The Analytical Mechanics of Consumption

In Mechanical and Economic Systems and Control

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Master of Science Thesis

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Abstract

The Utility Lagrangian and the Surplus Hamiltonian in economic engineering do not depend on consumption. Two theories are proposed to include the effect of consumption in the Utility Lagrangian and the Surplus Hamiltonian. The second of these two theories resolves the dissipation obstacle in port-Hamiltonian systems as an additional result.

The first theory includes consumption as a fractional-order derivative in the Fractional Utility Lagrangian, following an action principle for dissipative systems proposed in the literature. The principle of maximal utility from economic engineering results in a fractional Euler-Lagrange equation that relates a change in price to the accrued benefit less the accrued depreciation due to consumption. A Legendre transform of the Fractional Utility Lagrangian results in a Fractional Surplus Hamiltonian the reveals the effect of consumption on surplus. A drawback of this theory is that control formalisms of port-Hamiltonian systems theory cannot be applied to the Fractional Surplus Hamiltonian, since it is not canonical.

The second theory includes consumption in the Surplus Hamiltonian with complex state variables and —in general— dissipation in the Hamiltonian formalism. The theory of Complex Hamiltonians is developed to model damped harmonic oscillators as canonical Hamiltonian systems. The Complex Hamilton's equations result in the equations of motion of a damped harmonic oscillator and are equivalent to the canonical Poisson bracket between the complex state and the Complex Hamiltonian. Applying control formalisms from port-Hamiltonian systems theory to the Complex Hamiltonian bypasses the dissipation obstacle that in real-valued port-Hamiltonian systems stymies the control of dissipative systems.

Utilizing the analogies from economic engineering results in a Complex Surplus Hamiltonian. Evaluating the Complex Surplus Hamiltonian shows that the marginal propensity to consume is the economic analog of the damping ratio. Control formalisms from port-Hamiltonian systems can be applied to the Complex Surplus Hamiltonian.

As an additional result, it is shown that the fractional derivative can be used as a storage variable for heat generated by frictional dissipation; this results in an expression for dissipated energy of the same form as the familiar expressions for kinetic and potential energy.

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Preface

For the first time in my academic career I was asked not to just solve a problem, but to find a problem that is worthwhile to solve. Finding a problem meant studying the theory of economic engineering — developed the supervisor of this thesis— as well as studying the economic literature, and comparable approaches to economics developed by physicists.m However, this was the easy part. This thesis is the result of the difficult part: identifying and solving the problem.

I want to thank my supervisor dr.ir. M.B. Mendel. Not only for giving me the opportunity to do research in the field of economic engineering, but predominantly for putting me in control of my research and for allowing me to go beyond the standard curriculum of a systems and control engineer. Special thanks go to dr.ir. S. Boersma for the fruitful discussions on the theory of Complex Hamiltonians, developed in this thesis. Furthermore, I want to thank my peers in the economic engineering group at DCSC for the meetings and discussions that have had a great contribution to this thesis. Finally, I want to thank the members of the thesis committee for showing interest in this thesis. In particular, prof. dr. ir. B. De Schutter for preliminary feedback on the structure of this thesis and dr. ing. S. Grammatico for a brief, but fruitful correspondence on the Complex Hamiltonian.

Notes to the reader:

1. The references used in each chapter are listed at the end of that chapter
2. Important equations are emphasized by enclosing them in a cyan box
3. At the end of each chapter, the contributions made in that chapter are enumerated

" Study hard what interests you the most in the most undisciplined, irreverent and original manner possible. "

— *Richard Feynmann*

Chapter 1

Introduction

1-1 Economic Engineering

In economic engineering [1], economic systems are modeled as causal dynamical systems. The economic engineering theory is based on the analogies between the dynamics of economic phenomena on the one hand and mechanical (and electrical) phenomena on the other hand. Based on these analogies, fundamental dynamical relations in economic systems are derived from the dynamical relations in mechanical systems. An important advantage of deriving such relations is that it enables the application of control formalisms to control economic systems. The theories developed in economic engineering can be divided into two main approaches. The first approach uses Newtonian mechanics to describe changes in prices and quantities. The second approach uses analytical mechanics to model agents that maximize their utility or their surplus. This thesis focuses on the second approach.

