# VALIDATION OF THE INTERFACE-GMRES(R) SOLUTION METHOD FOR FLUID-STRUCTURE INTERACTIONS

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**Abstract.** The numerical solution of fluid-structure interactions with the customary subiteration method incurs numerous deficiencies. We validate a recently proposed solution method based on the conjugation of subiteration with a Newton-Krylov method, and demonstrate its superiority and beneficial characteristics.

# 1 INTRODUCTION

Fluid-structure interactions are of great relevance in aerospace, civil and offshore engineering and in biomechanics. Numerical methods for the aggregated fluid-structure equations customarily solve fluid and structure alternately subject to the complementary interface conditions; see, e.g., Ref. [1]. This process is typically repeated until convergence and commonly referred to as subiteration. Subiteration is a good solver for many problems, but it lacks robustness for large fluid-to-structure mass ratios; cf. Refs. [2, 3]. As subiteration operates in a time-integration process, it solves a sequence of similar problems. Since the method cannot reuse generated information, it is inefficient.

To overcome these deficiencies, we proposed in [4] a novel solution method that employs subiteration as a preconditioner to GMRES; see also the proceedings article [5] for a condensed presentation of the method. An error-amplification analysis of this method was presented in Ref. [6]. The conjugation of subiteration and GMRES requires only negligible computational resources, because the GMRES acceleration can be confined to the interface degrees-of-freedom, which is considerably cheaper than applying GMRES to the aggregated equations or to the Schur complement; see, e.g., Refs. [7, 8]. Thus, we refer to our method as Interface-GMRES(R), where the bracketed R indicates the possibility of reusing Krylov vectors in subsequent invocations of GMRES. Such reuse can yield substantial computational savings. Since Interface-GMRES(R) preserves the modularity of the underlying subiteration method, it can easily be implemented in codes which use subiteration as a solver. An investigation into efficient solution methods for fluid-structure interaction was also conducted in [9], and for multigrid in space/time applied to fluidstructure-interaction problems see [3].

In the present contribution, we validate the Interface-GMRES(R) method in a higherdimensional problem setting than the one adopted in [4, 5] to demonstrate the versatility of the method. To this end, we consider the prototypical panel fluid-structure interaction problem, viz., the interaction of an inviscid-fluid flow with a beam. Relevant features that distinguish the panel problem from the piston problem considered in Refs. [4, 5] are that it exhibits interface degrees-of-freedom pertaining to both space and time and, moreover, that it can display parameter-dependent stability behaviour such as flutter and divergence; see, e.g., Ref. [10].

To study the convergence behaviour of Interface-GMRES(R) in a systematic way, we explore first the physical parameter space of the fluid-structure system. In particular, we determine for which parameter settings the system is unstable, and which type of instability it exhibits. Next, we assess the convergence behaviour of Interface-GMRES(R) for representative settings of the physical and discretization parameters. We investigate the relation between the convergence behaviour of Interface-GMRES(R) and the stability of the problem. Numerical results are provided that demonstrate the performance and versatility of the Interface-GMRES(R) solution method.

This paper is organized as follows. Section 2 presents a problem statement of the panel problem. Section 3 concisely reviews the Interface-GMRES(R) solution method. Section 4 presents numerical results for the panel problem. Section 5 contains concluding remarks.

# 2 PROBLEM STATEMENT

Below, we present a concise description of the panel problem, for an elaboration we refer to Ref. [11]. The upper side of the panel is exposed to an airstream, and its lower side to a cavity with still air; see Fig. 1 for an illustration. We consider a panel with an infinite aspect ratio, which renders the problem essentially two-dimensional. The motion of the structure can then be described by the beam equation. Let x, y and t be spatial and temporal coordinates, respectively,  $\alpha(x,t)$  the y-coordinate position of the fluid-structure interface and L the length of the beam. The mathematical formulation of the fluid-structure system comprises the Euler equations on  $\Omega_{\alpha} := \{(x, y, t) : -\infty < x < \infty; \alpha(x, t) < y < \infty; 0 < t < T\}$  in connection with the beam equation at the interface  $\Gamma_{\alpha} := \{(x, y, t) : 0 < x < L; y = \alpha(x, t); 0 < t < T\}$ . We consider the Euler equations in



