized axial load $P/P_{\text{bif}}$ ($P_{\text{bif}}$ is the tangent modulus load of the perfect shell) is given as a function of the average axial strain $u_L/L$. The maximum load is slightly smaller than the tangent modulus load. In this figure the secondary bifurcation point has also been indicated. In figure 5.21 the radial displacements and the circumferential stress resultant at the point of initial yielding, the limit point and the bifurcation point are shown. Note that the characteristic wavelength of the deformation pattern is almost equal to the wavelength of the buckling mode of the perfect shell when edge effects are neglected. The asymmetric buckling mode is shown in fig. 5.22. This mode has been calculated by including the effect of a very small imperfection. Note that the maximum radial displacement is more or less at the place with the highest circumferential compressive force.

The effect of the shell length is that the bifurcation into an asymmetric mode is more and more delayed when the shell becomes shorter. For $L=5\ell_c$ we did not find a bifurcation point anymore. For the shells with $L/\ell_c = 7.5$, 10 and 15 we have found that bifurcation into an asymmetric mode with 8 full waves in circumferential direction occurred at an average axial strain of approximately 0.0045, 0.0041 and 0.0037, respectively. Tvergaard, using a corner theory, did find a bifurcation point for the shortest shell at an average axial strain of 0.0042. This difference with our results is probably due to the difference in the material description. It is known that use of a yield criterion with a smooth yield surface, such as the von Mises criterion, delays the occurrence of a bifurcation into an asymmetric mode [25].
6. DISCUSSION AND CONCLUSIONS

A semi-analytical formulation for the elastoplastic analysis of initially imperfect cylindrical shells under axial compression and/or lateral pressure has been formulated. Key element in this formulation is the approximation of the state variables, loads and imperfections in circumferential direction of the shell by truncated Fourier series. A set of coupled algebraic-ordinary differential equations has been derived, formulated in terms of the Fourier components of generalized stresses, displacements and plastic strains, that describe the mechanical behaviour of the shell. These equations are solved numerically with the method of quasi-linearization. A set of linearized equations that consistently takes into account both geometrical and material nonlinearities is therefore derived. In this approach all Fourier components are coupled, this in contrast to the pseudo-load approach used by Wunderlich et. al. [11]. Although this leads to the numerical integration of one large system of differential equations instead of a number of smaller ones, which is more efficient, our approach ensures a rapid convergence. This latter can be important because the integration of the constitutive equations can be rather expensive. In our approach it is also quite simple to include some kind of continuation procedure that enables us to trace the response of the shell through limit points. Here we have used the classical arc-length procedure of Riks [38]. Another advantage of analyzing a consistently linearized set of equations is that the increments in the incremental procedure can be quite large, both in the elastic and the elastoplastic region. From computations it followed that the step-size has little influence on the actual value of the limit load.

A number of elastic and elastoplastic buckling problems known from literature have been reexamined. A comparison of the results generally shows a good agreement. Only for the case of a clamped perfect shell that buckles in an asymmetric mode in the plastic region was a noticeable discrepancy detected. Bifurcation into an asymmetric pattern, that occurs after the limit point, took place for larger values of the end-shortening than found by Tvergaard. However this can be explained from the differences in the material description.

A number of improvements and extensions of the present formulation are possible. As stated in the introduction, main objective of the present study is to provide a simple computational tool that can analyze axially compressed elastoplastic shell taking into account the effects of both an axisymmetric and one asymmetric imperfection, and the effects of the boundary conditions. The computational costs of the present formulation can be reduced in two ways. First, the linear shooting method can be replaced by a simple finite difference method [31], e.g. a trapezoidal scheme. Advantage of the shooting method is that accurate results are obtained. However, if a large number of Fourier terms is included in the analysis then the number of differential equations that has to be integrated becomes very large. This number is proportional to N^2, where N is the number of Fourier terms included. But, if a simple finite difference is used then a larger number of mesh points would be needed to obtain the same accuracy. Second,
the layered approach can be replaced by the so-called global plasticity method. Advantage of this method is that one circumvents the integration of plastic strains (or stresses) and stiffnesses across the thickness of the shell. This will probably give a substantial decrease of the cpu-times. It is obvious however, that the spread of plastic zones in the shell will not be described as accurately as in the layered approach.

Extensions of the formulation are possible. First other plasticity models could be included. E.g. a model that includes kinematic hardening or a model that is more appropriate for bifurcation buckling problems. Second and more important is to include the effect of stiffening elements such as rings or stringers. As a first approximation a kind of smeared stiffener approach [8] could be used although this approach is usually only applied in the analysis of elastic shells. More accurate would be to consider rings or stringers as discrete elements so that local effects can be described. This is a subject of further investigation.

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REFERENCES


Appendix A: equilibrium equations in canonical form

Recall that equilibrium of the shell is governed by the principle of virtual work (2.2.2). As is well known equilibrium equations and the appropriate Kirchhoff boundary conditions can be obtained from this principle by means of variational calculus. These derivations are standard and we therefore just give the result. The equilibrium equations in nondimensional form are:

\[
\begin{align*}
\frac{\ddot{n}_x}{x} + n_{xy} \frac{\ddot{m}_y}{y} + p_x &= 0 \\
\frac{\ddot{n}_{xy}}{x} + n_{xx} \frac{\ddot{m}_x}{x} + n_{xy} \frac{\ddot{m}_y}{y} + n_x \left( \phi + \frac{\partial \tilde{\phi}}{\partial y} \right) + n_y \left( \phi + \frac{\partial \tilde{\phi}}{\partial x} \right) + p_y &= 0 \\
\frac{\ddot{m}_x}{x} + 2m_{xy} \frac{\ddot{m}_y}{y} + n_{xx} \frac{\ddot{m}_x}{x} + n_{xy} \frac{\ddot{m}_y}{y} + R \frac{h}{h} \left[ n_x \left( \phi + \frac{\partial \tilde{\phi}}{\partial y} \right) + n_y \left( \phi + \frac{\partial \tilde{\phi}}{\partial x} \right) \right] + p_x &= 0
\end{align*}
\]

(A.1)