In economics, utility and surplus are fundamental concepts [2, 3, 4, 5, 6]. Utility is a level of satisfaction, or happiness, that an agent obtains from an economic activity, see e.g. [5]. Surplus, is the amount of available funds that can generate economic activity, see e.g. [2]. Utility and Surplus are further summarized in Section 2-2.

However, both utility and surplus are inseparable from a third fundamental concept: Consumption [7, 8, 5, 6]. Consumption is the destruction of both utility and surplus [7]; consuming an economic asset destroys its utility and consuming an amount of surplus destroys its ability to perform further economic activity. Consumption is also further summarized in Section 2-2.

The inseparability of on the one hand utility and surplus and on the other hand consumption presents a problem for the analytical mechanical approach of economic engineering. Dissipation is the mechanical analog of consumption [1]. However, it is widely believed that dissipation cannot be included in analytical mechanics, see e.g. the introduction of [9] and references therein. By analogy, this implies that economic engineering theories of utility and surplus cannot include the effects of consumption.

In this thesis, two theories are developed to include consumption in the analytical mechanical approach of economic engineering:

1. The Mechanics of Consumption with Fractional Derivatives
2. The Mechanics of Consumption with Complex-Hamiltonian Systems

1-2 The Mechanics of Consumption with Fractional Derivatives

The first theory: *The Mechanics of Consumption with Fractional Derivatives* is developed in Chapter 3. This theory is based on the assumption in economic engineering that utility can be modeled analogous to the Lagrangian from mechanics [1]. Consumption is included in the Utility Lagrangian using the fractional calculus of variations, a method developed to include dissipative systems in the action principle [9, 10, 11, 12, 13]. A maximum utility principle is derived analogous to the action principle in mechanics by applying the fractional calculus of variations. The maximum utility principle reveals the dynamic relation between the marginal utilities from trading, consuming and holding assets.

It will appear that the theory of consumption with fractional derivatives cannot be included in the port-Hamiltonian [14, 15] description and therefore not in energy-based control formalisms, as shown in Section 3-6. For this reason a second theory is developed.

1-3 The Mechanics of Consumption with Complex-Hamiltonian Systems

The second theory, *The Mechanics of Consumption with Complex-Hamiltonian Systems*, is based on the assumption in economic engineering that surplus can be modeled analogous to the Hamiltonian from mechanics [1]. Consumption is included in the Surplus Hamiltonian by describing the agent with complex-valued state variables. This theory is developed in two steps. First the theory of *Complex-Hamiltonian Systems* is developed for the mechanical damped harmonic oscillator in Chapter 4. Then, the theory of Complex-Hamiltonian systems is applied to economic systems in Chapter 5, using the mechanical-economical analogs summarized in Section 2-3.

The concept of describing dissipative systems with a complex-valued Hamiltonian originates from quantum mechanics, see e.g. [16, 17]. The theory in Chapter 4 is developed by defining the equations of motion of a damped harmonic oscillator as the complex Poisson bracket between the state and the Complex Hamiltonian. Using this definition, the Complex Hamiltonian is derived by substituting the known equations of motion of a damped harmonic oscillator. The complex Hamiltonian is then incorporated in the port-Hamiltonian description of mechanical systems, resulting in the Complex-Hamiltonian systems theory.

The Complex-Hamiltonian theory is used to model economic systems in Chapter 5. The analogy between the Hamiltonian and surplus introduced by economic engineering is extended by including consumption in the complex Surplus Hamiltonian. The complex Hamiltonian will be used to model the saving-investment cycle from economics, [2]. It will be shown that the marginal propensity to consume from economics is equal to the damping ratio from mechanics. Chapter 5 furthermore shows that control methods from port-Hamiltonian theory can be applied to the complex-port-Hamiltonian description of consumers.

