Improved numerical solution of Dobrovol'skaya's boundary integral equations on similarity flow for uniform symmetrical entry of wedges

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ABSTRACT
Dobrovol'skaya [1] presented a similarity solution for the water entry of symmetrical wedges with constant velocity. The solution involves an integral equation that becomes increasingly harder to numerically solve as the deadrise angle decreases. Zhao and Faltinsen [2] were able to present reliable results for deadrise angles down to 4°. In this paper, Zhao and Faltinsen's results are improved and reliable results for deadrise angles down to 1° are confirmed by comparing to the asymptotic solutions at small deadrise angles and the solutions by the traditional boundary element method at relatively large deadrise angles. The present similarity solution results provide a reference solution in theoretical studies of water entry problems and in developing accurate numerical solvers for simulating strongly nonlinear wave-body interactions, which flows are governed by Laplace equation or Euler equation.

1. Introduction
Solid objects entering through a water (liquid) surface often involves large unsteady hydrodynamic loads and rapid deformation of free surface and is therefore of great interest to the design of ship bow, lifeboats, planning vessels, high-speed seaplanes, surface-piercing propellers and offshore or coastal structures. Wagner [3] studied water entry of wedges. He accounted for the local uprise of the water and presented details of the flow at the spray roots, which included predictions of maximum pressure. Wagner's first-order outer-domain solution does not include the details at the spray roots and overestimates the vertical hydrodynamic forces. For finite deadrise angles \( \theta \) as defined in Fig. 1, Wagner used a flat-plate approximation, which leads to pressure singularities at the plate edges. Cointe and Armand [4] and Howison et al. [5] used matched asymptotic expansions to combine Wagner's inner-flow-domain solution at a spray root with an outer-flow-domain solution. In that way, the pressure singularities at the spray roots are removed. Cointe [6] studied also the jet domain and presented predictions of the angle between the free surface and the body surface at the intersection point for water entry of a wedge with constant entry velocity. The theoretical model by Cointe and Armand [4] or Howison et al. [5] only gives satisfactory solution for small deadrise angles. Faltinsen [7] studied water entry of wedges with larger deadrise angles. He accounted for the deadrise angle in constructing the outer domain solution, which results in significant improvement of the asymptotic solution for larger deadrise angles. The previously mentioned theoretical models give approximate analytical solutions for two-dimensional problems. By neglecting gravity, Dobrovol'skaya [1] presented the similarity solution, which exactly represents the water entry of symmetrical semi-infinite wedges with constant velocity within the framework of potential flow of incompressible liquid. Semenov and Iafrati [8] obtained similarity solutions for the water entry of asymmetric wedges without flow separation. For three-dimensional problems, Faltinsen and Zhao [9] presented asymptotic solutions for water entry of axisymmetric bodies; Scolan and Korobkin [10] presented exact analytical solutions to the Wagner problem; Wu and Sun [11] found the existence of similarity solutions in the case of an expanding paraboloid entering water.
Dobrovol'skaya's similarity solution is applicable for any deadrise angle. Its existence and uniqueness has been proved by Fraenkel and Keady [12]. The similarity solution has been widely used for a reference solution in theoretical studies of water entry problems and also in developing accurate numerical solvers for simulating strongly nonlinear wave-body interactions, for instance, by Mei et al. [13], Söding [14], Semenov and Iafrati [8], Wu [15] and Wang and Faltinsen [16]. When used as a reference solution, the similarity solution results should be accurate. The similarity solution is represented by a nonlinear singular integral equation, which is very difficult to solve. The challenges increase with reducing the deadrise angle, because smaller deadrise angles...
result in thinner and longer jet flows. Dobrovol'skaya [1] only presented results for deadrise angles equal to and larger than 30°. Zhao and Faltinsen [2] pointed out that Dobrovol'skaya [1]'s results for the deadrise angle of 30° are not accurate. They improved Dobrovol'skaya's results and obtained results in a wider range of deadrise angles (down to 4°). However, there was non-negligible discrepancy in the pressure distribution on the wedge surface when compared with the results by the boundary element method (see [2], Fig. 6). Due to numerical challenges, results for deadrise angles smaller than 4° have not been obtained yet. In this paper, a nested iterative method based on quasi-dynamic under-relaxation is proposed to derive accurate results of Dobrovol'skaya's similarity solution. By employing this method, we successfully obtained similarity solution results for deadrise angles down to 1°. The numerical error of the present similarity solution results has been estimated. The accuracy of the results is further confirmed by comparing to the asymptotic solutions at small deadrise angles and the solutions by the traditional boundary element method at relatively large deadrise angles. The present similarity solution results agree well with the asymptotic solutions at small deadrise angles and the discrepancy between the two solutions tends to vanish with decreasing the deadrise angle, which are expected when comparing well-developed asymptotic solutions to exact solutions. At relatively large deadrise angles, the present results coincide with those obtained by the traditional boundary element method, which improve Zhao and Faltinsen [2]'s results. The assumptions of the similarity solution must be kept in mind, such as a semi-infinite wedge is considered. It is noted that, at small deadrise angles, the air flow will cause the free surface to raise at the chines if the wedge is rigid with a finite length. The consequence is that air cavities are formed under the wedge bottom. However, there is more to it than that. Hydroelasticity will in practice matter [17]. Furthermore, liquid compressibility can matter for small deadrise angles.

