approximate identification and control design
with application to a mechanical system

1. Fifty years ago Ziegler and Nichols [1] showed that experimentation with various controllers enables the selection of a good controller. From then system theory has had its own “Berlin wall” separating the fields of identification and control design. In case identification and control design techniques, that have been developed ever since, are used to accomplish a high performance control system, then an iterative scheme of repeated identification and control design is necessary, which involves experimentation with various controllers in a way much similar to that of Ziegler and Nichols [2]. The time has come to reunite the separate problems of identification and control design.


2. In the standard robust control design problem the model-error imposes limitations on the achievable performance. In the high performance control design problem the desired performance imposes limitations on the allowed extent of the model-error.

This dissertation.

3. Every stable linear feedback system is robustly stable.

4. The bridges over the Danube enable one to stroll from Buda to Pest and back. In contrast, the gap between models and reality will never be bridged completely¹, and a leap of faith always is needed to get across.

¹ “A bridge between models and reality” has been the motto of the 9th IFAC/IFORS Symposium on Identification and System Parameter Estimation, Budapest, Hungary, July 8-12, 1991.

5. Prejudices encountered in system identification can be obviated by letting the data speak [3]. Nevertheless the final word must rest with the objective of the modeling procedure.


6. In linear system theory the inability of deterministic models to precisely represent a set of measured data often is attributed to noise and to unmodelled dynamics.
Without a priori assumptions a lower bound on the unmodelled dynamics can be obtained by applying feedback control: the unmodelled dynamics can cause instabilities, whereas additive noise cannot.

7. No alternative makes all others superfluous.

8. Academic research is like many industrial activities: not only development but also profound marketing is needed to make a product sell well.

9. The advanced courses organized by the Dutch Graduate School of Systems and Control greatly help Ph.D. students to make their way through the diverse field of systems and control theory.

This dissertation.

10. A strike in public transportation indirectly inflicts a substantial speed limit upon regular motorists.

11. The Delft University of Technology has an extensive protocol for Ph.D. graduations. As several steps of the procedure are taken just pro forma, it is recommendable to rationalize this protocol; and not just pro forma.

12. “All votes are equal, but some are more equal than others.”

_The Economist_ on Britain’s general elections; March 14, 1992, p.36.

13. A problem is not fully understood until it has been solved.
Approximate Identification and Control Design
Approximate Identification and Control Design
with application to a mechanical system

PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus,
prof. drs. P.A. Schenck,
in het openbaar te verdedigen
ten overstaan van een commissie
aangewezen door het College van Dekanen
op dinsdag 19 mei 1992 te 10.00 uur

door

Rudolphus Johannes Petrus Schrama
Werktuigbouwkundig Ingenieur

geboren te Haarlem
Subject headings: system identification / control theory.
To Annette
Luck or not, I gladly took the break
The odds were low, the chances nearly zero
But a chance it was, that I had to take...

Chris Rea, “Steel River”
New Light Through Old Windows

Preface

There is a great body of literature on the subjects of identification and control design. There are, however, only very few contributions on the combined problem of identification-and-control-design. Bridging the gap between the separate fields of identification and control design is the central issue of the research reported here. To that end I have investigated identification from a control design point of view, and I have surveyed control theory with the (im-)possibilities of identification still in my mind.

During the research project I bothered myself (and others) again and again with the question: What’s the use? Right from the beginning I aimed at developing some theory that is applicable in practice. Thereby I found myself faced with the viewpoint of the practitioner:

“The theory is alright, but does it work?”

and with that of the theoretician:

“Indeed it works, but what is the underlying theory?”

Balancing between these two perspectives, turned out to be much like walking a high-wire: one moment of inattentiveness, and you fall off in either direction.

Finally, I apologize to the reader, as it will be hard to digest this dissertation. The only real excuse that I can bring forward is the motto: better a good plan today, than a perfect plan tomorrow.

March 30, 1992

Ruud Schrama
Delft
Summary

Modern control theory employs models consisting of two parts: a nominal model and an uncertainty bound around it. The nominal model approximately describes the dynamics of the system under investigation. The uncertainty bound and the nominal model together induce a whole family of systems that contains the system of concern. And the problem of designing an acceptable compensator for the system, is transformed into the more conservative problem of designing a compensator that works well with each member of this family.

Models can be built from measured data by means of system identification. Recent years have seen a growing interest in the use of identified models in control design problems. Much of this attention has been devoted to the estimation of the abovementioned uncertainty bounds. Such a bound is necessary in order that a certain performance for the controlled system can be guaranteed. However, a high performance for the system in question cannot be obtained without a highly suited nominal model. The problem of identifying nominal models suited for high performance control design is addressed in this dissertation.

On the one hand the quality of a nominal model depends on the controller that it gives rise to. On the other hand the quality of the controller hinges on the nominal model. Hence the identification of a suited nominal model and the design of a high performance controller make up one combined problem. If separate procedures for identification and control design are used to solve this combined problem, then an iterative scheme is necessary. We develop iterative schemes of repeated identification and control design. The identification uses data obtained from the system, while it operates under feedback control. The nominal model is used to design a controller, which is applied to the system prior to the next identification stage.

At each step in this iteration the stabilizing compensator is known. We use the knowledge of this compensator to represent the system by a dual Youla parameterization. Based on this coprime factor representation we formulate a new identification problem, which allows us to solve the problem of approximate identification from closed-loop data. The coprime factors of the system are identified using frequency response estimates. Thereby we can concentrate on the asymptotic distribution of the bias due to undermodelling.

Each identified nominal model is used to enhance the previously designed controller. For this controller enhancement we use a robust control design method, extended with a procedure to determine appropriate weighting functions. A procedure of explicit stability ascertainment is used to guarantee that the new controller will work well with the system, even before it is actually implemented.

Having developed the necessary tools, the proposed iterative schemes are tested by a simulation study and by practical experiments.
Acknowledgements

First of all I would like to thank my advisor Okko Bosgra and co-advisor Paul Van den Hof for the support and the freedom to pursue my own ideas. Their guidance proved to be of great value especially in the final stage of the research. I believe, with some sense of understatement, that this job has not been an easy one, as it took them only a three day’s workshop to introduce notions like Schramatics and Schramania.

Next, I thank Peter Bongers for the many useful and fruitful discussions as well as for his tailor-made software, which together greatly influenced the course of my research project. I acknowledge the sessions with Peter Heuberger, in which we attempted to arm ourselves against paralogisms. Further, I am indebted to Richard Hakvoort and Ralph Van Hof, who contributed to my research project as Master students. They carried out all experiments, while allowing me to drive them to extremities. At several stages of the project, my search for “the real questions” has been influenced by colleagues abroad including Fred Hansen, Dan Rivera, Roy Smith, Carl Nett, John Doyle, Bo Wahlberg, Michel Gevers and Jim Krause: Thanks!

Special thanks go also to my (former) “room-mates” Pepijn Wortelboer, who endured most of the past five years, and Gregor Van Baars with his unforgettable un-equalled aptitude to jest ad nauseam. And, of course, nothing would have been possible without supporting and organizing activities, for which I thank Sjoerd Dijkstra, Peter Valk, Cor Kremers, Rens de Keyser, Piet Ruinaard, Els Arkesteijn, Diana Van Dijk, Trinette Scholte, plus others too numerous to mention but just as important.

Apologies go to friends and relatives, who have not heard much of me lately, except that I was busy writing a thesis. I am most grateful to my parents for their many years of encouragement and support. And finally, as for Annette, my spouse, I know that the first two years of our marriage has not always been what you had expected. For I was frequently present and absent at the same time. Although an acknowledgement here is inadequate: thanks for your love and support.
Note to the Reader

The chapter number and chapter title are shown at the top of each left page. The section number and the section title are shown at the top of each right page. Figures and equations are numbered consecutively within a chapter.

Within each section, all statements like comments, definitions, facts, theorems, remarks, and so forth, are numbered by means of a numeral preceded by the chapter- and section-number. The same counter is used for all these statements, so that e.g. Conjecture 6.1.5 follows immediately after Proposition 6.1.4. For a clear distinction between the text-body and examples, a different type-setting is used for the latter, except for the examples that cover a whole section. For the purpose of easy referencing, notation is sometimes introduced by means of an assumption.

Literature is referred to by numbers enclosed by square brackets, e.g. [21,7,19]. The authors' names are sometimes included in the text for easy reading.

Symbols like $\hat{P}_1$ cannot be produced by matlab. In the figures such symbols are represented as Ph1, where h stands for "hat".
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Chapter 1

Introduction

This chapter begins with a motivation of the research reported here. Thereafter we explain the problem of concern, and we outline our approach.

1.1 Motivations and Background

Control action causes a system to operate in some desired fashion. Without this control action many systems would exhibit an unacceptable natural behavior. Examples of such physical systems are chemical processes, vehicles, aircraft, consumer electronics, power plants, space structures, industrial robots, and so on. Furthermore, there is an unremitting endeavour to accomplish such things as a higher accuracy, an increased safety, additional comfort, a reduced environmental burden, a cut-back in energy consumption and, last but not least, a larger economic profit.

The enhancement of some performance often implies the reduction of uncertainties and the attenuation of disturbances. Such improvements can be accomplished by applying a feedback compensator. In the early days of feedback control compensators were constructed by way of trial and error. Feedback control was greatly facilitated by the introduction of standard components like the PI and PID controllers and their accompanying “tuning rules” [268]. Notwithstanding their heuristic nature these classical design rules proved to be successful quite often. Therefore these controllers are still widely and effectively used in many industrial processes and products today.

More and more complex systems have to be controlled and high demands are made upon their performance. As the precise consequences of tuning get beyond grasp, an increasing number of control problems cannot be tackled properly by experience and rules of thumb alone. This explains the growing interest in model-based control design: it is easier to construct a compensator for a complex model by way of calculation than by way of tuning. Naturally, we have to model the system under consideration before these design techniques can be applied.

There are basically two ways of building models. One possibility is to derive equa-
tions from first principles that govern the behavior of the system in question. This procedure often leads to a very detailed model of all individual phenomena. The other approach is to develop a model from observed or collected data, which is called system identification. It is easy to retrieve those aspects of the system that bring forth large contributions to the data. Other aspects may be untraceable. Thus system identification “automatically” tends to emphasize the prevailing characteristics of the system’s behavior. The resulting models are usually less detailed than those obtained from first principles. These two modelling procedures should be used in a complementary fashion, since a physically motivated model provides insight and an identified model is justified by real data. Nevertheless we just focus on building models by means of system identification.

As fields of study, system identification and control design have received considerable attention in literature. However each of these fields has been developed irrespectively of the other until a few years ago. As a consequence the established techniques for identification and control design are not accommodated to each other. The aim of our investigations is to adapt identification and control design mutually, so that the control design employs a model that the identification can provide.

1.2 Problem Statement

The models used in system identification and control design traditionally consist of a differential or difference equation or a set of such equations. As these equations never precisely describe the dynamics of a real system we call this type of model a nominal model. Further, for ease of discussion we define a plant as some system that has to be controlled. We frequently speak of “a plant with uncertain dynamics” in order to stress that no exact model is available for the plant.

Ideally one could derive a nominal model by some identification method, and then use this nominal model to construct a controller. Sometimes the designed controller achieves a good performance for the plant, and sometimes the plant is destabilized. The latter happens if the controller does not anticipate the imperfections of the nominal model. Hence we have the following two prerequisites for achieving a good compensator:

1. The controller must anticipate the imperfections of the nominal model, i.e. the controller must be robust.

2. The nominal model must be suited for control design.

These prerequisites are complementary. On the one hand, exact modelling is too costly or impossible, so that imperfections of the nominal model always must be anticipated. On the other hand, a bad nominal model will not give rise to a good controller.

From here the research for control-relevant system identification branches into two directions. These directions are depicted in Fig. 1.1, which relates to the above two
points. The branch on the left illustrates the demand of robust control theory for a

<table>
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<th>Control Design</th>
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<td>robust</td>
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quantification of the so-called "model-error". The right\(^1\) branch, which is of interest here, concerns the identification of a nominal model that is suited for high performance control design. We elaborate these two branches separately beginning with the quantification of the model-error.

Much of the robust control theory rests on a model that consists of a nominal model and an "uncertainty bound" around it. The basic idea is, that a nominal model and such a bound together induce a family of systems that accounts for the plant. When a compensator stabilizes each member of this family, then it is guaranteed to stabilize the plant. This has motivated the development of identification techniques for the estimation of an upper bound on the difference between a plant and its nominal model. By now there exist probabilistic and deterministic approaches to the quantification of the model-error. Most of these quantifications are independent from the design objectives, so that they are generally not tailor-made for the compensator at hand.

The probabilistic techniques produce "soft" bounds, i.e. bounds which consist of confidence intervals around the nominal model. These methods employ prior assumptions on the distribution of the noise and on the distribution of the model-error [140, 121, 267, 87]. The deterministic methods produce "hard" bounds. These bounds are guaranteed to be larger than the model-error, provided that some assumptions on the plant and on the noise are satisfied. Such assumptions typically involve the smoothness of the frequency response of the plant and an upper bound on the noise. Many contributions in this area concern the estimation of an $H_{\infty}$- or $l_1$-bound on the difference

\(^1\)In the first place "right" is meant as the opposite of "left".
between the plant and its nominal model [104, 213, 214, 106, 146, 145, 148, 230, 44]. Others assume that this bound is given a priori and a nominal model is estimated in the presence of “bounded unmodelled dynamics” [125, 122, 258, 248, 126, 215, 124, 123]. A third class of techniques produce bounds in the parameter space of the set of candidate nominal models [70, 157, 166, 19, 181].

All these identification methods have in common that they assess the quality of the nominal model by characterizing its deviation from the modelled plant. This characterization is used in the subsequent control design, so that stability or even some performance can be guaranteed for the plant of concern. As the requirements of high performance and of robust stability are conflicting, the estimated upper bound on the model-error should be as tight as possible. However, by itself, a tight estimated upper bound is not sufficient to guarantee a successful control design: the bound cannot be smaller than the true model-error, and thus the latter limits the achievable performance. This is not a matter of a tight upper bound, but that of a suitable nominal model.

We question when an identified nominal model is suited for control design. The nominal model must of course give rise to a good controller for the modelled plant. In common practice the designer wants also to have confidence in the nominal model. Without this confidence he is all but eager to use it for control design. Thus the nominal model must look good and be good. In this perspective we interpret Norton’s notion of the validity of a model:

“A model is validated by answering two questions:
is it credible, and does it work?” [165, p.285]

A nominal model is credible, if it confirms some available background knowledge of the investigated plant. For instance the parameters representing physical quantities (impedance, mass, flow, etc.) can take plausible values. Furthermore, according to Ljung, one often has a lot of faith in an identified nominal model, if it explains the observed data almost completely:

“The assessment of model quality is typically based on how the models perform when they attempt to reproduce the measured data.” [138, p.7]

Nevertheless, whether or not an identified nominal model “works” depends rather on its intended use than on its capacity to reproduce the measured data. Here the intended use is control design, and the controller should achieve a high performance for the plant. Suppose that the same controller achieves a bad performance for the nominal model that we are confident about. In that case we lose all faith in this controller irrespective of its capacity to control the plant. In practice we would not risk to apply such a controller to the plant at hand. Thus the controller must work sufficiently well also with the nominal model.
1.2 Problem Statement

We consider the problem of designing a controller that achieves a high performance for some plant. In pursuing this performance for the plant we have to require a similar performance for its nominal model. A high nominal performance can usually be achieved only at the expense of relatively small robustness margins. The model-error must be small in view of these robustness margins, in order that the corresponding compensator robustly stabilizes the plant. Thus, we reach the following important observation:

The requirement of a high performance imposes limitations on the allowed shape and extent of the mismatch between a plant and its nominal model.

This is a conversion of the common starting point in robust control design, which says that the amount of “unmodelled dynamics” limits the achievable performance. Instead we say, that the desired performance limits the allowed amount of “unmodelled dynamics”. In other words, we have to shape the model-error conformably to the robustness margins of the high performance feedback system. In this light the problem studied in this thesis reads as follows:

Given a plant with uncertain dynamics, which are assumed to be linear finite dimensional and time-invariant, design a high performance compensator for this plant by means of identification and model-based control design.

Suppose that we have designed a compensator from some nominal model. This designed compensator must be robust, but not at the cost of a low performance. And it must achieve a high performance for the nominal model, but not at the cost of too little robustness. Hence, in view of the above problem formulation, we have to evaluate the designed compensator by answering two questions:

1. Does the compensator achieve a high performance for the nominal model?

2. Does the compensator achieve a high performance for the uncertain plant?

If the answer to one of these questions is negative, then we have not solved the high performance control design problem.

Whether there exists a compensator that complies with these requirements hinges on the nominal model used in the design. When such a compensator does not exist, then the nominal model is not suited for high performance control design. Thus the quality of a nominal model depends on the compensator that it gives rise to. Conversely, the compensator and its quality depend on the nominal model. Hence we have to treat approximate identification and model-based control design as a joint problem.

A nominal model and a compensator make a solution to the joint problem, if their high performance is robust in view of the modelled plant. This is accomplished if the feedback system composed of the nominal model and its own high performance
compensator approximately describes the feedback system containing the plant and the same compensator. This approximation of the plant by a nominal model is called the *tailor-made system approximation*, because the approximation problem depends on the model-based compensator.

We use system identification techniques to carry out this tailor-made system approximation. This approximate identification problem cannot be solved in a straightforward fashion, because the compensator is not available yet. The precise quality of an identified nominal model remains unknown until the corresponding model-based control design procedure has been completed. On the other hand the latter cannot commence without a nominal model. In consequence an iterative scheme is required to solve the high performance control design problem by means of individual procedures for approximate identification and model-based control design.

The joint problem of identification and control design has been under investigation before. Probabilistic schemes for simultaneous identification and control design have been proposed in [11, 114]. In [206] conditions have been established under which the use of one separate identification procedure and one control procedure results in an overall optimal control. The inseparability of approximation and control design has been pointed out in [210, 184, 186, 198, 201]. Besides, the need of an iterative scheme has also been advocated in philosophical terms in [7]. Our argument differs from others in that we raised the approximation of the whole feedback interconnection as a motivation, rather than the approximation of the plant itself. More about this is said in the next chapter.

Some iterative schemes of repeated identification and control design have been proposed in literature. In [187, 183] Rivera *et al.* used such an iteration to build prefilters for a “control-oriented” prediction-error identification from one open-loop dataset. Alternatively, we elaborate an iterative scheme, in which each identification is based on new data collected while the plant is controlled by the latest compensator. Hence we have to solve the problem of approximate identification from closed-loop data. In tackling this problem we use the knowledge of the compensator that stabilizes the plant during the identification experiment. With this compensator we can represent the plant by a coprime factorization, and we identify the coprime factors with a frequency-domain identification technique. Since we use identification as a means for system approximation we concentrate on the so-called “bias” due to undermodelling.

After each identification the new nominal model is used to improve the compensator. For this improvement of the compensator we make use of a robust control design method in a non-conventional way. Instead of utilizing a family of systems to account for the plant in question, the design is guided by frequency response estimates that, along with the nominal model, provide the required information about the plant. The plant is accounted for by a bounded set of “uncertain dynamics” after the construction of a new compensator. Before this new compensator is applied to the plant, we use
this bounded set to guarantee that the new control system will be stable.

The iterative scheme is closely related to adaptive control, but there is an essential difference. In the proposed iteration we can study the identification and control design procedures separately. In contrast, identification and control design are completely intertwined in adaptive control, which makes it less transparent and difficult to understand. In [22] Bitmead et al. formed an iterative scheme by combining prediction-error identification and LQG/LTR control design. The iterative schemes of Bitmead et al. in [23, 264, 265] and of Hakvoort et al. [94, 95] are based on the prediction-error identification method and an LQ-performance. Additional differences and similarities between the various approaches will be revealed in the next chapter.

1.3 Organization of the Thesis

The sequel of the thesis consists of four parts. In Part I, Chapters 2 through 7, we develop the tools for the individual procedures of identification and control design. Then in Part II we combine these tools to form two iterative schemes, which are applied to a simulation study (Chapter 8) and an experimental set-up (Chapter 9). In Part III we review our iterative solution to the problem of designing a high performance controller for a plant with uncertain dynamics. Finally Part IV contains additional material, references, etcetera. Below we outline the contents of the chapters of Part I and II.

Chapter 2: Analysis of Control Design and System Identification.

We begin our search for a link between identification and control design with surveying some potentials and limitations of these two fields. We explain why the bound on the "model-error" should be used as a design variable rather than as a prior constraint. Also, we relate the feedback properties of a controlled plant to those of its nominal model. To that end we express these feedback properties in terms of transfer functions. We declare the estimation of these transfer functions to be the task of system identification. Further, we use system approximation to link this estimation to control design. More precise, we examine what system approximation problems are solvable by identification techniques and, at the same time, are meaningful in view of feedback control.

Chapter 3: Algebraic Theory of Linear Feedback Systems.

In practice a system usually has to be stabilized before experimentation is allowed. We investigate the implications of the assumption that the plant of concern operates under a stabilizing feedback control. For this we employ the algebraic theory of co-prime factorizations. As a result we generalize the algebraic framework of Nett [162] and Desoer and Gündes [50, 90].

Chapter 4: Open-loop Identification of Plants Under Feedback.
The problem of identifying a plant that operates under feedback by a known stabilizing controller is transformed into a new open-loop identification problem. These results extend the dual fractional representation approach that has been applied by Hansen et al. [100, 102] for closed-loop experiment design. All existing open-loop results apply to the new identification problem. This enables a comprehensible and manageable approximate identification of feedback controlled plants.

Chapter 5: Identification in View of Feedback Control.

We examine the potentials of the new open-loop identification problem of Chapter 4 in regard of two feedback-relevant system approximation problems. At this stage we do not yet develop an identification technique to get "from data to model".

Chapter 6: Cautious Controller Enhancement.

The nominal model that results from the system approximation of Chapter 5 has to be used for an enhancement of the controller. We base this enhancement on a robust control design method that optimizes robustness in respect of normalized coprime factor perturbations. The new compensator is obtained as a small "perturbation" of the old compensator, so that the nominal model reliably predicts how the plant will respond to the new compensator.

Chapter 7: Frequency-domain Identification of Coprime Factors.

The two feedback-relevant system approximation problems of Chapter 5 are customized for the control design procedures of Chapter 6. The resulting approximation problems are solved by means of frequency-domain identification techniques.

Chapter 8: Iterative Schemes of Identification and Control Design.

The tools of the previous chapters are combined together to form two iterative schemes of repeated identification and control design. These schemes are illustrated and evaluated by means of a simulation study.

Chapter 9: Experimental Verifications of The Iterative Schemes.

The iterative schemes of Chapter 8 are applied to an experimental set-up. This illustrates the practical utility of our iterative high performance control design procedure.
Part I

Identification And Control Design
Chapter 2

Analysis of Control Design and System Identification

In this chapter we survey control design and system identification separately and we take our stand in each of both fields. After the preliminaries of Section 2.1 we focus on robust control theory in Section 2.2. We examine the utility of the common robust control paradigm of "bounded uncertainty" for our high performance control design problem. From this we will infer that an "uncertainty bound" should be regarded as a design variable and not as an a priori constraint. Further, we address the issue of performance in terms of feedback properties. These properties are embodied by some transfer functions. Via these transfer functions we can relate the feedback properties of the controlled plant to those of the controlled nominal model. As the modelling of feedback properties boils down to the modelling of transfer functions, we study transfer function estimation in Section 2.3. We point out some typical problems that are involved in approximate identification and closed-loop identification. Then in Section 2.4 we piece together identification and control design through the definition of several system approximation problems. The latter problems are relevant for feedback control and they are conceptually solvable by means of identification methods. In the final section we use the discussed material to give some sharper directions to our investigations.

2.1 Definitions and Notation

Everything in this chapter hinges on interconnections between a controller and either a plant or a nominal model. For our purposes we define these components as follows.

Definition 2.1.1 A plant, a nominal model and a compensator are linear time-invariant finite dimensional (LTIFD) systems. A nominal model and a compensator

\footnote{We use the designations "compensator" and "controller" interchangeably.}
are completely known systems. A plant is some system that has to be controlled, but whose dynamics are not known precisely.

Remark 2.1.2 The incomplete knowledge of the plant's dynamics reflects that it is impossible to exactly model the behavior of a real system. In simulation studies we pretend that we do not know e.g. the plant's dynamical order, or whether it has unstable zeros. If the plant is unstable, then we do not know its number of unstable poles. Our developments are based on continuous-time systems, but many of the results carry over directly to discrete-time systems.

As usual in the study of control systems we regard plants and compensators as operators that map inputs into outputs. Like in [31] we distinguish two types of inputs and two types of outputs. For a plant these are defined as follows.

Definition 2.1.3 An input is a signal or a vector of signals that drives the operation of a plant. A plant can have two types of inputs.

i. The controlled input, denoted $u$, is the manipulated input used to control the operation of the plant. This input is always available for identification.

ii. The uncontrolled input, denoted $w$, is the vector of all other signals that drive the system. This input is not necessarily available for identification.

Definition 2.1.4 An output is a signal or a vector of signals that displays the operation of a plant. A plant can have two types of outputs.

i. The measured output, denoted $y$, is available for control.

ii. The monitored output, denoted $z$, embodies the qualities to be controlled.

As a shorthand notation we often speak of e.g. "the output $y$" instead of "the measured output $y$". The inputs and outputs of a compensator are denoted $u_c, w_c, y_c$ and $z_c$. The plants and compensators of interest involve a controlled input, $u$ or $u_c$, and a measured output, $y$ or $y_c$, but not necessarily the other inputs and outputs. We accordingly distinguish the following operators

$$P_{TR}: \begin{pmatrix} w \\ u \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} \quad C_{TR}: \begin{pmatrix} w_c \\ u_c \end{pmatrix} \mapsto \begin{pmatrix} z_c \\ y_c \end{pmatrix}$$

$$P: \quad u \mapsto y \quad C: \quad u_c \mapsto y_c$$

which are associated with plants $(P_{TR}, P, C)$ and compensators $(C_{TR}, C, C)$. The index $T$ stands for Two vector-inputs and/or Two vector-outputs. Instead of "the input-output operator $P$ of the plant" we speak of "the plant $P$".
As common in literature we represent the input-output operators by their transfer function matrices \[31\] and we rewrite “\( P : u \rightarrow y \)” to “\( u = P(s)y \)”. Further we often drop the Laplace transform variable \((s)\) from the notation, and we frequently shorten “transfer function matrix” to “transfer matrix”. In summary, we use the symbol \( P_{TT} (P_T, \tau) \) to indicate a plant, its input-output operator and its transfer matrix. The precise meaning will be clear from the context.

Having established our notions of a plant and a compensator we can define their possible interconnections. These are called feedback systems or sometimes control systems.

![Diagram](image)

**Fig. 2.1:** General feedback system \( H(P_{TT}, C_{TT}) \).

**Definition 2.1.5** A feedback system is an interconnection of a plant and a compensator such that

\[
\begin{align*}
    u &= r_1 + y_c \\
    u_c &= r_2 - y
\end{align*}
\]

where \( r_1 \) and \( r_2 \) are called exogenous inputs. The general feedback system, denoted \( H(P_{TT}, C_{TT}) \) and depicted in Fig. 2.1, has as special cases the single-variate feedback system \( H(P, C) \) and the standard feedback system \( H(P_T, C) \).

The general feedback system \( H(P_{TT}, C_{TT}) \) of Fig. 2.1 is the most general interconnection of two distinct systems \[60, 162, 50\]. The standard feedback system \( H(P_T, C) \) lacks \( z, z_c \) and \( w_c \). This is the simplest configuration, in which typical problems of identification and control design are encountered simultaneously. The single-variate feedback system \( H(P, C) \) lacks also the disturbance input \( w \). This is the central configuration in stability analysis and control design. Besides, the symbols like \( H(P_{TT}, C_{TT}) \) do not refer to a particular input-output map of the feedback systems. Along these lines the feedback system \( H(P_{TT}, C_{TT}) \) is called\(^2\) stable if all its mappings from each of \( w, w_c, r_2, r_1 \) to each of the other signals in Fig. 2.1 is stable.

In accordance with this terminology we sometimes refer to \( P_{TT} \) and \( P_T \) as respectively the general plant and the standard plant. From the following definition it is apparent that \( P \) is a special case of \( P_T \) and in turn \( P_T \) is a special case of \( P_{TT} \).

---

\(^2\)A formal definition will be given at a later stage.
Definition 2.1.6 The plant $P_{TT}$ is decomposed commensurately with its inputs and outputs:

$$
\begin{pmatrix}
    z \\
    y
\end{pmatrix}
= P_{TT}
\begin{pmatrix}
    w \\
    u
\end{pmatrix}
= \begin{bmatrix}
    P_{zw} & P_{zu} \\
    P_{yw} & P
\end{bmatrix}
\begin{pmatrix}
    w \\
    u
\end{pmatrix}
$$

(2.2)

and the four distinct parts are termed the outer-loop plant $P_{zw}$, the monitor plant $P_{zu}$, the disturbance plant $P_{yw}$ and the inner-loop plant $P$.

The compensator $C_{TT}$ is partitioned analogously as

$$
\begin{pmatrix}
    z_e \\
    y_e
\end{pmatrix}
= C_{TT}
\begin{pmatrix}
    w_c \\
    u_c
\end{pmatrix}
= \begin{bmatrix}
    C_{zw} & C_{zu} \\
    C_{yw} & C
\end{bmatrix}
\begin{pmatrix}
    w_c \\
    u_c
\end{pmatrix}
$$

(2.3)

in which $C_{zw}$ should be read as $C_{zw, e}$, etcetera.

The inner-loop parts $P$ and $C$ have no indices because they correspond precisely to the single-variate feedback system $H(P, C)$. As the terms $P$ and $C$ are of major importance to stability analysis, we will concentrate on modelling the inner-loop part $P$. We consider only the nominal model $\hat{P}$, which is a simple description of the important characteristics of the inner-loop plant $P$. A nominal feedback system is taken to be the single-variate feedback system $H(\hat{P}, C)$. A feedback system that contains the plant is frequently called an actual feedback system, so that its distinction from the nominal feedback system is accentuated.

Remark 2.1.7 The literature on identification considers practically only the standard plant $P_T$. Further, in identification the parts $P$ and $P_{yw}$ are often called the transfer function respectively the noise model of the plant $P_T$ [138].

In the recent $H_2$- and $H_{\infty}$-control theory [60, 72, 61, 143, 222] the decomposition of (2.2) is frequently used without designations. The control objective is often formulated as the problem of finding a compensator that minimizes some norm of the map from $u$ to $z$ [72, 61, 31]. In this context $y$ is merely a signal that is available for control and that is contaminated with measurement noise. Besides, the feedback configuration $H(P_{TT}, C_{TT})$ is often rearranged such, that $r_1, r_2$ and $w_c$ are incorporated into the plant's uncontrolled input $u$. We choose not to do so, since, from an identification point of view, the uncontrolled plant input $w$ plays an essentially different part than the signals $r_1, r_2$ and $w_c$.

In control design the nominal model $\hat{P}$ has to serve more or less as a substitute for the (inner-loop) plant $P$. However, $\hat{P}$ never provides an exact description of $P$. The fact that the nominal model is not completely equivalent to the (inner-loop) plant is indicated as the deficiency of the nominal model. In order to study this deficiency it must be expressed in terms of the input-output operators $P$ and $\hat{P}$. Such an expression is often called a "model-error", but we feel rather like calling it a mismatch\(^3\). We formalize this notion of a mismatch as follows.

\(^3\)The best possible approximate nominal model can exhibit a mismatch, but there is nothing erroneous about it.
2.2 Feedback Control

Definition 2.1.8 A mismatch, denoted $M$, is an input-output operator that constitutes the deficiency of a nominal model $\hat{P}$ with respect to the modelled (inner-loop) plant $P$.

i. The additive mismatch $M_A$ is defined as

$$M_A \doteq P - \hat{P}. \tag{2.4}$$

ii. The multiplicative mismatch $M_M$ is defined through

$$P = (I + M_M)\hat{P} \tag{2.5}$$

for a square invertible nominal model $\hat{P}$.

Every nominal model $\hat{P}$ and every plant $P$ of equal dimensions induce an unique additive mismatch $M_A$. This does not hold for the multiplicative mismatch $M_M$. A mismatch is the basis of system approximation: an approximation problem is quantized by taking a norm of some mismatch.

Remark 2.1.9 A mismatch is a LTIFD operator. The space of all LTIFD operators is a linear space, and thus the use of a norm is justified [161].

The concept of a mismatch is of crucial importance to our development of a link between identification and control design. Towards the end of this chapter we will discuss mismatches that are meaningful to feedback control, and that can be minimized in some sense by means of identification techniques.

2.2 Feedback Control

In the first part of this section we discuss the utility of a common robust control paradigm in view of our high performance control design problem. The second part of this section addresses the notions of performance and feedback properties.

2.2.1 Robust Control Design

We survey the use of “unstructured dynamical uncertainties” in robust control theory. Thereafter we confront this approach with our aim of designing a compensator for some plant with uncertain dynamics.

Common Model-based Control Design

A common paradigm in robust stabilization is that of unstructured dynamical perturbations often called “unstructured uncertainty”. These perturbations represent deviations from the nominal model. There exist various kinds of dynamical perturbations,
such as the additive (dynamical) perturbation $\Delta_A$, which affects a nominal model $\hat{P}$ through
\begin{equation}
\hat{P} + \Delta_A,
\end{equation}
and the multiplicative (dynamical) perturbation $\Delta_M$, which affects $\hat{P}$ through
\begin{equation}
(I + \Delta_M)\hat{P}.
\end{equation}
Other frequently used classes of dynamical perturbations are coprime factor perturbations [243] and feedback perturbations, which involve linear fractional transformations [65, 213, 17]. In general, dynamical perturbations comply with the following definition.

**Definition 2.2.1** A dynamical perturbation, denoted $\Delta$, is a stable LTI dynamical system that turns a nominal model into another system with the same input-output configuration.

We have defined the dynamical perturbations to be stable so that we can apply the small gain theorem [57] at a later stage. A robust stabilization problem usually concerns a ball of dynamical perturbations. Centered around the nominal model this ball induces a whole family of plants. We formalize this for the general dynamical perturbation $\Delta$, before we provide an example in terms of the multiplicative perturbation $\Delta_M$.

**Definition 2.2.2**

i. A ball $B(b_\Delta)$ of dynamical perturbations is a set of bounded dynamical perturbations $\Delta$ defined as
\[ B(b_\Delta) = \{\Delta | \sigma_{\text{max}}(\Delta(j\omega)) < b_\Delta(\omega) \ \forall \omega \geq 0\} \]
in which $b_\Delta(\omega)$ is a real-valued positive scalar function of the frequency $\omega$.

ii. A perturbative family $\mathcal{P}_\Delta(\hat{P}, b_\Delta)$ is a set of LTI systems induced by the nominal model $\hat{P}$ and the ball $B(b_\Delta)$.

This notation is used for all kinds of dynamical perturbations, except that the subscript $\Delta$ of the upper bound $b_\Delta$ is replaced by e.g. $M$ to indicate multiplicative perturbations. The multiplicative perturbative family $\mathcal{P}_\Delta(\hat{P}, b_M)$ consists of all systems $(I + \Delta_M)\hat{P}$ with $\Delta_M$ strictly bounded above by $b_M$, i.e. $\Delta_M \in B(b_M)$. Notice that we use a subscript of the upper bound $b$ to indicate the class of dynamical perturbations in question.

In robust control theory a perturbative family $\mathcal{P}_\Delta(\hat{P}, b_\Delta)$ is always chosen such, that it directly relates to a sufficient condition for robust stability. For instance in [64] it was shown, that each member of the multiplicative perturbative family $\mathcal{P}_\Delta(\hat{P}, b_M)$ is stabilized by some compensator $C_\hat{P}$, if and only if
\begin{equation}
\sigma_{\text{max}}\left(\hat{P}C_\hat{P}(I + \hat{P}C_\hat{P})^{-1}(j\omega)\right) \leq b_M^{-1}(\omega)
\end{equation}
for all frequencies $\omega \in \mathbb{R}$ and provided that $C_{\hat{P}}$ stabilizes $\hat{P}$. The largest ball of multiplicative perturbations allowed in view of robust stability is $B(\rho_M)$ with $\rho_M$ defined as

$$\rho_M(j\omega) = 1/\sigma_{max}(\hat{P}C_{\hat{P}}(I + \hat{P}C_{\hat{P}})^{-1}(j\omega)).$$

The radius $\rho_M$ is called the multiplicative robustness margin of the nominal feedback system $H(\hat{P}, C_{\hat{P}})$. Any ball $B(b_M)$ with $b_M(j\omega) > \rho_M(j\omega)$ contains a multiplicative perturbation that destabilizes $H(\hat{P}, C_{\hat{P}})$. Similar results hold for the class of additive dynamical perturbations.

The usual approach in robust control design is to presume that some perturbative family $\mathcal{P}_\Delta(\hat{P}, b_\Delta)$ is available, and that the control problem is to construct some compensator $C_{\hat{P}}$ achieving certain properties for the whole family $\mathcal{P}_\Delta$. The same properties are guaranteed for the plant $P$ of concern, provided that $P$ belongs to this family $\mathcal{P}_\Delta$. Along these lines the control design objective can be resolved as follows [17, 16, 159].

**Definition 2.2.3** A compensator $C_{\hat{P}}$ achieves

i. nominal stability, if the nominal feedback system $H(\hat{P}, C_{\hat{P}})$ is stable.

ii. nominal performance, if the nominal feedback system $H(\hat{P}, C_{\hat{P}})$ satisfies certain performance specifications.

iii. robust stability, if each member of the family $\mathcal{P}_\Delta(\hat{P}, b_\Delta)$ is stabilized by $C_{\hat{P}}$.

iv. robust performance, if the feedback system composed of $C_{\hat{P}}$ and any particular member of the family $\mathcal{P}_\Delta(\hat{P}, b_\Delta)$ satisfies certain performance specifications.

A specification of the notion of performance is postponed till Section 2.2.2. Right now we make the general observation that the robust control design problems corresponding to above definitions can be framed in terms of optimization problems. We distinguish three classes of optimization problems, viz. worst-case\footnote{Here worst-case is meant with respect to all elements of a family $\mathcal{P}_\Delta(\hat{P}, b_\Delta)$, not with respect to e.g. a set of disturbances or noises.} optimizations, constrained optimizations and unconstrained optimizations.

From a control theoretical point of view the optimization of the worst-case performance for some perturbative family $\mathcal{P}_\Delta(\hat{P}, b_\Delta)$ is the ultimate design strategy. The concept is as follows. Let a candidate compensator be judged from its worst-case performance taken over the whole family $\mathcal{P}_\Delta(\hat{P}, b_\Delta)$. Then choose the compensator that has the best worst-case performance of all candidate compensators. This conceptual control design problem is attractive because it pursues the best attainable robust performance. At the current state of affairs there exist solutions to the analysis problem, but the synthesis problem is still an open question. That is, the worst-case performance...
can be calculated for a given compensator and for particular classes of dynamical perturbations (e.g. $H_2$, $H_\infty$-performance [60, 61, 159, 222] and $l_1$-performance [41]), but there does not (yet) exist a method that directly optimizes the worst-case performance for $\mathcal{P}_\Delta(\hat{P}, b_\Delta)$ over an uncountable set of candidate controllers.

A constrained optimization is used to search for the best possible nominal performance under the constraint that the compensator achieves certain prespecified properties for the whole family $\mathcal{P}_\Delta$. This includes the well-studied combined problem of robust stability and nominal performance. In the latter case the prespecified property is stability: the nominal performance is optimized over the set of all compensators that stabilize each member of the family $\mathcal{P}_\Delta$. One example is the weighted $H_\infty$-minimization\(^5\) of the sensitivity with robust stability in the face of the ball $B(b_M)$ of multiplicative dynamical perturbations, i.e.

$$\min_{C \in \mathcal{C}(\hat{P})} ||W(I+\hat{P}C)^{-1}||_\infty \wedge ||b_M\hat{P}C(I+\hat{P}C)^{-1}||_\infty < 1 \quad (2.9)$$

where $\mathcal{C}(\hat{P})$ is the class of all compensators that stabilize $\hat{P}$. The inequality on the right is necessary and sufficient for the robust stability of $H(\hat{P}, C_P)$ with respect to the ball $B(b_M)$ [64]. This problem formulation can be extended to incorporate robust performance specifications. Typical examples thereof have been given by Doyle et al. [65, 59], who transformed robust performance requirements into robust stability requirements by means of fictitious perturbations. The analysis problem of the constrained optimization is the same as that of the worst-case optimization, and existing solutions include the structured singular values approach [65, 60, 159, 222]. As for the synthesis, several promising constrained optimization methods have been proposed [179, 31]. Despite that, today practically all compensators are still designed via unconstrained optimizations.

An unconstrained optimization is often used as a numerically tractable substitute for a constrained optimization. For example the optimization of (2.9) is frequently replaced by the mixed-sensitivity problem

$$\min_{C \in \mathcal{C}(\hat{P})} \left\{ ||W(I+\hat{P}C)^{-1}||_\infty + ||V\hat{P}C(I+\hat{P}C)^{-1}||_\infty \right\} \quad (2.10)$$

introduced in [128]. The real rational weights $W(s)$ and $V(s)$ are used to manipulate the trade-off between a small sensitivity (nominal performance) and a small complementary sensitivity (robustness). The crucial step of the design strategy is the selection of the weights. As the optimization is not subjected to any constraint except that $C$ must stabilize $\hat{P}$, i.e. $C \in \mathcal{C}(\hat{P})$, there is no prior guarantee about the robustness that will be achieved [74]. If the achieved robustness is too small or if the achieved nominal performance is too poor, then the weights $W(s)$ and $V(s)$ must be adjusted. Hence the design of a robust controller with an unconstrained optimization is typically an

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\(^5\)See [261, 60, 72] for a definition and implications of the $H_\infty$-norm.
iterative procedure. First, one chooses design weights to build a compensator. Then this compensator is judged from the accomplished performance and robustness. If the compensator is rejected, then the iteration is continued with an adjustment of the weights and a new optimization [60, 63, 179, 17, 31, 222].

**Using Dynamical Perturbations as a Design Variable**

The above discussion on model-based control design concerns the construction of a compensator for the perturbative family of systems $\mathcal{P}_\Delta(\hat{P}, b_\Delta)$. Now we discuss the utility of this control paradigm for our high performance control design problem. From Section 1.2 we first recall that the model-based compensator must achieve similar high performances for the plant $P$ and the nominal model $\hat{P}$. As the dynamics of $P$ are uncertain the plant is accounted for by means of some perturbative family of systems. The usual way to do this, is to select a class of dynamical perturbations $\Delta$, and to determine the smallest ball of perturbations that just contains the conformable mismatch $M$ between $P$ and $\hat{P}$. As an example we can take the class of additive dynamical per-

![Diagram](image)

**Fig. 2.2:** Bounds of a typical perturbative family $\mathcal{P}_\Delta (\cdots)$ and an estimated frequency response of a plant $(\cdots)$.  

$\Delta A$. The corresponding mismatch is $M_A = P - \hat{P}$, and we can use $\sigma_{\max}(M_A)$ or $\|M_A\|_\infty$ as the radius\(^6\) $b_A$. The bounds of a typical perturbative family $\mathcal{P}_\Delta$ have

---

\(^6\)Actually $B(b_\Delta)$ is an open set and thus $b_\Delta$ must be strictly larger than the size of $M$. To simplify the discussion we neglect this technicality at this stage of the developments.
been depicted in Fig. 2.2. The rationale of this figure is that physical systems exhibit a roll-off at high frequencies. Due to a small signal-to-noise ratio no accurate information about the frequency response of the plant is available in this frequency range. On the other hand the frequency response measurements at the lower frequencies are quite accurate.

In general terms, the plant \( P \) is usually accounted for by means of a perturbative family \( \mathcal{P}_\Delta(\hat{P}, b_\Delta) \) that contains \( P \). And the control design problem is frequently taken to be the stabilization of each member of the family \( \mathcal{P}_\Delta(\hat{P}, b_\Delta) \) [179, 171]. This paradigm implicitly presumes that the plant of interest is an arbitrary member of the family \( \mathcal{P}_\Delta(\hat{P}, b_\Delta) \). In contrast our aim is to achieve similar (high) performances for \( P \) and \( \hat{P} \), and we are not truly interested in each member of \( \mathcal{P}_\Delta(\hat{P}, b_\Delta) \). As a consequence, in view of our high performance control design problem, the above common robust control paradigm can be extremely conservative in the following sense. It can easily happen that the performance achieved for the plant \( P \) is much better than the worst-case performance taken over the whole family \( \mathcal{P}_\Delta(\hat{P}, b_\Delta) \). This phenomenon has been verified experimentally by Balas ([16] and [17, p.79]), and we will have similar experiences in Chapters 8 and 9.

Fig. 2.3: Illustration of the conservatism of the robust control paradigm.

We add weight to the above statement by means of Fig. 2.3. In here two perturbative families \( \mathcal{P}_\Delta(\hat{P}, b_1) \) and \( \mathcal{P}_\Delta(\hat{P}, b_2) \) are represented by the balls \( \mathcal{B}(b_1) \) (—) and
$B(b_2)$ (---) both centered around the nominal model $\hat{P}$. First we \textit{“apply”} the common robust control paradigm to these two families $\mathcal{P}_\Delta(\hat{P}, b_1)$ and $\mathcal{P}_\Delta(\hat{P}, b_2)$. Thereafter we examine the consequences for the plant $P$.

Let a compensator $C_1$ be given such that some worst-case performance $\Pi_1$ is achieved for the family $\mathcal{P}_\Delta(\hat{P}, b_1)$ (---). Then there exists a perturbation just outside $B(b_1)$, which results in a performance that is inferior\footnote{This is an analogy of the necessary and sufficient conditions like (2.9) for robust stability in the face of balls of dynamical perturbations.} to $\Pi_1$. This does not hold for dynamical perturbations in all directions. Thus $\Pi_1$ is the worst-case performance for some neighborhood of $\hat{P}$, that is larger than the ball $B(b_1)$, but that has not the shape of a ball. In Fig. 2.3 we have represented such a neighborhood as the ellipsoid (---): $\Pi_1$ is the worst-case performance for the whole ellipsoid (---) and not just for the ball $B(b_1)$ (---). Notice that the achieved robustness is larger than the robustness required for $\mathcal{P}_\Delta(\hat{P}, b_1)$.

Let also a compensator $C_2$ be given such that some worst-case performance $\Pi_2$ is achieved for the family $\mathcal{P}_\Delta(\hat{P}, b_2)$ (---). As the ball $B(b_2)$ is smaller than $B(b_1)$, the corresponding control problem is easier. The compensator $C_2$ achieves a better worst-case performance than $C_1$, but only for a smaller neighborhood around the nominal model $\hat{P}$. Again the achieved robustness is larger than the required robustness, thus $\Pi_2$ is the worst-case performance for the whole ellipsoid (---).

Now we turn to the consequences for the plant $P$ and we visualize the conservatism of the robust control paradigm as follows. The common approach is to use a large family $\mathcal{P}_\Delta(\hat{P}, b_1)$, which contains the plant $P$ and which limits the achievable worst-case performance. Instead we can take a smaller family $\mathcal{P}_\Delta(\hat{P}, b_2)$, which does not contain the plant $P$, but which gives rise to a better worst-case performance $\Pi_2$. This $\Pi_2$ is the worst-case performance not only for $\mathcal{P}_\Delta(\hat{P}, b_2)$ but for a larger neighborhood (---), which contains $P$. Thus $C_2$ achieves $\Pi_2$ or a better performance for the plant $P$, despite that the family $\mathcal{P}_\Delta(\hat{P}, b_2)$ used for the design of $C_2$ does not contain $P$. At the same time it is possible that $C_2$ destabilizes some members of $\mathcal{P}_\Delta(\hat{P}, b_1)$. From this reasoning we claim that the radius $b_\Delta$ should be regarded rather as a design variable than as a prior constraint.

We want of course a guarantee that the improved worst-case performance $\Pi_2$ is achieved for the plant $P$. Providing such a guarantee is the actual task of a perturbative family. However neither $\mathcal{P}_\Delta(\hat{P}, b_1)$ nor $\mathcal{P}_\Delta(\hat{P}, b_2)$ can help us in the case of $C_2$ and $\Pi_2$. In essence we can try any kind of neighborhood around $\hat{P}$ for this purpose. We should obviously choose a neighborhood that incurs only little conservatism with respect to the plant $P$, the nominal model $\hat{P}$ and the compensator $C_2$. On the other hand such a neighborhood is of practical use only if it can be defined by some simple description. Therefore in Chapter 6 we will make use of the designed compensator to shape the ellipsoids like those of Fig. 2.3 into balls of perturbations around the coprime factors of the nominal model $\hat{P}$. The plant is consequently regarded as a particular
compensator-based dynamical perturbation of the nominal model.

Having suggested that $b_\Delta$ should be used as a design variable, we raise the question of how to select an appropriate $b_\Delta$. Suppose the controller is designed by means of a constrained optimization. Such a procedure requires the specification of a perturbative family, for which we choose a family containing $P$, e.g. $\mathcal{P}_\Delta(\hat{P}, b_1)$, to begin with. Then we take a slightly smaller ball of dynamical perturbations, we design the corresponding compensator, and we use compensator-based dynamical perturbations to ascertain a good performance for the plant $P$. By reiterating these procedures we can derive a good compensator for $P$ from a perturbative family that does not contain $P$.

On the other hand suppose the controller is designed by means of an unconstrained optimization. Then we need an iteration to determine the design weights anyway (recall the discussion at the end of the first part of this section). In adjusting these weights a common class of dynamical perturbations can straightforwardly be replaced with a class of compensator-based dynamical perturbations. We elaborate this issue further in Chapter 6.

Remark 2.2.4 Considering the robust control of the family $\mathcal{P}_\Delta(\hat{P}, b_1)$ as the true control objective, is equivalent to assuming that the modelled physical system can vary over the whole family $\mathcal{P}_\Delta(\hat{P}, b_1)$. In contrast we assume that the modelled physical system can be represented by one arbitrarily complex LTIFD plant $P$. The truth nearly always lies midway: some subset of $\mathcal{P}_\Delta(\hat{P}, b_1)$ around $P$, like the fictitious one (-----) of Fig. 2.3, will be needed to account for the physical system of concern. We claim that much of the discussion concerning the plant $P$ carries over directly to subsets like the one of Fig. 2.3. Such a subset includes certain non-linear and time-varying phenomena (see [260, 64, 38, 62, 31] and [143, p.105]) as well as dynamical perturbations due to component tolerances of mass-produced end-products. Lastly, problems of simultaneous stabilization [191, 158, 136] and multi-model control [149, 223] that are based on more than one nominal model are beyond the scope of the thesis.

2.2.2 Simultaneous High Performance Control Design

In the light of the problem posed in Chapter 1 and its accompanying explanation we use this section to discuss the following two topics. First we make precise what we mean by a high performance. After that we relate the performance achieved for a plant to the performance achieved for its nominal model.

**Performance and Feedback Properties**

In general terms, a feedback system has to produce an acceptable output from all its inputs, and it should remain doing so in the presence of dynamical perturbations. We are interested especially in those aspects of a feedback system that can be altered only
by changing the compensator. These properties are called feedback properties, which include disturbance rejection, noise attenuation, sensitivity, stability and robustness margins [193, 143]. It has been pointed out by several authors that a feedback system's response to commands is not a feedback property. For it can be changed by prefiltering the command signal [193], and thus it can be assigned by an appropriate prefilter $C_{zw}$ resulting in the so-called two degrees-of-freedom compensator (cf. Fig. 2.1, page 13) [178, 239].

The main point that we make here, is that the feedback properties of a general feedback system $H(P_{TT}, C_{TT})$ — and of all its special cases — depend solely on the inner-loop parts $C$ and $P$, whose interconnection $H(P, C)$ we sometimes refer to as the inner-loop feedback system. We provide some explanation based on Fig. 2.1. Like the command following of $w_c$, we can change the response $z_c$ by appropriate filters $C_{zw}$ and $C_{zu}$. These filters leave all other input-output relations of the feedback system invariant. On the other hand a small change of the inner-loop compensator $C$ affects all input-output relations of the feedback system, and thus $C$ is of importance to the feedback properties.

As for the plant, its input $w$ often represents a particular class of disturbances, noises and reference signals. The associated objective is to minimize some norm of the transfer function from $w$ to $z$. This transfer function, denoted $H_{zw}(P_{TT}, C_{TT})$, equals

$$H_{zw}(P_{TT}, C_{TT}) = P_{zw} - P_{zu}(I + CP)^{-1}CP_{yw}.$$ 

$C$ is clearly the only part of the compensator $C_{TT}$ that can be used to alter this transfer function $H_{zw}(P_{TT}, C_{TT})$. The stability of this transfer function depends in the first place on the inner-loop part $P$. Later we will see that the terms $P_{zw}, P_{zu}$ and $P_{yw}$ are of no importance to the stability of the feedback system $H(P_{TT}, C_{TT})$, provided that $P_{TT}$ can be stabilized at all. Accordingly, we attribute the feedback properties to the inner-loop feedback system $H(P, C)$. We regard the other plant terms $P_{zw}, P_{zu}$ and $P_{yw}$ merely as weighting functions that dictate the ideal shape of $(I + CP)^{-1}C$.

We will be concerned with all transfer functions that are associated with the inner-loop feedback system $H(P, C)$. These are the transfer functions that map the exogenous inputs $r_1, r_2$ into $u, y, u_c$ and $y_c$ (see Fig. 2.1, page 13). Due to the algebraic relations between $r_1, u$ and $y_c$ and between $r_2, y$ and $u_c$, it suffices to investigate the four-block transfer matrix $T(P, C)$ defined as

$$T(P, C) = \begin{bmatrix} P \\ I \end{bmatrix} (I + CP)^{-1} \begin{bmatrix} C & I \end{bmatrix},$$

which maps col$(r_2, r_1)$ into col$(y, u)$. We call this $T(P, C)$ the feedback (transfer function) matrix of $P$ and $C$. For convenience we decompose this feedback matrix into four blocks:

$$\begin{bmatrix} T_{11}(P, C) & T_{12}(P, C) \\ T_{21}(P, C) & T_{22}(P, C) \end{bmatrix} = \begin{bmatrix} P(I + CP)^{-1}C & P(I + CP)^{-1} \\ (I + CP)^{-1}C & (I + CP)^{-1} \end{bmatrix}$$

(2.12)
In here the partitioning on the right defines the dimensions of the blocks on the left. We call $T_{22}(P, C)$ and $T_{11}(P, C)$ respectively the sensitivity and the complementary sensitivity, although they are not necessarily each others complement in case of MIMO systems.

An ubiquitous control objective is to demand a small sensitivity $(I+CP)^{-1}$ over a prescribed frequency range $[0, \omega_p]$ called the operating band [261, 172]. This guarantees a good attenuation of (small) dynamical perturbations and disturbances that are active in the operating band [64]. Likewise a small complementary sensitivity $P(I+CP)^{-1}C$ is desired for the attenuation of measurement noise [64] and dynamical perturbations of the compensator [193]. Besides, robustness with respect to dynamical perturbations of the plant $P$ and the compensator $C$ are needed, because in practice we always have to deal with imperfections (review also the final remark at the end of Section 2.2.1).

Although there exist various ways to specify a desired performance, in essence the goal of feedback control is to give the feedback matrix $T(P, C)$ some desired shape. The latter desired feedback matrix is a stable LTIFD system, and thus it is defined by its frequency response [259]. However we cannot assign just any frequency response to $T(P, C)$. In the first place the four blocks of $T(P, C)$ are mutually dependent: the complementary sensitivity $T_{11}(P, C)$ and the sensitivity $T_{22}(P, C)$ add up to 1 for SISO systems. Further, a minimization of the sensitivity over the operation band may imply a very large sensitivity over some other frequency interval [261, 172]. This pertains to the Bode phase-gain relations [32, 75, 143], which say that the achievable performance is limited in the presence of time-delays, non-minimum phase zeros or unstable poles. Performance limitations due to “non-minimum phase” characteristics exist also for non-linear time-varying systems [208]. Besides, the plant $P$ is not precisely known and the same holds for its number of unstable zeros. Thus we cannot tell a priori what performance is attainable for the plant in question.

A compensator $C$ must be constructed by optimizing the feedback matrix in some sense. For this optimization we need some measure by which two compensators can be compared. We represent such a measure of performance by a (semi-)norm of the feedback matrix, denoted $\| T(P, C) \|$. Although it is practically impossible to transform all kinds of design objectives into one norm of the feedback matrix $T(P, C)$ [91, 30, 179, 31], its wide applicability may be clear from the example below\(^8\). Lastly, at the end of this section we will show that the concept of a norm may be relaxed to that of a distance function or metric.

Example 2.2.5 Let a desired performance be specified in terms of a desired sensitivity $T_{22_{\text{des}}}$. This sensitivity is pursued by requiring that $\| T_{22_{\text{des}}}^{-1} T_{22}(P, C) \|_{\infty}$ is small [261].

---

\(^8\)The other way round every stabilizing compensator can be obtained from a particular $l_1$- or $H_\infty$-optimization [46].
Thus the quality of the feedback system is

\[
\left\| T_{22,\text{des}}^{-1} \begin{bmatrix} 0 & I \end{bmatrix} T(P, C) \begin{bmatrix} 0 \\ I \end{bmatrix} \right\|_{\infty}
\]

provided that \( T(P, C) \) is stable.

Alternatively, let the performance be measured in terms of the weighted sum of the power spectral densities of the "tracking error" \( y-r_2 \) and the plant input \( u \), i.e.

\[
\int_{-\infty}^{\infty} (\Phi_{y-r_2}(j\omega) + \lambda^2 \Phi_u(j\omega)) \, d\omega.
\]

The signals \( y-r_2 \) and \( u \) can easily be expressed in terms of the inputs to the feedback system filtered by the transfer function matrix \( T(P, C) \). In turn, the performance is measured by \( \|T(P, C)\Phi\|_2 \), in which the weighting function \( \Phi \) depends on the spectra of the inputs to the feedback system.

\[\square\]

**Mergence of Nominal and Actual Feedback Properties**

We have transformed the requirement of good feedback properties of \( H(P, C) \) (or \( H(P_T, C_T) \), or \( H(P_T, C) \)) into a small norm of the corresponding (inner-loop) feedback matrix \( T(P, C) \). The smaller \( \|T(P, C)\| \), the better the feedback properties of \( H(P, C) \).

In model-based control design the aim is to accomplish certain feedback properties for the controlled plant by designing a compensator \( C_{\hat{P}} \) from the nominal model \( \hat{P} \). As explained in Section 1.2 this compensator \( C_{\hat{P}} \) must also work well with the nominal model \( \hat{P} \), in order that we can be confident about \( C_{\hat{P}} \). The objective of simultaneous high performances for \( P \) and \( \hat{P} \) boils down to the requirement of small norms \( \|T(P, C_{\hat{P}})\| \) and \( \|T(\hat{P}, C_{\hat{P}})\| \). We can relate these performances together as in the following proposition.

**Proposition 2.2.6 : Norm-based Identification & Control Evaluation**

Let the (semi-)norms \( \|T(P, C_{\hat{P}})\| \) and \( \|T(\hat{P}, C_{\hat{P}})\| \) represent the performances of the plant \( P \) and the nominal model \( \hat{P} \) each controlled by the compensator \( C_{\hat{P}} \). Then the inequalities

\[
\left| \|T(\hat{P}, C_{\hat{P}})\| - \|T(P, C_{\hat{P}}) - T(\hat{P}, C_{\hat{P}})\| \right| \leq \|T(P, C_{\hat{P}})\| \quad (2.13)
\]

constitute lower and upper bounds on the performance of the controlled plant.

---

\(^9\text{NICE: Norms are used to evaluate the couple } \hat{P}, C_{\hat{P}} \text{ as a candidate solution to the high performance control design problem of Section 1.2. This is an evaluation of the combined procedure of identifying } \hat{P} \text{ and designing } C_{\hat{P}}.\)
Proof: The upper bound follows from applying the triangle inequality for (semi-)norms \([190]\) to
\[
T(P, C_\hat{P}) = T(\hat{P}, C_\hat{P})(T(P, C_\hat{P}) - T(\hat{P}, C_\hat{P})).
\]
For the lower bound we apply the triangle inequality twice:
\[
\|T(P, C_\hat{P}) - T(\hat{P}, C_\hat{P})\| \leq \|T(P, C_\hat{P})\| + \|- T(\hat{P}, C_\hat{P})\|
\]
\[
\|T(\hat{P}, C_\hat{P})\| \leq \|T(P, C_\hat{P})\| + \|T(\hat{P}, C_\hat{P}) - T(P, C_\hat{P})\|,
\]
so that
\[
\|T(P, C_\hat{P}) - T(\hat{P}, C_\hat{P})\| \leq \|T(P, C_\hat{P})\|
\]
\[
\|T(\hat{P}, C_\hat{P})\| - \|T(P, C_\hat{P}) - T(\hat{P}, C_\hat{P})\| \leq \|T(P, C_\hat{P})\|
\]
because \([- T(\hat{P}, C_\hat{P})]\) = \|T(\hat{P}, C_\hat{P})\|. \qed

We call the upper bound of (2.13) the \textit{worst-case performance} or alternatively the \textit{robust performance}. — The robust performance is taken with respect to some neighborhood of \(\hat{P}\) that contains the plant \(P\). — The term \(\|T(P, C_\hat{P}) - T(\hat{P}, C_\hat{P})\|\) is referred to as the (worst-case) \textit{performance degradation}, that is due to the fact that the compensator \(C_\hat{P}\) is designed for the nominal model \(\hat{P}\) rather than for the plant \(P\). Later we will see that the performance degradation induces a compensator-based neighborhood around the nominal model as illustrated by the ellipsoids in Fig. 2.3.

A small nominal performance norm \(\|T(\hat{P}, C_\hat{P})\|\) is pursued in each constrained and unconstrained controller optimization. The resulting controller must also work acceptably well with the plant. In fact, the nominal feedback system \(H(\hat{P}, C_\hat{P})\) should reliably predict the feedback properties of the actual feedback system \(H(P, C_\hat{P})\). Hence the controller \(C_\hat{P}\) must be designed such that the performance degradation is relatively small, i.e.
\[
\|T(P, C_\hat{P}) - T(\hat{P}, C_\hat{P})\| \ll \|T(\hat{P}, C_\hat{P})\|. \tag{2.14}
\]
If this strong inequality holds, then the bounds of (2.13) are tight and, a fortiori, the feedback matrix \(T(\hat{P}, C_\hat{P})\) approximates \(T(P, C_\hat{P})\) in a sense that is related to an acceptable performance.

Remark 2.2.7 It is not sufficient that \(\|T(P, C_\hat{P})\| \approx \|T(\hat{P}, C_\hat{P})\|\), because the norms constitute aggregated qualities of the feedback matrices. The approximation is good only if (2.14) holds. \qed

In order to solve our problem of simultaneous high performance control design we need a control design method that pursues a small nominal performance norm \(\|T(\hat{P}, C_\hat{P})\|\) and an even smaller performance degradation \(\|T(P, C_\hat{P}) - T(\hat{P}, C_\hat{P})\|\). This is not a trivial control problem, because the plant \(P\) is not precisely known. In Chapter 6 we develop a procedure for this control problem that consists of two steps. In the
first step we use frequency response estimates to select the compensator $C_\hat{P}$. This $C_\hat{P}$ achieves an acceptable performance for the nominal model $\hat{P}$ and probably the same performance for the plant $P$. The latter is ensured in the second step before the compensator $C_\hat{P}$ is applied to the plant $P$. For this we use compensator-based dynamical perturbations as discussed in Section 2.2.1.

Remark 2.2.8 The axioms of a norm that we utilized are positivity ($\|x\| \geq 0$), the triangle inequality and symmetry ($\|x-y\| = \|y-x\|$). These axioms apply also to distance functions or metrics [161], and thus we can relax the concept of a norm to that of a distance function. Upper and lower bounds similar to those of (2.13) follow from the triangle inequality [161, p.46]. With the norm replaced by a distance function it is possible to measure the performance as the difference between the achieved feedback system and some desired feedback system. E.g. let $T_{\text{spec}}$ be a desired feedback matrix and let the performance of $H(P, C)$ be quantized as $\|T(P, C) - T_{\text{spec}}\|_\infty$. The corresponding upper bound is

$$\|T(P, C) - T_{\text{spec}}\|_\infty \leq \|T(\hat{P}, C) - T_{\text{spec}}\|_\infty + \|T(P, C) - T(\hat{P}, C)\|_\infty.$$ 

And with $||[I \ 0](T(P, C) - T_{\text{spec}}) \cdot \text{col}(I, 0)||_\infty$ we can judge just the complementary sensitivities as in [129].

2.3 Identification

Control design is our motivation to identify a plant. As explained in the previous section high performance control design demands a good nominal model $\hat{P}$ of the inner-loop plant $P$. Therefore we address the identification of the transfer function of $P$. From a short survey of this estimation problem we decide to study approximate identification and closed-loop identification. Thereafter we combine these two issues to discuss the approximate closed-loop identification.

2.3.1 Transfer Function Estimation

The plant configuration that is commonly used in system identification, is the standard plant $P_T$ of Fig. 2.4. Measurements of the output $y$ and the input $u$ are available for identification. All effects that are not caused by the measured input $u$, are modelled by an additive term $v$ at the output of the plant [138]. This additive output disturbance $v$ is represented by a standard signal $w$ (or a class of signals) and a “noise filter” $P_yw$.

We adopt the deterministic or non-statistical approach to identification [138, 256, 232]. Our main interest lies with the capacity of identification techniques to solve system approximation problems. Therefore we concentrate on undermodelling, i.e. the fact that the nominal model $\hat{P}$ is too simple to describes the plant $P$. Accordingly,
we pay a lot of attention to “asymptotic bias contributions”, and we neglect “variance contributions” and the effects due to finite data-sets.

Our aim is to identify the inner-loop plant $P$ from its input $u$ and output $y$. Identification essentially attempts to retrieve all information that is concealed in the data. Thus all what system identification can do, is to develop a model that reproduces the observed data as good as possible. We do not precisely know the plant, and thus we cannot tell whether a particular data-set represents all aspects of the inner-loop part $P$. In fact, according to Willems [254, 255], the family of all signals that a plant can exhibit, is needed to precisely define this plant.

In system theory it is well known that the intended use of an approximate model must be taken into account in the identification procedure [139, 138]. Our intended use is control design. All user’s choices like the model-error, the parameterization, the data-acquisition, the identification criterion, etc., are subservient to this purpose: each choice is marked good if the ultimate controller is good. In fact we wish that the identification selects the best nominal model for control design out of any particular model-set and from any particular data-set. Thus the most important choice that we have to make is that of the selection rule\textsuperscript{10}.

As explained in Section 2.2.2 the inner-loop plant $P$ is of major importance to feedback control design. Hence we need an accurate nominal model $\hat{P}$ of the transfer function of $P$. In the first place we care less about modelling the noise contribution $P_{yw} w$, because it does not affect the robustness margins of a control system that involves $P_T$. Therefore we want the estimate $\hat{P}$ to be asymptotically independent of the noise contribution $P_{yw} w$.

It depends on the situation at hand whether precautionary measures are needed to ensure that the asymptotic estimate of $P$ is independent of $P_{yw} w$. In our case we search for an approximate nominal model $\hat{P}$ of $P$. Furthermore, we will have to cope with closed-loop plant data, since we intend to use system identification in an iterative scheme together with the design and implementation of compensators. Therefore we concisely survey the fields of approximate identification and of closed-loop identi-

\textsuperscript{10}In contrast, Ljung [138] suggests that the choice of the model set is the most important one. Of course, both the selection rule and the model set must be appropriate for the situation at hand in order to derive an useful model.
2.3 Identification

fication. After that we address the combination of these two identification problems. Besides, in order to keep these particular discussion as simple as possible we will treat only the case of SISO systems.

2.3.2 Approximate Identification

We use the established asymptotic results of the widely applied prediction error method [137, 81, 247, 138] to expose a typical problem encountered in approximate identification. In order not to obscure the key concepts we discuss this material in a loose fashion. The discussion is confined to the identification of a nominal model using a least squares prediction error criterion and either an output-error model structure or an ARX model structure. The resulting nominal models are denoted respectively $\hat{P}_{OE}$ and $\hat{P}_{ARX}$.

Wahlberg and Ljung [247] have derived general expressions for the asymptotic bias distribution over frequencies resulting from the prediction error identification method. In case of a one-step ahead prediction from open-loop data and without prefiltering, the nominal models $\hat{P}_{OE}$ and $\hat{P}_{ARX}$ satisfy

$$\hat{P}_{OE} = \arg \min_{B/A \in \mathcal{P}(\theta)} \int_{\omega} |P - B/A|^2 \Phi_u \, d\omega$$

$$\hat{P}_{ARX} = \arg \min_{B/A \in \mathcal{P}(\theta)} \int_{\omega} \frac{|P - B/A|^2 \Phi_u + \Phi_v}{|1/A|^2} \, d\omega$$

in which $A, B$ are the usual parameterized polynomials [138] and $\mathcal{P}(\theta)$ is the parameterized set of candidate nominal models.

In view of our goal of identifying the inner-loop plant $P$, the ARX-identification suffers from the following two problems: the criterion is shaped by an a priori unknown weight $|A|^2$, and the asymptotic result $\hat{P}_{ARX}$ depends on the noise contribution $v = P_{yw} w$. Consequently, even if $P$ belongs to the model-set, then $\hat{P}_{ARX}$ generally differs from $P$. On the other hand $\hat{P}_{OE}$ is independent from the noise contribution, and the estimation is consistent if the model-set is rich enough [138, Theorem 8.4]. Thus we can separate the deterministic contribution $Pu$ from the noise contribution $P_{yw} w$ by means of an output-error model structure. In more general terms, the deterministic and noise contributions must be accounted for independently in the identification criterion. Only then we obtain an asymptotic estimate $\hat{P}$ of the inner-loop plant, that is independent of the noise.

For referencing at a later stage we outline the identification method of Steiglitz and McBride [221], which is seemingly an alternative for the non-linear optimization of the output-error identification. These authors proposed an iterative scheme to neutralize the weight $1/A$ of the "noise model" in the ARX identification. The $k$-th estimation
of this iteration, denoted $B_k/A_k$ minimizes

$$\int_{\omega} \frac{|1/A_{k-1}|^2 |P - B_k/A_k|^2 \Phi_u + \Phi_v}{|1/A_k|^2} d\omega.$$ 

This iteration does not always converge, — we return to this matter in Chapter 7, — but if it converges, i.e. $A_k \to A_{k-1}$, then the weight $1/A_k$ is neutralized in the criterion. Meanwhile the stationary point(s) of this iteration depend on $\Phi_v$. And numerical experience has shown that if the noise contribution $\Phi_v$ is large, then the Steiglitz-McBride method tends to produce a nominal model that differs significantly from $\hat{P}_{OB}$.

### 2.3.3 Closed-loop Identification

The problem of closed-loop identification is usually studied from the standard feedback system $H(P_r, C)$ depicted in Fig. 2.5. As common in literature the exogenous inputs $r_1$ and $r_2$ and the noise $w$ are assumed to be mutually uncorrelated. In comparison with open-loop identification the typical problems in closed-loop identification arise from the correlation between the plant input $u$ and the noise $w$. Because of this correlation some precaution is needed in order that the plant can be identified correctly from its input $u$ and output $y$.

![Fig. 2.5: Standard feedback system $H(P_r, C)$](image)

Closed-loop identification is a well-studied problem for the case that the inner-loop plant $P$ and the noise contribution $P_{yw}w$ belong to the model-set. In literature several identification schemes have been proposed ranging from direct applications of open-loop methods to the identification of the whole feedback system with a subsequent retrieval of the plant dynamics. For each of these approaches certain conditions on the compensator $C$ and on the feedback system inputs have to be met, in order that the input-output data of the plant is informative [138]. There are basically two means to accomplish such informative data: feedback by a compensator that is more complex than the plant, and excitation of the feedback system via $r_1$ and $r_2$ [216, 164, 92]. Here we consider only the latter possibility.

The prediction error method will consistently identify $P$ and $P_{yw}w$, provided that the feedback system is sufficiently excited by $r_1$ and $r_2$ and the feedthrough of the loop transfer function $PC$ is zero [138] or has certain structure properties [233]. The joint
input-output identification yields a consistent estimate of the whole feedback system, provided that the loop transfer function $PC$ has at least one delay [80, 1]. The plant dynamics can be derived from this estimate by matrix manipulations.

The above methods use measurements of only $u$ and $y$ to consistently identify the relation $y = Pu + P_{yw}w$. Other techniques additionally employ an exogenous feedback system input $r_1$ and/or $r_2$. Such identification methods are commonly based on the correlation between an exogenous input and the signals $u$ and $y$. As a consequence a consistent estimate of the inner-loop plant $P$ can be obtained without modelling the noise contribution $P_{yw}w$. These techniques include spectral estimation [250], the instrumental variable method [225, 217, 218] and all indirect methods that construct a nominal model $\hat{P}$ of $P$ from estimates of the transfer functions contained in $T(P, C)$ (cf. (2.12)).

In conclusion we state once more that all above identification schemes for closed-loop identification have been developed and analyzed in the context of consistent identification. Only little is known about approximate closed-loop identification.

### 2.3.4 Approximate Closed-loop Identification

Our goal is to identify the inner-loop plant $P$ in such a way that its asymptotic estimate $\hat{P}$ is independent of the noise contribution $P_{yw}w$. Therefore, as explained in Section 2.3.2, the deterministic and noise contributions must be accounted for independently in the identification criterion.

In open-loop identification the contributions $Pu$ and $P_{yw}w$ that make up $y$ can be accounted for independently by applying e.g. the prediction error method in combination with an output-error model structure, cf. (2.15). This is not sufficient for a closed-loop identification, because $u$ and $w$ are correlated due to the feedback. From the general expression for the asymptotic distribution of bias over frequencies [247, 81] it can be shown that the prediction error / output-error method minimizes

$$\int_{w} \begin{bmatrix} P - \hat{P}_{OE} & P_{yw} - 1 \end{bmatrix} \begin{bmatrix} \Phi_u & \Phi_{uw} \\ \Phi_{wu} & \Phi_w \end{bmatrix} \begin{bmatrix} P - \hat{P}_{OE} \\ P_{yw} - 1 \end{bmatrix} \mathrm{d}\omega$$

for SISO systems. As the input $u$ of Fig. 2.5 satisfies

$$u = \frac{1}{1 + PC}(r_1 + Cr_2) - \frac{CP_{yw}}{1 + PC}w,$$

the cross-spectrum $\Phi_{uw}$ is nonzero. Thus in closed-loop identification the asymptotic estimate $\hat{P}_{OE}$ depends on the noise $w$ and on the “error” $P_{yw} - 1$. The consistency of $\hat{P}_{OE}$ is no longer guaranteed, because the “noise-model” equals 1, which is usually too simple [138, p.368].

From the above exposition we infer that we cannot achieve our goal with a direct application of the prediction error method to $u$ and $y$. On the other hand the
correlation methods, which use an exogenous input, produce consistent estimates of the inner-loop plant, while the noise is not modelled at all. For instance the ratio of the cross-spectra between \( r_1 \) and \( u, y \), i.e. \( \Phi_{yr}, / \Phi_{ur} \), is an estimate of the frequency response of the inner-loop plant \( P \). If the true spectra are used, then this estimate is independent from the noise \( w \) [250].

**Remark 2.3.1** A direct application of the instrumental variable method to the closed-loop data \( u \) and \( y \) yields a consistent estimate of \( P \), provided that the instrument is built from e.g. \( r_1 \) [225, 218]. The resulting bias distribution is shaped by an a priori unknown weight just like that of \( \hat{P}_{ARX} \) of (2.16). This weight could be neutralized by the “mode 2” iteration of Steiglitz and McBride [221]. Unfortunately very little is known about the convergence properties of this method [68] and the resulting (approximate) estimate.

Henceforth we suggest three different scenarios for the identification of \( P \), that employ \( r_1 \) as an “instrumental variable”. These scenarios appeal to intuition and they give us the opportunity to outline the identification scheme that we will develop in this thesis.

Scenario 1 is similar to taking the ratio of two cross-spectra. We minimize e.g. the prediction errors \( y - \hat{H}_{yr} r_1 \) and \( u - \hat{H}_{ur} r_1 \) using output-error model structures. Since \( r_1, r_2 \) and \( w \) are mutually uncorrelated, this identification results in consistent estimates of the feedback system transfer functions \( H_{yr} \) and \( H_{ur} \), which map \( r_1 \) into \( y \) respectively \( u \). Since \( y \) satisfies

\[
y = \frac{P}{1 + PC}(r_1 + Cr_2) + \frac{P_{yw}}{1 + PC}w, \tag{2.18}
\]

the ratio of the consistent estimates, i.e. \( \hat{H}_{yr}/\hat{H}_{ur} \), is a consistent estimate of the inner-loop plant \( P \) (cf. (2.17)).

**Remark 2.3.2** It is tempting to rewrite e.g. (2.18) (with \( r_2 \equiv 0 \) for notational convenience) to

\[(1 + PC)y = Pr_1 + v\]

and to identify \( \hat{P} \) by a minimization of the error \((1 + \hat{P}C)y - \hat{P}r_1 \). Such an identification suffers from the same problems as the ordinary ARX-identification of (2.16). That is, the bias \( H_{yr} - \hat{H}_{yr} \) is shaped by an a priori unknown weight \(|1 + \hat{P}C|^2 \), and the asymptotic estimate \( \hat{H}_{yr} \) depends on the noise contribution \( v = \hat{P}_{yw} w \). An iterative procedure like the Steiglitz-McBride method (Section 2.3.2) will neutralize the weight. However, the stationary point(s) of the iteration still depends on the noise spectrum \( \Phi_v \), even when an output-error model structure is used for \( \hat{P} \).