1-4 Additional Results for Modeling Dissipative Systems

As a byproduct of both theories, additional contributions are obtained in the mechanical engineering domain. The first additional contribution follows from applying mechanics with fractional derivatives in the bond graph description of dynamical systems. Chapter 6 shows how the fractional derivative of position can be used as frictional energy storage variable. This results in a mechanical expression for dissipated energy of the same form as the expression for potential and kinetic energy.

The second, and most important, additional contribution is the theory of complex port-Hamiltonian systems in Chapter 4. Although this theory is developed as a tool to describe consumers in the framework of economic engineering, the theory itself contributes to the description of damped harmonic oscillators as port-Hamiltonian systems. As shown in Section 4-5, including dissipation in the complex Hamiltonian is a solution to the dissipation obstacle in port-Hamiltonian theory.

1-5 Thesis Outline

This thesis is structured as follows.

Chapter 2 presents the background knowledge used in this thesis: the concepts of utility, surplus, and consumption in economics, as well as the the theories developed by the economic engineering group, and an overview of port-Hamiltonian systems theory.

Chapter 3 presents the first theory: *The Mechanics of Consumption with Fractional Derivatives*. To be self-contained, the chapter summarizes the relevant theories of fractional calculus from the literature. The theory developed in Chapter 3 achieves the goal of modeling consumers using Lagrangian and Hamiltonian mechanics, but fails in applying control methods from port-Hamiltonian systems theory.

Chapter 4 presents the major contribution of this thesis: the theory of *Complex-Hamiltonian Systems and Control*. This chapter can be read separately from the rest of this thesis. Any pertinent background material on port-Hamiltonian system theory can be found in Section 2-4.

Chapter 5 applies the theory of complex port-Hamiltonian systems to economic systems. This results in the second economic engineering theory: *The Mechanics of Consumption with Complex-Hamiltonian Systems*. This theory is developed by combining the theory developed in Chapter 4 and the mechanical-economic analogs developed in economic engineering, summarized in Section 2-3.

Chapter 6 presents an additional contribution resulting from the work done in this thesis. It is shown how fractional calculus can be used in the bond graph description of systems to derive a function for heat from energy dissipation with fractional derivatives. Since the theory in Chapter 6 is not relevant for the rest of the thesis, it can be read separately from the thesis.

Chapter 7 makes concluding remarks on the thesis, discusses the work presented in this thesis, and identifies topics for further research.

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Chapter 2

Background

2-1 Introduction

This chapter provides background material from the literature that will be used throughout this thesis. Section 2-2 provides background of the economic concepts of consumption, utility and surplus and compares their role in economics to their role in this thesis. Section 2-3 summarizes the theories of utility and surplus introduced by economic engineering. Finally, Section 2-4 briefly introduces the theory of port-Hamiltonian systems and addresses the consequence of the dissipation obstacle on modeling economic systems as port-Hamiltonian systems.

2-2 Consumption

In the words of Adam Smith consumption is the sole end and purpose of all production [1]. Consumption is defined as using up a resource for the acquisition of utility [2]. Food, beverages, clothes, consumer electronics are typical examples of consumption goods, but also durable goods such as cars and machines can be consumed. The consumption of durable goods is typically measured in terms of depreciation [3]. Consumption is studied both in micro- and macroeconomics. Microeconomists study individual agents that manage their consumption so as to maximize their utility [4]. Macroeconomists study the consumption of aggregate economies, analyzing relation between consumption expenditure and disposable income [5],[6].

In economic engineering, consumption is modeled as the analog of dissipation. Utility and surplus, on the other hand, are modeled in economic engineering with Lagrangian and Hamiltonian mechanics, respectively [7]. This implies that the utility and the surplus function, cannot include consumption, see Section 2-3-3.