2. Mathematical model

2.1. Governing equation

To model the symmetrical entry of a semi-infinite wedge into the initially calm water, a Cartesian coordinate system is introduced: the x-axis is along the undisturbed water surface; the y-axis is along the body axis of symmetry and positive upwards. The coordinate system and sketch of the water entry problem are shown in Fig. 1.

The air flow is neglected. In case that the entry velocity is not high enough to make acoustic effects relevant, it is appropriate to assume that the water is incompressible after a very early stage [18]. Because of the short duration of impact, viscous effects are negligible provided that the Reynolds number is large. Further, the flow is irrotational as there is no initial vorticity. Therefore, a velocity potential \( \psi(x, y, t) \) of incompressible liquid satisfying Laplace's equation

\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0
\]

is introduced. The kinematic free-surface condition is that a water particle remains on the free surface. The dynamic free-surface condition is that the water pressure is equal to the constant atmospheric pressure (surface tension is neglected). On the body surface, the normal velocity of the water is equal to that of the wedge body.

2.2. Similarity solution

Dobrovol'skaya [1] has presented similarity solutions for the water entry of symmetrical wedges with constant velocity. In the similarity flow, the velocity potential has the form

\[
\psi(x, y, t) = v_0^2 \Phi(x, \eta),
\]

where \( v_0 \) is the velocity of the wedge, \( \xi = x/v_0 t, \eta = y/v_0 t \) and \( \Phi(x, \eta) \) is a time-independent harmonic function. The function \( \Phi(x, \eta) \) has to satisfy the kinematic free-surface condition, the dynamic free-surface condition and the body-surface condition, which are expressed as

\[
\frac{\partial \Phi}{\partial \eta} = \frac{\partial \Phi}{\partial \xi} = 0,
\]

\[
\Phi - \frac{\partial \Phi}{\partial \xi} - \frac{\partial \Phi}{\partial \eta} \frac{1}{2} \left( \frac{\partial \Phi}{\partial \eta} \right)^2 + \frac{1}{2} \left( \frac{\partial \Phi}{\partial \xi} \right)^2 = 0,
\]

\[
\frac{\partial \Phi}{\partial \xi} \cos \alpha_0 - \frac{\partial \Phi}{\partial \eta} \sin \alpha_0 = \sin \alpha_0,
\]

respectively. The flow under consideration is then represented as a boundary-value problem. By using Wagner's \( h \)-function [3], the boundary-value problem can be reduced and solved by finding the solution of the nonlinear singular integral equation

\[
f(s) = \frac{c^3}{\pi} \int_0^1 \frac{(1 - t)^{-1/2} \exp \left\{ - \frac{c^3}{\pi} \int_0^1 \frac{\delta(s \mid \delta t)}{\delta t} \, dt \right\} }{1 - t} \, dt,
\]

where \( \alpha = \alpha_0/\pi \) (see Fig. 1) and

\[
\delta(s) = \int_0^1 t^2 \left( 1 - r \right)^{-1} \exp \left\{ - \int_0^1 \frac{\delta(s \mid \delta t)}{\delta t} \, dt \right\} \, dr - \int_0^1 \frac{\delta(s \mid \delta t)}{\delta t} \, dt.
\]

Once the function \( f(s) \) is determined, the hydrodynamic problem can be considered as solved. Dobrovol'skaya [1] presented the free-surface elevation in terms of \( f(s) \):

\[
\begin{align*}
\xi(s) & = \left\{ \frac{c}{\eta} \right\} \cos \left( \frac{\eta}{\pi} \right) \left\{ \int_0^1 \frac{3}{2} \left( 1 - t \right)^{1/2} + \alpha \right\} \exp \left\{ - \frac{c^3}{\pi} \int_0^1 \frac{\delta(s \mid \delta t)}{\delta t} \, dt \right\} \, dr, \\
\eta(s) & = \left\{ \frac{c}{\eta} \right\} \sin \left( \frac{\eta}{\pi} \right) \left\{ \int_0^1 \frac{3}{2} \left( 1 - t \right)^{1/2} + \alpha \right\} \exp \left\{ - \frac{c^3}{\pi} \int_0^1 \frac{\delta(s \mid \delta t)}{\delta t} \, dt \right\} \, dr.
\end{align*}
\]

where the constants \( c \) and \( \eta_0 \) have the form.
Fig. 2. Relative error of numerical integration. The left subfigure shows the relative error of numerical integration of the integral \( \int_0^1 f(t) \, dt \); the right shows the maximum relative error of numerical \( \omega_n(t) \) over \([0, 1]\); the symbol ** corresponds to \( l_n = 1.0 \times 10^{-3} \); the symbol *** corresponds to \( l_n = 5.0 \times 10^{-4} \).

\[
\begin{align*}
\eta_8 &= c \int_0^1 t^{-3/2}(1-t)^{-1+\alpha} \exp \left[ -t \int_0^t \frac{f(r)}{\Gamma(2-(1/\alpha)-1)} \, dr \right] \sin[\alpha_0(t)] \, dt.
\end{align*}
\]