Figure 1: Illustration of the panel fluid-structure-interaction problem (interface region expanded for clarity).

conservative form:

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} + \frac{\partial \mathbf{g}(\mathbf{u})}{\partial y} = 0, \quad (x, y, t) \in \Omega_{\alpha},$$
(1a)

with

$$\mathbf{u} := \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) := \begin{pmatrix} \rho u \\ \rho u^2 + p(\mathbf{u}) \\ \rho u v \\ (p(\mathbf{u}) + E)u \end{pmatrix}, \quad \mathbf{g}(\mathbf{u}) := \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p(\mathbf{u}) \\ (p(\mathbf{u}) + E)v \end{pmatrix},$$
$$p(\mathbf{u}) := (\gamma - 1) \left( E - \frac{1}{2} \rho (u^2 + v^2) \right), \quad (1b)$$

and  $\gamma = 1.4$ . In Eq. (1b),  $\rho$ , u, v, E and p denote the density, the x- and y-component of the velocity, the total energy and the pressure of the fluid, respectively.

Eq. (1) must be supplemented with appropriate initial and boundary conditions. On  $\partial \Omega_{\alpha} \backslash \Gamma_{\alpha}$  these are prescribed by

$$\mathbf{u}(x, y, 0) = \mathbf{u}^{0}(x, y), \qquad -\infty < x < \infty, \quad \alpha(x, 0) < y < \infty, \qquad (2a)$$

$$(\rho v)(x, 0, t) = 0,$$
  $x < 0, x > L, 0 < t < T,$  (2b)

with  $\mathbf{u}^0(x, y)$  the given initial conditions. Condition (2b) translates into the impermeability condition of the rigid wall. Moreover, 'farfield boundary conditions' are imposed for  $x \to \pm \infty$  and for  $y \to \infty$ . The interface conditions, i.e., the conditions on  $\Gamma_{\alpha}$ , are specified below.

The governing equation for the beam is:

$$M\frac{\partial^2 z}{\partial t^2} + D\frac{\partial^4 z}{\partial x^4} = -\pi + \beta, \qquad 0 < x < L, \quad 0 < t < T, \qquad (3)$$

where z designates the beam displacement from its equilibrium position, and the constants  $M, D \in \mathbb{R}_+$  denote the mass and the bending stiffness of the beam, respectively. The righthand member of Eq. (3) is the forcing term which is composed of the traction  $\pi$  exerted by the fluid on the structure through the interface, and the constant pressure  $\beta$  in the cavity underneath the panel. The cavity pressure is equal to the freestream pressure. Eq. (3) is subject to the initial and boundary conditions

$$z(x,0) = z^{0}(x), \qquad \qquad \frac{\partial z}{\partial t}(x,0) = \dot{z}^{0}(x), \qquad 0 < x < L, \qquad (4a)$$

$$z(0,t) = z(L,t) = 0, \qquad \frac{\partial z}{\partial x}(0,t) = \frac{\partial z}{\partial x}(L,t) = 0, \qquad 0 < t < T, \qquad (4b)$$

with  $z^0(x), \dot{z}^0(x)$  the given initial conditions. The boundary conditions (4b) state that the beam is clamped on both sides.

The Euler equations and the beam equation are connected at the interface  $\Gamma_{\alpha}$  by the kinematic conditions

$$\begin{aligned} (\rho v)|_{\Gamma_{\alpha}} &= \rho|_{\Gamma_{\alpha}} \frac{\partial \alpha}{\partial t}(x,t) + (\rho u)|_{\Gamma_{\alpha}} \frac{\partial \alpha}{\partial x}(x,t) , \qquad 0 < x < L , \quad 0 < t < T , \qquad (5a) \\ \alpha(x,t) &= z(x,t) , \qquad 0 < x < L , \quad 0 < t < T , \qquad (5b) \end{aligned}$$