The first two partial differential equations are the in-plane equilibrium equations. The third one is the out-of-plane equilibrium equation. The boundary conditions at the edges of the cylindrical shell are such that one can prescribe

\[
\begin{align*}
u \text{ or } n_x; \quad v \text{ or } s_{xy}; \quad w \text{ or } h; \quad \tilde{\phi} \text{ or } m_x
\end{align*}
\]

(A.2)

where \( s_{xy} \) and \( h_x \) are reduced shear forces defined by

\[
\begin{align*}
s_{xy} &= n_{xy} - \frac{h}{R} \frac{m_{xy}}{x} \\
h_x &= \frac{h}{R} \frac{m_x}{x} + 2 \frac{h}{R} \frac{m_{xy}}{xy} - n_x \left( \phi + \frac{\partial \tilde{\phi}}{\partial y} \right) - n_y \left( \phi + \frac{\partial \tilde{\phi}}{\partial x} \right)
\end{align*}
\]

(A.3)

It is now our objective to formulate the equilibrium equations in terms of the variables that can be prescribed at the edges of the shell (these variables are called the basic variables in the main text). Elimination of \( n_{xy} \) from the in-plane equilibrium equations with (A.3) gives:

\[
\begin{align*}
\frac{\ddot{n}_x}{x} + \frac{\ddot{s}_{xy}}{y} + \frac{\ddot{m}_y}{y} + p_x &= 0 \\
\frac{\ddot{s}_{xy}}{x} + n_{xy} \frac{\ddot{m}_x}{x} + \left[ (s_{xy} + h_x)(\phi + \frac{\partial \tilde{\phi}}{\partial y}) + n_x (\phi + \frac{\partial \tilde{\phi}}{\partial x}) \right] + p_y &= 0
\end{align*}
\]

(A.4)
The rotations $\phi_y$, $\bar{\phi}_y$ can be expressed in terms of $v$, $w$ and $\bar{w}$ with the second equation of (2.1.2). This gives the first two equations of (2.3.1). Elimination of the axial derivative of $m_x$ and $n_{xy}$ from the out-of-plane equilibrium equation with the equations (A.3) yields:

$$h_{,x} + n_{,y} + h_{,x} = \left\{ \left( s_{xy} + \frac{h}{R} m_{xy} \right) \left( \phi_{,x} + \bar{\phi}_{,x} \right) + n_{,y} \left( \phi_{,y} + \bar{\phi}_{,y} \right) \right\} - p_2 = 0 \quad (A.5)$$

Elimination of $\phi_y$ and $\bar{\phi}_y$ yields the out-of-plane equilibrium equation in canonical form. The last equation from (2.3.1) is nothing else than the second equation of (A.3) from which $n_{xy}$ and $\phi_y$ have been eliminated.
Appendix B: partially inverted constitutive equations

The constitutive equations in total form for a material which elastic behaviour is linear isotropic can be formulated in terms of the nondimensional variables as follows:

\[
\begin{align*}
n_{x} &= d_{11}^{s}(e_{x} - e_{p}) + d_{12}^{s}(e_{y} - e_{p}) \\
m_{x} &= d_{11}^{b}(k_{x} - k_{p}) + d_{12}^{b}(k_{y} - k_{p}) \\
n_{y} &= d_{12}^{s}(e_{x} - e_{p}) + d_{22}^{s}(e_{y} - e_{p}) \\
m_{y} &= d_{12}^{b}(k_{x} - k_{p}) + d_{22}^{b}(k_{y} - k_{p}) \\
n_{xy} &= d_{33}^{s}(e_{xy} - e_{p}) \\
m_{xy} &= d_{33}^{b}(k_{xy} - k_{p})
\end{align*}
\]  \(B.1\)

where the nondimensional stretching and bending stiffnesses are:

\[
\begin{align*}
d_{11}^{s} &= (E/\sigma_{s})(h/R)/(1-\nu^{2}) \\
d_{11}^{b} &= 1/12 d_{11}^{s} \\
d_{12}^{s} &= \nu d_{11}^{s} \\
d_{12}^{b} &= \nu d_{11}^{b} \\
d_{33}^{s} &= (G/\sigma_{s})(h/R) \\
d_{33}^{b} &= 1/12 d_{33}^{s}
\end{align*}
\]  \(B.2\)

The stress resultant \(n_{xy}\) is eliminated from these equations by introduction of the reduced shear force \(s_{xy}\):

\[
\begin{align*}
s_{xy} &= n_{xy} - \frac{h}{R} m_{xy} \\
&= d_{33}^{s}(e_{xy} - e_{p}) + \frac{h}{R} d_{33}^{b}(k_{xy} - k_{p}) \\
&= \tilde{d}_{33}^{s}(e_{xy} - e_{p}) + \tilde{d}_{33}^{b}(k_{xy} - k_{p})
\end{align*}
\]  \(B.3\)

where we have introduced:

\[
\begin{align*}
\tilde{k}_{xy} &= k_{x} + \frac{h}{R} e_{x} = 2\phi_{x} + \frac{h}{R} u_{x} + \phi_{x} + \phi_{xy} + \phi_{xy}
\end{align*}
\]  \(B.4\)

and

\[
\tilde{d}_{33}^{s} = d_{33}^{s} + \left(\frac{h}{R}\right)^{2} d_{33}^{b}
\]
\[ d_{33} = -h d_{33} \]  

(B.5)

The constitutive equation for \( m_{xy} \) is rewritten as follows:

\[ m_{xy} = d^{-bs}_{33} (e_x e_y - e^p_x e^p_y) + d^b_{33} (\bar{k}_{xy} - \bar{k}^p_{xy}) \]  

(B.6)

Equations (B.1), (B.2) and (B.14) can immediately be written as follows:

\[
\begin{align*}
  e_x &= c_{11} n_x + c_{14} (e_y - e^p_y) + e^p_x \\
  e_{xy} &= c_{22} s_x + c_{26} (\bar{k}_{xy} - \bar{k}^p_{xy} - h \bar{k}_{xy}) + e^p_{xy} \\
  k_x &= c_{33} m_x + c_{35} (k - k^p_y) + k^p_x
\end{align*}
\]  

(B.7)

where the elastic constants \( c_{ij} \) are:

\[
\begin{align*}
  c_{11} &= 1 / d^s_{11} & c_{14} &= -d^s_{12} / d^s_{11} \\
  c_{22} &= 1 / d^s_{33} & c_{26} &= -d^s_{33} / d^s_{33} \\
  c_{33} &= 1 / d^b_{33} & c_{35} &= -d^b_{35} / d^b_{33}
\end{align*}
\]  

