A PRECONDITIONER FOR THE HELMHOLTZ EQUATION WITH PERFECTLY MATCHED LAYER

Yogi A. Erlangga

Technische Universität Berlin, Institut für Mathematik
Straße des 17. Juni 136, D-10623 Berlin, Germany
e-mail: erlangga@math.tu-berlin.de

Key words: Helmholtz equation, PML, Bi-CGSTAB, multigrid

Abstract. This paper discusses an iterative method for solving the Helmholtz equation with the perfectly matched layer (PML). The method consists of an outer and inner iteration process. The inner iteration is used to approximately solve a preconditioner, which in this case is based on a modified PML equation. The outer iteration is a Krylov subspace method (Bi-CGSTAB). The method explained here is identical with the method already discussed and proposed, e.g., in [Erlangga, Oosterlee, Vuijk, SIAM J. Sci. Comput., 27 (2006), pp. 1471-1492]. We show that the extension of the method to the PML equation is straightforward, and the performance for this type of problem does not degrade as compared to Helmholtz problems with, e.g. Engquist and Majda’s second order boundary condition.

1 INTRODUCTION

In this paper we are concerned with the iterative solution of the Helmholtz equation

\[-\Delta \phi - (\omega/c)^2 \phi = g, \quad \text{in } \Omega \in \mathbb{R}^2,\]

\[\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial \phi}{\partial r} - j (\omega/c) \phi \right) = 0, \quad \text{on } \partial \Omega,\]

which is obtained from Fourier transform of the full wave equation in the time domain. \(\omega = 2\pi f\) and \(c\) denote the angular frequency and the speed of sound, respectively, with \(f\) the frequency. This equation finds applications in many fields, e.g. in aeroacoustics and in geophysical surveys. We are particularly interested in the field solution of (1), obtained from the use of a finite difference discretization. This type of discretization is actually not a restriction of the iterative method explained here as the method can also be used in the finite element setting; see e.g. Turkel and Erlangga ([9], in this proceedings). In this respect we are not going to consider boundary element methods.

As the boundary condition (2) can only be satisfied at infinite distance, we usually truncate the domain at a finite distance in order to keep the computational work minimal.
As a consequence, in the truncated domain, the boundary condition (2) is no longer valid, and a boundary condition at finite distance should be constructed, which mimics the physical, non-reflecting condition at the boundary.

Since the work of Engquist and Majda [3], a quite number of absorbing boundary conditions has been formulated and proposed. In 1994, Berenger formulated the so-called Perfectly Matched Layer (or PML) for electromagnetic waves [2], which mathematically is absorbing for outgoing waves at any incidence angle. Later, Abarbanel and Gottlieb [1] provide analysis on the well-posedness of the PML equation. We refer the reader to, e.g., Applied Numerical Mathematics Vol. 27 (1998), which contains discussions only on the absorbing boundary conditions.

As PML nowadays gains popularity, we consider the PML equation in this paper and propose an iterative method for solving this equation. The method discussed here is mainly based on our work, previously published in [4], [6] and [7]. To make the presentation short we only provide numerical results, and skip some important analysis. The result from this analysis, however, is similar to that already presented in [4], [6] and [7].

2 THE HELMHOLTZ AND PML EQUATION

Denote the physical domain and the PML domain by $\Omega_p$ and $\Omega_d$, respectively, and $\Omega = \Omega_d \cup \Omega_p$, $\Omega_p \cap \Omega_d = \{0\}$. The perfectly matched layer for the Helmholtz equation is formulated as follows [8]:

$$A\phi := -\frac{\partial}{\partial x} \left( s_y \frac{\partial \phi}{\partial x} \right) - \frac{\partial}{\partial y} \left( s_x \frac{\partial \phi}{\partial y} \right) - \frac{\omega^2}{c^2} s_x s_y \phi = g, \quad \text{(3)}$$

with

$$s_x = 1 + \frac{\sigma_x}{j\omega}, \quad s_y = 1 + \frac{\sigma_y}{j\omega}. \quad \text{(4)}$$

In (4), $\sigma_x$ and $\sigma_y$ are real and non-negative, and are represented by piecewise smooth functions depending only on $x$ and $y$ respectively. In $\Omega_p$, $\sigma_x$ and $\sigma_y$ are equal to zero, and (3) reduces to the standard Helmholtz equation. In $\Omega_d$, a linear function is often sufficient for $\sigma_x$ and $\sigma_y$, for example

$$\sigma_x = \alpha_x x, \quad \sigma_y = \alpha_y y, \quad \alpha_x, \alpha_y \in \mathbb{R}_+. \quad \text{(5)}$$

Furthermore, the value of $c$ in $\Omega_d$ is set equal to the value of $c$ on $\partial \Omega_p$. In addition, the Dirichlet boundary condition $\phi = 0$ is imposed on $\partial \Omega$.