A drawback of scenario 1 is that the estimates \( \hat{H}_{yr} \) and \( \hat{H}_{ur} \) are of high order, and in approximate identification the redundant dynamics \((1 + \hat{P}C)^{-1} \) practically do not
cancel out. Scenario 2 obviates this problem by identifying $P$ in two steps: first the transfer function mapping $r_1$ into $u$ is identified, and then the "noise-less" part of $u$ is reconstructed and used to identify $P$. We explain this scenario for the case that $r_2 \equiv 0$. From $u$ and $r_1$ we determine the estimate $\hat{H}_{ur}$ of the transfer function $(1+PC)^{-1}$. We introduce
\[ u_r = \frac{1}{1+PC} r_1; \quad u = u_r + \frac{P_{yw}}{1+PC} w. \]
The map $H_{ur,r}$ from $r_1$ to $u_r$ equals $H_{ur}$ mapping $r_1$ into $u$. Hence $u_r$ can be derived approximately as $\hat{u}_r = \hat{H}_{ur} r_1$. Now (2.18) can be rewritten to
\[ y = P u_r + \frac{P_{yw}}{1+PC} w. \]
With $u_r$ replaced by $\hat{u}_r$, the identification of $P$ has become an open-loop identification problem, because $u_r$ and $\hat{u}_r$ are not correlated with $w$. Surely, the quality of the reconstructed $\hat{u}_r$ will affect the estimate $\hat{P}$. This two-step identification procedure is formalized in [234].

The third scenario consist of simultaneously identifying the transfer functions $H_{yr}$ and $H_{ur}$ with a customized model-structure. In precise terms, $\hat{P}$ is identified such that the (vector) prediction error
\[ \begin{pmatrix} y - \frac{\hat{P}}{1+PC} r_1 \\ u - \frac{1}{1+PC} r_1 \end{pmatrix}, \tag{2.19} \]
is the smallest (in a least-squares sense) over the set of candidate nominal models. This is a special output error identification problem, and thus the asymptotic estimate $\hat{P}$ is consistent and independent of the noise. A noteworthy difference with the two other scenarios is that only scenario 3 employs knowledge of the compensator $C$ in parameterizing the transfer functions under investigation. The corresponding optimization is not trivial, because $\hat{P}$ appears in the prediction error in a multiple and non-linear fashion. And as Remark 2.3.2 applies here as well, we truly have to solve the difficult non-linear optimization involved with (2.19).

The stage having been set, now we outline the scheme for approximate closed-loop identification that we will develop in this thesis. The scheme is based on coprime factor representations and it is apparently a mixture of the above three scenarios. We represent the plant $P$ by a ratio $ND^{-1}$ of stable transfer functions, and we transform the dynamic equations of the plant into
\[ \begin{align*}
y &= Nx + v_y \\
u &= Dx + v_u \tag{2.20}
\end{align*} \]
in which $v_y$ and $v_u$ are noises. We will identify $P$ by identifying its factors $N$ and $D$. As there exist many coprime factorizations of $P$, this representation is not (yet) unique. In
addition, the variable $z$ cannot be measured. In order to cope with these problems we assume that knowledge of the compensator $C$ is available. The compensator $C$ is used to represent $P$ by means of a dual Youla parameterization, which leads to an unique factorization $ND^{-1}$. Moreover, for this particular factorization we can reconstruct the variable $z$ from measurements of $u$ and $y$ using $C$. This reconstruction of $z$ is based on the relation

$$u + C y = r_1 + C r_2$$

(see (2.17) and (2.18)). We will show that the resulting $z$ is uncorrelated with $v_y$ and $v_u$ in (2.20). Hence we can approximately identify $N$ and $D$, and thus $P$, from $u, y$ and $z$. In this way we actually use the knowledge of $C$ to build the counterpart (i.e. $z$) of $r_1$ and $r_2$ used in the three scenarios proposed above.

Due to the reconstruction of $z$ our approach looks like scenario 2. Instead we will reconstruct $z$ exactly without a preceding identification step. Next, since $z$ serves as an “instrument” for the identification of $N$ and $D$, and since $P = ND^{-1}$, our approach looks like scenario 1, in which the “instrument” $r_1$ is used to identify the transfer functions $H_y$ and $H_u$ and a nominal model follows as the ratio $\tilde{H}_y/\tilde{H}_u$. However, in order to enforce cancellations in $ND^{-1}$, we will use a customized model structure for $N$ and $D$, so that the eventual identification will be very similar to scenario 3.

In summary we will use the compensator $C$ twice to identify the inner-loop plant $P$: for the dual Youla parameterization of the plant and for the reconstruction of the “instrument” $z$. The framework for this open-loop identification of the controlled inner-loop plant $P$ is developed in Chapter 4. The customized model structure(s) are introduced in Chapter 7, where the identification problem is solved by means of frequency domain techniques.

### 2.4 Identification and Control

We begin this section with illustrating that a good open-loop model can be bad for control and vice versa. Thereafter we concretize the tailor-made system approximation alluded to in Section 1.2. We show that this approximation problem links identification and control design to a joint problem. As we use identification as a means for system approximation we concentrate on the “asymptotic bias distribution” due to undermodelling. We suggest how an iterative solution to this problem should be brought about. The identification procedure used in this iteration must do without the precise knowledge of the subsequently designed compensator. Therefore we end up this section with a survey of system approximation problems, that are solvable by identification techniques, and that are meaningful in view of feedback control at the same time.

#### 2.4.1 The Illustrative Example, I

We introduce an example that will repeatedly be used to elucidate several ideas, concepts and methods. In this example the plant $P$ is a strictly proper continuous-time
SISO system of order 8. Its transfer function coefficients and its poles and zeros are listed in Table A.1. We identify nominal models from frequency response data. These data consist of 100 frequency response samples. The samples are uniformly distributed over a logarithmic interval ranging from 0.1 to 100 rad/s. We denote this set of frequencies as \( \Omega \). Besides, we use noise-free frequency response data in each simulation study, so that we can investigate the "asymptotic bias" due to undermodelling without problems caused by disturbances and the like.

![Nyquist plot](image)

Fig. 2.6: Nyquist plots of \( P \) (---) and \( \hat{P}_1 \) (---) of The Illustrative Example.

We begin with the identification of the nominal model \( \hat{P}_1 \) of order 5 from the frequency response samples \( P(\omega_i) \) as

\[
\hat{P}_1 = \arg \min_{\hat{P}} \sum_{\omega_i \in \Omega} \left| P(\omega_i) - \hat{P}(j\omega_i) \right|^2 .
\]  

(2.21)

The notation \( P(\omega_i) \) is used to distinguish the frequency response samples from their transfer function counterpart \( P(j\omega_i) \). The nominal model \( \hat{P}_1 \) is quite accurate as can be seen from Fig. 2.6: \( P \) and \( \hat{P}_1 \) have similar Nyquist plots and (2.21) measures the square of the distance between these two curves. The coefficients of the nominal models (and compensators) can be found in Appendix A.

Before we design a compensator from \( \hat{P}_1 \), we first gain some additional confidence in this nominal model. For this purpose we compare the responses of \( P \) and \( \hat{P}_1 \) to three signals:
Fig. 2.7: Time-domain responses of $P$ (—) and $\hat{P}_1$ (--) to various input signals (….) in open-loop and closed-loop.

- a: Open-loop response to $r_s$.
- b: Open-loop response to $r_n$.
- c: Open-loop response to $r_w$.
- d: Closed-loop tracking of $r_n$.

i. a step-signal or setpoint change $r_s$,

ii. a random process $r_n$ with a flat spectrum over 0-10 rad/s (see Appendix A),

iii. a random Gaussian white noise process $r_w$.

Fig. 2.7.a shows that the responses $Pr_s$ and $\hat{P}_1 r_s$ are very similar. Thus $\hat{P}_1$ is a good description of $P$ when it comes to step responses. In Fig. 2.7.b we see that $Pr_n$ and $\hat{P}_1 r_n$ make a very good match despite some small deflections. The difference between the responses to the white noise $r_w$ is somewhat more “noisy”.

We realize that practical data is always contaminated with noise. Thus if $\hat{P}_1$ had been identified from practical data, then we would be very confident about this nominal model. As $\hat{P}_1$ is a good nominal model, we use it for control design. The object is to design a control system $H(P, C)$ with a small sensitivity over a large operating band. Notice that we do not know the achievable performance, because the dynamics of the
plant $P$ are uncertain. We pursue a small sensitivity in the range 0-10 rad/s for the plant by each of the following control paradigms:

1. High nominal performance.


3. Robust performance.

The first paradigm pertains to the certainty equivalence principle. We will see that a high-performance compensator for the nominal model does not stabilize the plant. By the second paradigm we achieve a moderate nominal performance and the plant is just stabilized. The third paradigm leads to similar performances for $P$ and $\hat{P}_1$. However this robust performance is also a poor performance.

**High Nominal Performance**

The compensator $C_{hp}$ of order 2 is designed\textsuperscript{11} to achieve a high performance for the nominal model ($hp$ stands for high performance). This high performance is a small sensitivity, which corresponds to a complementary sensitivity of nearly 1 in the operating band. So we can investigate the object of a small sensitivity from the feedback system’s tracking properties. To that end we examine the response of $T_{11}(\hat{P}_1, C_{hp}) = \hat{P}_1 C_{hp}(1 + \hat{P}_1 C_{hp})^{-1}$ to the signal $r_n$. In Fig. 2.7.d we see that the response $T_{11}(\hat{P}_1, C_{hp})r_n$ (-) is very similar to the reference signal $r_n$ (---), which implies that a high nominal performance is achieved. On the other hand $C_{hp}$ destabilizes the plant $P$. The response $T_{11}(P, C_{hp})r_n$ has been plotted in Fig. 2.7.d only over a restricted time interval (---).

Despite the confidence that we had (!) in the nominal model $\hat{P}_1$, it turns out to perform poorly in predicting the operation of the plant $P$ under feedback by the compensator $C_{hp}$. We investigate how we have been deceived by the nice matches of $P$ and $\hat{P}_1$ in Fig. 2.6 and Fig. 2.7.a,b,c. To that end we inspect the Nyquist plots of the loop transfer functions $PC_{hp}$ and $\hat{P}_1 C_{hp}$, which are shown in Fig. 2.8.a. At first sight these loop transfer functions are very similar. This is all but a surprise in view of Fig. 2.6. We take the detail out of Fig. 2.8.a indicated by the dotted box, and we enlarge it to Fig. 2.8.b. In the latter plot $PC_{hp}$ and $\hat{P}_1 C_{hp}$ still seem to be very similar. A second close-up of the dotted box in Fig. 2.8.b yields Fig. 2.8.c. This displays the crucial difference between the two loop gains: the Nyquist curve of $PC_{hp}$ encircles the point -1, marked ‘+’, and the nominal gain $\hat{P}_1 C_{hp}$ not. This explains the instabilities of the actual feedback system $H(P, C_{hp})$ as opposed to the stability of the nominal feedback system $H(P, C_{hp})$.

Now it is clear why the quality of an open-loop identified nominal model is potentially meaningless in regard of control design. We had to zoom twice to reveal the

\textsuperscript{11}The design method is explained in Chapter 6.
crucial feedback-relevant difference between $P$ and $\hat{P}_1$. As this particular aspect of the disparity between $P$ and $\hat{P}_1$ is indiscernible in the Nyquist plot of Fig. 2.8.a, it hardly contributes to the identification criterion of (2.21). So we may not expect a very good nominal model for high performance control design from open-loop considerations alone.

For completeness we examine the Bode log-magnitude diagrams of $P$ and $\hat{P}_1$ depicted in Fig. 2.8.d. The plant and the nominal model show a very good match at the low frequencies up to 1.3 rad/s. The nominal model provides a very poor description of the plant above this frequency. It may seem that we should identify a nominal model through some relative error (Fig. 2.8.d) rather than through an additive error (Fig. 2.8.a). However, minimizing the relative error will turn out not to be the obvious...
Fig. 2.9: Tracking properties of $P$ (—) and $\hat{P}_1$ (--) under control by two robust compensators and for two references.

a: $C_{mp}$ and $r_s$.  

b: $C_{mp}$ and $r_n$. 

b: $C_{lp}$ and $r_s$. 

d: $C_{lp}$ and $r_n$. 

way for control-relevant identification either. For it is obvious that the additive error and the relative error must be small near the cross-over frequency, but for both error terms it is unclear how large the deviations may be at other frequencies.

**Nominal Performance and Robust Stability**

We moderate the design specifications, i.e. we require a smaller bandwidth, and we build a new compensator $C_{mp}$ of order 2 from $\hat{P}_1$ ($mp$ stands for moderate performance). This compensator "just" stabilizes the plant $P$ and, under this constraint, minimizes the sensitivity for $P$. Again we examine the tracking properties of the two feedback systems. The responses to the step $r_s$ and the coloured noise $r_n$ are drawn in Fig. 2.9.a and b. The response $T_{11}(\hat{P}_1, C_{mp}) r_n$ differs more from $r_n$ than $T_{11}(\hat{P}_1, C_{hp}) r_n$ of Fig. 2.7.d. This implies that $H(\hat{P}_1, C_{hp})$ has a smaller sensitivity in the operating band than $H(\hat{P}_1, C_{mp})$, and thus the nominal performance has dropped.
The plant is merely stabilized, which explains the oscillatory behavior (—).

Robust Performance

Recurring to Fig. 2.7 we conclude that \( P \) and \( \hat{P}_1 \) operate similarly in open-loop. We expect similar operations under feedback by a compensator with a (very) low gain. That is, we expect that a (very) little improvement of the performance of \( \hat{P}_1 \) upon its open-loop operation is robust in view of the plant \( P \). We accomplish a robust performance compensator by adjusting the trade-off between nominal performance and robust stability in favour of the latter, until the performances for \( P \) and \( \hat{P}_1 \) are similar. The resulting low performance compensator \( C_{ip} \) of order 2 achieves a robust performance: the responses of \( T_{11}(P, C_{ip}) \) and \( T_{11}(\hat{P}_1, C_{ip}) \) to the step \( r_s \) (Fig. 2.9.c) are very much alike. The same holds for the coloured noise \( r_n \) (Fig. 2.9.d). The frequency responses of \( T(P, C_{ip}) \) and \( T(\hat{P}_1, C_{ip}) \) are also very similar, and thus the actual and nominal feedback systems have almost the same feedback properties. Unfortunately, these feedback systems have rather poor tracking capacities. Thus we can use \( \hat{P}_1 \) to achieve some robust performance, but not a robust high performance.

Robust High Performance

In addition to \( \hat{P}_1 \) we have also the nominal model \( \hat{P}_Q \) (\( Q=\text{Quality} \)), whose parameters can be found in Table A.2. We design the compensator \( C_Q \) from \( \hat{P}_Q \), just like \( C_{hp} \) has been designed from \( \hat{P}_1 \). Thus \( C_Q \) pursues the same high performance for \( \hat{P}_Q \), that has been achieved by \( C_{hp} \) for \( \hat{P}_1 \). Unlike \( C_{hp} \), the new compensator \( C_Q \) stabilizes the plant \( P \). So we have at least a high nominal performance with robust stability. Recall that the constraint of robust stability has limited the attainable nominal performance for \( \hat{P}_1 \).

A comparison of the responses to the step \( r_s \) (Fig. 2.10.a) and to the coloured noise \( r_n \) (Fig. 2.10.b) makes clear, that \( H(P, C_Q) \) and \( H(\hat{P}_Q, C_Q) \) both have very good tracking properties. Moreover the actual and nominal feedback systems have almost the same feedback properties. Thus the nominal model \( \hat{P}_Q \) and the compensator \( C_Q \) together make a solution to the joint problem of approximation and model-based control design: a high nominal performance is achieved for \( P \) and for \( \hat{P}_Q \), and \( H(\hat{P}_Q, C_Q) \) is very well capable of predicting the behavior of \( H(P, C_Q) \). In fact, the feedback matrix \( T(\hat{P}_Q, C_Q) \) is a good approximation of \( T(P, C_Q) \) in the sense of (2.14).

From the responses of \( T_{11}(P, C_Q) \) and \( T_{11}(\hat{P}_Q, C_Q) \) we cannot detect a significant difference between \( P \) and \( \hat{P}_Q \). From one look at their Bode log-magnitude plots (Fig. 2.10.c) it's clear that the nominal model \( \hat{P}_Q \) is really different from the plant \( P \). The plant has two resonances in the lower frequency range, whereas the nominal model has only one resonance. The difference between the plant and its nominal model is even more striking: \( P \) is stable and \( \hat{P}_Q \) is not (cf. Table A.3). Despite these significant differences the nominal model has proven to be well-suited for high performance
2.4 IDENTIFICATION AND CONTROL

Fig. 2.10: Verification of the suitability of the nominal model $\hat{P}_Q$ for the design of a high performance compensator $C_Q$.

a: Tracking of $r_s$ (---).

b: Tracking of $r_n$ (---).

c: Bode log-magnitude plot of $P$ (---), $\hat{P}_Q$ (--), $C_Q$ (---).

d: Part of Nyquist plot of loop transfer functions $PC_Q$ (---), $\hat{P}_Q C_Q$ (--).

control design. On the other hand a minimization of the relative error would probably not have led to a nominal model like $\hat{P}_Q$, as may be clear from Fig. 2.10.c.

We end up with questioning how the nominal model $\hat{P}_Q$, that performs so badly in open-loop, can perform so well in predicting the closed-loop operation of the plant. In Fig. 2.10.d we have drawn the Nyquist curves of $PC_Q$ and $\hat{P}_Q C_Q$. These curves are almost indiscernible near the point $-1$ and they differ much at the lower frequencies. The magnitude of the loop transfer functions is large at low frequencies, so that in this range the complementary sensitivities $T_{11}(P, C_Q)$ and $T_{11}(\hat{P}_Q, C_Q)$ are near to 1 anyway. Conclusively, the nominal model $\hat{P}_Q$ provides a very good description of the plant $P$ around the cross-over frequency and, at least as important, the deviations at other frequencies do not impair the control design.
Discussion and References

In short, the example illustrates three issues that are important to the joint problem of approximation and control design. These issues are:

1. An open-loop plant response can be a poor guide for the construction of a nominal model, that has to be suited for high performance control design.

2. A nominal model, that is suited for high performance control design, does not necessarily provide a good description of the uncontrolled plant.

3. A nominal model, that is suited for high performance control design, describes the plant accurately in the cross-over frequency range.

These issues have been raised in literature before, but they are seldomly listed together. As for the first issue several authors have pointed out that a nominal model can be unfit for use in control design, even if it produces a very accurate match with open-loop data. Van Zee [235] showed this for the prediction error identification method, and Dailey and Lukich [42] did the same for the maximum likelihood estimator. We hasten to mention that this does not mean that an open-loop identified nominal model is useless. On the contrary, an open-loop identification often will be the first step in our intended iteration of repeated identification and control design. The open-loop model enables some or much improvement of the plant’s operation. For instance the robust poor performance of our example is better than the open-loop operation. Also, in [12, 13] Backx and Damen used an open-loop nominal model of a tube glass production process to design a compensator, that improved upon manual control. And Parsons [177] argues that open-loop identified nominal models of flexible structures can easily be better than analytical models.

In view of the second issue, Jacobsen et al. [116] compared several physically motivated models of a distillation column. Their conclusion is that the best model for control design does not provide the best open-loop step response. And thirdly, the need for a good fit around the cross-over frequency has been advocated by many authors (see e.g. [192, 211, 116, 229]). The deviations at other frequencies may be larger, as long as they do not impair the control design. The latter condition is all but evident considering the difference between $P$ and $\hat{P}_Q$ depicted in Fig. 2.10.c.

In summary we have made the following point. If the plant’s open-loop operation differs totally from a desired high performance operation, then open-loop modelling is not sufficient for high performance control design, and it is not clear a priori what error-term should be minimized instead. The key question is, how to tune the “model-error” towards the control design objective. In [11, 114] this question has been addressed using a probabilistic framework and a model set that is rich enough to contain the plant. More recently ad hoc solutions for control-relevant approximation have been accomplished e.g. for the suppression of vibrations over a prespecified frequency region [99, 15]. In general, the nominal model must give rise to a high nominal performance
that is robust in the face of the plant. Thus the nominal model has to accurately describe the plant in view of the model-based high-performance compensator. This is precisely what we have called the tailor-made system approximation, which is used below to establish the need of an iterative scheme for high performance control design.

2.4.2 Iterative High Performance Control Design

Let us recapitulate some material. In Section 1.2 we established the need of a controller that achieves similar high performances for the plant and for the nominal model. From control theory we know that nominal performance and robustness are conflicting requirements. Hence a high nominal performance involves relatively little robustness. This little robustness must be sufficient to guarantee a high performance for the plant, and thus the nominal model must be accurate in the sense of its own high performance. This requirement can be concretized as follows. We take the upper bound of (2.13),

\[ \|T(P, C_{\hat{P}})\| \leq \|T(\hat{P}, C_P)\| - \|T(P, C_P) + T(\hat{P}, C_P)\|, \tag{2.22} \]

and to streamline the discussion we assume that the norm is infinitely large if its argument is unstable. High performances for \(P\) and for \(\hat{P}\) correspond to small performance norms \(\|T(P, C_P)\|\) and \(\|T(\hat{P}, C_P)\|\). Moreover \(H(P, C_P)\) and \(H(\hat{P}, C_P)\) have similar feedback properties if the performance degradation \(\|T(P, C_P) - T(\hat{P}, C_P)\|\) is much smaller than the nominal performance norm \(\|T(\hat{P}, C_P)\|\). A couple \(\hat{P}, C_P\) possessing these properties is a solution to the high performance control design problem posed at the outset.

Suppose we use individual procedures of identification and control design to find such a couple \(\hat{P}, C_P\), that \(\|T(\hat{P}, C_P)\|\) is small, and \(\|T(P, C_P) - T(\hat{P}, C_P)\|\) is even much smaller. Then

- in the identification stage we (have to) select \(\hat{P}\) out of a set of candidate nominal models, and
- in the control design stage we construct \(C_P\) from the nominal model \(\hat{P}\).

As explained in Section 2.2.2 the control design always pursues a small nominal performance norm \(\|T(\hat{P}, C_P)\|\), so that causes no complications. The key issue is that the performance degradation \(\|T(P, C_P) - T(\hat{P}, C_P)\|\) must be relatively small in respect of \(\|T(\hat{P}, C_P)\|\) (cf. (2.14)), in order that \(H(\hat{P}, C_P)\) reliably predicts the feedback properties of \(H(P, C_P)\). This performance degradation depends on \(P\) and on \(C_P\), and thus it results from both the identification stage and the control design stage. Hence the tailor-made system approximation problem of minimizing \(\|T(P, C_P) - T(\hat{P}, C_P)\|\) is the link between identification and control design.

Our aim is to use individual procedures for identification and control design to derive a couple \(\hat{P}, C_P\) with the above properties. As advocated in Section 1.2 this
requires an iterative scheme. We elucidate this necessity of such an iteration in terms of performance norms. As $||T(\hat{P}, C_P)||$ is minimized in the control design stage, it appears that $||T(P, C_P) - T(\hat{P}, C_P)||$ should be minimized in the identification stage. However this is not possible without any knowledge of $C_P$. On the other hand, we have to estimate some $\hat{P}$ before we can design $C_P$. Thus we inevitably have to select $\hat{P}$ without knowing its precise quality as a solution to the joint problem. In turn we have to use $C_P$ in a non-trivial fashion to seek for a possibly better suited nominal model.

![Figure 2.11: Desired evolution of the robust performance in the primary iteration; performance norms $||T(P, C_i)||$ (×), $||T(\hat{P}, C_i)||$ (o) and bound (⋯).](image)

We develop two iterative schemes of repeated identification and control design. Here we outline the basic ideas underlying these schemes.

The first scheme, which is called the primary iterative scheme, is motivated as follows. An iterative scheme produces sequences of nominal models and compensators. Although our primary goal is that the eventual couple $\hat{P}, C_P$ solves the joint problem of approximation and control design, all other nominal models and compensators must also meet certain requirements. Each designed compensator is actually applied during the iteration. Because of numerous practical reasons these temporary compensators may not lead to unacceptably poor performances for the plant. In fact, a repeated replacement of the compensator is practically acceptable only if each new compensator
Fig. 2.12: Desired evolvement of the robust performance in the advanced iteration; performance norms $\|T(P, C_i)\|$ (⋆), $\|T(\hat{P}, C_i)\|$ (○) and bound (---).

Improves upon the previous one.

An ideal evolvement of the primary iterative scheme has been illustrated in Fig. 2.11. In here we see the evolution of the nominal performance norm $\|T(\hat{P}_1, C_i)\|$, marked 'o', from a high value at the start to a low value in the end. A similar gradual decrease of the upper and lower bounds of (2.13) is represented by the dotted line (---). In order that each new compensator improves the nominal performance as well as the robust performance, (i.e. the upper bound), the controller must be designed in agreement with the inequality

$$\|T(P, C_p) - T(\hat{P}, C_p)\| \ll \|T(\hat{P}, C_p)\|$$

of (2.14). This inequality condition limits the performance achievable for the nominal model $\hat{P}$ (recall the case of $\hat{P}_1$ and the robust but low performance compensator $C_{lp}$ of Section 2.4.1). Since each controller $C_i$ has to satisfy the strong inequality of (2.14), the bounds get tighter as the nominal performance improves. Thereby the guaranteed performance for the plant improves. For completeness we have indicated also the fictitious performance norms $\|T(P, C_i)\|$ for the plant by '⋆'. These can be anything in between the bounds, because we have no complete control over the plant's performance.

In each step of the primary iteration we improve the current nominal performance as much as is allowed in view of the above inequality constraint. The corresponding
control design procedure is called cautious controller enhancement\textsuperscript{12}, and it is developed in Section 6. In the identification stages of this primary iteration we determine a nominal model $\hat{P}$ such that it resembles the plant $P$ in respect of the "old" compensator $C_{i-1}$; i.e. $H(\hat{P},C_{i-1})$ resembles $H(P,C_{i-1})$. More about this identification procedure is said in Section 2.4.3.

In the second iterative scheme, called the advanced iterative scheme, we immediately focus on the desired performance, i.e. the high performance that, in the primary iterative scheme, is pursued only in the final step. In the advance iteration this high performance is pursued for each nominal model, which is illustrated in Fig. 2.12 by letting the nominal performance norms 'o' be constant during the iteration. We recall from Section 1.2 that a high nominal performance is one of the two requirements originating from our high performance control design problem. The other requirement is that of a high performance for the plant. We elaborate this second requirement to some detail.

In the example of Section 2.4.1 we have seen that if the high performance control paradigm is applied to a plain open-loop nominal model, then the resulting compensator can be destabilizing the plant. Hence precautionary measures are needed already in the first step of the iteration. With the control design being determined to pursue a high nominal performance (and nothing less), we have to take these measures already in the preceding identification stage. Ideally we would like to identify the first nominal model such, that the subsequent control design achieves a high performance for the plant by pursuing this performance for the nominal model. In other words, we would like to use the paradigm of robust performance as a basis for our first identification step. But this is precisely the joint problem, and thus (generally) it cannot be solved by one step of identification and control design. — The need of an iteration has been explained before. — Instead of robust performance we will base the first identification step on the paradigm of high nominal performance and robust stability.

Unlike the primary iterative scheme, the first step of the advanced iteration hinges on the utilized control design method. This method optimizes robustness against coprime factor perturbations, while it pursues the high nominal performance. It is an unconstrained optimization, so that it is a priori unknown how large the robustness margins will be. However, we know that the compensator will anticipate perturbations of the coprime factors $\hat{N},\hat{D}$ of the candidate nominal model $\hat{P}$. We model the plant as a perturbation of $\hat{N}$ and $\hat{D}$, and we estimate a nominal model by minimizing this perturbation. Thereby the identification and the control design both optimize robustness in regard of the same high nominal performance. Additional remarks about this identification procedure are made at the end of Section 2.4.3.

It can happen that the robustness achieved in the first step is not sufficient to

\textsuperscript{12}This control design procedure is similar in spirit to the cautious techniques used in adaptive control [115, 82].
2.4 Identification and Control

guarantee stability for the plant. Then of course we will not implement the compensator, but we have to return to the identification stage. On the other hand if the plant is only a small perturbation of the first nominal model with respect to the achieved robustness margin, then we may apply the first compensator to the plant. This case has been depicted in Fig. 2.12. As mentioned before we use the control paradigm of high nominal performance throughout the iteration. And in the second and preceding steps of this iteration we use the identification procedure of the primary iterative scheme. As a consequence the advanced iterative scheme proceeds, except for the first step, as a special case of the primary iteration. — There is no increase of performance requirements in the advanced iteration. — Therefore we lay emphasis on the primary iterative scheme, and we discuss the advanced iteration as an additional result.

In summary, the primary iterative scheme is meant to improve the robust performance (i.e. the upper bound on the plant’s performance) in each iteration step. This iterative improvement starts from the open-loop operation irrespectively of the ultimately desired performance. The advanced iterative scheme immediately focusses on the desired high (nominal) performance, and thereby it is suggestive of the certainty equivalence principle [195, 86]. However it is based on an optimization of robustness, and the designed controllers are implemented only if they will achieve an acceptable performance. Further, the presumption underlying the ideal evolutions of Fig. 2.11 and Fig. 2.12 is that there exists a controller for each robust performance level. This assumption can be violated if e.g. the order of the controller is restricted. Apart from such constraints, we need tools to carry out the approximate identification and the control design. Several paradigms of approximation for control design are discussed in the next section. This section is ended up with a survey of literature.

The need of an iteration to tune the “model-error” towards the control objective has been put forward earlier. Wilfert and co-workers [251, 252] studied frequency response estimation for the tuning of PI- and PID-controllers. They concluded that the required accuracy of the estimate depends on the compensator. Enns [67] addressed model reduction for control design, and he pointed out that the right nominal model cannot be found without any knowledge of the compensator. Skelton [210] argued that the plant’s input greatly influences the approximation problem, and thus input-output data must be collected while the plant operates under feedback. Rivera and Morari [184, 186] recognized that an approximation followed by control design can be highly suboptimal. Bitmead, Gevers and Wertz [21, 22] and Anderson and Kosut [7] suggested to use an iterative scheme of individual identification and control design stages as an alternative for adaptive control.

A few identification techniques and iterative schemes have been proposed and/or elaborated in literature. In [21, 22] Bitmead, Gevers and Wertz introduced an iteration that looks like our primary iterative scheme. The main difference is that each of their
identification steps has been based on robust stability instead of robust performance. This distinction will be clarified in Section 2.4.3. Bitmead and co-workers combined the prediction error identification method together with LQ design [23, 264, 265]. The results are similar in spirit to our primary iterative scheme except for the following difference. Each control step of their iteration starts from the desired high nominal performance, and then the control objective is moderated as much as is needed to accomplish sufficient robustness (like in The Illustrative Example of Section 2.4.1). Instead each control step of our primary iteration improves the current performance irrespective of the desired performance. The latter approach is also taken by Hakvoort et al. [94, 95], who combined the prediction error identification method with LQG design.

Another contribution is made by Liu and Skelton [134, 135], who transformed the problem of identifying a low order nominal model for control design into the problem of identifying a high order model of the whole feedback system with a subsequent iterative model reduction and control design problem. The latter iteration consists of intuitive tuning guided by experience and understanding of the plant under consideration.

2.4.3 System Approximation in View of Feedback Control

In this section we first survey the importance of coprime factorizations in feedback-relevant system approximation. Thereafter we distinguish four classes of such approximation problems. This will enable us to give sharper direction to our developments.

Vidyasagar and co-workers have demonstrated that a system approximation is meaningful for feedback control only if it is an approximation in the graph topology (see [244, 238] and [239, Sec.7.2]). An essential implication of this topology is as follows. The nominal model \( \hat{P} \) is represented by its coprime factorization\(^{13}\) \( \hat{N} \hat{D}^{-1} \). The stable dynamical perturbations \( \Delta_N, \Delta_D \) turn the coprime factorization \( (\hat{N}, \hat{D}) \) into

\[
\begin{align*}
\hat{N}_\Delta & = \hat{N} + \Delta_N \\
\hat{D}_\Delta & = \hat{D} + \Delta_D
\end{align*}
\]

which defines the perturbed nominal model \( \hat{P}_\Delta = \hat{N}_\Delta \hat{D}_\Delta^{-1} \). — This encompasses the additive dynamical perturbation \( \Delta_A \) as the special case \( \Delta_N = \Delta_A \hat{D}, \Delta_D = 0 \). — The stable nominal feedback system \( H(\hat{P}, C_\hat{P}) \) is robustly stable against small dynamical perturbations \( (\Delta_N, \Delta_D) \) of the coprime factors of \( \hat{P} \). Moreover the graph topology says that the perturbed feedback system \( H(\hat{P}_\Delta, C_\hat{P}) \) converges to \( H(\hat{P}, C_\hat{P}) \) if and only if \( \Delta_N \to 0 \) and \( \Delta_D \to 0 \). So for small perturbations \( (\Delta_N, \Delta_D) \) the perturbed feedback system \( H(\hat{P}_\Delta, C_\hat{P}) \) is guaranteed to be stable, and for even smaller perturbations we can guarantee a robust performance. On this basis of this distinction between robust

\(^{13}\)The factors \( \hat{N}, \hat{D} \) are stable transfer functions. Precise definitions will be given in Chapter 3.
stability and robust performance we divide feedback-relevant system approximation into the following categories.

**Definition 2.4.1**

i. Performance-approximation is the approximation of a plant \( P \) by a nominal model \( \hat{P} \), so that, for some given compensator \( C \), the feedback systems \( H(P, C) \) and \( H(\hat{P}, C) \) have similar feedback properties.

ii. Stability-approximation is the approximation of a plant \( P \) by a nominal model \( \hat{P} \), so that, for some given compensator \( C \), either the nominal feedback system \( H(\hat{P}, C) \) is robustly stable in the face of the plant \( P \), or the actual feedback system \( H(P, C) \) is robustly stable in the face of the nominal model \( \hat{P} \).

For a clarification of this distinction we turn to the common additive mismatch \( M_A = P - \hat{P} \). It follows from (2.6), (2.7) and (2.8) that \( H(P, C) \) is stable if

\[
\|(P - \hat{P})C(I + \hat{P}C)^{-1}\|_\infty < 1
\]

and \( H(\hat{P}, C) \) is stable. Such a nominal model \( \hat{P} \) is a stability-approximation of the plant \( P \) in view of the compensator \( C \). This approximation allows \( P \) to be near to the robustness margin, and thus the performance of \( H(P, C) \) can differ greatly from that of \( H(\hat{P}, C) \). In order to explain the performance-approximation we suppose that the performance is measured by some norm of the complementary sensitivity, which is just a special case of \( \|T(P, C)\| \) (see also Example 2.2.5). Then for robust performance the difference

\[
P C (I + PC)^{-1} - \hat{P} C (I + \hat{P}C)^{-1} = (I + PC)^{-1} PC - \hat{P} C (I + \hat{P}C)^{-1}
\]

\[
= (I + PC)^{-1} [PC(I + \hat{P}C) - (I + PC)\hat{P}C](I + \hat{P}C)^{-1}
\]

\[
= (I + PC)^{-1}(P - \hat{P})C(I + \hat{P}C)^{-1}
\]

must be small. In this performance-approximation the mismatch \( (P - \hat{P})C(I + \hat{P}C)^{-1} \) of the stability-approximation is additionally weighted by the sensitivity \( (I + PC)^{-1} \) of the plant.

The mismatch of the performance-approximation becomes infinitely large if just one of the two feedback system \( H(P, C) \) and \( H(\hat{P}, C) \) is almost unstable. In contrast the mismatch \( (P - \hat{P})C(I + \hat{P}C)^{-1} \) remains finite if only \( H(P, C) \) is nearly unstable. This has the following consequence. Suppose \( H(P, C) \) is nearly unstable. Then the performance-approximation results in a nominal feedback system \( H(\hat{P}, C) \) with a similar performance close to instability. The stability-approximation however pursues a small additive mismatch \( P - \hat{P} \) and a small nominal sensitivity \( (I + PC)^{-1} \). Hence the resulting nominal feedback system \( H(\hat{P}, C) \) is likely to have a good performance (small sensitivity) and a large robustness margin, which contrasts with the properties of the controlled plant.
Remark 2.4.2 The difference between the two approximation problems of Definition 2.4.1 has also been recognized in the area of controller reduction [185, 9, 3]. Surprisingly, controller reduction plays a role in some experiment design results. Gevers and Ljung [81] and Hansen et al. [101, 100, 102] compare an optimal high-order plant-based compensator $C_P$ with a low-order model-based compensator $C_P$, and they design an identification experiment from the “reduction-error” $C_P - C_P$.

Remark 2.4.3 In model reduction the weight $(I + PC)^{-1}$ can be calculated for a given compensator $C$. In identification we can either use an estimate of this sensitivity, or use $C$ for closed-loop experiments so that the weight $(I + PC)^{-1}$ is accounted for by the data.

The second distinction that we make concerns the controller in question.

Definition 2.4.4

i. Fixed-loop approximation is the approximation a plant $P$ by a nominal model $\hat{P}$ in view of a given compensator $C$.

ii. Design-oriented approximation is the approximation of a plant $P$ by a nominal model $\hat{P}$ in view of a compensator $C_P$, that has not been designed yet.

By now we have four types of feedback-relevant system approximation, which are discussed separately in the remainder of this section. Only the first and last type of approximation problems are fully elaborated in this thesis. As system approximation embeds approximate identification and model reduction [108], we pay attention to both topics in the discussions below. For notational convenience we often confine the discussion to complementary sensitivities.

**Performance-approximation by Fixed-loop Identification**

The starting point in this type of feedback-relevant approximate identification is a stable feedback system $H(P, C)$ with a fixed compensator $C$. The goal is to find a nominal model $\hat{P}$, such that the performance of $H(\hat{P}, C)$ resembles that of $H(P, C)$. In terms of complementary sensitivities this means a minimization of the mismatch

$$PC(I + PC)^{-1} - \hat{P}C(I + \hat{P}C)^{-1}.$$

The corresponding optimization problem is all but trivial, since $\hat{P}$ appears in the mismatch in a multiple and non-linear fashion. A simplification by transforming this mismatch is highly undesired, as that will greatly influence the approximation. For instance we may not postmultiply the mismatch by $(I + \hat{P}C)^{-1}$, and carry out the approximation by minimizing a norm of the new mismatch $(I + PC)^{-1}(P - \hat{P})C$. Further, as explained in Remark 2.3.2, the use of an iterative procedure with a mismatch like

$$PC(I + PC)^{-1} - \hat{P}_kC(I + \hat{P}_{k-1}C)^{-1}$$

(2.24)
yields an asymptotic estimate $\hat{P}$ that depends on the noise contributions.

A cost function based on the difference between complementary sensitivities or more generally on $T(P, C) - T(\hat{P}, C)$ will have many local minima. Hence we need a good initial estimate for a search procedure. In Chapter 7 we explain how coprime factorizations and frequency response data can be used to accomplish a good initial condition for the minimization $T(P, C) - T(\hat{P}, C)$ in an $L_2$-sense.

The iteration of (2.24) has been used in combination with the prediction error identification method by Bitmead and co-workers [23, 264, 265] and by Hakvoort et al. [94, 95]. Both developments rest on a small disturbance in the operating band, so that the iteration of (2.24) yields good results. Bitmead et al. minimize the mismatch $PC(I + PC)^{-1} - \hat{P}C(I + \hat{P}C)^{-1}$ and Hakvoort et al. identify a nominal model from $P(I+PC)^{-1} - \hat{P}(I+\hat{P}C)^{-1}$. Liu and Skelton [134] take an indirect approach to identify the plant: they identify the feedback system's transfer functions from covariance data and they extract a nominal model from that estimate. Lastly Lee, Anderson and Kosut [129] have proposed a conceptual scheme for the approximate identification of the complementary sensitivity. They employ the knowledge of the stabilizing compensator in a way that is based on the framework for identification introduced by Hansen [100]. As we use the dual of this framework, our approach to performance-approximation by fixed-loop identification is similar to the proposed concepts of Lee et al..

**Stability-approximation by Fixed-loop Identification**

Stated in terms of complementary sensitivities the fixed-loop stability-approximation problem concerns the minimization of the mismatch

$$(P - \hat{P})C(I + PC)^{-1}$$

or

$$(P - \hat{P})C(I + \hat{P}C)^{-1}$$

for some given compensator $C$. The first mismatch uses the knowledge that the compensator $C$ stabilizes the plant $P$. The corresponding optimization searches for a nominal model $\hat{P}$, whose additive mismatch $P - \hat{P}$ is hopefully smaller than the robustness margin of $H(P, C)$. In the second mismatch the plant $P$ and the nominal model $\hat{P}$ change roles.

In [21, 22] Bitmead, Gevers and Wertz have used the prediction error identification method to accomplish a fixed-loop stability-approximation. In essence their identification has been based on the mismatch $(P - \hat{P})(I + \hat{P}C)^{-1}$, which does not exploit the knowledge that $H(P, C)$ is stable.
Performance-approximation by Design-oriented Identification

The aim of this type of feedback-relevant approximation is to minimize the performance degradation in view of some future compensator. As this compensator is not available yet, the identification must be based on a prediction of the plant’s future performance. Suppose that \( T_{22.\text{des}} \) is a desired sensitivity function. Then for SISO systems we take (2.23) and we substitute \( T_{22.\text{des}} \) for \((I+PC)^{-1}\) and \((1 - T_{22.\text{des}})\) for \(\hat{P}C(I+\hat{P}C)^{-1}\). Thereby we obtain the mismatch

\[
T_{22.\text{des}} \frac{P - \hat{P}}{\hat{P}} (1 - T_{22.\text{des}}),
\]

from which \(\hat{P}\) can be identified without knowledge of the future compensator.

Surely, in the subsequent control design the desired sensitivity must be accomplished or at least nearly accomplished for both the plant \(P\) and the nominal model \(\hat{P}\). If that is not the case, then the utility of the approximation is unclear. Despite that, Rivera et al. have successfully applied such a scenario to control-relevant identification [187, 188, 183] and model-reduction [184, 186].

Stability-approximation by Design-oriented Identification

Like its fixed-loop counterpart this approximation problem concerns the control paradigm of nominal performance and robust stability. This kind of approximate identification is used in the first step of our advanced iterative scheme. We will base our approach to this type of approximation on a control design method that optimizes robustness against coprime factor perturbations. For each candidate nominal model \(\hat{P}\) we know the precise coprime factors \(\hat{N}, \hat{D}\) for which robustness is optimized in the control design. Therefore we will identify a nominal model by minimizing the difference between \(\hat{N}, \hat{D}\) and the corresponding coprime factors of the plant. If the mismatch between the coprime factorizations is small, then the plant is only a small coprime factor perturbation of \((\hat{N}, \hat{D})\), and thus \(P\) is stabilized by the subsequently designed controller.

This type of approximation has been advocated only by Enns [67], who studied model reduction for control design. Based on the stability condition of (2.8) he used the desired complementary sensitivity \(T_{11.\text{des}}\) to weight the multiplicative mismatch:

\[
\frac{P - \hat{P}}{\hat{P}} T_{11.\text{des}}
\]

underlies Enns' control-relevant model reduction.
2.5 Synopsis

With the material presented in this chapter we are able to give more precise directions to our subsequent developments.

Our basic assumption is that the uncertain plant of concern operates under a stabilizing feedback, and that the compensator is known. In Chapter 3 we use the compensator to represent the plant as an element of the class of all stabilized systems. This is based on the dual Youla parameterization, called the $R$-parameterization, and the plant is represented by its coprime factors. In Chapter 4 we show that the identification of these coprime factors is an open-loop identification problem. So we can identify the inner-loop plant $\tilde{P}$ via its coprime factors in such a way, that its asymptotic estimate $\hat{P}$ is independent from the noise contributions.

Next in Chapter 5 we shape the mismatch $(N - \hat{N}, D - \hat{D})$ of this open-loop identification problem to two different mismatches for feedback-relevant approximation. At this stage we do not yet minimize a norm of these mismatches. Firstly, we relate the mismatch $(N - \hat{N}, D - \hat{D})$ to the mismatch $T(P, C) - T(\hat{P}, C)$, which enables a fixed-loop performance-approximation. The second mismatch $(\hat{N} - NQ, \hat{D} - DQ)$ enables a design-oriented stability-approximation. The latter approximation anticipates the robust control design method that optimizes robustness against coprime factor perturbations.

The robust control design method is used in Chapter 6 to develop a procedure for cautious controller enhancement. The design method is an unconstrained optimization, so we first have to choose a design weight and thereafter we must ascertain stability of the new control system. In choosing the design weight we use additional information about the plant, along with its nominal model, in the form of frequency response estimates. The stability ascertainment is carried out with classes of compensator-based dynamical perturbations. Having made precise the control design stage, we minimize the feedback-relevant mismatches of Chapter 5 from frequency response data in Chapter 7.

In Chapter 8 we blend the developed tools together to form the primary and advanced iterative schemes of repeated identification and control design. These two iterative high performance control design procedures are applied to a simulation study in Chapter 8 and to an experiment set-up in Chapter 9.
Chapter 3

Algebraic Theory of Linear Feedback Systems

The fact that a known compensator stabilizes some plant conceals a lot of information about the dynamics of that plant. Such a plant is known to belong to the class of all systems, that are stabilized by the particular compensator. In this chapter we exploit such knowledge of the uncertain plant.

We use the algebraic theory of coprime factorizations to study the stability of linear feedback systems. The first section discusses basic ideas and applications. Then in Section 3.2 we adopt the algebraic framework from Gündes and Desoer [90], and we introduce some notation and definitions.

In Section 3.3 we study the stability of the single-variate feedback system $H(P, C)$. We represent the plant $P$ as an unique element of the set of all plants that are stabilized by the compensator $C$. This representation is called the $R$-parameterization of $P$, which is slightly more general than the well-known dual Youla parameterization of stabilizing compensators. The unique coprime factorization of $P$, that arises from the $R$-parameterization is called the associated coprime factorization of the plant $P$.

In Section 3.4 the $R$-parameterization is generalized to the case of $H(P_{TT}, C_{TT})$. This result is called the $(R, S)$-parameterization of the plant $P_{TT}$. Finally in Section 3.5 the $(R, S)$-parameterization is adapted to the standard feedback system $H(P_T, C)$, because the latter feedback system serves as a starting point in the next chapter.

3.1 Introduction

This section is composed of two parts. The first part exposes the basic idea behind the algebraic theory. The second part provides a concise survey of literature illustrating the wide applicability of the algebraic theory in system theory.
3.1.1 Basic Principles

The algebraic theory has its roots in the analysis and synthesis of linear single-variate feedback systems. The plant $P$ and compensator $C$ of such a single-variate feedback system $H(P, C)$ (Fig. 3.1) are represented as $P = ND^{-1}$ and $C = \tilde{D}_c^{-1}\tilde{N}_c$. These representations are called right and left fractional representations or factorizations. Substitution in (2.11) yields

$$T(P, C) = \begin{bmatrix} N \\ D \end{bmatrix} (\tilde{D}_c D + \tilde{N}_c N)^{-1} \begin{bmatrix} \tilde{N}_c & \tilde{D}_c \end{bmatrix},$$

which enables a characterization of all achievable single-variate feedback systems. In [257] and later in [178] such a characterization was developed for LTIFD systems

![Fig. 3.1: Single-variate feedback system $H(P, C)$.](image)

with $N$ and $D$ being polynomials. It was the algebraic formulation of [56], that greatly simplified the stability argument. The latter development is based on the following idea. The analysis or synthesis of a feedback system usually concerns desired properties of $H(P, C)$. The most basic desired property is stability, i.e. $T(P, C) \in \mathbb{R}H_\infty$. Accordingly the plant $P$ and the compensator $C$ are factorized in such a way, that $N, D, \tilde{N}_c$ and $\tilde{D}_c$ are stable. Then the feedback matrix $T(P, C)$ will be stable as well, provided that $(\tilde{D}_c D + \tilde{N}_c N)^{-1}$ is stable.

**Remark 3.1.1** As an alternative, one could choose the terms $N$ and $D$ e.g. to be functions that are analytic in $\text{Re} \{s\} \geq -1$. With $\tilde{N}_c$ and $\tilde{D}_c$ also being analytic in $\text{Re} \{s\} \geq -1$, the feedback matrix $T(P, C)$ is analytic in $\text{Re} \{s\} \geq -1$ if and only if $(\tilde{D}_c D + \tilde{N}_c N)^{-1}$ has the same property [56].

An algebraic theory for $H(P_{TT}, C_{TT})$ with $P_{TT}$ and $C_{TT}$ being LTI systems has been introduced in [162] and exposed in [50, 90]. These developments concerns the parameterization of the class of all compensators structured like $C_{TT}$ that stabilize the plant $P_{TT}$. From an identification point of view the dual parameterization is of interest. Such a parameterization has been employed in [100, 102] to study the identification of the plant $P_T$, while it is feedback controlled in the configuration $H(P_T, C)$. Our main contribution to the algebraic theory is the parameterization of the class of all LTIFD plants like $P_{TT}$, that are stabilized by the LTIFD compensator $C_{TT}$. This result is seemingly dual to the parameterization of the set of compensators that stabilize $P_{TT}$.
which has been developed in [162, 50, 90]. However, the latter work concerns proper controllers and strictly proper plants. In contrast, our result applies also when neither the plant nor the compensator is strictly proper.

3.1.2 Fields of Applications

The application of the algebraic theory lies primarily with the stability study of the single-variate feedback system \( H(P, C) \) of Fig. 3.1. The set of all stabilizing compensators can easily be characterized in terms of a coprime factorization of the plant \( P \) and one factor that ranges over the set of all stable transfer functions [56, 244, 239]. This parameterization is sometimes called the Youla parameterization or Q-parameterization [143]. A next step in the development of the algebraic theory is the extension to the so-called two-parameter compensator \( C_T \). This \( C_T \) equals \( C_{TT} \) of Fig. 2.1 except that it lacks the output \( z_c \). In [51, 239] it is shown, that each achievable transfer function from \( w_c \) to \( y \) can be accomplished simultaneously with each achievable feedback matrix \( T(P, C) \). Further, [47] studies the design of decoupling compensators by considering plants that have a second vector output \( z \). The stability of the general feedback system \( H(P_{TT}, C_{TT}) \) is treated in [162, 50, 90] and [196]. The latter work will be summarized in Section 3.4. A short overview of literature on the computation of coprime factorizations of LTIFD systems can be found in Appendix B.2.

All the above results have been derived in terms of rings of systems, which applies well to LTIFD systems. Our developments will also be framed in terms of LTIFD systems. And yet, we intend to apply the developed theory to a real system in Chapter 9. More specific, we will apply a LTIFD compensator derived from a ditto nominal model to a real system, that might be distributed, time-varying or non-linear. In Appendix E.1 our algebraic results are shown to be robust for the "mixed case", in which the real controlled system is not truly a LTIFD system.

The second field of applications concerns optimal control design. The above developments provide precisely the set of all stable feedback systems, that are attainable for a given plant. The corresponding parameterization has been used as a starting point in the search for an \( H_\infty \)-optimal controller (see e.g. [60, 72, 143, 26, 151, 152, 25, 222]). Coprime factorizations have been employed also in the development of the multi-objective control design techniques of [91, 30, 127, 179]. Other control design techniques, that use coprime factorizations, deal with reliable and simultaneous stabilizations [191, 245, 158, 169], decentralized control [49, 90] and observer design [58].

The third field of applications is system approximation in view of feedback control. In [262] the disparity between a plant \( P \) and its nominal model \( \hat{P} \) is related to the difference between their \( H(P, C) \) and \( H(\hat{P}, C) \). In [244] coprime factorizations were used to define the graph topology, which implies that \( H(\hat{P}, C) \) will remain stable in the presence of small dynamical perturbations of the coprime factors of \( \hat{P} \). Coprime
factorizations have been utilized in controller reduction [132, 8, 133, 28] and model reduction in view of feedback [153, 154, 29].

Coprime factorizations have been used to relate the stability of the nominal feedback system $H(\hat{P}, C)$ to the possible stability of $H(P, C)$. For instance in [111] necessary and sufficient conditions for the stability of $H(P, C)$ have been conceived in terms of coprime factors of $\hat{P}$ and $C_P$ and the transfer function $P - \hat{P}$. A generalization of this work to $H(PT, CT)$ is provided in [69]. A more common approach is to use an upper bound on the difference $P - \hat{P}$ rather than the transfer function itself. In these cases the stability of $H(P, C)$ is demonstrated by means of a robustness margin of $H(\hat{P}, C)$. Several robustness margins for LTI systems have been derived from coprime factor representations [243, 77, 40, 24, 203]. Fractional representations have also been used to study robust stability in the face of non-linear perturbations [38, 54].

All the above applications concern analysis and synthesis problems related to feedback systems. Hence the algebraic theory is likely to be of use to the joint problem of identification and control design. Indeed in [100, 102, 103, 125, 227, 112, 113] and in [197, 201, 198, 199, 200, 202] several aspects of the joint problem have been tackled using coprime factorizations. The latter references cover parts of the work reported here. A comparative treatise of the various contributions is postponed to Section 7.1.

### 3.2 Algebraic Framework

The basic concepts that underly coprime factor representations originate from ring theory. In this section we adopt some notation and definitions from this theory and we formalize the algebraic structure. Several additional terms are listed in Appendix B.1.

Like Desoer and Gündes [50, 90] we start building the algebraic structure with a principal ring $\mathcal{H}$. We complete the structure as by Vidyasagar et al. [244].

**Definition 3.2.1**

$\mathcal{H}$ is a principal ideal domain.

$\mathcal{F}$ is the field of fractions associated with $\mathcal{H}$, i.e. $\mathcal{F} = \{x/y \mid x, y \in \mathcal{H}, y \neq 0\}$

$\mathcal{J}$ is the group of units in $\mathcal{H}$, i.e. $\mathcal{J} = \{x \in \mathcal{H} \mid x^{-1} \in \mathcal{H}\}$.

The algebraic theory is applicable to various classes of systems [56]. We use this theory to study LTIFD continuous-time systems. In this context the domain $\mathcal{H}$ is identified with the set $\mathbb{R}H_\infty$, which consists of all stable proper real-rational systems. Accordingly $\mathcal{F}$ is the set of all real-rational systems (not necessarily proper or stable), and each element of $\mathcal{J}$ is stable and has a stable inverse.

**Remark 3.2.2** In [162, 50, 90] the algebraic structure is supplied with the multiplicative subset $I$ of $\mathcal{H}$, and the ring $\mathcal{G}$ of fractions of $\mathcal{H}$ associated with $I$. That is, $\mathcal{G} = \{x/y \mid x \in \mathcal{H}, y \in I\}$ or equivalently $\mathcal{I} = \{x \in \mathcal{H} \mid x^{-1} \in \mathcal{G}\}$. The various rings
are related as $J \subset I \subset H \subset G \subset F$. In addition, the above references make use also of the Jacobson radical $G_r$, which is defined as $G_r = \{ x \in G \mid (1 + xy)^{-1} \in G, \ \forall y \in G \}$.

For the special case of $H = IRH_\infty$ the elements of $I$ are stable and have a proper inverse. $G$ contains all proper systems and $G_r$ consists of all strictly proper systems. In [261] the Jacobson radical $G_r$ has been introduced to represent physical systems, because the latter do not anticipate an input nor do they respond instantaneously. From the viewpoint of system approximation it is undesired to constrain the candidate nominal model to be strictly proper (see also Fig. 2.2).

With some minor restrictions like the omission of the commutative property, the sets of Definition 3.2.1 can be extended to MIMO systems. A system with $p$ outputs and $m$ inputs is said to belong to $H^{p \times m} (F^{p \times m})$ if all the entries of its transfer function matrix belong to $H (F)$. Also, a system belongs to $J^{p \times p}$ if it belongs to $H^{p \times p}$ and its determinant is an element of $J$. Since dimensions play a minor role in the development of the algebraic theory, we denote $H^{p \times m}$ simply as $H$ and the like for $F$ and $J$.

The stage having been set, we define the notion of coprimeness over the ring $H$ and subsequently we state its relation to the so-called Bezout identity.

**Definition 3.2.3** ([239]) Two elements $A, B \in H$ are coprime if every greatest common divisor of $A$ and $B$ is a unit of $H$.

For our purposes coprimeness of $A$ and $B$ means that $A$ and $B$ belong to $IRH_\infty$ and they have no common unstable zeros.

**Fact 3.2.4** ([239])

i. Let $N, D \in H$, then the pair $(N, D)$ is right coprime if and only if there exist right Bezout factors\(^1\) $X, Y \in H$ such that

\[
XN + YD = I.
\]  
(3.1)

This equality is called the (right) Bezout identity.

ii. Let $\tilde{N}, \tilde{D} \in H$, then the pair $(\tilde{D}, \tilde{N})$ is left coprime if and only if there exist left Bezout factors $\tilde{X}, \tilde{Y} \in H$ such that

\[
\tilde{N}\tilde{X} + \tilde{D}\tilde{Y} = I.
\]  
(3.2)

This equality is called the (left) Bezout identity.

Notice that the right Bezout factors $X, Y$ make a left coprime pair and vice versa. Based on these notions we define coprime factorizations like in [239]. In Proposition B.2.2 it is shown that Bezout factors are not unique.

\(^1\)The notion of a Bezout factor is introduced for ease of discussion. It is not a common notion in the algebraic theory of coprime factorizations.
Definition 3.2.5 Let \( N, D \in \mathcal{H} \) and \( \tilde{N}, \tilde{D} \in \mathcal{H} \). Then \((N, D)\) is a right coprime factorization (rcf) of some plant \( P \in \mathcal{F} \) if \( \det(D) \neq 0 \), \( P = ND^{-1} \) and \((N, D)\) is right coprime. Analogously \((\tilde{D}, \tilde{N})\) is a left coprime factorization (lcf) of some plant \( P \in \mathcal{F} \) if \( \det(\tilde{D}) \neq 0 \), \( P = \tilde{D}^{-1}N \) and \((\tilde{D}, \tilde{N})\) is left coprime.

For the case of \( \mathbb{IR}H_\infty \) coprimeness of \((N, D)\) implies that there is no cancellation of unstable poles and zeros in the product \( ND^{-1} \). We end up this section with two basic facts of the algebraic theory of fractional representations. The first fact has been taken from [239] and it concerns the non-uniqueness of coprime factorizations. The second fact, which is elaborated in Section B.2, relates a left right coprime factorizations of some plant to a right coprime factorization.

Fact 3.2.6

i. Let \((N, D)\) and \((\tilde{D}, \tilde{N})\) be an rcf respectively an lcf. Then \((NQ, DQ)\) is right coprime and \((Q\tilde{D}, Q\tilde{N})\) is left coprime if and only if \(Q \in \mathcal{J}\).

ii. Let some plant \( P \in \mathcal{F} \) have a rcf \((N, D)\) and a lcf \((\tilde{D}, \tilde{N})\). Then there exist right Bezout factors \(X, Y\) of \((N, D)\) and left Bezout factors \(\tilde{X}, \tilde{Y}\) of \((\tilde{D}, \tilde{N})\) such that

\[
\begin{bmatrix}
X & \tilde{X} \\
\tilde{N} & \tilde{D}
\end{bmatrix}
\begin{bmatrix}
D & -\tilde{X} \\
N & \tilde{Y}
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}.
\]

This is called the double Bezout identity, and the factors \(X, Y, \tilde{X}, \tilde{Y}\) \(\in \mathcal{H}\) are called double Bezout factors.

In Proposition B.2.3 it is shown that double Bezout factors are not unique. However the pair \(\tilde{X}, \tilde{Y}\) is uniquely for given \(ND, \tilde{N}, \tilde{D}, X\) and \(Y\).

3.3 Stability of The Single-variate Feedback System

In this section we derive a necessary and sufficient condition for the stability of the single-variate feedback system \(H(P, C)\). This feedback system is composed of the plant \(P \in \mathcal{F}\) and the compensator \(C \in \mathcal{F}\) as depicted in Fig. 3.2. If \(H(P, C)\) is stable, then \(P\) belongs to the class of all systems, that are stabilized by \(C\). The latter class will be represented by the \(R\)-parameterization, which is a slight generalization of the dual Youla parameterization. The coprime factorization that corresponds to the plant \(P\) is called the associated rcf of \(P\). This associated rcf will play an important role throughout the thesis.

The following proposition formalizes the notion of stability of the single-variate feedback system \(H(P, C)\).

Proposition 3.3.1 The single-variate feedback system \(H(P, C)\) is called stable if and only if \(T(P, C) \in \mathcal{H}\) with \(T(P, C)\) as defined in (2.11).
3.3 Stability of the Single-variate Feedback System

![Diagram of a single-variate feedback system](image)

Fig. 3.2: Single-variate feedback system $H(P, C)$.

Remark 3.3.2 In [50, 90] an element of $\mathcal{H}$ is called $\mathcal{H}$-stable, because the ring $\mathcal{H}$ may differ from $\mathbb{R}H_\infty$. Nevertheless we use $\mathcal{H} = \mathbb{R}H_\infty$ and therefore we apply the common notion of stability.

For ease referencing we fix some notation in the next assumption. The subsequent lemma states the necessary and sufficient condition for the stability of $H(P, C)$.

**Assumption 3.3.3** The following coprime factorizations exist:

i. the plant $P \in \mathcal{F}$ has a rcf $(N, D)$ with Bezout factors $X, Y$ and a lcf $(\tilde{D}, \tilde{N})$ with Bezout factors $\tilde{X}, \tilde{Y}$.

ii. the compensator $C \in \mathcal{F}$ has a rcf $(N_c, D_c)$ with Bezout factors $X_c, Y_c$ and a lcf $(\tilde{D}_c, \tilde{N}_c)$ with Bezout factors $\tilde{X}_c, \tilde{Y}_c$.

**Lemma 3.3.4** Let Assumption 3.3.3 hold. Define $\Lambda$ and $\tilde{\Lambda}$ as

$$
\begin{bmatrix}
\Lambda & 0 \\
0 & \tilde{\Lambda}
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{D}_c & \tilde{N}_c \\
-\tilde{N} & \tilde{D}
\end{bmatrix}
\begin{bmatrix}
D & -N_c \\
N & D_c
\end{bmatrix}
$$

(3.3)

then the following statements are equivalent

i. $H(P, C)$ is stable,

ii. $\Lambda \in \mathcal{J}$,

iii. $\tilde{\Lambda} \in \mathcal{J}$.

**Proof:** We prove the equivalences i.$\iff$ii. and i.$\iff$iii., which together imply ii.$\iff$iii.

(i.$\iff$ii.) By Proposition 3.3.1 stability of $H(P, C)$ is equivalent to $T(P, C) \in \mathcal{H}$.

We write the transfer function $T(P, C)$ of (2.11) as

$$
T(P, C) = \begin{bmatrix}
N \\
D
\end{bmatrix}
\Lambda^{-1}
\begin{bmatrix}
\tilde{N}_c & \tilde{D}_c
\end{bmatrix}
$$

(3.4)
from which the implication $\Lambda \in \mathcal{J} \Rightarrow T(P, C) \in \mathcal{H}$ is immediate. This proofs ii.$\Rightarrow$i.

Further, stability of $T(P, C)$ implies stability of

$$
\begin{bmatrix}
X & Y
\end{bmatrix}
T(P, C)
\begin{bmatrix}
\tilde{X}_c \\
\tilde{Y}_c
\end{bmatrix}
= \Lambda^{-1},
$$

and thus $\Lambda \in \mathcal{J}$, since $\Lambda$ is stable by construction (3.3).

(i.$\Leftrightarrow$iii.) Now we examine the transfer function

$$
T(C, P) = \begin{bmatrix}
C(I+PC)^{-1}P & C(I+PC)^{-1} \\
(I+PC)^{-1}P & (I+PC)^{-1}
\end{bmatrix}
= \begin{bmatrix}
N_c \\
D_c
\end{bmatrix}
\tilde{\Lambda}^{-1}
\begin{bmatrix}
\tilde{D} & \tilde{N}
\end{bmatrix},
$$

which maps $\text{col}(-r_1, r_2)$ into $\text{col}(y_c, u_c)$. By a similar use of Bezout factors we obtain the implication $T(C, P) \in \mathcal{H} \Leftrightarrow \Lambda \in \mathcal{J}$. From the equalities $u = r_1 + y_c$ and $y = r_2 - u_c$ it can be straightforwardly shown that $T(C, P) \in \mathcal{H} \Leftrightarrow T(P, C) \in \mathcal{H}$. Finally, application of Proposition 3.3.1 completes the proof. \qed

In order to throw some light on the above stability theorem we substitute $ND^{-1}$ and $\tilde{D}_c^{-1}\tilde{N}_c$ for respectively $P$ and $C$ in Fig. 3.2. Simultaneously we relocate the exogenous inputs $r_1$ and $r_2$ by definition of $\xi = \tilde{D}_c r_1 + \tilde{N}_c r_2$. The result is depicted in Fig. 3.3. The transfer function from $\xi$ to $z$ is precisely $\Lambda^{-1}$. Bounded signals $r_1$ and $r_2$ cause a bounded signal $\xi$. This $\xi$ results in a bounded $x$ and also in bounded signals $u$ and $y$ provided that $\Lambda^{-1} \in \mathcal{H}$.

![Fig. 3.3](image-url)

Fig. 3.3: Coprime factor representation of the single-variate feedback system $H(P, C)$ with relocated inputs and outputs.
By Fact 3.2.6.i the stabilizing compensator $C$ has a lcf $(\Lambda^{-1}\tilde{D}_c, \Lambda^{-1}\tilde{N}_c)$ and a rcf $(N_c\tilde{\Lambda}^{-1}, D_c\tilde{\Lambda}^{-1})$. When these particular factorizations are substituted for $(\tilde{D}_c, \tilde{N}_c)$ and $(N_c, D_c)$ then the left hand side of (3.3) becomes identity. Hence the assumption that "$\Lambda = I$" and "$\tilde{\Lambda} = I$" implies that the corresponding factorization of $C$ is chosen in agreement with the factorization of $P$. This particular factorization of $C$ is used in the common Youla parameterization [56, 244, 239]. Dual to the Youla parameterization we want to parameterize the set $\mathcal{P}(C)$ of all systems that are stabilized by the compensator $C$. We could of course use the dual parameterization of $\mathcal{C}(P)$, but we want the factorizations of the plant $P$ and the compensator $C$ to be independent. Therefore we generalize the well-known case of $\Lambda = I$ to that of $\Lambda \neq I$.

We base the parameterization of the set $\mathcal{P}(C)$ on an auxiliary model denoted $P_\circ$. This auxiliary model $P_\circ$ is just some LTIFD system that is stabilized by $C$. It can be chosen from the set $\mathcal{P}(C)$ regardless of the plant $P$, but it may also be an available nominal model of $P$. The next assumption is introduced for ease of reference.

**Assumption 3.3.5** The auxiliary model $P_\circ \in \mathcal{F}$ has the following properties:

i. The auxiliary feedback system $H(P_\circ, C)$ is stable.

ii. $P_\circ$ has a rcf $(N_\circ, D_\circ)$ and a lcf $(\tilde{D}_\circ, \tilde{N}_\circ)$.

Similar to (3.3) we introduce the notation

$$
\begin{bmatrix}
\Lambda_\circ & 0 \\
0 & \tilde{\Lambda}_\circ
\end{bmatrix} = 
\begin{bmatrix}
\tilde{D}_c & \tilde{N}_c \\
-\tilde{N}_\circ & \tilde{D}_\circ
\end{bmatrix}
\begin{bmatrix}
D_\circ & -N_c \\
N_\circ & D_c
\end{bmatrix}
$$

(3.5)

so that, by Lemma 3.3.4, $H(P_\circ, C)$ is stable if and only if $\Lambda_\circ \in \mathcal{J}$, or equivalently, if and only if $\tilde{\Lambda}_\circ \in \mathcal{J}$.

**Theorem 3.3.6** Let Assumption 3.3.3 and Assumption 3.3.5 hold. Then the plant $P \in \mathcal{F}$ is stabilized by the compensator $C$ in the feedback configuration $H(P, C)$ of Fig. 3.2 if and only if $P$ has a rcf $(N^a, D^a)$ defined as

$$
D^a = D_\circ - N_c R \\
N^a = N_\circ + D_c R
$$

(3.6)

with $R \in \mathcal{H}$ such that $\det(D_\circ - N_c R) \neq 0$. The rcf $(N^a, D^a)$ is called associated to $(N_\circ, D_\circ)$ and $C$. $(N^a, D^a)$ is uniquely determined by $P, C$ and $(N_\circ, D_\circ)$; $R$ is uniquely determined by $P,(N_c, D_c)$ and $(N_\circ, D_\circ)$.

**Proof:**

($\Leftarrow$) First we prove that $(N^a, D^a)$ of (3.6) is a right coprime factorization. From Assumption 3.3.5 and Lemma 3.3.4 we know that $\Lambda_\circ$ of (3.5) belongs to $\mathcal{J}$, and thus
\( \Lambda_o^{-1}\tilde{D}_c \in \mathcal{H} \) and \( \Lambda_o^{-1}\tilde{N}_c \in \mathcal{H} \). Multiplication of these terms to the right by \( D^a \) respectively \( N^a \) and substitution of (3.6) yield

\[
\Lambda_o^{-1}(\tilde{D}_c D^a + \tilde{N}_c N^a) = I + (\tilde{N}_c D_c - \tilde{D}_c N_c)R = I.
\]

Hence by Fact 3.2.4.i \( \Lambda_o^{-1}\tilde{N}_c \), \( \Lambda_o^{-1}\tilde{D}_c \) are right Bezout factors of \((N^a, D^a)\), so that \((N^a, D^a)\) is right coprime (notice that \(N^a\) and \(D^a\) belong to \(\mathcal{H}\)). Further, the constraint \(\det(D_o - N_c R) \neq 0\) guarantees the existence of \((D^a)^{-1}\). According Definition 3.2.5 the pair \((N^a, D^a)\) is a right coprime factorization of the plant \(N^a(D^a)^{-1}\). From the above equation it follows that

\[
\tilde{D}_c D^a + \tilde{N}_c N^a = \Lambda_o \in \mathcal{J},
\]

and the stability of \(H(N^a(D^a)^{-1}, C)\) follows from Lemma 3.3.4. Thus if \(P\) has an associated rcf \((N^a, D^a)\), then \(H(P, C)\) is stable.

(\(\Rightarrow\)) Let a plant \(P\) be given such that \(H(P, C)\) is stable. We construct \(R \in \mathcal{F}\) from \(P\), \((N_c, D_c)\) and \((N_o, D_o)\), and we rewrite the expression as follows

\[
R = (D_c + PN_c)^{-1}(PD_o - N_o) = (\tilde{D}D_c + \tilde{N}N_c)^{-1}(\tilde{N}D_o - \tilde{D}N_o) = \tilde{\Lambda}^{-1}(\tilde{N}D_o - \tilde{D}N_o).
\]

The latter expression shows that \(R \in \mathcal{H}\), because \(H(P, C)\) is stable and thus \(\tilde{\Lambda} \in \mathcal{J}\) by Lemma 3.3.4. As a consequence \(N^a\) and \(D^a\) of (3.6) also belong to \(\mathcal{H}\). Substitution of (3.8) in (3.6) yields

\[
\begin{align*}
D^a &= D_o - C(I + PC)^{-1}(PD_o - N_o) \\
N^a &= N_o + (I + PC)^{-1}(PD_o - N_o).
\end{align*}
\]

This demonstrates that the associated rcf \((N^a, D^a)\) of the plant \(P\) is uniquely determined by \((N_o, D_o)\) and \(C\). Coprimeness of the pair \((N^a, D^a)\) has already been proven above. It remains to be shown that \(D^a\) has an inverse, and that \(P = N^a(D^a)^{-1}\). To that end we rewrite the expression for \(D^a\) of (3.10) to

\[
\begin{align*}
D^a &= D_o - C(I + CP)^{-1}(P - P_o)D_o \\
&= (I - CP(I + CP)^{-1} + (I + CP)^{-1}CP_o)D_o \\
&= (I + CP)^{-1}(D_o + CN_o) \\
&= D(\tilde{D}_c D + \tilde{N}_c N)^{-1}(\tilde{D}_c D_o + \tilde{N}_c N_o) \\
&= D\Lambda^{-1}\Lambda_o.
\end{align*}
\]

The latter equality shows that \(D^a\) has an inverse, viz. \(\Lambda_o^{-1}\Lambda D^{-1}\). This implies that \((N^a, D^a)\) is a coprime factorization and that \(\det(D_o - N_c R) \neq 0\). From a similar operation for \(N^a\) we get

\[
N^a = N_o - (I + PC)^{-1}(P - P_o)D_o
\]
\[ P(I + CP)^{-1}(D_o + CN_o) \]
\[ = N A^{-1} \Lambda_o. \]  
\[ \text{(3.13)} \]
\[ \text{(3.14)} \]

Taking (3.12) and (3.14) together it follows that \( N^a (D^a)^{-1} = ND^{-1} \) and thus \( (N^a, D^a) \) is a rcf of \( P \).

The importance of Theorem 3.3.6 is twofold. Firstly the plant \( P \) is known to have an associated rcf \((N^a, D^a)\), provided that it is stabilized by the compensator \( C \). Secondly for a given plant \( P \), a given compensator \( C \) and a fixed auxiliary rcf \((N_o, D_o)\) this rcf \((N^a, D^a)\) is unique. In the next chapter we will exploit these properties for the identification of feedback controlled plants. We can use Theorem 3.3.6 also for a characterization of the set \( \mathcal{P}(C) \) of all systems that are stabilized by \( C \).

**Corollary 3.3.7** Let Assumption 3.3.5 hold. Then the set \( \mathcal{P}(C) \) of all systems in \( \mathcal{F} \), that are stabilized by \( C \) in the single-variate feedback configuration of Fig. 3.2, is given by

\[ \mathcal{P}(C) = \{(N_o + D_c R_H)(D_o - N_c R_H)^{-1} \mid R_H \in \mathcal{H}, \det(D_o - N_c R_H) \neq 0\} \]

This is called the \( R \)-parameterization of the set \( \mathcal{P}(C) \).

**Remark 3.3.8** Since \( C \) belongs to \( \mathcal{F} \), the set \( \mathcal{P}(C) \) may contain non-proper systems. It was shown by Vidyasagar in [238] that if \( C \) is strictly proper, then all elements of \( \mathcal{P}(C) \) are proper. It can easily be shown that if \( C \) is proper, then most of the systems in \( \mathcal{P}(C) \) are proper as well. The dual of the latter justifies a posteriori the use of the Youla or Q-parameterization in control design: although the set \( C(P) \) of a proper plant \( P \) contains non-proper elements, it is used for the design of a proper compensator [60, 72, 143, 222].

There is a distinction between \( R \) in Theorem 3.3.6 and \( R_H \) in Corollary 3.3.7. The latter term \( R_H \) varies freely over the space \( \mathcal{H} \). Each \( R_H \) corresponds to some system that is stabilized by \( C \), and there exist no two \( R_H \)'s that induce the same system. So for a particular plant \( P \) (and particular \((N_c, D_c)\) and \((N_o, D_o)\)) there is one \( R_H \), which is precisely the term \( R \) of Theorem 3.3.6. This \( R \) embodies the “difference” between the plant \( P_o \) and the auxiliary model \( P_c \).

Perhaps we should call “\( P = (N_o + D_c R)(D_o - N_c R)^{-1} \)” the \( R \)-representation of \( P \), but with a slight abuse of notation we call it the \( R \)-parameterization of \( P \). This representation is depicted in Fig. 3.4. Notice that \( u = D^a x = (D_o - N_c R)x \) and \( y = N^a x = (N_o + D_c R)x \). These mappings from \( x \) to \( u \) and \( y \) are realized with a feedback of \( x \) over the auxiliary \( D_o^{-1} \) and with a bypass of \( x \) along \( N_o \). The term \( R \) needs to appear only once in the diagram. With this diagram we can visualize the utility of the coprime factorizations in feedback-relevant approximation the plant \( P \): if \( R \) tends to zero, then \( H(P_o, C) \rightarrow H(P, C) \). We return to this matter in Chapter 5.
Fig. 3.4: Block diagram of a stable single-variate feedback system $H(P, C)$ with $R$-parameterization of $P$.

The use of the $R$-parameterization in approximating the feedback system $H(P, C)$ is additionally motivated by replacing $(N, D)$ with $(N^a, D^a)$ in the expression for $T(P, C)$ of (3.4). With the simultaneous use of (3.7) we obtain

$$T(P, C) = \begin{bmatrix} N^a \\ D^a \end{bmatrix} \Lambda^{-1} \begin{bmatrix} \bar{N}_c & \bar{D}_c \end{bmatrix}$$

(3.15)

and with (3.6) we get

$$T(P, C) = T(P_0, C) + \begin{bmatrix} D_c \\ -N_c \end{bmatrix} R \Lambda^{-1} \begin{bmatrix} \bar{N}_c & \bar{D}_c \end{bmatrix}.$$  

(3.16)

These equations show that the closed-loop transfer function $T(P, C)$ is affine in $R$. This affine relation has been well studied for Youla parameterization of the class $C(P)$ [60, 239, 162, 72, 50, 30, 143, 90, 222]. Also, $T(P, C)$ is linear in $N^a$ and $D^a$, and $R$ and $(N^a, D^a)$ are the only terms in the right hand sides of (3.15) and (3.16) that depend on the plant $P$.