(B.8)

These equations can be used to eliminate \( e_x, e_{xy} \) and \( k_x \) from the constitutive equations for \( n_y, m_{xy} \) and \( m_y \), respectively. The resulting equations can be written as:

\[
\begin{align*}
  n_y &= c_{41} n_x + c_{44} (e_y - e^p_y) \\
  m_y &= c_{55} m_x + c_{55} (k - k^p_y) \\
  m_{xy} &= c_{62} s_{xy} + c_{66} (\bar{k}_{xy} - \bar{k}^p_{xy} - h \bar{k}_{xy}) \bar{k}^p_{xy}
\end{align*}
\]  

(B.9)

with
\[c_{41} = -c_{14} = \frac{d_s^{12}}{d_s^{11}}\]
\[c_{44} = d_s^{11} - \frac{(d_s^{12})^2}{d_s^{11}}\]
\[c_{53} = -c_{35} = \frac{d_b^{12}}{d_b^{11}}\]
\[c_{55} = d_b^{11} - \frac{(d_b^{12})^2}{d_b^{11}}\]
\[c_{62} = -c_{26} = \frac{d_{bs}^{12}}{d_{bs}^{11}}\]
\[c_{66} = d_{bs}^{11} - \frac{(d_{bs}^{12})^2}{d_{bs}^{11}}\]  
\[(B.10)\]

Equations (2.3.4, 2.3.5) are obtained from (B.7) and (B.9) by elimination of the strains $\varepsilon_y$, $k_y$, and $\kappa_{xy}$ with the strain-displacement relations (2.2.3) and (B.4). Although the derivation given here is restricted to a material whose elastic behaviour is isotropic it is not difficult to generalize the method to more general material behaviour, e.g. orthotropic or anisotropic behaviour.
Appendix C: the coupled set of nonlinear differential/algebraic equations

First the equations for the axisymmetric terms in the Fourier expansion will be given.

\[ (n_x^o)' = - p_x^o \]

\[ (h_x^o)' = - n_y^o + p_z^o \]

\[ (m_x^o)' = \frac{R}{h} h_x^o + \frac{R}{h} \sum_{j=0}^{N} c_{ij} x^o (\varphi_x^j + \varphi_y^j) \]

\[ (u_x^o)' = c_{11} n_x^o + c_{14} [e_y^o - (e_y^o)^o] + (e_x^o)^o - \frac{R}{2h} \sum_{j=0}^{N} c_{ij} \varphi_x^j (\varphi_x^j + 2\varphi_y^j) \]

\[ (w^o)' = - \frac{R}{h} \varphi_x^o \]

\[ (\phi_x^o)' = c_{33} m_x^o + (k_x^o)^o - c_{35} (k_y^o)^o \]  \hspace{1cm} (C.1)

The nonlinear algebraic equations are:

\[ n_y^o = c_{41} n_x^o + c_{44} [e_y^o - (e_y^o)^o] \]

\[ m_y^o = c_{53} m_x^o - c_{55} (k_y^o)^o \]  \hspace{1cm} (C.2)

The parameter \( c \) is defined as follows:

\[ c_j = 1 \quad \text{if} \quad j = 0 \quad \text{or} \quad c_j = 1/2 \quad \text{if} \quad j \neq 0 \]  \hspace{1cm} (C.3)

The strain \( e_y^o \) is a function of the displacements and is given by:

\[ e_y^o = - w^o + \frac{R}{4h} \sum_{j=1}^{N} \varphi_y^j (\varphi_y^j + 2\varphi_y^j) \]  \hspace{1cm} (C.4)

The terms depending on the generalized plastic strains will be defined later on.
The nonlinear differential equations for the asymmetric terms are as follows:

\[
\begin{align*}
(n)_x^{'} &= \frac{(i_s)^2}{i_{xy}} \frac{h}{1} \frac{m}{i_{xy}} - p \frac{i}{x} \\
(s_y)_x^{'} &= \frac{i}{y} \frac{h}{1 \frac{m}{i_{xy}}} \frac{\Sigma}{i_{xy}} \frac{b_{ikj}}{(s + \frac{h}{m} \frac{i}{x} + \frac{\phi_i}{x} + \frac{\phi_j}{x} + \frac{n}{(\phi_i + \phi_j)})} - p \frac{i}{y} \\
(h)_x^{'} &= -\frac{i}{y} \frac{h}{2} \frac{m}{1} \frac{\Sigma}{i_{xy}} \frac{b_{ikj}}{(s + \frac{h}{m} \frac{i}{x} + \frac{\phi_i}{x} + \frac{\phi_j}{x} + \frac{n}{(\phi_i + \phi_j)})} - p \frac{i}{y} \\
(m)_x^{'} &= \frac{R}{h} \frac{i}{x} \frac{2}{2m} \frac{\Sigma}{i_{xy}} \frac{a_{ijk}}{(\phi_i + \phi_j)} + \frac{R}{h} \frac{\Sigma}{i_{xy}} \frac{b_{ikj}}{(s + \frac{h}{m} \frac{i}{x} + \frac{\phi_i}{x} + \frac{\phi_j}{x} + \frac{n}{(\phi_i + \phi_j)})} - p \frac{i}{y} \\
(u)_x^{'} &= c_{i1} \frac{i}{x} + c_{i2} \frac{i}{x} - \frac{R}{h} \frac{\Sigma}{i_{xy}} \frac{a_{ijk}}{(\phi_i + \phi_j)} + \frac{R}{h} \frac{\Sigma}{i_{xy}} \frac{b_{ikj}}{(s + \frac{h}{m} \frac{i}{x} + \frac{\phi_i}{x} + \frac{\phi_j}{x} + \frac{n}{(\phi_i + \phi_j)})} - p \frac{i}{y} \\
(v)_x^{'} &= c_{i3} \frac{i}{x} + c_{i4} \frac{i}{x} - \frac{R}{h} \frac{\Sigma}{i_{xy}} \frac{a_{ijk}}{(\phi_i + \phi_j)} + \frac{R}{h} \frac{\Sigma}{i_{xy}} \frac{b_{ikj}}{(s + \frac{h}{m} \frac{i}{x} + \frac{\phi_i}{x} + \frac{\phi_j}{x} + \frac{n}{(\phi_i + \phi_j)})} - p \frac{i}{x} \\
(w)_x^{'} &= -\frac{R}{h} \frac{i}{x} \\
(a)_x^{'} &= c_{i5} \frac{i}{x} + c_{i6} \frac{i}{x} - \frac{R}{h} \frac{\Sigma}{i_{xy}} \frac{a_{ijk}}{(\phi_i + \phi_j)} + \frac{R}{h} \frac{\Sigma}{i_{xy}} \frac{b_{ikj}}{(s + \frac{h}{m} \frac{i}{x} + \frac{\phi_i}{x} + \frac{\phi_j}{x} + \frac{n}{(\phi_i + \phi_j)})} - p \frac{i}{x} \\
(C.4)
\end{align*}
\]