In this thesis, two theories are developed to include consumption in the utility and the surplus function in economic engineering. In Chapter 3, consumption is included in the Utility Lagrangian and the Surplus Hamiltonian by applying fractional calculus. In Chapter

5, consumption is included in the Surplus Hamiltonian by applying the theory of complex-Hamiltonian systems, developed in Chapter 4.

2-2-1 Utility

Utility is a key concept in economics [4, 8, 9, 10, 11]. Since its introduction by Daniel Bernoulli [12] economists have used utility as a level of satisfaction to model the behaviour of economic agents [11]. In the paradigm of the maximum utility problem, economic agents are assumed to manage their consumption so as to maximize their utility [4]. In neoclassical economics, the utility-maximizing agent is modeled as an optimization problem. Economists call this optimization problem the maximum utility problem. The maximum utility problem can be solved using Lagrangian optimization techniques, see e.g.[13]. The Lagrangian $L(x)$ represents the level of utility obtained from consuming a number of products $x(t)$, given a budget. The optimal consumption of $x(t)$ is found using Lagrange's theorem. The distinction between the consumption of non-durables and consumption of durables in utility functions is typically not made; utility is modeled as a function of only the non-durable consumption and not of holding durables [14].

In economic engineering, utility is also modeled as a Lagrangian. However, in economic engineering utility is modeled as the Lagrangian from Lagrangian mechanics, which is distinct from the Lagrangian used in optimization.

Instead of treating the utility-maximizing agent as an optimization problem, the maximization of utility by an agent is considered as a fundamental principle in economic engineering, see section 2-3-1. This principle is analogous the principle of least action for a mechanical system. By modeling the economic agent as a Lagrangian system instead of as an optimization problem, the differential equations that represent the internal dynamics of the agent can be retrieved from the Euler-Lagrange equation. These differential equations will reveal the price dynamics of the utility maximizing agent.

In Chapter 3, I extend this theory by including consumption in the Utility Lagrangian. This extension results in a Euler-Lagrange equation that includes the effect of depreciation due to consumption on the price dynamics, see Section 3-4.

2-2-2 Surplus

Economic surplus occurs when the amount of money reserved for a period, the disposable income, is different from the amount of money spent in that period: the consumption [2]. Surplus plays an important role in economics [5]. An agent that has a surplus can lend its money to a borrower so that the borrower can increase its productivity. In the saving-investment cycle it is assumed that the aggregate saving, the surplus, equals the aggregate investment [6]. In economic engineering, surplus and savings are interpreted as the economic analogs of energy, see Section 2-3-2. Based on this analogy, Hamiltonian mechanics is used to model the surplus of economic agents.

As with the Utility Lagrangian, the Surplus Hamiltonian in economic engineering cannot include consumption.

In this thesis, two theories that include consumption in the Hamiltonian formulation of surplus are developed. The first theory that includes consumption in the Hamiltonian is developed in Chapter 3. Fractional calculus is used to include consumption in the Utility Lagrangian, which is related to the Surplus Hamiltonian by a Legendre transform, see Section 2-3-2. The second theory is developed in Chapter 4 and Chapter 5. In Chapter 4, complex state variables are used to include mechanical dissipation in the Hamiltonian, resulting in what I call complex-Hamiltonian systems theory. This theory is applied to include consumption in the Surplus Hamiltonian in Chapter 5.

2-3 Economic Engineering

In an effort to make this thesis self-contained, this section presents the relevant theories introduced by dr. M. Mendel as the head the economic engineering group at DCSC [7].