**Remark 3.3.9** In [227] the parameter $R$ was interpreted as the frequency shaped difference between the transfer functions of $P$ and $P_0$:

$$R = D_c^{-1}(I + PC)^{-1}(P - P_0)D_c = \bar{\Lambda}^{-1}\bar{D}(P - P_0)D_c.$$

\[\square\]

### 3.4 Stability of The General Feedback System

In this section we generalize the algebraic theory for the single-variate feedback system $H(P, C)$ to the case of the general feedback system $H(P_T, C_T)$ of Fig. 3.5. This is the most general interconnection of two systems [162, 50, 90]. The latter references
treat the parameterization of the class of all proper two-input two-output compensators that stabilize the proper plant $P_{TT}$, whose inner-loop part $P$ is strictly proper. We are interested in the dual of this result, i.e. the parameterization of the set $\mathcal{P}_{TT}(C_{TT})$ of all two-input two-output systems that are stabilized by the compensator $C_{TT}$. A direct use of the abovementioned result would imply that we can handle only compensators, whose inner-loop part $C$ is strictly proper. This is undesirable because (models of) compensators are often not strictly proper; an example is the well-known PI-controller. Therefore we generalize the dual algebraic theory of [162, 50, 90]. Our results apply also to non-proper systems, which can be of use in the study of e.g. bilaterally coupled systems [202]. Our developments proceed along the same lines as those of Desoer and Gündes [50], except that the proofs are different.

The remainder of this section is divided into three parts. We first show that the general feedback system $H(P_{TT}, C_{TT})$ is stable if and only if its inner-loop feedback system $H(P, C)$ is stable and all instabilities of its components $P_{TT}$ and $C_{TT}$ are contained in their inner-loop parts$^2$ $P$ and $C$. If the latter condition on $P_{TT}$ holds, then $P_{TT}$ is called admissible. This property is exploited in the second part to derive structured coprime factorizations of $P_{TT}$ and $C_{TT}$. This particular step is lacking in the previous section, because $H(P, C)$ consists of only inner-loop parts. In the third part we derive a necessary and sufficient condition for the stability of $H(P_{TT}, C_{TT})$. Finally we generalize the $R$-parameterization of $\mathcal{P}(C)$ of Corollary 3.3.7 to the $(R, S)$-parameterization of $\mathcal{P}_{TT}(C_{TT})$.

\textbf{Coprime factorizations and stability}

As discussed in Section 3.2 every $P_{TT} \in \mathcal{F}$ has a rcf and a lcf. The next lemma relates the rcf $(N, D)$ of $P$ to a coprime factorization of $P_{TT}$.

$^2$The property that the disturbance plant $P_{yw}$ may contain the same instabilities as $P$, has been “built in” rather than brought forth in the joint input-output method for identification of stochastic feedback systems [80, 5, 6].
Lemma 3.4.1 Let Assumption 3.3.3 hold and let $P_{TT} \in \mathcal{F}$ be partitioned as in (2.2). Then $P_{TT} \in \mathcal{F}$ has a rcf $(N'_{TT}, D'_{TT})$ and a lcf $(\tilde{D}'_{TT}, \tilde{N}'_{TT})$ defined as

$$
(N'_{TT}, D'_{TT}) = \begin{pmatrix} N'_{11} & N'_{12} \\ N'_{21} & NL \end{pmatrix}, \quad \begin{bmatrix} D_{11} & 0 \\ D_{21} & DL \end{bmatrix}
$$

$$
(\tilde{D}'_{TT}, \tilde{N}'_{TT}) = \begin{pmatrix} \tilde{D}'_{11} & \tilde{D}'_{12} \\ 0 & L\tilde{D} \end{pmatrix}, \quad \begin{bmatrix} \tilde{N}'_{11} & \tilde{N}'_{12} \\ \tilde{N}'_{21} & L\tilde{N} \end{bmatrix}
$$

(3.17) (3.18)

with $N'_{11}, N'_{12}, N'_{21}, D'_{11}, D'_{21}, L \in \mathcal{H}$ and $\tilde{N}'_{11}, \tilde{N}'_{12}, \tilde{N}'_{21}, \tilde{D}'_{11}, \tilde{D}'_{12}, \tilde{L} \in \mathcal{H}$.

Proof: We prove the existence of $(N'_{TT}, D'_{TT})$ and comment on the existence of $(\tilde{D}'_{TT}, \tilde{N}'_{TT})$. Since $P_{TT}$ belongs to $\mathcal{F}$, it has a rcf $(A, B)$. As $\mathcal{H}$ is a principal ideal domain, the (square) $B \in \mathcal{H}$ can be decomposed into the Hermite form $B = D_{TT}U$, in which $D_{TT} \in \mathcal{H}$ is a lower triangular matrix and $U \in \mathcal{J}$ (see Appendix B.1). We define $N'_{TT} \doteq AU^{-1}$ so that $(N'_{TT}, D'_{TT})$ is a rcf of $P$ by virtue of Fact 3.2.6.i. We partition these $N'_{TT}$ and $D'_{TT}$ conformally with the inputs and outputs:

$$
N'_{TT} = \begin{bmatrix} N'_{11} & N'_{12} \\ N'_{21} & N'_{22} \end{bmatrix}, \quad D'_{TT} = \begin{bmatrix} D_{11} & 0 \\ D_{21} & D_{22} \end{bmatrix}
$$

Expanding the expression $N'_{TT}(D'_{TT})^{-1}$ yields $P = N'_{22}(D_{22})^{-1}$. With $P = ND^{-1}$ and $DD^{-1} = I$ we have $N'_{22} = ND^{-1}D_{22} \in \mathcal{H}$ and $D_{22} = DD^{-1}D_{22} \in \mathcal{H}$ and thus

$$
XN'_{22} + YD'_{22} = (XN + YD)D^{-1}D_{22} = D^{-1}D_{22} \in \mathcal{H}.
$$

By defining $L \doteq D^{-1}D_{22}$ we obtain $L \in \mathcal{H}$ and the desired expressions $D'_{22} = DL$ and $N'_{22} = NL$. This completes the proof for the rcf in (3.17). The proof of the dual lcf is analogous, except that the left associate Hermite form is to be used.

The above result holds for every system in $\mathcal{F}$, and thus $C_{TT}$ has a lcf $(\tilde{D}'_{cTT}, \tilde{N}'_{cTT})$ defined as

$$
(\tilde{D}'_{cTT}, \tilde{N}'_{cTT}) = \begin{pmatrix} \tilde{D}'_{c11} & \tilde{D}'_{c12} \\ 0 & L_c\tilde{D}_c \end{pmatrix}, \quad \begin{bmatrix} \tilde{N}'_{c11} & \tilde{N}'_{c12} \\ \tilde{N}'_{c21} & L_c\tilde{N}_c \end{bmatrix}
$$

(3.19)

without loss of generality. In (3.19) $(\tilde{D}_c, \tilde{N}_c)$ is the lcf of the inner-loop compensator $C$ and all entries of $D'_{cTT}$ and $N'_{cTT}$ belong to $\mathcal{H}$.

So far we have only related coprime factorizations of $P_{TT}$ and $C_{TT}$ to the coprime factorizations of their inner-loop parts $P$ and $C$. Now we formalize the notion of stability for $H(P_{TT}, C_{TT})$ in a way similar to that of Proposition 3.3.1.

Proposition 3.4.2 Let $H_{io} \in \mathcal{F}$ be the transfer function that maps the feedback system inputs $w, r_1, w_c$ and $r_2$ into $z, y, z_c$ and $y_c$, i.e.

$$
H_{io} : \text{col}(w, r_1, w_c, r_2) \mapsto \text{col}(z, y, z_c, y_c).
$$

Then the general feedback system $H(P_{TT}, C_{TT})$ is stable if and only if $H_{io} \in \mathcal{H}$. 
Besides the presence of \( w, w_c \) and \( z, z_c \) there is a minor difference between \( H_{io} \) and the feedback matrix \( T(P, C) \) used in Proposition 3.3.1. The latter concerns the signals \( y \) and \( u \), whereas \( H_{io} \) has \( y \) and \( y_c \) as its outputs. From inspection of Fig. 3.5 it is clear that \( u \) and \( y_c \) are equivalent in view of stability.

For the derivation of a stability condition from \( H_{io} \), we need a bico-prime factorization \( H_{io} = N_H D_H^{-1} \tilde{N}_H \) like in [50]. Bico-primeness of \( N_H D_H^{-1} \tilde{N}_H \) implies that \( (N_H, D_H) \) is right coprime and \( (D_H, \tilde{N}_H) \) is left coprime [48].

**Proposition 3.4.3** Let Assumption 3.3.3 hold. Further let \((N_{TT}', D_{TT}')\) of (3.17) and \((\tilde{D}_{cTT}', \tilde{N}_{cTT}')\) of (3.19) be a rcf of \( P_{TT} \in \mathcal{F} \) respectively a lcf of \( C_{TT} \in \mathcal{F} \). Then the transfer function \( H_{io} \) defined in Proposition 3.4.2 has a bico-prime factorization

\[
H_{io} = N_H D_H^{-1} \tilde{N}_H
\]

with

\[
N_H = \begin{bmatrix} N'_{TT} & 0 \\ 0 & I \end{bmatrix} \quad \tilde{N}_H = \begin{bmatrix} I & 0 \\ 0 & \tilde{N}'_{cTT} \end{bmatrix}
\]

\[
D_H = \begin{bmatrix} D_{TT}' \\ \tilde{D}_{cTT}' \end{bmatrix}
\]

\[
N'_{cTT} \begin{bmatrix} 0 & 0 \\ N_2 & N_L \end{bmatrix} \quad \tilde{N}'_{cTT}
\]

**Proof:** The equivalence \( H_{io} = N_H D_H^{-1} \tilde{N}_H \) is proven by construction. We define \( y_H = \text{col}(z, y, z_c, y_c) \) and \( u_H = \text{col}(w, r_1, w_c, r_2) \) so that \( y_H = H_{io} u_H \). Then we decompose the input-output map of \( P_{TT} \) into \( \text{col}(z, y) = N_{TT}' x_{TT} \) and \( D_{TT}' x_{TT} = \text{col}(w, u) \), which implicitly defines \( x_{TT} \). Likewise, the input-output relation of \( C_{TT} \) is written as \( \tilde{D}_{cTT}' \text{col}(z_c, y_c) = \tilde{N}_{cTT}' \text{col}(u_c, u) \). Now we define \( z_H = \text{col}(x_{TT}, z_c, y_c) \) and we substitute \( r_1 + y_c \) for \( u \) and \( r_2 - y \) for \( u_c \). Then the equalities representing the input-output relations of \( P_{TT} \) and \( C_{TT} \), can be combined into \( D_H x_H = \tilde{N}_H u_H \) and \( y_H = N_H x_H \). Consequently \( y_H = N_H D_H^{-1} \tilde{N}_H u_H \) and thus \( H_{io} = N_H D_H^{-1} \tilde{N}_H \).

It remains to be proven that the pairs \((N_H, D_H)\) and \((D_H, \tilde{N}_H)\) are coprime. Let \( X_{TT}', Y_{TT} \in \mathcal{H} \) be a pair of right Bezout factors of \((N_{TT}', D_{TT}')\). Then we define the stable terms \( X_H \) and \( Y_H \) as

\[
X_H \doteq \begin{bmatrix} X_{TT}' & Y_{TT}' \\ 0 & I \end{bmatrix} \quad Y_H \doteq \begin{bmatrix} Y_{TT}' & 0 \\ 0 & 0 \end{bmatrix}
\]

which satisfy \( X_H N_H + Y_H D_H = I \), so that \((N_H, D_H)\) is coprime by Fact 3.2.4.i. Similarly we let \( \hat{X}_{cTT}', \hat{Y}_{cTT} \in \mathcal{H} \) be a pair of left Bezout factors of \((\tilde{D}_{cTT}', \tilde{N}_{cTT}')\). Then the
stable terms $\tilde{X}_H$ and $\tilde{Y}_H$, defined as

$$
\tilde{X}_H = \begin{bmatrix}
I & \begin{bmatrix} 0 & 0 \\
0 & I \end{bmatrix} \\
0 & \tilde{X}'_{TT}
\end{bmatrix} \quad \tilde{Y}_H = \begin{bmatrix}
0 & 0 \\
0 & \tilde{Y}'_{TT}
\end{bmatrix},
$$

yield $\tilde{N}_H \tilde{X}_H + D_H \tilde{Y}_H = I$, and thus the pair $(D_H, \tilde{N}_H)$ is left coprime.

\begin{thm}
Let Assumption 3.3.3 hold. Further let $(N'_{TT}, D'_{TT})$ of (3.17) be a rcf of $P_{TT} \in \mathcal{F}$ and let $(\tilde{D}'_{cTT}, \tilde{N}'_{cTT})$ of (3.19) be a lcf of $C_{TT} \in \mathcal{F}$. Define

$$
d_P = \text{det } D'_{TT} / \text{det } D \quad (3.20)
$$

$$
d_C = \text{det } \tilde{D}'_{cTT} / \text{det } \tilde{D}_c \quad (3.21)
$$

Then the following statements are equivalent:

i. $H(P_{TT}, C_{TT})$ is stable

ii. $\text{det } D_H \in \mathcal{J}$

iii. $d_P \in \mathcal{J}$, $d_C \in \mathcal{J}$ and $\text{det } \Lambda \in \mathcal{J}$ with $\Lambda$ as defined in (3.3).

\begin{proof}
(i.$\Rightarrow$ii.) By Proposition 3.4.2 $H(P_{TT}, C_{TT})$ is stable if and only if $H_{io} \in \mathcal{H}$. By Fact B.1.3.i right coprimeness of $(N_H, D_H)$ and stability of $\tilde{N}_H$ together imply that $D_H^{-1} \tilde{N}_H \in \mathcal{H}$. Subsequently by Fact B.1.3.ii left coprimeness of $(D_H, \tilde{N}_H)$ and stability of $D_H^{-1} \tilde{N}_H$ together imply that $D_H^{-1} \in \mathcal{H}$. And thus by Fact B.1.2.ii $\text{det } D_H \in \mathcal{J}$.

(ii.$\Rightarrow$iii.) First we decompose $D_H$ into two matrices. Thereafter we calculate $\text{det } D_H$ as the product of two separate determinants. Since $(\tilde{D}_c, \tilde{N}_c)$ is a lcf, the inverse of $\tilde{D}_c$ exists. Hence we may write $D_H$ as

$$
D_H = \begin{bmatrix}
I & 0 & 0 & 0 \\
0 & \tilde{D}^{-1}_c & 0 & -\tilde{D}^{-1}_c \\
0 & I & 0 \\
0 & 0 & \tilde{L}_c
\end{bmatrix}
$$

$$
\begin{bmatrix}
D'_{11} & 0 & 0 \\
\Pi & \Lambda & 0 \\
\tilde{N}'_{c12} \tilde{N}'_{c21} & \tilde{N}'_{c12} NL & \tilde{D}'_{c11} \\
\tilde{N}'_{c22} \tilde{N}'_{c21} & \tilde{N}'_{c22} NL & \tilde{D}'_{c22}
\end{bmatrix}
$$

in which $\Pi = \tilde{D}_c D'_{21} + \tilde{N}_c N'_{21}$. Expanding the determinants of the matrices on the right hand side yields

$$
\text{det } D_H = \text{det } \tilde{D}^{-1}_c \text{det } \tilde{L}_c \text{det } D'_{11} \text{det } \Lambda \text{det } L \text{det } \tilde{D}'_{c11} \text{det } \tilde{D}_c.
$$

By (3.17) and (3.19) we can substitute $\text{det } D'_{TT}$ and $\text{det } \tilde{D}'_{cTT}$ for $\text{det } D'_{11}$ and $\text{det } L$ respectively for $\text{det } \tilde{D}'_{c11}$ and $\text{det } \tilde{D}_c$. As $\text{det } D^{-1} = 1 / \text{det } D$, etc., we can write

$$
\text{det } D_H = (\text{det } D'_{TT} / \text{det } D)(\text{det } \tilde{D}'_{cTT} / \text{det } \tilde{D}_c) \text{det } \Lambda = d_P \tilde{d}_C \text{det } \Lambda.
$$
\end{proof}
Repeated application of Fact B.1.2.i reveals that all three scalars $d_P$, $\tilde{d}_C$ and $\det \Lambda$ belong to $J$ if $\det D_H \in J$.

(iii.$\Rightarrow$i.) We already showed that $\det D_H = d_P \tilde{d}_C \det \Lambda$. So $d_P \in J \land \tilde{d}_C \in J \land \Lambda \in J$ $\Rightarrow$ $\det D_H \in J \Rightarrow D_H^{-1} \in H$ and since $N_H$ and $\tilde{N}_H$ also belong to $H$ the transfer function $H_{i_o}=N_H D_H^{-1} \tilde{N}_H$ is stable, which implies that $H(P_{TT}, C_{TT})$ is stable. $\square$

**Remark 3.4.5** The stability theorem is the same as in [50], but the proof is quite different. The reason is, that we consider plants and compensators in $\mathcal{F}$ instead of $\mathcal{G}$, so that we cannot use the specific relationship between $\mathcal{G}$ and $\mathcal{I}$.

Since $P_{TT}$ and $C_{TT}$ are completely dual in view of stability of $H(P_{TT}, C_{TT})$, we could also have formulated the stability condition in terms of a lcf of $P_{TT}$ and a rcf of $C_{TT}$. Therefore we state without further notice that the dual of Theorem 3.4.4 holds as well.

Theorem 3.4.4 says that if the three scalars $d_P$, $\tilde{d}_C$ and $\det \Lambda$ belong to $J$, then $H(P_{TT}, C_{TT})$ is stable. From (3.20) it is clear that it depends solely on $P_{TT}$ whether or not the condition $d_P \in J$ is satisfied. The same holds for $\tilde{d}_C$ and $C_{TT}$. Since we are interested only in stabilizable plants and compensators, we will characterize the class of stabilizable systems.

**Admissible systems and compensators**

Since $P_{TT}$ and $C_{TT}$ are dual in view of stability, we characterize only the class of plants that can be stabilized in the feedback configuration of $H(P_{TT}, C_{TT})$. The result can immediately be applied to characterize the class of compensators with the same property. We start with a formalization of what is meant by the paraphrase “stabilizable in the feedback configuration $H(P_{TT}, C_{TT})$”.

**Definition 3.4.6** A plant $P_{TT} \in \mathcal{F}$ is called admissible if there exists a compensator $C_{TT} \in \mathcal{F}$, so that $H(P_{TT}, C_{TT})$ of Fig. 3.5 is stable.

We relate the admissibility of $P_{TT}$ to a right and a left coprime factor representation.

**Proposition 3.4.7** Let $P_{TT} \in \mathcal{F}$ have a rcf $(N'_{TT}, D'_{TT})$ as in (3.17) and a lcf $(\tilde{D}'_{TT}, \tilde{N}'_{TT})$ as in (3.18). Then the following statements are equivalent:

i. $P_{TT}$ is admissible

ii. $d_P \in J$

iii. $\tilde{d}_P \in J$, with $\tilde{d}_P \doteq \det \tilde{D}'_{TT}/\det \tilde{D}$.

**Proof:**

(i.$\Rightarrow$ii.) By Definition 3.4.6 admissibility of $P_{TT}$ implies that the existence of a compensator $C_{TT}$ so that $H(P_{TT}, C_{TT})$, and by Theorem 3.4.4 that $d_P \in J$.

(ii.$\Rightarrow$iii.) We build $C_{TT}$ from $C_{sw} = 0$, $C_{sw} = 0$, $C_{yw} = 0$ and $C = Y^{-1}X$, where
$X, Y \in \mathcal{H}$ are the right Bezout factors of $(N, D)$. Then $(\text{diag}(I, Y), \text{diag}(0, X))$ is a lcf of $C_{TT}$, which gives rise to $\tilde{d}_C = 1$ and $\Lambda = XN + YD = I$. For this particular compensator the scalars $\tilde{d}_C$ and $\det \Lambda$ belong to $J$, and since $d_P \in J$ this $C_{TT}$ stabilizes $P_{TT}$ by Theorem 3.4.4.

(ii.$\Leftrightarrow$(iii.) Consequence of Fact B.1.3.iii.

**Theorem 3.4.8** Let Assumption 3.3.3 hold and let $P_{TT} \in \mathcal{F}$ be partitioned as in (2.2). Then $P_{TT}$ is admissible if and only if it has a rcf $(N_{TT}, D_{TT})$ and a lcf $(\tilde{D}_{TT}, \tilde{N}_{TT})$ satisfying

\begin{equation}
(N_{TT}, D_{TT}) \doteq \left( \begin{bmatrix} N_{11} & N_{12} \\
 \tilde{Y} N_{21} & N \end{bmatrix}, \left[ \begin{bmatrix} I & 0 \\
 -\tilde{X} N_{21} & D \end{bmatrix} \right] \right) \tag{3.22}
\end{equation}

\begin{equation}
(\tilde{D}_{TT}, \tilde{N}_{TT}) \doteq \left( \begin{bmatrix} I & -\tilde{N}_{12} X \\
 0 & \tilde{D} \end{bmatrix}, \left[ \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} Y \\
 \tilde{N}_{21} & \tilde{N} \end{bmatrix} \right] \right) \tag{3.23}
\end{equation}

in which $X, Y \in \mathcal{H}$ and $\tilde{X}, \tilde{Y} \in \mathcal{H}$ are double Bezout factors of $(N, D)$ and $(\tilde{D}, \tilde{N})$ and the other entries are related as $N_{11} = \tilde{N}_{11}, N_{12} = \tilde{N}_{12}$ and $N_{21} = \tilde{N}_{21}$.

**Proof:** Without loss of generality we let $(N'_{TT}, D'_{TT})$ of (3.17) be a rcf of $P_{TT}$ and we let $(\tilde{D}'_{TT}, \tilde{N}'_{TT})$ of (3.18) be a lcf of $P_{TT}$. First we show that $P_{TT}$ is admissible if and only if there exists a rcf with $D'_{11} = I$, $L = I$.

By Proposition 3.4.7 admissibility of $P_{TT}$ is equivalent to $d_P \in J$. From (3.17) and (3.20) we have $d_P = \det D'_{11} \det L$. By Fact B.1.2.i and Fact B.1.2.ii $d_P \in J \Leftrightarrow D'_{11} \in J \land L \in J$. The definition of $Q = \text{diag}((D'_{11})^{-1}, L^{-1}) \in J$ enables the construction of $(N'_{TT} Q, D'_{TT} Q)$. By Fact 3.2.6.i this is a rcf of $P_{TT}$, and its terms $D'_{11}$ and $L$ are equal to identity.

It can be shown analogously that $P_{TT}$ is admissible if and only if it has a lcf with $\tilde{D}'_{11} = I$ and $\tilde{L} = I$. Hence $P_{TT}$ is admissible if and only if it has the following coprime factorizations:

\begin{equation}
(N'_{TT}, D'_{TT}) \doteq \left( \begin{bmatrix} N'_{11} & N'_{12} \\
 N'_{21} & N \end{bmatrix}, \left[ \begin{bmatrix} I & 0 \\
 D'_{21} & D \end{bmatrix} \right] \right) \tag{3.22}
\end{equation}

\begin{equation}
(\tilde{D}'_{TT}, \tilde{N}'_{TT}) \doteq \left( \begin{bmatrix} I & \tilde{D}'_{12} \\
 0 & \tilde{D} \end{bmatrix}, \left[ \begin{bmatrix} \tilde{N}'_{11} & \tilde{N}'_{12} \\
 \tilde{N}'_{21} & \tilde{N} \end{bmatrix} \right] \right) \tag{3.23}
\end{equation}

which overrule (3.17) and (3.18) only during the sequel of this proof.

The coprime factorizations $(N_{TT}, D_{TT})$ of (3.22) and $(\tilde{D}_{TT}, \tilde{N}_{TT})$ of (3.23) are special cases of these $(N'_{TT}, D'_{TT})$ and $(\tilde{D}'_{TT}, \tilde{N}'_{TT})$. Thus if $P_{TT}$ has either of the coprime factorizations in the theorem, then admissibility is guaranteed. It remains to be proven that the above coprime factorizations can always be transformed into the factorizations of (3.22) and (3.23). This will establish the only-if part of the theorem.
As in [162, 50] we examine the trivial equality \( \tilde{D}_{TT}'N_{TT}' = \tilde{N}_{TT}'D_{TT}' \) to disclose a relation between \( N_{11}' \) and \( D_{21} \), and between \( \tilde{N}_{12}' \) and \( \tilde{D}_{12}' \). Besides the self-evident equivalence \( \tilde{N}D = \tilde{D}N \) we have

\[
\begin{align*}
N_{11}' + \tilde{D}_{12}'N_{21}' &= \tilde{N}_{11}' + \tilde{N}_{12}'D_{21}' \\
\tilde{N}_{12}'D - \tilde{D}_{12}'N &= N_{12}' \\
\tilde{D}N_{21}' - \tilde{N}D_{21}' &= \tilde{N}_{21}'.
\end{align*}
\] (3.24) (3.25) (3.26)

We rewrite (3.25) and implicitly define \( Z \in \mathcal{H} \) as

\[
\begin{bmatrix}
N_{12}' & Z
\end{bmatrix} = \begin{bmatrix}
\tilde{N}_{12}' & -\tilde{D}_{12}'
\end{bmatrix} \begin{bmatrix}
D & -\tilde{X} \\
N & \tilde{Y}
\end{bmatrix}.
\]

By the generalized Bezout identity of Fact 3.2.6.ii this is equivalent to

\[
\begin{bmatrix}
\tilde{N}_{12}' & -\tilde{D}_{12}'
\end{bmatrix} = \begin{bmatrix}
N_{12}' & Z
\end{bmatrix} \begin{bmatrix}
Y & X \\
-\tilde{N} & \tilde{D}
\end{bmatrix}.
\]

This equation encompasses the equality of (3.25) for every \( Z \in \mathcal{H} \), so that it actually provides the set of all \( \tilde{N}_{12}', \tilde{D}_{12}' \) which satisfy (3.25). A similar approach can be used to characterize all \( N_{21}', D_{21}' \) that satisfy (3.26):

\[
\begin{bmatrix}
D_{21}' \\
N_{21}'
\end{bmatrix} = \begin{bmatrix}
D & -\tilde{X} \\
N & \tilde{Y}
\end{bmatrix} \begin{bmatrix}
\tilde{Z} \\
\tilde{N}_{21}'
\end{bmatrix}
\]

with \( \tilde{Z} \in \mathcal{H} \). Substitution of both characterizations in \( (N_{TT}', D_{TT}') \) and \( (\tilde{D}_{TT}', \tilde{N}_{TT}') \) yields

\[
\begin{bmatrix}
N_{11}' & N_{12}' \\
N\tilde{Z} + \tilde{Y} \tilde{N}_{21}' & N
\end{bmatrix} \begin{bmatrix}
I & 0 \\
D\tilde{Z} - \tilde{X} \tilde{N}_{21}' & D
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
I & -N_{12}'X - Z\tilde{D} \\
0 & \tilde{D}
\end{bmatrix} \begin{bmatrix}
\tilde{N}_{11}' & N_{12}'Y - Z\tilde{N} \\
\tilde{Y} \tilde{N}_{21}' & \tilde{N}_{21}'
\end{bmatrix}
\]

which are factorizations of the admissible \( P_{TT} \). Next application of Fact 3.2.6.i with \( Q \) set to respectively \( \begin{bmatrix} I & 0 \\ -\tilde{Z} & I \end{bmatrix} \) and \( \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} \) results in

\[
\begin{bmatrix}
N_{11}' - N_{12}'\tilde{Z} & N_{12}' \\
\tilde{Y} \tilde{N}_{21}' & N
\end{bmatrix} \begin{bmatrix}
I & 0 \\
-\tilde{X} \tilde{N}_{21}' & D
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
I & -N_{12}'X \\
0 & \tilde{D}
\end{bmatrix} \begin{bmatrix}
\tilde{N}_{11}' + Z\tilde{N}_{21}' & N_{12}'Y \\
\tilde{N}_{21}' & \tilde{N}
\end{bmatrix}
\]
The coprime factorizations of (3.22) and (3.23) follow from the definitions

\[ N_{11} = \tilde{N}_{11} - N'_{12} \tilde{Z}, \quad N_{12} = N'_{12}, \quad N_{21} = \tilde{N}'_{21} \]
\[ \tilde{N}_{11} = \tilde{N}'_{11} + Z \tilde{N}'_{21}, \quad \tilde{N}_{12} = N'_{12}, \quad \tilde{N}_{21} = \tilde{N}'_{21} \]

Thus \( N_{12} = \tilde{N}_{12} \) and \( N_{21} = \tilde{N}_{21} \) hold simply by definition. Finally, with the above characterizations of \( N'_{21}, D'_{21}, N'_{12} \) and \( \tilde{D}'_{12} \) we get

\[
\begin{bmatrix}
\tilde{N}'_{12} & -\tilde{D}'_{12}
\end{bmatrix}
\begin{bmatrix}
D'_{12} \\
N'_{12}
\end{bmatrix}
= N'_{12} \tilde{Z} + Z \tilde{N}'_{21}
\]

which by use of (3.24) can be rearranged into

\[ N'_{11} - N'_{12} \tilde{Z} = \tilde{N}'_{11} + Z \tilde{N}'_{21} \]

and hence \( N'_{11} = \tilde{N}_{11} \).

**Remark 3.4.9** Theorem 3.4.8 shows that an admissible plant \( P_{TT} \in \mathcal{F} \) can be represented by a rcf with appearingly seven degrees of freedom, cf. \( (N_{TT}, D_{TT}) \) of (3.22). Four of these degrees depend on the inner-loop plant, knowingly \( N, D, \tilde{X} \) and \( \tilde{Y} \). The parts \( P_{sw}, P_{su} \) and \( P_{yw} \) are accounted for by the three terms \( N_{11}, N_{12} \) and \( N_{21} \). Remarkably these three terms appear in both the rcf \( (N_{TT}, D_{TT}) \) and the lcf \( (\tilde{D}_{TT}, \tilde{N}_{TT}) \).

In the proof of Theorem 3.4.8 \( (N_{TT}, D_{TT}) \) surprisingly originates from a rcf with six degrees of freedom. In Appendix B.2 it is shown that the double Bezout factors \( X, Y, \tilde{X}, \tilde{Y} \) are not unique. So the pair \( \tilde{X}, \tilde{Y} \) indeed constitutes a degree of freedom for \( (N_{TT}, D_{TT}) \). However \( \tilde{X}, \tilde{Y} \) do not appear as degrees of freedom in the input-output map of \( P_{TT} \), i.e.

\[
P_{TT} = N_{TT} D_{TT}^{-1}
= \begin{bmatrix}
N_{11} + N_{12} D^{-1} \tilde{X} N_{21} & N_{12} D^{-1} \\
\tilde{Y} N_{21} + N D^{-1} \tilde{X} N_{21} & N D^{-1}
\end{bmatrix}
= \begin{bmatrix}
N_{11} + N_{12} D^{-1} (\tilde{A} + DM) N_{21} & N_{12} D^{-1} \\
(\tilde{Y} + PX) N_{21} & N D^{-1}
\end{bmatrix}
= \begin{bmatrix}
N_{11} + N_{12} M N_{21} + N_{12} D^{-1} \tilde{A} N_{21} & N_{12} D^{-1} \\
\tilde{D}^{-1} N_{21} & N D^{-1}
\end{bmatrix}
\]

(3.27)

where \( \tilde{X} = \tilde{A} + DM \) has been taken from (B.2). A variation of \( \tilde{X}, \tilde{Y} \) does not affect \( P_{yw}, P_{su} \) or \( P \). Further, in accordance with Appendix B.2 the freedom in \( \tilde{X}, \tilde{Y} \) has been represented by some fixed \( \tilde{A} \) and a free transfer function \( M \in \mathcal{H} \). Hence \( P_{sw} \) consists of a possibly unstable term \( N_{12} D^{-1} \tilde{A} N_{21} \) and the two stable terms \( N_{12} M N_{21} \) and \( N_{11} \). Since \( N_{11} \) is not constrained, the part \( N_{12} M N_{21} \) is redundant, so that \( M \)}
can be discarded, which shows that the freedom in $\tilde{X}, \tilde{Y}$ is redundant in view of the class of all admissible plants.

The class of all plants in $\mathcal{F}$ is represented by the coprime factorization $(N'_{TT}, D'_{TT})$ of (3.17), which has eight degrees of freedom. By the admissibility property this has been reduced to five degrees of freedom. This number of degrees of freedom will be reduced further to four in the parameterization of all plants that are stabilized by some compensator $C_{TT}$.

Fig. 3.6: General feedback system $H(P_{TT}, C_{TT})$ with admissible $P_{TT}$ and $C_{TT}$.

As we are interested in stable feedback systems, we focus on admissible plants and compensators. By Theorem 3.4.8 an admissible pair of a plant $P_{TT}$ and a compensator $C_{TT}$ can be represented by the rcf $(N_{TT}, D_{TT})$ of (3.22) and the lcf $(\tilde{D}_{cTT}, \tilde{N}_{cTT})$, which is defined as

$$(\tilde{D}_{cTT}, \tilde{N}_{cTT}) = \left( \begin{bmatrix} I & -\tilde{N}_{c12}X_c \\ 0 & \tilde{D}_c \end{bmatrix}, \begin{bmatrix} \tilde{N}_{c11} & \tilde{N}_{c12}Y_c \\ \tilde{N}_{c21} & \tilde{N}_c \end{bmatrix} \right)$$

(3.28)

in accordance with (3.23). A block diagram of a feedback system $H(P_{TT}, C_{TT})$ with an admissible plant and compensator is shown in Fig. 3.6. The admissibility property of $P_{TT}$ is independent from $C_{TT}$, and vice versa. Besides, all entries of $P_{TT}$ shown in Fig. 3.6 are stable except possibly for $D^{-1}$. Hence all instabilities of $P_{TT}$ are contained in $P$, which is equivalent to $d_P \in \mathcal{J}$ according to Proposition 3.4.7. Thus the stability condition of Theorem 3.4.4 can be simplified for admissible plants and compensators.

**Corollary 3.4.10** Let Assumption 3.3.3 hold. Further let $(N_{TT}, D_{TT})$ of (3.22) be a rcf of $P_{TT} \in \mathcal{F}$ and let $(\tilde{D}_{cTT}, \tilde{N}_{cTT})$ of (3.28) be a lcf of $C_{TT} \in \mathcal{F}$. Then the feedback system $H(P_{TT}, C_{TT})$ is stable if and only if $\Lambda \in \mathcal{J}$, with $\Lambda$ as defined in (3.3).
The \((R, S)\)-parameterization

Corollary 3.4.10 states that stability of \(H(P_{TT}, C_{TT})\) is equivalent to stability of its inner-loop feedback system \(H(P, C)\), provided that \(P_{TT}\) and \(C_{TT}\) are admissible. The latter condition does not incur severe restrictions, since non-admissible plants or compensators never make a stable feedback system in the configuration of \(H(P_{TT}, C_{TT})\). Hence we can derive the following necessary and sufficient condition for a plant \(P_{TT}\) to be stabilized by the admissible compensator \(C_{TT}\). This result is a straightforward extension of Theorem 3.3.6.

**Theorem 3.4.11** Let Assumption 3.3.3 and Assumption 3.3.5 hold. Then the plant \(P_{TT} \in \mathcal{F}\) is stabilized by the admissible compensator \(C_{TT}\) in the feedback configuration \(H(P_{TT}, C_{TT})\) of Fig. 3.5 if and only if \(P_{TT}\) has a rcf \((N_{TT}^a, D_{TT}^a)\) defined as

\[
(N_{TT}^a, D_{TT}^a) = \begin{bmatrix}
S_{11} & S_{12} \\
D_c S & N^a \\
\end{bmatrix},
\begin{bmatrix}
I & 0 \\
-N_c S & D^a \\
\end{bmatrix}
\]  

(3.29)

with \(S_{11}, S_{12}, S \in \mathcal{H}\) and \((N^a, D^a)\) as in (3.6). The rcf \((N_{TT}^a, D_{TT}^a)\) is called associated to \((N_o, D_o)\) and \(C\). \((N_{TT}^a, D_{TT}^a)\) is uniquely determined by \(P_{TT}, C\) and \((N_o, D_o)\).

**Proof:**

\((\Leftarrow)\) Similar to the proof of Theorem 3.3.6 we first show that \((N_{TT}^a, D_{TT}^a)\) is a right coprime factorization. From Assumption 3.3.5 and Lemma 3.3.4 we know that \(\Lambda_o\) of (3.5) belongs to \(\mathcal{J}\) and thus \(\Lambda_o^{-1} \in \mathcal{H}\). So \(X_{TT}^a, Y_{TT}^a\), defined as

\[
X_{TT}^a = \begin{bmatrix}
0 & 0 \\
0 & \Lambda_o^{-1} \tilde{N}_c \\
\end{bmatrix},
Y_{TT}^a = \begin{bmatrix}
I & 0 \\
0 & \Lambda_o^{-1} \tilde{D}_c \\
\end{bmatrix}
\]

belong to \(\mathcal{H}\). By Fact 3.2.4.i the identity \(X_{TT}^a N_{TT}^a + Y_{TT}^a D_{TT}^a = I\) demonstrates the coprimeness of \((N_{TT}^a, D_{TT}^a)\). Further, since \((N^a, D^a)\) of (3.6) is a rcf, \(D^a\) has an inverse. Hence \(D_{TT}^a\) of (3.29) has an inverse, which makes \((N_{TT}^a, D_{TT}^a)\) a rcf of the plant \(N_{TT}^a (D_{TT}^a)^{-1}\).

Next we show that this plant \(N_{TT}^a (D_{TT}^a)^{-1}\) is admissible. As \((N^a, D^a)\) satisfies (3.6), the inner-loop feedback system \(H(P, C)\) is stable by virtue of Theorem 3.3.6. We define \(A = \tilde{A} S \in \mathcal{H}\) and we substitute \(\tilde{A}^{-1} A\) for \(S\) in (3.29). Thereby \((N_{TT}^a, D_{TT}^a)\) becomes a special case of \((N_{TT}, D_{TT})\) in (3.22): the particular \(\tilde{X}, \tilde{Y}, N_{11}, N_{12}, N_{21}, (N, D)\) are respectively \(N_c \tilde{A}^{-1}, D_c \tilde{A}^{-1}, S_{11}, S_{12}, A, (N^a, D^a)\). Hence by Theorem 3.4.8 \(N_{TT}^a (D_{TT}^a)^{-1}\) is admissible.

Conclusively, if \(P_{TT}\) has a rcf \((N_{TT}^a, D_{TT}^a)\) as in (3.29) then \(P_{TT}\) is admissible. As \(C_{TT}\) is also admissible, the stability of \(H(P, C)\), which follows from \(P = N^a (D^a)^{-1}\) and Theorem 3.3.6, guarantees the stability of \(H(P_{TT}, C_{TT})\) by Corollary 3.4.10.

\((\Rightarrow)\) By Theorem 3.4.4 stability of the feedback system \(H(P_{TT}, C_{TT})\) implies that \(d_P \in \mathcal{J}\) and \(\Lambda \in \mathcal{J}\). From Proposition 3.4.7 we have the implication \(d_P \in \mathcal{J} \Rightarrow P_{TT}\) is
admissible. Hence by Theorem 3.4.8 $P_{TT}$ has a rcf $(N_{TT}, D_{TT})$ as in (3.22). Meanwhile from Lemma 3.3.4 we know that if $\lambda \in J$ then also $\tilde{\lambda} \in J$. Thus $\tilde{N} N_c \tilde{A}^{-1} + \tilde{D} D_c \tilde{A}^{-1} = I$ and $A, B \in H$ defined as $A \equiv N_c \tilde{A}^{-1}$ and $B \equiv D_c \tilde{A}^{-1}$ are left Bezout factors of $(\tilde{D}, \tilde{N})$. Further by Theorem 3.3.6 $P$ has a rcf $(N^a, D^a)$ as in (3.6). Since the rcf $(N_{TT}, D_{TT})$ of (3.22) has been based on an arbitrary rcf of the inner-loop part $P$ of the admissible $P_{TT}$, we may substitute $(N^a, D^a)$ for $(N, D)$ in (3.22). With the additional substitutions of $A$ and $B$ for $\tilde{X}$ and $\tilde{Y}$, the rcf of $P_{TT}$ becomes

$$
\begin{bmatrix}
    N_{11} & N_{12} \\
    AN_{21} & N^a
\end{bmatrix}
\begin{bmatrix}
    I & 0 \\
    -BN_{21} & D^a
\end{bmatrix}
$$

By defining $S_{11} = N_{11}$, $S_{12} = N_{12}$, $S = \tilde{A}^{-1} N_{21} \in H$ we arrive at the rcf $(N_{TT}^a, D_{TT}^a)$ of (3.29). Thus stability of $H(P_{TT}, \mathcal{C}_{TT})$ implies that $P_{TT}$ has a rcf $(N_{TT}^a, D_{TT}^a)$.

From Theorem 3.3.6 we know that $(N^a, D^a)$ is uniquely determined by $(N_o, D_o)$ and $C$ (see (3.10)). Further we expand $P_{TT} = N_{TT}^a (D_{TT}^a)^{-1}$ into

$$
P_{TT} = \begin{bmatrix}
    S_{11} + S_{12} (D^a)^{-1} N_c S & S_{12} (D^a)^{-1} \\
    (D_c + N^a (D^a)^{-1} N_c) S & N^a (D^a)^{-1}
\end{bmatrix}
$$

from which we derive

$$
S_{12} = P_{zu} D^a \\
S_{11} = P_{zw} - P_{zu} C (I + PC)^{-1} P_{yw}
$$

This demonstrates the dependency of $S_{12}$ and $S_{11}$ on $(N_o, D_o)$ and $C$. We also have

$$
S = D_c^{-1} (I + PC)^{-1} P_{yw}
$$

(3.30)

showing that $S$ is completely determined by $(N_c, D_c)$. Finally since $S$ is premultiplied by $D_c$ and $N_c$ in respectively $N_{TT}^a$ and $D_{TT}^a$, the rcf $(N_{TT}^a, D_{TT}^a)$ is uniquely determined by $(N_o, D_o)$ and $C$.

The rcf $(N_{TT}^a, D_{TT}^a)$ has practically only four degrees of freedom, because $N^a$ and $D^a$ can be written as in (3.6) so that only the terms $R$, $S$, $S_{11}$ and $S_{12}$ depend on the plant $P$.

Like in the previous section the stability condition of Theorem 3.4.11 can be used to parameterize the set $\mathcal{P}_{TT}(C_{TT})$ of all plants that are stabilized by $C_{TT}$.

**Corollary 3.4.12** Let $C \in \mathcal{F}$ be the inner-loop part of the admissible compensator $C_{TT} \in \mathcal{F}$ and let Assumption 3.3.5 hold. Then the set $\mathcal{P}_{TT}(C_{TT})$ of all plants, that are stabilized by $C_{TT}$ in the general feedback configuration of Fig. 3.5, is given by

$$
\mathcal{P}_{TT}(C_{TT}) = \{ \begin{bmatrix}
    S_{11} & S_{12} \\
    D_c S_h & N_o + D_c R_h
\end{bmatrix}
\begin{bmatrix}
    I & 0 \\
    -N_c S_h & D_o - N_c R_h
\end{bmatrix}
\begin{bmatrix}
    I & 0 \\
    -N_c S_h & D_o - N_c R_h
\end{bmatrix}^{-1}
\mid R_h, S_h, S_{11}, S_{12} \in H, \ det(D_o - N_c R_h) \neq 0\}.
$$
This is called the \((R,S)\)-parameterization\(^3\) of the set \(\mathcal{P}_{TT}(C_{TT})\).

**Remark 3.4.13** Unlike the results in [162, 50] our parameterization of \(\mathcal{P}_{TT}(C_{TT})\) allows also inner-loop parts that are not strictly proper. This parameterization could not have been obtained by a simple redefinition of the rings \(\mathcal{G}\) and \(\mathcal{I}\) used in the above references. For in that case \(\mathcal{G} \equiv \mathcal{F}\), the multiplicative subset \(\mathcal{I}\) contains all units in \(\mathcal{F}\) meaning that \(\mathcal{I} = \mathcal{F} \setminus \{0\}\), and the Jacobson radical of \(\mathcal{G}\) becomes \(\mathcal{G}_s = \{x \in \mathcal{F} \mid (1+xy)^{-1} \in \mathcal{F}, \forall y \in \mathcal{F}\} = \{0\}\). The latter implies that a redefinition of \(\mathcal{G}\) and \(\mathcal{I}\) would result in a parameterization of the set of plants, that are "stabilized" by a compensator \(C_{TT}\) with \(C = 0\). This is precisely the set of stable plants. \(\square\)

![Block diagram](image)

**Fig. 3.7:** Double appearance of \(N_c, D_c\) in the representation of \(P\) and \(P_{yw}\) due to the \(R\)-parameterization of \(P\) and proper choices of \(\tilde{X}\) and \(\tilde{Y}\).

With the substitution of \((N^a, D^a)\) for \((N, D)\) we can incorporate the block-diagram of the \(R\)-parameterization of Fig. 3.4 in the representation of \(H(P_{TT}, C_{TT})\) in Fig. 3.6. The simultaneous substitution of \(D_s\) and \(N_s\) for \(\tilde{Y} N_{21}\) and \(\tilde{X} N_{21}\) results in a double appearance of \((N_c, D_c)\) as shown in Fig. 3.7. One of these pairs \((N_c, D_c)\) can be removed by taking the sum of \(R_x\) and \(Sw\) before these signals enter \(N_c\) and \(D_c\); we define the signal

\[
g = Rx + Sw
\]

which will prove useful in the next chapter. Now we can draw a block diagram of the stable general feedback system \(H(P_{TT}, C_{TT})\) in terms of the \((R,S)\)-parameterization as depicted in Fig. 3.8. Like in the block diagram of the \(R\)-parameterization in Fig. 3.4 the transfer functions \((D^a)^{-1}\) and \(N^a\) consist of a feedback over the auxiliary factor \(D_o^{-1}\) and a parallel connection along \(N_o\). So again the inner-loop auxiliary feedback system \(H(P_o, C)\) converges to the actual inner-loop feedback system \(H(P, C)\) if \(R\) tends to 0.

---

\(^3\)This designation was introduced in [100], where a similar parameterization based on a left coprime factorization was used to conduct experiment design for exact identification.
3.4 Stability of the General Feedback System

![Block diagram of a stable general feedback system](image)

Fig. 3.8: Block diagram of a stable general feedback system $H(P_{TT}, C_{TT})$ with $(R, S)$-parameterization of $P_{TT}$.

The transfer function $T(P_{TT}, C_{TT})$, which maps the inputs col$(w_c, w, r_2, r_1)$ of the feedback system into its signals col$(z_c, z, y, u)$, satisfies

$$T(P_{TT}, C_{TT}) = \begin{bmatrix}
C_{zw} - C_{zu} P(I+CP)^{-1} C_{yw} & -C_{zu} (I + PC)^{-1} P_{yw} \\
0 & P_{zw} (I+CP)^{-1} C_{yw} \\
P(I+CP)^{-1} C_{yw} & (I+PC)^{-1} P_{yw} \\
(I+CP)^{-1} C_{yw} & -(I+CP)^{-1} C_{yw} \\
0 & P_{zu} (I+CP)^{-1} \\
C_{zu} (I+PC)^{-1} & -C_{zu} P(I+CP)^{-1} \\
P_{zu} (I+CP)^{-1} & P(I+CP)^{-1} \\
(I+CP)^{-1} & (I+CP)^{-1}
\end{bmatrix}$$

(3.32)

Similar to $T(P, C)$ of (3.4) we express $T(P_{TT}, C_{TT})$ in terms of the associated rcf $(N^a_{TT}, D^a_{TT})$ of (3.29) and the lcf $(\hat{D}_{cTT}, \hat{N}_{cTT})$ of (3.28):

$$T(P_{TT}, C_{TT}) = \begin{bmatrix}
C_{zw} & -C_{zu} P_{yw} & C_{zu} & 0 \\
0 & P_{zw} & 0 & 0 \\
0 & P_{yw} & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
-C_{zu} P \\
P_{zu} \\
P \\
I
\end{bmatrix} (I+CP)^{-1} \begin{bmatrix}
C_{yw} & -CP_{yw} & C & I
\end{bmatrix}
\begin{bmatrix}
C_{zw} & -C_{zu} D_{c} S & C_{zu} & 0 \\
0 & S_{11} & 0 & 0 \\
0 & D_{c} S & 0 & 0 \\
0 & -N_{c} S & 0 & 0
\end{bmatrix} + \begin{bmatrix}
-C_{zu} N^a \\
S_{12} \\
N^a \\
D^a
\end{bmatrix} \Lambda_{\alpha}^{-1} \begin{bmatrix}
\hat{D}_{c} C_{yw} & 0 & \hat{N}_{c} & \hat{D}_{c}
\end{bmatrix}.$$

(3.33)
Apparently \( T(P_{TT}, C_{TT}) \) is the sum of two matrices, the first of which has entries linear in \( S \) and \( S_{11} \), and the rows of the second matrix are linear (or affine) in \( N^a, D^a \) or \( S_{12} \). Finally from (3.33) it becomes clear that a good approximation of \( N^a, D^a, S, S_{11} \) and \( S_{12} \) provides a good approximation of \( T(P_{TT}, C_{TT}) \).

### 3.5 Stability of The Standard Feedback System

The standard feedback system \( H(P_T, C) \) of Fig. 3.9 is the simplest feedback configuration, that combines typical problems encountered in identification and feedback control. For ease of reference we adopt the results of the previous section to the special case \( H(P_T, C) \). To this end we consider only the parts \( C \) and \( P_T = [P_{yw} \ P] \) of \( C_{TT} \) respectively \( P_{TT} \).

![Fig. 3.9: Standard feedback system \( H(P_T, C) \).](image)

**Theorem 3.5.1** Let Assumption 3.3.5 and Assumption 3.3.5 hold. Then the plant \( P_T \in \mathcal{F} \) is stabilized by the compensator \( C \) in the standard feedback configuration \( H(P_T, C) \) of Fig. 3.9 if and only if \( P_T \) has a rcf \((N_T^a, D_T^a)\) defined as

\[
(N_T^a, D_T^a) = \left( \begin{array}{cc} D_c S & N^a \\ \frac{I}{-N_c S} & D^a \end{array} \right)
\]

with \( S \in \mathcal{H} \) and \((N_a, D_a)\) as in (3.6). The rcf \((N_T^a, D_T^a)\) is called associated to \((N_o, D_o)\) and \( C \). \((N_T^a, D_T^a)\) is uniquely determined by \( P_T, C \) and \((N_o, D_o)\); \( R \) and \( S \) are uniquely determined by \( P_T, (N_c, D_c) \) and \((N_o, D_o)\).

The stability condition is employed to parameterize the set \( \mathcal{P}_T(C) \) of all plants that are stabilized by \( C \). We designate this parameterization as the \((R, S)\)-parameterization like in Corollary 3.4.12.

**Corollary 3.5.2** Let Assumption 3.3.5 hold. Then the set \( \mathcal{P}_T(C) \) of all plants in \( \mathcal{F} \), that are stabilized by \( C \) in the standard feedback configuration of Fig. 3.9, is given by

\[
\mathcal{P}_T(C) = \{ \left[ \begin{array}{cc} D_c S_H & N_o + D_c R_H \\ -N_c S_H & D_o - N_c R_H \end{array} \right]^{-1} \mid R_H, S_H \in \mathcal{H}, \det(D_o - N_c R_H) \neq 0 \},
\]

which is called the \((R, S)\)-parameterization of the set \( \mathcal{P}_T(C) \).
For completeness we point out that if $H(P_T, C)$ is known to be stable, then we may represent the plant $P_T$ by its associated rcf $(N_T^2, D_T^2)$. Further, a block diagram of the stable standard feedback system $H(P_T, C)$ can be drawn as in (3.10). $(R, S)$-parameterization of the stabilized plant $P_T$ is used for identification in the next chapter.
Chapter 4

Open-loop Identification of Plants Under Feedback

In this chapter we build a framework that enables an open-loop identification of the inner-loop plant $P$, while it operates under feedback by a known stabilizing compensator. With this framework $P$ can be identified in such a way, that its asymptotic estimate is independent of the disturbances and noises that affect the feedback system. As alluded to in Section 2.3.4 we will identify $P$ by estimating its right coprime factors. More precise, we use the $(R, S)$-parameterization of the previous chapter, and the resulting framework is designed for the identification of the associated rcf $(N^a, D^a)$.

Reverting to (2.20) we need an “instrumental variable” for the identification of $(N^a, D^a)$. This variable, called the intermediate, is constructed in Section 4.1. Then in Sections 4.2 and 4.3 we combine the $(R, S)$-parameterization with the intermediate to recast the standard closed-loop identification problem into two alternative open-loop identification problems. All existing open-loop results apply to the latter problems, so that they pave the way to a comprehensive and manageable approximate identification of feedback controlled plants. At the end of the chapter we provide some justification and a summary of merits and drawbacks of our approach.

4.1 Reconstruction of the Intermediate

Our basic assumption is that the plant operates under a known stabilizing feedback compensator. For the general case $H(P_{TT}, C_{TT})$ this implies, according to Theorem 3.4.11, that the plant $P_{TT}$ with uncertain dynamics can be represented by the $(R, S)$-parameterization of Fig. 3.8. In this block-diagram we indicated the variable $x$ right in between the auxiliary $D_0^{-1}$ and $N_0$. We call this $x$ the intermediate, and here we show how it can be derived from $u$ and $y$. No precise knowledge of the plant $P_{TT}$ nor any knowledge of its disturbance input $w$ is needed for this reconstruction of $x$.

We begin with an investigation into the more general cas of Fig. 3.6, which rep-
represents the interconnection between an admissible plant \( P_{TT} \) and an admissible compensator \( C_{TT} \). Notice that admissibility of \( P_{TT} \) is a weaker property than stability of \( H(P_{TT}, C_{TT}) \). Here we see the variable \( x_a \) in between \( D^{-1} \) and \( N \). We call this \( x_a \) the admissible intermediate. The intermediate \( x \) coincides with \( x_a \) for the special case that \( (N, D) \) equals the rcf \((N^a, D^a)\) associated to \( C \) and \((N_o, D_o)\). We examine the relation between \( u, y \) and \( x_a \) to disclose the following fact: \( x_a \) can be reconstructed from \( u \) and \( y \) without precise knowledge of the plant only if \( (N, D) \) is an associated rcf \((N^a, D^a)\).

With the admissible rcf \((N_{TT}, D_{TT})\) of (3.22) we can express the input-output relationships of \( P_{TT} \) as

\[
\begin{pmatrix}
  w \\
  u \\
  z \\
  y
\end{pmatrix} =
\begin{pmatrix}
  D_{TT} & N_{TT}
\end{pmatrix}
\begin{pmatrix}
  w \\
  x_a
\end{pmatrix} =
\begin{bmatrix}
  I & 0 \\
  -\bar{X}N_{21} & D
\end{bmatrix}
\begin{pmatrix}
  w \\
  x_a
\end{pmatrix}
\]

\[
(4.1)
\]

which is depicted in Fig. 3.6. In here \( (N, D) \) may be any particular rcf of \( P \). With these expressions we can relate \( x_a \) to the inner-loop plant input \( u \) and to the output \( y \) as stated below.

**Lemma 4.1.1** Let Assumption 3.3.3 hold. Then the admissible intermediate \( x_a \) appearing in Fig. 3.6 satisfies

\[
x_a = Yu + Xy,
\]

\[
(4.2)
\]

provided that the Bezout factors \( X, Y \) are the unique double Bezout factors associated with \( (N, D) \) and \( \bar{X}, \bar{Y} \).

**Proof:** The expressions for \( u \) and \( y \) from (4.1) can be rearranged to

\[
\begin{pmatrix}
  u \\
  y
\end{pmatrix} =
\begin{bmatrix}
  -\bar{X}N_{21} & D \\
  \bar{Y}N_{21} & N
\end{bmatrix}
\begin{pmatrix}
  w \\
  x_a
\end{pmatrix} =
\begin{bmatrix}
  D & -\bar{X} \\
  N & \bar{Y}
\end{bmatrix}
\begin{pmatrix}
  x_a \\
  N_{21}w
\end{pmatrix}.
\]

By Fact 3.2.6(ii) the two-by-two block matrix on the right hand side has an unique inverse. This defines \( X, Y \). Premultiplying both sides by this inverse yields

\[
\begin{pmatrix}
  x_a \\
  N_{21}w
\end{pmatrix} =
\begin{bmatrix}
  Y & X \\
  -\bar{N} & \bar{D}
\end{bmatrix}
\begin{pmatrix}
  u \\
  y
\end{pmatrix}.
\]

The result (4.2) is just the upper part thereof. \( \qed \)

This lemma implies that the (double) Bezout factors \( X, Y \) are needed to derive \( x_a \) from \( u \) and \( y \). Since \( X, Y \) depend on the unknown rcf \((N, D)\), they are generally not available. Fortunately there is one exception, namely the special case in which
4.1 Reconstruction of the Intermediate

$P_{TT}$ is represented by its rcf $(N^a_{TT}, D^a_{TT})$ of (3.29) associated to $(N_o, D_o)$ and $C$. In this representation $(N, D)$ is replaced with $(N^a, D^a)$, which by (3.7) has $\Lambda_o^{-1} \tilde{N}_c = (D_o + CN_o)^{-1} C$ and $\Lambda_o^{-1} \tilde{D}_c = (D_o + CN_o)^{-1}$ as Bezout factors. Thus the Bezout factors of the associated rcf $(N^a, D^a)$ are uniquely determined by the inner-loop compensator $C$ and the auxiliary rcf $(N_o, D_o)$. The compensator $C$ is known so that we can choose an auxiliary rcf $(N_o, D_o)$. Consequently, the Bezout factors are at our disposal, although the rcf $(N^a, D^a)$ is unknown.

Henceforth we confine the discussion to the associated rcf $(N^a_{TT}, D^a_{TT})$, because this is the only admissible rcf that enables a reconstruction of the intermediate $x$ without precise knowledge of the plant. In terms of $(N^a_{TT}, D^a_{TT})$ the input-output relations are

\[
\begin{pmatrix}
w \\
u
\end{pmatrix} = \begin{bmatrix}
I & 0 \\
-N_o S & D^a
\end{bmatrix}
\begin{pmatrix}
x
\end{pmatrix}
\]

\[
\begin{pmatrix}
z \\
y
\end{pmatrix} = \begin{bmatrix}
S_{11} & S_{12} \\
D_o S & N^a
\end{bmatrix}
\begin{pmatrix}
w \\
x
\end{pmatrix}
\]  

(4.3)

in which $x$ is the intermediate of Fig. 3.8. From these equations we can reconstruct the intermediate as follows.

**Proposition 4.1.2** Let Assumptions 3.3.3 and 3.3.5 hold. Let the plant $P_{TT} \in \mathcal{F}$ be stabilized by the admissible compensator $C_{TT} \in \mathcal{F}$. Then the intermediate $x$ appearing in the $(R, S)$-parameterization of Fig. 3.8 can be reconstructed from $u$ and $y$ by the stable filter operation

\[
x = \Lambda_o^{-1} (\tilde{D}_c u + \tilde{N}_c y)
\]

(4.4)

irrespective of $P_{TT}$.

**Proof:** Follows from substitution of $\Lambda_o^{-1} \tilde{N}_c, \Lambda_o^{-1} \tilde{D}_c$ for $X, Y$ in Lemma 4.1.1. \qed

**Remark 4.1.3** The filters that produce $x$ in (4.4) do not depend on $P_{TT}$. Hence they can be used even if $H(P_{TT}, C_{TT})$ is not stable. However, then we may no longer speak of an $(R, S)$-parameterization as defined in Corollary 3.4.12, and in that case $u$ and $y$ grow without bounds. This matter is elaborated further in Appendix E.2. \qed

The fact that the intermediate $x$ can be determined without exact knowledge of $P_{TT}$ attests to the practical utility of our approach. We will demonstrate another striking property of $x$, namely that $x$ is not related to the disturbance plant input $w$ despite the feedback loop.

**Assumption 4.1.4** The disturbance plant input $w$ is uncorrelated with all other inputs to the feedback system.

In the case of $H(P_{TT}, C_{TT})$ of Fig. 3.5 this assumption says that $w$ is uncorrelated with $r_1, r_2$ and $w_c$. 

Lemma 4.1.5 Let Assumption 4.1.4 and the conditions of Proposition 4.1.2 hold. Then the intermediate $x$ of (4.4) is uncorrelated with the disturbance plant input $w$.

**Proof:** The bottom two rows of $T(P_{TT}, C_{TT})$ in (3.33) map $\text{col}(w_c, w, r_2, r_1)$ into $\text{col}(y, u)$. From (4.4) we know that $\Lambda_o^{-1}[\tilde{N}_c \; \tilde{D}_c]$ maps $\text{col}(y, u)$ into $x$. Hence pre-multiplication of the former rows of $T(P_{TT}, C_{TT})$ by the latter term gives the transfer function from $\text{col}(w_c, w, r_2, r_1)$ to $x$:

$$
x = \Lambda_o^{-1}[\tilde{D}_c C_{yw} \; 0 \; \tilde{N}_c \; \tilde{D}_c] \cdot \text{col}(w_c, w, r_2, r_1)
= \Lambda_o^{-1}(\tilde{D}_c C_{yw}w_c + \tilde{N}_c r_2 + \tilde{D}_c r_1).
$$

(4.5)

Consequently $x$ is uncorrelated with each signal, that is uncorrelated with each of the three variables $r_1$, $r_2$ and $w_c$. 

Proposition 4.1.2 established that $x$ can be reconstructed without precise knowledge of the plant $P_{TT}$. The expression for $x$ in (4.5) even goes further: the intermediate $x$ does not depend on the plant $P_{TT}$ nor on the disturbance $w$. Thus the intermediate $x$ is independent from the factors $R$ and $S$ and from the disturbance $w$, although it appears right amidst the plant’s associated factors. In order to comprehend this phenomenon we study variable $q$ of (3.31). This variable $q$ appears in the $(R, S)$-parameterizations of $H(P_{TT}, C_{TT})$ in Fig. 3.8 and of $H(P_T, C)$ in Fig. 3.10. We examine $q$ in the latter feedback configuration in order to simplify notation.

We redraw the $(R, S)$-parameterization of Fig. 3.10 as in Fig. 4.1. The feedback of $N_c q$ over $D_o^{-1}$ of Fig. 3.10 has been replaced by the equivalent feedback $(N_c D_c^{-1}) D_c q = CD_c q$. This block-diagram discloses that the contribution $q' = D_c q$ is fed back twice.

![Block-diagram](image-url)

**Fig. 4.1:** Cancellation of contributions $Rx$ and $Sw$ right before $u$ enters the nominal model.
to the summing junction of $\alpha$: once via $+C$ within the plant representation, and once via $-C$ by the actual feedback path. Hence the contribution $q$ cancels out right before it would enter the auxiliary model factor $\tilde{D}_o^{-1}$, so that it cannot affect $x$. Hence the disturbance contribution $Sw$ and the “difference” between $P$ and $P_0$ embodied by $Rx$ cannot reach $x$.

The variable $q$ will be needed in Section 4.3. The following result states that $q$ can be reconstructed from $u$ and $y$ in a way analogous to the reconstruction of $x$.

**Proposition 4.1.6** Let Assumptions 3.3.3 and 3.3.5 hold. Let the plant $P_{TT} \in \mathcal{F}$ be stabilized by the admissible compensator $C_{TT} \in \mathcal{F}$. Then the variable $q$ appearing in the $(R, S)$-parameterization of Fig. 3.8 can be reconstructed from $u$ and $y$ by the stable filter operation

$$q = \tilde{\Lambda}_o^{-1}(\tilde{D}_o y - \tilde{N}_o u) \quad (4.6)$$

irrespective of $P_{TT}$.

**Proof:** We combine the expressions for $u$ and $y$ of (4.3) and simultaneously we replace $(N^a, D^a)$ by their equivalents of (3.6). This yields

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{bmatrix} \tilde{D}_o - N_c R & -N_c S \\ N_o + D_c R & D_c S \end{bmatrix} \begin{pmatrix} x \\ w \end{pmatrix}.$$  

Premultiplying both sides by $[-\tilde{N}_o \ \tilde{D}_o]$ results in

$$\tilde{D}_o y - \tilde{N}_o u = (\tilde{D}_o D_c + \tilde{N}_o N_c)(Rx + Sw).$$

Using (3.5) and (3.31) we obtain (4.6).

---

Fig. 4.2: Composition of $y$ and $u$ from $x$ and $q$. 
Unlike the intermediate $x$ the variable $q$ depends on the deficiency of the auxiliary model ($R \neq 0$) and on the disturbance contribution $Sw$. Thus a change of $Sw$ or $R$ will lead to a change of $q$, but not to a change of $x$. Nevertheless $R$ and $Sw$ are not needed for the reconstruction of $q$ from $u$ and $y$. This can be understood as follows. We modify the expression for $\text{col}(u, y)$ used in the above proof into

$$
\begin{pmatrix}
    u \\
    y
\end{pmatrix} =
\begin{bmatrix}
    D_o & -N_c \\
    N_o & D_c
\end{bmatrix}
\begin{pmatrix}
    x \\
    q
\end{pmatrix}.
$$

These relationships are depicted in Fig. 4.2 together with the variables that drive $x$ and $q$. This block-diagram reveals why $q$ depends, and why $x$ does not depend on $Sw$ and $R$. Finally, by (3.5) the map $\text{col}(x, q) \mapsto \text{col}(u, y)$ has an inverse, which is at our disposal.

### 4.2 The Equivalent Identification Problem

Here we use the reconstructed intermediate $x$ for the identification of the associated rcf $(N_o^a, D_o^a)$ of the inner-loop plant $P$. We treat the three feedback configurations of Chapter 3 in reversed order. First we derive the main result from the standard feedback system $H(P_T, C)$. Thereafter we show that the general case of $H(P_{TT}, C_{TT})$ requires only a minor extension. We comment on the single-variate feedback configuration $H(P, C)$ at the end of the section. Before proceeding we emphasize that we focus on the identification of the inner-loop plant $P$. The identification of $P_{yw}$ and $Sw$ is treated in the next section.

**Proposition 4.2.1** Let Assumption 4.1.4 hold. Let the plant $P_T \in \mathcal{F}$ be stabilized by $C \in \mathcal{F}$ in the standard feedback configuration $H(P_T, C)$ of Fig. 3.9. Then the following procedure enables the identification of a nominal model $\hat{P}$ of the inner-loop plant $P$ from $u, y$ and $C$, such that $\hat{P}$ is asymptotically independent of the disturbance $w$.

1. Select an auxiliary model $P_o$ with rcf $(N_o, D_o)$ such that $H(P_o, C)$ is stable.
2. Select a rcf $(N_c, D_c)$ and lcf $(\hat{D}_c, \hat{N}_c)$ of $C$.
3. Reconstruct the intermediate $x$ as in (4.4).
4. Use an identification technique to estimate models $\hat{N}, \hat{D}$ of $N_o^a, D_o^a$ from

$$
\begin{pmatrix}
    u \\
    y
\end{pmatrix} =
\begin{bmatrix}
    D_o^a & -N_c^a \\
    N_o^a & D_c^a
\end{bmatrix}
\begin{pmatrix}
    x \\
    q
\end{pmatrix} +
\begin{bmatrix}
    -N_c \\
    D_c
\end{bmatrix} Sw
$$

by considering $x, \text{col}(u, y)$ and $w$ respectively as input, output and disturbance. The open-loop identification of (4.7) is the equivalent identification problem.
4.2 The Equivalent Identification Problem

v. Construct an estimate \( \hat{P} \) of \( P \) from the estimates \( \hat{N} \) and \( \hat{D} \) to

\[
\hat{P} = \hat{N} \hat{D}^{-1}.
\]

(4.8)

This procedure is called the framework for open-loop identification of feedback controlled plants.

**Proof:** Steps i. and ii. are evident from Section 3.3 and Definition 3.2.1. By Theorem 3.5.1 \( P_T \) has an associated rcf \((N^a, D^a)\), which is unique since \((N_o, D_o)\) and \(C\) are fixed at this stage. There is no correlation between the input \( x \) and the additive disturbances \(-N_c Sw\) and \( D_c Sw\). Therefore the identification \( N^a \) and \( D^a \) is a typical open-loop identification problem, and thus asymptotic estimates \( \hat{N} \) and \( \hat{D} \) can be derived independently from \( w \) (see also Section 2.3). In that case \( \hat{P} \) constructed in step v. is also asymptotically independent of \( w \).

---

![Diagram](image)

**Fig. 4.3:** Block diagram of the equivalent open-loop identification problem.

The identification problem of (4.7) is equivalent to the identification of \( P \) in the sense that a nominal model \( \hat{P} \) can be derived directly from the estimates \((\hat{N}, \hat{D})\) of \((N^a, D^a)\).

Since the intermediate \( x \) is the input to the unknown factors \( N^a \) and \( D^a \), experiment design consists in manipulating \( x \). An experiment can readily be designed for the equivalent identification problem of (4.7), provided that the exogenous inputs \( r_1 \) and \( r_2 \) are at our disposal. This may be clear from the block-diagram of Fig. 4.3, which combines the relations of (4.7) and (4.5) with omission of \( w_e \).

**Proposition 4.2.2** Let Assumptions 3.3.3 and 3.3.5 hold. Let the stabilized plant \( P_T \in \mathcal{F} \) be represented by the \((R, S)\)-parameterization of Fig. 3.10. Then the intermediate \( x \) equals \( x_d \) if \( r_1 \) and \( r_2 \) are chosen as \( r_1 = D_ox_d \) and \( r_2 = N_ox_d \).
Proof: By (3.5) \( \Lambda_o^{-1}(D_c D_o + \tilde{N}_c N_o) = I \). Thus by (4.5) and with \( w_c \equiv 0 \) we obtain \( x = x_d \).

**Remark 4.2.3** The aim of experiment design is to maximize the amount of useful information in the data. From the above proposition it is clear that there is no information in \( x \) if \( r_1 = 0 \) and \( r_2 = 0 \). So the equivalent identification of Proposition 4.2.1 requires that \( r_1 \) and \( r_2 \) together are sufficiently informative.

The remainder of the section concerns the closed-loop identification for the configurations \( H(P_{TT}, C_{TT}) \) and \( H(P, C) \). In order to see that the identification of the plant \( P_{TT} \) in \( H(P_{TT}, C_{TT}) \) requires only a slight extension of the framework of Proposition 4.2.1 we reconsider the input-output relations of \( P_{TT} \) in (4.3). We may omit the top row in the first equivalence in (4.3) since it is trivial. A rearrangement of the remaining three equations yields

\[
\begin{align*}
  u &= D^a x - N_c S w \\
  y &= N^a x + D_c S w \\
  z &= S_{12} x + S_{11} w
\end{align*}
\]  

which obviously is an extension of (4.7). Since \( S_{12} \) and \( S_{11} \) do not appear in the top two equations, we can identify the inner-loop plant \( P \) by the procedure of Proposition 4.2.1. The identification of \( S_{12} \) and \( S_{11} w \) from the bottom equation constitutes a completely individual open-loop identification problem. Hence in view of typical closed-loop problems the identification of \( P_{TT} \) is of equal complexity as the identification of \( P \).

Finally, the single-variate feedback system lacks the disturbance \( w \). The identification of \( P \) has accordingly not the typical closed-loop character: there is no correlation between an input and a disturbance. So less precautions are necessary to identify \( P \). Nevertheless the approach of Proposition 4.2.1 can still be used to carry out the approximation of \( P \).

### 4.3 The Associated Identification Problem

In addition to the equivalent identification problem of the previous section we can recast the closed-loop identification of \( P_T \) in \( H(P_T, C) \) into another open-loop identification problem. In this alternative scheme the inner-loop plant \( P \) is identified not from the associated rcf \( (N^a, D^a) \) but, ceteris paribus, from the factor \( R \).

From (3.33) we can easily derive \( T(P_T, C) \) as

\[
T(P_T, C) = \left[ \begin{array}{c}
D_c S \\
- N_c S
\end{array} \right] [T(P, C)].
\]

Using the expression for \( T(P, C) \) of (3.16) we can represent \( T(P_T, C) \) as in Fig. 4.4. This
4.3 The Associated Identification Problem

Figure 4.4: Feedback system $H(P_T, C)$ as a perturbation of the auxiliary feedback system $H(P_o, C)$.

The figure shows $H(P_T, C)$ as a perturbation of the auxiliary feedback system $H(P_o, C)$. As compared with Fig. 3.10 and especially Fig. 3.7 the "bridge" centered around the auxiliary rcf $(N_o, D_o)$ has been transformed into a bypass over the whole auxiliary feedback system $H(P_o, C)$. The so-called $(R, S)$-system is the only part of Fig. 4.4 that depends on the plant. Similar to the equivalent identification problem we can identify the terms $R$ and $S$ to build an estimate of $P_T$.

**Proposition 4.3.1** Let Assumption 4.1.4 hold. Let the plant $P_T \in \mathcal{F}$ be stabilized by $C \in \mathcal{F}$ in the standard feedback configuration $H(P_T, C)$ of Fig. 3.9. Then the following procedure enables the identification of a nominal model $\hat{P}$ of the inner-loop plant $P$ from $u, y$ and $C$, such that $\hat{P}$ is asymptotically independent of the disturbance $w$.

1. Perform steps i., ii. and iii. of Proposition 4.2.1.

2. Reconstruct $q$ as in (4.6).

3. Use an identification technique to estimate models $\hat{R}, \hat{S}$ of $R, S$ from

$$q = Rr + Sw$$  (4.10)

by considering $x, q$ and $w$ respectively as input, output and disturbance. The open-loop identification of (4.10) is the associated identification problem.

---

1 This terminology has been taken from [100, 102].
iv. Construct an estimate $\hat{P}_T$ of $P_T$ from the estimates $\hat{R}$ and $\hat{S}$ as

$$\hat{P}_T = \begin{bmatrix} (D_c + \hat{N} \hat{D}^{-1} N_c) \hat{S} & \hat{N} \hat{D}^{-1} \end{bmatrix}$$ (4.11)

with $\hat{N} = N_o + D_c \hat{R}$ and $\hat{D} = D_o - N_c \hat{R}$.

**Proof:** Analogous to the proof of Proposition 4.2.1 with additional use of (3.30) for the construction of $\hat{P}_w$.

---

![Fig. 4.5: Block diagram of the associated open-loop identification problem.](image.png)

The associated identification problem can be represented as in Fig. 4.5. As $x$ and $w$ are uncorrelated, the identification of $R$ and $S$ is identical to the standard open-loop identification problem of Fig. 2.4. Furthermore, Fig. 4.4 provides additional insight in the mechanism that neutralizes the feedback loop. Notice that both $w$ and $x$ affect $q$, and that $q$ affects $u$ and $y$. Of course $u$ and $y$ are not fed back to the exogenous inputs $r_1, r_2$. Therefore $x$ does not depend on $R, S$ or $w$. All open-loop properties of the equivalent identification problem of Proposition 4.2.1 apply analogously to the associated identification problem of Proposition 4.3.1. One noteworthy difference between the two schemes is that $S$ appears only once in (4.10) and twice in (4.7). Hence in regard of the noise contribution the associated identification is a standard problem, whereas the equivalent identification is not. The latter requires specific parameterizations and algorithms, which makes it less suited for the identification of $S$.

### 4.4 Justification, Merits and Drawbacks

We begin this section with a summary of the merits of our approach to the problem of approximate identification from closed-loop data. We also compare the two schemes of Section 4.2 and Section 4.3. Thereafter we provide some justification, which builds on the material presented in Section 2.3.4. In the end we discuss some drawbacks and some open questions.

An important feature of the two approaches developed in this chapter is, that they allow the asymptotic estimates of the inner-loop plant $P$ to be independent of
the noise contribution. As a result the inner-loop plant can be identified with a high accuracy even if the noise model is too simple. This is a direct consequence of the open-loop character of the new identification problems, which allow the application of every identification technique. For instance in [199, 202] spectral analysis techniques were applied to the "input-output" pairs \((u, x)\) and \((y, x)\) to estimate the frequency responses of the associated factors \(D^a\) and \(N^a\). These estimates were used to construct an estimate of the frequency response of the inner-loop plant \(P\).

The equivalent and associated identification problems have similar properties, because \((N^a, D^a)\) and \(R\) appear both linearly in the feedback matrix \(T(P, C)\) (see (3.15) and (3.16)). We will focus on the identification of \((N^a, D^a)\), but many of the future results can also be derived in terms of the factor \(R\).

Remark 4.4.1 At this stage we point out a difference between our right coprime approach and the left coprime approach of Hansen [100, 102]. The left coprime factorization gives rise to dual versions of the \((R, S)\)-parameterization, the associated rcf \((N^a, D^a)\) and the associated identification problem. However there exists no dual equivalent identification problem (see Appendix C for details). Consequently, the associated left coprime factors have to be identified via their \(\hat{R}\)-parameter.

For a justification of the open-loop character of the equivalent identification problem we examine \(y_x = N^a x\) and \(u_x = D^a x\). These contributions originate from \(r_1\) and \(r_2\) only. Using the expressions (3.12) and (3.14) for \((N^a, D^a)\) and (4.5) for \(x\) we can write \(y_x, u_x\) as

\[
\begin{pmatrix}
y_x \\ u_x
\end{pmatrix} = \begin{bmatrix} N \\ D \end{bmatrix} \Lambda^{-1} \Lambda_c \tilde{D}_c \begin{pmatrix} r_1 + \tilde{N}_c r_2 \\ r_2 \end{pmatrix}
\]

and with (3.4) we obtain

\[
\begin{pmatrix}
y_x \\ u_x
\end{pmatrix} = T(P, C) \begin{pmatrix} r_2 \\ r_1 \end{pmatrix}.
\]

Hence the estimation of the transfer functions \(N^a, D^a\) is closely related to the estimation of the transfer function matrix \(T(P, C)\). In fact, the problem of identifying \(T(P, C)\) from measurements of \(u\) and \(y\) is a special case of the equivalent identification problem.