The algebraic equations become:

\[
\begin{align*}
n_y &= c_{i3} \frac{i}{x} + c_{i4} \frac{i}{x} - \frac{R}{h} \frac{\Sigma}{i_{xy}} \frac{a_{ijk}}{(\phi_i + \phi_j)} \\
m_y &= c_{i5} \frac{i}{x} + c_{i6} \frac{i}{x} - \frac{R}{h} \frac{\Sigma}{i_{xy}} \frac{b_{ikj}}{(s + \frac{h}{m} \frac{i}{x} + \frac{\phi_i}{x} + \frac{\phi_j}{x} + \frac{n}{(\phi_i + \phi_j)})} \\
m_{xy} &= c_{i7} \frac{i}{x} + c_{i8} \frac{i}{x} - \frac{R}{h} \frac{\Sigma}{i_{xy}} \frac{b_{ikj}}{(s + \frac{h}{m} \frac{i}{x} + \frac{\phi_i}{x} + \frac{\phi_j}{x} + \frac{n}{(\phi_i + \phi_j)})} \\
(C.5)
\end{align*}
\]

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In these expressions $\phi^i_j$ and $\phi^i_{xy}$ are functions of the displacements and are given by:

$$\phi^i_j = \phi^i_j(\phi^k_j + 2\phi^k_j)$$

$$\phi^i_{xy} = -2\phi^i_x$$

where $\phi^i_j$ and $\phi^i_{xy}$ are given by the exact expressions

$$\phi^i_y = v^i - w^i$$

The Fourier components of the plastic membrane and bending strains are:

$$e^{p,j}_x = \frac{1}{\pi d_i} \int_0^{2\pi} e^p_x \cos(1\tilde{y}) \, d\tilde{y}$$

$$k^{p,j}_x = \frac{1}{\pi d_i} \int_0^{2\pi} k^p_x \cos(1\tilde{y}) \, d\tilde{y}$$

$$e^{p,i}_y = \frac{1}{\pi d_i} \int_0^{2\pi} e^p_y \cos(1\tilde{y}) \, d\tilde{y}$$

$$k^{p,i}_y = \frac{1}{\pi d_i} \int_0^{2\pi} k^p_y \cos(1\tilde{y}) \, d\tilde{y}$$

$$e^{p,ij}_y = \frac{1}{\pi d_i} \int_0^{2\pi} e^p_{ij} \cos(1\tilde{y}) \, d\tilde{y}$$

$$k^{p,ij}_y = \frac{1}{\pi d_i} \int_0^{2\pi} k^p_{ij} \cos(1\tilde{y}) \, d\tilde{y}$$

$$d_i = 1/2 \quad \text{if} \quad i = 0$$

$$d_i = 1 \quad \text{if} \quad i \neq 0$$

where the plastic membrane- and bending strains are defined by (2.2.6). The integrals (C.8) are obtained by numerical integration in $\tilde{y}$- and $\tilde{z}$ direction. The formulation of the nonlinear first-order ordinary differential equations is completed by specifying $a_{ijk}$ and $b_{ijk}$ as follows:

$$a_{ijk} = \frac{1}{\pi} \int_0^{2\pi} \cos^2(1\tilde{y}) \cos^2(1\tilde{y}) \, d\tilde{y}$$

$$b_{ijk} = \frac{1}{\pi} \int_0^{2\pi} \sin^2(1\tilde{y}) \sin^2(1\tilde{y}) \, d\tilde{y}$$

(C.9)
Appendix D: derivation of equation (3.2.5)

Recall that the linearized relations between generalized plastic strains and generalized strains for a finite increment can be written as follows, see eqs.(2.2.9):

\[
\Delta \varepsilon_p^p = h_p^{(1)} \Delta \varepsilon + h_p^{(2)} \Delta k
\]

\[
\Delta k_p^p = 12 [ h_p^{(2)} \Delta \varepsilon + h_p^{(3)} \Delta k ]
\]

(D.1)

It will be convenient to represent these equations as follows:

\[
\Delta \varepsilon_p^p = k_{ij} \Delta \varepsilon + k_{ij} \Delta \varepsilon + k_{ij} \Delta \varepsilon + k_{ij} \Delta k + k_{ij} \Delta k + k_{ij} \Delta k
\]

(D.2)

where

\[
\varepsilon_p^p = \varepsilon_p^p ; \quad \varepsilon_p^p = \varepsilon_p^p ; \quad \varepsilon_p^p = \varepsilon_p^p ; \quad \varepsilon_p^p = \varepsilon_p^p ; \quad \varepsilon_p^p = \varepsilon_p^p
\]

(D.3)

and

\[
k_{ij} = [ h_p^{(1)} ]_{ij} \quad 1 \leq i \leq 3, 1 \leq j \leq 3
\]

\[
k_{ij} = [ h_p^{(2)} ]_{ij} \quad 1 \leq i \leq 3, 4 \leq j \leq 6
\]

\[
k_{ij} = 12 [ h_p^{(2)} ]_{ij} \quad 4 \leq i \leq 6, 1 \leq j \leq 3
\]

\[
k_{ij} = 12 [ h_p^{(3)} ]_{ij} \quad 4 \leq i \leq 6, 4 \leq j \leq 6
\]

(D.4)

The generalized total strain increments can be expressed in terms of generalized displacement increments.