(9)

The pressure distribution on the wetted wedge surface was expressed as

\[
p = p_0 - 2\Re \left[ V(r) - \overline{V}(\overline{f}(r)) \right] - \frac{1}{2} \pi c^2 \frac{\partial^2 \overline{V}(\overline{f}(r))}{\partial f^2} f(r)\frac{\partial f}{\partial f} |f=f(r)|^2 \leq r \leq 1,
\]

(11)

where \( p_0 \) is the pressure on the free surface, \( \rho \) is the density of water, \( V \) is the complex velocity potential of \( \Phi \). The complex coordinate \( \xi \) of points on the wetted wedge surface is given by formula

\[
\xi_0 = \xi_0 + \eta_0(r) = \left[ \xi_0 - c \sin \alpha_0 H_0(r) \right] + 1 \left[ \eta_8 - c \cos \alpha_0 H_0(r) \right].
\]

(12)

where

\[
H_0(r) = \int_0^1 t^{-3/2}(1-t)^{-1+\alpha} f(2t-1) \exp \left[ -t \int_0^t \frac{f(r)}{\Gamma(2-(1/\alpha)-1)} \, dr \right] dt.
\]

(13)

The wedge apex \( \zeta_0 \) corresponds to \( r = 0 \) and the intersection point \( \xi_0 \) to \( r = 1 \). The functions \( V(\xi)V(\overline{\xi}) \) and \( \Re \{ V(\xi)V(\overline{\xi}) \} \) in Eq. (11) have the form

\[
W(\xi) = \begin{cases} 
\frac{\Gamma(1+\alpha)}{\Gamma(2-(1/\alpha)-1)} \xi_0^{1/2} + \alpha - 2 \pi c^2, & \xi_0 \leq 1, \\
\frac{\Gamma(1+\alpha)}{\Gamma(2-(1/\alpha)-1)} \xi_0^{1/2} + \alpha - 2 \pi c^2, & \xi_0 > 1.
\end{cases}
\]

(21)

Further, we introduce the functions

\[
q(t) = \int_0^t f(t) - f(1) \, dt, \quad Q(t) = \exp[q(t) - w(0)],
\]

(24)

\[
R_0(t) = \frac{1}{2} \int_0^t \left[ (1+t)^{-3/2} \xi_0^{1/2} \right] Q(t) t^{-\alpha} (1-t)^{-1+\beta} \, dt,
\]

(26)

where

\[
\eta_8 = \frac{c^2}{\alpha} \sin \alpha_0 G_0(r) \left[ \xi_0 - c^2 \cos \alpha_0 G_0(r) \right]^2.
\]

(14)

2.3. Numerical method

We present a nested iterative method for accurately solving Eqs. (6) and (7). By introducing the functions

\[
w(t) = \int_0^1 f(t) - f(t) \, dt, \quad W(t) = \exp[w(t) - w(0)]
\]

(18)

in the domain \( t \in [0, 1] \), Eq. (6) can be converted to

\[
f(s) = \frac{1}{c^2} \int_0^t \int_0^s \frac{1}{\Gamma(2-(1/\alpha)-1)} \, dr \, dt.
\]

(19)

\[
\int_0^t f(t) \frac{\sin[\alpha_0(t)]}{\Gamma(2-(1/\alpha)-1)} \, dt.
\]

(22)

\[
\Re \{ V(\xi)V(\overline{\xi}) \} = \left[ \xi_0 - c \sin \alpha_0 H_0(r) \right] \Re \{ \overline{V}(\overline{f}(r)) \} - \left[ \eta_8 - c \cos \alpha_0 H_0(r) \right] \Re \{ V(\xi)V(\overline{\xi}) \}.
\]

(16)

\[
\Re \{ V(\xi)V(\overline{\xi}) \} = \left[ \xi_0 - c \sin \alpha_0 H_0(r) \right] \Re \{ \overline{V}(\overline{f}(r)) \} - \left[ \eta_8 - c \cos \alpha_0 H_0(r) \right] \Re \{ V(\xi)V(\overline{\xi}) \}.
\]

(20)

Over \([0, 1]\), \( f(s) \) is a monotone increasing function, which has the following representation [1]

\[
f(1) = 1/2 + \alpha - \beta,
\]

(23)

\[
f(s) = Q(s^{3/2}) \text{ when } s \to 0,
\]

(24)

\[
f(s) = f(1) - y(1-s)^{3/2-\beta} \text{ when } s \to 1.
\]

Further, we introduce the functions

\[
\tilde{G}_0(t) = \int_0^t (1-t)^{-1+\beta} (2t-1)^{-1+\alpha} \, dt,
\]

(15)

\[
\tilde{G}_0(t) = \int_0^t (1-t)^{-1+\alpha} \, dt.
\]

(27)
It is easy to verify that \( H_0(s) \) and \( G_0(s) \) are consistent with \( H_0(r) \) and \( G_0(r) \), respectively. Furthermore, \( \frac{k^2}{c^2} = H_0(0)/G_0(0) \).