In this identification problem \(r_1\) and \(r_2\) are not measured, so that \(r_1 + Cr_2\) must be derived as \(u + Cy\) (see also Section 2.3.4). In practice \(C\) must be stable for the calculation of \(Cy\). A stable \(C\) has a lcf \((I, C)\), and it allows the auxiliary model and its rcf to be chosen as \(P_0 = 0\) and \((0, I)\). Then we get \(\Lambda_c = I, x = r_1 + Cr_2, N^a = P(I + CP)^{-1}\) and \(D^a = (I + CP)^{-1}\). An estimate of \(P\) is obtained as the ratio of the estimates of \(P(I + CP)^{-1}\) and \((I + CP)^{-1}\), which are transfer functions from \(r_1 + Cr_2\) to \(y\) and \(u\).
In the general equivalent identification problem of Proposition 4.2.1 the auxiliary model \( P_o \) is not zero and \((N^a, D^a)\) satisfies
\[
N^a = P(I+CP)^{-1}(D_o+CN_o) \\
D^a = (I+CP)^{-1}(D_o+CN_o)
\]  
(4.12)
according to (3.11) and (3.13). The advantages of the “generalization” concerns both conceptual and numerical aspects. The conceptual advantages are due to the kinship with several control design methods and robustness margins. This will be clarified in Chapters 5 and 7. Furthermore, in Chapter 7 we will demonstrate that the coprime factorizations are of great use in finding a solution to the non-linear optimization problem of estimating \( \hat{P} \) while it appears in expressions like \( \hat{P}(I+CP)^{-1} \).

A serious drawback of identifying \( P \) via \((N^a, D^a)\) over a direct estimation is the increased complexity of the dynamics that are involved: If the plant \( P \) is not (precisely) known, then the choice of the auxiliary \((N_o, D_o)\) will generally not lead to cancellations in the expressions for \( N^a \) and \( D^a \) of (4.12). In other words, we cannot identify just any rcf of \( P \), but the order of the rcf \((N^a, D^a)\) that has to be identified is generally equal to the sum of the orders of \( P, P_o \) and \( C \). As we only want to identify \( P \) there are a lot of redundant dynamics present in \((N^a, D^a)\). In order to account for these redundant dynamics we represent the plant \( \hat{P} \) by its rcf \((\hat{N}, \hat{D}) \) instead of \((\hat{N}, \hat{D}) \).

We can control the order of \( \hat{P} = \hat{N} \hat{Q}(\hat{D} \hat{Q})^{-1} \) through the orders of \( \hat{N} \) and \( \hat{D} \), because \( \hat{Q} \) cancels out anyway. The mismatch of interest becomes
\[
\begin{bmatrix}
N^a \\
D^a
\end{bmatrix} - \begin{bmatrix}
\hat{N} \\
\hat{D}
\end{bmatrix} \hat{Q}
\]  
(4.13)
where \( \hat{Q} \) may, at least conceptually, be of arbitrary complexity.

Conclusively, we have established a framework for the desired approximate identification from closed-loop data at the expense of non-trivial constraints on the set of candidate nominal models. A nominal model \( P \) is represented as \((\hat{N}, \hat{Q})(\hat{D}, \hat{Q})^{-1} \), where \((\hat{N}, \hat{D}) \) is a rcf and \( \hat{Q} \in J \). The problem of parameterizing \((\hat{N}, \hat{D})\) and \( \hat{Q} \) remains an open question at this stage. In the next chapter we make some specific choices for \( \hat{Q} \). We return to the parameterization problem in Chapter 7.

Lastly we point out that our framework essentially rests on two assumptions: the plant belongs to the principal ideal domain \( \mathcal{H} \), and the stabilizing compensator used in the experiment is known. It is reassuring to know that this framework is robust in view of either of these two assumptions. In Appendix E.1 we demonstrate that the coprime factorizations approach still applies if \( \mathcal{H} = \mathbb{R} \mathbb{H}_{\infty} \) and the system under investigation is infinite dimensional, time-varying or non-linear. The only prerequisite is that the plant is stabilized by a real rational compensator. Further, for the identification of \( P \) we only need \( C \) and measurements of \( u \) and \( y \). The compensator \( C \) is used to filter
the intermediate $x$ out of $u$ and $y$. This $x$ is used as in “instrumental variable” in the identification procedure. In Appendix E.2 we show that a small error in the knowledge of the actually applied compensator incurs only a small error in the identified model.
Chapter 5

Identification in View of Feedback Control

We use system approximation as the key to interconnect identification and control design. In Section 5.1 we discuss that every feedback-relevant system approximation is necessarily an approximation in sense of the graph topology. In the subsequent sections we use the equivalent identification problem of Chapter 4 to set up mismatches that have the following two important properties. On the one hand these mismatches can serve as a basis for approximate identification. On the other hand these mismatches induce the graph topology when they are supplied with a suitable norm.

5.1 Approximation in the Graph Topology

In Section 2.4.3 we surveyed several types of system approximation, that are potentially meaningful in view of feedback control design. All these approximations aim at a nominal model that lies in a sufficiently small neighbourhood of the plant, and thus they rest on some robustness property [73]. We quote Vidyasagar on the issue of robustness.

"There are two aspects to the study of robustness. The first is qualitative and pertains to the nature of the permissible perturbations. Mathematically, this is the problem of defining a topology on the set of plant-compensator pairs. The second is quantitative, and is concerned with the extent of the permissible perturbations. This is the problem of defining a metric on the set of plant-compensator pairs." [239, p.230]

Vidyasagar et al. [244] showed that a system approximation is meaningful for feedback control only if it is an approximation in the graph topology. We discuss this topology for LTIFD systems and we let $\mathcal{H} = \mathbb{R}H_\infty$ be equipped with the $H_\infty$-norm. The key concept underlying the graph topology is the notion of a graph. The graph $\mathcal{G}(P)$ of a plant $P \in \mathcal{F}$ with $m$ inputs and $p$ outputs is defined on the $(m+p)$-dimensional
Lebesgue space \(^1\) \(L_{2,T}^{m+p}\) as the subspace

\[\mathcal{G}(P) = \{(u, y) \in L_{2,T}^{m+p} \mid y = Pu\}.\]

Thus, the graph \(\mathcal{G}(P)\) consists of all input-output pairs that correspond to \(P\) and that are bounded in an \(L_2\)-sense \([238]\). If \(P\) is unstable then not every \(u \in L_{2,T}^{m}\) is mapped into \(L_{2,T}^{p}\). The graph \(\mathcal{G}(P)\) can be represented as

\[\mathcal{G}(P) = \{\sigma(x_a, N x_a) \mid x_a \in L_{2,T}^{m}\}\]

(5.1)

where \((N, D)\) is a rcf of \(P\). We remark that \(x_a\) is the auxiliary intermediate of Fig. 3.6. Now we can adopt the definition of the graph topology from \([238]\).

**Definition 5.1.1** A sequence of nominal models \(\{\hat{P}_i\}\) in \(\mathcal{F}\) converges to \(P \in \mathcal{F}\) in the graph topology if there exist rcf's \((\hat{N}_i, \hat{D}_i)\) of \(\hat{P}_i\) and \((N, D)\) of \(P\) such that \(\hat{N}_i \to N\), \(\hat{D}_i \to D\) in \(\text{IRH}_\infty\).

This topology is known to be the *weakest* topology in which feedback stability is a robust property. That means that if \(\hat{P}_i\) converges to \(P\) in the graph topology, then the nominal feedback matrix \(T(\hat{P}_i, C)\) converges to \(T(P, C)\) in \(\text{IRH}_\infty\) for some stabilizing compensator \(C\), and vice versa. Hence the feedback system \(H(\hat{P}_i, C)\) will be stable if \(\hat{P}_i\) is sufficiently "close" to \(P\). The quantization of this "closeness" requires a metric.

By now it is well-known that the graph topology is induced by various metrics, in particular by the gap metric and graph metric. The gap metric was introduced in the control literature by Zames and El-Sakkary \([262, 66]\). For the computation of this metric Georgiou \([76]\) proposed the use of normalized rcf's.

**Definition 5.1.2** The pairs \((N_n, D_n)\) and \((\hat{D}_n, \hat{N}_n)\) are called respectively a normalized rcf (ncrf) and a normalized lcf (nclf) of a plant \(P \in \mathcal{F}\) if \((N_n, D_n)\) is a rcf of \(P\), \((\hat{D}_n, \hat{N}_n)\) is a lcf of \(P\), and in addition

\[
\begin{align*}
D_n^T(-s)D_n(s) + N_n^T(-s)N_n(s) & = I \\
\hat{D}_n(s)\hat{D}_n^T(-s) + \hat{N}_n(s)\hat{N}_n^T(-s) & = I
\end{align*}
\]

(5.2)

for all \(s\).

Normalized factorizations are unique up to multiplication of the factors by a constant orthogonal matrix. A definition of the gap can be found in e.g. \([262, 66, 77]\). Here we merely state Georgiou's result on the calculation of the gap between two systems.

**Proposition 5.1.3** \([76]\) The gap between the systems \(P\) and \(\hat{P}\) in \(\mathcal{F}\), denoted as \(\delta(P, \hat{P})\), satisfies

\[
\delta(P, \hat{P}) = \max\{\bar{\delta}(P, \hat{P}), \bar{\delta}(\hat{P}, P)\}
\]

\(^1\)The graph can be given an interpretation also in terms of the space \(L_{\infty,T}\) of essentially bounded signals. For more details see \([240, 242, 40, 230]\).
where $\tilde{\delta}(\hat{P}, P)$ is the directed gap for which holds that

$$
\tilde{\delta}(\hat{P}, P) = \inf_{Q_\Delta \in \mathcal{H}} \left\| \begin{bmatrix} \hat{N}_n \\ \hat{D}_n \end{bmatrix} - \begin{bmatrix} N \\ D \end{bmatrix} Q_\Delta \right\|_\infty
$$

with $(N, D)$ a rcf of $P$ and $(\hat{N}_n, \hat{D}_n)$ a normalized rcf of $\hat{P}$.

The expression for the directed gap is the $H_\infty$-norm of the smallest stable perturbation of the nominal rcf $(\hat{N}_n, \hat{D}_n)$ that induces the plant $P$. Notice that in (5.3) the rcf $(N, D)$ of the plant does not have to be normalized due to the free term $Q_\Delta$. The computation of the (other) directed gap $\tilde{\delta}(P, \hat{P})$ of course does require a normalized rcf $(N_n, D_n)$ of $P$.

The graph metric has been defined by Vidyasagar [238] in a similar fashion except that the infimum over $Q_\Delta \in \mathcal{H}$ is taken with the additional constraint that $\|Q_\Delta\|_\infty \leq 1$. Due to this constraint the graph metric is difficult to evaluate [76, 77]. The equivalence of the gap and graph metrics has been demonstrated in [76, 173]. Both the gap and graph metrics have been used to derive sufficient conditions for robust stability. More about this and other robustness margins is said in Section 6.3.

Remark 5.1.4 Although we discussed the graph topology for real rational systems, it has a wider applicability. Here we merely summarize a few extensions beyond $\text{IRH}_\infty$.

In [118] it has been shown that the topologies induced by the gap and graph metrics are equivalent when placed on an algebra of unstable linear distributed systems. Further, in [242] it has been shown, that if a linear distributed system can be approximated by a lumped system in the sense of the graph topology, then it can also be stabilized by a lumped compensator. A general comparison of the topologies induced by the graph and gap metrics can be found in [266]. Geometric approaches to robust stabilization based on the graph are discussed in [71, 168].

5.2 Performance-approximation by Fixed-loop Identification

Performance-approximation concerns the derivation of a nominal model $\hat{P}$ that describes the behaviour of the plant $P$, when $P$ and $\hat{P}$ are feedback controlled by identical compensators. This implies that $\hat{P}$ is a solution to the performance-approximation only if it is an approximation of $P$ in the graph topology (cf. Section 5.1). Here we work out the fixed-loop identification, which will be used later in our primary iterative scheme.

In fixed-loop performance-identification the mismatch of interest is the additive feedback matrix mismatch $M_T$ defined as

$$
M_T = T(P, C) - T(\hat{P}, C).
$$

With this mismatch we can restate the convergence properties of the previous section as follows.
Proposition 5.2.1 Let $H(P, C)$ with $P, C \in \mathcal{F}$ be stable. Then $\hat{P} \to P$ in the graph topology if and only if $M_T \to 0$ in $\mathcal{RH}_\infty$.

Proof: By (5.4) $\|M_T\|_\infty \to 0$ implies that $T(\hat{P}, C) \to T(P, C)$ in $\mathcal{RH}_\infty$ and the result follows from Theorem 2.2 in [238].

By this proposition we can use the norm $\|M_T\|_\infty$ to metrise the graph topology in the neighborhood of $P$. — This metric does not induce the whole graph topology, because it is not defined for those plants in $\mathcal{F}$, that are not stabilized by $C$. — In case the compensator is stable, then the convergence of $M_T \to 0$ can be simplified to $T_{12}(\hat{P}, C) \to T_{12}(P, C)$.

Proposition 5.2.2 Let $H(P, C)$ with $P \in \mathcal{F}, C \in \mathcal{K}$ be stable. Then $\hat{P} \to P$ in the graph topology if and only if $\hat{P}(I+C\hat{P})^{-1} \to P(I+CP)^{-1}$ in $\mathcal{RH}_\infty$.

Proof: We express $M_T$ of (5.4) in terms of $\hat{P}(I+C\hat{P})^{-1}$ and $P(I+CP)^{-1}$:

$$M_T = \begin{bmatrix} P(I+CP)^{-1} - \hat{P}(I+C\hat{P})^{-1} \\ (I+CP)^{-1} - (I+C\hat{P})^{-1} \end{bmatrix} \begin{bmatrix} C & I \\ I & -C \end{bmatrix} \begin{bmatrix} P(I+CP)^{-1} - \hat{P}(I+C\hat{P})^{-1} \\ -CP(I+CP)^{-1} + C\hat{P}(I+C\hat{P})^{-1} \end{bmatrix} \begin{bmatrix} C & I \end{bmatrix}.$$

From this we derive an upper bound on $\|M_T\|_\infty$:

$$\|M_T\|_\infty \leq k \cdot \|P(I+CP)^{-1} - \hat{P}(I+C\hat{P})^{-1}\|_\infty$$

where $k = \|\begin{bmatrix} I \\ -C \end{bmatrix}\|_\infty \cdot \|\begin{bmatrix} C & I \end{bmatrix}\|_\infty \geq 1$. Further since $\begin{bmatrix} I & 0 \end{bmatrix} M_T \begin{bmatrix} 0 \\ I \end{bmatrix}$ equals $P(I+CP)^{-1} - \hat{P}(I+C\hat{P})^{-1}$ it is easy to obtain the lower bound

$$\|P(I+CP)^{-1} - \hat{P}(I+C\hat{P})^{-1}\|_\infty \leq \|M_T\|_\infty.$$ 

Due to the two bounds $M_T \to 0$ in $\mathcal{RH}_\infty$ if and only if $\hat{P}(I+C\hat{P})^{-1} \to P(I+CP)^{-1}$ in $\mathcal{RH}_\infty$. Application of the previous proposition completes the proof.

So far we have established that $P$ is approximated by $\hat{P}$ in the sense of the graph topology, if the approximation is based on the mismatch $M_T$ and if $M_T$ belongs to $\mathcal{RH}_\infty$. We want to conduct fixed-loop performance-approximation by means of the equivalent identification problem of Section 4.2. The latter enables the identification of $P$ from the mismatch in (4.13), which can be transformed into the mismatch $M_T$ of (5.4) by the following weighting functions.
Proposition 5.2.3 Let Assumptions 3.3.3 and 3.3.5 hold. Let $H(P, C)$ be stable and let nominal model $\hat{P} \in \mathcal{F}$ have a rcf $(\hat{N}, \hat{D})$. Then the mismatch $M_T$ of (5.4) satisfies

$$M_T = \begin{bmatrix} N^a \\ D^a \end{bmatrix} - \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} Q_T \begin{bmatrix} D_o + CN_o \end{bmatrix}^{-1} \begin{bmatrix} C & I \end{bmatrix}$$

(5.5)

if and only if $Q_T = (\hat{D} + C\hat{N})^{-1}(D_o + CN_o)$. In addition $M_T \in \mathcal{H}$ if and only if $Q_T \in \mathcal{J}$.

Proof: By Assumption 3.3.5 $H(P_o, C)$ is stable and thus the inverse $(D_o + CN_o)^{-1} = \Lambda^{-1}_{o} \bar{D}_c$ exists. By (3.15) we can write $T(P, C)$ as

$$T(P, C) = \begin{bmatrix} N^a \\ D^a \end{bmatrix} (D_o + CN_o)^{-1} \begin{bmatrix} C & I \end{bmatrix}$$

(5.6)

and due to (3.4) $T(\hat{P}, C)$ satisfies

$$T(\hat{P}, C) = \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} (\hat{D} + C\hat{N})^{-1} \begin{bmatrix} C & I \end{bmatrix}. $$

With substitution of these expression into (5.4) we obtain

$$M_T = \begin{bmatrix} N^a \\ D^a \end{bmatrix} (D_o + CN_o)^{-1} \begin{bmatrix} C & I \end{bmatrix} - \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} (\hat{D} + C\hat{N})^{-1} \begin{bmatrix} C & I \end{bmatrix}. $$

(5.7)

Introducing $(D_o + CN_o)(D_o + CN_o)^{-1}$ into the term on the right yields

$$M_T = \begin{bmatrix} N^a \\ D^a \end{bmatrix} (D_o + CN_o)^{-1} \begin{bmatrix} C & I \end{bmatrix} - \begin{bmatrix} \hat{N} \\ \hat{D} \end{bmatrix} (\hat{D} + C\hat{N})^{-1} (D_o + CN_o) \begin{bmatrix} D_o + CN_o \end{bmatrix}^{-1} \begin{bmatrix} C & I \end{bmatrix}. $$

This reveals that (5.5) holds if $Q_T = (\hat{D} + C\hat{N})^{-1}(D_o + CN_o)$. The only-if part is proven as follows. Let (5.5) hold; (5.7) holds by definition. Then multiply (5.5) and (5.7) to the left by the Bezout factors of $(\hat{N}, \hat{D})$ and to the right by col$(0, I)(D_o + CN_o)$. This results in

$$Q_T = (\hat{D} + C\hat{N})^{-1}(D_o + CN_o). $$

For the second part of the proof we write $Q_T$ as

$$Q_T = (\bar{D}_c \hat{D} + \bar{N}_c \hat{N})^{-1}(\bar{D}_c D_o + \bar{N}_c N_o) = \hat{\Lambda}^{-1}_{o} \Lambda_{o}$$

in which $\hat{\Lambda} = \bar{D}_c \hat{D} + \bar{N}_c \hat{N}$. Then by the stability of both $H(P, C)$ and $H(P_o, C)$ we have $M_T \in \mathcal{H} \Leftrightarrow T(\hat{P}, C) \in \mathcal{H} \Leftrightarrow \hat{\Lambda} \in \mathcal{J} \Leftrightarrow Q_T \in \mathcal{J}$, where we used Lemma 3.3.4 and Fact B.1.2.i for the latter two implications. □
In Section 4.4 we mentioned that $\hat{Q}$ does not affect the order of $\hat{P}$. The only task $\hat{Q}$ has in (4.13) is to account for the redundant dynamics in $(N^a, D^a)$. As $\hat{Q}$ cancels out in $\hat{N}Q(\hat{D}Q)^{-1}$ we suggested that $\hat{Q}$ may be any element in $J$. This enables an approximate identification of $P$ via the approximate identification of its associated rcf $(N^a, D^a)$.

In this section we want to use this approximate identification problem to solve the performance-approximation problem. To that end we have to weight the mismatch of (4.13) by $(D_o+CN_o)^{-1}$ and by $[C \ I]$ as shown in Proposition 5.2.3. Moreover, the resulting approximation problem equals the performance-approximation problem based on $M_T$ of (5.4) only if $\hat{Q}$ takes the specific form of $Q_T$ in Proposition 5.2.3. For an explanation of the constraint we have to revert to the $R$-parameterization of Section 3.3.

We let the auxiliary rcf $(N_o, D_o)$ and the compensator $C$ be given. Then, according to Corollary 3.3.7, every $\hat{P} \in P(C)$ — i.e. $\hat{P}$ is stabilized by $C$ — has an unique associated rcf $(\hat{N}^a, \hat{D}^a)$. Like in (5.6) this rcf has the property that

$$T(\hat{P}, C) = \left[ \begin{array}{c} \hat{N}^a \\ \hat{D}^a \end{array} \right] (D_o+CN_o)^{-1} \left[ \begin{array}{cc} C & I \end{array} \right].$$

Using this together with (5.6) we can write $M_T$ as

$$M_T = \left[ \begin{array}{c} N^a \\ D^a \end{array} \right] - \left[ \begin{array}{c} \hat{N}^a \\ \hat{D}^a \end{array} \right] (D_o+CN_o)^{-1} \left[ \begin{array}{cc} C & I \end{array} \right].$$

Thus $M_T$ equals a weighted difference between the (unique) associated rcf’s of $P$ and the candidate $\hat{P}$. Similar to (4.12) the associated rcf $(\hat{N}^a, \hat{D}^a)$ satisfies

$$\left[ \begin{array}{c} \hat{N}^a \\ \hat{D}^a \end{array} \right] = \left[ \begin{array}{c} P \\ I \end{array} \right] (I+CP)^{-1}(D_o+CN_o)$$

or equivalently

$$\left[ \begin{array}{c} \hat{N}^a \\ \hat{D}^a \end{array} \right] = \left[ \begin{array}{c} \hat{N} \\ \hat{D} \end{array} \right] (\hat{D}+C\hat{N})^{-1}(D_o+CN_o)$$

for any rcf $(\hat{N}, \hat{D})$ of the candidate $\hat{P}$.

We conclude that $Q_T$ has two tasks: it must account for the redundant dynamics in $(N^a, D^a)$ just like $\hat{Q}$, and it must shape $(\hat{N}Q_T, \hat{D}Q_T)$ to an associated rcf of $\hat{P}$, so that a simple weighting yields $M_T$. Furthermore the constraint on $Q_T$ makes clear that it is quite complex to represent the set of candidate models. We return to this matter in Section 7.3.

With $\hat{Q}$ replaced by $Q_T$ and $(\hat{N}Q_T, \hat{D}Q_T)$ replaced by $(\hat{N}^a, \hat{D}^a)$ the equivalent identification problem of Section 4.2 can be arranged such, that the estimation of $\hat{P}$ effectively pursues the minimization of some norm of the mismatch $M_T$. For instance if
one uses time-series and the prediction error method, then the prediction error (vector) is
\[
\varepsilon_{eq} = \left[ \begin{bmatrix} N^a \\ D^a \end{bmatrix} - \begin{bmatrix} \hat{N}^a \\ \hat{D}^a \end{bmatrix} \right] x + \begin{bmatrix} D^c \\ -N^c \end{bmatrix} Sw,
\]
provided that an output error model structure is used for \( \text{col}(\hat{N}^a, \hat{D}^a) \). According to [247] the prediction error criterion can be adjusted to match the \( L_2 \)-norm of \( M_T \) by choosing the spectrum of \( x \) in agreement with \((D_o+CN_o)^{-1}[C I]\) (cf. (5.5)). Instead in Chapter 7 we will identify a parametric nominal model from frequency response estimates of \( N^a \) and \( D^a \).

**Remark 5.2.4** Instead of the equivalent identification problem of Section 4.2, we could also have used the associated identification problem of Section 4.3 for the fixed-loop performance-approximation. Substitution of \((N_o+D_c R, D_o-N_c R)\) and \((N_o+D_c \hat{R}, D_o-N_c \hat{R})\) for \((N^a, D^a)\) respectively \((\hat{N}^a, \hat{D}^a)\) in the expression for \( \varepsilon_{eq} \) yields
\[
\varepsilon_{eq} = \begin{bmatrix} D^c \\ -N^c \end{bmatrix} \left( [R-\hat{R}]x + Sw \right).
\]
The sum \([R-\hat{R}]x+Sw\) is precisely the prediction error that corresponds to the output error identification of \( R \) from the associated identification problem of (4.10). Hence fixed-loop performance-approximation can be carried out also via the associated identification problem of Section 4.3, provided that the spectrum of the intermediate \( x \) accounts for the weight \((D_o+CN_o)^{-1}[C I]\), and that the prediction error \([R-\hat{R}]x+Sw\) is filtered by \( \text{col}(D^c, -N^c) \).

\[\square\]

### 5.3 Stability-approximation by Design-oriented Identification

Design-oriented identification requires information on the future compensator. All information that we have got consists of the control objectives and the control design method. The utilized control design method must provide sufficient information about the future controller, in order that this controller can be anticipated in the preceding identification stage. A method that meets this requirement is the robust control design method of Bongers and Bosgra [26, 25]. Here we exploit the typical nature of this particular robust control design technique to set up the design-oriented stability-approximation alluded to in Section 2.4.3.

The design method of [26, 25] optimizes robustness against normalized coprime factor perturbations. That is, the compensator \( \hat{P} \) will anticipate perturbations of the nrcf \((\hat{N}_n, \hat{D}_n)\) of the nominal model \( \hat{P} \). This property of \( \hat{P} \) is known, even when the nominal model \( \hat{P} \) has not been identified yet! We anticipate this property in the identification stage as follows. We model the plant \( P \) as a perturbation of the nrcf \((\hat{N}_n, \hat{D}_n)\) of the candidate \( \hat{P} \). Then we estimate \( \hat{P} \) by minimizing this particular
coprime factor perturbation. The goal is that the minimized mismatch between $P$ and $\hat{P}$ is so small, that it lies within the ball of robustly stabilized dynamical perturbations, that is maximized in the subsequent control design stage.

The resulting stability-approximation does not depend explicitly on the compensator and therefore it can be solved prior to the control design stage. Moreover we will show that the mismatch of the equivalent identification problem of Section 4.2 can easily be modified to match the mismatch of this design-oriented stability-approximation. Here we concisely summarize those aspects of the method that we need for the design-oriented approximation problem. More details will be provided in Chapter 6. We begin with a formalization of the notion of a coprime factor perturbation.

**Definition 5.3.1** A coprime factor perturbation $(\Delta_N, \Delta_D)$ is a stable dynamical perturbation that affects the rcf $(\hat{N}, \hat{D})$ of the nominal model $\hat{P}$ as

$$\hat{N} + \Delta_N$$

$$\hat{D} + \Delta_D.$$  \hfill (5.8)

In accordance with Definition 2.2.2 a ball $B(b_{ND})$ of coprime factor perturbations is defined as

$$B(b_{ND}) = \left\{ \begin{bmatrix} \Delta_N \\ \Delta_D \end{bmatrix} \mid \sigma_{\max} \left( \begin{bmatrix} \Delta_N(j\omega) \\ \Delta_D(j\omega) \end{bmatrix} \right) < b_{ND}(\omega) \quad \forall \omega \geq 0 \right\}. \hfill (5.9)$$

In Section 6.3 we will discuss several robustness margins that are conceived in terms of coprime factor perturbations. For now suffice it to say, that every compensator $C$ that stabilizes $\hat{P}$, gives rise to some bound $b_{ND}$, such that $H(\hat{P}, C)$ remains stable for all perturbations $(\Delta_N, \Delta_D) \in B(b_{ND})$.

The control design method developed by Bongers [26, 25] derives the controller $C_\hat{P}$ from a nominal model $\hat{P}$ as

$$C_\hat{P} = \arg \min_C \|T(\hat{P}, C)\|_\infty. \hfill (5.10)$$

This compensator $C_\hat{P}$ is optimally robust in the following sense. It stabilizes the system $(\hat{N}_n + \Delta_N)(\hat{D}_n + \Delta_D)^{-1}$ for each coprime factor perturbation $(\Delta_N, \Delta_D)$ that satisfies

$$\left\| \begin{bmatrix} \Delta_N \\ \Delta_D \end{bmatrix} \right\|_\infty < b_{ND\infty} \hfill (5.11)$$

for some positive number $b_{ND\infty}$, and there exists no other compensator that allows a larger ball of coprime factor perturbations. In other words the designed controller maximizes in an $H_\infty$-sense the ball of normalized coprime factor perturbations that are allowed in view of robust stability.

Notice that the coprime factor perturbations of (5.8) and the control design method of (5.10) are based solely on $\hat{P}$, they do not involve the plant $P$. In order to relate
these matters to the plant $P$ under investigation, we introduce the concept of a coprime factor mismatch $M_{ND}(Q)$. This mismatch is defined as

$$\begin{align*}
M_{ND}(Q) &= \begin{bmatrix} \tilde{N}_n \\ \tilde{D}_n \end{bmatrix} - \begin{bmatrix} N \\ D \end{bmatrix} Q 
\end{align*}$$

(5.12)

in which $(\tilde{N}_n, \tilde{D}_n)$ is a nrcf of $\hat{P}$, $(N, D)$ is a rcf of $P$ and $Q \in \mathcal{J}$ so that $(NQ, DQ)$ is a rcf of $P$ as well. Notice that $Q$ is a free “parameter” of $M_{ND}(Q)$, which corresponds to the fact that the pair $(NQ, DQ)$ is a rcf of $P$ for all $Q \in \mathcal{J}$. A comparison of $M_{ND}(Q)$ with $(\Delta N, \Delta D)$ of (5.8) makes clear that the plant can be regarded as the particular perturbation $-M_{ND}(Q)$ of the normalized coprime factors $(\tilde{N}_n, \tilde{D}_n)$ of $\hat{P}$. If there exists some $Q \in \mathcal{J}$ such that the norm $\|M_{ND}(Q)\|_\infty$ is smaller than the optimal robustness margin $b_{ND}$ of (5.11), then the plant $P$ is robustly stabilized by the model-based compensator $C_\hat{P}$. In fact we only need that $P$ is a stable coprime factor perturbation of $\hat{P}$. Thus $Q \in \mathcal{J}$ of (5.12) may be replaced by $Q_\Delta \in \mathcal{H}$. The resulting $M_{ND}(Q_\Delta)$ corresponds to the expression for the directed gap of (5.3).

Now we have linked both the plant $P$ and the future compensator $C_\hat{P}$ to the nrcf $(\tilde{N}_n, \tilde{D}_n)$ of $\hat{P}$. On the one hand the control design method optimizes robustness against normalized coprime factor perturbations, and on the other hand the plant is regarded as a normalized coprime factor perturbation. In this context the optimal approximation that can precede the control design of (5.10) is

$$\hat{P} = \arg \min_{\hat{P}, Q_\Delta} \left\| \begin{bmatrix} \tilde{N}_n \\ \tilde{D}_n \end{bmatrix} - \begin{bmatrix} N \\ D \end{bmatrix} Q_\Delta \right\|_\infty$$

(5.13)

where $Q_\Delta \in \mathcal{H}$, and $(\tilde{N}_n, \tilde{D}_n)$ is a nrcf of $\hat{P}$. This approximation minimizes the deficiency of the nominal model $\hat{P}$ in terms of a coprime factor perturbation of its own nrcf $(\tilde{N}_n, \tilde{D}_n)$.

Now we can easily establish a link between the above robust control design objectives and the equivalent identification problem of Section 4.2. The latter identification problem is based on the mismatch of (4.13). The question is how to shape this mismatch into that of (5.13). For this purpose we may multiply the mismatch by weights, which amounts to filtering of data. We multiply (4.13) to the right by $Q_\Delta$ and we choose the weight $Q_\Delta$ equal to $\hat{Q}^{-1}$. Further we constrain the rcf of the candidate nominal model $\hat{P}$ to be a normalized rcf $(\tilde{N}_n, \tilde{D}_n)$. Thereby we obtain the mismatch

$$M_{ND}(Q_\Delta) = \begin{bmatrix} \tilde{N}_n \\ \tilde{D}_n \end{bmatrix} - \begin{bmatrix} N^a \\ D^a \end{bmatrix} Q_\Delta,$$

so that the free “parameter” $\hat{Q}$ (4.13), which is meant to account for the redundant dynamics, has been replace by another free “parameter” $Q_\Delta$ in a way that is possible from an identification point of view. In comparison with the performance-approximation of
the previous section we have started from the same mismatch (4.13), but we have chosen a completely different \( \mathcal{Q} \) and different weights to arrive at the mismatch of (5.13).

The mismatch \( M_{ND}(Q_\Delta) \) is completely in agreement with that of (5.13), if we take \( Q_\Delta \) from \( \mathcal{H} \) instead of from \( \mathcal{J} \). The remaining question is how to derive \( (\hat{N}_n, \hat{D}_n) \) and \( Q_\Delta \) from data, so that \( \| M_{ND}(Q_\Delta) \|_\infty \) is small. In Section 7.4 we use frequency response estimates of the associated rcf \( (N^a, D^a) \) for this identification problem.
Chapter 6

Cautious Controller Enhancement

This chapter concerns the following control design problem. We suppose that the plant $P$ with uncertain dynamics is stabilized by some known compensator $C$. Further, we presume that a nominal model $\hat{P}$ is available, whose feedback system $H(\hat{P}, C)$ approximately describes the actual feedback system $H(P, C)$. The object is to enhance the compensator $C$.

In Section 6.1 we use the Youla-parameterization to represent the candidate enhanced compensator as a coprime factor perturbation of the compensator $C$. We study also how these perturbations affect the feedback systems $H(\hat{P}, C)$ and $H(P, C)$.

Thereafter we search for some “perturbation” of $C$ that improves the nominal and robust performances. For this purpose we use an unconstrained controller optimization. As explained in Section 2.2.1 an unconstrained optimization can be manipulated through design weights, but there is no prior guarantee about the robustness that will be achieved. After the unconstrained design we have to assure that the resulting compensator will stabilize the plant. The design weight is selected in Section 6.2 in such a cautious fashion, that the nominal model reliably predicts the response of the plant to the controller enhancement. In Section 6.3 we discuss two classes of compensator-based dynamical perturbations, by which we can ascertain the stability of the new control system before the enhanced compensator is actually applied.

6.1 Controller Enhancement through Compensator Perturbations

The compensator $C$, which is known to stabilize the plant $P$ and the nominal model $\hat{P}$, is called the available compensator. The (candidate) enhanced compensator, denoted $C_E$, has to be designed from the nominal model $\hat{P}$, and thus it must at least stabilize $\hat{P}$. We study the problem of controller enhancement for the uncertain plant $P$ by combining the following characterizations of $C_E$ and $P$.

- The candidate enhanced compensator $C_E$ is represented as a coprime factor perturbation of the available compensator $C$. For this we use the Youla param-
eterization or dual $R$-parameterization of the set of all controllers that stabilize $\hat{P}$.

- The plant $P$ is represented as a coprime factor perturbation of the nominal model $\hat{P}$. For this we use the dual Youla-parameterization or $R$-parameterization of the set of all systems that are stabilized by $C$.

Together these coprime factor perturbations turn the available nominal feedback system $H(\hat{P}, C)$ into the enhanced actual feedback system $H(P, C_E)$. In this way we link the enhancement of the controller $C$ to the deficiency of the nominal model $\hat{P}$.

We begin with the characterization of the enhanced compensator $C_E$. We let $(\hat{N}, \hat{D})$ and $(N_c, D_c)$ be rcf’s of $\hat{P}$ and $C$. Then by virtue of the dual of Theorem 3.3.6 $C_E$ has an associated rcf $(N_E^a, D_E^a)$ defined as

$$N_E^a \triangleq N_c + \hat{D} R_c$$
$$D_E^a \triangleq D_c - \hat{N} R_c.$$

In here $R_c$ satisfies $R_c = (\hat{D} + C_E \hat{N})^{-1}(C_E D_c - N_c)$ in accordance with (3.8). This settles the connection between $\hat{P}$, $C$ and $C_E$.

For the characterization of the plant $P$ we let the nominal model $\hat{P}$ play the role of the auxiliary model $P_o$, which has been used to set up the $R$-parameterization of Section 3.3. We denote $R_n$ the $R$-parameter, that follows when $(\hat{N}, \hat{D})$ is substituted for $(N_o, D_o)$ in (3.8). Then the pair $(N^n, D^n)$, defined as

$$N^n \triangleq \hat{N} + D_c R_n$$
$$D^n \triangleq \hat{D} - N_c R_n,$$

is the rcf of $P$, that is associated to $C$ and the rcf $(\hat{N}, \hat{D})$ of $\hat{P}$. This settles the connection between $P$, $\hat{P}$ and $C$. The combination of the two $R$-parameterizations $P$ and $C_E$ is called the double $R$-parameterization. A block diagram of this double $R$-parameterization has been depicted in Fig. 6.1.

At this point we emphasize that the above term $R_n$ differs in character from the $R$-parameter used in the previous chapters. The latter depends on the auxiliary model $P_o$, which only required to be stabilized by $C$. The resulting framework enabled the fixed-loop performance-identification of Section 5.2. Thus the representation with the $R$-parameter is used to derive the nominal model $\hat{P}$. The latter is not just stabilized by $C$, but it actually describes the plant $P$. In this section we use $\hat{P}$ to represent the plant $P$ once more by an associate rcf. Here the incentive is to construct a better compensator $C_E$ from $\hat{P}$, and not to improve the nominal model $\hat{P}$. In this perspective the block-diagram of Fig. 6.1 links the nominal model's deficiency to the enhancement of the compensator.
6.1 Controller Enhancement through Compensator Perturbations

![Diagram](image)

Fig. 6.1: Double $R$-parameterization of feedback system $H(P, C_E)$.

We consider four feedback systems built from the components $C$, $C_E$, $\hat{P}$ and $P$, viz. the two available feedback systems $H(P, C)$ and $H(\hat{P}, C)$, and the two enhanced feedback systems $H(P, C_E)$ and $H(\hat{P}, C_E)$. In the sequel of this section we relate their feedback matrices together by expressing them in terms of $R_n$ and $R_c$. We begin with defining the "filter" $F$ as

$$F = \begin{bmatrix} D_c & \hat{N} \\ -N_c & \hat{D} \end{bmatrix}$$

whose inverse $F^{-1}$ satisfies

$$F^{-1} = \begin{bmatrix} (D_c + \hat{P}N_c)^{-1} & 0 \\ 0 & (\hat{D} + C\hat{N})^{-1} \end{bmatrix} \begin{bmatrix} I & -\hat{P} \\ C & I \end{bmatrix}.$$ 

Since $H(\hat{P}, C)$ is stable, it can easily be shown that both $F$ and $F^{-1}$ are stable. We substitute these expressions for $F$ and $F^{-1}$ in the trivial equality

$$T(P, C_E) = FF^{-1}T(P, C_E)FF^{-1}.$$ 

By straightforward but tedious manipulations we obtain

$$T(P, C_E) = F \begin{bmatrix} R_n \\ I \end{bmatrix} (I + R_c R_n)^{-1} \begin{bmatrix} R_c \\ I \end{bmatrix} F^{-1}$$

which shows that the enhanced actual feedback system $H(P, C_E)$ is closely related to the feedback system $H(R_n, R_c)$, i.e. to an interconnection of the nominal model's deficiency and the compensator enhancement. In fact, the stability of $H(P, C_E)$ is dictated by the stability of $H(R_n, R_c)$. We state this stability result as a corollary, because it is derived in a wider setting in Appendix E.2, Theorem E.2.4.

**Corollary 6.1.1** Let $P, C, \hat{P}, C_E \in \mathcal{F}$ and let $H(P, C_E)$ be represented by the double $R$-parameterization of Fig. 6.1. Then $H(P, C_E)$ is stable if and only if $H(R_n, R_c)$ is stable.
Remark 6.1.2 In Appendix E.2 this stability result is derived for the more general feedback system $H(P_T, C_T)$, which is composed of two vector-input one vector-output systems. Similar results were obtained for the single-variate feedback system $H(P, C)$ by Tay et al. [227] and for certain non-linear plant-compensator pairs by Paice and Moore [174].

In establishing (6.1) we have expressed $C_E$ and $P$ in terms of $R_n$, $R_c$ and the coprime factors of $\hat{P}$ and $C$. We only have to make $R_n$ and/or $R_c$ zero in (6.1) to obtain

$$T(P, C) = F \begin{bmatrix} 0 & R_n \\ 0 & I \end{bmatrix} F^{-1}$$

$$T(\hat{P}, C_E) = F \begin{bmatrix} 0 & 0 \\ R_c & I \end{bmatrix} F^{-1}$$

$$T(\hat{P}, C) = F \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} F^{-1}. \tag{6.2}$$

With these expressions we can readily describe how the available feedback systems $H(P, C)$ and $H(\hat{P}, C)$ are affected by the compensator enhancement:

$$T(P, C_E) - T(P, C) = F \begin{bmatrix} R_n \\ I \end{bmatrix} (I + R_c R_n)^{-1} \begin{bmatrix} R_c \\ -R_c R_n \end{bmatrix} F^{-1}$$

$$T(\hat{P}, C_E) - T(\hat{P}, C) = F \begin{bmatrix} 0 & 0 \\ R_c & 0 \end{bmatrix} F^{-1}. \tag{6.3}$$

The next result states that the controller enhancement is a robust property in respect of the deficiency of the nominal model.

Proposition 6.1.3 Let the assumptions of Corollary 6.1.1 hold. Then the enhancement of the nominal feedback system converges to that of the actual feedback system, i.e.

$$T(\hat{P}, C_E) - T(\hat{P}, C) \to T(P, C_E) - T(P, C)$$

in $\mathbb{RH}_\infty$, if and only if $\hat{P} \to P$ in the graph topology.

Proof: From Proposition 5.2.1 and (6.2) it follows that $\hat{P} \to P$ in the graph topology $\Leftrightarrow R_n \to 0$ in $\mathbb{RH}_\infty$. Since $R_c$ and $R_n$ are stable, $T(R_n, R_c)$ will be stable for a sufficiently small $R_n$ by virtue of the small gain theorem [57]. So $R_n \to 0$ in $\mathbb{RH}_\infty$ \Leftrightarrow

$$\begin{bmatrix} R_n \\ I \end{bmatrix} (I + R_c R_n)^{-1} \begin{bmatrix} R_c \\ -R_c R_n \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ R_c & 0 \end{bmatrix}$$

in $\mathbb{RH}_\infty$. The results follows from (6.3) because $F$ and $F^{-1}$ are stable. \hfill \Box
In addition to the enhancement of the feedback systems we can also write the performance degradation for \( C \) and \( C_E \) in terms of \( R_n \) and \( R_c \):

\[
T(P, C_E) - T(\hat{P}, C_E) = F \begin{bmatrix} R_n \\ -R_c R_n \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} (I + R_c R_n)^{-1} \begin{bmatrix} R_c & I \\ R_c & I \\ 0 & 0 \end{bmatrix} F^{-1} 
\]

(6.4)

\[
T(P, C) - T(\hat{P}, C) = F \begin{bmatrix} R_n \\ -R_c R_n \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F^{-1}.
\]

The next proposition states that the performance degradation is a robust property in respect of the controller enhancement.

**Proposition 6.1.4** Let the assumptions of Corollary 6.1.1 hold. Then the enhanced performance degradation converges to the available performance degradation, i.e.

\[
T(P, C_E) - T(\hat{P}, C_E) \to T(P, C) - T(\hat{P}, C)
\]

in \( \mathbb{RH}_\infty \), if and only if \( C_E \to C \) in the graph topology.

**Proof:** Dual to the proof of Proposition 6.1.3 it holds that \( C_E \to C \) in the graph topology \( \iff R_c \to 0 \) in \( \mathbb{RH}_\infty \).

\[
\begin{bmatrix} R_n \\ -R_c R_n \end{bmatrix} (I + R_c R_n)^{-1} \begin{bmatrix} R_c & I \\ R_c & I \end{bmatrix} \to \begin{bmatrix} 0 & R_n \\ 0 & 0 \end{bmatrix}
\]

in \( \mathbb{RH}_\infty \). The result follows from (6.4) because \( F \) and \( F^{-1} \) are stable. \( \square \)

Proposition 6.1.3 implies that an enhancement of \( H(\hat{P}, C) \) results in a similar enhancement of \( H(P, C) \), provided that the nominal model’s deficiency (or the “model-error”) tends to zero in the graph topology. Proposition 6.1.4 says that the performance degradation for \( C_E \) is similar to the performance degradation for \( C \), provided that the compensator enhancement tends to zero in the graph topology. In summary these results relate the controller enhancement to the deficiency of the nominal model for the case that either \( R_n \to 0 \) or \( R_c \to 0 \). By the following conjecture we join these propositions together for finite coprime factor perturbations of both \( \hat{P} \) and \( C \), i.e. \( R_n \not\to 0 \) and \( R_c \not\to 0 \), by the following conjecture.

**Conjecture 6.1.5** The enhancement of the performance of the nominal feedback system \( H(\hat{P}, C) \) induces a similar enhancement of the performance of the actual feedback system \( H(P, C) \), provided that, in the sense of the graph topology, the nominal model \( \hat{P} \) approximates the plant \( P \) and the enhanced compensator \( C_E \) approximates the available compensator \( C \).

**Remark 6.1.6** This conjecture implies a stronger claim than that of robustness in view of simultaneous perturbations of \( \hat{P} \) and \( C \). It concerns controller enhancement
and thus it concerns "favourable compensator perturbations" that improve the nominal performance. Furthermore this improvement must be robust in the face of the nominal model's deficiency. The conjecture does not relate a whole class of bounded perturbations of the available compensator $C$ to a whole class of bounded perturbations of the nominal model $\hat{P}$.

Notice that the conjecture connects the nominal and actual feedback systems $H(\hat{P}, C)$ and $H(P, C)$ via robustness properties. Hence it is more or less a cross-fertilization between simultaneous stabilization and robust stabilization with the requirement of stability replaced by that of performance. So far we have analyzed the problem of controller enhancement. In the next section we use this analysis to develop a procedure for the synthesis of an enhanced controller.

Remark 6.1.7 In [227, 226] Tay et al. proposed to identify $R_n$ and to construct an enhanced compensator through the design of an appropriate $R_c$. We do neither of these two for the following reasons. We want to apply the controller enhancement in an iterative scheme, by which the actual feedback system has to evolve from a (very) low performance to a (very) high performance control system. Suppose we would accomplish this evolution of the controller by redesigning $R_c$ over and over again. Then, in effect, the final $R_c$ has to replace the initial compensator by the high performance controller, and the eventual compensator $(N_c + \hat{D}R_c)(D_c - \hat{N}R_c)^{-1}$ would have redundant dynamics. A second reason involves the identification of $R_n$, which is no longer an open-loop identification if $R_c \neq 0$ (see Appendix E.2 for details). Therefore Tay et al. apply some adaptive control scheme, which involves the usual drawback that the initial compensator should be sufficiently close to the eventual compensator [226]. Instead we wish to start our iterative control design from open-loop.

6.2 Improvement of Robust Performance

We have just seen that an enhanced compensator $C_E$ can be regarded as a favourable "perturbation" of the available compensator $C$, provided that the nominal model $\hat{P}$ is sufficiently accurate. Here we develop a procedure, that enables a systematic search for such a favourable perturbation. The development of this procedure is divided into four parts

In the first part we adopt the control design method of Bongers and Bosgra [26, 25], which optimizes robustness in respect of coprime factor perturbations. This design method is based on an unconstrained optimization of the controller. The design procedure consists accordingly of selecting design weights, carrying out the optimization, and verifying whether the required robustness is achieved. Here we treat the selection of the design weights. The ascertainment of the robustness is dealt with in Section 6.3.

In the second part of this section we survey the use of the abovementioned method as a loop shaping control design technique. In the third part we apply this technique to
The Illustrative Example in order to disclose its utility in respect of Conjecture 6.1.5: the enhancement of the nominal performance can be related qualitatively to the enhancement of the performance achieved for the plant. Based on these experiences and the acquired understanding we set up a scenario for cautious controller enhancement at the end of this section. To this end we establish a quantitative link between the enhancement of the nominal performance and the enhancement of the plant’s performance. This link is reasoned rather than proven, and its implications are not fully known at the current state of affairs.

6.2.1 Robust Stabilization of Normalized Coprime Factors

The problem of robust stabilization of coprime factor representations has been put forward by Vidyasagar and Kimura [243]. We state this problem in terms of the right coprime factors $\tilde{N}, \tilde{D}$ of the nominal model $\tilde{P}$. Every system in $\mathcal{F}$ can be written as the perturbed nominal model $(\tilde{N} + \Delta N)(\tilde{D} + \Delta D)^{-1}$ for some $\Delta N, \Delta D \in \mathcal{H}$. It is possible to define particular families of systems around the nominal model $\tilde{P}$ by placing restrictions on the rcf $(\tilde{N}, \tilde{D})$ and on the coprime factor perturbation $(\Delta N, \Delta D)$.

The robust stabilization problem considered by Bongers [26, 25] optimizes robustness against perturbations of a normalized rcf $(\tilde{N}_n, \tilde{D}_n)$ of $\tilde{P}$. Further, the coprime factor perturbations $(\Delta N, \Delta D)$ of concern belong to the ball $\mathcal{B}(b_{ND_{\infty}})$, which is defined as

$$\mathcal{B}(b_{ND_{\infty}}) = \left\{ \begin{bmatrix} \Delta N \\ \Delta D \end{bmatrix} \mid \left\| \begin{bmatrix} \Delta N \\ \Delta D \end{bmatrix} \right\|_{\infty} < b_{ND_{\infty}} \right\}$$

with $b_{ND_{\infty}}$ being some positive real number. The corresponding perturbative family of systems is

$$\mathcal{P}_\Delta(\tilde{P}, b_{ND_{\infty}}) = \left\{ (\tilde{N}_n + \Delta N)(\tilde{D}_n + \Delta D)^{-1} \mid (\Delta N, \Delta D) \in \mathcal{B}(b_{ND_{\infty}}) \right\}.$$  

The robust stabilization problem is to construct a compensator, that stabilizes every system in the family $\mathcal{P}_\Delta(\tilde{P}, b_{ND_{\infty}})$. McFarlane and Glover have studied the dual robust stabilization problem, which is conceived in terms of a left coprime factorization of the nominal model [83, 84, 151]. A lot of these results carry over directly to the right coprime case.

McFarlane [151] has shown that the robust stabilization problem has a solution if and only if

$$b_{ND_{\infty}}^2 \leq 1 - \left\| \begin{bmatrix} \tilde{N}_n \\ \tilde{D}_n \end{bmatrix} \right\|_H^2 \doteq \rho_{ND_{\infty}}^2$$

where $\left\| . \right\|_H$ denotes the Hankel norm. The term $\rho_{ND_{\infty}}$ is the largest attainable robustness margin in the sense of $H_{\infty}$-bounded coprime factor perturbations. This optimal
robustness margin $\rho_{ND\infty}$ is achieved when the compensator is derived from the nominal model $\hat{P}$ through

$$C_{\hat{P}} = \arg \min_{C \in \mathcal{C}(\hat{P})} ||T(\hat{P}, C)||_{\infty}$$  \hspace{1cm} (6.5)$$

in which $\mathcal{C}(\hat{P})$ is again the set of all compensators that stabilize $\hat{P}$. Bongers [27] related the minimization of $||T(\hat{P}, C)||_{\infty}$ to the directed gap via the expression of (5.3). The resulting sufficient condition for robust stability reads as follows.

**Corollary 6.2.1** ([27]) Let $\hat{P}, C \in \mathcal{F}$ be such that $H(\hat{P}, C)$ is stable. Let $\hat{P}_\Delta \in \mathcal{F}$ be such that $\delta(\hat{P}, \hat{P}_\Delta) < b_{dg}$. Then $H(\hat{P}_\Delta, C)$ is stable if $||T(\hat{P}, C)||_{\infty} \leq 1/b_{dg}$.

Lastly, Georgiou and Smith [77, 78] have analyzed the optimization of robustness against normalized coprime factor perturbations, which turned out to be equivalent to robustness optimization in the gap metric.

### 6.2.2 Loop Shaping

The unconstrained minimization of (6.5) optimizes robustness against normalized coprime factor perturbations and, at the same time, it constitutes a trade-off between nominal performance and robust stability. The compensator $C_{\hat{P}}$ is designed in accordance with the usual design specifications concerning the sensitivity and the complementary sensitivity [150]. In case $P$ is a SISO plant, then $T_{11}(\hat{P}, C)$ is the complement of the sensitivity $T_{22}(\hat{P}, C)$. These terms contribute both to the $H_\infty$-norm of $T(\hat{P}, C)$. Thus the minimization of (6.5) trades off the magnitude of the sensitivity $T_{22}(\hat{P}, C)$ against the magnitude of the complementary sensitivity $T_{11}(\hat{P}, C)$. This trade-off is not as clear-cut as that of the mixed sensitivity optimization of (2.10).

Weighting functions can be used to adjust the trade-off between nominal performance and robust stability. The control design of (6.5) is extended with weights as follows. First two weights $W_N$ and $W_D$ are combined with the nominal model $\hat{P}$ to form the weighted nominal model

$$G = W_N \hat{P} W_D^{-1}.$$  

Then a controller $C_G$ is designed from this weighted nominal model $G$ according to

$$C_G = \arg \min_{C \in \mathcal{C}(G)} ||T(G, C)||_{\infty}.$$  \hspace{1cm} (6.6)$$

This controller $C_G$ is applied to $G = W_N \hat{P} W_D^{-1}$, which implies that $\hat{P}$ is controlled by $C_{\hat{P}, W} = W_D^{-1} C_G W_N$. The weights $W_N$ and $W_D$ can affect the order of $C_{\hat{P}, W}$ twice. Firstly, the order of $G$ will be larger than that of $\hat{P}$, which results in a high order compensator $C_G$. Secondly the order of $C_{\hat{P}, W} = W_D^{-1} C_G W_N$ is generally larger than that of $C_G$. 
The controller $C_{P,W}$ achieves optimal robustness against dynamical perturbations of the normalized coprime factors of $G = W_N \hat{P}W_D^{-1}$. Recall that the controller $C_P$ of (6.5) does the same for $\hat{P}$ instead of for $W_N \hat{P}W_D^{-1}$. From

$$T(G, C_G) = \begin{bmatrix} W_N \hat{P} \\ W_D \end{bmatrix} (I + C_{P,W} \hat{P})^{-1} \begin{bmatrix} C_{P,W} W_N^{-1} \\ W_D^{-1} \end{bmatrix}$$

it follows that the design of $C_{P,W}$ via $G$ and $C_G$ boils down to the weighted $H_\infty$-minimization

$$C_{P,W} = \arg \min_{C \in \mathcal{C}(\hat{P})} \left\| \begin{bmatrix} W_N & 0 \\ 0 & W_D \end{bmatrix} T(\hat{P}, C) \begin{bmatrix} W_N^{-1} \\ 0 \end{bmatrix} \right\|_\infty.$$ (6.7)

McFarlane and Glover proposed a “loop shaping” design procedure on the basis of this weighted minimization [150, 151]. They suggested to use the weights $W_N$, $W_D$ to define the desired shape of the loop $\hat{P}C_{P,W}$ as follows. The weighted nominal model $G = W_N \hat{P}W_D^{-1}$ should be given the desired properties of $\hat{P}C_{P,W}$, i.e. a large minimum singular value $\sigma_{\text{min}}(G(j\omega))$ in the low frequency range and a small maximum singular value $\sigma_{\text{max}}(G(j\omega))$ at the higher frequencies. If this weighted nominal model $G$ is subjected to the design of (6.6), then the resulting “loop-shape” $GC_G$ turns out to be conformable to $G = W_N \hat{P}W_D^{-1}$. Consequently, $H(G, C_G)$ will have its bandwidth close to the cross-over frequency of $G$. In the next example we elucidate why the cross-over frequency of the designed feedback system is close to the cross-over frequency of the utilized nominal model. The idea behind this example is due to Bongers [25].

**Example 6.2.2** We consider the SISO plant

$$\frac{1}{s^2 + 0.4s + 2}$$

whose magnitude and phase plots are shown in Fig. 6.2.a,b. We examine the normalized ncf of this plant before and after constant scaling by 100. Before scaling, the plant’s cross-over frequency is near 0.3 rad/s (see Fig. 6.2.a). The magnitudes of the normalized right coprime factors are drawn in Fig. 6.2.c and d (---). They are smaller than one by definition (see (5.2)). This explains why, at low frequencies, the large plant’s magnitude is reflected invertedly in the denominator term, while the numerator is about one. The converse holds for the high frequencies, where the plant’s magnitude is small. Thus in the high and low frequency ranges the numerator and denominator are either constant (no phase shift) or have a very small magnitude. As a consequence the dynamics of the ncf are significant near the cross-over frequency of the open-loop plant. Scaling by 100 makes the cross-over frequency move to 4.5 rad/s (see Fig. 6.2.a). An inspection of Fig. 6.2.c and d reveals that the corresponding ncf (---) indeed has its significant dynamics around the same frequency of 4.5 rad/s.
The control design method of (6.7) optimizes robustness against perturbations of a nrcf. Thus the compensator is effective especially at those frequencies, where the dynamics of the nrcf are significant. This is in agreement with the fact that the controller is effective near the bandwidth of the feedback system.

An increase of the bandwidth of the nominal feedback system involves of course a reduction of robustness. McFarlane and Glover [150] showed that the robustness margin can become unacceptably small if the performance requirements are high in regard of e.g. a right half-plane zero.

The “loop shaping” control design procedure contains some counter-intuitive aspects. Let us consider SISO systems, so that we may omit the weight $W_D$ from $G=W_N P W_D^{-1}$. Then we can rewrite $T(G, C_G)$ into

$$T(G, C_G) = \begin{bmatrix} W_N & 0 \\ 0 & I \end{bmatrix} T(\hat{P}, \hat{C}_p, \hat{W}) \begin{bmatrix} W_N^{-1} & 0 \\ 0 & I \end{bmatrix}$$
\[
= \begin{bmatrix}
\hat{P}(I+C_{\hat{P},W}\hat{P})^{-1}C_{\hat{P},W} & W_N\hat{P}(I+C_{\hat{P},W}\hat{P})^{-1} \\
(I+C_{\hat{P},W}\hat{P})^{-1}C_{\hat{P},W}W_N^{-1} & (I+C_{\hat{P},W}\hat{P})^{-1}
\end{bmatrix}
\]

Here we see that the \(W_N\) does not weigh the contributions \((I+C_{\hat{P},W}\hat{P})^{-1}\) and \(\hat{P}(I+C_{\hat{P},W}\hat{P})^{-1}\) and \(\hat{P}(I+C_{\hat{P},W}\hat{P})^{-1}\) of the sensitivity and the complementary sensitivity appear in the weighted \(H_\infty\)-minimization of (6.7) in exactly the same fashion as in the unweighted minimization of (6.5). Meanwhile, when \(W_N\) is large, then \(\hat{P}(I+C_{\hat{P},W}\hat{P})^{-1}\) is emphasized and \((I+C_{\hat{P},W}\hat{P})^{-1}\) is de-emphasized. The former requires a larger control effort, which is allowed by the latter. In conclusion the sensitivity and complementary sensitivity are influenced in some indirect fashion, and the one weight \(W_N\) embodies the trade-off between nominal performance and robust stability. For comparison, we mention that the mixed sensitivity problem is based on two weights, which judge the nominal performance and robust stability in direct and separate ways (cf. (2.10)).

Only little is known about the "indirect" loop shaping of the sensitivity and complementary sensitivity. In fact, the very few contributions in literature concern only constant weightings [150, 151]. Therefore we choose to use also only a constant scalar positive weight \(\alpha\), and we design the compensator \(C_{\hat{P}}\) according to

\[
C_{\hat{P}} = \arg\min_{C \in \mathcal{C}(\hat{P})} ||T(\alpha\hat{P}, C/\alpha)||_\infty. \tag{6.8}
\]

The designed feedback system has the same cross-over frequency as \(\alpha\hat{P}\). The weight \(\alpha\) can be used to "prescribe" the designed bandwidth: an increase of \(\alpha\) leads to an increase of the cross-over frequency of \(\alpha\hat{P}\), provided that \(\hat{P}\) rolls off at high frequency.

The above discussion on loop shaping concerns only robustness and performance for the nominal feedback system \(H(\hat{P}, C_{\hat{P}})\). We have to link these results to the actual feedback system \(H(P, C_{\hat{P}})\). Suppose we have a nominal model \(\hat{P}\) and a ball \(B(b_{ND\infty})\) of \(H_\infty\)-bounded nrcf perturbations, such that the family \(\mathcal{P}_\Delta(\hat{P}, b_{ND\infty})\) contains the plant \(P\). Further, suppose that the compensator \(C_{\hat{P}}\) has been derived as in (6.5) or equivalently as in (6.8) for \(\alpha = 1\). Then \(C_{\hat{P}}\) achieves the largest possible robustness margin \(\rho_{ND\infty}\) in respect of balls of \(H_\infty\)-bounded perturbations of the nrcf of \(\hat{P}\). It might happen that the optimal robustness margin is smaller than the ball of dynamical perturbations, i.e. \(\rho_{ND\infty} < b_{ND\infty}\). Then there does not exist a compensator that stabilizes all members of \(\mathcal{P}_\Delta(\hat{P}, b_{ND\infty})\).

Fortunately the stabilization of each member of \(\mathcal{P}_\Delta(\hat{P}, b_{ND\infty})\) is not the true design objective: we want to find an appropriate compensator for the plant \(P\). The next step would be to adjust the trade-off between nominal performance and robust stability in favour of the latter by choosing some \(\alpha < 1\) in (6.8). Then we can verify whether or not the resulting designed nominal feedback system has the required robustness properties. This reduction of \(\alpha\) is repeated until the required robustness is achieved. Conversely, when the achieved nominal performance is poor for \(\alpha = 1\), then we can iteratively search for some larger \(\alpha\), that yields a better nominal performance together with the
required robustness. We will propose a procedure for such an iterative increase of $\alpha$ in Section 6.2.4. But first we examine the utility of the design weight $\alpha$ by a repeated application of the control design method of (6.8) to The Illustrative Example.

### 6.2.3 The Illustrative Example, II

Here we apply the control design method of (6.8) to The Illustrative Example. The goal of this investigation is to discover some qualitative relationship between the design weight $\alpha$ on the one hand, and the nominal performance, the actual performance and the robust (or worst-case) performance on the other hand. More precise, we know that an increase of $\alpha$ leads to an improvement of the nominal performance, but the question is how far $\alpha$ may be increased, so that the plant's performance and the robust performance are improved as well.

We begin with making precise the systems considered in this example. At a later stage we explain why this particular case has been chosen. The plant $P$ is the SISO system of order 8, whose parameters can be found in Table A.1. We derive the plant-based compensator

$$C_p = \frac{0.517s^4+5.8983s^3+59.432s^2+218.11s+230.07}{s^4+24.133s^3+310.66s^2+277.10s+988.35}$$

by substituting $P$ for $\hat{P}$ in (6.8) with $\alpha=0.4$. We connect this compensator $C_p$ to the plant, and we derive the (low order) nominal model

$$\hat{P} = \frac{0.0167s^5+0.28388s^4+39.860s^3+2455.1s^2+1577.7s+6477.3}{s^5+12.678s^4+154.06s^3+700.83s^2+205.48s+360.16}$$

by a fixed-loop performance identification. The particular identification method is explained in Section 7.3. For now it is sufficient to know, that the feedback matrices $T(P,C_P)$ and $T(\hat{P},C_P)$ are very much alike. Consequently $T(P,C_P)$ and $T(\hat{P},C_P)$ will respond similarly to small compensator perturbations. This makes $\hat{P}$ suited for a small controller enhancement.

As $\hat{P}$ has been derived in the presence of a plant-based compensator with $\alpha=0.4$, we expect this nominal model to be suited for the design of (6.8) with the same $\alpha$. We take 15 $\alpha$'s to design as many controllers of order 4. These $\alpha$'s are

$$\begin{align*}
\alpha_1 &= 0.0239 & \alpha_4 &= 0.0800 & \alpha_7 &= 0.2675 & \alpha_{10} &= 0.8944 & \alpha_{13} &= 2.9907 \\
\alpha_2 &= 0.0358 & \alpha_5 &= 0.1196 & \alpha_8 &= 0.4 & \alpha_{11} &= 1.3375 & \alpha_{14} &= 4.4721 \\
\alpha_3 &= 0.0535 & \alpha_6 &= 0.1789 & \alpha_9 &= 0.5981 & \alpha_{12} &= 2.0000 & \alpha_{15} &= 6.6874 
\end{align*}$$

which are equally spaced on a logarithmic scale and centered around $\alpha_8=0.4$. With (6.8) we construct a compensator $C_i$ for each $\alpha_i$, $i = 1, \ldots, 15$.

The weighted $H_\infty$-norm of the feedback matrix $T(P,C_i)$, that is minimized in (6.8), embodies for each $\alpha$ a particular trade-off between nominal performance and
robustness. For instance $\|T(\alpha_8 \hat{P}, C_8/\alpha_8)\|_\infty$ and $\|T(\alpha_9 \hat{P}, C_9/\alpha_9)\|_\infty$ concern different compensators at different trade-offs. Comparing these quantities is not meaningful. Therefore we introduce the criterion

$$J_{\infty}(\hat{P}, C_i, \alpha_k) \doteq \|T(\alpha_k \hat{P}, C_i/\alpha_k)\|_\infty.$$  

We can use the criterion $J_{\infty}(\hat{P}, C_i, \alpha_k)$ to compare all compensators $C_i, i = 1, \ldots, 15$ in regard of the trade-off corresponding to some $\alpha_k$. We are actually not interested in the value of each criterion $J_{\infty}(\hat{P}, C_i, \alpha_k), i = 1, \ldots, 15$, but we just want to order the compensators $C_i$ according to their criteria for some $\alpha_k$. As this is a relative property we may normalize the criteria to

$$J_{\infty,n}(\hat{P}, C_i, \alpha_k) = \frac{J_{\infty}(\hat{P}, C_i, \alpha_k)}{\min\{J_{\infty}(\hat{P}, C_j, \alpha_k), j = 1, \ldots, 15\}}.$$  

This means that the minimum of the criteria $J_{\infty,n}(\hat{P}, C_j, \alpha_k), i = 1, \ldots, 15$ equals one for each $\alpha_k$.

We apply all compensators successively to the nominal model $\hat{P}$ and we calculate the criteria $J_{\infty,n}(\hat{P}, C_i, \alpha_k)$ for all compensators $C_i, i = 1, \ldots, 15$ and all weights $\alpha_k, k = 1, \ldots, 15$. The criteria $J_{\infty,n}(\hat{P}, C_i, \alpha_8), i = 1, \ldots, 15$ have been plotted in Fig. 6.3.a (---). In here the horizontal axis corresponds to the index $i = 1, \ldots, 15$. The curve is minimal for $i = 8$, which means that $C_8$ is the best controller in view of the trade-off corresponding to $\alpha_8$. This result is all but surprising, because $C_8$ has been derived as the minimizing argument of $\|T(\alpha_8 \hat{P}, C/\alpha_8)\|_\infty = J_{\infty}(\hat{P}, C, \alpha_8)$. Fig. 6.3.a shows also the curves that belong to $\alpha_1$ and $\alpha_{15}$. It is reassuring to know that these curves are minimal for respectively $C_1$ and $C_{15}$.

The same property holds for all $\alpha_k, k = 1, \ldots, 15$. This has been depicted in Fig. 6.3.b, where each criterion $J_{\infty,n}(\hat{P}, C_k, \alpha_k)$ has been marked 'o'. These criteria are all equal to 1, which implies that $C_k$ is the best compensator in view of the trade-off corresponding to $\alpha_k$ for all $k = 1, \ldots, 15$.

The curves of Fig. 6.3.b are represented by a 3-D plot in Fig. 6.3.c. The zero-level of this picture is indicated by the strips at the left and right "fronts". The left front of the "sculpture" corresponds to the curve $J_{\infty,n}(\hat{P}, C_i, \alpha_{15})$, i.e. the dotted line of Fig. 6.3.a. The right back of the "sculpture" corresponds to the curve $J_{\infty,n}(\hat{P}, C_i, \alpha_1)$, i.e. the dashed line of Fig. 6.3.a. All curves are drawn with an upper limit of 10, which explains the leveled pieces at the upper side. This upper limit brings about the image of a channel. The bottom of this channel coincides with the coordinates $(i, i), i = 1, \ldots, 15$, which is in agreement with the markers of Fig. 6.3.b. This is amplified once more by the contour plot of Fig. 6.3.d: the lowest level coincides with the diagonal $(C_i, \alpha_i)$ from the upper left corner to the bottom right corner. The other two corners correspond to the level of 10, and the contours are equidistant on a logarithmic scale. The markers
in Fig. 6.3.d will be of use in a minute.

We apply all compensators successively to the plant $P$, and we evaluate the normalized control criteria $J_{\infty,n}(\hat{P}, C_i, \alpha_k)$ for $i = 1, \ldots, 15$ and $k = 1, \ldots, 15$. The solid line in Fig. 6.4.a represents the series of criteria $J_{\infty,n}(P, C_i, \alpha_k), i = 1, \ldots, 15$. This curve is minimal for $C_8$, which implies that $C_8$ is the best compensator for the plant in view of the trade-off corresponding to $\alpha_k$. — By “the best compensator” we mean the best out of $\{C_1, \ldots, C_{15}\}$. — From the conformity of the solid curves in Fig. 6.3.a and Fig. 6.4.a we draw the following conclusion: the feedback systems $H(\hat{P}, C_8)$ and $H(P, C_8)$ respond in a similar fashion to the small perturbations of $C_8$ that turn it into e.g. $C_9$, $C_{10}$ or $C_7$. Moreover we can regard $C_8$ as an approximate solution to the problem of designing an optimal compensator from the nominal model $\hat{P}$ for the plant $P$ at the
Fig. 6.4: Normalized control criteria for the plant \( P \).

\( a: J_{\infty,n}(P, C_i, \alpha_8) \) \((-\), \( J_{\infty,n}(P, C_i, \alpha_1) \) \((-\), \( J_{\infty,n}(P, C_i, \alpha_{15}) \) \(\cdots\).  

\( c: 3\text{-D representation.} \)

\( b: J_{\infty,n}(P, C_i, \alpha_k), k = 1, \ldots, 15. \)

\( d: \text{Contour-plot.} \)

trade-off level \( \alpha_8 \). In a formula this reads

\[
C_8 = \arg \min_{C \in \mathcal{C}(\hat{P})} \| T(\alpha_8 \hat{P}, C/\alpha_8) \|_{\infty} \approx \arg \min_{C \in \mathcal{C}(P, \hat{P}, \alpha)} \| T(\alpha_8 P, C/\alpha_8) \|_{\infty}
\]
and \( C(P, \hat{P}, \alpha) \) signifies the class of all compensators that stabilize \( P \), and that can be designed from \( \hat{P} \) by (6.8) for some positive real number \( \alpha \).

On the other hand the courses of the criteria \( J_{\infty,n}(P, C_i, \alpha_1) \) and \( J_{\infty,n}(P, C_i, \alpha_{15}) \) differ from their counterparts for the nominal model \( \hat{P} \) (cf. Figures 6.3.a and 6.4.a). The controllers \( C_1 \) and \( C_{15} \) are not the best for the plant at the respective performance levels \( \alpha_1 \) and \( \alpha_{15} \). Instead \( C_3 \) and \( C_{14} \) are the best compensators for the plant \( P \) at those trade-offs. Thus, unlike \( C_8 \), neither \( C_1 \) nor \( C_{15} \) is an approximate solution to the problem of designing a compensator from the nominal model \( \hat{P} \) for the plant \( P \) at the respective design weights.
Remark 6.2.3 $C_1$ and $C_{15}$ are also not an approximate solution to the problem of designing a compensator from $P$ instead of from $\hat{P}$. That is

$$C_1 \not= \arg \min_{C \in \mathcal{C}(P)} \|T(\alpha_1 P, C/\alpha_1)\|_\infty$$

and the like for $C_{15}, \alpha_{15}$. The reason is that $C_1, C_{15}$ are not (nearly) optimal over $\mathcal{C}(P, \hat{P}, \alpha)$, and thus they are also not (nearly) optimal over the wider class $\mathcal{C}(P)$. The converse is not necessarily true for the compensator $C_8$: it is not guaranteed that $C_8$ is an approximate solution to $\min_{C \in \mathcal{C}(P)} \|T(\alpha_8 P, C/\alpha_8)\|_\infty$. This is not a severe drawback, because in practice $P$ is not precisely known, and a minimization over $\mathcal{C}(P, \hat{P}, \alpha)$ is the best that we can do. \hfill \Box

Remark 6.2.4 The plots in Fig. 6.4 reveal that the nominal model $\hat{P}$ is suited to design controllers for $\alpha$’s around 0.4. They also show that $P$ and $\hat{P}$ respond differently for relatively large and small values of $\alpha$. This “symmetry” had not been discovered if we would have used an open-loop model, because the open-loop operation is the limit for $\alpha \to 0$. \hfill \Box

The nominal model $\hat{P}$ has been custom-made for the combination of the plant $P$, the control objective of (6.8) and $\alpha = \alpha_8$. No wonder that $\hat{P}$ does not lead to satisfactory results for $\alpha_1$ and $\alpha_{15}$. It turns out that $\hat{P}$ is of good use at $\alpha = \alpha_8$. We remark that we have not given any mathematical evidence of this fact. For we have the following facts:

i. the plant-based compensator $C_P$ is optimal for $P$ in view of $\alpha_8$,

ii. the nominal model $\hat{P}$ describes $P$ in respect of $C_P$,

iii. the model based compensator $C_8$ is optimal for $\hat{P}$ in view of $\alpha_8$.

Facts i. and ii. together imply that $C_P$ is good for $\hat{P}$ in view of $\alpha_8$. However these 3 facts do not imply that $C_8$ is the best compensator for $P$ that can be designed from $\hat{P}$.

We assume that the whole setting has certain robustness properties, from which we expect that if $\alpha$ is close enough to $\alpha_8$, then in view of $\alpha$ the corresponding compensator is (almost) the best compensator for $P$ that can be designed from $\hat{P}$. In a formula this reads as

$$\arg \min_{C \in \mathcal{C}(\hat{P})} \|T(\alpha \hat{P}, C/\alpha)\|_\infty \approx \arg \min_{C \in \mathcal{C}(P, \hat{P}, \alpha)} \|T(\alpha P, C/\alpha)\|_\infty$$

just like in case of $\alpha = \alpha_8$. We examine such a behavior from Fig. 6.4.b, where the criteria $J_{\infty,n}(P, C_i, \alpha_k), i = 1, .., 15$ have been plotted for each $\alpha_k, k = 1, .., 15$, and each pair $(C_k, \alpha_k)$ has been marked ‘o’. Unlike in Fig. 6.3.b some markers are not at level 1, and the corresponding compensators are not optimal for the plant $P$ in view of their “own” trade-offs. On the other hand the compensators $C_i, i = 6, .., 10$ may be regarded as an approximate solution in the above sense.
Fig. 6.5: Upper bounds $J_\infty(\hat{P}, C_i, \alpha_k) + J_\Delta(\hat{P}, C_i, \alpha_k)$ normalized as in (6.11).

a: For $\alpha_8$ (—), $\alpha_1$ (—), and $\alpha_{15}$ (…).
b: Upper bounds for $\alpha_k$, $k = 1, \ldots, 15$.
c: 3-D representation.
d: Contour-plot.

In order to find a qualitative relationship between $\alpha$ and the various performances, we represent $J_{\infty,n}(P, C_i, \alpha_k), i = 1, \ldots, 15, k = 1, \ldots, 15$ by a 3-D plot and a contour plot just as for the nominal model $\hat{P}$. The markers ‘o’ of Fig. 6.4.b imply that for the first and last $\alpha_k$’s the diagonal $(C_k, \alpha_k)$ has not the smallest criterion. Indeed Fig. 6.4.c and Fig. 6.4.d show that the “channel” does not run parallel to this diagonal. In order to detect some qualitative property we urge the reader to visualize the following “experiment”:

We start in the middle of the channel of Fig. 6.3 at $(C_8, \alpha_8)$. This point has been marked ‘+’ in Fig. 6.3.d. Then we move towards $(C_{15}, \alpha_{15})$ along the diagonal, and we inspect the ratios

$$J_{\infty,n}(\hat{P}, C_8, \alpha_k)/J_{\infty,n}(\hat{P}, C_k, \alpha_k)$$

at each point $(C_k, \alpha_k)$. For instance we divide the criterion of $(C_8, \alpha_{14})$, marked ‘x’ in Fig. 6.3.d, by that of $(C_{14}, \alpha_{14})$, marked ‘o’ in the same
contour plot. While the \((C_k, \alpha_k)\)'s are all on the same level, the criterion of \((C_8, \alpha_k)\) increases more and more.

Next let us also move along the diagonal of Fig. 6.4. Again we start in the middle, i.e. at '+' in Fig. 6.4.d. Now not all \((C_k, \alpha_k)\)'s are on the bottom of the “channel”, but the criterion increases (see e.g. \((C_{14}, \alpha_{14})\) marked 'o' in Fig. 6.4.d). Meanwhile the levels of the \((C_8, \alpha_k)\)'s, \(k > 8\), increase less than in Fig. 6.3.d. These two effects make that, as \(\alpha_k\) increases, the ratio \(J\_{\infty, n}(\hat{P}, C_8, \alpha_k)/J\_{\infty, n}(\hat{P}, C_k, \alpha_k)\) gets ahead of the ratio \(J\_{\infty, n}(P, C_8, \alpha_k)/J\_{\infty, n}(P, C_k, \alpha_k)\). With \(C_k\) regarded as an enhanced compensator this means, that the ratio of the available and enhanced nominal performance norms grows faster than the ratio of the available and enhanced performance norms of the plant.