), 
$$0 < x < L$$
,  $0 < t < T$ , (5b)

and the dynamic condition

$$p(\mathbf{u}|_{\Gamma_{\alpha}}) = \pi(x, t), \qquad \qquad 0 < x < L, \quad 0 < t < T.$$
(5c)

The condition (5a) constitutes a 'slip' boundary condition, which translates into the tangency of the flow to the moving beam and renders the interface impermeable. The condition (5b) identifies the interface position and the beam position. The condition (5c) implies equilibrium of the forces exerted on the interface by the fluid and the structure. Note that the interface conditions are imposed on the moving boundary  $\Gamma_{\alpha}$ .

Upon suitable non-dimensionalization, we can identify the following dimensionless parameters that govern the behaviour of the panel fluid-structure system:

$$\lambda = \frac{LC_0^{-1}}{M^{1/2}L^2D^{-1/2}}, \quad \mu = \frac{\rho_0 L}{M}, \quad Ma = \frac{V_0}{C_0}, \tag{6}$$

where  $C_0$  denotes the speed of sound,  $\rho_0$  is the reference density and  $V_0$  is the freestream velocity. The parameter  $\lambda$  can be identified as the ratio of characteristic time scales of the fluid and the structure, the parameter  $\mu$  constitutes the ratio of characteristic fluid mass to characteristic structure mass, and the parameter Ma is the Mach number.

A distinct property of the panel problem is its ability to exhibit parameter-dependent stability behaviour. That is, the fluid-structure system can display instabilities such as flutter and divergence for certain parameter settings, whereas other parameter settings

yield stable behaviour; cf. Ref. [10]. Instability of the fluid-structure system is a property that is shared by many fluid-structure-interaction problems and that is of significant practical importance. Since flutter and divergence can induce the failure of the structure, the analysis and prediction of such instabilities plays a crucial role in engineering design. For instance, in aerospace engineering, flutter and divergence impose constraints on the allowable operating conditions of aircraft. Hence, they need to be controlled by an adequate design; see, e.g., Ref. [12].

### 3 THE INTERFACE-GMRES(R) SOLUTION METHOD

For self-containedness of this paper, we review in this section the Interface-GMRES(R) method that was recently proposed in [4] and analysed in [6]. Since the Interface-GMRES(R) method builds on the customary subiteration method, we shall first recall the subiteration method.

#### 3.1 The subiteration method

The interconnection between the state variables and their domain of definition complicates the numerical treatment of fluid-structure interaction problems. This complication can be bypassed through an iterative solution procedure often referred to as *subiteration*: Given an initial approximation  $z_0(x, t)$ , for  $j = 1, 2, \ldots$  repeat until convergence

- (S1) Solve the kinematic condition: find  $\alpha_j$  such that  $\alpha_j(x,t) = z_{j-1}(x,t)$ .
- (S2) Solve the fluid on  $\Omega_{\alpha_j}$  subject to  $u_3(x, \alpha_j, t) = u_1(x, \alpha_j, t) \frac{\partial \alpha_j}{\partial t}(x, t) + u_2(x, \alpha_j, t) \frac{\partial \alpha_j}{\partial x}(x, t)$ on  $\Gamma_{\alpha_j}$  to obtain  $\mathbf{u}_j$ .
- (S3) Solve the dynamic condition: find  $\pi_j$  such that  $\pi_j(x,t) = p(\mathbf{u}_j(x,\alpha_j(x,t),t))$ .
- (S4) Solve the structure problem with right member  $-\pi_i(x,t) + \beta$  to obtain  $z_i(x,t)$ .

This procedure obviates the simultaneous treatment of fluid and structure. Subiteration can be conceived as a mapping  $C : z_j \mapsto z_{j+1}$ , and essentially constitutes a fixed-point iteration z : Cz = z, with C the operator associated with subiteration. The subiteration process is formally stable if the spectral radius of C is smaller than unity. However, despite formal stability, transient divergence can occur for large fluid-to-structure mass ratios or large time steps. This non-monotonous convergence is caused by nonnormality of C (cf. [2]) and can even lead to failure of the iterative method. Hence, it constitutes an essential drawback of subiteration.