\[
\Delta \varepsilon_p^p = k_{ij} [ \Delta u + \Delta v + \Delta w ] + k_{ij} [ \Delta v + \Delta w - \Delta u - \Delta w - ( \phi_x + \phi_y ) ( \Delta v + \Delta w ) ] +
\]

\[
k_{ij} [ \Delta u + \Delta v + \Delta w ] + k_{ij} [ \Delta u + \Delta v + \Delta w ] + k_{ij} [ \Delta v + \Delta w ] + k_{ij} [ \Delta v + \Delta w ] -
\]

\[
\frac{h}{R} k_{ij} ( \Delta v + \Delta w ) + k_{ij} ( 2 \Delta \phi_x - \frac{h}{R} \Delta v )
\]

(D.5)
The displacement variables are described by the truncated Fourier series (2.4.2). Substitution of these series into the above expression yields after some reordering:

\[
\Delta e^p_i = \sum_{j=0}^{N} k_j \cos \bar{y} (\Delta u^j) + \sum_{j=1}^{N} (k_{13} - \frac{h}{R} k_{25}) \sin \bar{y} (\Delta v^j) + \sum_{j=0}^{N} k_{14} \cos \bar{y} (\Delta \phi^j) - \\
\sum_{j=1}^{N} k_{13} \sin \bar{y} \Delta u^j + \\
\sum_{j=1}^{N} \left( (k_{12} - \frac{h}{R} k_{15}) \cos \bar{y} - [k_{12}(\phi + \bar{\phi}_y) + k_{13}(\phi + \bar{\phi}_x)] \sin \bar{y} \right) \Delta v^j - \\
k_{12} \Delta w^o - \\
\sum_{j=1}^{N} \left( (k_{12} - \frac{h}{R} k_{15}) \cos \bar{y} - [k_{12}(\phi + \bar{\phi}_y) + k_{13}(\phi + \bar{\phi}_x)] \sin \bar{y} \right) \Delta w^j + \\
\frac{R}{h} \left[ k_{11}(\phi + \bar{\phi}_x) + k_{13}(\phi + \bar{\phi}_y) \right] \Delta \phi^o + \\
\sum_{j=1}^{N} \left( \frac{R}{h} [k_{11}(\phi + \bar{\phi}_x) + k_{13}(\phi + \bar{\phi}_y)] \cos \bar{y} - 2k_{15} \sin \bar{y} \right) \Delta \phi^j_x \tag{D.6}
\]

The required relations (3.2.5) are obtained by substitution of this expression into

\[
(\Delta e^p_i)^o = \frac{1}{2\pi} \int_{0}^{2\pi} \Delta e^p_i \, d\bar{y} \quad i = 1, 2, 4, 5
\]

\[
(\Delta e^p_i)^j = \frac{1}{\pi} \int_{0}^{2\pi} \Delta e^p_i \cos \bar{y} \, d\bar{y} \quad i = 1, 2, 4, 5
\]

\[
(\Delta e^p_i)^j = \frac{1}{\pi} \int_{0}^{2\pi} \Delta e^p_i \sin \bar{y} \, d\bar{y} \quad i = 3, 6 \tag{D.7}
\]

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Substitution of (D.6) into the first equation of (D.7) yields for $i=1,2,4,5$:

\[
(\Delta e^i_1)^o = \frac{1}{2} \sum_{j=0}^{N} k^j_{11} (\Delta u^j)^o + \frac{1}{2} \sum_{j=1}^{N} k^j_{13} \Delta u^j + \frac{1}{2} \sum_{j=0}^{N} k^j_{14} (\Delta \phi^j_x)^o - \frac{1}{2} \sum_{j=1}^{N} k^j_{12} \Delta w^j - \frac{1}{2} \sum_{j=1}^{N} (k^j_{12} - \frac{1}{R} k^j_{15}) -
\]

\[
\frac{1}{2} \sum_{j=1}^{N} \left[ \sum_{k=1}^{N} (\phi^j_y + \phi^j_y)(k^j_{12} - k^j_{12}) + \sum_{k=0}^{N} (\phi^j_x + \phi^j_x)(k^j_{13} + k^j_{13}) \right] \Delta \phi^j + \frac{1}{2h} \sum_{j=1}^{N} \left[ \sum_{k=0}^{N} (\phi^j_y + \phi^j_y) k^j_{11} + \sum_{j=0}^{N} (\phi^j_y + \phi^j_y) k^j_{16} \right] \Delta \phi^j_x - \frac{1}{2} \sum_{j=1}^{N} (2i k^j_{16}) - \frac{1}{2h} \sum_{k=0}^{N} \left[ \sum_{i=1,4,5}^{N} (\phi^i x + \phi^i x)(k^i_{11} + k^i_{11}) + \sum_{i=1,4,5}^{N} (\phi^i y + \phi^i y)(k^i_{13} - k^i_{13}) \right] \Delta \phi^i_x
\]

\[(D.8)\]

The elements $k^j_{ik}$ are defined as:

\[
\begin{align*}
\frac{2\pi}{\pi} \int_{Y_j} k_{ik} \cos(\Omega_j \gamma) \, d\gamma &= i = 1, 2, 4, 5 \text{ and } k = 1, 2, 4, 5 \text{ or } i = 3, 6 \text{ and } k = 3, 6 \\
\frac{2\pi}{\pi} \int_{Y_j} k_{ik} \sin(\Omega_j \gamma) \, d\gamma &= i = 1, 2, 4, 5 \text{ and } k = 3, 6 \text{ or } i = 3, 6 \text{ and } k = 1, 2, 4, 5
\end{align*}
\]

\[(D.9)\]
Substitution of D.6 into the second equation of D.7 yields for $i = 1,2,4,5$ and $j = 1,..,N$:

\[
(\Delta u_j^i) = \frac{1}{2} \sum_{k=0}^{N} (k_{i1}^{j+k} + k_{i1}^{j-k})(\Delta u_k^i) + \frac{1}{2} \sum_{k=1}^{N} \left[ k_{i3}^{j+k} - k_{i3}^{j-k} \right] \left[ h_{i6}^{k+j} - h_{i6}^{k-j} \right] (\Delta v_k^i) + \frac{1}{2} \sum_{k=0}^{N} (k_{i4}^{j+k} + k_{i4}^{j-k})(\Delta x_k^i) - \frac{1}{2} \sum_{k=1}^{N} (k_{i3}^{j+k} - k_{i3}^{j-k}) h_{i5}^{j+k} \Delta u_k^i + \frac{1}{4} \sum_{m=0}^{N} \left( \phi_x^m + \phi_y^m \right) \left( k_{i3}^{j+k+m} + k_{i3}^{j-k-m} + k_{i3}^{k+m-j} + k_{i3}^{k-j-m} \right)
\]

\[
\cdot \Delta v_k^i - k_{i2}^{j+k} \Delta w^i
\]

\[
\sum_{k=1}^{N} \left[ \frac{1}{2} (k_{i2}^{j+k} + k_{i2}^{j-k}) h_{i5}^{j+k} \right] + \frac{1}{4} \sum_{m=0}^{N} \left( \phi_x^m + \phi_y^m \right) \left( k_{i3}^{j+k+m} + k_{i3}^{j-k-m} + k_{i3}^{k+m-j} + k_{i3}^{k-j-m} \right) h_{i5}^{j+k} \Delta v_k^i + \frac{1}{4} \sum_{m=1}^{N} \left( \phi_x^m + \phi_y^m \right) \left( k_{i2}^{j+k+m} + k_{i2}^{j-k-m} - k_{i2}^{j+k-m} - k_{i2}^{j-k-m} \right) h_{i5}^{j+k} \Delta w^i
\]

\[
\sum_{m=0}^{N} \left( \phi_x^m + \phi_y^m \right) \left( k_{i1}^{j+m} + k_{i1}^{j-m} \right)
\]

\[
\sum_{m=1}^{N} \left( \phi_x^m + \phi_y^m \right) \left( k_{i3}^{j+m} + k_{i3}^{j-m} \right)
\]

\[
- \sum_{k=1}^{N} \left[ \frac{1}{2} (k_{i1}^{j+k} + k_{i1}^{j-k}) \right]
\]

\[
\sum_{m=0}^{N} \left( \phi_x^m + \phi_y^m \right) \left( k_{i1}^{j+k+m} + k_{i1}^{j-k-m} + k_{i1}^{j+k-m} + k_{i1}^{j-k-m} \right)
\]

\[
\sum_{m=1}^{N} \left( \phi_x^m + \phi_y^m \right) \left( k_{i3}^{j+k+m} + k_{i3}^{j-k-m} + k_{i3}^{j+k-m} + k_{i3}^{j-k-m} \right)
\]

\[
(\text{D.11})
\]
Substitution of D.6 into the third equation of D.7 yields for $i = 3,6$ and $j = 1,\ldots,N$:

\[
\begin{aligned}
(\Delta e^p_j)_{i1} &= \frac{1}{2} \sum_{k=0}^{N} \left( k_{i1}^{j+k} + k_{i1}^{i-k} \right) (\Delta u^k)_{i1} + \frac{1}{2} \sum_{k=1}^{N} \left( k_{i3}^{i-k} - k_{i3}^{j+k} - \frac{h}{R} (k_{i6}^{j-k} - k_{i6}^{i+j+k}) \right) (\Delta v^k)_{i1} + \\
&\frac{1}{2} \sum_{k=0}^{N} \left( k_{i4}^{j+k} + k_{i4}^{i-k} \right) (\Delta \phi^k)_{i1} - \frac{1}{2} \sum_{k=1}^{N} \left( k_{i3}^{j-k} - k_{i3}^{j+k} \right) l_k \Delta u^k + \\
&\left( \frac{1}{2} [ k_{i2}^{i+k} + k_{i2}^{i-k} - h (k_{i5}^{j+i+k} + k_{i5}^{j-i-k}) ] \right) l_k - \\
&\frac{1}{4} \sum_{m=0}^{N} \left( \phi^m_x + \phi^m_y \right) (k_{i3}^{m-j+k} + k_{i3}^{m+j-k} - k_{i3}^{m-j-k} - k_{i3}^{m+j+k}) - \\
&\frac{1}{4} \sum_{m=1}^{N} \left( \phi^m_y + \phi^m_y \right) (k_{i2}^{j+k-m} + k_{i2}^{j-k+m} - k_{i2}^{j+k-m} - k_{i2}^{j-k+m}) \Delta v^k - \frac{1}{4} [ k_{i2}^{j+k} + k_{i2}^{j-k} - h \frac{1}{2} (k_{i5}^{j+i+k} + k_{i5}^{j-i-k}) ] + \\
&\frac{1}{4} \sum_{m=0}^{N} \left( \phi^m_x + \phi^m_y \right) (k_{i3}^{m-j+k} + k_{i3}^{m+j-k} - k_{i3}^{m-j-k} - k_{i3}^{m+j+k}) l_k + \\
&\frac{1}{4} \sum_{m=1}^{N} \left( \phi^m_y + \phi^m_y \right) (k_{i2}^{j+k-m} + k_{i2}^{j-k+m} - k_{i2}^{j+k-m} - k_{i2}^{j-k+m}) \Delta w^k + \\
&\frac{1}{2} \sum_{m=0}^{N} \left( \phi^m_x + \phi^m_y \right) (k_{i1}^{j+m} + k_{i1}^{j-m}) + \sum_{m=0}^{N} \left( \phi^m_x + \phi^m_y \right) (k_{i3}^{j-m} - k_{i3}^{j+m}) \Delta \phi^k_{i1} + \\
&\sum_{k=1}^{N} \left( -1 \Delta (k_{i6}^{j-k} - k_{i6}^{i+j+k}) + \\
&\frac{1}{4} \sum_{m=0}^{N} \left( \phi^m_x + \phi^m_y \right) (k_{i1}^{j+k+m} + k_{i1}^{j-m} + k_{i1}^{j+k-m} + k_{i1}^{j-k-m}) + \\
&\frac{1}{4} \sum_{m=1}^{N} \left( \phi^m_x + \phi^m_y \right) (k_{i3}^{j+k-m} + k_{i3}^{k+m-j} - k_{i3}^{j-m} - k_{i3}^{j+k+m}) \Delta \phi^k_x \\
&\frac{1}{4} \sum_{m=1}^{N} \left( \phi^m_x + \phi^m_y \right) (k_{i3}^{j+k-m} + k_{i3}^{k+m-j} - k_{i3}^{j-m} - k_{i3}^{j+k+m}) \Delta \phi^k_{x} + \\
&\frac{1}{4} \sum_{m=1}^{N} \left( \phi^m_x + \phi^m_y \right) (k_{i3}^{j+k-m} + k_{i3}^{k+m-j} - k_{i3}^{j-m} - k_{i3}^{j+k+m}) \Delta \phi^k_{x} \\
\end{aligned}
\]
<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>$n_x$</th>
<th>$s_{xy}$</th>
<th>$h_x$</th>
<th>$m_x$</th>
<th>$u$</th>
<th>$v$</th>
<th>$w$</th>
<th>$\phi_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>S2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>S3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>S4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>C1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C3</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Note: 0 - variable is free at the edge of the shell  
1 - variable is prescribed at the edge of the shell