From the difference between the two ratios in the above “experiment” we make the following observation. Suppose that \(C_8\) is applied to the plant, and suppose that we want to enhance this compensator. We design \(C_{\hat{P}}\) from \(\hat{P}\) via (6.8) for some large \(\alpha\). Then the ratios are significantly different meaning that the feedback systems \(H(P, C_8)\) and \(H(\hat{P}, C_8)\) respond differently to the pair of \(\alpha\) and \(C_{\hat{P}}\). Because of this “disharmony” we reject this enhanced compensator \(C_{\hat{P}}\), and we must find another \(\alpha\) or probably another nominal model \(\hat{P}\).

Lastly we comment on the robust performance. From (2.13) we recall that the performance norm of the plant is upper bounded by the sum of the nominal performance norm and the performance degradation. We reflect the performance degradation by the criterion

\[
J^{\Delta}_\infty(\hat{P}, C_i, \alpha_k) = ||T(\alpha_k P, C/\alpha_k) - T(\alpha_k \hat{P}, C/\alpha_k)||_{\infty},
\]

so that the upper bound can be written as

\[
J_\infty(P, C_i, \alpha_k) \leq J_\infty(\hat{P}, C_i, \alpha_k) + J^{\Delta}_{\infty}(\hat{P}, C_i, \alpha_k).
\]

The criterion \(J^{\Delta}_{\infty}(\hat{P}, C_i, \alpha_k)\) is calculated for all \(i = 1, \ldots, 15\) and \(k = 1, \ldots, 15\), and the upper bounds are normalized to

\[
\frac{J_\infty(\hat{P}, C_i, \alpha_k) + J^{\Delta}_{\infty}(\hat{P}, C_i, \alpha_k)}{\text{min}(J_\infty(\hat{P}, C_j, \alpha_k) + J^{\Delta}_{\infty}(\hat{P}, C_j, \alpha_k), j = 1, \ldots, 15)}.
\]

Fig. 6.5 displays these normalized robust performances in exactly the same way as the previous two figures. A comparison of each plot in Fig. 6.5 with its counterpart in Fig. 6.4 reveals that the robust performance and the plant's performance are similar in their relation to the design weight \(\alpha\) and to the nominal performance. Thus a “disharmony” between the robust performance enhancement and the nominal performance enhancement also indicates, that the corresponding enhanced compensator should be rejected. Below we use this experience to set up a scenario for cautious controller enhancement.
6.2.4 Scenario for Cautious Controller Enhancement

Here we develop a scenario for cautious controller enhancement. We intend to design the enhanced compensator $C_E$ from the nominal model $\hat{P}$ by the optimization of (6.8). Our main point of concern is the selection of an appropriate design weight $\alpha$.

The scenario is customized for the primary iterative scheme of repeated identification and control design proposed in Section 2.4, which justifies the following assumptions. The uncertain plant $P$ is stabilized by a known compensator $C$, that has been designed accordingly to (6.8) for $\alpha = \alpha_0$ and from some old nominal model. Further the available newly identified nominal model $\hat{P}$ is assumed to be accurate in the sense of (2.13), i.e.

$$||T(\alpha_0 P, C/\alpha_0) - T(\alpha_0 \hat{P}, C/\alpha_0)||_{\infty} \leq ||T(\alpha_0 \hat{P}, C/\alpha_0)||_{\infty} \approx ||T(\alpha_0 P, C/\alpha_0)||_{\infty}. \quad (6.12)$$

We use this nominal model $\hat{P}$ to design the compensator $C_E$ by means of the method of (6.8). The question is how to choose $\alpha \geq \alpha_0$ such that the resulting nominal feedback matrix $T(\hat{P}, C_\rho)$ reliably predicts the performance of $H(P, C_\rho)$. As $T(\hat{P}, C_\rho)$ improves upon $T(P, C)$ it is likely that $T(P, C_\rho)$ improves upon $T(P, C)$.

From Proposition 5.2.1 it follows that $||T(\alpha X, C/\alpha) - T(\alpha \hat{P}, C/\alpha)||_{\infty}$ metrises the graph topology for all systems $X \in \mathcal{F}$ in some neighborhood of $\hat{P}$. In other words, the compensator $C$ achieves some robust performance in a neighborhood of $\hat{P}$, which contains the plant $P$. The objective is to design an enhanced compensator $C_E$ that improves the robust performance for a possibly smaller neighborhood of $\hat{P}$, which also must of course contain the plant $P$. This brings us back to Conjecture 6.1.5, by which we claimed that an enhancement of the nominal performance induces a similar enhancement of the plant's performance, provided that the controller enhancement is small in regard of the deficiency of the nominal model $\hat{P}$. Thus it depends on the quality of the nominal model $\hat{P}$, whether or not an enhancement is possible at all.

Suppose we design the compensator $C_{\hat{P}, \alpha_0}$ by (6.8) from $\hat{P}$ for $\alpha = \alpha_0$. The Illustrative Example, II, Section 6.2.3, has learned that $H(P, C)$ and $H(\hat{P}, C)$ respond similarly to the replacement of $C$ by $C_{\hat{P}, \alpha_0}$, provided that the nominal model is sufficiently accurate in the sense of (6.12). Hence if $C_{\hat{P}, \alpha_0}$ deteriorates the plant's performance or the robust performance, then $\hat{P}$ is not good enough for controller enhancement. In such a case we must first improve the quality of the nominal model, until the model-based compensator $C_{\hat{P}, \alpha_0}$ changes $H(P, C)$ about as much as $H(\hat{P}, C)$. Such a nominal model exists by virtue of Proposition 6.1.3.

Now suppose that $C_{\hat{P}, \alpha_0}$ works well with the plant $P$. Then we attempt to enhance the controller by choosing a larger weight $\alpha$. This will improve, inter alia, the bandwidth of the nominal feedback system and hopefully also that of the controlled plant. However, if $\alpha$ is much larger than $\alpha_0$, then the trade-off is too much in favour of the nominal performance. This will result in a change of the robust performance, that is no longer similar to the improvement of the nominal performance. So we have to find
out how much $\alpha$ may be enlarged to improve the nominal performance as well as the robust performance.

In finding an appropriate design weight $\alpha$ we build on The Illustrative Example, II. We have seen that a compensator designed for some $\alpha$ is optimal for the nominal model, only if the performance is measured by the $H_\infty$-norm of the corresponding $\alpha$-weighted feedback matrix

$$T(\alpha \hat{P}, C_E/\alpha) = W_\alpha T(\hat{P}, C_E) W_\alpha^{-1}$$

with $W_\alpha \doteq \text{diag}(\alpha I, I)$. In order to monitor various phenomena as a function of frequency we judge the controller enhancement from the frequency dependent maximum singular value $\sigma_{\max}(W_\alpha T(\hat{P}, C_E)(j\omega) W_\alpha^{-1})$.

Reverting to the "disharmonies" discussed at the end of Section 6.2.3 we base the selection on the ratios of available and enhanced performances. For notational convenience we define

$$\Pi(P)(j\omega) \doteq \frac{\sigma_{\max}(W_\alpha T(P, C)(j\omega) W_\alpha^{-1})}{\sigma_{\max}(W_\alpha T(\hat{P}, C_E)(j\omega) W_\alpha^{-1})}$$

$$\Pi(\hat{P})(j\omega) \doteq \frac{\sigma_{\max}(W_\alpha T(\hat{P}, C)(j\omega) W_\alpha^{-1})}{\sigma_{\max}(W_\alpha T(P, C_E)(j\omega) W_\alpha^{-1})}$$

$$\Pi_\Delta(C_E)(j\omega) \doteq 1 + \frac{\sigma_{\max}(W_\alpha \left[ T(P, C_E)(j\omega) - T(\hat{P}, C_E)(j\omega) \right] W_\alpha^{-1})}{\sigma_{\max}(W_\alpha T(\hat{P}, C)(j\omega) W_\alpha^{-1})}$$

$$\Pi_\Delta(C)(j\omega) \doteq 1 + \frac{\sigma_{\max}(W_\alpha \left[ T(P, C)(j\omega) - T(\hat{P}, C)(j\omega) \right] W_\alpha^{-1})}{\sigma_{\max}(W_\alpha T(\hat{P}, C)(j\omega) W_\alpha^{-1})}$$

The function $\Pi(P)$ reflects the ratio of the available and enhanced performances achieved for the plant. In terms of The Illustrative Example, II, it equals e.g. the ratio $J_{\infty,n}(\hat{P}, C_8, \alpha_{14})/J_{\infty,n}(P, C_{14}, \alpha_{14})$ or equivalently the ratio of the values marked 'x' and 'o' in Fig. 6.4.d. Likewise $\Pi(\hat{P})$ represents the ratio of the available and enhanced nominal performances.

In Section 6.2.3 we established that these two ratios are almost equal for those pairs $C_\hat{P}$ and $\alpha$, that make an approximate solution to the problem of designing a compensator for the plant and from the nominal model. In turn, in selecting an appropriate $\alpha$ we demand that the ratio's $\Pi(P)$ and $\Pi(\hat{P})$ are alike, i.e. that the ratio $\Pi(P)/\Pi(\hat{P})$ is nearly 1. As we cannot deduce how near the latter ratio must be to 1, we simply dictate that $\alpha$ must be selected such that the inequalities

$$0.7 \leq \frac{\Pi(P)(j\omega)}{\Pi(\hat{P})(j\omega)} \leq 1.3$$

are satisfied for each frequency $\omega \in \mathbb{R}$. At the same time we demand that

$$0.7 \leq \frac{\Pi_\Delta(C_E)(j\omega)}{\Pi_\Delta(C)(j\omega)} \leq 1.3$$
is satisfied for each frequency $\omega \in \mathbb{R}$. The latter ratio relates the robust performances of Fig. 6.5 to the nominal performances of Fig. 6.3 in a way just like $\Pi(P)/\Pi(\hat{P})$ relates the performances of Fig. 6.4 and Fig. 6.3 together.

The 30\% bounds of (6.13) and (6.14) are based on the observations made in Section 6.2.3. Only little is known about their validity in other control problems. Nevertheless it is clear that these bounds determine the allowed extent of controller enhancement. In the experimental verification of Chapter 9 we use 5\% bounds in order to be more cautious.

Remark 6.2.5 The inequalities of (6.13) and (6.14) can conceptually always be met. For if $C_E = C$, then $\Pi(P) = 1$, $\Pi(\hat{P}) = 1$ and $\Pi_\Delta(C_E) = \Pi_\Delta(C)$. However the controller $C_E$, that results from the method of (6.8) for $\hat{P}$, will always differ from $C$. In fact, due to the design method, there is a minimum “difference” between $C$ and $C_E$. We elaborate this somewhat further.

The nominal model $\hat{P}$ used in the control design stage of an iterative scheme has been based on the available controller $C$. Hence $C$ has not been designed from $\hat{P}$. As the weight $\alpha$ is the only design tool used in the controller enhancement, the class of compensators that can be brought about by the method of (6.8) is quite restricted. It will be much smaller than the class $C(\hat{P})$ of all compensators that stabilize $\hat{P}$. In general the “old” compensator $C$ does not belong to this restricted class and there exists no sequence of $\alpha$’s such that the controllers designed from $\hat{P}$ converge to $C$. Hence the controller enhancement cannot be made arbitrarily small.

In case the controller $C$ has been designed from a nominal model that differs much from $\hat{P}$, then the minimal enhancement will be large. Then we need quite an accurate nominal model, because $H(P, C)$ and $H(\hat{P}, C)$ have to respond similarly to a large perturbation of $C$. Otherwise the conditions of (6.13) and (6.14) are not satisfied. Hence these inequalities indirectly impose a condition on the accuracy of the nominal model $\hat{P}$.

Remark 6.2.6 It can be reasoned but not (yet) proven that the conditions in (6.13) and (6.14) pertain to the Propositions 6.1.3 and 6.1.4.

So far we have based the controller enhancement on singular values of transfer functions that involve the plant $P$. This works for a simulation study, in which the uncertain plant $P$ is actually available. In order to make the controller enhancement applicable to real systems we replace the transfer functions with estimated frequency responses. For this purpose we use the double $R$-parameterization of Section 6.1.

We define the weighted filter $F_\alpha := FW_\alpha$. With the expressions of (6.1) and (6.2)
for the feedback matrices we can rewrite the ratios used in (6.13) and (6.14) into

\[
\Pi(P)(j\omega) = \sigma_{\text{max}}(F_{\alpha} \begin{bmatrix} 0 & R_n \\ 0 & I \end{bmatrix} F_{\alpha}^{-1}) / \sigma_{\text{max}}(F_{\alpha} \begin{bmatrix} R_n & 0 \\ I & I \end{bmatrix} (I + R_c R_n)^{-1} \begin{bmatrix} R_c & I \\ 0 & I \end{bmatrix} F_{\alpha}^{-1})
\]

\[
\Pi(\hat{P})(j\omega) = \sigma_{\text{max}}(F_{\alpha} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} F_{\alpha}^{-1}) / \sigma_{\text{max}}(F_{\alpha} \begin{bmatrix} 0 & 0 \\ R_c & I \end{bmatrix} F_{\alpha}^{-1})
\]

and

\[
\Pi_{\Delta}(C_E)(j\omega) = 1 + \frac{\sigma_{\text{max}}(F_{\alpha} \begin{bmatrix} R_n \\ -R_c R_n \end{bmatrix} (I + R_c R_n)^{-1} \begin{bmatrix} R_c & I \\ 0 & I \end{bmatrix} F_{\alpha}^{-1})}{\sigma_{\text{max}}(F_{\alpha} \begin{bmatrix} 0 & 0 \\ R_c & I \end{bmatrix} F_{\alpha}^{-1})}
\]

\[
\Pi_{\Delta}(C)(j\omega) = 1 + \sigma_{\text{max}}(F_{\alpha} \begin{bmatrix} 0 & R_n \\ 0 & 0 \end{bmatrix} F_{\alpha}^{-1}) / \sigma_{\text{max}}(F_{\alpha} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} F_{\alpha}^{-1})
\]

in which all the entries \( R_n, R_c \) and \( F_{\alpha} \) must be evaluated as a function of frequency. The frequency response of \( R_c \) can be calculated from \( R_c = (\hat{D} + C_E \hat{N})^{-1} (C_E D_c - N_c) \) in which \( (\hat{N}, \hat{D}) \) and \( (N_c, D_c) \) are rcf's of \( \hat{P} \) and \( C \). The frequency response of \( R_n \) can be estimated from measurements as explained in Section 4.3.

In summary our procedure for cautious controller enhancement consists of the control design of (6.8) and the selection of a design weight \( \alpha \) such that the inequalities in (6.13) and (6.14) are satisfied. For the evaluation of these inequalities we use frequency responses or their estimates of \( R_n \) and \( R_c \) as design indicators. The selection of \( \alpha \) is carried into effect by an iterative search. The design weight \( \alpha \) is initially set to \( \alpha_e \), which has been used in the design of the available compensator \( C \). Subsequently \( \alpha \) is stepwise increased until one of the inequalities is violated. Then \( \alpha \) is reset to its penultimate value and the corresponding compensator is taken as the enhanced compensator \( C_E \). We illustrate this procedure by the following example.
Fig. 6.6: Cautious controller enhancement based on frequency dependent "disharmonies".

a: $\Pi(P)/\Pi(\hat{P})$ for $\alpha = 6.65$ (---), and 0.5, 3 and 4.5 (3x(...)).

b: $\Pi_\Delta(C_E)/\Pi_\Delta(C)$ for $\alpha = 6.65$ (---), and 0.5, 3 and 4.5 (3x(...)).

c: $\Pi(P)$ (---), $\Pi(\hat{P})$ (---), $\Pi_\Delta(C_E)$ (---), $\Pi_\Delta(C)$ (---).

d: Sensitivity functions for $P, C_E$ (---), $\hat{P}, C_E$ (---), $P, C$ (---), $\hat{P}, C$ (---).
Example 6.2.7 We take the plant $P$ and the nominal model $\hat{P}$ from The Illustrative Example, II, Section 6.2.3. Further we let $C_8$ of that example be the available compensator. Then we increase $\alpha$ stepwise from 0.4 to 6.65. The resulting compensator is
\[ C_P = \frac{29.153s^4+645.33s^3+6588.5s^2+6159.2s+17900}{s^4+101.58s^3+3046.1s^2+2074.8s+7949.2}. \]

The interesting question is, how do the ratios and "disharmonies" look like? Fig. 6.6.a shows the "disharmony" of (6.13), i.e. the ratio $\Pi(P)$ of the available and enhanced plant's performances divided by the ratio $\Pi(\hat{P})$ of the available and enhanced nominal performances. In Fig. 6.6.b we see the "disharmony" of (6.14), i.e. the ratio $\Pi_\Delta(C_E)$ of the enhanced robust and nominal performances divided by the ratio $\Pi_\Delta(C)$ of the available robust and nominal performances. Besides, the dotted lines in these two plots correspond to the designs for $\alpha = 0.5, 3$ and 4.5. Further, the following ratios have been depicted in Fig. 6.6.c:

- $\Pi(P) = ||T(P, C)||/||T(P, C_E)||$ ($\rightarrow$),
- $\Pi(\hat{P}) = ||T(\hat{P}, C)||/||T(\hat{P}, C_E)||$ ($\rightarrow$),
- $\Pi_\Delta(C_E) = 1 + ||T(P, C_E) - T(\hat{P}, C_E)||/||T(\hat{P}, C_E)||$ ($\leftarrow$),
- $\Pi_\Delta(C) = 1 + ||T(P, C) - T(\hat{P}, C)||/||T(\hat{P}, C)||$ ($\leftarrow$).

where $||X||$ is used as a shorthand for $\sigma_{\text{max}}(W_\alpha X(j\omega)W_\alpha^{-1})$. Finally Fig. 6.6.d displays the utility of the cautious controller enhancement: the enhanced sensitivities for the plant ($\rightarrow$) and for the nominal model ($\leftarrow$) make a significant improvement upon the available sensitivities, respectively ($\rightarrow$) and ($\leftarrow$).

In conclusion we address the predictive character of the cautious controller enhancement. The procedure rests on an estimate of the frequency response of $R_\alpha$. This frequency response estimate is usually accurate up to some bound of uncertainty (see e.g. Fig. 2.2). We have not incorporated this uncertainty in the evaluation of $\Pi(P), \Pi_\Delta(C_E)$ or $\Pi_\Delta(C)$. Moreover we use the frequency response estimate essentially to predict frequency responses of the possibly enhanced actual feedback system $H(P, C_E)$. Thus the improvement of the performance for the plant is predicted in an $L_\infty$-sense. Before the enhanced controller is applied, we have to ensure that the predicted performance holds in an $H_\infty$-sense as well. For this purpose we use the stability ascertainment of the next section.
6.3 Robustness Margins

Before a newly designed compensator\(^1\) \(C\) is applied to the plant \(P\), we want to make sure that the new control system \(H(P, C)\) will be stable. For this purpose we use robustness margins of the designed stable nominal feedback system \(H(\hat{P}, C)\) that are based on the small gain theorem. A robustness margin guarantees that all members of a family of systems around the nominal model \(\hat{P}\) are stabilized by the compensator \(C\). In the first part below we expose that robustness margins are generally conservative in the sense that the above family around \(\hat{P}\) is only a subset of the class \(\mathcal{P}(C)\) of all systems that are stabilized by \(C\). Different robustness margins guarantee stability for different subsets of \(\mathcal{P}(C)\). Hence stability of \(H(P, C)\) cannot be demonstrated by just any robustness margin. Instead we look for robustness margins with little conservatism in regard of the plant \(P\), the nominal model \(\hat{P}\) and the compensator \(C\). In Section 6.3.2 we begin our search with a condition for coprimeness of dynamically perturbed coprime factors. Then in Section 6.3.3 we use this result to retrieve the stability condition for gap-ball perturbations and the closely related result of Bongers [24]. Thereafter we introduce a new robustness margin that is tailor-made for our iterative scheme of repeated identification and control design: we use an “old” stabilizing compensator to show that the plant is stabilized by some “new” compensator. Finally in Section 6.3.5 we compare the various robustness margins and we show their capacities.

6.3.1 Conservatism of the Small Gain Theorem

An important implication of the small gain theorem [57] is that a stable system with a magnitude smaller than 1 remains stable under unity feedback. This theorem is well-suited to derive sufficient conditions for stability in the presence of dynamical perturbations. We expose this capacity for the additive dynamical perturbation \(\Delta_A\).

\[\Delta_A\]

\[\hat{P}\]

\[C\]

\[\rightarrow\]

\[\Delta_A\]

\[C(I+\hat{P}C)^{-1}\]

Fig. 6.7: Robust stability in the presence of unstructured additive perturbations.

\(^1\)In terms of the previous section the new compensator is the enhanced compensator \(C_E\). Here we denote the new compensator just by \(C\) for ease of notation.
by $\Delta_A$, then the closed-loop of the feedback system is affected in a way shown at the left of Fig. 6.7. The unperturbed closed-loop transfer function $C(I+\hat{P}C)^{-1}$ maps the output of $\Delta_A$ into the input of $\Delta_A$ as depicted at the right of Fig. 6.7. In this latter loop the “system” $\Delta_A C(I+\hat{P}C)^{-1}$ operates under unity feedback. As $C(I+\hat{P}C)^{-1}$ and $\Delta_A$ are stable we may apply the small gain theorem, and stability of the perturbed feedback system $H(\hat{P}+\Delta_A, C)$ is guaranteed if

$$\sigma_{\max} \left( \Delta_A(j\omega)C(j\omega)(I+\hat{P}(j\omega)C(j\omega))^{-1} \right) < 1,$$ \hspace{1cm} (6.15)

for each frequency $\omega \in \mathbb{R}$. In turn, the closed-loop transfer function $C(I+\hat{P}C)^{-1}$ allows only certain sets of additive perturbations. This pertains to the field of robustness analysis.

The aim of a robustness analysis of $H(\hat{P}, C)$ is to determine the largest family of systems around the nominal model $\hat{P}$, whose members are all stabilized by the compensator $C$. This analysis is composed of the following two steps.

1. Choose a class of dynamical perturbations $\Delta$.

2. Assess the corresponding robustness margin $\rho_\Delta$ of $H(\hat{P}, C)$.

The first step defines the class of systems that is considered at all. For instance in case of the additive dynamical perturbation $\Delta_A$ we consider the family $\{\hat{P}+\Delta_A \mid \Delta_A \in \mathcal{H}\}$. In the second step we determine the largest ball of additive dynamical perturbations that is allowed in view of stability. The radius $\rho_A$ of this ball is the additive robustness margin, which satisfies

$$\rho_A \approx 1/\sigma_{\max} \left( C(j\omega)(I+\hat{P}(j\omega)C(j\omega))^{-1} \right).$$

The largest ball-shaped family of robustly stabilized systems is $\mathcal{P}_\Delta(\hat{P}, \rho_A)$ (cf. Definition 2.2.2).

**Remark 6.3.1** The robustness margins $\rho_M$ and $\rho_A$ have been derived also for perturbations $\Delta_M$ and $\Delta_A$ that are not necessarily stable [64, 237]. These results rest on an evaluation of the loop gain $\Delta_A C(I+\hat{P}C)^{-1}$ on the standard Nyquist ’D’-contour. The perturbed nominal model and the unperturbed nominal model must have the same number of unstable poles. For our purposes this means that the plant $P$ must have as many unstable poles as the nominal model $\hat{P}$. On the other hand the plant $P$ is not precisely known, so that we can never verify its number of unstable poles unless that number is zero.  

The above remark shows that the stability of $H(P, C)$ can be ascertained with the robustness margin $\rho_M$ or $\rho_A$ of $H(\hat{P}, C)$ practically only if both $P$ and $\hat{P}$ are stable. In that case the family

$$\{\hat{P}+\Delta_A \mid \Delta_A \in \mathcal{H}\}$$
equals precisely the set $\mathcal{H}$ of all open-loop stable systems. On the other hand each non-trivial compensator $C$ stabilizes some unstable system. Thus all members of $\mathcal{H}$ that are stabilized by $C$ make a subset of $\mathcal{P}(C)$. — Recall that $\mathcal{P}(C)$ is the set of all systems that are stabilized by $C$. — In turn this subset has as a subset the ball-shaped family $\mathcal{P}_\Delta(\hat{P}, \rho_A)$ of robustly stabilized system. In summary $\mathcal{P}_\Delta(\hat{P}, \rho_A) \subset \mathcal{P}(C)$, which implies that the robustness margin is conservative. The only exception occurs for the zero-compensator $C_0 = 0$. For this compensator $\mathcal{P}(\hat{P}, \rho_A(C_0))$ and $\mathcal{P}(C_0)$ coincide with the above family $\mathcal{H}$. Thus the additive robustness margin $\rho_A$ is non-conservative only for $C_0$. We say that the dynamical perturbation $\Delta_A$ is based on (or tailor-made for) the zero-compensator.

In order to ascertain that $C$ robustly stabilizes the plant $P$, we only have to verify whether $P$ belongs to some family $\mathcal{P}_\Delta(\hat{P}, \rho_\Delta)$. Thus we only have to complete the above two steps as follows.

3. Express the deficiency of the nominal model, i.e. the “model-error”, by the mismatch $M$ that is conformable to the dynamical perturbation $\Delta$.

4. Verify whether the magnitude of $M$ is smaller than the robustness margin $\rho_\Delta$.

In summary, we have to express the plant as a particular dynamical perturbation of the nominal model. Stability is guaranteed if this dynamical perturbation belongs to $\mathcal{P}_\Delta(\hat{P}, \rho_\Delta)$. The various families $\mathcal{P}_\Delta$ that are reminiscent of the small gain theorem, include those that are based on the structured singular value [65], stable factor perturbations [243], graph metric perturbations [238, 243] and gap metric perturbations [79, 77]. The families $\mathcal{P}_\Delta(\hat{P}, \rho_\Delta)$ are conservative in the sense that $C$ stabilizes many systems lying outside $\mathcal{P}_\Delta(\hat{P}, \rho_\Delta)$. This conservatism is inherent to the small gain theorem [57, 93]. Nevertheless much of this conservatism can be obviated if the stability ascertainment is customized for the triple $P, \hat{P}$ and $C$.

The stability ascertainment hinges on the selection of the class of dynamical perturbations: the choice of dynamical perturbations (step 1) determines — for given $P, \hat{P}$ and $C$ — the robustness margin (step 2), the mismatch (step 3) and the outcome of the stability test (step 4). If the stability of $H(P, C)$ cannot be ascertained by one class of dynamical perturbations, then it may be useful to try another class of perturbations. For instance if the additive mismatch $M_A = P - \hat{P}$ is larger than the corresponding robustness margin $\rho_A$, then we can still try to demonstrate stability of $H(P, C)$ by modelling $P$ as a coprime factor perturbation of $\hat{P}$.

The stability ascertainment can be made less conservative by using information on the controller in selecting the class of dynamical perturbations. In Section 6.3.3 we discuss dynamical perturbations that are based on the newly designed compensator. In Section 6.3.4 we use dynamical perturbations based on an “old” compensator. These results are special cases of the general robust stability result introduced in the next section.
6.3.2 Coprime Factor Perturbations

Here we study robust stability in the face of coprime factor perturbations \((\Delta_N, \Delta_D)\) as defined in (5.8). For ease of reference we first fix some notation by means of the following assumption.

**Assumption 6.3.2** The nominal model \(\hat{P} \in \mathcal{F}\) and the compensator \(C \in \mathcal{F}\) have the following properties.

i. \(\hat{P}\) has a rcf \((\tilde{N}, \tilde{D})\).

ii. \(C\) has a rcf \((N_c, D_c)\) and a lcf \((\tilde{D}_c, \tilde{N}_c)\).

iii. \(H(\hat{P}, C)\) is stable or equivalently \(\hat{\Lambda} \in \mathcal{J}\), with \(\hat{\Lambda} \triangleq \tilde{D}_c \tilde{D} + \tilde{N}_c \tilde{N}\).

The equivalence in iii. follows directly from Lemma 3.3.4. As explained earlier any system in \(\mathcal{F}\) can be expressed as the perturbed nominal model \((\tilde{N} + \Delta_N)(\tilde{D} + \Delta_D)^{-1}\).

Although \((\tilde{N}, \tilde{D})\) is a rcf, the pair \((\tilde{N} + \Delta_N, \tilde{D} + \Delta_D)\) is not necessarily coprime. The next lemma provides a sufficient condition for the coprimeness of \((\tilde{N} + \Delta_N, \tilde{D} + \Delta_D)\).

**Lemma 6.3.3** Let Assumption 6.3.2 hold. Then the pair \((\tilde{N} + \Delta_N, \tilde{D} + \Delta_D)\) is coprime if

\[
\sigma_{\text{max}} \left( \Lambda^{-1}(j\omega) \begin{bmatrix} \tilde{N}_c(j\omega) & \tilde{D}_c(j\omega) \end{bmatrix} \begin{bmatrix} \Delta_N(j\omega) \\ \Delta_D(j\omega) \end{bmatrix} \right) < 1
\]

for all frequencies \(\omega \in \mathbb{R}\).

**Proof:** For ease of notation we introduce

\[
\Lambda_{\Delta} \triangleq \tilde{D}_c(\tilde{D} + \Delta_D) + \tilde{N}_c(\tilde{N} + \Delta_N).
\]

Thereby we can rewrite the inequality of the lemma to

\[
\sigma_{\text{max}} \left( \Lambda^{-1}(j\omega)[\Lambda_{\Delta}(j\omega) - \hat{\Lambda}(j\omega)] \right) < 1,
\]

in which \(\hat{\Lambda}, \hat{\Lambda}^{-1}\) and \(\Lambda_{\Delta}\) belong to \(\mathcal{H}\). If \((\Delta_N, \Delta_D)\) is such that this inequality holds, then \(||\hat{\Lambda}^{-1}[\Lambda_{\Delta} - \hat{\Lambda}]/||_{\infty} < 1\). Hence \(\hat{\Lambda}^{-1}[\Lambda_{\Delta} - \hat{\Lambda}]\) is a contraction, and it is stable under unity feedback by virtue of the small gain theorem. Consequently \(I + \hat{\Lambda}^{-1}[\Lambda_{\Delta} - \hat{\Lambda}]\) has a stable inverse or equivalently \((I + \hat{\Lambda}^{-1}[\Lambda_{\Delta} - \hat{\Lambda}])\) belongs to \(\mathcal{J}\). Further

\[
\hat{\Lambda} \cdot (I + \hat{\Lambda}^{-1}[\Lambda_{\Delta} - \hat{\Lambda}]) = \Lambda_{\Delta}
\]

and since both terms at the left hand side belong to \(\mathcal{J}\), application of Fact B.1.2.i yields \(\Lambda_{\Delta} \in \mathcal{J}\). Reverting to the definition of \(\Lambda_{\Delta}\) it is easy to obtain the identity

\[
(\Lambda_{\Delta}^{-1} \tilde{N}_c)(\tilde{N} + \Delta_N) + (\Lambda_{\Delta}^{-1} \tilde{D}_c)(\tilde{D} + \Delta_D) = I.
\]

As all four terms in parentheses are stable, the pair \((\tilde{N} + \Delta_N, \tilde{D} + \Delta_D)\) is right coprime according to Fact 3.2.4.i. \(\square\)
This condition relates the coprimeness of the pair \( (\hat{N}+\Delta_N, \hat{D}+\Delta_D) \) to a compensator \( C \) that stabilizes the nominal model \( \hat{P} = \hat{N} \hat{D}^{-1} \). Due to this relation the same condition can be used to guarantee stability in the presence of coprime factor perturbations.

**Lemma 6.3.4** Let Assumption 6.3.2 hold and define \( \hat{P}_\Delta = (\hat{N}+\Delta_N)(\hat{D}+\Delta_D)^{-1} \). The feedback system \( H(\hat{P}_\Delta, C) \) is stable if

\[
\sigma_{\text{max}} \left( \hat{\Lambda}^{-1}(j\omega) \begin{bmatrix} \tilde{N}_c(j\omega) & \tilde{D}_c(j\omega) \end{bmatrix} \begin{bmatrix} \Delta_N(j\omega) \\ \Delta_D(j\omega) \end{bmatrix} \right) < 1
\]

for all frequencies \( \omega \in \mathbb{R} \).

**Proof:** By the proof of the previous lemma the above condition guarantees, that the pair \( (\hat{N}+\Delta_N, \hat{D}+\Delta_D) \) is a rcf of \( \hat{P}_\Delta \) and that

\[
\Lambda_\Delta = \tilde{N}_c(\hat{N}+\Delta_N) + \tilde{D}_c(\hat{D}+\Delta_D)
\]

belongs to \( \mathcal{J} \). By Lemma 3.3.4 these two facts make up a sufficient condition for the stability of \( H(\hat{P}_\Delta, C) \).

**Remark 6.3.5** Strictly speaking the condition in Lemma 6.3.4 does not guarantee that \( \hat{D}+\Delta_D \) has an inverse. If this inverse does not exist, then the pair \( (\hat{N}+\Delta_N, \hat{D}+\Delta_D) \) is not a factorization. Nevertheless if the condition in Lemma 6.3.4 is met, then the pair is coprime, and \( H(P, C) \) is robustly stable against this perturbation. The point is that this particular coprime factor perturbation cannot arise from a perturbation in the form of a perturbed plant \( P_\Delta \), which can be explained as follows.

Suppose that the coprime factor perturbation \( (\Delta_N, \Delta_D) \) is such that the condition in Lemma 6.3.4 is met. Then \( \tilde{D}_c(\hat{D}+\Delta_D)+\tilde{N}_c(\hat{N}+\Delta_N) \) has a stable inverse, and thus

\[
\begin{bmatrix} \hat{N}+\Delta_N \\ \hat{D}+\Delta_D \end{bmatrix} (\hat{D}_c(\hat{D}+\Delta_D)+\tilde{N}_c(\hat{N}+\Delta_N))^{-1} \begin{bmatrix} \tilde{N}_c \\ \tilde{D}_c \end{bmatrix}
\]

is stable. This operator equals the feedback matrix \( T((\hat{N}+\Delta_N)(\hat{D}+\Delta_D)^{-1}, C) \), i.e.

\[
\begin{bmatrix} (\hat{N}+\Delta_N)(\hat{D}+\Delta_D)^{-1} \\ I \end{bmatrix} (I+C(\hat{N}+\Delta_N)(\hat{D}+\Delta_D)^{-1})^{-1} \begin{bmatrix} C \\ I \end{bmatrix}
\].

In case the inverse of \( \hat{D}+\Delta_D \) does not exist, then the former operator is stable, but it simply cannot be given the latter interpretation of a feedback system.

6.3.3 A Robustness Margin with Reduced Conservatism

In the interest of a good appreciation of the stability condition of Lemma 6.3.4, we show that two established robustness margins are induced by this result as a special
case. These are the robustness margins in the gap metric sense and the closely related less conservative stability condition by Bongers [24, 25]. These results are conceived in terms of coprime factorizations of the nominal model \( \hat{P} \) and a plant \( P_\Delta \in \mathcal{F} \), whereas the condition of Lemma 6.3.4 is framed in terms of \( \hat{P} \) and the coprime factor perturbation \((\Delta_N, \Delta_D)\). As suggested by Bongers [24] and Georgiou [77] we let \((N_\Delta, D_\Delta)\) be a rcf of \( P_\Delta \), and we define \((\Delta_N, \Delta_D)\) as

\[
\Delta_N = N_\Delta Q_\Delta - \hat{N} \\
\Delta_D = D_\Delta Q_\Delta - \hat{D},
\]

in which \( Q_\Delta \) is stable. The factor \( Q_\Delta \) must belong to \( \mathcal{H} \) in order that \((\Delta_N, \Delta_D)\) is a stable coprime factor perturbation of \((\hat{N}, \hat{D})\). In this way every particular perturbed plant \( P_\Delta \) can be represented as a stable coprime factor perturbation of the nominal model \( \hat{P} \):

\[
(\hat{N} + \Delta_N)(\hat{D} + \Delta_D)^{-1} = (N_\Delta Q_\Delta)(D_\Delta Q_\Delta)^{-1} = N_\Delta D_\Delta^{-1} = P_\Delta.
\]

The above coprime factor perturbation \((\Delta_N, \Delta_D)\) turns \( \hat{P} \) into one and the same perturbed nominal model \( P_\Delta \) for every invertible \( Q_\Delta \in \mathcal{H} \). Moreover \( Q_\Delta \) is at our discretion. So for some given \( \hat{P} \) and \( P_\Delta \) we can be exploit \( Q_\Delta \) to make the coprime factor perturbation \((\Delta_N, \Delta_D)\) small in an appropriate sense. This brings us to the first special case of Lemma 6.3.4.

**Proposition 6.3.6** Let Assumption 6.3.2 hold and let \( P_\Delta \) belong to \( \mathcal{F} \). Then the feedback system \( H(P_\Delta, C) \) is stable if

\[
\delta(\hat{P}, P_\Delta) < \frac{1}{\|T(\hat{P}, C)\|_\infty}
\]

where \( \delta(\hat{P}, P_\Delta) \) is the directed gap.

**Proof:** We let \((N_\Delta, D_\Delta)\) be a rcf of \( P_\Delta \). We show that the condition of the proposition is stronger than that of Lemma 6.3.4. For notational convenience we omit the argument \((j\omega)\) in the following derivations. With \((N_\Delta Q_\Delta - \hat{N}, D_\Delta Q_\Delta - \hat{D})\) substituted for \((\Delta_N, \Delta_D)\) in the inequality of Lemma 6.3.4 we get

\[
\sigma_{\text{max}} \left( \hat{\Lambda}^{-1} \left[ \begin{array}{c} \hat{N}_e \\ \hat{D}_e \end{array} \right] \frac{N_\Delta Q_\Delta - \hat{N}}{D_\Delta Q_\Delta - \hat{D}} \right) \\
\leq \sigma_{\text{max}} \left( \hat{\Lambda}^{-1} \left[ \begin{array}{c} \hat{N}_e \\ \hat{D}_e \end{array} \right] \right) \cdot \sigma_{\text{max}} \left( \left[ \begin{array}{c} \hat{N} - N_\Delta Q_\Delta \\ \hat{D} - D_\Delta Q_\Delta \end{array} \right] \right) \\
\leq \left\| \hat{\Lambda}^{-1} \left[ \begin{array}{c} \hat{N}_e \\ \hat{D}_e \end{array} \right] \right\|_\infty \cdot \left\| \left[ \begin{array}{c} \hat{N} \\ \hat{D} \end{array} \right] - \left[ \begin{array}{c} N_\Delta \\ D_\Delta \end{array} \right] Q_\Delta \right\|_\infty.
\]
6.3 Robustness Margins

These relations hold for each rcf \((\hat{N}, \hat{D})\) and any \(Q_\Delta \in \mathcal{H}\). The following step consists of choosing some particular \((\hat{N}, \hat{D})\) and \(Q_\Delta\). We choose a rcf \((\hat{N}_n, \hat{D}_n)\) for \((\hat{N}, \hat{D})\), so that \(\hat{A} = \hat{D}_c \hat{D}_n + \hat{N}_c \hat{N}_n\) and

\[
\|T(\hat{P}, C)\|_\infty = \left\| \begin{bmatrix} \hat{N}_n \\ \hat{D}_n \end{bmatrix} \hat{A}^{-1} \begin{bmatrix} \hat{D}_c & \hat{N}_c \end{bmatrix} \right\|_\infty = \left\| \hat{A}^{-1} \begin{bmatrix} \hat{D}_c & \hat{N}_c \end{bmatrix} \right\|_\infty
\]

because \(\text{col}(\hat{N}_n, \hat{D}_n)\) is inner [77, 24]. Further, from (5.3) we have

\[
\bar{\delta}(\hat{P}, P_\Delta) = \inf_{Q_\Delta \in \mathcal{H}} \left\| \begin{bmatrix} \hat{N}_n \\ \hat{D}_n \end{bmatrix} - \begin{bmatrix} N_\Delta \\ D_\Delta \end{bmatrix} Q_\Delta \right\|_\infty.
\]

By Theorem 6.1 of [72] the infimum is actually achieved for some \(Q_\Delta \in \mathcal{H}\). We denote this particular \(Q_\Delta\) as \(Q_\delta\); i.e.

\[
\bar{\delta}(\hat{P}, P_\Delta) = \left\| \begin{bmatrix} \hat{N}_n \\ \hat{D}_n \end{bmatrix} - \begin{bmatrix} N_\Delta \\ D_\Delta \end{bmatrix} Q_\delta \right\|_\infty.
\]

For the case of \((\hat{N}_n, \hat{D}_n)\) and \(Q_\delta\) we can rewrite the above inequality to

\[
\sigma_{\max} \left( \hat{A}^{-1} \begin{bmatrix} \hat{N}_c & \hat{D}_c \end{bmatrix} \begin{bmatrix} N_\Delta Q_\delta - \hat{N}_n \\ D_\Delta Q_\delta - \hat{D}_n \end{bmatrix} \right) \leq \|T(\hat{P}, C)\|_\infty \cdot \bar{\delta}(\hat{P}, P_\Delta).
\]

Hence if the condition of the proposition holds, then

\[
\sigma_{\max} \left( \hat{A}^{-1} \begin{bmatrix} \hat{N}_c & \hat{D}_c \end{bmatrix} \begin{bmatrix} N_\Delta Q_\delta - \hat{N}_n \\ D_\Delta Q_\delta - \hat{D}_n \end{bmatrix} \right) < 1
\]

and thus by Lemma 6.3.4 \(H(P_\Delta, C)\) is stable.

This proposition can be used a fortiori to show that \(H(P_\Delta, C)\) is stable for all \(P_\Delta\) such that the gap \(\delta(\hat{P}, P_\Delta) < 1/\|T(\hat{P}, C)\|_\infty\) (see also [77]). Thus the inverse of the \(H_\infty\)-norm of the feedback matrix \(T(\hat{P}, C)\) is a robustness margin in a gap-metric sense. In a similar fashion we can retrieve the robustness margin introduced by Bongers [24].

**Proposition 6.3.7** Let Assumption 6.3.2 hold and let \(P_\Delta\) have a rcf \((N_\Delta, D_\Delta)\). Further let \((\hat{D}_{nc}, \hat{N}_{nc})\) be a rncf of \(C\) and define \(\hat{A}_{nc} = \hat{D}_{nc} \hat{D} + \hat{N}_{nc} \hat{N}\). Then the feedback system \(H(P_\Delta, C)\) is stable if

\[
\inf_{Q_\Delta \in \mathcal{H}} \left\| \begin{bmatrix} \hat{N} \hat{A}_{nc}^{-1} \\ \hat{D} \hat{A}_{nc}^{-1} \end{bmatrix} - \begin{bmatrix} N_\Delta \\ D_\Delta \end{bmatrix} Q_\Delta \right\|_\infty < 1.
\]
Proof: The proof evolves analogously to that of the previous proposition. Once more we the inequality
\[
\sigma_{\text{max}} \left( \Lambda^{-1} \begin{bmatrix} \tilde{N}_c & \tilde{D}_c \end{bmatrix} \begin{bmatrix} N_{\Delta} Q_{\Delta} - \tilde{N} \\ D_{\Delta} Q_{\Delta} - \tilde{D} \end{bmatrix} \right) \\
\leq \| \tilde{\Lambda}^{-1} \begin{bmatrix} \tilde{N}_c & \tilde{D}_c \end{bmatrix} \|_\infty \cdot \| \begin{bmatrix} \tilde{N} \\ \tilde{D} \end{bmatrix} - \begin{bmatrix} N_{\Delta} \\ D_{\Delta} \end{bmatrix} \|_{\infty}.
\]
but now we choose another \( Q_{\Delta} \) and other coprime factorizations \((\tilde{D}_c, \tilde{N}_c)\) and \((\tilde{N}, \tilde{D})\).

Again by Theorem 6.1 of [72] the infimum of the proposition is achieved for some \( Q_{\Delta} \). Accordingly we define
\[
Q_{\delta c} = \arg \min_{Q_{\Delta} \in \mathcal{H}} \| \begin{bmatrix} \tilde{N}_{\Lambda^{-1}_{nc}} \\ \tilde{D}_{\Lambda^{-1}_{nc}} \end{bmatrix} - \begin{bmatrix} N_{\Delta} \\ D_{\Delta} \end{bmatrix} \|_{\infty}.
\]
As \( H(\hat{P}, C) \) is stable, the term \( \Lambda_{nc} \) belongs to \( \mathcal{J} \), so that \((\tilde{N}_{\Lambda^{-1}_{nc}}, \tilde{D}_{\Lambda^{-1}_{nc}})\) is a ref of \( \hat{P} \) by Fact 3.2.6.i. Now we substitute \((\tilde{D}_{nc}, \tilde{N}_{nc})\) for \((\tilde{D}_c, \tilde{N}_c)\), \((\tilde{N}_{\Lambda^{-1}_{nc}}, \tilde{D}_{\Lambda^{-1}_{nc}})\) for \((\tilde{N}, \tilde{D})\) and \( Q_{\delta c} \) for \( Q_{\Delta} \) in the above inequality. Thereby \( \tilde{\Lambda} \) is replaced with \( \tilde{D}_{nc} \tilde{\Lambda}_{nc}^{-1} + \tilde{N}_{nc} \tilde{\Lambda}_{nc}^{-1} = I \) and we get
\[
\sigma_{\text{max}} \left( \begin{bmatrix} \tilde{N}_{nc} & \tilde{D}_{nc} \end{bmatrix} \begin{bmatrix} N_{\Delta} Q_{\delta c} - \tilde{N}_{\Lambda^{-1}_{nc}} \\ D_{\Delta} Q_{\delta c} - \tilde{D}_{\Lambda^{-1}_{nc}} \end{bmatrix} \right) \\
\leq \| \begin{bmatrix} \tilde{N}_{nc} & \tilde{D}_{nc} \end{bmatrix} \|_\infty \cdot \| \begin{bmatrix} \tilde{N}_{\Lambda^{-1}_{nc}} \\ \tilde{D}_{\Lambda^{-1}_{nc}} \end{bmatrix} - \begin{bmatrix} N_{\Delta} \\ D_{\Delta} \end{bmatrix} \|_{\infty}.
\]
Since \((\tilde{D}_{nc}, \tilde{N}_{nc})\) is normalized, \( \| \begin{bmatrix} \tilde{N}_{nc} & \tilde{D}_{nc} \end{bmatrix} \|_\infty = 1 \), and the right hand side of the latter inequality equals the infimum of the proposition. If this infimum is smaller than 1, then stability of \( H(P_{\Delta}, C) \) follows from application of Lemma 6.3.4.

In [24] Bongers has demonstrated that the robustness margin of Proposition 6.3.7 with compensator-based dynamical perturbations is less conservative than the robustness margin in the gap-metric sense of Proposition 6.3.6, which does not employ compensator-based dynamical perturbations. This points to some conservatism of the control design of Section 6.2, which optimizes robustness in the gap-metric sense [77]. Notice that only the robustness margin of Proposition 6.3.7 employs compensator-based dynamical perturbations: the mismatch involves \( \Lambda_{nc}^{-1} \) and thus it involves the compensator \( C \). The dynamical perturbation of Proposition 6.3.6 is based on the directed gap \( \delta(\hat{P}, P_{\Delta}) \), which does not involve the compensator \( C \).

Remark 6.3.8 The above two propositions exhibit the following Kafkaesque attribute. The infimum is taken over \( Q_{\Delta} \in \mathcal{H} \). Thus this \( Q_{\Delta} \) is not required to have a stable
inverse. Now suppose that the condition of either proposition is met. Then, as shown in the respective proofs, the coprime factor perturbation satisfies the condition of Lemma 6.3.4 and thus also the condition of Lemma 6.3.3. The latter implies that $(\hat{N} + \Delta_N, \hat{D} + \Delta_D)$ is coprime. Hence $(N_\Delta Q_\Delta, D_\Delta Q_\Delta)$ is a rcf, and since $(N_\Delta, D_\Delta)$ is also a rcf, $Q_\Delta$ must belong to $\mathcal{J}$ by virtue of Fact 3.2.6.i. In summary, $Q_\Delta$ is not required to belong to $\mathcal{J}$, but if the infimum is such that $H(P_\Delta, C)$ is guaranteed to be stable, then the minimizing $Q_\Delta$ does belong to $\mathcal{J}$. □

6.3.4 A Robustness Margin without Conservatism

The robustness margins of Proposition 6.3.6 and Proposition 6.3.7 use respectively no controller information and information on the new compensator. In the light of the iterative scheme of repeated identification and control design we have available an “old” compensator $K$ with a rcf $(N_k, D_k)$, which is known to stabilize the plant $P$ and the nominal model $\hat{P}$. Here we use this particular knowledge to to define another class of compensator-based dynamical perturbations. For this purpose we use the $R$-parameterization or dual Youla parameterization.

As we know that $P$ belongs to $\mathcal{P}(K)$, i.e. the set of all systems stabilized by $K$, we build a sufficient condition for robust stability that concerns only this family $\mathcal{P}(K)$. To that end we introduce the compensator-based coprime factor perturbation

$$\begin{align*}
\Delta_N &= D_k \Delta_R \\
\Delta_D &= -N_k \Delta_R
\end{align*}$$

(6.16)
in which $\Delta_R \in \mathcal{H}$. The class of all systems considered is

$$\{P_\Delta \mid P_\Delta = (\hat{N} + D_k \Delta_R)(\hat{D} - N_k \Delta_R)^{-1}, \Delta_R \in \mathcal{H}\} = \mathcal{P}(K).$$

This equals the $R$-parameterization of Corollary 3.3.7 except that $\Delta_R$ has been substituted for $R_K$. The latter has been done to emphasize that we are dealing with an unstructured dynamical perturbation.

Having settled the class of dynamical perturbations, we now establish a sufficient condition for robust stability of $H(\hat{P}, C)$ in the face of these “old-compensator-based” dynamical perturbations.

**Proposition 6.3.9** Let Assumption 6.3.2 hold, let $P_\Delta \in \mathcal{P}(K)$ and let $K$ have a rcf $(N_k, D_k)$. Define $\Delta_R \in \mathcal{H}$ and $\rho_R \in \mathbb{R}$ as

$$\Delta_R \doteq (D_k + P_\Delta N_k)^{-1}(P_\Delta - \hat{P})\hat{D}$$

$$\rho_R \doteq \|(\hat{D} + C\hat{N})^{-1}(C - K)D_k\|_\infty^{-1}.$$

Then the feedback system $H(P_\Delta, C)$ is stable if $\|\Delta_R\|_\infty < \rho_R$. 
Proof: From

\[ P_\Delta = (\tilde{N} + D_k \Delta_R)(\tilde{D} - N_k \Delta_R)^{-1} \in \mathcal{P}(K) \]

it is easy to obtain the expression for \( \Delta_R \) given in the proposition. Stability of \( \Delta_R \) follows from

\[ \Delta_R = (\tilde{D}_\Delta D_k + \tilde{N}_\Delta N_k)^{-1}(\tilde{N}_\Delta \tilde{D} - \tilde{D}_\Delta \tilde{N}) \]

where \((\tilde{D}_\Delta, \tilde{N}_\Delta)\) is a lcf of \( P_\Delta \), and \((\tilde{D}_\Delta D_k + \tilde{N}_\Delta N_k)^{-1} \) is stable because \( H(P_\Delta, K) \) is stable (Lemma 3.3.4).

Next we apply Lemma 6.3.4 to the compensator \( C \) and the plant \( P_\Delta \in \mathcal{P}(K) \). That is, we replace \((\Delta_N, \Delta_D)\) by \((D_k \Delta_R, -N_k \Delta_R)\). Then stability of \( H(P_\Delta, C) \) is guaranteed if

\[ \sigma_{\max} \left( \tilde{\Lambda}^{-1}(j\omega)[\tilde{N}_c(j\omega)D_k(j\omega) - \tilde{D}_c(j\omega)N_k(j\omega)]\Delta_R(j\omega) \right) < 1. \]

This inequality holds if the stronger condition

\[ \left\| \tilde{\Lambda}^{-1}(\tilde{N}_c D_k - \tilde{D}_c N_k) \right\|_\infty \cdot \| \Delta_R \|_\infty < 1 \]

is satisfied. The result follows from the substitution

\[ \tilde{\Lambda}^{-1}(\tilde{N}_c D_k - \tilde{D}_c N_k) = (\tilde{D}_c^{-1}\tilde{\Lambda})^{-1}\tilde{D}_c^{-1}(\tilde{N}_c D_k - \tilde{D}_c N_k) = (\tilde{D} + C\tilde{N})^{-1}(CD_k - N_k). \]

\[ \qed \]

This stability condition induces the family \( \mathcal{P}_\Delta(\hat{P}, \rho_R) \) of robustly stabilized systems. The class of dynamical perturbations used in Proposition 6.3.9 are based on the compensator \( K \). The stability condition is non-conservative if \( C = K \); \( \rho_R(K) = \infty \) and \( \mathcal{P}_\Delta(\hat{P}, \rho_R(K)) = \mathcal{P}(K) \). Along the same lines we have said in Section 6.3.1 that the additive dynamical perturbation \( \Delta_A \) is based on (and non-conservative for) the zero-compensator \( C_0 \).

In conclusion we compare the families and dynamical perturbations used in Proposition 6.3.9 and in Proposition 6.3.7. The latter proposition employs plain coprime factor perturbations and the corresponding perturbative family consists of all real rational system (i.e. \( \mathcal{F} \)). Thus this stability ascertainment is applicable to all real rational systems. On the other hand Proposition 6.3.9 considers only the systems in \( \mathcal{P}(K) \subset \mathcal{F} \), so that this stability ascertainment cannot be applied to systems that are not stabilized by \( K \). However we cannot conclude that one stability condition is more conservative than the other. The “weighted” gap of Proposition 6.3.7 will generally include systems in \( \mathcal{F} \setminus \mathcal{P}(K) \), and conversely, the family \( \mathcal{P}_\Delta(\hat{P}, \rho_R) \) will generally contain a larger subset of \( \mathcal{P}(K) \) than the “weighted” gap family. In plain terms, for both robustness margins there exists a robustly stabilized perturbation, such that the corresponding perturbed plant lies outside the stability region of the other margin.
Remark 6.3.10 The stability condition of Proposition 6.3.9 can readily be employed to capture simultaneous perturbations of \( \hat{P} \) and \( K \). For that purpose we define \( \Delta_K \) as

\[
\Delta_K = (\hat{D} + C\hat{N})^{-1}(C - K)D_k
\]

so that

\[
K_\Delta = (N_k + \hat{D}\Delta_K)(D_k - \hat{N}\Delta_K)^{-1}.
\]

By congruence of \( \Delta_R \) and \( (\hat{P}, K) \) the feedback system \( H(P_\Delta, K_\Delta) \) is stable for all perturbations \( \Delta_R, \Delta_K \in \mathcal{H} \) such that

\[
\|\Delta_R\|_\infty \cdot \|\Delta_K\|_\infty < 1,
\]

provided that \( H(\hat{P}, K) \) is stable. This result is closely related to the developments of Section 6.1.

Remark 6.3.11 The family of systems

\[
\{P_\Delta \mid P_\Delta = (\hat{N} + D_k\Delta_R)(\hat{D} - N_k\Delta_R)^{-1}, \ \Delta_R \in \mathcal{H}\}
\]

has been addressed also by Sefton et al. [207]. These authors proposed to decompose the usual coprime factor perturbation as

\[
\begin{bmatrix}
\Delta_N \\
\Delta_D
\end{bmatrix} =
\begin{bmatrix}
\hat{D}_n & -N_k \\
\hat{N}_n & D_k
\end{bmatrix}
\begin{bmatrix}
\Delta_X \\
\Delta_R
\end{bmatrix},
\]

with \( \Delta_X \in \mathcal{H} \). The contribution \( \Delta_X \) is not needed to represent all systems that are stabilized by \( K \). In [207] the term \( \Delta_X \) has been used to investigate the conservatism of the robustness margin for the fixed compensator \( K \). These perturbations were not used to investigate robust stability for the case that \( K \) is replaced by another compensator.

6.3.5 The Illustrative Example, III

We use the robustness margins of \( H(\hat{P}, C) \) to ascertain the stability of \( H(P, C) \). We will consider various nominal models and compensators that form part of the primary iteration of Section 8.2. This iteration consists of repeated identification and cautious controller enhancement, and it evolves from an open-loop identified model to a high performance compensator. All the models have been derived through fixed-loop performance-identification as indicated in Section 5.2. The precise optimization procedure used for this identification is discussed in Chapter 7. Nevertheless we already use these models and compensators to illustrate the utility of the various robustness margins for stability ascertainment.
The plant $P$ under consideration is that of Appendix A, Table A.1. The precise coefficients, poles and zeros of the nominal models and compensators can be found in the tables A.6-A.9. For three cases we verify whether at least one of the numbers

$$
\Gamma_C \doteq \inf_{Q_\Delta \in \mathcal{H}} \left\| \begin{bmatrix} \tilde{N} \hat{A}_{nc}^{-1} \\
\tilde{D} \hat{A}_{nc}^{-1} \end{bmatrix} - \begin{bmatrix} N \\
D \end{bmatrix} Q_\Delta \right\|_{\infty}
$$

$$
\Gamma_A \doteq (P - \hat{P})/\rho_A
$$

$$
\Gamma_R \doteq \Delta_R/\rho_R
$$

is smaller than 1. Here we replace the $H_\infty$-norms by the frequency dependent maximum singular values in order to display the frequency dependent behavior of these quantities. For completeness we mention that $\Gamma_C < 1$, $\Gamma_A < 1$ and $\Gamma_R < 1$ correspond respectively to the robustness margin of Proposition 6.3.7, the common additive robustness margin, and the robustness margin of Proposition 6.3.9.

The first case that we consider is that of nominal model $\hat{P}_2$ and compensator $C_2$. For this pair $\Gamma_C = 1.058$, which does not ascertain the stability of $H(P, C_2)$. In Fig. 6.8.a $\Gamma_C$ has been plotted as a function of frequency ($\omega$). On the other hand $\Gamma_A < 1$ and $\Gamma_R < 1$ for $\hat{P}_2, C_2$. Thus the additive robustness margin and the stability condition of Proposition 6.3.9 both guarantee the stability of $H(P, C_2)$.

As the iteration started in open-loop, $P$ must be stable. Further $\hat{P}_2$ is also stable. In ascertaining the stability of $H(P, C_2)$ with the robustness margin of Proposition 6.3.9 we correspondingly used dynamical perturbations based on the class $\mathcal{P}(0)$. This shows that this stability margin reduces precisely to the additive robustness margin $\rho_A$ for $K=0$: $\Gamma_A (\rho_A)$ coincides with $\Gamma_R (\rho_R)$ as a function of frequency for the triple $\hat{P}_2, C_2$ and $K=0$ (see Fig. 6.8.a).

The second case concerns $\hat{P}_3$ and $C_3$. Again the robustness margin of Proposition 6.3.7 fails: $\Gamma_C = 1.118$. A lower bound of $\Gamma_C$ has been drawn in Fig. 6.8.b. In the same plot we see that $\Gamma_A > 1$ ($\omega$). Actually the additive robustness margin may not be used for this case, since $\hat{P}_3$ is unstable and $P$ is not. Fortunately $\Gamma_R < 1$ ($\omega$) and thus stability of $H(P, C_3)$ is guaranteed. The latter ascertainement is based on $K=C_3$; i.e. stability has been examined for systems that are stabilized by $C_2$. Of course $P$ and $\hat{P}_3$ belong to $\mathcal{P}(C_2)$.

In the final case we examine $\hat{P}_4$ and $C_4$. Now $\Gamma_C < 1$ and thus stability of $H(P, C_4)$ is guaranteed. Based on $P, \hat{P}_4 \in \mathcal{P}(C_3)$, i.e. $K=C_3$, we draw the same conclusion from $\Gamma_R < 1$. The additive robustness margin may be applied, because $\hat{P}_4$ is stable. However $\Gamma_A > 1$ as shown in Fig. 6.8.c.

In conclusion we point out that we could not have accepted the compensator $C_3$, if we had not used the stability condition of Proposition 6.3.9. The advantage of this stability condition over a gap-metric-like robustness margin is that the former utilizes more detailed information about the coprime factor perturbation ($\Delta_N, \Delta_D$): the mutual dependency between the numerator and denominator perturbations $D_e \Delta_R$
Fig. 6.8: Frequency-dependent robustness margins $\Gamma_C$ (---), $\Gamma_A$ (--) and $\Gamma_R$ (--) for
$\hat{P}, C$ and $\mathcal{P}(K)$.

a: $\hat{P}=\hat{P}_2$, $C=C_2$, $\mathcal{P}(K)=\mathcal{P}(0)$.

b: $\hat{P}=\hat{P}_3$, $C=C_3$, $\mathcal{P}(K)=\mathcal{P}(C_2)$.

c: $\hat{P}=\hat{P}_4$, $C=C_4$, $\mathcal{P}(K)=\mathcal{P}(C_3)$.

d: 

and $-N_c\Delta_R$ is exploited, i.e. phase information about $\Delta_N$ and $\Delta_D$ is taken into
account. This phase information is profitable provided that the new compensator
is sufficiently close to the old compensator.
Chapter 7

Frequency-domain Identification of Coprime Factors

This chapter deals with the identification of the coprime factors of a SISO plant. First we review some literature concerning this identification problem in order to put our developments in some perspective. Then in Section 7.2 we discuss a common SISO open-loop frequency-domain identification technique. Thereafter we modify this identification technique three times in accordance with our needs. With the first modification we can solve the problem of fixed-loop performance-approximation (Section 7.3). The second modification is meant for design-oriented stability-approximation (Section 7.4). The third and last application of this frequency domain identification technique lies with the ascertainment of stability of a new control system (Section 7.5).

7.1 Identification of Coprime Factors

For our purposes the task of identification is to select a nominal model, that describes all properties of the plant that are relevant for feedback control design. The control design paradigm of interest is that of optimizing the feedback properties for the controlled plant with uncertain dynamics. As elucidated in Chapter 2, feedback properties are embodied by the frequency response of the feedback matrix $T(P, C)$. Thus a nominal model $\hat{P}$ describes the feedback properties of the plant $P$ under feedback by $C$, if the frequency response of $T(\hat{P}, C)$ resembles that of $T(P, C)$. We will accordingly identify a nominal model from frequency response data. More specific, we will tackle the feedback-relevant approximation problems of Chapter 5 by the frequency-domain identification of the plant's associated coprime factors $(N^a, D^a)$.

Only a few contributions in literature address the application of coprime factorizations to system identification problems. To our knowledge, all contributions made so far concern the time-domain identification of a left coprime factorization. In contrast we treat the frequency-domain identification of a right coprime factorization. The
reader is referred to Appendix C for details about the differences between the identification of left and right coprime factorizations. Here we merely put our developments in some perspective by reviewing some literature.

In [101, 100, 102, 103] Hansen has analyzed the problem of exact identification from closed-loop data. He took the full advantage of the coprime factor representations to develop a closed-loop experiment design procedure. This experiment design optimizes the variance of the nominal model estimator in regard of an optimal plant-based compensator. A significant difference with our work is, that Hansen addresses the variance distribution in exact identification, whereas we focus on the bias distribution in approximate identification.

In [125] Krause uses an lcf for approximate identification in the presence of bounded noise. In this work the emphasis lays on identifying a nominal model, such that its deficiency belongs to an a priori specified ball of dynamical perturbations. Thereby it belongs to the left branch of Fig. 1.1, i.e. it provides a quantification of the "model-error”.

Iglesias [112, 113] used left coprime factor representations to analyze the stability and robustness of an indirect adaptive control scheme. The identification part of this scheme is based on the prediction error method. It utilizes the plant output together with an exogenous feedback system input, and thus it comes under the head of performance-approximation. However, the model set is parameterized such, that the asymptotic estimate \( \hat{\mathcal{P}} \) of the inner-loop plant \( \mathcal{P} \) depends on the disturbance and the noise.

Finally, Tay et al. [227, 226] addressed the problem of identifying a plant in the case that a priori information on this plant is available in the form of a nominal model. They proposed to use the \( R \)-parameterization for this purpose. The nominal model is used as the auxiliary model \( P_0 \) of Section 3.3, and the \( R \)-parameter is identified according to the associated identification of Section 4.3. One difference with our work is that Tay does not update the auxiliary model used in the \( R \)-parameterization. For additional details the reader is referred to the remarks of Section 6.1.

### 7.2 Identification in the Frequency-domain

By frequency-domain identification we mean the estimation of a nominal model from a finite number of possibly noise corrupted frequency response samples. This estimation problem is addressed in the first part below. An estimated nominal model approximately accounts for the frequency response samples. These samples represent the frequency response of the plant only at a finite number of frequencies. Nevertheless it is tempting to regard the complete frequency response of the nominal model as an approximation of the plant's frequency response. We touch on this issue in the second part of this section.
7.2.1 Transfer Function Estimation from Frequency Response Data

Frequency-domain identification consists of deriving frequency response samples and constructing a nominal model. A frequency response estimate can be obtained from time-series [180, 250], sine-wave testing [250, 138] or other periodic testing. Although this estimation is relatively time-consuming, it has proven its utility in the identification of e.g. flexible structures [231, 14, 18, 160] and rotorcrafts [228, 204].

We choose this particular identification scheme for a variety of reasons. The main reason is that we can concentrate on the asymptotic bias distribution. We assume that an accurate frequency response estimate is available. For the simulation study of Chapter 8 we take noise free frequency response data, and in the benchmark problem of Chapter 9 we use sinewave testing to obtain accurate frequency response estimates at a finite number of frequencies. Accordingly we may develop estimation algorithms neglecting small potential difference between the available frequency domain data and the true underlying frequency response.

A second important reason is that frequency response samples of the plant $P$ can be used to "predict" the behavior of a new control system: from the samples we can calculate a frequency response estimate of the loop gain $PC_E$ (Section 6.2) and the samples are of use in ascertaining the stability of the new control system (Section 6.3). This explains to a large extent why frequency-domain identification has prevailed over time-domain methods in identification for the purpose of control design (see e.g. [42, 15, 229]).

There is also an algorithmic pay-offs: in the frequency-domain, “filtering” of the data boils down to a simple multiplication by frequency responses. A convolution of impulse responses or a filtering of time-series are not needed.

We indicate the (estimated) frequency response of the plant $P$ by $P(\omega)$. This frequency response is assumed to be known for the frequencies $\omega_i, i=1,..,N$. In the simulation studies we calculate $P(\omega_i)=P(j\omega_i)$, and in the benchmark problem $P(\omega_i)$ is an estimated frequency response sample. We define $\Omega$ to be the vector $\text{col}(\omega_1,..,\omega_N)$ of frequencies in question, and $P_i, i=1,..,N$ signifies the available data set.

We parameterize the set of candidate nominal models by a numerator polynomial $b(s)$ and a denominator polynomial $a(s)$ defined according to

$$b(s) = b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0$$
$$a(s) = s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0.$$

The collection of parameters is represented by

$$\theta = \text{col}(b_n, \ldots, b_1, b_0, a_{n-1}, \ldots, a_1, a_0)$$

and the model set under consideration is

$$\mathcal{P}(\theta) \doteq \{ \frac{b(s)}{a(s)} \mid \theta \in \mathbb{R}^{2n+1} \}.$$
The vector of estimated parameters is denoted $\hat{\theta}$, and associated with $\hat{\theta}$ is the nominal model $\hat{P} \in \mathcal{P}(\theta)$ formed by replacing $\theta$ with $\hat{\theta}$ in the obvious manner.

We want to estimate the nominal model $\hat{P}$ as

$$\hat{P} = \arg \min_{P \in \mathcal{P}(\theta)} \sum_{i=1}^{N} |P_i - \hat{P}(j\omega_i)|^2,$$  \hspace{1cm} (7.1)

which involves a non-linear optimization. Such a procedure is numerically expensive and it may get stuck in a local minimum at an early stage of the optimization. In literature various approaches have been suggested to obviate the non-linear character of the optimization in (7.1) over the parameters $\theta$. We use such a method to find a reasonably good initial estimate for the non-linear optimization of (7.1).

The initial estimate is derived by the iterative search introduced by Sanathanan and Koerner [194], which is the frequency-domain counterpart of the time-domain identification method of Steiglitz and McBride [221]. In the iterative scheme the $k$-th parameter vector $\theta_k$ satisfies

$$\theta_k = \arg \min_{s \in \mathbb{R}^{n+1}} \sum_{i=1}^{N} \left| \frac{a(j\omega_i, \theta_k)}{a(j\omega_i, \theta_{k-1})} b(j\omega_i, \theta_k) - a(j\omega_i, \theta_{k-1}) \right|^2.$$  

In here $a(s, \theta_{k-1})$ signifies the denominator polynomial associated with the previously estimated parameter vector $\theta_{k-1}$. When this iteration has converged, then $a(s, \theta_k) \approx a(s, \theta_{k-1})$ and the resulting criterion equals the summation of (7.1). There is no guarantee that this iteration converges, and also the conditions under which it converges are not known [220]. Numerical experience has revealed, that a failure of this iteration usually corresponds to a poor approximation. In such cases an increase of the model order generally leads to a convergence of the algorithm and an improvement of the estimate. Besides, the algorithm works also with slightly noise corrupted frequency response data.

When the Sanathanan-Koerner iteration has converged, then the criterion of (7.1) is minimized further by means of a Newton-Raphson optimization. Details can be found in Appendix F.

Finally we make some comments about the precise choice of the vector $\Omega$ of frequency points. We take the frequencies of $\Omega$ uniformly distributed over a logarithmic frequency range like in [130, 194]. This makes the criterion in (7.1) suggestive of the logarithmic frequency axis commonly used in Bode plots. We do this, because feedback properties are often specified on a logarithmic frequency scale [64, 143, 31]. This causes the lower frequency range to be emphasized. Notice that the higher frequencies still have to be taken into account in order to enforce robust stability.

We recall from The Illustrative Example, I (Section 2.4.1), that an essential disparity between a plant and its nominal model may be indiscernable unless their frequency
responses are plotted on a logarithmic magnitude scale (see Fig. 2.8). For this reason some authors have suggested to replace the absolute value in (7.1) by the logarithm of $P(\omega_i) - \hat{P}(j\omega_i)$ [141, 209, 220]. This makes sense in case some kind of multiplicative error has to be minimized. However, we intend to generalize the above criterion, so that it can be used to minimize the additive mismatch $T(P,C) - T(\hat{P},C)$. Moreover, when we compare, for instance, two sensitivity functions, then we are much more concerned about a difference of 100 to 1, than about a difference of 0.1 to 0.001. Only the former indicates, that just one of the feedback systems is possibly close to instability.

Remark 7.2.1 The algorithms have not been optimized in a numerical sense. Also the parameterization $a(s)/b(s)$ may be not the most suited one. Other parameterizations that have been used before are parallel connections [220, 209] and series connections [141] of first and second order systems. No comparison has been made with such parameterizations.

7.2.2 Frequency Response Interpolation

The criterion of (7.1) is based on the seminorm

$$||X(s)||_{2,\Omega} = \left( \sum_{i=1}^{N} |X(j\omega_i)|^2 \right)^{\frac{1}{2}}$$

(7.2)

(see e.g. [131]), which is closely related to the usual operator norm

$$||X(s)||_2 = \left( \int |X(j\omega_i)|^2 d\omega \right)^{\frac{1}{2}}.$$ 

A typical property of a semi-norm is that it can be zero for non-zero arguments. Consequently, even if $P_i, i = 1, .., N$ are exact frequency response samples of the plant $P$, and if the criterion of (7.1) is made zero, then the identified nominal model can still differ from the plant $P$. It is a well-known result in interpolation theory, that a finite number of frequency response samples does not uniquely define the underlying system [10, 4]. Without some assumption, nothing can be said really about the plant’s frequency response in between each two adjacent samples. In recent years this issue has received a lot of attention especially in view of robust control and robust stability. The various approaches rest on the assumption that the "degree of stability" of the plant is known; a discrete-time system is assumed to be analytic in a disc with a radius larger than 1, and the real parts of the poles of a continuous-time system are presumed to be smaller than some given negative number. The difference between an interpolating nominal model and any system, that satisfies this assumption, can be guaranteed to be bounded over all frequencies [104, 105, 175, 144, 44]. This provides a guaranteed bound on the nominal model’s deficiency provided that the plant under investigation has the assumed degree of stability. Similar assumptions have been used to prove,
that if the number of samples tends to infinity, then an interpolating nominal model approximates the plant uniformly over all frequencies \cite{176, 89, 146}.

Surely, we want the nominal model and the plant to be "close", which requires a closeness of frequency responses at all frequencies \cite{238}. However we have available only a frequency response estimate of the plant at a finite number of frequencies. From the above discussion on interpolation it is clear that assumptions are needed in order that anything can be said about the plant at all other frequencies. But then again, the achieved "closeness" between the nominal model and the plant has its origin in the basic assumption (e.g. the degree of stability of the plant). Our confidence in the latter assumption dictates our confidence in the "closeness". So we might as well use any other assumption, that we can be confident about. For ease of referencing we loosely formalize the "leap of faith" that we make. An explanation is given afterwards.

**Assumption 7.2.2** The vector $\Omega = \col(\omega_1, \ldots, \omega_N)$ of frequencies is chosen such, that the frequency response samples $P(j\omega_i)$, $\omega_i \in \Omega$ are representative of the frequency response $P(j\omega)$, $\omega \in \mathbb{R}$.

This general assumption has various interpretations and applications including the following three. Firstly, suppose that the plant $P$ has order 1. Then 1 frequency response sample is representative of $P$. If $P$ is of higher order than more samples are needed. Secondly suppose we describe the plant $P$ with an approximate low order nominal model $\hat{P}$. Let $\{\hat{P}_i\}$ be a sequence of such nominal models (of the same order) that are derived from an increasing number of frequency response samples of $P$. Then for a large enough $i$ it is practically of no consequence whether the number of samples is increased further or not. Then the frequency response data $P(j\omega_i)$, $\omega_i \in \Omega$ used to determine $\hat{P}_i$ is considered to be representative for $P$ in the approximation problem. Thirdly suppose that we want to ascertain the stability of $H(P, C)$ using the additive robustness margin $\rho_A = C(I + \hat{P}C)^{-1}$. That is, we have to verify whether $(P - \hat{P})(j\omega)$ is smaller than $\rho_A$ all frequencies $\omega \in \mathbb{R}$. Let $(P - \hat{P})(j\omega_i)$ be considerably smaller than $\rho_A$ for all frequencies $\omega_i \in \Omega$. Then $(P - \hat{P})(j\omega_i)$, $\omega_i \in \Omega$ is representative of $P - \hat{P}$ if it gives us sufficient confidence about the stability of $H(P, C)$.

### 7.3 Performance-approximation by Fixed-loop Identification

Here we use the optimization of (7.1) to develop a workable identification procedure for fixed-loop performance-approximation. We recall from Section 5.2 that the mismatch of interest is

$$M_T = T(P, C) - T(\hat{P}, C).$$

In here the compensator $C$ is known and $T(P, C)$ is stable. From Proposition 5.2.3 we know that this mismatch can be related to the equivalent identification problem of
Section 4.2 by the expression

$$M_T = \left[ \begin{bmatrix} N^a \\ D^a \end{bmatrix} - \begin{bmatrix} \tilde{N} \\ \tilde{D} \end{bmatrix} Q_T \right] (D_o + CN_o)^{-1} \begin{bmatrix} C & I \end{bmatrix}$$

with $Q_T = (\tilde{D} + CN_o)^{-1}(D_o + CN_o)$ and $(\tilde{N}, \tilde{D})$ is a rcf of the nominal model $\hat{P}$. Furthermore in Chapter 6 we have seen that for control design the feedback matrices $T(P, C)$ and $T(\hat{P}, C)$ should be compared in regard of some design weight $\alpha$. For compatibility\footnote{More about the compatibility of the identification and control objectives is said in Chapter 8.} we use this design weight also in the identification of $\hat{P}$. Hence we consider the mismatch

$$M_T(\alpha) = \left[ \begin{bmatrix} \alpha N^a \\ D^a \end{bmatrix} - \begin{bmatrix} \alpha \tilde{N} \\ \tilde{D} \end{bmatrix} Q_T \right] (D_o + CN_o)^{-1} \begin{bmatrix} C/\alpha & I \end{bmatrix}.$$ 

For the parameterization of the set of candidate models we define

$$d(s) = d_n s^n + d_{n-1} s^{n-1} + \cdots + d_1 s + d_0$$

$$\theta_d = \text{col}(b_n, b_1, b_0, a_{n-1}, a_1, a_0, d_n, d_1, d_0)$$

in which $b_i$ and $a_i$ originate from $b(s)$ and $a(s)$ of Section 7.2. For notational convenience we omit the indeterminant $s$ from the polynomials $a(s)$, $b(s)$ and $d(s)$. We represent the candidate right coprime factorization by $(b/d, a/d)$ so that we can write $M_T(\alpha)$ as

$$M_T(\theta_d, \alpha) = \left[ \begin{bmatrix} \alpha N^a \\ D^a \end{bmatrix} - \begin{bmatrix} \alpha b/d \\ a/d \end{bmatrix} Q_T(\theta_d) \right] (D_o + CN_o)^{-1} \begin{bmatrix} C/\alpha & I \end{bmatrix} \tag{7.3}$$

with $Q_T(\theta_d) = (a/d + C \cdot b/d)^{-1}(D_o + CN_o)$. Actually $d$ can be eliminated from (7.3), which yields

$$M_T(\theta, \alpha) = \left[ \begin{bmatrix} \alpha N^a \\ D^a \end{bmatrix} (D_o + CN_o)^{-1} - \begin{bmatrix} \alpha b/a \\ I \end{bmatrix} (I + Cb/a)^{-1} \right] \begin{bmatrix} C/\alpha & I \end{bmatrix}.$$ 

The candidate nominal model appears in the form of $b/a$.