# 3.2 The Interface-GMRES(R) method

The Interface-GMRES(R) method essentially constitutes a Newton-Krylov method [13] applied to the interface degrees-of-freedom. To solve the nonlinear fixed-point problem by a Newton-Krylov method, we reformulate it as z : Rz = 0 with R := C - I the residual

operator. Correspondingly, the residual of an iterate  $z_i$  is  $r_i := Rz_i = (C - I)z_i = z_{i+1} - z_i$ . For a given initial guess  $z_0$ , Newton's method generates a sequence of approximate solutions according to

$$z_0 \leftarrow z_0 + z'_0 = z_0 - \mathsf{R}'^{-1} \mathsf{R} z_0, \tag{7}$$

with  $\mathsf{R}' = \partial \mathsf{R}/\partial z$  and  $z'_0$  a perturbation around the linearization state  $z_0$ . Each Newton step requires the solution of a linear problem of the form

$$Rz_0 + R'z_0' = 0. (8)$$

Substituting into (8) the ansatz  $z'_0 \in \mathcal{K}^m := \operatorname{span}\{z_j - z_0\}_{j=1}^{j=m}$  with  $\mathcal{K}^m$  the Krylov space associated with (8) and using finite-difference approximation, we obtain

$$\mathsf{R}z_0 + \mathsf{R}' \sum_{j=1}^{j=m} \alpha_j (z_j - z_0) = r_0 + \sum_{j=1}^{j=m} \alpha_j (r_j - r_0) + O(\|\sum_{j=1}^{j=m} \alpha_j (z_j - z_0)\|^2) = 0, \quad (9)$$

with  $\mathcal{R}^m := \operatorname{span}\{r_j - r_0\}_{j=1}^{j=m}$  the residual space corresponding to  $\mathcal{K}^m$ . The coefficients  $\alpha_j$  for the redefinition  $z_0 \leftarrow z_0 + \sum_{j=1}^{j=m} \alpha_j (z_j - z_0)$  are determined by solving (9) in a least-squares sense

$$\bar{\boldsymbol{\alpha}} = \arg \min \|r_0 + \sum_{j=1}^{j=m} \alpha_j (r_j - r_0)\|_2, \qquad \xi := \|r_0 + \sum_{j=1}^{j=m} \bar{\alpha}_j (r_j - r_0)\|_2, \qquad (10)$$

with  $\xi$  the norm of the residual of the linear problem. The latter constitutes an estimate for the norm of the residual of the nonlinear problem.

 $\mathcal{K}^m$  coincides with span  $\{\zeta_j - z_0\}_{j=1}^{j=m}$  with  $\zeta_j$  the *j*-th subiteration iterate. The minimalresidual property of GMRES implies that the subiteration residuals form an upper bound for the GMRES residuals and that, in contrast to the subiteration iterates, the GMRES iterates must form a non-increasing sequence. However, this implies faster Newton-Krylov convergence only for problems which are sufficiently linear. For strongly nonlinear problems, the linearization in the Newton-Krylov method can hamper convergence.

Provided with an initial approximation  $z_0(x, t)$ , Algorithm 1 summarizes the Interface-GMRES method, endowed with Gram-Schmidt orthonormalization (lines 6a–f) and underrelaxation with an appropriate constant  $\nu$  (line 6e). The former improves the robustness, the latter facilitates the subiteration process and allows the combination of GMRES with subiteration even if subiteration is formally unstable. The fluid solution can be extracted from the subiteration process on line 1 or 13. The convergence tolerances for the nonlinear and the linear problem are denoted by  $\epsilon_0$  and  $\epsilon_1$ , respectively. We set  $\epsilon_1 = \kappa ||r_i||$ with  $r_i$  the residual in the current Newton step *i* and  $\kappa < 1$  an appropriate scalar. In contrast to methods which apply GMRES to the aggregated equations or to the Schur complement, see Refs. [7, 8], the proposed Newton-Krylov method is confined to the interface degrees-of-freedom and, therefore, the storage requirements for the Krylov space and