Table 1: Definition of the boundary conditions according to Yamaki [34].

<table>
<thead>
<tr>
<th>Boundary condition</th>
<th>Mode</th>
<th>Yamaki [34]</th>
<th>Rotter [35]</th>
<th>Present Results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Donnell's theory</td>
<td>Modified Flugge</td>
<td></td>
</tr>
<tr>
<td>S1</td>
<td>9(S)</td>
<td>0.866</td>
<td>0.854</td>
<td>0.867</td>
</tr>
<tr>
<td>S2</td>
<td>1(S)</td>
<td>0.505</td>
<td>0.495</td>
<td>0.501</td>
</tr>
<tr>
<td>S3</td>
<td>8(A)</td>
<td>0.843</td>
<td>0.830</td>
<td>0.836</td>
</tr>
<tr>
<td>S4</td>
<td>1(S)</td>
<td>0.503</td>
<td>0.491</td>
<td>0.500</td>
</tr>
<tr>
<td>C1</td>
<td>9(S)</td>
<td>0.925</td>
<td>0.918</td>
<td>0.925</td>
</tr>
<tr>
<td>C2</td>
<td>9(S)</td>
<td>0.925</td>
<td>0.920</td>
<td>0.918</td>
</tr>
<tr>
<td>C3</td>
<td>8(A)</td>
<td>0.907</td>
<td>0.899</td>
<td>0.906</td>
</tr>
<tr>
<td>C4</td>
<td>8(A)</td>
<td>0.906</td>
<td>0.897</td>
<td>0.904</td>
</tr>
</tbody>
</table>

Table 2: Results of the bifurcation buckling analysis for an axially compressed elastic cylindrical shell.
<table>
<thead>
<tr>
<th>Imperfection</th>
<th>N=1</th>
<th>N=2</th>
<th>ref. [3]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.5\cos(2\pi x/L) + 0.05\cos(\pi x/L)\cos(13y/R)$</td>
<td>0.971</td>
<td>0.958</td>
<td>0.97</td>
</tr>
<tr>
<td>$-0.5\cos(2\pi x/L) + 0.05\sin(\pi x/L)\cos(13y/R)$</td>
<td>0.666</td>
<td>0.664</td>
<td>0.667</td>
</tr>
<tr>
<td>$0.5\cos(2\pi x/L) + 0.05\sin(\pi x/L)\cos(13y/R)$</td>
<td>0.962</td>
<td>0.946</td>
<td>0.952</td>
</tr>
</tbody>
</table>

Table 3: Normalized limit loads $P/P_{cl}$ of a clamped elastic shell ($R/h = 1000$, $L/R = 1$). $N$ is the number of Fourier terms used to describe the asymmetric response.

<table>
<thead>
<tr>
<th>$\epsilon$</th>
<th>$\sigma/\sigma_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25000E-2</td>
<td>0.10000E+1</td>
</tr>
<tr>
<td>0.26572E-2</td>
<td>0.10500E+1</td>
</tr>
<tr>
<td>0.28984E-2</td>
<td>0.11000E+1</td>
</tr>
<tr>
<td>0.32614E-2</td>
<td>0.11500E+1</td>
</tr>
<tr>
<td>0.37979E-2</td>
<td>0.12000E+1</td>
</tr>
<tr>
<td>0.45783E-2</td>
<td>0.12500E+1</td>
</tr>
<tr>
<td>0.56965E-2</td>
<td>0.13000E+1</td>
</tr>
<tr>
<td>0.72766E-2</td>
<td>0.13500E+1</td>
</tr>
<tr>
<td>0.94814E-2</td>
<td>0.14000E+1</td>
</tr>
<tr>
<td>0.12521E-1</td>
<td>0.14500E+1</td>
</tr>
<tr>
<td>0.16666E-1</td>
<td>0.15000E+1</td>
</tr>
<tr>
<td>0.29738E-1</td>
<td>0.16000E+1</td>
</tr>
<tr>
<td>0.52650E-1</td>
<td>0.17000E+1</td>
</tr>
<tr>
<td>0.25825E+0</td>
<td>0.20000E+1</td>
</tr>
</tbody>
</table>

Table 4: Points used to define the stress-strain curve (5.3.2)
<table>
<thead>
<tr>
<th>Imperfection</th>
<th>Limit point</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Bifurcation point</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$u_L/L$</td>
<td>$P/P_{blf}$</td>
<td>$u_L/L$</td>
<td>$P/P_{blf}$</td>
<td>$u_L/L$</td>
<td>$P/P_{blf}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\bar{e}_1$</td>
<td>$\bar{e}_2$</td>
<td>present</td>
<td>ref. [26]</td>
<td>present</td>
<td>ref. [26]</td>
<td>present</td>
<td>ref. [26]</td>
<td>present</td>
<td>ref. [26]</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.00386</td>
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*) Data is for the secondary bifurcation point

Table 5: Comparison of results using Tvergaard's long elastoplastic shell ( $E/\sigma_0 = 0.0025$, $R/h = 100$, $n = 10$ ) with a 2-mode imperfection $\bar{w} = \bar{e}_1 \cos(2\pi x/L) + \bar{e}_2 \cos(\pi x/L) \cos(ny/R)$. 
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Table 6: Error norm $\epsilon_d$ for iteration procedure based on the consistent and continuum elastoplastic moduli.
Fig 2.1: Cylinder geometry and coordinate system

Stress resultants

Moment resultants

Fig 2.2: Positive stress and moment resultants
Fig. 4.1: Linear set of equations following from the shooting procedure
Fig. 5.1: Normalized load versus end-shortening curve for Arbocz & Sechler's shell, R/h = 1000, 
L/R = 1 with a 2-mode imperfection \( w = -0.5\cos(2\pi x/L) + 0.05\cos(\pi x/L)\cos(13y/R) \).
Fig. 5.2: Calculated radial displacement for Arbocz & Sechler's shell, $R/h = 1000$, $L/R = 1$ with a 2-mode imperfection $\bar{w} = 0.5 \cos(2\pi x/L) + 0.05 \cos(\pi x/L) \cos(13y/R)$. The response of the shell is described with a one term Fourier expansion.