The starting point of economic engineering is modeling economic systems analogous to mechanical systems. The analogs between mechanical and economic systems are similar to the analogs between mechanical, electrical, hydraulic and pneumatic systems, extensively used in bond graph theory [15]. We start by defining the state-space variables of the economic agent

$$q := \text{asset stock} \qquad [\#] \tag{2-1}$$

$$p := \text{price per unit asset} \qquad \left[\frac{\$}{\#}\right] \tag{2-2}$$

Asset stock is taken as the amount of a certain asset[1] held or desired by the economic agent. The agent's asset stock is positive when the agent owns the amount of assets (long in finance terms) and negative if the agent owes an amount of assets they do not own (short). Asset stocks are measured in physical units, where a continuous scale of units is consumed, e.g. weight or metric volume. Price is defined as the dollar price per unit of the corresponding asset, i.e for every asset, q_k, there is a corresponding price, p_k, where $k \quad N$ and N is the space of unique assets. The state of an economic agent is given by the state vector $x = \begin{pmatrix} q_1 & \dots & q_N & p_1 & \dots & p_N \end{pmatrix}^{|} \quad \mathsf{R}^{2N}$. From the units of the states, it follows that a volume on the state space has the units of money, M [\$]. By time derivation it follows that the economic analog of energy is ***surplus***, Σ [$\frac{\$}{\text{yr}}$], and ***growth***, G [$\frac{\$}{\text{yr}^2}$], is the economic analog of power.

A change in the state variables with respect to time is

$$\dot{q} := \text{flow of assets} \qquad \left[\frac{\#}{\text{yr}}\right] \tag{2-3}$$

$$\dot{p} := \text{price movement} \qquad \left[\frac{\$}{\#\text{yr}}\right] \tag{2-4}$$

Asset flow is the amount of assets transferred in physical units per time. The standard unit of time (second, day, year, etc.) depends on the type of asset and economic system. The price movement is the change of price per unit of time signaled between agents. Following bond graph terminology, it is said that the asset flow, $\dot{q}$ is the economic flow variable, $f(t)$, and the price movement, $\dot{p}$, is the economic effort variable, $e(t)$.

[1] Assets are any tangible or intangible results from past economic transactions or activities [2]

2-3-1 The Utility Lagrangian

Following Lagrangian mechanics, see e.g. Landau [16], the state of an agent at any given time can be derived if the asset stock at time t, $q(t)$, and at what rate the stock is changing, $\dot{q}(t)$, are known. Let us introduce he running, or intertemporal, Utility Lagrangian

$$L(\dot{q}, q) \tag{2-5}$$

In economic engineering, it is assumed that over a time interval $[a, b]$ economic agents changes their asset stock from $q(a)$ to $q(b)$, such that the integral

$$S = \int_a^b L(\dot{q}, q) dt \tag{2-6}$$

takes the highest possible value. That is, the agent changes their stock so as to maximize the total utility S.

Following the calculus of variations, the total utility is maximized if, assuming a concave utility function [11], the first variation of S vanishes

$$\delta S = 0 \tag{2-7}$$

The condition in (2-7) will be referred to as the principle of maximum utility. In contrast to economists that model the maximization of utility as a Lagrangian optimization problem, the maximization of utility is considered in economic engineering as a fundamental principle of the economic agent. Evaluating the principle of maximum utility with the calculus of variations results in the Euler-Lagrange equation:

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \left(\frac{\partial L}{\partial q}\right) = 0 \tag{2-8}$$

From classical mechanics we have that the utility of an agent is maximum if the Euler-Lagrange equation is satisfied [16].

The marginal utilities in the Euler-Lagrangian equation (2-8) are characterized as follows.

Price The marginal utility from to a change in a asset flow is defined as the price, p, determined by the agent; it is the price the agent is willing to pay (receive) for an increasing (decreasing) flow of assets

$$\frac{\partial L}{\partial \dot{q}} \equiv p \tag{2-9}$$

Benefit (Cost) The Euler-Lagrange equation entails that the flow of price is equal to the instantaneous marginal utility from holding an asset, the benefit, B

$$\dot{p} = \frac{\partial L}{\partial q} = B \tag{2-10}$$

The benefit is the opposite of a cost and is expressed in units $\left[\frac{\$}{\#yr}\right]$.