To evaluate \( H_0(0) \) and \( G_0(0) \), numerical integration should be used. Directly applying a standard quadrature rule for singular integrals may result in significant errors. To overcome this drawback, we adopt analytical methods for the integration in the vicinity of singularities. For example, to evaluate \( \int_0^1 t^{-\alpha} dt \), we split the domain into small numbers. The integral is evaluated analytically over \( [0 \varepsilon] \) and by a numerical quadrature rule over \( [\varepsilon 1] \). In the present study, the singular integrals have the form

\[
\int_0^1 g(t)t^{-\alpha}(1-t)^{-\beta} dt,
\]

where \( g(t) \) is a smooth function. For integration of this kind of integrals, we propose a quadrature method: the domain \([0 1]\) is divided into three subdomains, i.e., \([0 \varepsilon]\), \([\varepsilon 1 - \varepsilon]\), and \([1 - \varepsilon 1]\); the subdomains \([\varepsilon 1]\) and \([1 - \varepsilon]\) are divided into a number of smaller elements; the functions \( I(t) = g(t)(1-t)^{-\beta} \) and \( R(t) = g(t)t^{-\alpha} \) are assumed to have a linear variation over all elements in \([0 \varepsilon]\) and \([1 - \varepsilon]\), respectively and then the analytical integration is performed; the trapezoidal rule is adopted for numerical integration over \([\varepsilon 1 - \varepsilon]\). As long as element sizes over \([0 \varepsilon]\) and \([1 - \varepsilon]\) are small enough, the assumption of a piecewise linear variation of \( I(t) \) over \([0 \varepsilon]\) and \( R(t) \) over \([1 - \varepsilon]\) results in a negligible error. The numerical integration over \([1 - \varepsilon]\) dominates the error of the integration over \([0 1]\). Once \( H_0(0) \) and \( G_0(0) \) are obtained, all numerical node values of the functions \( H_0(s) \) and \( G_0(s) \) become known.

The proposed quadrature method is also applied to evaluate

\[
I_2(t) = \int_0^1 [\tau(T)(1 - T)'(1 - T)]W^{-1}(T)\frac{1}{(1 - T)^{1+\beta}} dT,
\quad (28)
\]

which is equal to the denominator of the integrand in (20). It is noticed that \( I_2(0) = \infty \) and the first integrand of (20) has only one singularity (which locates at \( t = 1 \)). So the trapezoidal rule is suitable for the quadrature over \([0 1]\) and the analytical method over \([1 - \varepsilon]\). By using the expression

\[
I_2(t) = \tilde{a}(t)(1-t)^{\beta} \quad \text{with} \quad \tilde{a}(1) = \frac{1}{B}W^{-1}(1),
\quad (29)
\]

Eq. (20) can be integrated analytically over \([1 - \varepsilon]\).

Before the integration of Eq. (20), \( H_0(z) \) and \( G_0(z) \), the functions \( W(t) \) and \( Q(t) \) should be evaluated. On a given grid system, \( 0 = t_1 < t_2 < \ldots < t_N < t_{N+1} = 1 \), \( f(t) \) is assumed to have a linear variation over any element \([t_i, t_{i+1}]\). Then, the function \( W(t) \) can be expressed as

\[
w(t) = \sum_{i=1}^{N} \left[ f(t_i) - f(t) + \frac{(t - t_i)}{t_{i+1} - t_i} (f(t_{i+1}) - f(t_i)) \right] t_{i+1}^{-1} + f(t_{i+1}) - f(t_i), \quad (30)
\]

Accounting for the asymptotic behavior of (23) over \([t_N, t_{N+1}]\), we modify \( w(t) \):

\[
w^*(t) = w(t) + \left( \frac{1}{(1/2 - \alpha)} - 1 \right) f(t) - f(t_N), \quad (31)
\]

\( g(t) \) is evaluated in the similar way for \( t > \frac{1}{2} \) and it is evaluated by the trapezoidal rule for \( t \leq \frac{1}{2} \). It should be noted that \( W(t) \) is equal to \( Q(t) \).

Dobrovol'skaya [1] showed that \( f(t) \) should satisfy the condition:

\[
\frac{1}{4} + \alpha < f(t) < \frac{1}{2} + \alpha.
\]

So we set the initial guess of \( f(t) \) to be

\[
f(t) = (3/8 + \alpha)t. \quad (32)
\]

A nested iterative method is proposed to solve Eqs. (6) and (7) for \( f(t) \):

1. \( W(t) \) and \( Q(t) \) are evaluated based on the \( k \)-th approximation of \( f(t) \), i.e., \( f_k(t) \).
2. The right-hand side of Eq. (20) is denoted as \( f_k(t), W(t), Q(t) \).
An iteration is performed: \( f_k'(t) = f_k'(t), W(t), Q(t) \).
3. Compute the error \( e_{\text{out}} := \max_{0 \leq t \leq 1} |f_k(t) - f_k(t)| \). If \( e_{\text{out}} \) is smaller than the prescribed value, \( f(t) \) is approximated as \( f_k(t) \) and the iteration is completed. Otherwise, set \( f_{k+1}(t) = (1 - u)f_k(t) + uf_k(t) \), where \( u \) is a under-relaxation factor. Go to inner iterations.
4. Perform the \( i \)-th inner iteration: \( f_{k+1}(t) = f_{k+1}(t), W(t), Q(t) \).
Compute the error \( e_{\text{in}} := \max_{0 \leq t \leq 1} |f_{k+1}(t) - f_{k+1}(t)| \). If \( e_{\text{in}} < \min(0.1e_{\text{out}}, 0.01) \), set \( f_{k+1}(t) = f_{k+1}(t) \), exit the inner iterations and go to step (1). Otherwise, set \( f_{k+1+1}(t) = (1 - u)f_{k+1}(t) + uf_{k+1}(t) \) and perform the \((i+1)\)-th inner iteration.