Since $T(P, C)$ is stable, we want $T(\hat{P}, C)$ to be stable as well ($\alpha$ does not affect the stability). In the mismatch $M_T(\theta, \alpha)$ the nominal model $\hat{P}$ is represented by $b(s)/a(s)$. Hence we are interested in polynomials $a(s)$ and $b(s)$ such that $T(b(s)/a(s), C)$ is stable. We could try to constrain the parameters such that this stability requirement is met for all candidate nominal models. Literature provides such constraints, which involve Lyapunov equations and state space transformations \[36, 88\]. It goes without saying that these parameterizations are not readily applicable, and that they potentially hinder a smooth numerical optimization. We wish not to deal with such problems yet, so
we take up a more pragmatic approach. We take $\theta \in \mathbb{R}^{2n+1}$, and if the resulting nominal model is stabilized by $C$, then apparently additional constraints are not needed. This approach turns out to work well, except for some cases of extreme undermodelling.

So far we have only parameterized the mismatch $M_T$. The plant $P$ contributes to this mismatch through its associated rcf $(N^c, D^c)$. The frequency response of $(N^c, D^c)$ can be estimated by means of the framework of Proposition 4.2.1. This data is denoted $N_i^c$ and $D_i^c$, which correspond to the frequencies $\omega_i$ of $\Omega$. From these frequency response data we estimate the nominal model $\tilde{P}$ by minimizing the seminorm $\|M_T(\theta, \alpha)\|_{2, \Omega}$ over $\mathcal{P}(\theta)$. For MIMO systems this seminorm is defined as

$$
\|X(s)\|_{2, \Omega} = \left( \sum_{i=1}^{N} \text{tr}\{X(-j\omega_i)^T \cdot X(j\omega_i)\} \right)^{1/2}
$$

in accordance with the usual operator norm $\|X(s)\|_2$ (see e.g. [143]). The trace operator sums the squares of the singular values of $X(j\omega_i)$ for each frequency $\omega_i$. Hence it may be replaced with the square of the Frobenius norm [85]. With this replacement the derivation of the nominal model $\tilde{P}$ can be rewritten to

$$
J_{T,i} = \left[ \begin{array}{c}
\alpha N_i^c \\
D_i^c
\end{array} \right] (D_o(j\omega_i)+C(j\omega_i)N_o(j\omega_i))^{-1} - \left[ \begin{array}{c}
\alpha \tilde{P}(j\omega_i) \\
I
\end{array} \right] (I+C(j\omega_i)\tilde{P}(j\omega_i))^{-1}
$$

$$
\tilde{P} = \arg \min_{\tilde{P} \in \mathcal{P}(\theta)} \sum_{i=1}^{N} J_{T,i} \cdot (\|C(j\omega_i)/\alpha\|^2 + 1).
$$

This minimization problem is solved by the Newton-Raphson method, details of which can be found in Appendix F.

The above optimization does not involve a coprime factorization of the candidate nominal model. Moreover if the frequency responses of $T_{12}(P, C)$ and $T_{22}(P, C)$ had been estimated, then we could have used those frequency response samples instead of the data $N_i^c(D_o(j\omega_i)+C(j\omega_i)N_o(j\omega_i))^{-1}$ and $D_i^c(D_o(j\omega_i)+C(j\omega_i)N_o(j\omega_i))^{-1}$. In that case the algebraic theory would be completely out of the picture. Nevertheless we need coprime factorizations to derive a good initial estimate for the optimization of (7.5).

The candidate nominal model appears in the cost function $J_{T,i}$ of (7.5) in a multiple and non-linear fashion. Consequently the criterion generally has local minima and the outcome of the optimization will depend on the initial vector of parameters. Hence we have to select a very good initial estimate in order that the minimization of (7.5) produces an useful nominal model $\tilde{P}$. We obtain such an initial estimate by the procedure below.
7.3 Performance-approximation by Fixed-loop Identification

Derivation of an Initial Estimate

We observe once more that the associated rcf \((N^a, D^a)\) is of higher complexity than the plant \(P\), cf. (4.12). We could identify \((\hat{N}_0, \hat{D}_0)\) from the mismatches \(N^a - \hat{N}_0\) and \(D^a - \hat{D}_0\), and calculate the initial estimate \(\hat{P}_0 = \hat{N}_0\hat{D}_0^{-1}\). However the low order \((\hat{N}_0, \hat{D}_0)\) are bad representatives of the high order \((N^a, D^a)\), so that \(\hat{P}_0\) will not be a good initial estimate. We recall from Section 4.4, that an additional term is needed to account for the redundant dynamics in \((N^a, D^a)\). In (4.13) we incorporated such a term as

\[
\begin{bmatrix}
N^a \\
D^a
\end{bmatrix} - \begin{bmatrix}
\hat{N}_0 \\
\hat{D}_0
\end{bmatrix} \hat{Q}.
\]

The term \(\hat{Q}\), which accounts for the redundant dynamics, depends on the candidate initial estimate \((\hat{N}_0, \hat{D}_0)\) and vice versa. In order to obviate this mutual dependency we take an other approach. First we remove the redundant dynamics directly from \((N^a, D^a)\), and thereafter we use the result to derive an initial estimate \((\hat{N}_0, \hat{D}_0)\). We outline this procedure in terms of transfer functions, and thereafter we explain how it can be put into practice using frequency response data.

We let \((N_n, D_n)\) be a normalized rcf of the plant \(P\). This nrcf \((N_n, D_n)\) is of the same complexity (has the same order) as \(P\). This is the lowest complexity that a coprime factorization of \(P\) can have (see Appendix B.3). We substitute \(N_n D_n^{-1}\) for \(P\) in (4.12) which yields

\[
N^a = N_n (D_n + CN_n)^{-1}(D_o + CN_o)
\]
\[
D^a = D_n (D_n + CN_n)^{-1}(D_o + CN_o).
\]

We define \(Q_n = (D_n + CN_n)^{-1}(D_o + CN_o)\) so that \((N^a, D^a)\) equals \((N_n Q_n, D_n Q_n)\). Notice that \(Q_n\) belongs to \(\mathcal{F}\), because \((N^a, D^a)\) and \((N_n, D_n)\) are both rcf's of \(P\) (cf. Fact 3.2.6.i). Now the redundant dynamics in \((N^a, D^a)\) are represented by \(Q_n\).

For SISO systems the "spectrum" associated with \((N^a, D^a)\) is

\[
N^a(-j\omega)N^a(j\omega) + D^a(-j\omega)D^a(j\omega)
= Q_n(-j\omega) \left[ N_n(-j\omega)N_n(j\omega) + D_n(-j\omega)D_n(j\omega) \right] Q_n(j\omega)
= Q_n(-j\omega)Q_n(j\omega)
\]

by virtue of (5.2). Hence by calculating the above spectrum from \((N^a, D^a)\) we have access to the redundant dynamics. \(Q_n\) can be identified as the minimum phase stable factor of the spectrum associated with \((N^a, D^a)\). Next we use \(Q_n\) to determine the nrcf \((N_n, D_n)\) from \((N^a, D^a)\):

\[
(N_n, D_n) = (N^a Q_n^{-1}, D^a Q_n^{-1}),
\]
by which \((N^a, D^a)\) is transformed into dynamics of low complexity without loss of information about \(P\).

We could try to derive an initial \((\hat{N}_0, \hat{D}_0)\) from the mismatches \(N_n - \hat{N}_0\) and \(D_n - \hat{D}_0\). These two mismatches do not depend on the compensator \(C\), whereas we wish to customize the initial estimate for the mismatch \(M_T\). To this end we write \(T(P, C)\) as

\[
\begin{bmatrix}
N^a \\
D^a
\end{bmatrix} (D_o + CN_o)^{-1} \begin{bmatrix} C & I \end{bmatrix} = \begin{bmatrix} N^n \\
D^n
\end{bmatrix} Q_n (D_o + CN_o)^{-1} \begin{bmatrix} C & I \end{bmatrix}
\]

and we determine the initial \((\hat{N}_0, \hat{D}_0)\) from the mismatch

\[
\begin{bmatrix} N_n - \hat{N}_0 \\
D_n - \hat{D}_0
\end{bmatrix} Q_n (D_o + CN_o)^{-1} \begin{bmatrix} C & I \end{bmatrix}.
\]

The initial estimate for (7.5) is then obtained as \(\hat{N}_0 \hat{D}_0^{-1}\).

The same procedure is applicable in combination with a scalar design weight \(\alpha\), because constant scaling of \(P\) does not affect its complexity. We let \(Q_{n,\alpha}\) be the spectral factor associated with the weighted associated rcf \((\alpha N^a, D^a)\), i.e.

\[(\alpha N^a(-j\omega))(\alpha N^a(j\omega)) + D^a(-j\omega)D^a(j\omega) = Q_{n,\alpha}(-j\omega)Q_{n,\alpha}(j\omega)\]

and we define

\[(N_{n,\alpha}, D_{n,\alpha}) = (\alpha N^a Q_{n,\alpha}^{-1}, D^a Q_{n,\alpha}^{-1}).\]

Notice that the rcf \((N_{n,\alpha}, D_{n,\alpha})\) is normalized and that \(N_{n,\alpha} D_{n,\alpha}^{-1}\) equals \(\alpha P\).

Hitherto the discussion has been based on transfer functions. Now we can rephrase the whole procedure in terms of the frequency response data \(N^a\) and \(D^a\), which makes it applicable in practice. First we calculate the spectral data

\[Q_i^* Q_i = (\alpha N_i^a)^* \cdot (\alpha N_i^a) + (D_i^a)^* \cdot D_i^a\]

and we estimate a spectral factor \(Q_{n,\alpha}(s)\) such that

\[Q_{n,\alpha}(-j\omega_i) \cdot Q_{n,\alpha}(j\omega_i) \approx Q_i^* Q_i, \quad i = 1, \ldots, N.\]

For this estimation we use the method of Appendix F.2. Next we calculate the frequency responses

\[N_{n,\alpha,i} = \alpha N_i^a Q_{n,\alpha}^{-1}(j\omega_i)\]
\[D_{n,\alpha,i} = D_i^a Q_{n,\alpha}^{-1}(j\omega_i).\]

As the spectrum \(Q_{n,\alpha}^* Q_{n,\alpha}\) will approximately describe the spectral data \(Q^* Q\), the data \(N_{n,\alpha}, D_{n,\alpha}\) are only "nearly normalized". Hence the data \(N_{n,\alpha}, D_{n,\alpha}\) represent
a rcf, which is only a slight distortion of some normalized rcf. Thus the significant
dynamics represented by $N_{n,\alpha}, D_{n,\alpha}$ are of the same complexity as the plant.

Building on (7.6) we determine the initial estimate $(\hat{N}_0, \hat{D}_0)$ in such a way that

$$\sum_{i=1}^{N} \left[ \begin{array}{c} N_{n,\alpha,i} \\ D_{n,\alpha,i} \end{array} \right] - \left[ \begin{array}{c} \alpha \hat{N}_0(j\omega_i) \\ \hat{D}_0(j\omega_i) \end{array} \right] \hat{Q}(j\omega_i)^2 \cdot |W_n(j\omega_i)|^2$$

is minimal. In here $W_n = Q_{n,\alpha}(D_0 + CN_0)^{-1}[C/\alpha \cdot 1]$. This criterion is optimized
by the method of Appendix F.4, which produces $\hat{N}_0, \hat{D}_0$ and the additional term $\hat{Q}$.
The term $\hat{Q}$ is introduced in order that $\hat{P}_0 = \hat{N}_0 \hat{Q}(D_0 \hat{Q})^{-1}$ is a small coprime factor
perturbation of $(N_{n,\alpha}, D_{n,\alpha})$. Using the expressions for $N_{n,\alpha,i}, D_{n,\alpha,i}$ and $W_n$ we can
rewrite the argument of the above summation into

$$\left[ \begin{array}{c} \alpha N^a \\ D^a \end{array} \right] - \left[ \begin{array}{c} \alpha \hat{N}_0 \\ \hat{D}_0 \end{array} \right] \hat{Q} Q_{n,\alpha} (D_0 + CN_0)^{-1} \left[ \begin{array}{c} C/\alpha \\ I \end{array} \right].$$

This equals the mismatch $M_T(\theta_d, \alpha)$ of (7.3), except that $\hat{Q} Q_{n,\alpha} \neq Q_T(\theta_d)$. Practical
experience has demonstrated that $\hat{Q} Q_{n,\alpha} \approx Q_T(\theta_d)$, unless the approximation of
$H(P, C)$ by $H(\hat{P}, C)$ is very poor due to severe undermodelling. Conclusively $\hat{N}_0 \hat{D}_0^{-1}$ is
determined so that its mismatch $M_T(\theta_d, \alpha)$ or equivalently $M_T(\theta, \alpha)$ is small. Since the
optimization of (7.5) minimizes $M_T(\theta, \alpha)$, the nominal model $\hat{N}_0 \hat{D}_0^{-1}$ is a well-suited
initial estimate.

**Example 7.3.1** Getting ahead of Chapter 8 we mention that in an iteration of identifica-
tion and controller enhancement we can use the nominal model $\hat{P}_{i-1}$ as an initial estimate
for the identification of the next nominal model $\hat{P}_i$. Quite often this works well and in
those cases we do not need the initial estimation procedure described above. Nevertheless
at times the results are greatly improved by constructing an appropriate initial estimate.
Here we show an example thereof.

We consider the combination of the plant $P$ and the compensator $C_3$ of the primary
iteration of The Illustrative Example (see Appendix A for detailed information). First we
determine the frequency response data of the plant's rcf that is associated with a nrcf
of the nominal model $\hat{P}_3$ and the compensator $C_3$. The Bode log-magnitude plots of $P$
(—) and of $\hat{P}_3$ and $C_3$ (—) are drawn in Fig. 7.1 (the dotted line corresponding to $\hat{P}_3$
almost coincides with (—) of $\hat{P}_{4x}$). We calculate the two nominal models $\hat{P}_4$ and $\hat{P}_{4x}$
by the optimization of (7.5). For the identification of $\hat{P}_4$ we use the above initial estimation
procedure. $\hat{P}_{4x}$ is derived with $\hat{P}_3$ as an initial estimate. The Bode plots of $\hat{P}_4$ (—) and
$\hat{P}_{4x}$ (—) differ greatly: in the lower frequency range $\hat{P}_{4x}$ provides a better description of
the plant $P$, and at the higher frequencies this holds for $\hat{P}_4$.

The improvement of $\hat{P}_4$ upon $\hat{P}_{4x}$ is most apparent from the singular value plots of the
feedback matrices and their differences. In Fig. 7.2 we have drawn $\sigma_{\text{max}}(T(P, C_3)(j\omega))$
(—) as a reference. The maximum singular values of the difference $T(P, C_3) - T(\hat{P}_{4x}, C_3)$
Fig. 7.1: Advantage of the initial estimation procedure. Bode log-magnitude diagrams of \( P (-) \), \( \hat{P}_3 (\cdots) \) and \( C_3 (\cdots) \), \( \tilde{P}_4 (\cdots) \) and the additional estimate \( \tilde{P}_{4\xi} (\cdots) \).

\( \cdots \) is small in the low frequency range, but gets relatively large around 10 rad/s. The converse holds for \( T(P, C_3) - T(\hat{P}_4, C_3) \). Notice also the similarity between \( \tilde{P}_{4\xi} \) and \( \hat{P}_3 \), which indicates that \( \hat{P}_3 \) was already near a local minimum of (7.5). For completeness we mention that the identification criterion corresponding to \( \hat{P}_4 \), i.e. the sum of the plotted singular values of \( T(P, C_3) - T(\hat{P}_4, C_3) \), is much smaller than the criterion corresponding to \( \tilde{P}_{4\xi} \). Finally we remark that \( \tilde{P}_4 \) is also much better than \( \tilde{P}_{4\xi} \) in an \( H_\infty \)-sense: the maximum of \( \cdots \) is much larger than that of \( \cdots \).

**Remark 7.3.2** We could have used the associated identification problem of Section 4.3 instead of the equivalent identification problem of (4.2) to arrive at the optimization of (7.5). The required frequency response data of \( T(P, C) \) can be derived from (3.16) and a frequency response estimate of \( R \). However we have no means to remove the redundant dynamics from \( R \).

### 7.4 Stability-approximation by Design-oriented Identification

In Section 5.3 we combined the mismatch of the equivalent identification problem of Section 4.2 with the control design method of Section 6.2 to establish a design-oriented
7.4 Stability-approximation by Design-oriented Identification

Fig. 7.2: Advantage of the initial estimation procedure. Maximum singular value plots of $T(P, C_3)$ (---), $T(P, C_3) - T(\hat{P}_4, C_3)$ (--), $T(P, C_3) - T(\hat{P}_{4\pi}, C_3)$ (-----) and $T(P, C_3) - T(\hat{P}_3, C_3)$ (---).

...stability-approximation problem. This approximation is design-oriented in the sense that it precisely anticipates the maximum robustness margin that is achieved in the subsequent control design. In Section 6.2 we exposed that the robustness margin can be manipulated by means of loop-shaping. The resulting control design

$$C_{\hat{P}} = \arg \min_{C \in C(\hat{P})} \|T(\alpha \hat{P}, C/\alpha)\|_{\infty}$$

optimizes robustness against perturbations of the normalized coprime factors of $\alpha \hat{P}$. The optimal approximation problem preceding this control design is to model $\alpha \hat{P}$ as a normalized coprime factor perturbation of $\alpha \hat{P}$ and to minimize this particular perturbation. Thereby the approximation anticipates the robustness margin that is maximized in the subsequent control design stage.

We let $(\hat{N}, \hat{D})$ be a rcf of the nominal model $\hat{P}$ and we define $(\hat{N}_{n, \alpha}, \hat{D}_{n, \alpha})$ as a normalized rcf of the loop-shaped nominal model $\alpha \hat{P}$. These two rcf's can be related as follows. We introduce the notation

$$Q^*(s)Q(s) = (\alpha \hat{N}(s))^*(\alpha \hat{N}(s)) + \hat{D}^*(s)\hat{D}(s)$$

(7.7)
with \( Q(s) \in \mathcal{J} \). As explained in the previous section we can use the minimum phase stable spectral factor \( Q(s) \) to normalize \((\alpha \hat{N}, \alpha \hat{D})\) to
\[
(\hat{N}_{n, \alpha}, \hat{D}_{n, \alpha}) = (\alpha \hat{N} Q^{-1}, \hat{D} Q^{-1}).
\] (7.8)

Now we obtain the loop shaped design-oriented stability-approximation from the approximation problem of (5.13) by replacing \( \hat{P} \) by \( \alpha \hat{P} \), \((\bar{N}, \bar{D})\) by \((\hat{N}_{n, \alpha}, \hat{D}_{n, \alpha})\) and \((N, D)\) by \((\alpha N, \alpha D)\). Further we replace \((\hat{N}_{n, \alpha}, \hat{D}_{n, \alpha})\) by the above \((\alpha \hat{N} Q^{-1}, \hat{D} Q^{-1})\). The resulting mismatch becomes
\[
\left[ \begin{array}{c}
\alpha \hat{N} \\
\hat{D}
\end{array} \right] - \left[ \begin{array}{c}
\alpha N \\
D
\end{array} \right] Q_{\Delta} \right] Q^{-1}.
\]

We also have replaced \( Q_{\Delta} \) of (5.13) by \( Q_{\Delta} Q^{-1} \). This is just a matter of notation since \( Q_{\Delta} \) is a free parameter. The resulting approximation problem is
\[
\hat{P} = \arg \min_{\hat{P}, Q_{\Delta}} \left\| \left[ \begin{array}{c}
\alpha \hat{N} \\
\hat{D}
\end{array} \right] - \left[ \begin{array}{c}
\alpha N \\
D
\end{array} \right] Q_{\Delta} \right\|_{\infty}.
\] (7.9)

where \( Q_{\Delta} \in \mathcal{H}, \) \( \hat{P} = \hat{N} \hat{D}^{-1} \), and \( \hat{Q} \) is the minimum phase stable factor such that \((\alpha \hat{N} \hat{Q}, \hat{D} \hat{Q})\) is normalized (cf. (7.8)).

As there exists no identification technique that can be used to solve the latter approximation problem, we replace the \( H_\infty \) (or \( L_\infty \)) approximation by the \( L_2 \) approximation
\[
\hat{P} = \arg \min_{\hat{P}, Q_{\Delta}} \left\| \left[ \begin{array}{c}
\alpha \hat{N} \\
\hat{D}
\end{array} \right] - \left[ \begin{array}{c}
\alpha N \\
D
\end{array} \right] Q_{\Delta} \right\|_2.
\] (7.10)

It is true that the resulting \( \hat{P} \) will not be the minimizing argument of the \( H_\infty \)-criterion in (7.9). Nevertheless the \( L_2 \) approximation yields a reasonably good nominal model in an \( L_\infty \)-sense, provided that the resulting mismatch is sufficiently smooth. Thus \( \hat{P} \) of (7.10) is probably not the best nominal model in the sense of (7.9), but it will be a very good approximate solution to the corresponding optimization problem. Consequently \( \hat{P} \) of (7.10) will be suited to design a compensator by (6.8) for the plant \( P \).

This observation is backed-up by the following fact. If the \( L_2 \)-norm of (7.10) tends to zero for some sequence of nominal models, then the \( L_\infty \)-norm of (7.9) will also tend to zero, provided that some smoothness conditions are satisfied (see [33, 147] and also [22, p.228]).

The approximation of (7.10) can readily be translated into a frequency domain identification problem. We let \( N^a_i, D^a_i, i = 1, \ldots, N \) be the available frequency response data of the associated rcf \((N^a, D^a)\) of the plant \( P \). Further we represent the rcf of the candidate nominal model by \((b/d, a/d)\) as in the previous section. For each frequency
7.4 Stability-approximation by Design-oriented Identification

\[ \omega_i \text{ of } \Omega \text{ we define} \]

\[ J_{r,i} = \left[ \begin{array}{c} \frac{\alpha b(j\omega_i)}{d(j\omega_i)} \\ \frac{a(j\omega_i)}{d(j\omega_i)} \end{array} \right] - \left[ \begin{array}{c} \alpha N^a_i \\ D^a_i \end{array} \right] \left( j\omega_i \right) \left( \frac{a(j\omega_i)}{d(j\omega_i)} \right)^2 \left( \frac{(\alpha b(-j\omega_i)) \cdot (\alpha b(j\omega_i)) + a(-j\omega_i) \cdot a(j\omega_i)}{d(-j\omega_i) \cdot d(j\omega_i)} \right)^{-1} \quad (7.11) \]

The inverted term stands for \( \hat{Q}^* \hat{Q} \), which can be expressed in terms of \( \alpha \) and \( \left( \hat{N}, \hat{D} \right) \) as in (7.7). Thus we do not need an explicit spectral factorization in order to determine \( J_{r,i} \). The resulting frequency domain identification problem is

\[ \hat{\theta}_d = \arg \min_{\theta_d \in \mathbb{R}^{3n+2}, Q_\Delta \in \mathcal{H}} \sum_{i=1}^{N} J_{r,i} \quad (7.12) \]

and \( \hat{P} \) is obtained from \( \hat{\theta}_d \) in the obvious way.

For the minimization of (7.12) we use the following iterative search procedure. We let \( Q_{\Delta,k} \) and \( \theta_{d,k} \) be the values that \( Q_\Delta \) and \( \theta_d \) take after the \( k \)-th step of the iteration. Then we first (try to) improve the parameter vector \( \theta_d \) by means of a Newton-Raphson optimization, details of which are provided in the final section of Appendix F. This produces the new parameters \( \theta_{d,k+1} \), while \( Q_{\Delta,k} \) remains fixed. Then we (attempt to) improve the factor \( Q_{\Delta,k} \) by the procedure of Section F.6, while \( \theta_{d,k+1} \) remains fixed. This alternate optimization is terminated when the sum of (7.12) is no longer decreased significantly.

Again we have to find a good initial estimate. The procedure that we use is much like that of the previous section. First we “normalize” the \( \alpha \)-weighted rcf \((\alpha N^a, D^a)\) which yields \((N_{n,\alpha}, D_{n,\alpha})\). Notice that if we had known the plant \( P \), then the control design would have been based on this rcf \((N_{n,\alpha}, D_{n,\alpha})\). Then we use the method of Appendix F.4 to find a triple of \( \hat{N}_{n,\alpha}, \hat{D}_{n,\alpha} \) and \( \hat{Q}_\alpha \), that minimizes the (semi-) \( L_2 \)-norm of

\[ \left[ \begin{array}{c} N_{n,\alpha} \\ D_{n,\alpha} \end{array} \right] - \left[ \begin{array}{c} \hat{N}_{n,\alpha} \\ \hat{D}_{n,\alpha} \end{array} \right] \hat{Q}_\alpha. \]

The resulting \( \hat{N}_{n,\alpha}, \hat{D}_{n,\alpha} \) are used as initial values for \( \alpha \cdot b/d \) and \( a/d \) and \( \hat{Q}_\alpha^{-1} \) serves as an initial estimate of \( Q_\Delta \). The inversion of \( \hat{Q}_\alpha \) is needed to transform the above mismatch into that of (7.11). The former is reminiscent of the directed gap \( \delta(\alpha P, \alpha \hat{P}) \), whereas the latter pertains to \( \delta(\alpha \hat{P}, \alpha P) \).
7.5 Experimental Stability Ascertainment

As explained in Section 7.2 we consider the case in which information about the plant is available only at the frequencies of \( \Omega \). We accordingly conceived several identification procedures in terms of vectors of sampled frequency response data. Here we do the same for the ascertainment of stability.

In Section 6.3 we derived various conditions that can guarantee the stability of the enhanced actual feedback system \( H(P, C_E) \) from the robustness margins of the enhanced nominal feedback system \( H(\hat{P}, C_E) \). We evaluate these conditions at the frequencies \( \omega_i, i = 1, \ldots, N \). This stability test per frequency is related in spirit to the pointwise criterion for robustness of Schumacher [205]. We do not make some explicit assumption on the plant \( P \) in order to guarantee that the stability condition holds for all frequencies. Instead we invoke Assumption 7.2.2, and thus we will conclude that \( H(P, C_E) \) is stable if some stability condition is satisfied for all frequencies of \( \Omega \).

This pointwise stability ascertainment can be unreliable when the stability condition is almost violated at the finite number of frequencies \( \Omega \). Therefore we require that the robustness margin is much larger than the corresponding mismatch. If that is not the case, then we reject the compensator. This rejected compensator might perhaps just stabilize the plant, but we want a good performance as well.

The stability condition of Proposition 6.3.9 can be applied straightforwardly to a frequency response estimate of \( R \). The latter can be constructed from frequency response data of \( (N^a, D^a) \), cf. (3.6). Examples of this stability test have been provided at the end of Section 6.3.

The stability condition of Proposition 6.3.7 requires the identification of a stable factor \( Q_{\Delta} \). We do not need to achieve the infimum of Proposition 6.3.7, but we only have to find some \( Q_{\Delta} \), for which the stability condition holds. A procedure for estimating such a \( Q_{\Delta} \) is discussed in Appendix F.6. Again, examples can be found in Section 6.3.
Part II

Iterative High-performance Control Design
Chapter 8

Iterative Schemes of Identification and Control Design

In this chapter we blend all preceding developments together to form two iterative schemes of repeated identification and control design. In the first section below we outline the design objectives that are pursued by both schemes. In Section 8.2 we design a high performance control system by repeated applications of the fixed-loop identification of Section 7.3 and the cautious controller enhancement of Section 6.2. This is called the primary iterative scheme. Then in Section 8.3 we perform a similar iteration with the exception, that only the first identification step is carried out by the design-oriented identification method Section 7.4. This will significantly speed up the iterative high performance control design procedure. In the last section we discuss the interaction between the individual identification and control design stages and we compare the iterative scheme with adaptive control.

8.1 Iterative High Performance Control Design

We recapture Section 2.4 in a nutshell. The true objective is to use system identification and model-based control design to construct a controller that achieves a high performance for the plant. A practical prerequisite for a successful control design is a satisfactory nominal performance. Otherwise we have no confidence in the nominal model and/or the compensator. This leads to the joint problem of finding a nominal model $\hat{P}$ and a compensator $C_\rho$ so that the designed $H(\hat{P}, C_\rho)$ exhibits the desired performance and $H(P, C_\rho)$ has similar feedback properties. An iteration of repeated identification and control design is necessary in order to arrive at a suitable pair of $\hat{P}$ and $C_\rho$.

An iteration produces a series of nominal models $\{\ldots, \hat{P}_{i-1}, \hat{P}_i, \hat{P}_{i+1}, \ldots\}$ and an associated series of compensators $\{\ldots, C_{i-1}, C_i, C_{i+1}, \ldots\}$. A whole iteration step consists of the update of the nominal model and the subsequent update of the compensator.
We evaluate the couple \(\hat{P}_{i-1}, C_{i-1}\) in terms of the NICE Proposition 2.2.6, i.e.
\[
\|T(P, C_{i-1})\| \leq \|T(\hat{P}_{i-1}, C_{i-1})\| + \|T(P, C_{i-1}) - T(\hat{P}_{i-1}, C_{i-1})\|. \tag{8.1}
\]
The nominal model \(\hat{P}_{i-1}\) is possibly not the best description of \(P\) under feedback by \(C_{i-1}\). We identify \(\hat{P}_i\) by minimizing the performance degradation. Then we have the inequality
\[
\|T(P, C_{i-1})\| \leq \|T(\hat{P}_i, C_{i-1})\| + \|T(P, C_{i-1}) - T(\hat{P}_i, C_{i-1})\|. \tag{8.2}
\]
Next \(\hat{P}_i\) is used for the design of \(C_i\), and an evaluation of the new couple \(\hat{P}_i, C_i\) reads
\[
\|T(P, C_i)\| \leq \|T(\hat{P}_i, C_i)\| + \|T(P, C_i) - T(\hat{P}_i, C_i)\|. \tag{8.3}
\]
The upper bounds of (8.1) and (8.3) measure the achieved robust performance. The series of upper bounds should ideally decrease at each iteration step so that\(^1\)

1. there is no distortion of the plant's performance during the iteration, and
2. in the end a high performance is guaranteed for the plant.

This desired improvement of the robust performance has been illustrated in Fig. 2.11. In the next section we show that such an evolution of the nominal, actual and robust performances can be achieved by means of the techniques developed in the previous chapters.

In this chapter the simulation studies concern The Illustrative Example. Most of the numerical results are listed in Appendix A. Here we mention merely that the plant, the nominal models and the compensators have orders 8, 5 and 4. The low order approximation has been chosen to emphasize the effects of undermodelling. Finally we wish to make clear that also other simulation studies have been carried out, in which the same procedures have led to similar results.

### 8.2 Primary Iterative Scheme

The \(i\)-th step of the primary iterative scheme consists of the following 3 stages. In the identification stage we determine the frequency responses of the plant's coprime factors, that are associated with \(C_{i-1}\) and the nrcf \((\tilde{N}_{n,i-1}, \tilde{D}_{n,i-1})\) of \(\hat{P}_{i-1}\). This compensator \(C_{i-1}\) and the nominal model \(\hat{P}_{i-1}\) are available, and \(\hat{P}_{i-1}\) serves as the auxiliary model \(P_o\) that has been used extensively in Chapters 3 and 4. In practice the compensator \(C_{i-1}\) has to be applied to the plant \(P\), and the required frequency response data can be obtained through the identification framework of Proposition 4.2.1. To these data

\(^{1}\)The first requirement is stipulated in view of practical applications; it is not impossible that the temporary use of a destabilizing compensator speeds up the iteration.
we apply the approximate fixed-loop performance-identification of Section 7.3. The resulting nominal model \( \hat{P} \) is such that

\[
J_{2,\Omega}^A(\hat{P}, C_{i-1}, \alpha_{i-1}) \doteq \|T(\alpha_{i-1}P, C_{i-1}/\alpha_{i-1}) - T(\alpha_{i-1}\hat{P}, C_{i-1}/\alpha_{i-1})\|_{2,\Omega}
\]

is small. Notice the similarity between this criterion \( J_{2,\Omega}^A(X) \) and \( J_{\infty}(X) \) of (6.10).

In the second stage the cautious controller enhancement of Section 6.2 is applied to \( \hat{P} \). That is, we first determine the frequency responses of the plant's coprime factors, that are associated with \( C_{i-1} \) and the nrcf \((\tilde{N}_{n,i}, \tilde{D}_{n,i}) \) of the new nominal model \( \hat{P} \) (instead of the nrcf of the "old" \( \hat{P}_{i-1} \)). The reader is referred to Section 6.1 for details about the difference between the associated plant factors that are used in the identification and control design stages. From these data and from \( \hat{P} \) we design an enhanced compensator \( C_i \) by increasing \( \alpha_{i-1} \) up to the new design weight \( \alpha_i \).

The third stage concerns the ascertainment of the stability of \( H(P, C_i) \) from the robustness margins of \( H(\hat{P}, C_i) \). This procedure has been explained and illustrated in Section 6.3. We omit the stability ascertainment from further discussions except for the following comments. All cautiously enhanced controllers have been demonstrated to stabilize the plant \( P \) prior to their application. Especially the robustness margin of Proposition 6.3.9 proved to be very useful for this purpose: as the "old" control system \( H(P, C_{i-1}) \) is known to be stable, we can ascertain the stability of \( H(P, C_i) \) with \( K = C_{i-1} \). Without this stability test we could not have accepted for instance the controller \( C_3 \) (see the end of Section 6.3 for details).

### 8.2.1 Simulation Study

The iteration begins with the open-loop operation of the plant \( P \), and thus with an open-loop identification. The resulting nominal model \( \hat{P}_1 \) has already been employed in The Illustrative Example, I (Section 2.4.1). The whole iteration consists of 5 identification and control design stages. The applied design weights range from 0.113 to 20 (see also Table A.8). Fig. 8.1.a shows the Bode log-magnitude plots of the plant \( P \) (—), and the nominal models \( \hat{P}_1 \) (--) , \( \hat{P}_4 \) (...) and \( \hat{P}_5 \) (—). The Bode diagrams of \( \hat{P}_2 \) and \( \hat{P}_3 \) have not been plotted, because they are hardly distinguishable from that of \( \hat{P}_1 \). We see that \( \hat{P}_1 \) provides a good description of the plant \( P \) at the lower frequencies, but not at the higher frequencies. The converse holds for the final nominal model \( \hat{P}_5 \).

The frequency responses of the compensators \( C_i, i = 1,..,5 \) have been drawn in Fig. 8.1.b. These curves reveal that the control effort is gradually increased over all frequencies in each step of the iteration. In this plot we have also drawn the frequency response of the plant-based compensator \( C_P \) (—). This 4-th order compensator \( C_P \) has been derived by the design method of (6.8) with \( \hat{P} \) and \( \alpha \) replaced by \( P \) and \( \alpha_5 \). Thus \( C_P \) is the optimal compensator for the plant \( P \) and \( \alpha = \alpha_5 \). In Fig. 8.1.b the frequency responses of the plant-based compensator \( C_P \) (—) and the model-based compensator \( C_5 \) (—) are indiscernible. Also \( C_P \) and \( C_5 \) have similar transfer function coefficients (Table A.8) and similar poles and zeros (Table A.9). Hence the iteratively
designed high performance compensator $C_5$ is almost identical with the compensator $C_P$, that can be derived only with exact knowledge of the plant. We stress that in the iterative design procedure we only used frequency response samples of the plant's coprime factors! Neither exact knowledge of the plant $P$ nor any information about the compensator $C_P$ have been employed in deriving $C_5$. Accordingly, we have not built-in a mechanism that explicitly minimizes $C_5-C_P$. It just turns out that if the performance degradation $||T(P, C_P)-T(\hat{P}, C_P)||$ is much smaller than the nominal performance norm $||T(\hat{P}, C_P)||$, then $C_5$ is not only optimal for $\hat{P}$ but also nearly optimal for $P$.

Fig. 8.1: Results of the primary iteration.

a: Bode log-magnitude plots of $P (-)$, $\hat{P}_1 (-)$, $\hat{P}_4 (-)$, $\hat{P}_5 (-)$.

b: Bode log-magnitude plots of the compensators.

c: Improvement of $\hat{P}_5$ upon $\hat{P}_4$: max. sing. values of $T(\alpha_4 P, C_4/\alpha_4)$ (-), and $T(\alpha_4 P, C_4/\alpha_4)-T(\alpha_4 P_1, C_4/\alpha_4)$ with $i=4$ (-) and $i=5$ (-).

d: Logarithm of control cost functions evaluated at $\alpha_5$. 

We illustrate the identification stage with the estimation of \( \hat{P}_5 \) from \( T(P, C_4) \). The frequency dependent maximum singular value \( \sigma_{\text{max}}(T(\alpha_4 P, C_4/\alpha_4)) \) has been depicted in Fig. 8.1.c (---) as a reference. This figure displays also the singular values of \( T(\alpha_4 P, C_4/\alpha_4) - T(\alpha_4 \hat{P}_4, C_4/\alpha_4) \) (---) and \( T(\alpha_4 P, C_4/\alpha_4) - T(\alpha_4 \hat{P}_5, C_4/\alpha_4) \) (----). In view of the design weight \( \alpha_5 \) the nominal model \( \hat{P}_5 \) exhibits a relatively large deficiency at the higher frequencies. It is clear from the figure that \( \hat{P}_5 \) is much better than \( \hat{P}_4 \) in terms of the semi-norm \( \| . \|_{\infty, \Omega} \), although \( \hat{P}_5 \) has been derived by the minimization of the semi-norm \( \| . \|_{2, \Omega} \) of the above mismatch, cf. (7.5).

All compensators \( C_i, i = 1, \ldots, 5 \) have been evaluated in terms of the cost functions \( J_{\infty, \Omega} \) and \( J_{\infty, \Omega}^{\Delta} \), which equal \( J_{\infty} \) of (6.9) and \( J_{\infty}^{\Delta} \) of (6.10) except that \( \| . \|_{\infty} \) is replaced by \( \| . \|_{\infty, \Omega} \). Further we judge all compensators in respect of the final design weight \( \alpha_5 \). Thus for instance \( J_{\infty, \Omega}(\hat{P}_3, C_3, \alpha_5) \) measures the performance of \( H(\hat{P}_3, C_3) \) as the maximum singular value of \( T(\alpha_5 \hat{P}_3, C_3/\alpha_5) \) over all frequencies in \( \Omega \). Likewise \( J_{\infty, \Omega}^{\Delta}(\hat{P}_3, C_3, \alpha_5) \) quantizes the performance degradation \( T(P, C_3) - T(\hat{P}_3, C_3) \) at the trade-off corresponding to \( \alpha_5 \). With these cost functions the performance inequalities of (2.13) become

\[
J_{\infty, \Omega}(P, C_i, \alpha_5) \geq |J_{\infty, \Omega}(\hat{P}_i, C_i, \alpha_5) - J_{\infty, \Omega}^{\Delta}(\hat{P}_i, C_i, \alpha_5)|
\]

\[
J_{\infty, \Omega}(P, C_i, \alpha_5) \leq J_{\infty, \Omega}(\hat{P}_i, C_i, \alpha_5) + J_{\infty, \Omega}^{\Delta}(\hat{P}_i, C_i, \alpha_5).
\]

In Fig. 8.1.d the logarithm of these cost functions have been depicted for each compensator. The actual performances \( J_{\infty, \Omega}(P, C_i, \alpha_5) \) are indicated by 'x', and the nominal performances \( J_{\infty, \Omega}(\hat{P}_i, C_i, \alpha_5) \) are marked 'o'. The upper and lower bounds of the above inequalities are represented by dotted lines. A comparison with Fig. 2.11 reveals that we have accomplished the desired iterative improvement of the robust performance. In both pictures the nominal performance, 'o', and the upper bound on the plant's performance are improved at each iteration step. The decrease of the upper bound "forces" the actual performance of the plant, 'x', to decrease as well. More about these cost functions is said in the discussion at the end of this section.

The ultimate result is depicted in Fig. 8.2. This figure shows the frequency responses of the feedback matrices \( T(P, C_5) \) and \( T(\hat{P}_5, C_5) \). The achieved sensitivity is small in the low frequency range and the complementary sensitivity tends to zero at the higher frequencies. The performances for \( P \) and \( \hat{P}_5 \) are almost identical, despite the perceivable differences in Fig. 8.2.c and d. These differences hardly contribute to the performance degradation, because they are neglectable on a linear scale. Besides, the elements represented in Fig. 8.2.b and c are multiplied by 20 respectively 1/20 in the cost functions based on \( \alpha_5 \).

Finally Fig. 8.3 displays the achieved sensitivities. Observe that the sensitivity \( T_{22}(P, C_1) \) (---) does not differ much from unity except at the resonances of \( P \). This means that the pair \( \hat{P}_1, C_1 \) yields only a minor improvement upon the open-loop operation. Indeed in The Illustrative Example, I, of Section 2.4.1 we have already seen that \( \hat{P}_1 \) is not suited to design a high performance compensator for the plant \( P \).
As the compensator evolves from $C_1$ to $C_3$ the sensitivity is gradually reduced at the lower frequencies without causing an unacceptable growth at the higher frequencies. In fact the ultimate sensitivity $T_{22}(P, C_3)$ practically equals the “ideal” sensitivity $T_{22}(P, C_P)$ corresponding to the plant-based compensator $C_P$. It is true that it is easy to control the plant $P$ in the sense that it has no open right half-plane poles or zeros. However we control a plant of order 8 with a compensator of order 4. The associated attainable performance is limited, which is a possible explanation of the right half plane zeros of $\hat{P}_4$ and $\hat{P}_3$.

### 8.2.2 The Mechanism of the Iterative Scheme

We analyze how the iterative design procedure accomplishes a high-performance controller for the plant. To that end we investigate the individual identification and control design stages as well as their interaction.
The series of design weights is of great importance for the evolution of the iteration. In the cautious controller enhancement these $\alpha_i$'s are chosen such, that the designed nominal feedback system $H(\hat{P}_i, C_i)$ is compatible with the actual feedback system $H(P, C_i)$ in respect of the trade-off that corresponds with $\alpha_i$. Hence the nominal performance of $H(\hat{P}_i, C_i)$ is expected to predict the performance of $H(P, C_i)$. Conceived in terms of cost functions the aim of the control design is that

$$J(P, C_i, \alpha_i) \approx J(\hat{P}_i, C_i, \alpha_i)$$

in the sense that the performance degradation is much smaller than the nominal performance, i.e.

$$J^\Delta(\hat{P}_i, C_i, \alpha_i) \ll J(\hat{P}_i, C_i, \alpha_i)$$

(see also (2.13), (2.14) and the accompanying discussion). In other words the task of the control design is to improve the nominal performance by replacing $C_{i-1}$ by $C_i$ without incurring a large performance degradation.

The nominal model $\hat{P}_i$ has been derived without knowledge of $\alpha_i$ and $C_i$. Hence $\hat{P}_i$ will not provide the best possible description of the behavior of $H(P, C_i)$. The aim of the identification is to derive a nominal model $\hat{P}_{i+1}$ that improves upon $\hat{P}_i$ in describing $P$ in respect of $C_i$ and $\alpha_i$. This is done by making the performance degradation $J^\Delta(\hat{P}_{i+1}, C_i, \alpha_i)$ as small as possible. Thus the task of the identification is to tighten the upper and lower bounds on the plant's performance. The plant's performance itself
is not changed in the identification stage.

The eventually achieved compensator $C_5$ yields a small nominal cost function $J(\hat{P}_5, C_5, \alpha_5)$ and an even much smaller performance degradation $J^\Delta(\hat{P}_5, C_5, \alpha_5)$. Hence we can regard the whole iteration as the design of an optimal compensator for the trade-off corresponding to $\alpha_5$. Therefore we call the corresponding $J$'s the global cost functions similar to e.g. [23]. The other cost functions depending on $\alpha_1$ to $\alpha_4$ are called local cost functions. Several local cost functions have been plotted in Fig. 8.4 and the global cost functions $J_{\infty, \Omega}$ and $J_{2, \Omega}$ have been depicted in Fig. 8.5 respectively Fig. 8.6.

![Fig. 8.4](image)

Fig. 8.4: Local cost functions $J(P, C_j, \alpha)$ (•), $J(\hat{P}_i, C_j, \alpha)$ (○) and corresponding bounds. The pairs at the horizontal axes indicate the subscripts $i,j$ of $\hat{P}_i, C_j$.

- a: $J_{\infty, \Omega}$, $\alpha = \alpha_3$.
- b: $J_{\infty, \Omega}$, $\alpha = \alpha_4$.
- c: $J_{2, \Omega}$, $\alpha = \alpha_3$.
- d: $J_{2, \Omega}$, $\alpha = \alpha_4$.

We examine Fig. 8.4.a, which shows the cost functions $J_{\infty, \Omega}$ at $\alpha = \alpha_3$ for the nominal models $\hat{P}_2, \hat{P}_3, \hat{P}_4$ and the controllers $C_2, C_3$. The indices $i,j$ at the horizontal
8.2 Primary Iterative Scheme

Fig. 8.5: Logarithm of the global cost functions $J_{\infty, \Omega}(P, C_i, \alpha_5)$ (⋆), $J_{\infty, \Omega}(\hat{P}_i, C_i, \alpha_5)$ (○) and the performance bounds (---).

axis of the figure are the subscripts of the corresponding $\hat{P}_i$ and $C_j$. Thus from the left to the right the stars '⋆' correspond to $H(P, C_2)$, $H(P, C_3)$, $H(P, C_3)$ and $H(P, C_3)$. Similarly the markers '○' stand for $H(\hat{P}_2, C_2)$, $H(\hat{P}_3, C_2)$, $H(\hat{P}_3, C_3)$ and $H(\hat{P}_4, C_3)$. In this figure we see that the identification discharged its task properly: the bounds on the plant’s performance are tightened in the transition form $\hat{P}_3$ to $\hat{P}_4$, although the cost function $J_{\infty, \Omega}^A$ has not been minimized explicitly. This tightening of the bounds is hardly/not visible for the transition from $\hat{P}_2$ to $\hat{P}_3$. Besides, the latter identification took place at $\alpha_2$, whereas its result is now evaluated at $\alpha_3$. Likewise the improvement of $\hat{P}_4$ upon $\hat{P}_3$ is evaluated at $\alpha_4$ in Fig. 8.4.b. The latter tightening of the bounds is clearly noticeable at both $\alpha_3$ and $\alpha_4$, but the enveloping bounds develop differently. We return to this subject at a later stage. Further Fig. 8.4.b also displays the improvement of $\hat{P}_5$ upon $\hat{P}_4$ in respect of $\alpha_4$ and $C_4$.

The plots in Fig. 8.4.c and d display the evolvement in terms of the cost functions $J_{2, \Omega}$. Notice that the courses of Fig. 8.4.b and d are quite similar, and that Fig. 8.4.a and c exhibit some differences. Anyway the identification tightens the performance bounds in Fig. 8.4.c and d as well. This is not surprising, since $\hat{P}_4$ and $\hat{P}_5$ have been derived by minimizing the cost functions $J_{2, \Omega}^A(\hat{P}_4, C_3, \alpha_3)$ and $J_{2, \Omega}^A(\hat{P}_5, C_4, \alpha_4)$.

Especially Fig. 8.4.c discloses that the quality of the approximate nominal model has to be evaluated from the bounds and not from the actual cost functions alone.
Fig. 8.6: Logarithm of the global cost functions $J_{2,\Omega}(P, C_1, \alpha_5) \, (\ast)$, $J_{2,\Omega}(\hat{P}_1, C_i, \alpha_5) \, (\circ)$ and the performance bounds (---).

This figure shows that

$$J_{2,\Omega}(P, C_2, \alpha_3) \approx J_{2,\Omega}(\hat{P}_2, C_2, \alpha_3)$$
$$J_{2,\Omega}(P, C_3, \alpha_3) \approx J_{2,\Omega}(\hat{P}_3, C_3, \alpha_3)$$
$$J_{2,\Omega}(P, C_3, \alpha_3) \approx J_{2,\Omega}(\hat{P}_4, C_3, \alpha_3)$$

but only the bounds that correspond to $\hat{P}_4$ are really tight. The consequences for the control design stage are significant. In Fig. 8.4.b and d the compensator $C_4$ improves a lot upon the compensator $C_3$ in every possible sense: at $\alpha_4$ the nominal, actual and robust performances for $\hat{P}_4, C_4$ are all much better than $\hat{P}_3, C_3$. From Fig. 8.5 and Fig. 8.6 it is clear that also the global objectives are very much improved with $C_4$ (bear in mind that the latter two plots show the logarithm of the cost functions). On the other hand the nominal model $\hat{P}_3$ is not that a good approximation of $P$ under feedback by $C_2$. In Fig. 8.4.a the lower bound for $\hat{P}_3, C_2$ is almost zero, which means that the performance degradation is almost as large as the nominal performance itself. The controller $C_3$ improves the nominal performance ('o' at $\hat{P}_3, C_3$ in Fig. 8.4.a,c) but not the worst case performance. The upper bound on the local cost functions are larger for $\hat{P}_3, C_3$ than for $\hat{P}_3, C_2$. This is a critical stage in the iteration, which deserves some extra attention.
A Critical Point of the Iteration

From the Figures 8.4, 8.5 and 8.6 we may conclude that, in respect of the cost functions $J_{2, \Omega}$ and $J_{\infty, \Omega}$ evaluated at $\alpha_3, \alpha_4$ and $\alpha_5$, the nominal feedback system $H(\hat{P}_4, C_3)$ provides a good description of the actual feedback system $H(P, C_3)$. This indicates that $H(\hat{P}_4, C_3)$ and $H(P, C_3)$ have similar feedback matrices, and thus similar sensitivities to perturbations of the compensator $C_3$. Hence $\hat{P}_4$ is suited for controller enhancement, and indeed $C_4$ greatly improves upon $C_3$ in every sense.

Now why all this special attention for $\hat{P}_4$? We use this particular stage of the iteration to explain that not only the quality of $\hat{P}_4$, but also the quality of $\hat{P}_3$ is of importance for design of $C_4$! In fact a (very) low quality of $\hat{P}_{i-1}$ brings forth the requirement of a (very) high quality of $\hat{P}_i$. This phenomenon is (partly) due to the fact that the utilized loop shaping design procedure involves a minimum controller enhancement (see also Remark 6.2.5). Hence the enhancement cannot be arbitrarily cautious, and a large minimum controller enhancement involves little caution so that a highly accurate nominal model is required. We investigate this minimum controller enhancement for $\hat{P}_4$.

We design the controller $C_{4x}$ from $\hat{P}_4$ with $\alpha = \alpha_3$. The difference between $C_3$ (—) and $C_{4x}$ (—) is apparent from their Bode log-magnitude curves shown in Fig. 8.7.b. If another design weight different from $\alpha_3$ is used, then the difference becomes larger. Thus replacing $C_3$ by $C_{4x}$ is the minimum controller enhancement for $H(\hat{P}_4, C_3)$. This enhancement is not very cautious: the difference between $C_3$ and $C_{4x}$ is significant. Fortunately $\hat{P}_4$ is of sufficient quality to predict the behavior of $P$ under feedback control by $C_{4x}$. This can be seen from Fig. 8.7.a,c, which contains copies of Fig. 8.4.a,c extended with the cost functions corresponding to $\hat{P}_4, C_{4x}$. These figures show how much $C_{4x}$ improves upon $C_3$: the nominal performance (‘o’), the robust performance (….) and the plant’s performance (‘*’) for $H(\hat{P}_4, C_{4x})$ (indexed 4,4x) are better than their counterparts for $H(\hat{P}_3, C_3)$ (indexed 4,3).

We relate the design of $C_{4x}$ to the quality of the nominal model $\hat{P}_3$ as follows. From Fig. 8.7.a,c we see that the performance bounds involved with $\hat{P}_3$ and $C_2$ (indexed 3,2) are not very tight. Consequently $H(P, C_3)$ and $H(\hat{P}_3, C_2)$ react differently if $C_2$ is replaced with $C_3$: the nominal performance is improved (‘o’), whereas the performance degradation is increased. As $\hat{P}_3$ is of poor quality, the controller $C_3$ differs much from the optimal controller for $P$ and $\alpha = \alpha_3$. As $\hat{P}_4$ resembles $P$, we can say that $\hat{P}_3$ is not a good description of $\hat{P}_4$ either. Consequently the optimal controller for $\hat{P}_4$ differs from the optimal controller for $\hat{P}_3$, which explains the difference between $C_3$ and $C_{4x}$. The larger the difference between $H(P, C_{i-1})$ and $H(\hat{P}_{i-1}, C_{i-1})$, the larger the difference between $\hat{P}_{i-1}$ and $\hat{P}_i$, and consequently the larger the minimum controller enhancement. The larger the minimum controller enhancement, the more accurate $\hat{P}_i$ must be in order to reliably predict the behavior of the plant $P$ under feedback by the enhanced compensator.

We review the replacement of $C_2$ with $C_3$ once more from Fig. 8.7.a,c. This replace-
Fig. 8.7: Analysis of the fourth step of the primary iteration.

a: $J_{\infty, \Omega}$, $\alpha = \alpha_3$.

b: Frequency responses of $C_3$ (---), $C_4$ (--), and $C_{4z}$ (-----).

c: $J_{2, \Omega}$, $\alpha = \alpha_3$.

d: Sensitivities of $P, C_2$ (---), $\hat{P}_3, C_2$ (--), $\hat{P}_4, C_3$ (-----), $PC_3$ (-----).

ment causes an increase of the local performance degradation, and thus $C_3$ enlarges the difference between the nominal and actual feedback systems. This has been visualized in Fig. 8.7d. In here we see the sensitivities $(I + C_2 P)^{-1}$ (---) and $(I + C_3 \hat{P}_3)^{-1}$ (--), which differ above 1.5 rad/s. The new sensitivity $(I + C_3 P)^{-1}$ (-----) is smaller than the old sensitivity $(I + C_2 P)^{-1}$ (---) in the low frequency range at the expense of an increased “push-pop” effect around 10 rad/s. This compensator $C_3$ makes the plant’s dynamics around 10 rad/s more dominant, so that the sensitivity $(I + C_3 \hat{P}_4)^{-1}$ (-----) of the new nominal model $\hat{P}_4$ has an excellent match near this frequency. The controller $C_3$ apparently provides a sufficient excitation for the identification of the high frequency dynamics of the plant $P$.

Meanwhile the local robust performances have been temporarily worsened with $C_3$. In Fig. 8.7a,c the local upper bounds for $H(\hat{P}_3, C_2)$ is better than upper bounds for $H(\hat{P}_3, C_3)$. We shortly investigate how this could have happened. An enhanced
controller $C_E$ improves the robust performance upon $C$ if
\[ ||T(\hat{P}, C)|| + ||T(P, C) - T(\hat{P}, C)|| \geq ||T(\hat{P}, C_E)|| + ||T(P, C_E) - T(\hat{P}, C_E)||. \]

This equation can straightforwardly be rewritten to
\[ \frac{\Pi_{\Delta}(C_E)}{\Pi_{\Delta}(C)} \leq \frac{||T(\hat{P}, C)||}{||T(P, C_E)||} \]

in which $\Pi_{\Delta}(C_E)$ and $\Pi_{\Delta}(C)$ are the ratios used in the condition (6.14) for cautious controller enhancement. The latter condition requires that the ratio is less than 1.3. The above inequality shows that the ratio of $\Pi_{\Delta}$'s must be less than the ratio of the available and enhanced nominal performances, in order that the robust performance is improved. These requirements are not fully compatible, although the ratio of nominal performances is usually larger than one. Due to this incompatibility an improvement of the local robust performance is not guaranteed for the cautious controller enhancement.

**Remark 8.2.1** It is tempting to suggest that the cautious controller enhancement should be based on the improvement of the local robust performance, i.e. on the upper bound on the plant's performance. However this is not sufficient due to the phenomenon of minimum controller enhancement. The latter requires that the performance degradation $||T(P, C_i) - T(\hat{P}_i, C_i)||$ is relatively small in respect of $||T(\hat{P}_i, C_i)||$. Otherwise $\hat{P}_{i+1}$ differs much from $\hat{P}_i$, which causes a possibly too large minimum controller enhancement. Thus an additional inequality constraint like
\[ ||T(P, C_i) - T(\hat{P}_i, C_i)|| \leq \gamma ||T(\hat{P}_i, C_i)|| \]
is needed. Unfortunately it is all but transparent which scalar $\gamma \in \mathbb{R}$ is appropriate. Therefore we base the cautious enhancement on the "disharmonies" of Section 6.2. □

### 8.3 Advanced Iterative Scheme

A drawback of the primary iterative scheme is that the identification of the nominal model $\hat{P}_i$ is focussed solely on the available controller $C_{i-1}$. There is no link with the future compensator $C_i$, except that $T(\hat{P}_i, C_{i-1})$ approximately describes the sensitivity to small perturbations of $C_{i-1}$. The iteration is likely to speed up, when the identification of $\hat{P}_i$ is based on all available information about the future $C_i$. Of course the compensator $C_i$ itself is not known yet, so that we can exploit only the design method and the objectives. For this purpose we use the identification technique of Section 7.4. This identification technique is custom-made for the control design method of Section 6.2 in the sense that these two procedures anticipate the same perturbations of the normalized coprime factors of the weighted nominal model.
8.3.1 Simulation Study

The identification method of Section 7.4 requires the frequency response estimate of a coprime factorization of the plant $P$. We first estimate a nominal model from the open-loop frequency response of $P$. This yields $\hat{P}_1$ of the primary iteration. Then $\hat{P}_1$ is used as the auxiliary model $P_0$ in the framework of Proposition 4.2.1. We substitute 0 for $C$ because $\hat{P}_1$ has been identified in open-loop. Then we determine the corresponding frequency response data $N_i^\alpha, D_i^\alpha$, $i=1,..,N$, from which we estimate the nominal model $\hat{P}_{21}$ by the method introduced in Section 7.4.

![Diagram](image)

Fig. 8.8: Results of the advanced iteration.

a: $\hat{D}Q_{21}^{-1}$, $D^\alpha Q_{\Delta,21}Q_{21}^{-1}$, $\alpha N_{21}^{-1}$, $\alpha N^\alpha Q_{\Delta,21}Q_{21}^{-1}$.
b: $\hat{D}$, $D^\alpha Q_{\Delta,21}$, $\alpha \hat{N}$.
c: see a.
d: $Q_{\Delta,21}$.

Before displaying the identification result we recall from (7.9) and (7.10), that the nominal model $\hat{P}_{21}$ should be such that the mismatch

$$\left[ \begin{array}{c} \alpha \hat{N}_{21} \\ \hat{D}_{21} \end{array} \right] - \left[ \begin{array}{c} \alpha N^\alpha \\ D^\alpha \end{array} \right] Q_{\Delta,21}$$

$$Q_{21}^{-1}$$
is small. That this actually has been accomplished is illustrated in Fig. 8.8. In Fig. 8.8.a and c we have plotted the Bode diagram of $\hat{D}_{21}Q_{21}^{-1}$ (—) and the data $D_{i}^{a}Q_{\Delta,21}(j\omega_{i})Q_{21}^{-1}(j\omega_{i})$ (—) as well as the frequency response of $\alpha\hat{N}_{21}Q_{21}^{-1}$ (—) and the data $\alpha N_{i}^{a}Q_{\Delta,21}(j\omega_{i})Q_{21}^{-1}(j\omega_{i})$ (—). The two denominator terms as well as the two numerator terms are very much alike. Hence the two contributions to the mismatch are small. Observe that the perceivable difference between the curves (—) and (—) is negligible on a linear scale.

Fig. 8.8.b shows the same data, except that $Q_{21}^{-1}$ has been omitted: $\hat{D}_{21}$ (—), $D_{i}^{a}Q_{\Delta,21}$ (—), $\alpha\hat{N}_{21}$ (—) and $\alpha N_{i}^{a}Q_{\Delta,21}$ (—). A comparison of the Figures 8.8.a and b reveals that $Q_{21}^{-1}$ emphasizes the denominator terms and de-emphasizes the numerator terms at the higher frequencies. Besides the stable term $Q_{\Delta,21}$ causes only a minor correction, which can be seen from the log-magnitude of its frequency response (Fig. 8.8.d).

$\hat{P}_{21}$ is used to design the compensator $C_{21}$ by the method of (6.8) for $\alpha = \alpha_{5}$, i.e. $\alpha = 20$. The compensator optimizes robustness in view of perturbations of the nrcf of $\alpha \hat{P}_{21}$. Thus the robustness anticipates dynamical perturbations of $\alpha \hat{N}_{21}Q_{21}^{-1}$ and $\hat{D}_{21}Q_{21}^{-1}$. In Fig. 8.8.a,c we see that the plant $P$ is only a very small dynamical perturbation of the nrcf ($\alpha \hat{N}_{21}Q_{21}^{-1}, \hat{D}_{21}Q_{21}^{-1}$). Accordingly the stability of $H(P,C_{21})$ can easily be demonstrated by the robustness margin of Proposition 6.3.7.

The nominal model $\hat{P}_{21}$ differs only a little from the final nominal model $\hat{P}_{5}$ of the primary iteration. A comparision of Table A.6 with Table A.10 and of Table A.7 with Table A.11 discloses that $\hat{P}_{21}$ and $\hat{P}_{5}$ have almost the same transfer function coefficients, poles and zeros. Correspondingly the compensator $C_{21}$ is very similar to the compensator $C_{5}$ of the primary iterative scheme (cf. Tables A.8, A.12, A.9 and A.13). As a consequence $C_{21}$ and $C_{5}$ achieve similar performances. For instance they cause similar sensitivities. In Fig. 8.9 we have drawn the sensitivities $(I+C_{5}P)^{-1}$ (—) and $(I+C_{21}P)^{-1}$ (—) on a logarithmic scale and a linear scale. The logarithmic plots are almost indiscernable and from the linear plots we can distinguish only a small difference between $(I+C_{5}P)^{-1}$ (—) and $(I+C_{21}P)^{-1}$ (—).

The controller $C_{21}$ is applied to the plant, and a new nominal model $\hat{P}_{22}$ is identified by means of the fixed-loop feedback-system identification of Section 7.3 just as in the primary iterative scheme. Subsequently a compensator $C_{22}$ is designed from $\hat{P}_{22}$ also for $\alpha = \alpha_{5}$. The pair $\hat{P}_{22},C_{22}$ is almost identical with the result $\hat{P}_{5},C_{5}$ of the primary iteration (cf. Tables A.6 through A.13). The linear and logarithmic plots of Fig. 8.9 show that the sensitivity $(I+C_{22}P)^{-1}$ (—) practically equals the sensitivity $(I+C_{5}P)^{-1}$ (—). In conclusion the advanced iterative scheme achieves the same high performance for the plant in two steps, whereas the primary iterative scheme needed 5 steps.

### 8.3.2 The Mechanism of the Design-oriented Identification

In the identification of $\hat{P}_{21}$ the nominal model's deficiency is made as small as possible in respect of the robustness margins of the feedback system $H(\hat{P}_{21}, C_{21})$. There is no
Fig. 8.9: Sensitivities achieved for the plant $P$ on linear and logarithmic scales (the vertical axis corresponds to the linear scale). $(I+C_5P)^{-1} (-)$, $(I+C_{22}P)^{-1} (\cdots)$, $(I+C_{21}P)^{-1} (\cdots)$.

Prior guarantee that the identification will yield a mismatch that is small enough for $C_{21}$ to robustly stabilize the plant $P$. For at that stage $C_{21}$ is not known yet. For the same reason it is also not guaranteed a priori, that the nominal feedback system $H(\hat{P}_{21}, C_{21})$ approximately describes the actual feedback system $H(P, C_{21})$. After the control design it turns out that the pair $\hat{P}_{21}, C_{21}$ is a good approximation in both senses.

The weight $\alpha$ is used in both stages of identification and control design. From Section 6.2 we know that $\alpha$ represents a trade-off between nominal performance and robust stability. Hence the identification of (7.9) and the control design of (6.8) constitute a joint solution to the problem of approximate identification and control design in the sense that both processes optimize the same robustness in view of the same nominal performance.

We end up this section with a few short comments. The use of $\alpha = \alpha_5$ is of course based on the result of the primary iterative scheme. On the other hand, when we really replace the primary iteration with the advanced iteration, then we do not know which design weight leads to an acceptable performance. For instance we could take the plant’s frequency response data $P_i$ and determine $\alpha$ so that $|\alpha P_i|$ is about 0dB near the desired bandwidth (see Section 6.2 for details). However at the outset we do not
know whether this desired bandwidth is robustly attainable for the plant $P$, because we do not know whether $P$ has unstable zeros. Nor do we know how complex the nominal model and/or the compensator should be. Furthermore, it can be impossible to identify the feedback-relevant dynamics when the open-loop data are contaminated with noise. Based on these observations we suggest that a combination of the primary and advanced iteration appears to be needed in general.

8.4 Discussion and Suggestions

In this section we discuss the results of The Illustrative Example in a more general context. First we address the required compatibility of the individual identification and control design stages. Then we recapitulate the iterative learning of the feedback-relevant properties of the plant. In the end we place our iterative scheme in the context of adaptive control.

8.4.1 The Compatibility of Identification and Control Design

The evolvement of an iteration of repeated identification and control design can be explained from the inequalities of (8.1), (8.2) and (8.3). The task of the $i$-th identification is to produce a nominal model $\hat{P}_i$ so that the new performance degradation is smaller than the old performance degradation, i.e.

$$\|T(P, C_{i-1}) - T(\hat{P}_i, C_{i-1})\| \leq \|T(P, C_{i-1}) - T(\hat{P}_{i-1}, C_{i-1})\|.$$ 

This inequality is guaranteed for the $\alpha$-weighted norm $\|\cdot\|_{2,\Omega}$, but not for the $\|\cdot\|_{\infty,\Omega}$-norm. The next compensator has to improve the robust (worst-case) performance, which implies that the inequality i.e.

$$\|T(\hat{P}_i, C_i)\| + \|T(P, C_i) - T(\hat{P}_i, C_i)\| \leq \|T(\hat{P}_{i-1}, C_{i-1})\| + \|T(P, C_{i-1}) - T(\hat{P}_{i-1}, C_{i-1})\|$$ 

must be satisfied. We have not explicitly used this inequality in the cautious controller enhancement (recall Remark 8.2.1), but we examine this inequality now the iteration has been completed. Reverting to Figures 8.5 and 8.6 we conclude that these requirements have been accomplished at each stage of the iteration for each of the two seminorms $\|\cdot\|_{2,\Omega}$ and $\|\cdot\|_{\infty,\Omega}$. Moreover the courses of these two seminorms are quite similar. On the other hand the applied identification technique is based on the seminorm $\|\cdot\|_{2,\Omega}$ and the cautious controller enhancement rests on on the semi-norm $\|\cdot\|_{\infty,\Omega}$. Thus the identification does not optimize the control design objective, and conversely the control design does not precisely account for the identification objective. The question is, why do these identification and control design stages co-operate so well despite the incompatibility of their individual cost functions?

The identification and control design have been made as compatible as possible\(^2\). Although the norms are different, at least the mismatches are compatible, since the

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\(^2\)Similar utterances are found in [21, 22].
processes are linked together through the mismatch
\[ T(P, C_{i-1}) - T(\hat{P}_i, C_{i-1}). \]

This mismatch is minimized in the identification in an \( \| \cdot \|_2, \Omega \)-sense, and the result dictates the subsequent cautious controller enhancement. The minimization of a 2-norm does of course not lead to the minimum achievable \( H_\infty \)-norm\(^3\). However we are satisfied if the mismatch \( T(P, C) - T(\hat{P}, C) \) has a small \( H_\infty \)-norm. We can use the 2-norm minimization to accomplish a small \( H_\infty \)-norm of the mismatch, since \( L_2 \) estimators are \( L_\infty \) consistent, provided that some smoothness conditions are satisfied [33, 147]. Indeed the \( L_2 \) optimal nominal models of The Illustrative Example turned out to be fit for use in control design.

It remains very difficult, if possible at all, to guarantee a successful \( \| \cdot \|_\infty, \Omega \)-controller enhancement from an upper bound on the cost function \( \| T(P, C) - T(\hat{P}, C) \|_2, \Omega \) used in the identification. Instead it is much easier to monitor all cost functions for each new nominal model and for each new compensator. In the Illustrative Example we have used visual inspections of the Bode plots of the feedback matrices and of the achieved robustness margins. We suggest that these inspections can be replaced with decision rules or verifiable criteria like the following two.

- After the \( i \)-th identification it is verified whether
  \[ \| T(P, C_{i-1}) - T(\hat{P}_i, C_{i-1}) \|_{\infty, \Omega} \leq \| T(P, C_{i-1}) - T(\hat{P}_{i-1}, C_{i-1}) \|_{\infty, \Omega}. \]

  Otherwise \( \hat{P}_i \) is rejected, and an increase of the nominal model order is necessary. In the light of Assumption 7.2.2 these norms must be evaluated over two or more independent data-sets.

- After the \( i \)-th control design it is verified whether the inequality
  \[ \| T(\hat{P}_i, C_i) \| + \| T(P, C_i) - T(\hat{P}_i, C_i) \| \]
  \[ \leq \| T(\hat{P}_{i-1}, C_{i-1}) \| + \| T(P, C_{i-1}) - T(\hat{P}_{i-1}, C_{i-1}) \| \]

  holds for \( \| \cdot \|_2, \Omega \) as well as for \( \| \cdot \|_{\infty, \Omega} \). Otherwise \( C_i \) is rejected and an increase of the compensator order is necessary; or \( \hat{P}_i \) is rejected and an increase of the order of the nominal model is necessary; or we have to conclude that under these circumstances a better performance for the plant is not attainable.

This proposed evaluation of each step of the iteration in terms of both the semi-norms \( \| \cdot \|_2, \Omega \) and \( \| \cdot \|_{\infty, \Omega} \) is rather uncommon. However in our opinion such a procedure could be very useful in view of the “true” high-performance control design objectives. In order to appreciate this standpoint, the reader should realize that an \( H_\infty \)-norm

\(^3\)It is quite a question whether a global minimum can be achieved at all when a cost function has local minima.
or an $H_2$-norm is only an aggregated quantity that (hopefully) represents various objectives. Besides, the need of an combined $H_2/H_\infty$ control design method has been motivated along the same lines [20, 224].

Finally we mention that the above scenario did not employ an identification method that minimizes the $H_{\infty}$-norm of the nominal model’s deficiency. So apparently we do not need an $H_{\infty}$-identification method to find suitable nominal models for $H_{\infty}$-control design. This implies that we do not need an $H_{\infty}$-identification method to solve the right branch of Fig. 1.1. However from the nominal model alone we cannot conclude stability for the plant. For this purpose we need techniques like $H_{\infty}$-identification, i.e. techniques that solve the left branch of Fig. 1.1.

### 8.4.2 Iterative Learning

During the iteration the plant is controlled by a sequence of compensators, that gradually improve the performance. At each step of the identification the nominal model is used to accomplish this improvement of the performance. Thus the nominal model is used to predict the behavior of the plant under feedback by the next compensator. This prediction is not perfect, i.e. the response of the plant to the implementation of the new compensator differs slightly from that of the nominal model. This difference is corrected in the next identification stage.

The actual feedback system differs from the nominal feedback system due to the disparity between the plant and its nominal model. This mismatch between the two feedback systems provides additional information about the plant. If this mismatch becomes large, then the plant dynamics are apparently more complex, than that we had expected at the outset (see Fig. 8.7.d). We learn about the feedback-relevant complexity of the plant and the required complexity of the compensator.

Another very important property of the plant that we learn during the iteration is the achievable performance. We learn also which design weight trades off the feedback properties in accordance with the achievable performance. For a good appreciation of the latter aspect, we consider the design objective of minimizing

$$E\{y^2 + \lambda u^2\},$$

in which $E\{\cdot\}$ is the expectation operator. Now suppose that

- it is more difficult to control the plant, than that we had expected at the outset.
  Then we need more control effort to accomplish a certain quality at the output.
  Accordingly the design weight $\lambda$ has to be decreased.

- or that it is easier to control the plant, than that we had expected at the outset.
  Then we are willing to make the controller more cost effective by increasing $\lambda$.

From this we infer that it is not truly possible to specify the global cost function a priori. Therefore we use local design criteria, which focus on the current operation of the
plant. The objective is to improve this current operation measured by local criteria. While the iteration proceeds, the sequence of local cost functions converges to a global cost function that is custom-made for the plant. Furthermore, the first controllers are not designed to achieve high performance, so that the identification must be based on the local cost functions as well. Otherwise the identified nominal model would describe the plant in regard of some high-performance trade-off, that is not yet pursued in the subsequent control design.

In conclusion we address the accuracy of the nominal model that is required for a successful controller enhancement. The identified nominal feedback system \( H(\hat{P}_i, C_{i-1}) \) and the actual feedback system \( H(P, C_{i-1}) \) have approximately the same cost functions, i.e.

\[
J(P, C_{i-1}) \approx J(\hat{P}_i, C_{i-1}).
\]

The controller enhancement starts with a small perturbation of the compensator \( C_{i-1} \). The two feedback systems will improve similarly if they have similar sensitivities to compensator perturbations. In other words, the performances of the feedback systems \( H(P, C_{i-1}) \) and \( H(\hat{P}_i, C_{i-1}) \) must have the same gradients with respect to the compensator \( C_{i-1} \). With some sloppy notation this can be expressed as

\[
\frac{\partial J(P, C_{i-1})}{\partial C_{i-1}} \approx \frac{\partial J(\hat{P}_i, C_{i-1})}{\partial C_{i-1}}.
\]

In our iterative scheme we learn about these gradients after the design and/or the application of the compensator \( C_i \). In this perspective it is not possible to guarantee prior to the control design that some nominal model \( \hat{P}_i \) is sufficiently accurate for performance enhancement. On the other hand \( H(P, C_{i-1}) \) and \( H(\hat{P}_i, C_{i-1}) \) have similar sensitivities to small perturbations of \( C_{i-1} \), because the mismatch

\[
T(P, C_{i-1}) - T(\hat{P}_i, C_{i-1})
\]

has been made small in the sense of the cost function \( J \). This is not equivalent to an approximation of the gradient, but it provides a good approximation of the gradient. The direct approximation of the gradient remains an open question, for which probing techniques seem to be an appropriate answer [115].

8.4.3 Quasi-adaptive Control

In conclusion of this chapter we draw a parallel between our iterative schemes and adaptive control. Let us first question, what is an adaptive controller? We quote Zames and Wang.

"A nonadaptive controller is one designed on the basis of a priori information, i.e., which is available at the outset. An adaptive controller makes
use of a posteriori information to achieve better performance than could be obtained with a nonadaptive one." [263].

In this perspective an iteration of repeated identification and control design appears to be an adaptive controller. The iterative high-performance control design procedure is at least not a nonadaptive controller in the above sense. Moreover, according to Goodwin [86] the central feature of an adaptive controller is the potential to perform an on-line replacement of an existing controller by a better one. He immediately adds a few conditions: the controller is to be replaced only if

1. the current controller should indeed be improved,

2. sufficient information is available for the re-design, and

3. the new controller will actually yield an improved performance.

In view of our iterative schemes point 1. is obvious, point 2. pertains to the quality of the nominal model, which cannot be dissociated from point 3. A new controller may turn out not to improve the performance at all. Then we can decide to reject this controller and to put the old compensator into place again. Meanwhile we have gained valuable information about the gradient discussed earlier.

The above points raised by Goodwin [86] are rather untraditional in the field of adaptive control. The adaptive control paradigm is commonly taken to be some continuous or recursive process, that converts data to a time-varying controller. Proving that the time-varying compensator converges to some limiting solution is an unremitting challenge. A limiting solution may even not exist, and thus convergence cannot be proven under all circumstances⁴.

An adaptive control scheme usually hinges on a vector of parameters. These parameters induce the set of candidate controllers or nominal models in a direct respectively indirect adaptive control scheme [195, 170, 21, 112]. Proving convergence of an adaptive scheme boils down to proving convergence of the parameter vector to (a neighborhood of) its optimal value. From an identification standpoint, convergence of the parameter vector requires a persistent excitation. This is basically a condition on some exogenous input to the feedback system that contains the plant [195, p.156]. However, even when the model set encompasses the plant and there are no disturbances, then it is still hard to prove convergence (see [170] and many references therein).

The control design rule of an indirect adaptive control scheme must be some continuous function of the nominal model parameters in order that convergence can be proven. For instance the identifier used by Bitmead et al. [21] requires local smoothness of the controller behavior: when a parameter varies slowly, then the compensator must vary slowly. Iglesias [112] presumes that a continuous variation of the parameter vector

⁴When convergence can be guaranteed, then a substantial safety network is often still needed to guarantee a proper operation [170].
maps into a continuous variation of the compensator in the graph topology. For many control design rules it is hard to verify any such continuity property. Nevertheless under these conditions convergence to a limiting solution can be guaranteed, provided that such a solution exists, and provided that the initial value of the parameter vector lies in a sufficiently small neighborhood around the limiting solution.

The latter prerequisite concerning the initial parameter vector implies that the adaptive controller can improve the performance, provided that it is already good. This relates to the advanced iterative scheme of Section 8.3. This scheme aims directly at a good performance by means of the design-oriented identification of Section 7.4 (recall Fig. 2.12). The first nominal model $\hat{P}_{21}$ is close to the "limiting" nominal model $\hat{P}_{22}$ (cf. the transfer function coefficients in Table A.10). The resulting compensator $C_{21}$ is near to the optimal plant-based compensator $C_P$ (cf. Table A.12 and Table A.8). Thus there is a kinship between, on the one hand, the usual problem of proving convergence of an adaptive control scheme and, on the other hand, the problem of proving convergence of our iterative scheme for some fixed design weight $\alpha$, i.e. for some fixed trade-off between the feedback properties.

In The Illustrative Example the primary iterative scheme evolves from open-loop operation to a high performance control system. The trade-off between the feedback properties evolves during the iteration. Neither the nominal model parameters nor the controller parameters evolve continuously. A convergence from low to high performance can most probably not be proven in terms of parameter vectors. This holds especially if the order of the nominal model or the compensator would have to be changed at some stage of the iteration.