Fig. 5.3: Calculated radial displacement for Arbocz & Sechler's shell, $R/h = 1000$, $L/R = 1$ with a 2-mode imperfection $\bar{w} = 0.5 \cos(2\pi x/L) + 0.05 \cos(\pi x/L) \cos(13y/R)$. The response of the shell is described with a two term Fourier expansion.
Fig. 5.4: Calculated radial displacement for Arbocz & Sechler's shell, \( R/h = 1000, \frac{L}{R} = 1 \) with a 2-mode imperfection \( \bar{w} = -0.5\cos(2\pi x/L) + 0.05\sin(\pi x/L)\cos(13y/R) \). The response of the shell is described with a one term Fourier expansion.

Fig. 5.5: Calculated radial displacement for Arbocz & Sechler's shell, \( R/h = 1000, \frac{L}{R} = 1 \) with a 2-mode imperfection \( \bar{w} = -0.5\cos(2\pi x/L) + 0.05\sin(\pi x/L)\cos(13y/R) \). The response of the shell is described with a two term Fourier expansion.
Fig. 5.6: Calculated radial displacement for Arbocz & Sechler's shell, $R/h = 1000$, $L/R = 1$ with a 2-mode imperfection $w = 0.5\cos(2\pi x/L) + 0.05\sin(\pi x/L)\cos(13y/R)$. The response of the shell is described with a one term Fourier expansion.

Fig. 5.7: Calculated radial displacement for Arbocz & Sechler's shell, $R/h = 1000$, $L/R = 1$ with a 2-mode imperfection $w = 0.5\cos(2\pi x/L) + 0.05\sin(\pi x/L)\cos(13y/R)$. The response of the shell is described with a two term Fourier expansion.
Fig. 5.8: Piecewise linear approximation of the stress–strain curve (5.3.2), \( E/\sigma_s = 0.0025, n = 10 \).
Fig. 5.9: Elastoplastic collapse of an axially compressed cylinder into a periodic buckling mode, neglecting edge effects \((R/h = 100, L/R = 0.3211, E/\sigma_s = 0.0025, n = 10)\).

Fig. 5.10: Calculated radial displacement of a perfect elastoplastic shell, neglecting the edge effects, at the first bifurcation point 1) the limit point 2) and the secondary bifurcation point 3), \((R/h = 100, L/R = 0.3211, E/\sigma_s = 0.0025 \text{ and } n = 10)\).
Fig. 5.11: Calculated circumferential stress resultant of a perfect elastoplastic shell at the limit point 2) and the secondary bifurcation point 3). Edge effects are neglected and \( R/h = 100, L/R = 0.3211, E/\sigma_s = 0.0025 \) and \( n = 10 \)

Fig. 5.12: Calculated radial displacement at \( \Delta u_L = 0.0050 \) for an elastoplastic cylindrical shell with \( \bar{w} = 10^{-4} \cos(2\pi x/L) - 10^{-4} \cos(4\pi x/L) \cos(9y/R) \). Edge effects are neglected and \( R/h = 100, L/R = 0.3211, E/\sigma_s = 0.0025 \) and \( n = 10 \).
Fig. 5.13: Nondimensional axial load versus average axial strain for a long axially compressed elastoplastic shell with \( \bar{w} = \bar{\xi}_1 \cos(2\pi x/L) \).

Fig. 5.14: Nondimensional axial load versus average axial strain for a long axially compressed elastoplastic shell with \( \bar{w} = \bar{\xi}_1 \cos(2\pi x/L) + \bar{\xi}_2 \cos(\pi x/L) \cos(9y/R) \).
Fig. 5.15: Calculated solution obtained with prescribed end-displacement $\Delta u = 0.004$ and $\Delta u = 0.032$.

The percentages give the number of points in which the material is loading. This case is for the same shell as in fig. 5.14 with $\varepsilon_1 = 0.1$ and $\varepsilon_2 = -0.1$.

Fig. 5.16: Nondimensional axial load versus average axial strain calculated with different number of Fourier terms $N$. This case is for the same shell as in fig. 5.14 with $\varepsilon_1 = 0.1$ and $\varepsilon_2 = -0.1$. 
Fig. 5.17: Radial displacement for the shell of fig. 5.15 at $u_L = 0.168$, calculated with prescribed end-displacement $\Delta u = 0.004$ and $\Delta u = 0.032$.

Fig. 5.18: Circumferential stress resultants for the shell of fig. 5.15 at $u_L = 0.168$, calculated with prescribed end-displacement $\Delta u = 0.004$ and $\Delta u = 0.032$. 

- 75 -
Fig. 5.19: Nondimensional load versus average axial strain curve for a perfect shell. One edge of the shell is clamped and $R/h = 100$, $L = 5L_c$, $n = 10$ and $E/\sigma_s = 0.0025$.

Fig. 5.20: Radial displacement for the clamped shell at the point of initial yielding ($u_L = 0.36$), the limit point ($u_L = 0.54$) and the bifurcation point ($u_L = 0.68$).
Fig. 5.21: Circumferential stress resultant for the clamped shell at the point of initial yielding ($u_L = 0.36$), the limit point ($u_L = 0.54$) and the bifurcation point ($u_L = 0.68$).

Fig. 5.22: Calculated radial displacement component $w^8$ for the clamped shell at $u_L = 0.68$ with an imperfection $\tilde{w} = 10^{-4} \cos(nx/L) \cos(8y/R)$.