Through the iteration process, a constant under-relaxation factor may result in divergent results. So it should be changing (dynamically) during iterations. Initially, the under-relaxation factor is set to a prescribed value \( u_0 \), which should be less than 1. We denote the present approximation of \( f(t) \) as \( f_{k+1}(t) \), the intermediate approximation \( f(t) = f_{k+1}(t) \), \( W(t) \), \( Q(t) \) and the new approximation \( f_{k+1}(t) = (1 - u)f_{k+1}(t) + uf_{k+1}(t) \). Further, we introduce the function

\[
B(t) = 1/2 + \alpha - f(t). \quad (33)
\]

\( B_{\text{out}}(t) \), \( B_{\text{in}}(t) \) and \( B_{\text{new}}(t) \) correspond to \( f_{k+1}(t), f_k(t) \) and \( f_{k+1}(t) \) respectively. Physically speaking, \( B(t) \) denotes the angle between the wedge surface and the free surface. The dynamic under-relaxation factor is determined by the following criterion:

\[
\max_{t \geq 1/2} \frac{|B_{\text{new}}(t) - B_{\text{out}}(t)|}{|B_{\text{in}}(t)|} = \Gamma, \quad (34)
\]

where \( \Gamma \) is a prescribed value less than 1. Immediately, the dynamic under-relaxation factor can be obtained:

\[
\alpha = \left( \Gamma \right)^{-1} \max_{t \geq 1/2} \frac{|f(t) - f_{\text{out}}(t)|}{1/2 + \alpha - f_{\text{in}}(t)}. \quad (35)
\]

This criterion represents that the change of the angle between the wedge surface and the free surface should be small after an iteration. For large deadrise angles, \( \Gamma = 0.1 \) is used for the present study. For small deadrise angles, smaller \( \Gamma \) is used. It should be noted that frequently changing the under-relaxation factor will also result in divergent results. In the present study, we use a quasi-dynamic under-relaxation factor: if the intermediate under-relaxation factor \( \alpha \) determined by (35) is close to the present under-relaxation factor \( u_{\text{old}} \): (for instance \( |u_{\text{old}} - u|/u_{\text{old}} < 0.2 \), the under-relaxation factor remains unchanged; if \( u_{\text{old}} \) is significantly smaller than \( u_{\text{old}} \): (for instance \( u_{\text{old}} < 0.2 \), the under-relaxation factor is changed to \( u_{\text{new}} = 0.8u_{\text{old}} \); once \( n \) (for instance 1000) inner iterations, the under-relaxation factor is updated as \( u_{\text{new}} \) = max \( \{0, 0.8u_{\text{old}}, 0.8u(\text{min}[1.2u_{\text{old}}])\} \); the under-relaxation factor has a upper limit, which is set to be \( u_0 \).

2.4. Grid and accuracy

In the present study, we use the grid system, which is symmetrical about \( t = \frac{1}{2} \). 1000 elements are uniformly distributed over \([0 \varepsilon 1]\).
[Image 0x0 to 595x842]

Fig. 3. Definitions of parameters characterizing slamming pressure during water entry of a wedge. \( C_p \) = pressure coefficient.

\[ \epsilon \], where \( \epsilon \) is set to \( 10^{-11} \). From \( \epsilon \) to \( \frac{1}{2} \), the element length is geometrically increasing, i.e. the ratio between successive elements is constant. The size of the smallest element, \( h_0 \), is approximately \( 10^{-2}\epsilon \). The largest element is next to \( \frac{1}{2} \) and its size is denoted as \( h_L \). Through Eq. (21), we know that the accuracy of \( \int \) directly influences that of \( \beta \). Cointe [6] has proposed

\[ \beta = \frac{1}{2} \left( \frac{\theta}{\pi} \right)^2 \]  

(36)

for small deadrise angles. For example, \( \beta \) equal to \( \frac{\pi}{6} \) corresponds to the asymptotic value of \( \beta \) equal to \( 1.54 \times 10^{-5} \). The numerical error of less than 1% of \( \beta \) requires that the numerical error of \( \int \) should be less than \( 10^{-7} \) approximately. The numerical error of \( \int \) can be estimated by assessing the accuracy of the numerical integration. To estimate the accuracy of numerical integration of the integral

\[ \int_x^{1/2} t^{-1+a} \int_x^t d(t) dt \quad 0 < a < 1, \]

we assume that \( d(t) \) can be expanded in the Taylor series

\[ d(t) = b_0 + b_1 t + b_2 t^2 + \ldots \]

It is possible to assess the accuracy of numerical integration of

\[ \int_x^{1/2} t^{-1+a} t^n dt \quad 0 < a < 1 \quad n = 0, 1, 2, \ldots, \]

because this integral can be evaluated analytically. In practice we only need to assess a few lowest order terms, since they are usually dominant.