1:  $i = 0; z_1 = Cz_0; r_0 = z_1 - z_0$ 2: while  $||r_i|| > \epsilon_0$  do  $j = 0; \ \xi = ||r_i||$ 3: while  $\xi > \epsilon_1$  do 4: j = j + 15:  $z_j' = z_j - z_0$ 6:  $z_{j} = z_{j} = z_{0}$   $z_{j+1} = \mathsf{C}z_{j}$   $r'_{j} = (z_{j+1} - z_{j}) - r_{i}$   $\bar{\boldsymbol{\alpha}} = \arg\min\|r_{i} + \sum_{k=1}^{k=j} \alpha_{k}r'_{k}\|$   $\boldsymbol{\xi} = \|r_{i} + \sum_{k=1}^{k=j} \bar{\alpha}_{k}r'_{k}\|$ 7: 8: 9: 10: end while 11:  $z_0 = z_0 + \sum_{k=1}^{k=j} \bar{\alpha}_k z'_k$   $i = i + 1; \ z_1 = C z_0; \ r_i = z_1 - z_0$ 12:13:14: end while

6a: 
$$z'_j = z_j - z_0$$
  
6b: **for**  $k = 1, ..., j - 1$  **do**  
6c:  $z'_j = z'_j - z'_k (z'_j \cdot z'_k) / ||z'_k||^2$   
6d: **end for**  
6e:  $z'_j = \nu z'_j / ||z'_j||$   
6f:  $z_j = z_0 + z'_j$ 

1:  $i = 0; j = 0; z_1 = Cz_0; r_0 = z_1 - z_0$ 3a:  $\bar{\alpha} = \arg \min \| r_i + \sum_{k=1}^{k=j} \alpha_k r'_k \|$ 3b:  $\xi = \| r_i + \sum_{k=1}^{k=j} \bar{\alpha}_k r'_k \|$ 3c:  $z_{j+1} = z_1$ 

Algorithm 1: The Interface-GMRES(R) method for solving z : Cz = z; the basic algorithm (*left*), modifications to enable Gram-Schmidt orthonormalization and underrelaxation (*right top*) and modifications to enable reuse of Krylov vectors within a time step (*right bottom*).

the computational expense for the solution of the least-squares problem (10) are much lower. Accordingly, we refer to this solution method as *Interface-GMRES*.

Reuse of Krylov vectors only requires minor modifications; see Algorithm 1. The inner loop then augments instead of overwrites the available spaces  $\mathcal{K}^m$  and  $\mathcal{R}^m$ . Depending on the reduction of the updated nonlinear residual in  $\mathcal{R}^m$ ,  $\mathcal{K}^m$  is further augmented or another Newton update is carried out.

In addition to reuse within a single time step, reuse is also possible within subsequent time steps. In the latter case, the available spaces  $\mathcal{K}$  and  $\mathcal{R}$  are transferred from one time interval to the next. Such reuse can substantially increase the efficiency of the method; however, it comes at the expense of robustness and therefore has to be exercised with some caution. We refer to the Interface-GMRES method with reuse as Interface-GMRESR. Finally, let us remark that the Interface-GMRES(R) solution method is generic and that it is easily implemented in existing codes which use subiteration as a solver.

#### 4 NUMERICAL EXPERIMENTS

To demonstrate the versatility of the Interface-GMRES(R) method, we assess its convergence behaviour on the panel problem. In particular, we investigate the effect of physical instability due to flutter on Interface-GMRES(R) convergence and on the effectiveness of reuse of the Krylov space. For reference purposes, we include comparisons with standard subiteration.

#### 4.1 Experimental setup

We consider the panel problem stated in Section 2. The infinite-dimensional domain with  $x \to \pm \infty$  and  $y \to \infty$  is modeled by a truncated domain. In particular, in the x-direction inflow and outflow fluid boundary conditions are prescribed with the flow going from the left to the right, and in the y-direction the domain is bounded by a solid wall at a distance of one from the panel. The distance of the solid wall to the panel is sufficiently large to ensure that the wall does not significantly influence the solution and the convergence behaviour of the solution methods.