On the other hand the frequency responses of the designed compensators and of the achieved sensitivities change more or less gradually (see Figures 8.1.b, 8.3 and 8.7.b,d). We do not require a continuous change of the frequency responses, but we impose conditions on the discrete change of frequency responses. A small difference between two frequency responses may involve a large difference between poles and zeros or transfer function coefficients [109, 110].

Conclusively it appears to be very difficult to prove convergence of the iterative high-performance control design procedure even under favourable circumstances. Instead it will probably be much easier to monitor the achieved performances, and to take action when some performance is not improved. Such a monitoring function could be extended by conditions like those introduced by Ma and Vidyasagar [142], who proved that BIBO stability of a time-varying feedback system is guaranteed, provided that the pair of nominal model and compensator vary only within a sufficiently small neighborhood.
Chapter 9

Practical Experiments with the Iterative Schemes

In this chapter we apply the iterative schemes of Chapter 8 to a practical experimental set-up. The results amplify those of the previous chapter, which attests to the practical utility of the developed theory.

9.1 The Experimental Set-up

9.1.1 The XY-table

The experimental set-up is the so-called XY-table shown in Fig. 9.1. The table is composed of a stone plate and a steel support, which is mounted to the floor. The table weighs over 420 kg. A steel block of 50 kg. lies on the table. Pressured air is fed through small holes in the block, which makes the mass float on a film of air. This air-bearing practically eliminates the friction between the block and the stone plate. Consequently small disturbances make the block drift off the table.

The configuration is depicted in Fig. 9.2, which provides a top view of the XY-table. The block, denoted $B$, is attached to the table by three hydraulic servo drives. The x- and y-axes correspond to the horizontal plane. The servo drives $S_i, i=1,\ldots,3$ are connected to the block $B$ and to the table by means of ball bearings. The length of the arms $\tau_i, i=1,\ldots,3$ of the servo drives can be controlled through changes in the oil-supply. Hereby we can manipulate the three degrees of freedom of $B$.

We want to carry out a SISO experiment, so we eliminate two degrees of freedom. The block $B$ is adjusted such that servo drive $S_1$ is positioned parallel with the x-axis. In this position the arms $\tau_2$ and $\tau_3$ are “frozen” by a local feedback loop over the servo drives $S_2$ and $S_3$. Small variations of the arm $\tau_1$ will make the block $B$ move practically only in the x-direction. The setpoint of the valve of the servo drive $S_1$ is taken as the input of the experimental set-up. The output is the arm $\tau_1$ and the control problem is to position the block $B$ relatively to the servo drive $S_1$. 
Fig. 9.1: Experimental XY-table.
9.1 The Experimental Set-up

9.1.2 Implementation of the Compensators

The designed controllers are implemented on a fixed-point digital signal processor. In order to minimize round-off errors the discretized compensators are balanced prior to the implementation. A typical example of the implementation of the compensators is illustrated in Fig. 9.3. The frequency response of the implemented compensator has been determined by means of sine-wave experiments and a frequency analyzer. The figure shows that the achieved frequency response (---) is similar to the calculated frequency response of the designed continuous-time compensator (—). The magnitude of the difference between the compensator and its implemented version is smaller than 10% over all frequencies.

9.1.3 Stabilization of the Experimental Set-up

The servo drive $S_1$ acts as an integrator, since a constant oil-supply causes a constant increase of the arm $r_1$. Hence the experimental set-up is open-loop unstable. Despite this instability the frequency response of the plant $P$ has been determined by an open-loop sine-wave experiment. The measured frequency response data consist of 400 samples uniformly distributed on a logarithmic scale over the range from 1 to 20 Hz.

We estimated the nominal model $\hat{P}_{30}$ of order 3 by the least-squares frequency-domain method of Section 7.2. The Bode-plots of the plant $P$ (measured) and the nominal model $\hat{P}_{30}$ are depicted in Fig. 9.4,a and c.

The quality of $\hat{P}_{30}$ is clearly doubtful around 100 rad/s. We could have tried to weight the data in order to improve the nominal model at the higher frequencies. Instead we use $\hat{P}_{30}$ to design a stabilizing compensator. Much robustness is required at
the higher frequencies, so that the robust performance is rather poor. An improvement of the nominal model for the purpose of high-performance control design is pursued by means of the iterations.

The controller $C_{30}$ of order 2 is designed from $\hat{P}_{30}$ by the robust control design method of Chapter 6 with $\alpha_{30} = 4$ (cf. (6.8)). The magnitude of $C_{30}$ is shown in Fig. 9.4.b. Theoretically spoken there is no possibility to guarantee the stability of $H(P, C_{30})$ from that of $H(\hat{P}_{30}, C_{30})$. For we know only that $P$ is unstable, but we cannot be 100% sure about its number of unstable poles. Nevertheless we draw a great confidence from the Bode-plots of $P$ and $\hat{P}_{30}$ (Fig. 9.4.a and c): we assume that $C_{30}$ will stabilize the plant and we implement it on the dsp. The achieved feedback matrices $T(P, C_{30})$ and $T(\hat{P}_{30}, C_{30})$ are indeed very similar. The sensitivities and complementary sensitivities thereof have been represented in Fig. 9.4.d. Now the plant operates under a stabilizing compensator and the set-up is ready for experimentation.

9.1.4 Frequency Response Measurements of Coprime Factors

The framework of Proposition 4.2.1 enables the estimation of the frequency responses of the associated coprime factors $N^a$ and $D^a$ of the plant $P$. The frequency response samples $N^a$, $D^a$ can be obtained from samples of $u$ and $y$ after reconstruction of the intermediate $x$. As explained in Proposition 4.2.2 we can let the intermediate $x$ take the value $x_d$ by choosing $r_1 = D_o x_d$ and $r_2 = N_o x_d$. Together with the designed compensator we also implement the filters $N_o$ and $D_o$ on the digital signal processor. The
9.1 The Experimental Set-up

Fig. 9.4: Stabilization of the experimental set-up.

a: Bode-plots of $P$ (---) and $\hat{P}_{30}$ (---).

b: Bode-plot of $C_{30}$.

c: see a.

d: $PC_{30}(I+PC_{30})^{-1}$ (---), $\hat{P}_{30}C_{30}(I+\hat{P}_{30}C_{30})^{-1}$ (---), $(I+PC_{30})^{-1}$ (---), $(I+\hat{P}_{30}C_{30})^{-1}$ (---).

input to these filters is $x_d$, which we may use instead of $x$ for the identification of $(N^a, D^a)$. Noise contributions are neglectable. Now $u, y$ and $x_d$ are on-line available, and we can determine the frequency response data $N_i^a, D_i^a$ by means of sine-wave experiments and a frequency analyzer.

Before estimating a nominal model we can check the quality of the frequency
response data $N_i^a, D_i^a$. According to (3.7) the equation

$$D^a + CN^a = D_o + CN_o$$

holds by definition. By an examination of the differences

$$(D_i^a + C(j\omega_i)N_i^a) - (D_o(j\omega_i) + C(j\omega_i)N_o(j\omega_i))$$
and

$$(D_i^q + C(j\omega_i)N_i^q)^{-1} - (D_o(j\omega_i) + C(j\omega_i)N_o(j\omega_i))^{-1}$$

we may detect bad samples in the data. These differences were usually negligibly small.

![Bode plots](image)

**Fig. 9.5:** Results of the primary iteration.

a: Bode-plots of $N_i^q/D_i^q$ associated to $(\hat{N}_{n,i}, \hat{D}_{n,i})$ and $C_i$ for $i=0$ (---), 2 (--), 5 (---) and 6 (-----).

b: Bode-plots of $\hat{P}_{31}$ (---), $\hat{P}_{33}$ (-----), $\hat{P}_{36}$ (---) and $\hat{P}_{37}$ (-----).

c: see a.

d: see b.

### 9.2 Primary Iterative Scheme

We apply exactly the same procedures as in Section 8.2 with only two exceptions. Firstly, we do not emphasize undermodelling aspects in the real experiments, because the physical system cannot be exactly represented by a LTIFD nominal model anyway. Secondly we wish to be extra careful for safety reasons. For, unlike in the simulation
study of the previous chapter, we are dealing with imperfect frequency response samples and an imperfect implementation of the designed compensators. Although the frequency domain identification techniques (Chapter 7) and the framework for open-loop identification (Appendix E.2) are robust in the face of these small imperfections, additional cautiousness is desired, because no quantization of the combined robustness is available yet. To that end we substitute 0.95 and 1.05 for 0.7 and 1.3 in (6.13) and (6.14).

Fig. 9.6: Sensitivities achieved for the plant $P$ with the compensators $C_{3j} \ (j$ is indicated in the plot).

The iteration starts with $\hat{P}_{30}$, $C_{30}$ and $\alpha_{30} = 4$ and it produces the nominal models $\hat{P}_{31}, \ldots, \hat{P}_{37}$ of order 3 and the compensators $C_{31}, \ldots, C_{37}$ of order 2. The design weights $\alpha_{31}, \ldots, \alpha_{37}$ are respectively 5.5, 8.3, 12.5, 20, 30.5, 50 and 70.

While the iteration evolves the plant operates under changing conditions. Using the pair $\hat{P}_k, C_k$ we determine the frequency response data $N_k^i, D_k^i$ of the plant’s rcf $(N^k, D^k)$ that is associated with $C_k$ and the normalized rcf of $\hat{P}_k$. In Fig. 9.5.a and c we have drawn the ratios $N_k^i/D_k^i$, $k = 0, 2, 5, 6$, which give an indication of the frequency response of the plant $P$. The frequency response data of the various experiments are very similar. Hence the plant $P$ is (almost) linear in regard of the various operational conditions.

The data $N_k^i, D_k^i$ and the compensator $C_k$ are used in the identification of $\hat{P}_{k+1}$. The frequency responses of the nominal models $\hat{P}_1, \hat{P}_3, \hat{P}_6, \hat{P}_7$ have been depicted in
Fig. 9.5.b,d. A comparison of the phase plots c and d reveals that the small high-
frequence deficiency of $\hat{P}_1$ is gradually transformed into the equally small low-frequency
deficiency of $\hat{P}_7$. This is in agreement with the fact that the mismatch must be small
near the gradually increasing cross-over frequency. The cross-over frequencies of the
loop transfer functions $\hat{P}_{31}C_{31}$, $\hat{P}_{33}C_{33}$ and $\hat{P}_{36}C_{36}$ are respectively 1.2, 1.75 and 1.94
log(rad/s).

Finally the improved performance has been illustrated in Fig. 9.6, which shows
the sensitivities that are achieved for the plant $P$. In the lower frequency range the
sensitivity is reduced from 100% to 10%. Meanwhile the “push-pop” effect is held
under control: the sensitivity is not unacceptably large at the higher frequencies.

9.3 Advanced Iterative Scheme

Similar to the previous section we apply the same procedures as in Section 8.3. We
identify the nominal model $\hat{P}_{41}$ of order 3 with the method of Section 7.4. We use
$\alpha=50$ and the frequency response data $N_i^0,D_i^0$ that is associated with $C_{36}$ and the nrcf
of $\hat{P}_{30}$. The value of $\alpha$ corresponds with $\hat{P}_{36}$ of the primary iteration, and not with $\hat{P}_{37}$.
We use 50 instead of 70, because the latter design weight involves an ill-conditioning
of the digital signal processor.

The nominal model $\hat{P}_{41}$ is compared with $\hat{P}_{36}$ and $P$ in Fig. 9.7.a and c. There
is a clear mismatch between $\hat{P}_{41}$ on the one hand and $\hat{P}_{36}$ and $P$ on the other hand.
This mismatch is relatively large at those frequencies where the magnitudes of all
three frequency responses are large. Nevertheless $\hat{P}_{41}$ is suitable for the design of a
high-performance compensator $C_{41}$ for the plant $P$. Fig. 9.7.d shows that $C_{41}$ and
$C_{36}$ achieve similar sensitivities for the plant $P$. The relative match (log-scale!) is
good especially at the higher frequencies. This corresponds to the similarity between
the controllers $C_{41}$ and $C_{36}$ shown in Fig. 9.7.b. It is noteworthy that the similarities
between $\hat{P}_{41}$ and $\hat{P}_{36}$, between $C_{41}$ and $C_{36}$, and between $(I+C_{41}P)^{-1}$ and $(I+C_{36}P)^{-1}$
are most apparent near the cross-over frequency of the achieved feedback system.
Fig. 9.7: Results of the advanced iteration.

a: Bode-plots of $\hat{P}_{41}$ (---), $\hat{P}_{36}$ (--) and $N_0^2/D_0^2$ (...).

c: see a.

b: Bode-plots of $C_{41}$ (---) and $C_{36}$ (---).

d: $(I+C_{41}P)^{-1}$ (---) and $(I+C_{36}P)^{-1}$ (---).
Part III

Epilogue
Chapter 10

Conclusions and Perspectives

In this chapter we review our iterative scheme of repeated identification and control design in the light of the high performance control design problem posed at the outset.

10.1 Contributions of the Thesis

We have developed a procedure that combines approximate identification and model-based control design to achieve a high performance control system. We demonstrated that the approximate identification and the control design have to be treated as a joint problem. Only then it is possible to search systematically for an appropriate nominal model and a ditto compensator. This fact is supported by the following two observations.

- An open-loop judgement of the quality of a nominal model can be very misleading if the nominal model is to be used for high performance control design.

- The input-output maps of a control system are non-linear functions of the input-output maps of the plant. Therefore an open-loop weighting function cannot account for some desired closed-loop operation of a plant, unless the corresponding feedback system is known in much detail.

The joint problem has been tackled by an iterative scheme of repeated identification and control design. The identification is used to derive a nominal model, that reliably predicts the plant’s response to perturbations of the controller. In the subsequent control design the latter perturbation is directed in such a way that the performance is improved.

Coprime factorizations play a prominent part in our iterative scheme. First of all they are profitable in the individual identification and control design stages of the primary iteration. For, on the one hand, the use of fractional representations brings about a good numerical conditioning and a suited initial estimate for the problem
of approximately identifying a feedback system. On the other hand, in the control
design the plant is represented as a perturbation of the coprime factors of the available
nominal model. This perturbation is used to guide the controller enhancement and
to ascertain the stability of the new control system prior to the application of the
enhanced controller.

The utility of the coprime factor representations follows a forteriori from the ad-
vanced iteration. The latter commences with the design objective of a high nominal
performance and robust stability. Coprime factorizations have been the key to a joint
solution in the sense, that the developed methods for identification and control design
both optimize robustness in regard of the same high nominal performance.

At this stage it is pertinent to summarize some aspects that greatly facilitate the
iterative high performance control design procedure. The identification and control
design have been made as compatible as possible. Especially the link between identi-
fication and control established in the NICE Proposition 2.2.6 has been of paramount
importance: it has been the starting point for the development of identification and
control design procedures that are mutually adapted, so that the control design em-
loys a model that the identification can provide. More specific, in the identification
stage the candidate nominal model is judged from its capacity to describe the perfor-
mane of the plant under feedback control, and the difference between the "real" and
"model" feedback systems is kept small in the subsequent control design stage.

Further, the stability of each new control system is demonstrated prior to the
implementation of the new controller. To this end the "model-error" is accounted for by
a ball of unstructured dynamical perturbations. The class of dynamical perturbations
considered is based on the compensator. Hence the stability condition rests on a
robustness margin and on a dynamical perturbation, that are both custom-made for
the controller at hand. This condition is less conservative than e.g. the usual stability
conditions for additive or multiplicative dynamical perturbations, because the latter
perturbations are not custom-made for the designed controller.

Another important aspect is, that the parameters of the nominal model and of
the compensator are allowed to evolve discontinuously. Hence the identification and
control design are allowed to cause small changes in frequency responses and/or cost
functions through large parameter variations.

The iteration discloses several important properties of the plant. We learn about the
plant's dynamics that are relevant for feedback control. We learn about the required
complexity of the nominal model and that of the compensator. Also, we discover what
performance is attainable for the plant at all. And on top of that, we find out what
weighting function is needed to accomplish this particular attainable performance. All
these properties are unknown or uncertain at the outset.
Finally we have touched on the issue of convergence. If possible at all, it will be very difficult to derive verifiable conditions that a priori guarantee a convergence of an automated iterative design procedure. It appears to be much easier to enforce this convergence by monitoring the iteration and by taking action if needed.

10.2 Open Research Areas

There are numerous evident issues that deserve further investigation. We mention just a few of them. As for the identification stage of the developed iterative scheme one can think of an extension of the algorithms of Chapter 7 to MIMO systems\textsuperscript{1}, or of a numerical optimization and “robustification” of the algorithms. As frequency response estimation is often impossible, similar investigations are needed in term of time-domain identification techniques. Such investigations can be based on the same framework for open-loop identification. As for the control design stage, it is desirable to use more advanced loop shaping or even completely different control objectives and different design methods.

Then there are several related matters that we hardly touched on. For instance, we did not mathematically support the issue of how many frequency response samples are needed for the identification of a nominal model fit for use in control design. Nor did we pay attention to experiment design, except that we repeatedly applied new compensators. Another interesting topic is the separation of the residual data, that are unexplained by the nominal model, into contributions caused by disturbances and “unmodelled dynamics”.

At the current stage of affairs these contributions cannot be separated without making assumptions on either of them. An unexplored possibility is to estimate the “unmodelled dynamics” of the plant using the knowledge that this plant is stabilized by several known controllers. Such an estimation method could be applied to the sequence of controllers designed in our iterative scheme.

Finally, we raise a point that might broaden the applicability of iterative high performance control design schemes like ours. We suggested to reject a nominal model if its compensator turns out not to improve the performance. This would make a sufficient condition to guarantee that the iteration does not diverge. Nevertheless we would like to have a prior guarantee that some improvement of the performance for the nominal model results in an improvement of the performance for the plant. For this purpose we have to know how the plant will respond to perturbations of the controller. Hence it might be profitable to use small controller perturbations as an additional source of excitation in the identification experiment. This would make the iterative scheme even more suggestive of an adaptive controller.

\textsuperscript{1}All developments preceding Chapter 7 concern MIMO systems.
10.3 Retrospection — What have we gained?

In conclusion of this final chapter we put our results in a "historical" perspective of the established system theory.

We have combined the separate problems of identification and control design into one joint problem. This joint problem requires a joint solution, i.e. an optimization over models and compensators simultaneously. However standard methods for identification or control design are not up to this task: they can either optimize models or optimize compensators.

The need of a joint approach has been observed before, and a few solutions have been proposed in literature [11, 114]. These approaches rest on the assumption that there exist parameters that exactly represent the plant under investigation. Further the utilized controlled is conceived in terms of the model parameters, which enables a simultaneous optimization.

Instead, we have reunited the individual procedures of identification and control design by the NICE Proposition 2.2.6, which evaluates the couple of model and compensator in terms of performance bounds. This particular relationship has been the key to the development of our iterative procedure, in which identification and control design alternate in search of an effectual compensator. This procedure is similar in spirit to the classical tuning rules that have been introduces decades ago by e.g. Ziegler and Nichols [268]:

Several compensators are applied to some plant in order to obtain relevant information, which enables the selection of a compensator suited to control this plant.

Having found a way to achieve a high performance compensator for a plant with uncertain dynamics, the intriguing question is whether there exist a shorter way.
Part IV

Addenda
Appendix A

The Illustrative Example

The Illustrative Example was introduced in Section 2.4.1 to elucidate various ideas, concepts and solutions concerning the joint problem of approximate identification and control design. Detailed information about this example is provided here. The plant $P$ is a SISO system of order 8, and its transfer function coefficients as well as its poles and zeros are listed in Table A.1. The nominal models are of order 5 and the model-based compensators have order 2 or 4. The transfer function coefficients, the poles and the zeros of all nominal models and the compensators can be found in this appendix. All compensators have been designed according to (6.8). The design weight $\alpha$ used in the construction of each controller has been added to the tables of the transfer function coefficients. Lastly, the low-pass filter that has been used to generate the coloured random signal $r_n$ from normally distributed white noise, is an 8-th order SISO system with transfer function $(0.05s + 1)^{-8}$. 
<table>
<thead>
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<th>denominator</th>
<th>poles</th>
<th>zeros</th>
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<tr>
<td>( s^0 )</td>
<td>0</td>
<td>(-3.000 \pm 9.539j)</td>
<td>(-0.3485 \pm 0.3334j)</td>
</tr>
<tr>
<td>( s^7 )</td>
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<td>(-0.0100 \pm 1.000j)</td>
<td>(-0.2589 \pm 1.389j)</td>
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<tr>
<td>( s^6 )</td>
<td>30.000</td>
<td>(-0.0015 \pm 0.3141j)</td>
<td>0</td>
</tr>
<tr>
<td>( s^5 )</td>
<td>3020.0</td>
<td>(-10.00 \pm 5.28 \cdot 10^{-13}j)</td>
<td>0</td>
</tr>
<tr>
<td>( s^4 )</td>
<td>30538</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s^3 )</td>
<td>40373</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s^2 )</td>
<td>74041</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s^1 )</td>
<td>41972</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s^0 )</td>
<td>12467</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table A.1: Specification of plant \( P \) of The Illustrative Example.

<table>
<thead>
<tr>
<th>( s^5 )</th>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{P}_1 ) ( n(s) )</td>
<td>(-0.40036)</td>
<td>12.172</td>
<td>13.459</td>
<td>28.933</td>
<td>15.794</td>
</tr>
<tr>
<td>( d(s) )</td>
<td>1</td>
<td>3.8155</td>
<td>1.1851</td>
<td>4.1731</td>
<td>1.1754</td>
</tr>
<tr>
<td>( \hat{P}_Q ) ( n(s) )</td>
<td>(1.638 \cdot 10^{-3})</td>
<td>(-0.13261)</td>
<td>38.769</td>
<td>2250.9</td>
<td>1447.4</td>
</tr>
<tr>
<td>( d(s) )</td>
<td>1</td>
<td>11.397</td>
<td>156.58</td>
<td>604.42</td>
<td>42.466</td>
</tr>
</tbody>
</table>

Table A.2: Coefficients of the numerators \( n(s) \) and the denominators \( d(s) \) that make up the nominal models \( n(s)/d(s) \).

<table>
<thead>
<tr>
<th>Poles and Zeros</th>
<th>( \hat{P}_1 )</th>
<th>( \hat{P}_Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>poles</td>
<td>(-0.0015 \pm 0.3141j)</td>
<td>(-0.0365 \pm 0.7436j)</td>
</tr>
<tr>
<td>zeros</td>
<td>(-0.0098 \pm 1.000j)</td>
<td>(-0.3215 \pm 0.3288j)</td>
</tr>
<tr>
<td>(-0.2483 \pm 1.319j)</td>
<td>(-3.291 \pm 10.67j)</td>
<td>(-3.063 \pm 1.428j)</td>
</tr>
</tbody>
</table>

Table A.3: Poles and zeros of some illustrative nominal models.
Table A.4: Coefficients of the numerators $n(s)$ and the denominators $d(s)$ that make up the compensators $n(s)/d(s)$; scalar design weights $\alpha_i$.

<table>
<thead>
<tr>
<th></th>
<th>$s^2$</th>
<th>$s^1$</th>
<th>$s^0$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{hp}$ $n(s)$</td>
<td>1.1823</td>
<td>.72553</td>
<td>2.8746</td>
<td>10</td>
</tr>
<tr>
<td>$d(s)$</td>
<td>1</td>
<td>.47833</td>
<td>2.2438</td>
<td></td>
</tr>
<tr>
<td>$C_{mp}$ $n(s)$</td>
<td>.55898</td>
<td>.51362</td>
<td>1.3877</td>
<td>0.8</td>
</tr>
<tr>
<td>$d(s)$</td>
<td>1</td>
<td>.36650</td>
<td>2.2589</td>
<td></td>
</tr>
<tr>
<td>$C_{lp}$ $n(s)$</td>
<td>.09752</td>
<td>.38623</td>
<td>.12246</td>
<td>0.15</td>
</tr>
<tr>
<td>$d(s)$</td>
<td>1</td>
<td>1.0320</td>
<td>3.7365</td>
<td></td>
</tr>
<tr>
<td>$C_Q$ $n(s)$</td>
<td>37.375</td>
<td>528.48</td>
<td>5625.0</td>
<td>10</td>
</tr>
<tr>
<td>$d(s)$</td>
<td>1</td>
<td>110.01</td>
<td>1741.6</td>
<td></td>
</tr>
</tbody>
</table>

Table A.5: Poles and zeros of some illustrative compensators.
<table>
<thead>
<tr>
<th>$P_i$</th>
<th>$n(s)$</th>
<th>$d(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{P}_1$</td>
<td>$n(s) = -0.40036$</td>
<td>$d(s) = 1$</td>
</tr>
<tr>
<td>$\hat{P}_2$</td>
<td>$n(s) = -0.44773$</td>
<td>$d(s) = 1$</td>
</tr>
<tr>
<td>$1$</td>
<td>3.8155</td>
<td>1.1851</td>
</tr>
<tr>
<td>$\hat{P}_3$</td>
<td>$n(s) = -0.46143$</td>
<td>$d(s) = 1$</td>
</tr>
<tr>
<td>$1$</td>
<td>3.7161</td>
<td>1.2405</td>
</tr>
<tr>
<td>$\hat{P}_4$</td>
<td>$n(s) = 0.01361$</td>
<td>$d(s) = 1$</td>
</tr>
<tr>
<td>$1$</td>
<td>12.766</td>
<td>150.45</td>
</tr>
<tr>
<td>$\hat{P}_5$</td>
<td>$n(s) = 0.0088$</td>
<td>$d(s) = 1$</td>
</tr>
<tr>
<td>$1$</td>
<td>13.258</td>
<td>156.27</td>
</tr>
</tbody>
</table>

Table A.6: Coefficients of the numerators $n(s)$ and the denominators $d(s)$ of the nominal models $\hat{P}_i = n(s)/d(s)$ of the primary iteration.

<table>
<thead>
<tr>
<th>$P_i$</th>
<th>Poles</th>
<th>Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{P}_1$</td>
<td>$-0.0015 \pm 0.3141j$</td>
<td>$-0.0098 \pm 1.000j$</td>
</tr>
<tr>
<td></td>
<td>$-0.3215 \pm 0.3288j$</td>
<td>$-0.2483 \pm 1.319j$</td>
</tr>
<tr>
<td>$\hat{P}_2$</td>
<td>$-0.0003 \pm 0.3131j$</td>
<td>$-0.0103 \pm 1.005j$</td>
</tr>
<tr>
<td></td>
<td>$-0.3020 \pm 0.3283j$</td>
<td>$-0.2132 \pm 1.334j$</td>
</tr>
<tr>
<td>$\hat{P}_3$</td>
<td>$0.0008 \pm 0.3138j$</td>
<td>$-0.0172 \pm 1.011j$</td>
</tr>
<tr>
<td></td>
<td>$-0.2759 \pm 0.3336j$</td>
<td>$-0.1896 \pm 1.357j$</td>
</tr>
<tr>
<td>$\hat{P}_4$</td>
<td>$-3.283 \pm 0.807j$</td>
<td>$-0.1814 \pm 0.8016j$</td>
</tr>
<tr>
<td></td>
<td>$12.81 \pm 0.6256j$</td>
<td>$-0.2981 \pm 1.624j$</td>
</tr>
<tr>
<td>$\hat{P}_5$</td>
<td>$-3.479 \pm 0.968j$</td>
<td>$-0.0353 \pm 0.7285j$</td>
</tr>
<tr>
<td></td>
<td>$57.71 \pm 207.5j$</td>
<td>$-0.3194 \pm 1.529j$</td>
</tr>
</tbody>
</table>

Table A.7: Poles and zeros of the nominal models of the primary iteration.
<table>
<thead>
<tr>
<th>$C_i$</th>
<th>$n(s)$</th>
<th>$d(s)$</th>
<th>$s^4$</th>
<th>$s^3$</th>
<th>$s^2$</th>
<th>$s^1$</th>
<th>$s^0$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>$n(s)$</td>
<td>1.2511</td>
<td>.46641</td>
<td>.10498</td>
<td>.26361</td>
<td>.00634</td>
<td></td>
<td>0.113</td>
</tr>
<tr>
<td></td>
<td>$d(s)$</td>
<td>1</td>
<td>2.7520</td>
<td>4.0565</td>
<td>2.4871</td>
<td>.65093</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_2$</td>
<td>$n(s)$</td>
<td>.19951</td>
<td>.75048</td>
<td>.36437</td>
<td>.43155</td>
<td>.04225</td>
<td></td>
<td>0.201</td>
</tr>
<tr>
<td></td>
<td>$d(s)$</td>
<td>1</td>
<td>1.8427</td>
<td>3.7518</td>
<td>2.1386</td>
<td>.61797</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_3$</td>
<td>$n(s)$</td>
<td>.26140</td>
<td>.88085</td>
<td>.68114</td>
<td>.56147</td>
<td>.08944</td>
<td></td>
<td>3.19</td>
</tr>
<tr>
<td></td>
<td>$d(s)$</td>
<td>1</td>
<td>1.1892</td>
<td>3.1354</td>
<td>1.6285</td>
<td>.51056</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_4$</td>
<td>$n(s)$</td>
<td>36.692</td>
<td>868.01</td>
<td>9219.4</td>
<td>8099.6</td>
<td>25180</td>
<td></td>
<td>8.18</td>
</tr>
<tr>
<td></td>
<td>$d(s)$</td>
<td>1</td>
<td>109.94</td>
<td>3448.1</td>
<td>2312.6</td>
<td>9214.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C_5$</td>
<td>$n(s)$</td>
<td>71.407</td>
<td>2182.1</td>
<td>28718</td>
<td>23854</td>
<td>68457</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>$d(s)$</td>
<td>1</td>
<td>129.16</td>
<td>4829.0</td>
<td>3344.1</td>
<td>11571</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$CP$</td>
<td>$n(s)$</td>
<td>68.059</td>
<td>2069.6</td>
<td>27650</td>
<td>19546</td>
<td>66633</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>$d(s)$</td>
<td>1</td>
<td>121.49</td>
<td>4687.5</td>
<td>2676.9</td>
<td>11329</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table A.8: Coefficients of the numerators $n(s)$ and the denominators $d(s)$ of the compensators $C_i = n(s)/d(s)$ of the primary iteration; design weights $\alpha_i$.

<table>
<thead>
<tr>
<th>Poles and Zeros</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$ poles</td>
</tr>
<tr>
<td>zeros</td>
</tr>
<tr>
<td>$C_2$ poles</td>
</tr>
<tr>
<td>zeros</td>
</tr>
<tr>
<td>$C_3$ poles</td>
</tr>
<tr>
<td>zeros</td>
</tr>
<tr>
<td>$C_4$ poles</td>
</tr>
<tr>
<td>zeros</td>
</tr>
<tr>
<td>$C_5$ poles</td>
</tr>
<tr>
<td>zeros</td>
</tr>
<tr>
<td>$CP$ poles</td>
</tr>
<tr>
<td>zeros</td>
</tr>
</tbody>
</table>

Table A.9: Poles and zeros of the compensators of the primary iteration.
Table A.10: Coefficients of the numerators \( n(s) \) and the denominators \( d(s) \) of the nominal models \( \hat{P}_i = n(s)/d(s) \) of the advanced iteration.

<table>
<thead>
<tr>
<th>( \hat{P}_{21} )</th>
<th>( s^5 )</th>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(s) )</td>
<td>0.00111</td>
<td>-0.08270</td>
<td>35.834</td>
<td>2410.5</td>
<td>1852.8</td>
<td>5393.5</td>
</tr>
<tr>
<td>( d(s) )</td>
<td>1</td>
<td>13.072</td>
<td>155.53</td>
<td>759.14</td>
<td>40.098</td>
<td>417.39</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \hat{P}_{22} )</th>
<th>( s^5 )</th>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(s) )</td>
<td>0.00046</td>
<td>-0.02373</td>
<td>32.886</td>
<td>2552.3</td>
<td>1635.2</td>
<td>6163.7</td>
</tr>
<tr>
<td>( d(s) )</td>
<td>1</td>
<td>13.574</td>
<td>158.78</td>
<td>711.43</td>
<td>137.37</td>
<td>365.79</td>
</tr>
</tbody>
</table>

Table A.11: Poles and zeros of the nominal models of the advanced iteration.

<table>
<thead>
<tr>
<th>( P_{21} )</th>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>poles</td>
<td>-3.188 + 10.13j</td>
<td>.0298 ± 7.399j</td>
<td>-6.755</td>
<td></td>
<td></td>
</tr>
<tr>
<td>zeros</td>
<td>65.02 + 187.9j</td>
<td>-3.716 ± 1.458j</td>
<td>-54.51</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( P_{22} )</th>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>poles</td>
<td>-3.658 + 9.930j</td>
<td>-0.0392 + 7.260j</td>
<td>-6.179</td>
<td></td>
<td></td>
</tr>
<tr>
<td>zeros</td>
<td>60.53 + 275.9j</td>
<td>-0.3071 + 1.530j</td>
<td>-68.92</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table A.12: Coefficients of the numerators \( n(s) \) and the denominators \( d(s) \) of the compensators \( C_i = n(s)/d(s) \) of the advanced iteration; design weights \( \alpha_i \).

<table>
<thead>
<tr>
<th>( C_{21} )</th>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(s) )</td>
<td>72.688</td>
<td>2196.1</td>
<td>28516</td>
<td>262358</td>
<td>62609</td>
<td>20.00</td>
</tr>
<tr>
<td>( d(s) )</td>
<td>1</td>
<td>133.25</td>
<td>4793.2</td>
<td>3786.8</td>
<td>10616</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( C_{22} )</th>
<th>( s^4 )</th>
<th>( s^3 )</th>
<th>( s^2 )</th>
<th>( s^1 )</th>
<th>( s^0 )</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n(s) )</td>
<td>69.422</td>
<td>2143.0</td>
<td>28536</td>
<td>22943</td>
<td>67956</td>
<td>20.00</td>
</tr>
<tr>
<td>( d(s) )</td>
<td>1</td>
<td>124.30</td>
<td>4812.3</td>
<td>3208.4</td>
<td>11523</td>
<td></td>
</tr>
</tbody>
</table>

Table A.13: Poles and zeros of the compensators of the advanced iteration.
Appendix B

Algebraic Theory

B.1 Algebraic Structure and Facts

This section lists some terminology and several noteworthy facts from ring theory. Most definitions summarized here have been taken from [239]. Some of the concepts are explained in terms of real rational continuous-time systems. These illustrations are enclosed by brackets.

**ring** A ring \( \mathcal{R} \) is a nonempty set \( \mathcal{R} \) together with the binary operations + (addition) and \( \cdot \) (multiplication) such that the following axioms are satisfied

- the additive group \( (\mathcal{R},+) \) is commutative; i.e. \( \forall x, y, z \in \mathcal{R} \)
  i. \( x + (y + z) = (x + y) + z \),
  ii. there exist an element 0 \( \in \mathcal{R} \) such that \( x + 0 = x \),
  iii. there exist an element \( -x \in \mathcal{R} \) such that \( x + (-x) = 0 \),
- the multiplicative group \( (\mathcal{R}, \cdot) \) is a semi-group;
  i.e. \( \forall x, y, z \in \mathcal{R} \)
  i. \( x \cdot (y \cdot z) = (x \cdot y) \cdot z \),
  ii. \( x \cdot (y + z) = x \cdot y + x \cdot z \),
  iii. \( (x + y) \cdot z = x \cdot z + y \cdot z \).

**commutative ring** A ring \( \mathcal{R} \) is commutative if \( xy = yx \) \( \forall x, y \in \mathcal{R} \).

**identity** A ring \( \mathcal{R} \) has an identity if there exists an element 1 \( \in \mathcal{R} \) such that \( 1 \cdot x = x \cdot 1 = x \), \( \forall x \in \mathcal{R} \).

**domain** A ring \( \mathcal{R} \) is a domain or integral domain if for \( x, y \in \mathcal{R} \), \( xy = 0 \) implies \( x = 0 \lor y = 0 \).

**unit** \( x \in \mathcal{R} \) is a unit of \( \mathcal{R} \) if it has an inverse in \( \mathcal{R} \).

[\( x \in \mathbb{R}H_{\infty} \) is a unit of \( \mathbb{R}H_{\infty} \) if it is a proper but non-strictly proper minimum phase system.]
unimodular A matrix \( X \in \mathcal{R}^{n \times n} \) is unimodular if \( X \) has an inverse in \( \mathcal{R}^{n \times n} \) (i.e. it is a unit in \( \mathcal{R}^{n \times n} \)).

left (right) associate Let \( \mathcal{R} \) be a principal ideal domain. Then \( X \in \mathcal{R}^{p \times m} \) is a left (right) associate of \( Y \in \mathcal{R}^{p \times m} \) (\( Z \in \mathcal{R}^{p \times m} \)) if there exist an unimodular \( U \in \mathcal{R}^{p \times p} \) \( (V \in \mathcal{R}^{m \times m}) \) such that \( X = UY \) \( (X = ZV) \).

Hermite form Let \( \mathcal{R} \) be a principal ideal domain. Then every \( X \in \mathcal{R}^{n \times n} \) is a left associate of some \( Y \in \mathcal{R}^{n \times n} \) and a right associate of some \( Z \in \mathcal{R}^{n \times n} \), that both are lower triangular.

Remark B.1.1 We point out three facts related to the Hermite form.

- The Hermite form of \( X \) is the decomposition \( X = UY \) with \( Y \) a lower (or upper) triangular matrix and \( U \) an unimodular matrix.

- The existence of the Hermite form is based on the particular property of a principal ideal ring \( \mathcal{R} \), that a greatest common divisor \( d \in \mathcal{R} \) of \( x_1, \ldots, x_n \in \mathcal{R} \) can be expressed as \( d = q_1 x_1 + \ldots + q_n x_n \) with \( q_i \in \mathcal{R} \).

- The Hermite form of a stable square system \( X \in \mathbb{R}^{n \times n} \) is the decomposition into the product of a stable system \( U \) which has a stable inverse, and a stable lower triangular system \( Y \).

A concise introduction to the axiomatic ring theory can be found in [56] and [244]. A more comprehensive background is provided in [239]. For a profound treatise on the theory of abstract algebra the interested reader is referred to [117].

The sequel of this section contains two series of facts, which are used in the development of the \((R, S)\)-parameterization of plants \( P_{TT} \) (Section 3.4). The following two properties have been adopted from [50] respectively [239].

Fact B.1.2

i. Let \( A, B \in \mathcal{H} \), then \( AB \in \mathcal{J} \) \( \iff \) \( A \in \mathcal{J} \), \( B \in \mathcal{J} \).

ii. Let \( A \in \mathcal{H} \), then \( A \in \mathcal{J} \) \( \iff \) \( \det A \in \mathcal{J} \).

The second property reads that a stable system is an unimodular element of \( \mathcal{H} \) if and only if its determinant is a unit in \( \mathcal{H} \) (i.e. the inverse of the determinant is stable). The following facts concern coprime factorizations of plants belonging to \( \mathcal{H} \) and \( \mathcal{J} \).

Fact B.1.3

i. Let \( C \in \mathcal{H} \) and let \( P \in \mathcal{F} \) have a \textsc{rcf} \((N, D)\) and a \textsc{lcf} \((\mathcal{D}, \mathcal{N})\). Then

\[ PC \in \mathcal{H} \iff D^{-1}C \in \mathcal{H} \]

\[ CP \in \mathcal{H} \iff \mathcal{D}C^{-1} \in \mathcal{H} \]
ii. Let $P \in \mathcal{F}$ have a ref $(N, D)$ and a lcf $(\bar{D}, \bar{N})$. Then

$$P \in \mathcal{H} \iff D^{-1} \in \mathcal{H} \iff \bar{D}^{-1} \in \mathcal{H}.$$ 

iii. Let plant $P \in \mathcal{F}$ have a ref $(N, D)$ and a lcf $(\bar{D}, \bar{N})$. Then there exist a scalar $d_D \in \mathcal{F}$ such that $\det(D) = d_D \det(\bar{D})$.

Proof: The first equivalence of fact i. is derived as follows. Since $\mathcal{H}$ is closed under multiplication and addition and since $N \in \mathcal{H}$, the product $ND^{-1}C \in \mathcal{H}$ if $D^{-1}C \in \mathcal{H}$. The opposite result is less trivial. Let $X, Y$ be Bezout factors of $(N, D)$. Then we have the following implications. $X, Y, C, PC \in \mathcal{H} \Rightarrow XPC + YC \in \mathcal{H} \Rightarrow (XN + YD)D^{-1}C \in \mathcal{H} \Rightarrow D^{-1}C \in \mathcal{H}$ where in the final implication use has been made of the Bezout identity of (3.1).

Fact ii. follows from fact i. with $B = I$. Fact iii. has been proven in [239].

### B.2 The Bezout Identity

This section provides some additional background to the Bezout factors and double Bezout factors defined in Fact 3.2.4 and Fact 3.2.6.ii. First we show that double Bezout factors $X, Y, \bar{X}, \bar{Y}$ exist for any couple of right and left coprime factorizations $(N, D)$ and $(\bar{D}, \bar{N})$ of a plant $P$.

**Lemma B.2.1** Let $P \in \mathcal{H}$ have a ref $(N, D)$ and a lcf $(\bar{D}, \bar{N})$. Let $X, Y \in \mathcal{H}$ be right Bezout factors of $(N, D)$. Then there exist unique left Bezout factors $\bar{X}, \bar{Y} \in \mathcal{H}$ of $(\bar{D}, \bar{N})$ such that

$$\begin{bmatrix} Y & X \\ -\bar{N} & \bar{D} \end{bmatrix} \begin{bmatrix} D & -\bar{A} \\ N & \bar{B} \end{bmatrix} = \begin{bmatrix} I & M \\ 0 & I \end{bmatrix}$$

(B.1)

which is called the double Bezout identity.

Proof: By Fact 3.2.4.ii $(\bar{D}, \bar{N})$ has left Bezout factors $\bar{A}, \bar{B} \in \mathcal{H}$ and thus

$$\begin{bmatrix} Y & X \\ -\bar{N} & \bar{D} \end{bmatrix} \begin{bmatrix} D & -\bar{A} \\ N & \bar{B} \end{bmatrix} = \begin{bmatrix} I & M \\ 0 & I \end{bmatrix}$$

which defines $M \in \mathcal{H}$. The two identities hold by definition and the appearance of the zero is trivial. The right hand side has a stable inverse, which is used for postmultiplication:

$$\begin{bmatrix} Y & X \\ -\bar{N} & \bar{D} \end{bmatrix} \begin{bmatrix} D & -\bar{A} - DM \\ N & \bar{B} - NM \end{bmatrix} = I.$$ 

The result follows from $\bar{X} \doteq \bar{A} + DM$, $\bar{Y} \doteq \bar{B} - NM$. Finally uniqueness of $\bar{X}, \bar{Y}$ for given $(N, D)$, $(\bar{D}, \bar{N})$ and $X, Y$ is immediate by the existence of an inverse of the leftmost block matrix in (B.1).
Now we examine right and left Bezout factors separately. The result below characterizes the class of all right Bezout factors of \((N, D)\) and the class of all left Bezout factors of \((\tilde{D}, \tilde{N})\).

**Proposition B.2.2** Let a plant \(P \in \mathcal{F}\) have a rcf \((N, D)\) with Bezout factors \(X, Y \in \mathcal{H}\) and a lcf \((\tilde{D}, \tilde{N})\) with Bezout factors \(\tilde{X}, \tilde{Y} \in \mathcal{H}\). Then \(A, B \in \mathcal{H}\) are right Bezout factors of \((N, D)\), i.e. \(AN + BD = I\), if and only if there exists some \(M \in \mathcal{H}\) such that

\[
A = X + MD\tilde{D}, \quad B = Y - M\tilde{N},
\]

and \(\tilde{A}, \tilde{B} \in \mathcal{H}\) are left Bezout factors of \((\tilde{D}, \tilde{N})\) if and only if there exists some \(\tilde{M} \in \mathcal{H}\) such that

\[
\tilde{A} = \tilde{X} + D\tilde{M}, \quad \tilde{B} = \tilde{Y} - N\tilde{M}.
\]

**Proof:** We prove only the case of right Bezout factors. The proof of the other part is completely dual.

\((\Leftarrow)\) Almost trivial: by definition \(A\) and \(B\) belong to \(\mathcal{H}\) and \(AN + BD = XN + YD + M\tilde{D}(P - P)D = I\).

\((\Rightarrow)\) Let \(\tilde{U}, \tilde{V} \in \mathcal{H}\) be such that \(X, Y, \tilde{U}, \tilde{V}\) are double Bezout factors of \((N, D)\) and \((\tilde{D}, \tilde{N})\); such \(\tilde{U}, \tilde{V}\) always exist by Lemma B.2.1. Then let \(M \in \mathcal{H}\) be defined via

\[
\begin{bmatrix}
B & A \\
I & M
\end{bmatrix}
\begin{bmatrix}
D & -\tilde{U} \\
N & \tilde{V}
\end{bmatrix} = I,
\]

which that the right Bezout factors \(A, B\) can be written as proposed. \(\square\)

Not every couple of right Bezout factors of \((N, D)\) and left Bezout factors of \((\tilde{D}, \tilde{N})\) make a quadruple of double Bezout factors. Based on one such quadruple we can characterize the whole class of double Bezout factors.

**Proposition B.2.3** Let \(P \in \mathcal{F}\) have a rcf \((N, D)\) and a lcf \((\tilde{D}, \tilde{N})\). Let the stable \(X, Y, \tilde{X}, \tilde{Y}\) be double Bezout factors of \((N, D)\) and \((\tilde{D}, \tilde{N})\) as in Lemma B.2.1. Then the quadruple \((A, B, \tilde{A}, \tilde{B})\) are double Bezout factors of \((N, D)\) and \((\tilde{D}, \tilde{N})\) if and only if there exists a factor \(M \in \mathcal{H}\) such that

\[
(A, B, \tilde{A}, \tilde{B}) = (X + MD\tilde{D}, Y - M\tilde{N}, \tilde{X} + DM\tilde{Y}, \tilde{Y} - NM).
\]

(B.2)
B.3 State Space Formulas

Proof:

(\(\Leftrightarrow\)) Substitution of (B.2) in (B.1) yields the identity.

(\(\Rightarrow\)) \(A, B\) and \(\tilde{A}, \tilde{B}\) are right respectively left Bezout factors of \((N, D)\) and \((\tilde{D}, \tilde{N})\).

By Proposition B.2.2 they can be written as

\[
A = X + M \tilde{D}, \quad B = Y - M \tilde{N} \\
\tilde{A} = \tilde{X} + D \tilde{M}, \quad \tilde{B} = \tilde{Y} - N \tilde{M}
\]

with \(M, \tilde{M} \in \mathcal{H}\). After expanding (B.1) the only non-trivial element is the upper right term \(X \tilde{Y} - Y \tilde{X} = 0\). A similar term \(A \tilde{B} - B \tilde{A}\) appears if \(X, Y, \tilde{X}, \tilde{Y}\) are replaced by \(A, B, \tilde{A}, \tilde{B}\). So in order that the latter quadruple are double Bezout factors, they have to satisfy \(A \tilde{B} - B \tilde{A} = 0\). That is

\[
0 = (X + M \tilde{D})(\tilde{Y} - N \tilde{M}) - (Y - M \tilde{N})(\tilde{X} + D \tilde{M}) \\
= (X \tilde{Y} - Y \tilde{X}) + M(\tilde{N} D - \tilde{D} N) \tilde{M} + M(\tilde{N} \tilde{X} + \tilde{D} \tilde{Y}) - (X N + Y D) \tilde{M}.
\]

Since \(\tilde{N} D = \tilde{D} N\), and since \(X, Y, \tilde{X}, \tilde{Y}\) are double Bezout factors, \(A, B, \tilde{A}, \tilde{B}\) are double Bezout factors if and only if \(M = \tilde{M}\). \(\Box\)

By the latter result the two degrees of freedom, that are present in the separate classes of right and left Bezout factors, is reduced to one degree of freedom in the class of double Bezout factors.

B.3 State Space Formulas

In this section we list some state space formulas, that are used in the identification procedures of Appendix F. Meanwhile we survey some literature concerning the computation of coprime factorizations.

In [120] it was shown that a strictly proper system admits a stable right factorization if and only if it admits a reachable and detectable realization. The development of this result implicitly encompassed an algorithm for computing a factorization from such a realization. In [163] explicit formulas for the calculation of a doubly coprime fractional representation as in Fact 3.2.6.ii and (B.1) were provided in terms of a stabilizable and detectable state space realization of a real rational strictly proper continuous time transfer function. Let \(P(s)\) be such a transfer function with state space realization \((A, B, C, 0)\). Henceforth a state space realization is assumed to be minimal. Then a rcf \((N(s), D(s))\) of \(P(s)\) with Bezout factors \(X(s), Y(s)\) can be defined as

\[
D(s) = I - K_r(sI - A + BK_r)^{-1} B
\]

\(^1\)Throughout this section we use the notation \((s)\) to indicate transfer functions, so that the feedthrough matrix \(D\) of a realization \((A, B, C, D)\) cannot be mistaken for the "denominator" \(D(s)\) of a rcf \((N(s), D(s))\).
\[
N(s) = C(sI - A + BK_r)^{-1}B
\]
\[
Y(s) = I + K_r(sI - A + K_e C)^{-1}B
\]
\[
X(s) = K_r(sI - A + K_e C)^{-1}K_e
\]

where \(K_r, K_e\) are real matrices such that all eigenvalues of \(A - BK_r\) and \(A - K_e C\) are contained in the open left-half complex plane. A normalized rcf can be obtained from the above formulas by choosing \(K_r\) as the solution to the LQ regulator problem [156]. This result has been extended to proper but not necessarily strictly proper transfer functions in [241]. For completeness we list the computational algorithm proposed in the latter reference. We let \(Q = \text{are}(F, G, H)\) denote the solution \(Q = Q^T\) of the algebraic Riccati equation \(F^TQ + QF - QGQ + H = 0\), which is such that all eigenvalues of \(F - GQ\) are contained in the open left-half complex plane. Then a normalized rcf \((N_n(s), D_n(s))\) of \(P(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\) can be calculated as

1. define \(L = I + D^T D, \quad V = I - DL^{-1}D^T\)
\[
M = I + DD^T, \quad U = I - DT M^{-1} D
\]

2. solve \(\Phi = \text{are}(A - BL^{-1}D^T C, BL^{-1}B^T, C^T VC)\)
\(\Psi = \text{are}(A - BD^T M^{-1}C, C^T M^{-1}C, BU B^T)\)

3. construct \(K_r = L^{-1}(D^T C + B^T \Phi)\)
\(K_e = (\Psi C^T + BD^T) M^{-1}\)

4. construct the nrcf \((N_n(s), D_n(s))\) as
\[
\begin{bmatrix} D_n(s) \\ N_n(s) \end{bmatrix} = \begin{bmatrix} A - BK_r & BL^{-\frac{1}{2}} \\ -K_r & L^{-\frac{1}{2}} \\ C - DK_r & DL^{-\frac{1}{2}} \end{bmatrix}
\]

The expression in (B.3) represents the transfer function matrix of the graph operator \(G_n\) associated with \((N_n(s), D_n(s))\). In addition to the nrcf of (B.3) we need a pair of Bezout factors \((X_n(s), Y_n(s))\) for the computations in Appendix F.4. The next result shows how such a pair of Bezout factors can be calculated from the matrices used in the above algorithm.

**Proposition B.3.1** Let \((N_n(s), D_n(s))\) of (B.3) be a nrcf of the transfer matrix \(P(s)\) with state space realization \((A, B, C, D)\). Define \(X_n(s), Y_n(s) \in \mathbb{R}H_\infty\) as
\[
\begin{bmatrix} Y_n(s) & X_n(s) \end{bmatrix} = \begin{bmatrix} A - K_e C & B - K_e D & K_e \\ L^\frac{1}{2} K_r & L^\frac{1}{2} & 0 \end{bmatrix}
\]
with $K_e$ as in the algorithm for $(N_n(s), D_n(s))$. Then $X_n, Y_n$ are right Bezout factors of $(N_n(s), D_n(s))$.

**Proof:** We demonstrate the identity $Y_n(s)D_n(s) + X_n(s)N_n(s) = I$ by relating the normalized rcf of (B.3) of the plant induced by $(A, B, C, D)$ to the rcf of the strictly proper $(A, B, C, 0)$, that has been given above. Notice that $N_n(s)(D_n(s))^{-1} \neq N(s)(D(s))^{-1}$. We take $A, B, C$ and $K_r$ from (B.3) to build $(N_n(s), D_n(s))$ like in the above equations:

$$
\begin{bmatrix}
D(s) \\
N(s)
\end{bmatrix}
= \begin{bmatrix}
A - BK_r & B \\
-K_r & I \\
C & 0
\end{bmatrix}
$$

Then we rewrite (B.3) into

$$\begin{align*}
\begin{bmatrix}
D_n(s) \\
N_n(s)
\end{bmatrix}
&= \begin{bmatrix}
-K_r(sI - A + BK_r)^{-1}BL^{-\frac{1}{2}} + L^{-\frac{1}{2}} \\
(C - DK_r)(sI - A + BK_r)^{-1}BL^{-\frac{1}{2}} + DL^{-\frac{1}{2}}
\end{bmatrix} \\
&= \begin{bmatrix}
I & 0 \\
D & I
\end{bmatrix}
\begin{bmatrix}
-K_r(sI - A + BK_r)^{-1}B + I \\
C(sI - A + BK_r)^{-1}B
\end{bmatrix}L^{-\frac{1}{2}}
\end{align*}$$

Analogously we obtain

$$
\begin{bmatrix}
Y_n(s) \\
X_n(s)
\end{bmatrix}
= L^{\frac{1}{2}} \begin{bmatrix}
Y(s) & X(s)
\end{bmatrix}
\begin{bmatrix}
I & 0 \\
-D & I
\end{bmatrix}
$$

Expanding $Y_n(s)D_n(s) + X_n(s)N_n(s)$ with the above expressions for $(N_n(s), D_n(s))$ and $X_n(s), Y_n(s)$ yields the identity. Finally $X_n(s), Y_n(s)$ are stable by the choice of $K_e$ in the algorithm.

We end up this section with mentioning some additional literature concerning the computation of coprime factorizations. In [249] the above results on the rcf $(N(s), D(s))$ with $X(s), Y(s)$ for strictly proper transfer matrices were extended to transfer matrices described by general state space realizations. In [27] a state space algorithm was developed for the calculation of normalized coprime factorizations of generalized dynamical systems. Finally analogous results for discrete-time systems were derived in [29, 155].
Appendix C

The Dual Associated Identification Problem

The open-loop identification problems of Chapter 4 rest on the representation of the plant by a right coprime factorization. A dual representation of $H(P_T, C)$ was used by Hansen to conduct experiment design for closed-loop identification [101, 100, 102]. This left coprime factorization gives rise to an associated identification problem, which is dual to that of Section 4.3. We will show that there exists no dual equivalent identification problem. Besides, because of the duality we do not formalize and prove all results.

The left counterpart of the associated rcf $(N^a, D^a)$ of (3.6) is

$$
\begin{align*}
\tilde{D}^a & \doteq \tilde{D}_o - \tilde{R}\tilde{N}_c \\
\tilde{N}^a & \doteq \tilde{N}_o + \tilde{R}\tilde{D}_c
\end{align*}
$$

(C.1)

with $(\tilde{D}_o, \tilde{N}_o)$ a lcf of the auxiliary model $P_o$, $(\tilde{D}_c, \tilde{N}_c)$ a lcf of the stabilizing compensator $C$, and $\tilde{R} \in \mathcal{H}$ such that $\det(\tilde{D}_o - \tilde{R}\tilde{N}_c) \neq 0$. With $(\tilde{D}^a, \tilde{N}^a)$ substituted for $(\tilde{D}, \tilde{N})$ in the admissible lcf of (3.23) we can represent the input-output relations of $P_T$ as

$$
\tilde{D}^a y = \tilde{N}^a u + \tilde{S} w
$$

(C.2)

where $\tilde{S}$ is substituted for $\tilde{N}_{21}$. By expressing $(\tilde{D}^a, \tilde{N}^a)$ as in (C.1) we arrive at the dual $(R, S)$-parameterization, which is called the $(\tilde{R}, \tilde{S})$-parameterization in Fig. C.1.

If anything at all, then (C.2) is the dual of (4.7), i.e. the equation that embodies the equivalent identification problem. They have in common that the associated factors account for the deterministic part and that $Sw$ appears as a noise contribution. The essential difference is, that there is no counterpart of $x$ present in (C.2). Consequently no use is made of the knowledge of $C$ in the identification of $(\tilde{D}^a, \tilde{N}^a)$ from $u$ and $y$. In fact (C.2) could have been derived directly from

$$
y = Pu + P_{yw} w
$$
if both sides had been premultiplied by $\tilde{D}$:

$$\tilde{D}y = \tilde{N}u + \tilde{D}P_{yu}w$$

which "defines" $\tilde{S}$. So (C.2) is a kind of generalized ARMAX representation of $P_T$ and its identification has no open-loop character at all. Moreover, in [100] Hansen showed that if the prediction error method is used to estimate models, then the identification from the latter equation is identical to the direct identification of $P_T$ from the closed-loop data.

The dual associated identification problem is derived by rearranging (C.2) after substitution of (C.1) for $(\tilde{D}^a, \tilde{N}^a)$. This yields

$$\tilde{D}_o y - \tilde{N}_o u = \tilde{R}(\tilde{N}_e y + \tilde{D}_e u) + \tilde{S}w$$

and with the definitions

$$\tilde{q} = \tilde{D}_o y - \tilde{N}_o u$$

$$\tilde{z} = \tilde{N}_e y + \tilde{D}_e u$$

(see also Fig. C.1) we get

$$\tilde{q} = \tilde{R}\tilde{z} + \tilde{S}w.$$  

The duality with (4.10) is obvious. The identification of $\tilde{R}$ and $\tilde{S}$ constitutes an open-loop identification problem, because $\tilde{z} = \tilde{D}_e (u + Cy) = \tilde{D}_e (r_1 + Cr_2)$ and thus $\tilde{z}$ is uncorrelated with $w$ by Assumption 4.1.4.

In order to express $T(P_T, C)$ in terms of the $(\tilde{R}, \tilde{S})$-parameterization we first establish the following relation\(^1\) between $R$ and $\tilde{R}$.

---

\(^1\)We are grateful to Fred Hansen for pointing out this result.
Proposition C.0.2 Let Assumptions 3.3.3 and 3.3.5 hold. Further let the \((R, S)\)-parameterization of Fig. 3.10 and the \((\tilde{R}, \tilde{S})\)-parameterization of Fig. C.1 both represent the stabilized plant \(P_T \in \mathcal{F}\). Then \(\tilde{R}\) and \(\tilde{R}\) are related as

\[
\tilde{R} \Lambda_o = \tilde{\Lambda}_o R
\]

with \(\Lambda_o\) and \(\tilde{\Lambda}_o\) as defined in (3.5).

Proof: Substitution of (3.6) and (C.1) in the trivial equivalence \(\tilde{N}^a D^a = \tilde{D}^a N^a\) yields

\[
\tilde{N}_c D_o + \tilde{R} \tilde{D}_c D_o - \tilde{R} \tilde{D}_c N_c R - \tilde{N}_o N_c R = \tilde{D}_o N_o - \tilde{R} \tilde{N}_c N_o - \tilde{R} \tilde{N}_c D_c R + \tilde{D}_o D_c R.
\]

Since \(\tilde{N}_o D_o = \tilde{D}_o N_o\) and \(\tilde{N}_c D_c = \tilde{D}_c N_c\) four terms cancel out. By taking the remaining terms together we obtain

\[
\tilde{R}(\tilde{D}_c D_o + \tilde{N}_o N_o) = (\tilde{D}_o D_c + \tilde{N}_o N_c) R
\]

and the result follows with (3.5).

---

Fig. C.2: Feedback system \(H(P_T, C)\) as a perturbation of the auxiliary feedback system \(H(P_o, C)\).

From Section 4.3 we know that the transfer function from \(\text{col}(w, r_2, r_1)\) to \(\text{col}(y, u)\) can be written as

\[
T(P_T, C) = \begin{bmatrix} D_c S & \end{bmatrix} \begin{bmatrix} T(P, C) \end{bmatrix}.
\]
By the above proposition we may substitute $\tilde{\Lambda}_o^{-1}\tilde{R}$ for $RA_o^{-1}$ in (3.16), which gives

$$T(P, C) = T(P_o, C) + \begin{bmatrix} D_c \\ -N_c \end{bmatrix} \tilde{\Lambda}_o^{-1}\tilde{R} \begin{bmatrix} \tilde{N}_c \\ \tilde{D}_c \end{bmatrix}.$$ 

Further, with the expression (3.30) for $S$ and $P_{yw} = (\tilde{D}^a)^{-1}\tilde{S}$ we achieve

$$\begin{bmatrix} D_c \\ -N_c \end{bmatrix} S = \begin{bmatrix} D_c \\ -N_c \end{bmatrix} (D_c + PN_c)^{-1} P_{yw}$$

$$= \begin{bmatrix} D_c \\ -N_c \end{bmatrix} (\tilde{D}^a D_c + \tilde{N}^a N_c)^{-1} \tilde{D}^a (\tilde{D}^a)^{-1} \tilde{S}$$

$$= \begin{bmatrix} D_c \\ -N_c \end{bmatrix} \tilde{\Lambda}_o^{-1}\tilde{S}$$

and thus $T(P_T, C)$ satisfies

$$T(P_T, C) = \begin{bmatrix} 0 & T(P_o, C) \end{bmatrix} + \begin{bmatrix} D_c \\ -N_c \end{bmatrix} \tilde{\Lambda}_o^{-1} \begin{bmatrix} \tilde{S} & \tilde{R}\tilde{N}_c & \tilde{R}\tilde{D}_c \end{bmatrix}$$

which is depicted in Fig. C.2. Comparing Fig. 4.4 and Fig. C.2 we see that the “right” and “left” associated identification problems differ only in their filter operations $\Lambda_o^{-1}$ and $\tilde{\Lambda}_o^{-1}$. Moreover one could choose the coprime factorizations $(N_o, D_o)$, $(\tilde{D}_o, \tilde{N}_o)$, $(N_c, D_c)$ and $(\tilde{D}_c, \tilde{N}_c)$ such that $\Lambda_o = I$ and $\tilde{\Lambda}_o = I$. Then the two associated identification problems are identical.
Appendix D

Noise Versus Unmodelled Dynamics

In Section 2.3 we have argued that identification techniques try to explain the observed data, while we are interested in the approximation of the transfer function of the inner-loop plant $P$. The portion of the data that is not accounted for by the inner-loop nominal model $\hat{P}$ is

$$y - \hat{y} = (P - \hat{P})u + P_{yw}w.$$ 

So both the deficiency of $\hat{P}$, which is often called "unmodelled dynamics", and the disturbance $w$, frequently denoted as "noise", give rise to some unexplained data. In view of control design these contributions are preferably separated. However, it was conjectured by Smith [213, 214] that such a separation is impossible without further assumptions either on the deficiency or on the disturbance. We subscribe to that conjecture and here we provide some support by means of the $(R, S)$-parameterization.

We assume that the nominal model $\hat{P}$ is a fairly good approximation of $P$ in the sense that the stable inner-loop feedback system $H(P, C)$ is approximately described by $H(\hat{P}, C)$. We substitute $\hat{P}$ and its rcf $(\hat{N}, \hat{D})$ for the auxiliary $P_0$ and $(N_0, D_0)$ in the $(R, S)$-parameterization of Fig. 3.10. Since $\hat{P}$ and $P$ are alike, the rcf $(\hat{N}, \hat{D})$ is similar to the associated rcf $(\bar{N} + D_cR, \bar{D} - N_cR)$ of $P$. Further, in Section 4.1 we already alluded to the fact that the deficiency and disturbance contribute to the feedback system as $Rx$ and $Sw$. Hence the separation of "noise" and "unmodelled dynamics" boils down to the separation of the contributions $Rx$ and $Sw$.

The block-diagram of Fig. 4.1 illustrates that $Rx$ and $Sw$ affect the feedback system in exactly the same way. This already suggests that these contributions cannot be separated without additional information. We will strengthen this suggestion as follows. We use two different nominal models to set up two $(R, S)$-parameterizations. In here the disturbance contributions will turn out to be identical, and they can be cancelled out. This will immediately cancel out the deficiency as well.

We let $\hat{P}_1$ and $\hat{P}_2$ be two nominal models of $P$ such that each of $P, \hat{P}_1$ and $\hat{P}_2$ is stabilized by $C$. For each of $\hat{P}_1$ and $\hat{P}_2$ we set up the $(R, S)$-parameterization. By (3.8) the $R$-parameter depends on the auxiliary rcf, thus we have different factors $R_1$ and
In contrast, by (3.30) the $S$-parameter does not depend on the auxiliary model. So $S$ is the same for the $(R, S)$-parameterizations corresponding to $\hat{P}_1$ and $\hat{P}_2$. Thereby we obtain the two $(R, S)$-systems

$$q_1 = R_1 x_1 + Sw$$
$$q_2 = R_2 x_2 + Sw$$

where $q_1, q_2$ and $x_1, x_2$ correspond to $q$ of (4.6) and $x$ of (4.4). By subtraction we can cancel out the disturbance contribution $Sw$:

$$q_1 - q_2 = R_1 x_1 - R_2 x_2.$$ 

The appearance of $R_1$ and $R_2$ is deceiving, since the equation does not conceal any information about the deficiency of either $\hat{P}_1$ or $\hat{P}_2$. In order to demonstrate the latter we re-examine $q_1$ and $R_1 x_1$ using (4.6), (3.8) and (4.4):

$$R_1 x_1 = (D_c + PN_c)^{-1}(P - \hat{P}_1)(I + C\hat{P}_1)^{-1}(u + Cy)$$
$$= D_c^{-1}[P(I + CP)^{-1} - \hat{P}_1(I + C\hat{P}_1)^{-1}](u + Cy)$$

and

$$q_1 = (D_c + \hat{P}_1 N_c)^{-1}(y - \hat{P}_1 u).$$

Similar relations hold for $R_2 x_2$ and $q_2$. Thus

$$R_1 x_1 - R_2 x_2 = D_c^{-1}[\hat{P}_2(I + C\hat{P}_2)^{-1} - \hat{P}_1(I + C\hat{P}_1)^{-1}](u + Cy)$$

and an analogous expression can be derived for $q_1 - q_2$. Consequently $q_1 - q_2$ and $R_1 x_1 - R_2 x_2$ merely contain information about the difference between $\hat{P}_1$ and $\hat{P}_2$, which implies that all information about $P$ has been cancelled out.
Appendix E

Robustness of the Framework for Open-loop Identification

In this appendix we investigate the robustness of the framework for open-loop identification of feedback controlled plants. This framework has been developed in Chapter 4 for real rational plants using exact knowledge of the stabilizing compensator. On the other hand a physical system is never truly a real rational system. It is reassuring to know, that this does not affect the practical utility of the fractional representations approach. In Section E.1 we will show that the framework for identification is still applicable if the system under investigation is infinite dimensional, time-varying or non-linear. For any such system the intermediate $z$ is a valid concept, it can be reconstructed from $u$ and $y$, and the resulting identification problem has an open-loop character. The only prerequisite is that the system in question is stabilized by a real rational controller. Unlike $(N^a, D^a)$ of (3.6) the associated rcf of a "non-real-rational" system is also non-real-rational. In such a case we need an identification technique that can deal with time-varying or non-linear systems.

In the subsequent section we examine the consequences of possibly inexact knowledge of the compensator. The compensator used in the reconstruction of $z$ may differ from the compensator, that actually stabilizes the plant during the experiment. We will show that a small difference between these two compensators incurs only small errors in the identification process. Besides, we have used the results of this analysis in Chapter 6 to develop a scenario for cautious controller enhancement.

We wish to state clearly that the material in this appendix is all but exhaustive and that several questions will be left unresolved. Meanwhile it serves its purpose in that the robustness of our framework for identification will be displayed either for inexact knowledge of the compensator or for time-varying non-linear systems. Similar result are likely to hold in case both perturbations occur simultaneously, but no proof of this presumptions is provided.
E.1 Infinite Dimensional, Time-varying and Non-linear Systems

For our purposes we have taken $\mathcal{H}_\infty$ as the principal ideal domain $\mathcal{H}$ of Definition 3.2.1. The resulting algebraic structure guarantees the existence of a coprime factorization of each system that belongs to $\mathcal{F}$. That is, each LTIFD system has a lcf and a rcf. This does not hold for infinite dimensional systems or time-varying non-linear systems in general. Therefore we question what the consequences are if the plant\(^1\) under consideration is an infinite dimensional system, denoted $P_{ID}$, or a non-linear possibly time-varying system, denoted $P_{NL}$. Meanwhile the compensator $C$ and the auxiliary model $P_o$ are still taken to be LTIFD systems. We confine the discussion to the single-variate feedback configuration as of Fig. 3.2 and we comment on the more general configurations in the end.

Several authors have investigated certain classes of systems that do have (coprime) factorizations. Here we will use the result that, loosely speaking, a plant has a (coprime) factorization if it is stabilized by a compensator that has a coprime factorization. The latter condition is in accordance with the basic assumption underlying the developments of Chapters 3 and 4: the plant is assumed to be stabilized by the LTIFD compensator $C$, which is known to have a coprime factorization. Consequently if $C \in \mathcal{F}$ stabilizes the plant under investigation, then the framework for identification of Chapter 4 is applicable.

**Infinite Dimensional Systems**

Various conditions guarantee the existence of a stable factorization of an infinite dimensional plant $P_{ID}$. For instance in [120] it was shown that a strictly proper plant admits a stable right factorization if and only if it admits a reachable and detectable strictly causal realization. According to [244] the factorability of the block-diagonal transfer matrix diag($P_{ID}, C_{ID}$) is guaranteed, if the feedback system $H(P_{ID}, C_{ID})$ is stable. In general this does not yet guarantee the existence of a factorization of either $P_{ID}$ or $C_{ID}$. An exception is the case in which $P_{ID}$ and $C_{ID}$ belong to the quotient field associated with $H_\infty$; then stability of $H(P_{ID}, C_{ID})$ does guarantee the individual factorability of $P_{ID}$ and $C_{ID}$ [212]. Similar results hold for linear finite dimensional time-varying systems (see e.g. [182]). Notice that the factorability of non-stabilizable systems remains unaddressed, but such systems are in conflict with our basic assumption anyway.

So far we reviewed some literature concerning the feedback system $H(P_{ID}, C_{ID})$. Now we discuss the implications for a feedback system $H(P_{ID}, C)$ composed of an infinite dimensional plant $P_{ID}$ and a LTIFD compensator $C$. First notice that a well-defined algebra\(^2\) of distributed systems encompasses the class of LTIFD systems as

---

\(^1\)We slightly abuse the notion of a plant as stated in Definition 2.1.1, because here the plant is not necessarily a LTIFD system.

\(^2\)A discussion on various algebras of linear distributed systems is beyond the scope of this robustness
a subset [34]. This means that any stabilizing LTIFD compensator has right and left coprime factorizations over such an algebra of linear distributed systems. Hence the stability of \( H(P_{ID}, C) \) implies, that the plant \( P_{ID} \) has right and left coprime factorizations by virtue of Lemma 8.3.2 of [239]. From here it is easy to generalize the \( R \)-parameterization of the set of all stabilized LTIFD plants to the case of infinite dimensional systems. In such a representation \( P \) is a LTIFD system just like \( C \), and the factor \( R \) of the associated rcf \( (N^a, D^a) \) is replaced by an infinite dimensional factor \( R_{ID} \), which gives rise to an infinite dimensional associated rcf \( (N_{ID}^a, D_{ID}^a) \). Consequently in the block-diagram of the \( R \)-parameterization (see Fig. 3.4) we only have to substitute \( R_{ID} \) for \( R \).

In Section 4.1 we demonstrated that the intermediate \( x \) does not depend on the term \( R \). In fact the signal \( Rx \) was cancelled out due to the positive feedback via \( C \) within the representation of \( P \) (recall Fig. 4.1). This cancellation is not altered if \( R \) is replaced by \( R_{ID} \). Hence we can still can construct \( x \) in case of an infinite dimensional plant \( P_{ID} \). Whether or not the resulting identification problem can be solved by means of standard identification methods depends on the situation at hand. This matter pertains to the problem of approximating an infinite dimensional system by a LTIFD system. For instance one can estimate a LTIFD model from \( u, y \) and \( x \) and thereby approximate the plant \( P_{ID} \).

**Time-varying, Non-linear Systems**

In order to appreciate the difficulties typically encountered in the analysis of non-linear feedback systems, we briefly examine a single-variate feedback system composed of non-linear operators. Let \( H(P_{NL}, C_{NL}) \) be such a feedback system with \( P_{NL} \) and \( C_{NL} \) non-linear operators interconnected as in the configuration of Fig. 3.2. For ease of notation we let \( r_2 \) be zero and thus \( u_c = -y \). Then the feedback system is described by

\[
\begin{align*}
  u &= r_1 - C_{NL} y \\
  y &= P_{NL} u.
\end{align*}
\]

In case we are interested in the plant output \( y \) it is tempting to write

\[
y = P_{NL}(r_1 - C_{NL} y).
\]

But from this equation we cannot solve \( y \) explicitly, since \( P_{NL}(r_1 - C_{NL} y) \) is generally unequal to \( P_{NL}r_1 - P_{NL}C_{NL} y \). Instead we should use \( u = r_1 - C_{NL} P_{NL} u \) to establish that \( u = (I + C_{NL} P_{NL})^{-1} r_1 \) and subsequently \( y = P_{NL} (I + C_{NL} P_{NL})^{-1} r_1 \). Notice that in view of the feedback system \( H(P_{NL}, C_{NL}) \) the expression \( (I + P_{NL} C_{NL})^{-1} P_{NL} \) has no meaning at all.

study. The interested reader is directed to [34, 35, 239, 39]. The property of interest here is that series, parallel and feedback interconnections of transfer functions in the algebra remain in the algebra.
The above conflict is a direct consequence of the fact that non-linear systems fail to satisfy the right-distributive property: \( A(B+C) \neq AB+AC \) [253]. On the other hand, non-linear systems satisfy all other axioms for a ring with identity [56]. Therefore parts of the algebraic theory for \( H(P,C) \) of Section 3.3 can be extended to non-linear systems by using a left-distributive algebra\(^3\) instead of a ring \( \mathcal{H} \) [56, 244]. The corresponding analysis of a non-linear feedback system goes as follows. Let \( K_{NL} = P_{NL}C_{NL} \) have a rcf \( (N_K, D_K) \) with right Bezout factors \( X_K, Y_K \). We wish to construct a rcf \( (N_T, D_T) \) of the “complementary sensitivity”, i.e.

\[
N_T D_T^{-1} = K_{NL} (I + K_{NL})^{-1} = N_K (D_K + N_K)^{-1}
\]

The second equivalence results from the substitution of \( N_K D_K^{-1} \) for \( K_{NL} \) and the fact that \((D_K + N_K) D_K^{-1})^{-1} = D_K (D_K + N_K)^{-1}\). The latter equality has been taken from [253, Section 2.5]. It appears natural to choose \( N_T = N_K \) and \( D_T = D_K + N_K \), but proving right coprimeness requires right-distributivity [56]. In case all terms are linear, then coprimeness follows from

\[
X_K N_K + Y_K D_K + Y_K N_K - Y_K N_K = I
\]

\[
(X_K - Y_K) N_K + Y_K (D_K + N_K) = I,
\]

so that \((X_K - Y_K)\) and \(Y_K\) are Bezout factors of \(N_K\) and \(D_K + N_K\). However in the non-linear case the latter identity is false because

\[
X_K N_K - Y_K N_K \neq (X_K - Y_K) N_K.
\]

Furthermore right-distributivity is also required for the \( R \)-parameterization of Section 3.3, and thus this parameterization does not carry over directly to non-linear systems. Nevertheless it was suggested in [56] that such results should hold, at least in part, for non-linear systems with an appropriate modification of the theory.

The difference between linear and non-linear systems is also reflected in the notion of stability. Stability and boundedness of linear systems imply that the corresponding transfer matrices have all their poles in the open left half-plane. In contrast there exist various notions of boundedness or stability for non-linear operators. Besides, these notions are equivalent in case the operator in question is linear [253]. Since the property of boundedness and/or stability underlies the concept of a coprime factorization (Definition 3.2.5), the latter is not uniquely defined in the non-linear case.

It may be clear that a general treatise on the factorization of non-linear systems is untractable. In fact literature provides several results for specific classes of non-linear systems\(^4\). Quite some attention has been payed to the coprime factorization of non-linear systems with certain state-space descriptions [52, 219, 37]. Further, in a series

\(^3\)An algebra \( A \) is left-distributive if \((A+B)C = AC + BC\) and \((\alpha A)B = \alpha (AB)\) for all \( A, B, C \in A \) and all scalars \( \alpha \) [253].

\(^4\)Classes of non-linear systems contain linear systems as special cases. The notification is thus a slight contradiction in terms.
of papers [96, 97, 98] Hammer studied a class of non-linear discrete-time systems, that admit a right coprime factorization. However, the utilized notion of stability does not relate to the non-linear analog of the feedback matrix \( T(P, C) \).

As in the case of linear distributed systems we do not require general results on non-linear feedback systems. Instead we are in search of a stability analysis, that is tailormade to the feedback system \( H(P_{NL}, C) \), which consists of a non-linear plant \( P_{NL} \) and a LTIFD compensator \( C \). Such results are provided by Desoer and Kabuli [53, 55] and Verma [236], who examined the dual feedback system \( H(P, C_{NL}) \) comprising a non-linear compensator and a linear plant. We adopt the results of Desoer and Kabuli to the case of \( H(P_{NL}, C) \) without discussing extended spaces. Also we assume that all systems are causal and we let \( \mathcal{H} \) be identical to \( \mathbb{R} \mathcal{H}_\infty \). Accordingly \( \mathcal{F} \) is the set of all LTIFD systems. The next definition establishes the notion of stability.