Similarly, to estimate the accuracy of the numerical \( w(t) \), we assess the numerical integration of

\[ \int_0^1 f_n(t) \frac{1}{t-t^m} dt \quad 0 < m < 1 \quad n = 0, 1, 2, \ldots \]

or equivalently \( f_n(t) = \sqrt{t} t^n \).

[Images of graphs showing pressure distribution around the maximum pressure for different angles]
Fig. 5. Comparison of the pressure distribution on the wetted wedge surface and the free surface elevation for wedges with deadrise angles of 20°, 30° and 40° symmetrically entering into calm water.
In the above expression, we have used Eq. (23) and the fact that $\beta$ is much smaller than 1/2. It can be shown that

$$w_n(t) = \int_0^1 \sqrt{\pi x^n - \sqrt{\pi t^n}} \, dx$$

$$= 2t \left[ 1 - \sqrt{\ln\left(\sqrt{1/t} + 1\right)} \right] + \sum_{n=0}^{N-1} \frac{1}{(l + 3/2)}. \quad (37)$$

The numerical tests are performed for the integrals $\int_0^{1/2} t^{-1+a} \, dt$ and $w_n(t)$ on the proposed grid system, where $l_m = 1.0 \times 10^{-3}$ and $5.0 \times 10^{-4}$ are used. The accuracy is represented as the relative error, $\epsilon_r$, which is calculated by comparing with the exact solution. $\epsilon_{\text{max}}$ denotes the maximum relative error of $w_n(t)$ over $[0,1]$. The results are shown in Fig. 2. For $a < 1$ and $a = 3$, the relative error of $O(10^{-7})$ is obtained. We note that the trapezoid rule gives exact results for $a = 1$ and 2. Therefore, the relative error of the numerical integration of the three lowest terms,

$$\int_0^{1/2} t^{-1+a} \, dt \quad (0 < a < 1, \quad n = 0, 1 \text{ and } 2),$$

is estimated to be $O(10^{-7})$. Because the lowest order terms of $I(t)$ are regarded to be dominant, the relative error of the numerical integration of $\int_0^{1/2} t^{-1+a} I(t) \, dt$ is also estimated to be $O(10^{-7})$ on the given grid systems. Similarly, the relative error of the numerical $w(t)$ is estimated to be $O(10^{-7})$. It is noted that the accuracy can be controlled or improved by modifying the grid system.

2.5. Asymptotic formula for small deadrise angles

For small deadrise angles, the hydrodynamic problem can be solved by matched asymptotic expansions (Cointe and Armand [4], Cointe [6] and Howison et al. [5]). The detailed theories will not be repeated here. Our attention is to present the asymptotic formulas, which will be used in the following section to compare with numerical results of the similarity solution.

Cointe [6] has proposed Eq. (36) for the contact angle between the free surface and the wetted water surface at small deadrise angles. Based on the similar asymptotic analysis, Zhao and Faltinsen [2] gave that

$$y_B = \sqrt{\pi} \left(\frac{\pi}{2} - 1\right), \quad (38)$$

where $y_B$ is the $y$-coordinate of the intersection point between the free surface and the wedge body, $c(t) = \frac{\pi}{2} \sqrt{\pi} \cot \theta$ and $|r|$ is related to $x$ by

$$x - c = \left[\delta/\pi \right] \left[ -\ln |r| - 4/|r|^{1/2} - |r| + 5 \right]. \quad (41)$$

$\delta$ is the jet root thickness and it is expressed as $\delta = \pi \sqrt{2} \frac{dc}{dt} \frac{3}{|r|^{1/2}}$. Eqs. (39) and (40) represent the pressure distribution on the wetted wedge surface. When $|r| = 1$, i.e. $x = c$, the maximum value of $p$ occurs:

$$p_{\text{max}} = p_0 - 2 \rho \left(\frac{dc}{dt}\right)^2 \left(\frac{|r|^{1/2}}{1 + |r|^{1/2}}\right)^{3/2} \quad (42)$$

The corresponding $y$-coordinate is

$$y_{\text{max}} = \sqrt{\pi} \left(\pi/2 - 1\right). \quad (43)$$

By integrating the pressure along the wetted wedge surface, we can obtain the vertical hydrodynamic force

$$F_y = 2 \pi^2 \rho \sqrt{\pi} \cot \theta \left[ \sqrt{r_e/(\pi \cot \theta)} + \pi/2 - \sqrt{2} \right], \quad (44)$$

where $r_e$ is the root of the equation $\ln r + 4r^{1/2} + 5 - 2\pi^2 \cot \theta$ and corresponds to $x = 0$. Korobkin [20] also presented the pressure formulations, which are asymptotically equivalent to the present ones.