We discretize (3) and the corresponding boundary condition with the nine-point finite difference stencil, as advocated by Tsynkov and Turkel in [8]. This ends up with a linear system

$$Au = b, \quad \text{(6)}$$

2
where $A$ is a sparse, complex-valued, symmetric but indefinite matrix for high frequencies. Since the Krylov iteration converges very slowly or even diverges for high wavenumber, we precondition (6) with a matrix $M$ such that

$$AM^{-1}v = b, \quad Mu = v,$$

is suitable for Krylov subspace acceleration.

In [4], [6], [7], we show that a preconditioner based on the discretization of a modified Helmholtz equation is effective in accelerating the convergence of the Krylov iteration. Since this preconditioning matrix still has to be inverted and the cost of inverting this matrix is as expensive as the cost of solving the original problem, the inversion is approximated by performing a small number of steps with an iterative method. In [7], multigrid has been applied to the preconditioner. Hence, the complete numerical method consists of an outer and an inner iteration, with a Krylov subspace method as the outer iteration and multigrid performing the inner iteration. An extensive study suggests that Bi-CGSTAB [10] is more suitable for the Helmholtz equation than other Krylov subspace algorithms [5].

For the Helmholtz equation, the optimal convergence of both the Krylov iteration and multigrid is achieved if we choose the modified Helmholtz operator of the form

$$-\Delta - (1 - 0.5\jmath) \left( \frac{\omega}{c} \right)^2.$$

In the context of PML, we propose a straightforward extension of (8), and formulate the “modified” PML operator

$$\mathcal{M} := -\frac{\partial}{\partial x} \left( s_y \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial y} \left( s_x \frac{\partial}{\partial y} \right) - s_x s_y (1 - 0.5\jmath) \left( \frac{\omega}{c} \right)^2.$$

as the preconditioner. The resultant preconditioning matrix is obtained from the same discretization method used for (3), with Dirichlet boundary conditions imposed.

3 RESULTS

For the numerical test we consider a 2D wedge problem, illustrated in Figure 1. This mimics a forward modelling of the Earth’s subsurface having three layers of different properties. A source is generated slightly below the midpoint of the upper physical boundary. A PML region is added to the physical domain and is used to absorb the outgoing waves. To approximately solve the preconditioning matrix, one “geometric” multigrid iteration is used, with $F(1,1)$ cycle. Jacobi iteration with relaxation ($\omega_{\text{relax}} = 0.5$) is used as the smoother. The Bi-CGSTAB (outer) iteration is terminated if the relative residual has been reduced by seven orders of magnitude. The numerical performance is shown in Table 1.
We compare the numerical performance from solving the PML equation, with the numerical performance from solving the Helmholtz equation with different boundary conditions: the second-order Engquist-Majda boundary condition (denoted by EM2), and the so-called “sponge” layer (denoted by ABC). We measure the CPU time on a 2.4 GHz Pentium 4 machine, with 2 GByte of RAM and LINUX operating system.

<table>
<thead>
<tr>
<th>$f$ (Hz)</th>
<th>Iter</th>
<th>CPU</th>
<th>Iter</th>
<th>CPU</th>
<th>Iter</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>15</td>
<td>0.4</td>
<td>16</td>
<td>0.4</td>
<td>16</td>
<td>0.2</td>
</tr>
<tr>
<td>20</td>
<td>25</td>
<td>2.8</td>
<td>24</td>
<td>2.7</td>
<td>23</td>
<td>1.3</td>
</tr>
<tr>
<td>30</td>
<td>34</td>
<td>12.9</td>
<td>35</td>
<td>13.9</td>
<td>32</td>
<td>4.6</td>
</tr>
<tr>
<td>40</td>
<td>45</td>
<td>30.3</td>
<td>39</td>
<td>25.0</td>
<td>41</td>
<td>11.6</td>
</tr>
<tr>
<td>50</td>
<td>77</td>
<td>95.1</td>
<td>57</td>
<td>78.3</td>
<td>56</td>
<td>37.0</td>
</tr>
<tr>
<td>60</td>
<td>63</td>
<td>145.6</td>
<td>53</td>
<td>117.7</td>
<td>59</td>
<td>59.9</td>
</tr>
</tbody>
</table>

From the table we find that the use of PML does not hamper the convergence of the iterative method, compared to, e.g., the Helmholtz equation with EM2. In fact, the number of iterations to reach convergence for Helmholtz problems with any type of boundary conditions tested here is similar. The number of iteration behaves more or less linearly with respect to $f$. The increase in CPU time in case of PML and ABC is mainly due to the additional grid points that are used in the PML or “sponge” region.
4 CONCLUSIONS

In this paper we iteratively solve the two dimensional PML equation with an inner-outer iteration process. For the outer iteration, Bi-CGSTAB is employed. The inner iteration process is accomplished by a multigrid method, and is applied to a modified PML equation. By using this process, a fast convergence is observed.

In the same line with this study, an application of the method to an exterior Helmholtz problem with the Bayliss-Gunzburger-Turkel (BGT) boundary condition can be found in [9], which also shows fast convergence.

REFERENCES