**Definition E.1.1** The non-linear plant \( P_{NL} \) is said to be stabilized by \( C \in \mathcal{F} \) in the single-variate feedback configuration \( H(P_{NL}, C) \) as in Fig. 3.2, if and only if \( H(P_{NL}, C) \) is well-posed and there exist continuous non-decreasing functions \( \Phi, \Phi_c : \mathbb{R}_+ \to \mathbb{R}_+ \) such that

\[
\|u\| \leq \Phi(\|r_1\| + \|r_2\|) \\
\|u_c\| \leq \Phi_c(\|r_1\| + \|r_2\|),
\]

(E.1)

in which \( \|\cdot\| \) is the norm of the respective vector (signal-)spaces.

The essence of this definition is that bounded exogenous inputs \( r_1 \) and \( r_2 \) should cause bounded \( u, u_c \) and hence bounded \( y, y_c \). Henceforth stability of a non-linear operator is meant in the sense of (E.1). The following lemma guarantees the existence of a rcf of \( P_{NL} \), provided that it is stabilized by \( C \).

**Lemma E.1.2** ([55]) The feedback system \( H(P_{NL}, C) \) with \( C \in \mathcal{F} \) is stable if and only if \( P_{NL} \) has a rcf \( (N_{NL}, D_{NL}) \) such that

\[
\tilde{N}_c N_{NL} + \tilde{D}_c D_{NL} = I
\]

in which \( (\tilde{D}_c, \tilde{N}_c) \) is a lef of \( C \).

Like in the linear case the rcf \( (N_{NL}, D_{NL}) \) satisfies the following properties: \( N_{NL} \) and \( D_{NL} \) are stable, \( D_{NL} \) is bijective and has a causal inverse, and \( P_{NL} = N_{NL} D_{NL}^{-1} \). The class of all rcf's of a stabilizable non-linear plant \( P_{NL} \) can be characterized like in Fact 3.2.6.i.

**Lemma E.1.3** ([55]) Let \( (N_{NL}, D_{NL}) \) be a rcf of \( P_{NL} \). Then \( (N_{NL}Q_{NL}, D_{NL}Q_{NL}) \) is a rcf of \( P_{NL} \) if and only if \( Q_{NL} \) is causal and stable and it has a causal and stable inverse.

We use the above Bezout identity and the class of all rcf's to extend the notion of an associated rcf to the non-linear case. Further we formulate a necessary and sufficient condition for stability of \( H(P_{NL}, C) \).
Theorem E.1.4 Let Assumptions 3.3.3 and 3.3.5 hold. Then the non-linear plant $P_{NL}$ is stabilized by the compensator $C \in \mathcal{F}$ in the single-variate feedback configuration $H(P_{NL}, C)$ as in Fig. 3.2, if and only if $P_{NL}$ has an associated rcf $(N^a_{NL}, D^a_{NL})$

\[
D^a_{NL} = D_o - N_o R_{NL} \\
N^a_{NL} = N_o + D_c R_{NL}
\]

(E.2)

with $R_{NL}$ a stable possibly non-linear operator such that $(D^a_{NL})^{-1}$ is causal.

Proof:

$(\Leftarrow)$ Completely analogous to the proof of Theorem 3.3.6 by virtue of the left-distributivity of non-linear operators.

$(\Rightarrow)$ By Lemma E.1.2 stability of $H(P_{NL}, C)$ implies the factorability of $P_{NL}$ into $(N^a_{NL}, D^a_{NL})$ such that $\tilde{N}_o N_{NL} + \tilde{D}_c D_{NL} = I$. We use $(N^a_{NL}, D^a_{NL})$ to choose $(N^a_{NL}, D^a_{NL})$ as

\[
D^a_{NL} = D_{NL} \Lambda_o, \quad N^a_{NL} = N_{NL} \Lambda_o.
\]

By Lemma E.1.3 this pair $(N^a_{NL}, D^a_{NL})$ is a rcf of $P_{NL}$ because $\Lambda_o \in \mathcal{J}$. It remains to be proven that $R_{NL}$ of (E.2) is stable and causal. To this end we substitute (E.2) for $(N^a_{NL}, D^a_{NL})$ in the difference $\tilde{D}_o N^a_{NL} - \tilde{N}_o D^a_{NL}$. This yields

\[
\tilde{D}_o N^a_{NL} - \tilde{N}_o D^a_{NL} = \tilde{D}_o (N_o + D_c R_{NL}) - \tilde{N}_o (D_o - N_c R_{NL}).
\]

Next we use the fact that the linear operators $\tilde{D}_o, \tilde{N}_o$ are not only left-distributive but also right-distributive; e.g.

\[
\tilde{D}_o (N_o + D_c R_{NL}) = \tilde{D}_o N_o + \tilde{D}_o D_c R_{NL}.
\]

Hereby we can write $R_{NL}$ as

\[
R_{NL} = \tilde{\Lambda}_o^{-1} (\tilde{D}_o N^a_{NL} - \tilde{N}_o D^a_{NL}) \Lambda_o,
\]

which shows that $R_{NL}$ is indeed stable and causal. 

The above results say that the $R$-parameterization of Fig. 3.2 is applicable to a plant $P_{NL}$, provided that it is stabilized by a LTIFD compensator. Such an $R$-parameterization of a non-linear plant $P_{NL}$ is the same as in the linear case, except that the LTIFD factor $R$ is replaced by some non-linear causal stable map $R_{NL}$. Like in the case of infinite dimensional systems the intermediate $z$ can be reconstructed, and the identification of $N^a_{NL}$ and $D^a_{NL}$ from $u, y$ and $z$ has an open-loop character. If $u, y$ and $z$ are used to estimate linear models, then the outcome is simply a linearization of the measured behavior. Depending on the situation at hand such a linear model may be appropriate for its purpose. For example in Chapter 9 linear models have been derived for a physical system, which is not truly linear, and the models turned out to be suitable for control design.
Finally we comment on the feedback configurations \( H(P_T, C) \) and \( H(P_{TT}, C_{TT}) \). At the current state of affairs there exists no algebraic theory for non-linear feedback systems with a configuration other than the single-variate feedback system. A straightforward generalization of the algebraic theory for linear feedback systems seems to be impossible, because there exists no Hermite form for the class of non-linear systems. This form has been used in Section 3.4 to derive a structured coprime factorization of the plant \( P_{TT} \). On the other hand all instabilities of the plant must eventually be contained in the loop in order that the feedback system is stable. Hence in some sense also non-linear plants have to be admissible in order to be stabilizable in the configuration of \( H(P_T, C) \) or \( H(P_{TT}, C_{TT}) \). For now, we simply assume that a non-linear plant \( P_{TNL} \) under investigation admits an \((R, S)\)-parameterization as in Fig. 3.4, except that \( R \) and \( S \) are replaced by some non-linear stable maps \( R_{NL} \) and \( S_{NL} \). It is easy to show that the framework for identification of Proposition 4.2.1 is applicable to this class of plants. It remains unclear whether or not there exist two-input one-output plants that are stabilized by some LTID compensator in the standard feedback configuration, but that do not allow an \((R, S)\)-parameterization.

### E.2 Inexact Knowledge of the Compensator

The \((R, S)\)-parameterization of Chapter 3 and the corresponding open-loop identification problems of Chapter 4 rest on the availability of the compensator, that stabilizes the plant during the experiment. This assumption does not incur severe restrictions in case we have designed the compensator ourselves. Nevertheless there may be a small disparity between the designed compensator and its implemented version. For instance in Chapter 9 the designed continuous-time compensators were implemented on a digital signal processor at a high sampling-rate.

Here we analyse how inexact knowledge of the compensator affects the equivalent identification problem of Section 4.2. First we represent both the actual plant and the actual compensator by means of an auxiliary model and an auxiliary compensator. This is called the double \((R_o, S_o)\)-parameterization. Thereafter we will show that the framework for open-loop identification of Proposition 4.2.1 is robust in the sense that a small "error" of the compensator used in the \((R, S)\)-parameterization causes only small errors in the estimated model.

#### The double \((R_o, S_o)\)-parameterization

We perform our analysis in terms of the feedback system \( H(P_T, C_T) \) of Fig. E.1. In view of the closed-loop identification problem this feedback system is of equal complexity as \( H(P_{TT}, C_{TT}) \), but \( H(P_T, C_T) \) is a lot simpler in notation. Further, \( H(P_T, C_T) \) encompasses the standard feedback system \( H(P_T, C) \) as a special case. The latter unsuited to reveal the duality between the plant and the compensator.
If the stabilizing compensator $C_T$ had been known, then we could use its inner-loop part $C$ to parameterize the plant $P_T$ with uncertain dynamics. Instead we only have a “model” of this compensator. We denote the inner-loop part of this “model” as the auxiliary compensator $C_o$. This auxiliary compensator will turn out to be completely dual to the auxiliary model $P_o$. Further, we emphasize on the difference between $C_o$ and $C$ (or $C_T$) by calling the latter the actual compensator. As $C_T$ and $P_T$ are uncertain or unknown, the only thing that we can do, is to represent them in terms of “$(R, S)$”-parameterizations based on $P_o$ and $C_o$. We will fix some properties of $P_o$ and $C_o$ for ease of reference.

**Assumption E.2.1** The auxiliary model $P_o \in \mathcal{F}$ and the auxiliary compensator $C_o \in \mathcal{F}$ have the following properties.

i. $H(P_o, C_o)$ is stable.

ii. $P_o$ has a rcf $(N_o, D_o)$ and a lcf $(\tilde{D}_o, \tilde{N}_o)$.

iii. $C_o$ has a rcf $(N_{co}, D_{co})$ and a lcf $(\tilde{D}_{co}, \tilde{N}_{co})$.

Analogous to (3.5) we define

$$\begin{bmatrix} \Lambda_{oco} & 0 \\ 0 & \hat{\Lambda}_{oco} \end{bmatrix} = \begin{bmatrix} \tilde{D}_{co} & \tilde{N}_{co} \\ -\tilde{N}_o & \tilde{D}_o \end{bmatrix} \begin{bmatrix} D_o & -N_{co} \\ N_o & D_{co} \end{bmatrix}$$

(E.3)

and by Assumption E.2.1 and Lemma 3.3.4 $\Lambda_{oco} \in \mathcal{J}$ and $\hat{\Lambda}_{oco} \in \mathcal{J}$.

Now we can express the plant $P_T$ in terms of the factorization $(N_T^{ao}, D_T^{ao})$, i.e.

$$(N_T^{ao}, D_T^{ao}) \doteq \begin{bmatrix} I & 0 \\ -N_{co}S_o & D^{ao} \end{bmatrix},$$

(E.4)

which is equivalent to the associated rcf $(N_T^o, D_T^o)$ of (3.34), except that all terms concerning $C$ have been replaced by those corresponding to $C_o$. Similarly we obtain the following expressions for the various entries of $(N_T^{ao}, D_T^{ao})$. Like (3.6) we define $(N^{ao}, D^{ao})$ as

$$D^{ao} \doteq D_o - N_{co}R_o$$

$$N^{ao} \doteq N_o + D_{co}R_o$$

(E.5)
and \((N^{ao}, D^{ao})\), \(R_o\) and \(S_o\) are related to \(P\), \(C_o\), \((N_{co}, D_{co})\) and \((N_o, D_o)\) through
\[
\begin{align*}
D^{ao} &= (I + C_o P)^{-1}(D_o + C_o N_o) \\
N^{ao} &= P(I + C_o P)^{-1}(D_o + C_o N_o) \\
R_o &= D_{co}^{-1}(I + PC_o)^{-1}(PD_o - N_o) \\
S_o &= D_{co}^{-1}(I + PC_o)^{-1}P_yw
\end{align*}
\] (E.6)

which follow straightforwardly from (3.11), (3.13), (3.8) and (3.30). A dual treatment of the compensator results in the factorization \((N_{ct}^{ao}, D_{ct}^{ao})\):
\[
(N_{ct}^{ao}, D_{ct}^{ao}) = \begin{bmatrix} D_o S_{co} & N_c^{ao} \end{bmatrix} \begin{bmatrix} I & 0 \\ -N_o S_{co} & D_c^{ao} \end{bmatrix}
\] (E.7)

with
\[
\begin{align*}
D_c^{ao} &= D_{co} - N_o R_{co} \\
N_c^{ao} &= N_{co} + D_o R_{co}
\end{align*}
\] (E.8)

and all terms are related to \(C\), \(P\), \((N_o, D_o)\) and \((N_{co}, D_{co})\) through
\[
\begin{align*}
D_c^{ao} &= (I + P_c C)^{-1}(D_{co} + P_c N_{co}) \\
N_c^{ao} &= C(I + P_c C)^{-1}(D_{co} + P_c N_{co}) \\
R_{co} &= D_o^{-1}(I + CP_o)^{-1}(CD_{co} - N_{co}) \\
S_{co} &= D_o^{-1}(I + CP_o)^{-1}C_yw
\end{align*}
\] (E.9)

This double \((R_o, S_o)\)-parameterization of \(P_T\) and \(C_T\) has been depicted in Fig. E.2. How can we interpret this block-diagram? For a start, if \(R_{co} = 0\) then Fig. E.2 is similar to the \((R, S)\)-parameterization of Section 3.5. That is, if we replace \(R_o\) and \(S_o\)
by $R_{\mathcal{H}}$ and $S_{\mathcal{H}}$, then, still with $R_{co}=0$, we obtain the parameterization of all plants that are stabilized by $C_o$. The other way round, if the “plant-parameter” $R_o$ equals 0 and we replace the compensator terms $R_{co}$ and $S_{co}$ by $R_{\mathcal{H}}$ and $S_{\mathcal{H}}$, then we get the parameterization of all compensators that stabilize $P_o$. Both these properties are lost of $R_o \neq 0$ and $R_{co} \neq 0$.

Of course we will choose the auxiliary model $P_o$ and the auxiliary compensator $C_o$ such that $H(P_o, C_o)$ is stable. However, this auxiliary feedback system $H(P_o, C_o)$ does not necessarily relate to the actual feedback system $H(P_T, C_T)$! The reason is that we can freely choose $P_o$ and $C_o$ as long as $H(P_o, C_o)$ is stable. Without additional assumptions there is no guarantee that either of the feedback systems $H(P_T, C_o)$ and $H(P_o, C_T)$ is stable. In fact these feedback systems might even be ill-posed, i.e. inverses like $(I + P_o C)^{-1}$ possibly do not exist. On the one hand this would make the above expressions in (E.6) and (E.9) meaningless. On the other hand this is a rather technical problem with little practical significance. Therefore we exclude ill-posedness by the following assumption.

**Assumption E.2.2** The actual inner-loop plant $P$ and compensator $C$ and the auxiliary model $P_o$ and compensator $C_o$ satisfy

$$\det(I + P_C C)^{-1} \neq 0, \quad \det(I + C P_o)^{-1} \neq 0$$

$$\det(I + P C_o)^{-1} \neq 0, \quad \det(I + C_o P)^{-1} \neq 0.$$

Notice that this assumption is weaker than the usual assumption of well-posedness, which requires the determinants to be non-zero at infinite frequency [60]. By Assumption E.2.2 we can construct the terms in (E.6) and (E.9) and compose the pairs $(N_T^{ao}, D_T^{ao})$ and $(N_T^{co}, D_T^{co})$ of (E.4) and (E.7). Despite their resemblance to $(N_T^{ao}, D_T^{ao})$ of (3.34) the pairs $(N_T^{ao}, D_T^{ao})$ and $(N_T^{co}, D_T^{co})$ are possibly not right coprime. We demonstrate this fact for the pair $(N_T^{ao}, D_T^{ao})$; the pair $(N_T^{co}, D_T^{co})$ can be treated analogously.

According to (E.3) the rcf $(N_o, D_o)$ has $X_o = \Lambda_{ao}^{-1} \tilde{N}_{co}$ and $Y_o = \Lambda_{ao}^{-1} \tilde{D}_{co}$ as right Bezout factors. With these $X_o, Y_o$ we can build $X_T, Y_T$ as

$$X_T = \begin{bmatrix} 0 \\ X_o \end{bmatrix}, \quad Y_T = \begin{bmatrix} I & 0 \\ 0 & Y_o \end{bmatrix}$$

and straightforward calculation shows that

$$X_T N_T^{ao} + Y_T D_T^{ao} = I.$$

This looks like the Bezout identity of (3.1) and thus the pair $(N_T^{ao}, D_T^{ao})$ seems to be a right coprime factorization, because $X_T$ and $Y_T$ belong to $\mathcal{H}$. However also $N_T^{ao}$ and

---

5As in Chapter 3 the subscript $\mathcal{H}$ is taken to mean that the corresponding factor is a free “param- eter” in the space $\mathcal{H}$, possibly except for some singular points.
$D^o_T$ should belong to $\mathcal{H}$, and that is not true if $H(P_T, C_o)$ is unstable. This can be seen as follows. If $P_T$ is admissible and its inner-loop part $P$ has a rcf $(N, D)$, then by Corollary 3.4.10 instability of $H(P_T, C_o)$ implies that $\Lambda_{co}$, defined as

$$\Lambda_{co} = \tilde{D}_{co} D + \tilde{N}_{co} N,$$

has no stable inverse. Further, the expression for $D^{ao}$ and $N^{ao}$ of (E.6) can be rewritten as

$$D^{ao} = D\Lambda_{co}^{-1}\Lambda_{occo},$$
$$N^{ao} = N\Lambda_{co}^{-1}\Lambda_{occo}.$$

Now instability of $(N^{ao}, D^{ao})$ can be demonstrated by contradiction: let $X, Y$ be the Bezout factors of $(N, D)$, then

$$(YD^{ao} + XN^{ao})\Lambda_{occo}^{-1} = \Lambda_{co}^{-1}$$

should be stable since $X, Y, \Lambda_{occo}^{-1} \in \mathcal{H}$. However we just established that $\Lambda_{co}^{-1} \notin \mathcal{H}$.

In conclusion we infer that the four terms $D^{ao}$, $N^{ao}$, $R_o$ and $S_o$ of (E.6) are potentially unstable and thus the representation of $P_T$ in Fig. E.2 is not an $(R, S)$-parameterization in the sense of Corollary 3.5.2 if $H(P_T, C_o)$ is unstable. A similar conclusion can be drawn for the representation of $C_T$ if $H(P_o, C_T)$ is unstable.

Notwithstanding the possibility that $H(P_o, C_o)$ differs from $H(P_T, C_T)$, the double $(R_o, S_o)$-parameterization of Fig. E.2 can be related to the stability of $H(P_T, C_T)$. To that end we introduce the intermediate feedback system $H([S_o, R_o], [S_{co}, R_{co}])$, which is composed of the "$R$" and "$S$" parameters of Fig. E.2. A block-diagram of this feedback system is given in Fig. E.3. A comparison of the latter block-diagram with that of Fig. E.1 reveals that $H([S_o, R_o], [S_{co}, R_{co}])$ is just another feedback system with a configuration like $H(P_T, C_T)$. It is easy to verify from (3.33) that its transfer function $T([S_o, R_o], [S_{co}, R_{co}])$ mapping $\text{col}(w_c, w, \zeta_2, \zeta_1)$ into $\text{col}(q_o, x_o)$ equals

$$T([S_o, R_o], [S_{co}, R_{co}]) =$$
\[
\begin{bmatrix}
0 & S_o & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
R_o \\
I
\end{bmatrix} (I + R_{co} R_o)^{-1} \begin{bmatrix}
S_{co} & -R_{co} S_o & R_{co} & I
\end{bmatrix}.
\] (E.10)

Conformably with Proposition 3.4.2 the intermediate feedback system is stable if and only if its transfer function \( T([S_o, R_o], [S_{co}, R_{co}]) \) is stable. The next proposition discloses the connection between the double \((R_o, S_o)\)-parameterization and the intermediate feedback system.

**Proposition E.2.3** The double \((R_o, S_o)\)-parameterization of Fig. E.2 is related to the intermediate feedback system \( H([S_o, R_o], [S_{co}, R_{co}]) \) of Fig. E.3 through

\[
\begin{pmatrix}
\zeta_2 \\
\zeta_1
\end{pmatrix} = \begin{bmatrix}
D_{co} & N_o \\
-N_{co} & D_o
\end{bmatrix}^{-1} \begin{pmatrix}
r_2 \\
r_1
\end{pmatrix}.
\] (E.11)

**Proof:** From Fig. E.2 we derive

\[
\begin{pmatrix}
y \\
u
\end{pmatrix} = \begin{bmatrix}
D_{co} & N_o \\
-N_{co} & D_o
\end{bmatrix} \begin{pmatrix}
q_o \\
x_o
\end{pmatrix},
\] (E.12)

and

\[
\begin{pmatrix}
y \\
u
\end{pmatrix} = \begin{pmatrix}
r_1 \\
r_2
\end{pmatrix} + \begin{bmatrix}
D_{co} & N_o \\
-N_{co} & D_o
\end{bmatrix} \begin{pmatrix}
-x_{co} \\
q_{co}
\end{pmatrix}.
\] (E.13)

Elimination of \( \text{col}(y, u) \) and premultiplication by \( \begin{bmatrix} \tilde{D}_{o} & -\tilde{N}_{o} \\ \bar{N}_{co} & \bar{D}_{co} \end{bmatrix} \) yields

\[
\begin{bmatrix}
\tilde{A}_{oco} & 0 \\
0 & \Lambda_{oco}
\end{bmatrix} \begin{pmatrix}
q_o \\
x_o
\end{pmatrix} = \begin{bmatrix}
\tilde{D}_{o} & -\tilde{N}_{o} \\
\bar{N}_{co} & \bar{D}_{co}
\end{bmatrix} \begin{pmatrix}
r_2 \\
r_1
\end{pmatrix} + \begin{bmatrix}
\Lambda_{oco} & 0 \\
0 & \Lambda_{oco}
\end{bmatrix} \begin{pmatrix}
-x_{co} \\
q_{co}
\end{pmatrix},
\]

which can be rewritten into

\[
\begin{pmatrix}
x_{co} \\
x_o
\end{pmatrix} = \begin{bmatrix}
\tilde{A}_{oco}^{-1} & 0 \\
0 & \Lambda_{oco}^{-1}
\end{bmatrix} \begin{bmatrix}
\tilde{D}_{o} & -\tilde{N}_{o} \\
\bar{N}_{co} & \bar{D}_{co}
\end{bmatrix} \begin{pmatrix}
r_2 \\
r_1
\end{pmatrix} + \begin{pmatrix}
-x_{co} \\
q_{co}
\end{pmatrix},
\] (E.14)

and making use of Fig. E.3 we obtain

\[
\begin{pmatrix}
\zeta_2 \\
\zeta_1
\end{pmatrix} = \begin{bmatrix}
\tilde{A}_{oco}^{-1} & 0 \\
0 & \Lambda_{oco}^{-1}
\end{bmatrix} \begin{bmatrix}
\tilde{D}_{o} & -\tilde{N}_{o} \\
\bar{N}_{co} & \bar{D}_{co}
\end{bmatrix} \begin{pmatrix}
r_2 \\
r_1
\end{pmatrix} + \begin{pmatrix}
-x_{co} \\
q_{co}
\end{pmatrix},
\] (E.15)

which completes the proof.
Now we can derive the following necessary and sufficient condition for stability of the feedback system $H(P_T, C_T)$.

**Theorem E.2.4** Let Assumptions E.2.1 and E.2.2 hold. Let $P_T, C_T \in \mathcal{F}$ be represented by the double $(R_o, S_o)$-parameterization of Fig. E.2. Then the feedback system $H(P_T, C_T)$ is stable if and only if the corresponding intermediate feedback system $H([S_o, R_o], [S_{co}, R_{co}])$ of Fig. E.3 is stable.

**Proof:** By Proposition 3.4.2 it is sufficient to prove that $T(P_T, C_T)$ is stable if and only if $T([S_o, R_o], [S_{co}, R_{co}])$ of (E.10) is stable. Therefore we express the former in terms of the latter. $T(P_T, C_T)$ maps $\text{col}(w_c, w, r_2, r_1)$ into $\text{col}(y, u)$. By (E.12) $\text{col}(y, u)$ is the image of $\text{col}(q_o, x_o)$. The latter equals

$$T([S_o, R_o], [S_{co}, R_{co}]) \cdot \text{col}(w_c, w, \zeta_2, \zeta_1),$$

i.e. the output of the intermediate feedback system. Combining this with (E.11) we can write $T(P_T, C_T)$ as

$$T(P_T, C_T) = \begin{bmatrix} D_{co} & N_o \\ -N_{co} & D_o \end{bmatrix} T([S_o, R_o], [S_{co}, R_{co}]) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 \\ \begin{bmatrix} D_{co} & N_o \\ -N_{co} & D_o \end{bmatrix}^{-1} \end{bmatrix}$$

and the inverse of the block-matrix on the right is stable by virtue of (E.15).

$(\Leftarrow)$ If $T([S_o, R_o], [S_{co}, R_{co}]) \in \mathcal{H}$ then all three elements in the above expression are stable and thus $T(P_T, C_T) \in \mathcal{H}$.

$(\Rightarrow)$ Define

$$A \doteq \begin{bmatrix} D_{co} & N_o \\ -N_{co} & D_o \end{bmatrix}^{-1}, \quad B \doteq \text{diag}(I, I, \begin{bmatrix} D_{co} & N_o \\ -N_{co} & D_o \end{bmatrix}),$$

so that $T([S_o, R_o], [S_{co}, R_{co}]) = A \cdot T(P_T, C_T) \cdot B$. By (E.15) both terms $A, B$ belong to $\mathcal{H}$. So $T(P_T, C_T) \in \mathcal{H}$ implies that $T([S_o, R_o], [S_{co}, R_{co}]) \in \mathcal{H}$. □

**Remark E.2.5** Theorem E.2.4 generalizes Theorem 2.1 of Tay et al. [227], who studied the interconnection of $H(P, C)$ instead of the more general $H(P_T, C_T)$. Their use of this theorem is commented upon in Chapter 6.

The connection between $H(P_T, C_T)$ and $H([S_o, R_o], [S_{co}, R_{co}])$ becomes even more clear if $T(P_T, C_T)$ is represented as a perturbation of the so-called double-auxiliary feedback system $H(P_o, C_o)$. To this end we substitute $x_o = q_{co} + \zeta_1$ in (E.12) and obtain

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} D_{co} & N_o \\ -N_{co} & D_o \end{bmatrix} \begin{bmatrix} q_o \\ q_{co} \end{bmatrix} + \begin{bmatrix} N_o \\ D_o \end{bmatrix} A_{co}^{-1} \begin{bmatrix} \tilde{N}_{co} & \tilde{D}_{co} \end{bmatrix} \begin{bmatrix} r_2 \\ r_1 \end{bmatrix} = \begin{bmatrix} D_{co} & N_o \\ -N_{co} & D_o \end{bmatrix} \begin{bmatrix} q_o \\ q_{co} \end{bmatrix} + T(P_o, C_o) \begin{bmatrix} r_2 \\ r_1 \end{bmatrix}$$
Fig. E.4: The feedback system $H(P_T, C_T)$ as a perturbation of the double-auxiliary feedback system $H(P_o, C_o)$.

by using (E.14) for $\zeta_1$ and (3.4) for $T(P_o, C_o)$. We can take $\text{col}(q_o, q_{co})$ as the output of the intermediate feedback system, by which $H(P_T, C_T)$ is decomposed into the sum of the double-auxiliary feedback system $H(P_o, C_o)$ and the pre- and post-multiplied intermediate feedback system $H([S_o, R_o], [S_{co}, R_{co}])$. This decoupling is depicted in Fig. E.4.

Presumed Open-loop Identification

The equivalent and associated identification problems of Chapter 4 have an open-loop character provided that the actual compensator $C$ is used for the reconstruction of the intermediate. However there will practically always be some difference between the designed compensator and its implemented version. As the designed compensator is at our disposal, we use it for the parameterization of the uncertain plant. Hence this
compensator plays the role of the auxiliary compensator $C_0$ in the double $(R_o, S_o)$-parameterization of Fig. E.2. The implemented compensator is the actual compensator $C_T$, which is represented as a perturbation of the auxiliary $C_0$. We analyze qualitatively and subsequently quantitatively how the open-loop identification is affected by the use of $C_o$ rather than $C$.

Setting up the equivalent and associated identification problems we can construct $q_o$ and $x_o$ in the same way as $q$ in (4.6) and $x$ in (4.4). That is, we can use the inverse of (E.15) and the relationship in (E.12) to derive

$$
\begin{pmatrix}
q_o \\
x_o
\end{pmatrix} = \begin{bmatrix}
\tilde{\Lambda}_o^{-1} & 0 \\
0 & \Lambda_o^{-1}
\end{bmatrix} \begin{bmatrix}
\tilde{D}_o & -\tilde{N}_o \\
\tilde{N}_o & \tilde{D}_o
\end{bmatrix} \begin{pmatrix}
y \\
u
\end{pmatrix}.
$$

Besides, from (E.13) and (E.14) it follows that none of the variables $x_{co}, q_{co}, \zeta_2$ or $\zeta_1$ can be reconstructed from $u, y$ without additional information on $r_1$ and/or $r_2$.

The associated identification problem concerns the identification of $S_o$ and $R_o$ from

$$q_o = R_o x_o + S_o w. \tag{E.16}$$

With a glance at Fig. E.3 it becomes clear that $x_o$ and $w$ are correlated due to the feedback by $R_{co}$. Hence the identification of $R_o$ and $S_o$ from $x_o$ and $q_o$ is a typical closed-loop identification problem unless $R_{co} = 0$. Of course this holds also for the identification of $(N^{ao}, D^{ao})$ from $y, u$ and $x$ and

$$
y = N^{ao} x_o + D^{ao} S_o w \quad \text{and} \quad u = D^{ao} x_o - N^{ao} S_o w.
$$

Thus the open-loop character of the identification problems in Sections 4.2 and 4.3 is not a robust property in view of inexact knowledge of the compensator. On the other hand in the identification community it is well-known that a small feedback contribution does not deteriorate the identification process. That is, if $x_o \approx \zeta_1$ in Fig. E.3, than the model estimated from $x_o$ and $q_o$ is a good approximation of the model that would have been derived if $R_{co} = 0$, i.e. $x_o = \zeta_1$.

We quantify the above qualitative observation in terms of a spectral analysis of a SISO plant-compensator pair. From (E.12) and (E.16) we derive

$$
\begin{pmatrix}
y \\
u
\end{pmatrix} = \begin{bmatrix}
N^{ao} \\
D^{ao}
\end{bmatrix} x_o + \begin{bmatrix}
D_{co} \\
-N_{co}
\end{bmatrix} S_o w.
$$

With (E.10) and (E.11) we can write $x_o$ as

$$
x_o = (I + R_{co} R_o)^{-1} \left\{ S_{co} w_c - R_{co} S_o w + \begin{bmatrix}
R_{co} \\
1
\end{bmatrix} \begin{bmatrix}
D_{co} & N_o \\
-N_{co} & D_o
\end{bmatrix}^{-1} \begin{pmatrix}
r_2 \\
r_1
\end{pmatrix} \right\}.
$$
Defining
\[ \rho = S_{co}w_c + \begin{bmatrix} R_{co} & I \end{bmatrix} \begin{bmatrix} D_{co} & N_o \\ -N_{co} & D_o \end{bmatrix}^{-1} \begin{bmatrix} r_2 \\ r_1 \end{bmatrix}, \]
we get
\[ \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} N^{ao} \\ D^{ao} \end{bmatrix}(I + R_{co}R_o)^{-1}\rho + \begin{bmatrix} D^{ao}_c \\ -N^a_o \end{bmatrix}(I + R_oR_{co})^{-1}S_ow \]
\[ x_o = (I + R_{co}R_o)^{-1}(\rho - R_{co}S_ow). \]

Notice that \( \rho \) is uncorrelated with \( w \), but \( x \) is correlated with \( w \) if \( R_{co} \neq 0 \). Presuming that we have an open-loop identification problem, we would derive models \( \hat{N}^{ao}, \hat{D}^{ao} \) for the factors \( N^{ao}, D^{ao} \) as
\[ \hat{N}^{ao} = \Phi_{y|x_o}/\Phi_{x_o|x_o} \]
\[ \hat{D}^{ao} = \Phi_{u|x_o}/\Phi_{x_o|x_o} \]
where \( \Phi_{y|x_o}, \Phi_{u|x_o} \) are cross-spectra and \( \Phi_{x_o|x_o} \) is an auto-spectrum [180]. In order to express these spectra in terms of \( \rho \) and \( w \) we invoke the spectral analysis of bivariate linear processes of [119]. This analysis says that if
\[ \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \]
with \( \nu_1, \nu_2 \) being uncorrelated white noise processes with unit variances, then
\[ \Phi_{\tau_1\tau_1} = |K_{11}|^{-1} + |K_{12}|^{-1} \]
\[ \Phi_{\tau_2\tau_2} = |K_{21}|^{-1} + |K_{22}|^{-1} \]
\[ \Phi_{\tau_1\tau_2} = K_{22}^* K_{12} + K_{21}^* K_{11} \]
where all entries are functions of frequency. We express \( \rho \) and \( w \) in terms of some filtered uncorrelated white noise processes \( \eta_\rho \) and \( \eta_w \):
\[ \rho = F_\rho \eta_\rho \]
\[ w = F_w \eta_w. \]

Before applying these tools we filter \( y, u \) and \( x_o \) by \( (I + R_{co}R_o) \), which does not affect the asymptotic estimates \( \hat{N}^{ao}, \hat{D}^{ao} \), but which simplifies notation somewhat. The filtered output, input and intermediate are
\[ \begin{bmatrix} y' \\ u' \\ x_o' \end{bmatrix} = \begin{bmatrix} N^{ao}F_\rho \\ D^{ao}F_\rho \\ F_\rho \end{bmatrix} \eta_\rho + \begin{bmatrix} D^{ao}_c S_o F_w \\ -N^a_o S_o F_w \\ -R_{co}S_o F_w \end{bmatrix} \eta_w. \]
By straightforward calculation we achieve

\[ \Phi'_{y_2z} = |F_\rho|^2 N^{a_0} + (-R_{co} S_\omega F_\omega)^* (D^{a_0}_{c} S_\omega F_\omega) \]
\[ \Phi'_{u_2z} = |F_\rho|^2 D^{a_0} + (-R_{co} S_\omega F_\omega)^* (-N^{a_0}_c S_\omega F_\omega) \]
\[ \Phi'_{z_2z} = |F_\rho|^2 + |R_{co} S_\omega F_\omega|^2 \]

and thus

\[ \widehat{N}^{a_0} = \frac{N^{a_0} - R_{co}^* (D_{co} - N_\omega R_{co}) |S_\omega F_\omega|}{1+|R_{co}|^2 |S_\omega F_\omega|^2/|F_\rho|^2} \]
\[ \widehat{D}^{a_0} = \frac{D^{a_0} + R_{co}^* (N_{co} + D_\omega R_{co}) |S_\omega F_\omega|}{1+|R_{co}|^2 |S_\omega F_\omega|^2/|F_\rho|^2} . \]

Clearly \( \widehat{N}^{a_0} = N^{a_0} \) and \( \widehat{D}^{a_0} = D^{a_0} \) if \( R_{co} = 0 \). Moreover even if \( R_{co} \) is large at frequencies where the signal-to-noise ratio \( |F_\rho|^2/|S_\omega F_\omega|^2 \) is large, then still \( \widehat{N}^{a_0} \approx N^{a_0} \) and \( \widehat{D}^{a_0} \approx D^{a_0} \). And finally if \( R_{co} \) tends to zero in a sensible topology, then \((\widehat{N}^{a_0}, \widehat{D}^{a_0})\) converges to \((N^{a_0}, D^{a_0})\). We formalize this property by the following conjecture.

**Conjecture E.2.6** The asymptotic bias distribution of the models estimated by means of the framework for open-loop identification of Proposition 4.2.1 is robust in view of inexact knowledge of the compensator used in the experiments.

This conjecture claims that the inner-loop plant can be estimated with an arbitrary asymptotic accuracy provided that the difference between the coprime factors of the designed and implemented compensators is made arbitrarily small. Finally we remark that in addition to \( u \) and \( y \) information on \( r_1 \) and/or \( r_2 \) is required to find out from Fig. E.2 whether or not there is a difference between the designed and implemented compensators. Of course one could also "identify" the implemented compensator from open-loop measurements before hooking it on the actual plant.
Appendix F

Least Squares
Frequency Response Identification

F.1 Iterative Search Method for Non-linear Criteria

The non-linear criteria discussed in Chapter 7 are all optimized via the same strategy. We explain this strategy from the criterion $J(\theta)$, which depends on the vector $\theta$ of parameters in $\mathbb{R}$. Our solution is based on the Newton-Raphson (or quasi-Newton) method supplied with a backtracking line-search [45].

The Newton-Raphson method is founded on the second order polynomial function

$$f(x) = a_2 x^2 + a_1 x + a_0$$

and its derivatives

$$f'(x) = 2a_2 x + a_1$$
$$f''(x) = 2a_2.$$  

The three parameters $a_0, a_1$ and $a_2$ can be determined if $f(x)$, $f'(x)$ and $f''(x)$ are known for (at least) one value of $x$. We denote the latter $x_o$. From $f(x_o)$, $f'(x_o)$ and $f''(x_o)$ we can determine

$$a_2 = 0.5f''(x_o)$$
$$a_1 = f'(x_o) - x_o f''(x_o)$$
$$a_0 = f(x_o) - x_o f'(x_o) + 0.5 x_o^2 f''(x_o).$$

Since $f(x)$ takes its minimum if $f'(x) = 0$, its minimizing argument can be determined as

$$x_{min} = \frac{-a_1}{2a_2} = x_o - f'(x_o)/f''(x_o).$$

In summary the minimum of the function $f(x)$ can be localized from any $x_o$ and its associated first and second derivatives.
In the Newton-Raphson minimization method the above procedure is applied also to functions, that are not quadratic in the indeterminate. We let \( g(x) \) be such a function and for some \( x_0 \) we determine \( g'(x_0) \) and \( g''(x_0) \). We calculate the three parameters \( a_0, a_1 \) and \( a_2 \) as in the case of \( f(x) \), and we define

\[
\hat{g}(x) = a_2 x^2 + a_1 x + a_0.
\]

The function \( \hat{g}(x) \) is called a *model* of the function \( g(x) \) at \( x_0 \) [45]. This function takes its minimum for

\[
x_{\text{min}} = x_0 - g'(x_0)/g''(x_0).
\]

This minimizing argument \( x_{\text{min}} \) of \( \hat{g}(x) \) is used as an approximation of the minimizing argument of the function \( g(x) \). Repeated application yields the iterative search procedure

\[
x_{k+1} = x_k - g'(x_k)/g''(x_k),
\]

which converges to a (local) minimum of \( g(x) \) provided that \( x_0 \) belongs to a convergence region.

The model \( \hat{g}(x) \) often is not appropriate, when \( x_0 \) is not close to the minimizing argument of \( g(x) \). In that case \( g(x_{k+1}) \) may be larger than \( g(x_k) \), which implies a divergence of the algorithm. In order to preclude such a divergence we use the following backtracking line-search for each \( x_k \):

\[
m = 0
\]

while \( g(x_{k+1}) \geq g(x_k) \) and \( m \leq 50 \)

\[
x_{k+1} = x_k - (0.78)^m \cdot g'(x_k)/g''(x_k)
\]

\[
m = m + 1
\]

end

so that the “update” \( x_{k+1} \) is guaranteed to improve upon \( x_k \). If an improvement has not been reached at \( m = 50 \), then the search is terminated. Otherwise we continue with the backtracking procedure

\[
\text{repeat}
\]

\[
x_{\text{min}} = x_{k+1}
\]

\[
\hat{g}_{\text{min}} = g(x_{k+1})
\]

\[
x_{k+1} = x_k - (0.78)^m \cdot g'(x_k)/g''(x_k)
\]

\[
m = m + 1
\]

until \( g(x_{k+1}) > \hat{g}_{\text{min}} \) or \( m > 50 \)

\[
x_{k+1} = \hat{\hat{x}}_{\text{min}}
\]

which enables a further improvement upon \( x_k \).
The above procedure is applied to the multi-dimensional optimization of the criterion \( J(\theta) \), where \( \theta_k \in \mathbb{R}^n \) is the vector of \( n \) parameters, that have been established at the \( k \)-th iteration step. For each element \( \theta_{k,m} \) of \( \theta_k \) we calculate \( \partial J/\partial \theta_{k,m} \) and \( \partial^2 J/\partial \theta^2_{k,m} \). The parameters are updated accordingly to

\[
\theta_{k+1,m} = \theta_{k,m} - \frac{\partial J}{\partial \theta_{k,m}} \frac{\partial^2 J}{\partial \theta^2_{k,m}}
\]

and the backtracking line-searches of all parameters are applied simultaneously.

Lastly, we mention that this search is executed only for those parameters \( \theta_{k,m} \), for which the second partial derivative \( \partial^2 J/\partial \theta^2_{k,m} \) is positive. — Notice that \( g(x) \) has a maximum instead of a minimum if \( g''(x_0) < 0 \). — All remaining parameters are subjected to a steepest decent search method afterwards. The whole search procedure is terminated when

\[
\frac{g(x_{k+1}) - g(x_k)}{g(x_k)} < 0.5\%.
\]

See [45] for background information on stopping criteria.

**F.2 SISO Frequency Response Modeling**

The parameterization of the model set by \( b(s)/a(s) \) may lead to problems that are numerically ill-conditioned. This happens especially when the frequency range involves a few decades. This ill-conditioning can be reduced by the following scaling of the frequency vector \( \Omega \). We define \( \omega_{\text{max}} \) as the largest frequency of \( \Omega \) and \( \tilde{\omega}_i = \omega_i / \omega_{\text{max}} \). With \( \omega_i = \omega_{\text{max}} \cdot \tilde{\omega}_i \) the parameterization \( b(j\omega)/a(j\omega) \) can be replaced by

\[
\tilde{b}_n(j\tilde{\omega})^n + \cdots + \tilde{b}_1(j\tilde{\omega}) + \tilde{b}_0 \quad \text{and} \quad (j\tilde{\omega})^n + \cdots + \tilde{a}_1(j\tilde{\omega}) + \tilde{a}_0
\]

and the parameters of \( \theta \) are retrieved by

\[
b_m = \tilde{b}_m \cdot \omega_{\text{max}}^{-m}, \quad m = 0, \ldots, n
\]

\[
a_m = \tilde{a}_m \cdot \omega_{\text{max}}^{-m}, \quad m = 0, \ldots, n - 1.
\]

This relationship is due to the fact that \( a_n = 1 \) and \( \tilde{a}_n = 1 \).

For the Newton-Raphson optimization we need first and second derivatives. We provide explicit expressions of these derivatives for the calculation of \( \theta_{k+1} \). Fore ease of notation we replace the vector \( \theta_k \) and its element \( \theta_{k,m} \) by \( \theta \) and \( \theta_m \). We write the criterion of (7.1) as

\[
J_f(\theta) = \sum_{i=1}^{N} |P_i - \frac{b(j\omega_i)}{a(j\omega_i)}|^2.
\]
Differentiation and summation may be interchanged, so that
\[ \frac{\partial J_f(\theta)}{\partial \theta_m} = \sum_{i=1}^{N} \frac{\partial |P_i - b(j\omega_i)/a(j\omega_i)|^2}{\partial \theta_m} \]
and a similar expression holds for the second derivative. We define
\[ J_{f,i} = (P_i - \frac{b(j\omega_i)}{a(j\omega_i)})^* \cdot (P_i - \frac{b(j\omega_i)}{a(j\omega_i)}) \]
for each \( i = 1, \ldots, N \) and
\[ F_m(s) = \frac{s^m}{a(s)}, \quad \tilde{P}(s) = \frac{b(s)}{a(s)}. \] (F.1)

Then the derivatives can be obtained from
\[ \frac{\partial J_{f,i}}{\partial a_m} = 2 \text{Re} \left( (P_i - \tilde{P}(j\omega_i))^* \cdot \tilde{P}(j\omega_i) F_m(j\omega_i) \right) \]
\[ \frac{\partial^2 J_{f,i}}{\partial a_m^2} = 2|\tilde{P}(j\omega_i)|^2 |F_m(j\omega_i)|^2 - 4 \text{Re} \left( (P_i - \tilde{P}(j\omega_i))^* \cdot \tilde{P}(j\omega_i) (F_m(j\omega_i))^2 \right) \]
\[ \frac{\partial J_{f,i}}{\partial b_m} = -2 \text{Re} \left( (P_i - \tilde{P}(j\omega_i))^* \cdot F_m(j\omega_i) \right) \]
\[ \frac{\partial^2 J_{f,i}}{\partial b_m^2} = 2|F_m(j\omega_i)|^2. \]

F.3 SISO Spectral Factorization

We let \( Q_i^*Q_i \) be a spectrum sampled at the frequencies \( \Omega \). We search for a minimal phase spectral factor by the method of [2]. That is, we first search for some transfer function \( Z(s) \) such that \( Z(-j\omega_i) + Z(j\omega_i) \approx Q_i^*Q_i, i = 1, \ldots, N \). Thereafter we derive a spectral factor \( Q(s) \) from \( Z(s) \). For each individual frequency \( \omega_i \) we define the criterion
\[ J_{s,i} = \left( Q_i^*Q_i - \frac{b(-j\omega_i)}{a(-j\omega_i)} - \frac{b(j\omega_i)}{a(j\omega_i)} \right)^2 \]
and we estimate the parameters as
\[ \hat{\theta} = \arg \min_{\theta \in \mathbb{R}^{n+1}} \sum_{i=1}^{N} J_{s,i} \]
in which \( \theta = \text{col}(b_1, b_1, b_0, a_{n-1}, \ldots, a_1, a_0) \). The transfer function \( Z(s) \) is obtained by substitution of \( \hat{\theta} \) for \( \theta \) in \( b(s)/a(s) \). The criterion is optimized with the Newton-Raphson method using the derivatives
\[ \frac{\partial J_{s,i}}{\partial a_m} = 4(Q_i^*Q_i - \tilde{Z}(-j\omega_i) - \tilde{Z}(j\omega_i)) \cdot \text{Re} \left( \tilde{Z}(j\omega_i) F_m(j\omega_i) \right) \]
\[
\frac{\partial^2 J_{s,i}}{\partial \sigma_m^2} = 8\text{Re} \left( \tilde{Z}(j\omega_i) F_m(j\omega_i) \right)^2 - 8(Q_i^* Q_i - \tilde{Z}(-j\omega_i) - \tilde{Z}(j\omega_i)) \cdot \text{Re} \left( \tilde{Z}(j\omega_i) (F_m(j\omega_i))^2 \right)
\]
\[
\frac{\partial J_{s,i}}{\partial b_m} = -4(Q_i^* Q_i - \tilde{Z}(-j\omega_i) - \tilde{Z}(j\omega_i)) \cdot \text{Re} \left( F_m(j\omega_i) \right)
\]
\[
\frac{\partial^2 J_{s,i}}{\partial b_m^2} = 4\text{Re} \left( F_m(j\omega_i) \right)^2
\]

with \( F_m(s) \) as in (F.1) and \( \tilde{Z} = b(s)/a(s) \) with the current parameter values substituted for \( \theta \).

From the estimated \( Z(s) \) we derive a stable minimum phase spectral factor \( Q(s) \) such that
\[
Q^*(-s)Q(s) = Z(s) + Z(-s).
\]

We let \( X = \text{are}(A, B, C) \) be the solution \( X = X^T \) of the algebraic Riccati equation \( A^TX + XA - XBX + C = 0 \), which is such that all eigenvalues of \( A - BX \) are stable. Then we can construct \( Q(s) \) from \( Z(s) = J + H(sI - F)^{-1}G \) as follows:

1. define \( R = J^T + J \)
2. solve \( \Phi = \text{are}(F - GR^{-1}H, -GR^{-1}GT, H^TR^{-1}H) \)
\[
\Psi = \text{are}(F^T, -K_c^TRK_c, 0)
\]
3. construct \( K_c = R^{-1} \Phi \)
\[
K_f = \Psi K_c^TR
\]
4. construct the minimum phase spectral factor \( Q(s) = \begin{bmatrix} F + K_fK_c & G + K_f \\ R^\frac{1}{2}K_c & R^\frac{1}{2} \end{bmatrix} \)

Finally we sketch how we determine an initial estimate for the Newton-Raphson optimization of \( \sum J_{s,i} \). The magnitude of \( Z(s) \) should be close to \( A \simeq \frac{1}{2}Q^*Q \). Therefore we derive an initial estimate from this vector \( A \) of amplitude data. First we generate a phase \( \phi \) for the amplitude data \( A \) by means of the function \texttt{genphase.m} of the Matlab \( \mu \)-toolbox. (This function is based on the complex-cepstrum.) Then we subject the phase \( \phi \) together with the amplitude \( A \) to the SISO transfer function identification procedure of the previous section. This produces a model, whose phase and amplitude only approximate \( \phi \) and \( A \). From this model we take its phase \( \psi \) and we combine it with the original amplitude data \( A \). The phase \( \psi \) is goes with the amplitude data \( A \) much better than the previously determined phase \( \phi \). We subject the pair \( \psi \) and \( A \) to the same identification procedure and from the result we calculate the initial values for the parameters \( \theta \).
F.4 SISO Plant Coprime Factors Identification

We consider the problem of identifying a rcf \((\hat{N}, \hat{D})\) such that

\[
\sum_{i=1}^{N} \left| \begin{bmatrix} N_i \\ D_i \end{bmatrix} - \begin{bmatrix} \hat{N}(j\omega_i) \\ \hat{D}(j\omega_i) \end{bmatrix} \right| Q(j\omega_i) W(j\omega_i) \right|^2
\]

(F.2)

is minimized. In here \(N_i, D_i, i = 1, \ldots, N\) are the given frequency response data. \(Q(s)\) is some fixed transfer function and \(W(s)\) is some real valued positive weighting function. We represent the candidate coprime factorization by \((b(s)/d(s), a(s)/d(s))\) just as in Section 7.3. Theoretically speaking we should impose restrictions on the parameter vector \(\theta_d\) in order that \(b(s)/d(s), a(s)/d(s)\) is a right coprime factorization. However we apply an unconstrained optimization, because the estimates turn out to be right coprime factorizations anyway.

Similar to Section 7.2 we begin the identification with an iterative procedure like the Sanathanan-Koerner iteration. The \(k\)-th vector of parameters \(\theta_{d,k}\) is derived uniquely through

\[
\theta_{d,k} = \arg \min_{\theta_d \in \mathbb{R}^{n+3}} \sum_{i=1}^{N} \left| \begin{bmatrix} \frac{d(j\omega_i)}{d_{k-1}(j\omega_i)} N_i \\ \frac{d(j\omega_i)}{d_{k-1}(j\omega_i)} D_i \end{bmatrix} - \begin{bmatrix} b(j\omega_i) \\ a(j\omega_i) \end{bmatrix} \right| Q(j\omega_i) W(j\omega_i) \right|^2
\]

In here \(d_{k-1}(s)\) signifies the denominator polynomial of the coprime factors, that have been determined in step \(k - 1\). When this iterative search has converged, then \(d_k(s) \approx d_{k-1}(s)\) and thus the minimized criterion corresponds to the “output-error” criterion of (F.2). For additional comments about this iteration we refer to Section 7.2.

The eventual estimate that results from the Sanathanan-Koerner iteration is used as an initial estimate for the Newton-Raphson optimization of (F.2). With the notation

\[
G_m(j\omega_i) = \frac{(j\omega_i)^m}{d(j\omega_i)}
\]

\[
\hat{N}(j\omega_i) = \frac{b(j\omega_i)}{d(j\omega_i)}
\]

\[
\hat{D}(j\omega_i) = \frac{a(j\omega_i)}{d(j\omega_i)}
\]

\[
J_{ND,i} = \left| \begin{bmatrix} \frac{d(j\omega_i)}{d_{k-1}(j\omega_i)} N_i \\ \frac{d(j\omega_i)}{d_{k-1}(j\omega_i)} D_i \end{bmatrix} - \begin{bmatrix} b(j\omega_i) \\ a(j\omega_i) \end{bmatrix} \right| Q(j\omega_i) W(j\omega_i) \right|^2
\]

the required derivatives can be obtained from

\[
\frac{\partial J_{ND,i}}{\partial a_m} = -2\text{Re} \left( (D_i - \hat{D}(j\omega_i) Q(j\omega_i))^* Q(j\omega_i) G_m(j\omega_i) \right)
\]
\[ \frac{\partial^2 J_{ND,i}}{\partial a_{m}^2} = 2|Q(j\omega_i)|^2 |G_m(j\omega_i)|^2 \]
\[ \frac{\partial J_{ND,i}}{\partial b_m} = -2\text{Re} \left( (N_i - \tilde{N}(j\omega_i)Q(j\omega_i))^* Q(j\omega_i)G_m(j\omega_i) \right) \]
\[ \frac{\partial^2 J_{ND,i}}{\partial d_m^2} = 2|Q(j\omega_i)|^2 |G_m(j\omega_i)|^2 \]
\[ \frac{\partial J_{ND,i}}{\partial d_m} = 2\text{Re} \left( \left( (D_i - \tilde{D}(j\omega_i)Q(j\omega_i))^* \tilde{D}(j\omega_i) + (N_i - \tilde{N}(j\omega_i)Q(j\omega_i))^* \tilde{N}(j\omega_i) \right) \times G_m(j\omega_i)Q(j\omega_i) \right) \]
\[ \frac{\partial^2 J_{ND,i}}{\partial d_m^2} = 2|Q(j\omega_i)|^2 |G_m(j\omega_i)|^2 \left( |\tilde{D}(j\omega_i)|^2 + |\tilde{N}(j\omega_i)|^2 \right) \]
\[ -4\text{Re} \left( \left( (D_i - \tilde{D}(j\omega_i)Q(j\omega_i))^* \tilde{D}(j\omega_i) + (N_i - \tilde{N}(j\omega_i)Q(j\omega_i))^* \tilde{N}(j\omega_i) \right) Q(j\omega_i)G_m^2(j\omega_i) \right) . \]

Next we consider the problem of identifying a rcf \( \hat{Q}(s) \) together with a stable factor \( \hat{Q}(s) \) such that
\[
\sum_{i=1}^{N} \left[ \begin{array}{c} N_i \\ D_i \end{array} \right] - \left[ \begin{array}{c} \hat{N}(j\omega_i) \\ \hat{D}(j\omega_i) \end{array} \right] \hat{Q}(j\omega_i) \right] W(j\omega_i) \]
is minimized. For this we use the following iterative procedure. At some stage \( \hat{Q} \) takes the value \( \hat{Q}_{k-1} \). With this value of \( \hat{Q} \) substituted for \( Q \) in (F.2) we execute the above Sanathanan-Koerner iteration. The result is a rcf \( (\hat{N}_k, \hat{D}_k) \) which makes the sum of (F.2) small for the particular \( \hat{Q}_{k-1} \). Then we normalize the rcf \( (\hat{N}_k, \hat{D}_k) \) by means of some \( Q_n \in \mathcal{J} \) such that
\[
\hat{N}_k = \hat{N}_{n,k}Q_n \\
\hat{D}_k = \hat{D}_{n,k}Q_n
\]
in which \( (\hat{N}_{n,k}, \hat{D}_{n,k}) \) is a normalized rcf, and we construct \( \hat{Q}_k = Q_n\hat{Q}_{k-1} \). We repeat the Sanathanan-Koerner iteration and the normalization of the estimate until \( (\hat{N}_k, \hat{D}_k) \) is almost identical with \( (\hat{N}_{n,k}, \hat{D}_{n,k}) \). The eventual values of this rcf and that of \( \hat{Q}_k \) are taken as the desired estimates \( (\hat{N}, \hat{D}) \) and \( \hat{Q} \).

### F.5 Fixed-loop Approximate Performance-identification

For the Newton-Raphson optimization of Section 7.3 we need the partial derivatives \( \partial J_{T,i}/\partial \theta_m \) and \( \partial^2 J_{T,i}/\partial \theta_m^2 \) with \( J_{T,i} \) as defined in (7.4). For ease of notation we
introduce

\[ T_{12,i} = N_i^o(D_o(j\omega_i) + C(j\omega_i)N_o(j\omega_i))^{-1} \]
\[ T_{22,i} = D_i^o(D_o(j\omega_i) + C(j\omega_i)N_o(j\omega_i))^{-1} \]
\[ \tilde{T}_{12,i} = T_{12}(\tilde{P}, C)(j\omega_i) \]
\[ \tilde{T}_{22,i} = T_{22}(\tilde{P}, C)(j\omega_i) \]

and we let \( F_m(s) \) and \( \tilde{P}(s) \) be given by (F.1). Then the required derivatives can be written as

\[ \frac{\partial J_{T,i}}{\partial a_m} = 2 \text{Re} \left( \left[ \alpha(T_{12,i} - \tilde{T}_{12,i})^* - (T_{22,i} - \tilde{T}_{22,i})^* \frac{C(j\omega_i)}{\alpha} \right] \alpha F_m(j\omega_i) \tilde{T}_{22,i}^2 \right) \]
\[ \frac{\partial^2 J_{T,i}}{\partial a_m^2} = 2(1 + \frac{|C(j\omega_i)|^2}{\alpha}) |\alpha F_m(j\omega_i)|^2 |\alpha \tilde{P}(j\omega_i)|^2 |\tilde{T}_{22,i}|^4 \]
\[ + 4 \text{Re} \left( \left[ (T_{22,i} - \tilde{T}_{22,i})^* C(j\omega_i) / \alpha - \alpha(T_{12,i} - \tilde{T}_{12,i})^* \right] \alpha F_m(j\omega_i) \tilde{T}_{22,i}^2 \right) \]
\[ \frac{\partial J_{T,i}}{\partial b_m} = 2 \text{Re} \left( \left[ (T_{22,i} - \tilde{T}_{22,i})^* \frac{C(j\omega_i)}{\alpha} - \alpha(T_{21,i} - \tilde{T}_{21,i})^* \right] \alpha F_m(j\omega_i) \tilde{T}_{22,i}^2 \right) \]
\[ \frac{\partial^2 J_{T,i}}{\partial b_m^2} = 2(1 + \frac{|C(j\omega_i)|^2}{\alpha}) |\alpha F_m(j\omega_i)|^2 |\tilde{T}_{22,i}|^4 + 4 \text{Re} \left( \left[ \alpha(T_{12,i} - \tilde{T}_{12,i})^* - (T_{22,i} - \tilde{T}_{22,i})^* C(j\omega_i) / \alpha \right] (\alpha F_m(j\omega_i))^2 \tilde{T}_{22,i}^3 \right) \].

### F.6 Frequency-Gap of Two SISO Plants

The identification problem of interest is the estimation of a stable factor \( Q(s) \) such that

\[ \sum_{i=1}^{N} \left| \begin{bmatrix} N_i \\ D_i \end{bmatrix} - \begin{bmatrix} \hat{N}(j\omega_i) \\ \hat{D}(j\omega_i) \end{bmatrix} \right| q_i \]

is minimized. In here \( N_i, D_i \) represent available frequency response data and \((\hat{N}, \hat{D})\) is a given rcf. In order to obtain a factor \( Q(s) \) we first calculate a scalar \( q_i \in \mathbb{C} \) for each frequency \( \omega_i \) of \( \Omega \) such that

\[ \left| \begin{bmatrix} N_i \\ D_i \end{bmatrix} - \begin{bmatrix} \hat{N}(j\omega_i) \\ \hat{D}(j\omega_i) \end{bmatrix} \right| q_i \]

is minimal. This is a standard least squares problem and the global minimum is achieved for

\[ q_i = \frac{D_i^* \hat{D}(j\omega_i) + N_i^* \hat{N}(j\omega_i)}{|\hat{D}(j\omega_i)|^2 + |\hat{N}(j\omega_i)|^2}. \]
As the vector of $q_i$'s minimizes the distance between $(N_i, D_i)$ and $(\tilde{N}, \tilde{D})$ for each individual frequency, it gives rise to a frequency dependent lower bound on the directed gap (see also Section 5.1). Finally, we estimate the factor $Q(s)$ from the "frequency response data" $q_i$ by means of the method described in Section 7.2.

Sometimes the identification of $Q$ from the vector of $q_i$'s yields an unstable factor. Hence it would be profitable to parameterize $Q$ such that it is guaranteed stable. Such parameterizations exist [167, 107], but they are all but readily applicable. Therefore we do not constrain the parameters of the candidate stable factor. Instead we check afterward whether $Q(s)$ is stable or not. If not, then we repeat the estimation procedure with a model structure of one order lower than the estimated $Q(s)$. If necessary the order of $Q(s)$ is reduced stepwise to zero.

F.7 Design-oriented Approximate Stability-identification

For the Newton-Raphson optimization of (7.12) over $\theta_d$ we need the partial derivatives of $J_{r,i}$ of (7.11). For notational convenience we introduce

\[ \tilde{N}(j\omega_i) = \frac{b(j\omega_i)}{d(j\omega_i)} \]
\[ \tilde{D}(j\omega_i) = \frac{a(j\omega_i)}{d(j\omega_i)} \]
\[ G_m(j\omega_i) = \frac{(j\omega_i)^m}{d(j\omega_i)} \]
\[ Z(j\omega_i) = \alpha^2|\tilde{N}(j\omega_i)|^2 + |\tilde{D}(j\omega_i)|^2 \]
\[ E(j\omega_i) = \frac{|D_i^e Q_{\Delta}(j\omega_i) - \tilde{D}(j\omega_i)|^2 + \alpha^2|N_i^e Q_{\Delta}(j\omega_i) - \tilde{N}(j\omega_i)|^2}{Z(j\omega_i)} \]

The required derivatives are

\[ \frac{\partial J_{r,i}}{\partial a_m} = -\frac{2}{Z(j\omega_i)} \text{Re} \left( (D_i^e Q_{\Delta}(j\omega_i) - \tilde{D}(j\omega_i))^* G_m(j\omega_i) \right) - \frac{\partial Z(j\omega_i)}{\partial a_m} \frac{E(j\omega_i)}{Z^2(j\omega_i)} \]
\[ \frac{\partial^2 J_{r,i}}{\partial a_m^2} = \frac{Z(j\omega_i) - E(j\omega_i)}{Z^2(j\omega_i)} \frac{\partial^2 Z(j\omega_i)}{\partial a_m^2} + 2 \left( \frac{\partial Z(j\omega_i)}{\partial a_m} \right)^2 \frac{E(j\omega_i)}{Z^3(j\omega_i)} \]
\[ \quad \quad + \frac{4}{Z^2(j\omega_i)} \text{Re} \left( (D_i^e Q_{\Delta}(j\omega_i) - \tilde{D}(j\omega_i))^* G_m(j\omega_i) \right) \frac{\partial Z(j\omega_i)}{\partial a_m} \]
\[ \frac{\partial Z(j\omega_i)}{\partial a_m} = 2 \text{Re} \left( \tilde{D}^*(j\omega_i) G_m(j\omega_i) \right) \]
\[ \frac{\partial^2 (j\omega_i)}{\partial a_m^2} = 2 |G_m(j\omega_i)|^2 \]
\[ \frac{\partial J_{r,i}}{\partial b_m} = -\frac{2\alpha^2}{Z(j\omega_i)} \text{Re} \left( (N_i^e Q_{\Delta}(j\omega_i) - \tilde{N}(j\omega_i))^* G_m(j\omega_i) \right) - \frac{\partial Z(j\omega_i)}{\partial b_m} \frac{E(j\omega_i)}{Z^2(j\omega_i)} \]
\[
\frac{\partial^2 J_{r,i}}{\partial b_m^2} = \frac{Z(j\omega_i) - E(j\omega_i)}{Z^2(j\omega_i)} \frac{\partial^2 Z(j\omega_i)}{\partial b_m^2} + 2 \left( \frac{\partial Z(j\omega_i)}{\partial b_m} \right)^2 \frac{E(j\omega_i)}{Z^3(j\omega_i)}
+ \frac{4\alpha^2}{Z^2(j\omega_i)} \text{Re} \left( (N_t^s Q_\Delta(j\omega_i) - \bar{N}(j\omega_i))^* G_m(j\omega_i) \right) \frac{\partial Z(j\omega_i)}{\partial b_m}
\]

\[
\frac{\partial Z(j\omega_i)}{\partial b_m} = 2\alpha^2 \text{Re} (\bar{N}^* (j\omega_i) G_m(j\omega_i))
\]

\[
\frac{\partial^2 (j\omega_i)}{\partial b_m^2} = 2\alpha^2 |G_m(j\omega_i)|^2
\]

\[
\frac{\partial J_{r,i}}{\partial d_m} = \frac{2}{Z(j\omega_i)} \text{Re} \left( (D_t^s Q_\Delta(j\omega_i) - \bar{D}(j\omega_i))^* D_t^s Q_\Delta(j\omega_i) G_m(j\omega_i) \right)
+ \alpha^2 (N_t^s Q_\Delta(j\omega_i) - \bar{N}(j\omega_i))^* N_t^s Q_\Delta(j\omega_i) G_m(j\omega_i))
\]

\[
\frac{\partial^2 J_{r,i}}{\partial d_m^2} = \frac{2 |G_m(j\omega_i)|^2}{Z(j\omega_i)} (|D_t^s Q_\Delta(j\omega_i)|^2 + \alpha^2 |N_t^s Q_\Delta(j\omega_i)|^2)
\]
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Glossary of Symbols

**Matrices**

$A_{ij}$ (block) element of (block) matrix $A$ at row $i$ and column $j$

$A^T$ transpose of matrix $A$

$A^*$ complex-conjugate transpose of matrix $A$

$\sigma(A)$ singular value of matrix $A$

$\sigma_{\text{max}}(A)$ maximum singular value of matrix $A$

$\det A$ determinant of matrix $A$

$\text{col}(a_1, \ldots, a_n) = [a_1^T, \ldots, a_n^T]^T$

$\text{diag}(a_1, \ldots, a_n)$ $n \times n$ (block) matrix $A$ with $A_{ii} = a_i$ and $A_{ij} = 0$ for $i \neq j$

**General sets and norms**

$\mathbb{N}$ set of natural numbers

$\mathbb{R}$ set of real numbers

$\mathbb{R}^{p \times q}$ set of $p \times q$ real matrices

$\mathbb{C}$ set of complex numbers

$\mathcal{H}$ principal ideal domain, 58

$\mathcal{F}$ field of fractions associated with $\mathcal{H}$, 58

$\mathcal{J}$ group of units in $\mathcal{H}$, 58

$\mathbb{RH}_\infty$ set of real rational stable transfer functions

$L_{2,T}$ time-domain Lebesgue space

$\varepsilon$ is an element of

$\mathcal{C}$ is a subset of

$\mathcal{A} \setminus \mathcal{B}$ exclusion of $\mathcal{B}$ from $\mathcal{A}$

$|a|$ Euclidian norm of a vector $a$

$\|A\|_2$ norm of a matrix $A$; subscripts indicate particular norms

$\sigma_{\text{max}}(A)$ with $A \in \mathbb{C}^{p \times m}$

$\|A(s)\|_2$ $H_2$-norm of transfer function $A(s)$

$\|A(s)\|_\infty$ $H_\infty$-norm of transfer function $A(s)$

$\|A(s)\|_{\infty, \Omega}$ max $\omega \in \Omega \{\sigma_{\text{max}}(A(j\omega))\}$, 167

$\|A(s)\|_{2, \Omega}$ $\sum_{\omega \in \Omega} \|A(j\omega)\|_F^2$, 149

$J_p(\hat{P}, C, W)$ $\|T(\hat{P}W, W^{-1}C)\|_p$, $p$ indicates a particular norm

$J_p^S(\hat{P}, C, W)$ $\|T(PW, W^{-1}C) - T(\hat{P}W, W^{-1}C)\|_p$, $p$ indicates a particular norm
Signals

\( u, u_c \) controlled input of plant/compensator, 12
\( w, w_c \) uncontrolled input of plant/compensator, 12
\( y, y_c \) measured output of plant/compensator, 12
\( z, z_c \) monitored output of plant/compensator, 12
\( r_1, r_2 \) exogenous inputs, 13
\( x \) intermediate variable, 85
\( v \) additive disturbance at the output, 28
\( q \) auxiliary variable in the \((R, S)\)-parameterization, 78
\( \Phi_u \) spectrum (power spectral density) of \( u \)
\( \Phi_{yu} \) cross spectrum between \( y \) and \( u \)

Systems and models

\( C, C_T, C_{TT} \) compensators with inputs and outputs like \( P, P_T, P_{TT}, 12 \)
\( C_{\hat{P}} \) (inner-loop) model-based compensator
\( C_P \) (inner-loop) plant-based compensator
\( C_{yw}, C_{zw}, C_{zu} \) see \( P_{yw}, P_{zw}, P_{zu} \)
\( \mathcal{C}(\hat{P}) \) set of all inner-loop compensators stabilizing \( \hat{P} \), 18
\( H(P, C) \) single-variante feedback system, 13, 61
\( H(P_T, C) \) standard feedback system, 13, 80
\( H(P_{TT}, C_{TT}) \) general feedback system, 13, 67
\( \mathcal{P}(C) \) set of all inner-loop plants stabilized by \( C \), 65
\( \mathcal{P}_T(C) \) set of all standard plants stabilized by \( C \), 80
\( \mathcal{P}_{TT}(C_{TT}) \) set of all general plants \( P_{TT} \) stabilized by admissible \( C_{TT} \), 77
\( P \) one vector-input one vector-output plant, 12; inner-loop plant, 14
\( P_T; P_{TT} \) standard plant, 12; general plant, 12
\( P_{yw}, P_{zw}, P_{zu} \) disturbance plant, outer-loop plant, monitor plant, 14
\( P_0 \) auxiliary model of (inner-loop) plant \( P \), 230
\( \hat{P} \) nominal model of inner-loop plant \( P \), 14
\( \hat{P}_1, \hat{P}_i \) 1st and i-th nominal models
\( P(s) \) transfer function of \( P \)
\( P^*(s) \) \( P^T(-s) \)
\( P(\omega) \) (estimated) frequency response of \( P \) at frequency \( \omega \), 35
\( P_i \) \( P(\omega_i) \)
\( T(P, C) \) feedback matrix of \( P \) and \( C \), 23
\( T(P_{TT}, C_{TT}) \) transfer function matrix of the feedback system \( H(P_{TT}, C_{TT}) \), 79
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = C(sI-A)^{-1}B + D
\]
Glossary of Symbols

Parameterization of model sets

\( \mathcal{P}(\theta) \) model set parameterized in terms of \( \theta \), 148
\( \theta \) vector of parameters, each of which is an indeterminate in \( \mathbb{R} \), 148
\( \hat{\theta} \) vector of estimated parameters, each of which is an element in \( \mathbb{R} \), 148

Dynamical perturbations and mismatches

\( \Delta \) dynamical perturbation of a nominal model, 16
\( \Delta_M \) multiplicative dynamical perturbation, 16
\( \Delta_A \) additive dynamical perturbation, 16
(\( \Delta_N, \Delta_D \)) coprime factor perturbation, 104
\( \Delta_R \) \( R \)-parameter perturbation, 139
\( b_\Delta \) upper bound on dynamical perturbations, 16
\( \mathcal{B}(b_\Delta) \) ball of dynamical perturbations, 16
\( \rho_\Delta \) robustness margin, 131
\( \mathcal{P}_\Delta(\hat{P}, b_\Delta) \) (perturbative) family of plants, 16
\( \mathcal{P}_S(\hat{P}, \rho_\Delta) \) family of robustly stabilized systems, 132
\( M_M \) multiplicative mismatch, 15
\( M_A \) additive mismatch, 15
\( M_T \) additive feedback matrix mismatch \( T(P, C) - T(\hat{P}, C) \), 99
\( M_{ND}(Q_\Delta) \) coprime factor mismatch, 105

Fractional representations

\( G(P) \) graph of \( P \), 98
(\( N, D \), (\( \hat{D}, \hat{N} \)) right/left coprime factorization of \( P \), 59, 61
(\( N_{TT}, D_{TT} \)) rcf of \( P_{TT} \), 68, 72
(\( N^a, D^a \)) rcf of (inner-loop) plant \( P \) associated to compensator \( C \) and the rcf
(\( N_o, D_o \)) of the auxiliary model \( P_o \), 63
(\( N^a_T, D^a_T \)) idem for standard plant \( P_T \), 80
(\( N^a_T, D^a_T \)) idem for general plant \( P_{TT} \), 76
(\( N, D_n \)) normalized right coprime factorization, 98
(\( N, D_o \)) right coprime factorization of auxiliary model \( P_o \), 63
(\( N_c, D_c \)) right coprime factorization of compensator \( C \), 61
(\( \hat{N}, \hat{D} \)) right coprime factorization of \( \hat{P} \), 101
\( N^a, D^a \) (estimated) frequency response of \( N^a, D^a \), see \( P \)
\( R \) transfer function in \( R \)-parameterization, 63
\( S, S_{11}, S_{12} \) transfer functions in \( (R, S) \)-parameterization, 76
\( \hat{R}, \hat{S} \) analogs of \( R, S \) in a left coprime plant description, 217
$X, Y$ right Bezout factors corresponding to $(N, D)$, 59
$\tilde{Y}, \tilde{X}$ left Bezout factors corresponding to $(\tilde{D}, \tilde{N})$, 59
$\Lambda, \tilde{\Lambda}$ central transfer function in stability analysis, 61

**Special symbols and notation**

$\min_{a \in A} f(a)$ minimization of $f(a)$ over all element $a$ of $A$
$\text{arg min}_{a \in A} f(a)$ element $a$ of $A$ that minimizes $f(a)$
$I$ identity matrix
$\omega$ frequency, vector of frequencies
$\omega_i$ $i$-th element of the vector of frequencies $\omega$
$\Omega$ finite set of frequencies
$j$ $\sqrt{-1}$
$\text{Re}(s)$ real part of $s$
$\text{Im}(s)$ imaginary part of $s$
$\square$ end of proof, end of remark, end of example
$\equiv$ equals by definition
$\approx$ is approximately equal to, approximates
$\to$ tends to
$\exists$ there exist(s)
$\forall$ for all
$\land$ and
$\leftarrow$ is implied by
$\Rightarrow$ implies
\texttt{matlab}TM trademark of The Mathworks, Inc.

**Acronyms and abbreviations**

ARX Auto Regressive eXogenous
lcf left coprime factorization, 60
LTI Linear Time-Invariant
LTIFD Linear Time-Invariant Finite Dimensional
MIMO Multi-Input Multi-Output
nlcf normalized left coprime factorization, 98
nrzf normalized right coprime factorization, 98
rcf right coprime factorization, 60
SISO Single-Input Single-Output
Samenvatting

Benaderende Identificatie en Regelaar Ontwerp
met Toepassing op een Mechanisch Systeem

Aan veel systemen worden eisen gesteld die met behulp van een regelaar moeten worden verwezenlijkt. In de moderne robuuste regeltheorie wordt gebruik gemaakt van modellen die bestaan uit een nominale deel en een onzekerheidsbeschrijving. Het nominale model is een eenvoudige representatie van de systeem dynamica. De verschillen tussen het nominale model enerzijds en het werkelijke systeem anderzijds worden veelal gekarakteriseerd met een bovengrens op de "modelfout". Te zamen vormen het nominale model en de onzekerheidsbeschrijving een klasse van systemen die het te regelen systeem omvat. Een acceptabele regeling van de gehele klasse garandeert een acceptabele regeling van het werkelijke systeem.

Modellen kunnen worden gecreëerd aan de hand van meetgegevens, hetgeen systeem identificatie wordt genoemd. Het gebruik van geïdentificeerde modellen voor regelaarontwerp geniet de laatste jaren een toenemende aandacht. Er is met name veel moeite gestoken in het afschatten van de bovenvermelde "modelfout". Een goede bovengrens is nodig om een zeker regelgedrag van het werkelijke systeem te kunnen garanderen. Het ontwerp van een zeer goede regelaar vereist echter een zeer nauwkeurige beschrijving van de systeemdynamica, dus een zeer nauwkeurig nominale model. Met name de identificatie van een geschikt nominaal model krijgt hier de aandacht.

Enerzijds wordt de kwaliteit van een nominaal model bepaald door de regelaar, die daarmee kan worden ontworpen. Anderzijds hangt de kwaliteit van de regelaar af van het nominale model. Dus het identificeren van een geschikt nominaal model en het ontwerpen van een goede regelaar vormen een gecombineerd probleem. Indien afzonderlijke procedures voor identificatie en regelaarontwerp worden gebruikt om dit gecombineerde probleem op te lossen, dan is een iteratieve aanpak met herhaalde identificatie en regelaarontwerp noodzakelijk.

In de ontwikkelde iteratieve oplossing wordt elke identificatie uitgevoerd met meetgegevens, die verkregen zijn terwijl de meest recente regelaar is toegepast op het onderhavige systeem. Om de benaderende identificatie in de aanwezigheid van een gesloten regelkring te kunnen uitvoeren wordt gebruik gemaakt van de kennis omtrent de stabiliserende regelaar. Met behulp van coprieme factorrepresentaties wordt het gesloten-lus identificatie probleem omgevormd tot een open-lus identificatie probleem. De coprieme factoren van het geregelde systeem worden geïdentificeerd met behulp van frequentie responsies. In de ontwikkeling van deze identificatie techniek wordt de nadruk worden gelegd op de asymptotische afwijkingen tussen het eenvoudige nominale model en het complexere systeem. Met elk nieuw nominaal model wordt een verbe-

Na de ontwikkeling van de benodigde gereedschappen wordt de voorgestelde iteratieve aanpak getoetst aan de hand van een gesimuleerd voorbeeld. Vervolgens wordt dezelfde procedure toegepast op een praktische experimentele opstelling. Daaruit blijkt dat de iteratieve oplossingsmethode voor het verkrijgen van een goed nominaal model en een goede regelaar ook praktisch toepasbaar is.
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You're a fool if you think it is over
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Chris Rea, “Fool”
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