Integrating the Euler-Lagrange equation results in a price dynamics equation, stating that the reservation price at time t is equal to the initial price plus the accrued benefit

$$p(t) = p(0) + \int^t B dt \tag{2-11}$$

2-3-2 The Surplus Hamiltonian

Whereas utility is analogous to the Lagrangian, surplus analogous to the Hamiltonian [7]. In mechanics, the Hamiltonian is a function that represents the mechanical energy in a system. In economic engineering, the Hamiltonian represents the total surplus. The Surplus Hamiltonian is related to the Utility Lagrangian by a Legendre transform from asset flow, $\dot{q}$, to price, p.

$$H(p, q) = p\dot{q} - L(q, \dot{q}) \tag{2-12}$$

Analogous to the Hamiltonian of classical mechanics, the Surplus Hamiltonian for a closed system is a conserved quantity, $\frac{dH}{dt} = 0$. Furthermore, surplus is the generator of economic activity by means of the Hamilton equations

$$\frac{\partial H}{\partial q} = -\dot{p}, \qquad \frac{\partial H}{\partial p} = \dot{q} \tag{2-13}$$

A change in surplus due to an increase in asset stock thus results in a negative price-movement. The second Hamilton's equation states that a change in surplus due to a change in price results in a positive asset flow. This equation represents the price effect of scarcity and abundance of products.

2-3-3 Dissipation as Analog of Consumption

The final economic analogy summarized in this section is the analogy between dissipation and consumption [7]. In mechanical systems, dissipation is the conversion of mechanical energy into heat [17]. Dissipation is an irreversible process; once mechanical energy is converted into heat it cannot be converted back into mechanical energy without the use of additional work. In economic systems, consumption has an analogous effect on surplus and utility; once economic surplus or utility is consumed, it cannot be reversed [18].

The analogy between consumption and dissipation introduces a problem for the analogies between utility and the Lagrangian and surplus and the Hamiltonian. Lagrangian and Hamiltonian mechanics can only be applied to conservative systems, i.e. systems that do not dissipate energy, see the introduction of [19] and references therein. This implies that the Utility Lagrangian and the Surplus Hamiltonian cannot be applied to economic systems with consumption.

In this thesis, two theories are developed to include consumption in the Utility Lagrangian and the Surplus Hamiltonian. The first theory, developed in Chapter 3, includes consumption in the Utility Lagrangian. The main result of this theory is analytically showing that consuming assets decreases their value, resulting in a decrease in price caused by depreciation, see Section 3-4.

Consumption does not only have an effect on the (to utility related) price; it also has an effect on surplus [6]. In Chapter 5, the complex-Hamiltonian theory, which itself is developed in Chapter 4, is applied to include the effect of consumption on surplus. The main results of this theory are showing that the marginal propensity to consume is analogous to the damping ratio in mechanics (Section 5-2-3) and that complex port-Hamiltonian can be used to model and control economic agents, see Section 5-3.

2-4 Port-Hamiltonian Systems

Economic engineering seeks for control methods that leverage energy. From the analogies between the Lagrangian and utility on one hand and the Hamiltonian and surplus on the other hand, energy-based control methods are preferred over signal-based methods, since energy-based methods can include utility and surplus as a quantity. Using signal-based methods requires utility and surplus to be accounted for separately from the signal dynamics.

In this thesis I will explore the use of the control formalisms introduced by port-Hamiltonian (pH) systems theory [20, 21] . In this section, I give an overview of pH systems theory, its control formalisms and a major open problem for applying its control formalisms to dissipative systems: the dissipation obstacle.

pH system theory [21] combines the Hamiltonian dynamics of a system and a geometric interconnection structure that models the input-output behaviour of the system and its connected environment, see Figure 2-1. Key feature of pH systems is leveraging the Hamiltonian as energy function. Furthermore, its geometric interconnection structure enables the modelling of many (complex) physical systems [22].