3. Numerical results

By the present method, we successfully obtained numerical results of the similarity solution for deadrise angles down to 1°. The largest element $l_m$ in the numerical integration procedure is as follows. $l_m = 1.0 \times 10^{-3}$ is used for deadrise angles $\leq 4^\circ$ and $l_m = 5.0 \times 10^{-4}$ for deadrise angles $\leq 4^\circ$. Table 1 shows the slamming parameters predicted by the present method, Zhao and Faltinsen [2]'s method and the asymptotic method. The discrepancy between the present similarity solution and the asymptotic solution tends to vanish with decreasing the deadrise angle. At small deadrise angles, the asymptotic solution of the pressure distribution on the wedge
Table 1
Comparison of slamming parameters during water entry of a wedge with constant vertical velocity \( V_0 \), \( \theta \) = deadrise angle; \( C_{p\text{max}} \) = pressure coefficient at maximum pressure; \( y_{\text{max}} \) = y-coordinate of maximum pressure; \( \Delta S \) = spatial extent of slamming pressure (see Fig. 3); \( c = 0.5 \text{m} \text{rad} \text{cot} \theta \); \( \phi \) = total vertical hydrodynamic force on the wedge; \( \theta \) = angle between the free surface and the wedge surface at the intersection point (see Fig. 13); \( y_{\text{max}} \) = y-coordinate of intersection point; Pres. = present similarity solution; Zhao = Zhao and Faltinsen [2]'s similarity solution; Asym. = asymptotic solution; Disc. = \( \text{Pres.} - \text{Asym.} \).

<table>
<thead>
<tr>
<th>( \theta^\circ )</th>
<th>( C_{p\text{max}} )</th>
<th>Zhao</th>
<th>Asym.</th>
<th>Disc.</th>
<th>Zhao</th>
<th>Asym.</th>
<th>Disc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.809759</td>
<td>-</td>
<td>0.809355</td>
<td>0.11%</td>
<td>0.5702</td>
<td>-</td>
<td>0.5708</td>
</tr>
<tr>
<td>2</td>
<td>2018.211</td>
<td>-</td>
<td>2023.355</td>
<td>0.25%</td>
<td>0.5694</td>
<td>-</td>
<td>0.5708</td>
</tr>
<tr>
<td>3</td>
<td>894.490</td>
<td>-</td>
<td>898.356</td>
<td>0.43%</td>
<td>0.5685</td>
<td>-</td>
<td>0.5708</td>
</tr>
<tr>
<td>4</td>
<td>501.441</td>
<td>503.03</td>
<td>504.606</td>
<td>0.63%</td>
<td>0.5673</td>
<td>0.5656</td>
<td>0.5708</td>
</tr>
<tr>
<td>5</td>
<td>77.699</td>
<td>77.847</td>
<td>79.360</td>
<td>2.09%</td>
<td>0.5544</td>
<td>0.5556</td>
<td>0.5708</td>
</tr>
<tr>
<td>6</td>
<td>17.735</td>
<td>17.774</td>
<td>18.626</td>
<td>4.78%</td>
<td>0.5072</td>
<td>0.5087</td>
<td>0.5708</td>
</tr>
<tr>
<td>7</td>
<td>6.895</td>
<td>7.602</td>
<td>7.402</td>
<td>6.68%</td>
<td>0.4254</td>
<td>0.4243</td>
<td>0.5708</td>
</tr>
<tr>
<td>8</td>
<td>3.253</td>
<td>3.268</td>
<td>3.254</td>
<td>7.19%</td>
<td>0.2897</td>
<td>0.2866</td>
<td>0.5708</td>
</tr>
</tbody>
</table>

Table 2
Maximum non-dimensional curvature of the free surface during water entry of a wedge with constant vertical velocity. \( \theta \) = deadrise angle; \( \kappa_{\max} \) = maximum non-dimensional curvature of the free surface.

<table>
<thead>
<tr>
<th>( \theta^\circ )</th>
<th>( \kappa_{\max} )</th>
<th>( \kappa_{\max} )</th>
<th>( \kappa_{\max} )</th>
<th>( \kappa_{\max} )</th>
<th>( \kappa_{\max} )</th>
<th>( \kappa_{\max} )</th>
<th>( \kappa_{\max} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.000956</td>
<td>-</td>
<td>0.000982</td>
<td>1.66%</td>
<td>25061.855</td>
<td>-</td>
<td>25261.387</td>
</tr>
<tr>
<td>2</td>
<td>0.003806</td>
<td>-</td>
<td>0.003935</td>
<td>3.28%</td>
<td>2604.311</td>
<td>-</td>
<td>2605.478</td>
</tr>
<tr>
<td>3</td>
<td>0.008450</td>
<td>-</td>
<td>0.008578</td>
<td>4.83%</td>
<td>2970.351</td>
<td>-</td>
<td>2762.352</td>
</tr>
<tr>
<td>4</td>
<td>0.014843</td>
<td>0.015049</td>
<td>0.015042</td>
<td>6.31%</td>
<td>1498.933</td>
<td>1503.638</td>
<td>5179.87</td>
</tr>
<tr>
<td>5</td>
<td>0.008093</td>
<td>0.009088</td>
<td>0.010389</td>
<td>14.24%</td>
<td>212.959</td>
<td>213.980</td>
<td>213.980</td>
</tr>
<tr>
<td>6</td>
<td>0.413474</td>
<td>0.4118</td>
<td>0.557616</td>
<td>26.23%</td>
<td>42.272</td>
<td>42.485</td>
<td>50.640</td>
</tr>
<tr>
<td>7</td>
<td>20</td>
<td>-</td>
<td>14.620</td>
<td>14.139</td>
<td>18.748</td>
<td>25.22 %</td>
<td>38.324</td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td>-</td>
<td>5.447</td>
<td>5.477</td>
<td>8.322</td>
<td>25.22%</td>
<td>38.324</td>
</tr>
</tbody>
</table>

is compared to the present similarity solution, which is shown in Fig. 4. Good agreement is obtained, except that there is a small discrepancy (also indicated in Table 1) in the position of maximum pressure. These verify the asymptotic theories, since the asymptotic solution should approach the exact solution when the deadrise angle goes to zero.