We use initial conditions for the beam according to its first mode shape. The initial conditions for the fluid are determined as the steady-state solution of the flow over a beam that is deflected according to its first mode shape. The system parameters are given in Table 1, where  $\tau$  denotes the length of the solution time interval. With Ma = 1.5, the flow is supersonic.

Case	$\lambda$	$\mu$	Ma	au
Ι	0.25	*	1.5	0.05
II	*	10	1.5	0.05

Table 1: System parameters for the panel problem (\* indicates a variable parameter).

The fluid-structure system is discretized by the space/time finite-element method with piecewise-polynomial base functions that are discontinuous in time and continuous in space. As base functions for the structure discretization we use Legendre polynomials, and enforce  $C^1$ -continuity in space by means of Lagrange multipliers. The base functions for the fluid are of modal type in conformity with Ref. [14, ch.3].

The time-discontinuous Galerkin discretization implies that displacement and velocity of the structure are discontinuous from one time slab to the next. However, since the fluid-boundary representation assumes a continuous displacement, the discontinuity in the structure displacement needs to be controlled. To render the discontinuity in the structure displacement and velocity negligible, we use polynomials of sufficiently high degree for the approximation space of the structure.

We remark that the considered discretization does not maintain the conservation properties at the fluid-structure interface; cf. Ref. [15]. To render the error pertaining to the lack of conservation negligible, we choose a discretization for fluid and structure that is sufficiently fine.

The discretization parameters are given in Table 2, where the polynomial degree of the approximation spaces associated with  $\mathbf{u}$ ,  $\alpha$ , z and  $\pi$  are, respectively,  $(P_{\mathbb{U}}^x, P_{\mathbb{U}}^y, P_{\mathbb{U}}^t)$ ,  $(P_{\mathbb{A}}^x, P_{\mathbb{A}}^t)$ ,  $(P_{\mathbb{Z}}^x, P_{\mathbb{Z}}^t)$  and  $(P_{\mathbb{P}}^x, P_{\mathbb{P}}^t)$ , and the number of elements, N, is denoted accordingly. The number of elements in the x-direction is specified over the length of the beam. The discretization time step is equal to the length of the solution time interval. The discretization is sufficiently fine to ensure that the results are essentially mesh independent.

$N_{\mathbb{U}}$	$N_{\mathbb{A}}$	$N_{\mathbb{Z}}$	$N_{\mathbb{P}}$	$P_{\mathbb{U}}$	$P_{\mathbb{A}}$	$P_{\mathbb{Z}}$	$P_{\mathbb{P}}$
(16, 24, 1)	(16, 1)	(16, 1)	(16, 1)	(2, 2, 2)	(1, 1)	(6, 6)	(2, 2)

Table 2: Discretization parameters for the panel problem, test cases I and II.

In each time slab, we provide an initial approximation of the structure displacement based on a linear extrapolation of the initial conditions conforming to

$$z_0(x,t) = z^0(x) + \dot{z}^0(x)t, \qquad 0 \le x \le L, \quad 0 \le t \le \tau.$$
(11)

We set the convergence tolerance to  $\epsilon_0 = 10^{-4} ||r_0||$ , i.e., we require a reduction of the initial residual by four orders of magnitude. In addition, we specify for the Newton-Krylov method the tolerance for the GMRES iteration according to  $\epsilon_1 = 10^{-1} ||r_i||$ , i.e., we use a relative tolerance for the convergence in the inner loop of the acceleration; cf. Section 3.2. Moreover, the underrelaxation parameter is set to  $\nu = 10^{-2} ||r_0||$  for the Interface-GMRES method with reuse and to  $\nu = 10^{-2} ||r_i||$  for the method without reuse.

#### 4.2 Numerical results

In the first test case, we study the convergence of the Interface-GMRES(R) method and subiteration for three distinct settings of the problem with parameters as given in Table 1, case I and  $\mu = 1, 50, 100$ . We remark that the spectral radius of the subiteration-operator derivative scales with  $\mu$ ; see also Ref. [2].