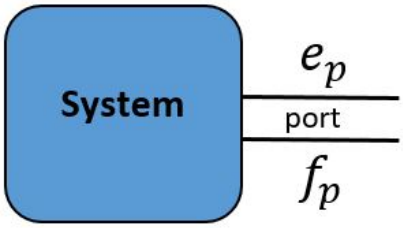

Figure 2-1: Illustrative system with external port. The port is always constructed by an effort signal, $e(p)$, and a flow signal $f(p)$ so that the product of the signal is power.

2-4-1 Input-State-Output Port-Hamiltonian System

An interesting feature of the port-based approach is that *effort* and *flows* are used as the natural input and output of a system. This means that if a mechanical system is driven by a force, the velocity of the system is the output. Or, if an electrical network has a voltage input, the current will be the output. For an economic system, this implies that a pair of asset flow and price movement represents the input-output pair of a system. The input-state-output description of pH systems [23]

$$\begin{aligned} \dot{x} &= [J - R]\nabla H + g(x)u \\ y &= g^{\top}(x)\nabla H \end{aligned} \tag{2-14}$$

Here we have that $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m, m \leq n$ is the input of the system, $y \in \mathbb{R}^m$ the output of the system, $H : \mathbb{R}^n \Rightarrow \mathbb{R}$ is the Hamiltonian, $g(x) : \mathbb{R}^n \Rightarrow \mathbb{R}^{n \times m}$ denotes the input matrix, $R : \mathbb{R}^n \Rightarrow \mathbb{R}^{n \times n}$ is a semi-definite matrix representing the dissipative elements, $J : \mathbb{R}^n \Rightarrow \mathbb{R}^{n \times n}$ is the interconnection matrix and the ∇ operator denotes the vector of partial derivatives: $\nabla = [\frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_n}]^{\top}$. This interconnection matrix $J = -J^{\top}$ represents the symplectic structure of the pH system. This is an important feature, since this is the structure on which Hamiltonian dynamics are defined. This structure is violated when damping is present in the pH theory, $R = 0$.

2-4-2 Economic Port-Hamiltonian System

With the analogies between mechanical and economic systems, economic systems can be modeled as a pH system. In fact, simple macroeconomic models have been formalized in the pH framework in the work of Macchelli [24]. An important difference between the work of Macchelli and the economic engineering approach applied in this thesis, however, is the dynamical interpretation of price. Whereas in economic engineering, price is the analog of momentum, in the work of Macchelli price is the analog of a force. As a result, the Hamiltonian of Macchelli does not have the units of income or cash flow, but of money itself. Furthermore, Macchelli does not consider the supply and demand curves to be analogous to inertia, but to dissipation.

In the economic engineering framework, the pH formulation of a simple autonomous economic systems with asset stock q, price p, and surplus function H is

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = [J - C] \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} \tag{2-15}$$

Here, C is the consumption matrix as analog of the dissipation matrix in (2-14), $\dot{q}$ is the rate of change in asset stock, $\dot{p}$ is the rate of change in price, and J is again the interconnection matrix. In this formulation, however, consumption is not considered as an endogenous property of the surplus of an agent. Since the Surplus Hamiltonian and the Utility Lagrangian are related by a Legendre transform, handling consumption outside the Surplus Hamiltonian implies that the agent does not obtain utility from consumption. Conceptually, this is not in line with the utility functions in economics, which in general are a function of consumption.

This conceptual problem with economic pH systems is dealt with in Chapter 4 and Chapter 5. In Chapter 4, the pH framework is reformulated with complex coordinates so that dissipation is endogenous to the Hamiltonian. In Chapter 5, this reformulation is used to model economic systems.

However, there is not only a conceptual problem regarding dissipation and consumption in pH systems theory. Several of the developed control methods in pH systems theory cannot be applied to dissipative systems. This is known as the dissipation obstacle.