At relatively large deadrise angles, the present similarity solution is checked against the boundary element method [16,19], which solves the boundary integral equation transformed from Eq. (1) and tracks the evolution of the free surface by the second order Runge-Kutta method. At the start of the numerical simulation by the boundary element method, a small penetration \( \eta_0 \) of the wedge into the water is given, the velocity potential is set to zero and the free-surface elevation is prescribed by the Wagner's outer-domain solution

\[
y(x) = \eta_0 \sqrt{\frac{x}{\gamma(x) \arcsin \left( \frac{c(t_0)}{x} \right) - \eta_0 \phi}},
\]

where \( c(t_0) = \gamma \eta_0 \cot \theta \) is the \( x \)-coordinate of the intersection point between the free surface and the wedge surface. As time increases, the solution by the boundary element method should approach the exact solution. Fig. 5 compares the present similarity solution to the converged boundary-element-method solution. Except for the free surface of jet flow, perfect agreement has been obtained between the two methods. The discrepancy in the free surface of jet flow is due to the cut-off of jet flow used in the boundary element method for stabilizing the numerical solution.

Zhao and Faltinsen [2] also compared their similarity solution with a boundary element method. It is observed that there are some discrepancies between their two solutions (see [2], Fig. 6). It is indicated that Zhao and Faltinsen [2]'s results slightly overestimate the boundary-element-method solution. Further, the convergence history of the boundary-element-method solution is illustrated in Fig. 6, which shows the results of a wedge with the deadrise angles of 30° symmetrically entering into calm water. It can be seen that the boundary-element-method solution converges to the present similarity solution and improves Zhao and Faltinsen [21's results.

It is observed that the maximum curvature of the free surface, \( \kappa_{\max} \), occurs at the root of the jet. This parameter increases as the deadrise angle decreases, which brings an attention to the effect of surface tension which may matter locally at small deadrise angles. Based on \( \xi = x/\eta_0 \) and \( \eta = y/\eta_0 \), the curvature of the free surface can be expressed as

\[
\kappa = \frac{r}{\gamma(x)},
\]

\[
r = r_1^{2} - \eta^{2}/(1 + \eta^{2})^{3/2}.
\]

Inserting equation (8) into (47), we can show

\[
r(s) = \frac{c^{2}}{c^{2}} \frac{1 - 2\gamma c(1 - s)\gamma^{2(1 - 2\gamma)}}{2\gamma W(1 - 2\gamma)}/I_{2}(s),
\]
where, \( s = 1 \) corresponds to the intersection point between the free surface and the wedge body and \( s = 0 \) corresponds to the free surface at infinity. Then, the maximum non-dimensional curvature of the free surface is

\[
\tilde{k}_{\text{max}} = \max_{0 \leq \psi < \pi} \tilde{k}(\psi).
\]

Table 2 shows the maximum non-dimensional curvature versus the deadrise angle. The maximum curvature of the free surface,

\[
k_{\text{max}} = \frac{k_{\text{max}}}{V_0 U},
\]

can be large at the very initial stage and/or at small deadrise angles, which implies that surface tension may matter locally.

The present similarity solution results of the pressure distribution on the wetted wedge surface, the free surface configuration and the curvature of the free surface are given in Appendix A.

4. Conclusions

A reliable and accurate method is developed for solving the similarity flow of a wedge symmetrically entering into calm water with constant velocity. By using the present method, exact solutions of the similarity flow for deadrise angles less than \( 4^\circ \) are first obtained. The accuracy of the numerical results is estimated. The numerical results show that the asymptotic theories proposed by Cointe and Armand [4], Howison et al. [5] and Zhao and Faltinsen [2] are consistent with the present method at small deadrise angles; with decreasing the deadrise angle, the discrepancy between the two solutions of slamming parameters tends to vanish; the pressure distribution on the wetted wedge surface obtained by the present method agrees well with the asymptotic solution. At relatively large deadrise angles, the present similarity solutions agree almost perfectly with the traditional boundary element method and improve Zhao and Faltinsen [2]'s results. All these demonstrate that the present similarity solution is accurate. It can be used as a reference solution in theoretical studies of water entry problems and in developing accurate numerical solvers for simulating strongly nonlinear wave-body interactions. Finally, the curvature of the free surface has been investigated. The maximum curvature occurs at the root of the jet, where the surface tension matters at the very initial stage and/or at small deadrise angles.

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Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.apor.2017.05.006.

References