Fig. 2 plots the displacement of the beam in space/time. For all considered settings, the oscillation of the structure attenuates with time, indicating that the fluid-structure system is stable. Moreover, it is apparent that the beam deflection is downwind according to the direction of the flow. The convergence behaviour of the Newton-Krylov method with and without reuse and of the subiteration method is displayed in Fig. 3 for time steps 1 and 50 for exemplification. In addition, we plot in Figs. 4 and 5 the dimension of the Krylov space and the cumulative number of iterations versus the time-step counter, respectively. The cumulative number of iterations specifies the total number of iterations required for convergence up to and including the time step under consideration. Fig. 3 illustrates that if reuse is applied, initially most iterations of the Newton-Krylov method are spent on generating the Krylov space. However, in subsequent time steps, increasingly fewer Krylov vectors need to be added to the space due to reuse; see also Fig. 4. This results in a decreasing number of iterations per time step and manifests in the gradually changing slope of the cumulative-iteration-count curve; see Fig. 5. In contrast, the number of iterations required by subiteration hardly changes in subsequent time steps. We infer from these results that reuse can render the Newton-Krylov method computationally cheaper than subiteration even under conditions that are favorable for the convergence of subiteration; see Figs. 3 and 5 (*left*) with  $\mu = 1$ . Subiteration convergence deteriorates significantly with increasing  $\mu$ , in contrast to Newton-Krylov convergence. Hence, a discrepancy in computational cost for larger  $\mu$  emanates. For  $\mu = 100$ , subiteration diverges. Note that the Newton-Krylov method attains convergence despite the instability of the



Figure 2: Test case I: Space/time displacement of the beam (colour bars) for system parameters according to Table 1 and  $\mu = 1$  (*left*),  $\mu = 50$  (*center*) and  $\mu = 100$  (*right*).

underlying subiteration method.

For reference, we have included in Figs. 3 and 5 the results for the Newton-Krylov method without reuse of the Krylov space. A comparison to the method with reuse clearly demonstrates the significant savings in computational cost that can be obtained by reusing the Krylov space.

To put our results into context, we remark that for an initial amplitude of the beam deflection of approximately  $10^{-4}$  the system behaviour is close to linear. Preliminary studies indicate that for nonlinear system behaviour corresponding to larger initial amplitudes the performance of the Newton-Krylov method degrades only moderately. Moreover, we remark that our results are in good agreement with the results obtained on the piston model problem; cf. [4].

In the second test case, we investigate the effect of physical instability on convergence and on the effectiveness of reusing the Krylov space. To this end, we consider the fluidstructure system with parameters according to Table 1, case II and two representative settings of  $\lambda$ , viz.,  $\lambda = 0.1$  and  $\lambda = 0.25$ . The discretization parameters are specified in Table 2.

Fig. 6 plots the numerical solution of the beam displacement in space/time for the unstable system (*left* figure) and the stable system (*right*). Whereas for  $\lambda = 0.1$  the oscillation amplifies which indicates flutter, for  $\lambda = 0.25$  the oscillation attenuates, indicating stability of the fluid-structure system. Fig. 7 (*left*) plots the cumulative number of iterations versus the time-step counter for the Newton-Krylov method and for subiteration as a reference. In addition, Fig. 7 (*right*) plots the dimension of the Krylov space



**Figure 3:** Test case I: Residual reduction in the  $L^2$  norm versus iteration number in time steps 1 (*top*) and 50 (*bottom*) for the Newton-Krylov method with reuse (——) and without reuse (——) and for subiteration (…); residual estimates and true residuals of the Newton-Krylov method are indicated by  $\circ$  and  $\Box$ , respectively, and residuals of subiteration by  $\triangle$ ;  $\mu = 1$  (*left*),  $\mu = 50$  (*center*) and  $\mu = 100$  (*right*). y-axis in  $\log_{10}$ -scale.

versus the time-step counter. We remark that these figures plot upto a time step of n = 200 corresponding to computational time t = 10, whereas Fig. 6 plots only upto n = 100 (t = 5). Note that the instability becomes increasingly pronounced with time. Fig. 7 (*left*) displays a slight change in slope of the cumulative-iteration-count curve of the Newton-Krylov method with reuse for the unstable system setting. To explain this change in slope, we consider the evolution of the Krylov-space dimension plotted in Fig. 7 (*right*). The figure exhibits that, after the initial construction of a sufficiently large Krylov space, the dimension of the space remains essentially constant upto a time step of approximately 100. Henceforth, the dimension of the Krylov space further increases in the case of the unstable system, which means that additional Krylov vectors need to be added to the space to attain convergence. This indicates a mild degradation in the effectiveness of the



Figure 4: Test case I: Dimension of the Krylov space versus the time-step counter for the Newton-Krylov method with reuse in subsequent time steps;  $\mu = 1$  (*left*),  $\mu = 50$  (*center*) and  $\mu = 100$  (*right*).



Figure 5: Test case I: Cumulative number of iterations versus the time-step counter for the Newton-Krylov method with reuse (—) and without reuse (—) and for subiteration (…);  $\mu = 1$  (*left*),  $\mu = 50$  (*center*) and  $\mu = 100$  (*right*).

reused Krylov space which can be attributed to the significant change in the solution induced by flutter. However, this effect appears to be minor in that reuse remains beneficial and renders Newton-Krylov convergence faster than subiteration convergence; see Fig. 7 (*left*). This result underlines that the improvement in efficiency that can be gained by reuse is not restricted to stable fluid-structure systems only but also applies to systems undergoing flutter.

In conclusion, the test cases indicate that the Interface-GMRES method generally outperforms subiteration. Settings corresponding to a relatively weak coupling in the fluid-structure-interaction problem, e.g. due to small  $\mu$ , are favorable for the subiteration method. For such settings, the convergence behaviour of subiteration and Interface-GMRES is comparable. For larger  $\mu$  and, accordingly, a stronger coupling, Interface-GMRES converges much faster than subiteration. Even if the coupling is so strong that the subiteration method separately diverges, the Interface-GMRES method still dis-



Figure 6: Test case II: Space/time displacement of the beam (colour bars): Solution computed with system parameters according to Table 1 with  $\lambda = 0.1$  (*left*) and  $\lambda = 0.25$  (*right*).

plays adequate convergence behavior. Moreover, if the reuse option is exercised, then the Interface-GMRESR method converges in just a few iterations, independent of the strength of the coupling.



Figure 7: Test case II: Cumulative number of iterations versus the time-step counter for the Newton-Krylov method with reuse in subsequent time steps ( $\Box$ ) and without reuse ( $\circ$ ) and for subiteration ( $\triangle$ ) (*left*), and dimension of the Krylov space versus the time-step counter for the Newton-Krylov method with reuse (*right*); system parameters according to Table 1 with  $\lambda = 0.1$  (----) and  $\lambda = 0.25$  (---).

#### 5 CONCLUSIONS

In this paper we have assessed the convergence behaviour of the recently proposed Interface-GMRES(R) solution method on the prototypical panel fluid-structure-interaction problem. This model problem exhibits parameter-dependent stability behaviour, admitting instabilities such as flutter and divergence.

Our numerical experiments demonstrate that the Interface-GMRESR method with reuse of the Krylov space generally converges faster than the customary subiteration method. For the Interface-GMRES method without reuse, however, this is not always the case. If the coupling in the fluid-structure-interaction problem is weak, e.g. due to a small fluid-to-structure mass ratio, then the subiteration method can display slightly better convergence. For strongly-coupled problems, the Interface-GMRES method clearly outperforms the subiteration method. Moreover, the Interface-GMRES method even converges in cases where the underlying subiteration method diverges, e.g. for large fluidto-structure mass ratios.

Our results indicate that physical instability in the form of flutter can induce a mild degradation of the effectiveness of reuse of the Krylov space. However, this effect appears to be minor and reuse remains beneficial. These findings underline the versatility of the method.

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