2-4-3 The Dissipation Obstacle

Besides modeling system with respect to the energy function, control methods have been developed in pH systems theory that "put energy back in control" [20]. Three of these methods are summarized in Appendix C. However, these control methods suffer from what is known as the dissipation obstacle, see Appendix C-0-5.

In essence, the dissipation obstacle is due to the damping matrix, R, in the input-state-output description (2-14), see [20, 21, 25]. This implies that control formalisms from pH systems cannot be applied to systems with dissipation ($R > 0$). For the implementation of pH systems theory on economic systems, this implies that economic systems with consumption will suffer from the dissipation obstacle, following from the analogy between dissipation and consumption, see Section 2-3-3.

In Chapter 4, the dissipation obstacle will be resolved by omitting the need for a dissipation matrix. By formulating a Hamiltonian with complex coordinates, the Complex Hamiltonian, the effect of dissipation can be accounted for in the Hamiltonian itself. In Section 4-5, control methods from pH systems theory are redefined for the Complex Hamiltonian. Using the redefined methods, controllers are designed for dissipative systems without being affected by the damping obstacle. The methods developed in Chapter 4 are applied to economic systems in Chapter 5.

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Chapter 3

The Mechanics of Consumption with Fractional Derivatives

3-1 Introduction

In the framework of economic engineering, utility is a function of holding and trading assets, see Section 2-3-1. Using the calculus of variations to maximize the total utility over a period, the differential equations that describe the dynamics of the economic agent are revealed in the Euler-Lagrange equation. Neoclassical economists assume that individuals maximize their utility, see e.g. [1]. Economists consider utility primarily as a function of consumption and, in the view of some economists, secondarily as a function of holding assets [2]. In economic engineering, however, consumption is modeled as the dynamical analog of dissipation, see Section 2-3-3. Since dissipation is not included in the classical sense of the Lagrangian formalism [3], it follows by analogy that consumption cannot be included in the Utility Lagrangian from economic engineering. The approach of economic engineering to utility thus does not correspond to that of economists.

Summarizing, this chapter addresses the following problem

Problem

The Utility Lagrangian of economic engineering is not a function of consumption.

In mechanics, several techniques have been introduced in order to include dissipative systems in the Lagrangian formalism. These techniques include Rayleigh's dissipation function, multiplying the Lagrangian by an auxiliary exponent to account for dissipation, and directly including the microscopic behaviour of frictional forces in the Lagrangian, see the works of Riewe [4, 5] and references therein. In his work on Lagrangian mechanics for dissipative systems, Riewe indicated that although all these techniques are correct non of them provide a general method of including dissipation in classical Lagrangian mechanics.

Riewe showed that a general method of including dissipation in Lagrangian mechanics can be introduced by using fractional derivatives. In Riewe's method, a fractional derivative is introduced as an independent variable for frictional energy in the Lagrangian. Using variational calculus, this results in the desired velocity dependent frictional force in the equations of motion. Riewe's work was the beginning of the theory of fractional calculus of variations. This theory has been further developed by several authors over the past two decades. An overview of these developments is provided in [6]. Furthermore, use of fractional calculus in economics has been researched in e.g. [7].

In this chapter, I use the fractional calculus of variations to include consumption in the Utility Lagrangian in economic engineering. Analogous to Riewe's frictional energy, utility from consumption is modeled with fractional derivatives. Following the steps of the fractional calculus of variations, I obtain an economic meaningful Euler-Lagrange equation. The Euler-Lagrange equation reveals the dynamical relation between the marginal utilities from trading, holding and consuming commodities.

Solution

Include consumption as an independent variable in the Utility Lagrangian of economic engineering by using the *fractional calculus of variations*.

The remainder of this chapter is structured as follows. Section 3-2 presents the relevant mathematical background of the fractional calculus used in this chapter. Section 3-3 presents the main result of this chapter: the Fractional Utility Lagrangian. Section 3-4 elaborates on the economic interpretation of the Fractional Utility Lagrangian. Section 3-5 presents the effect of consumption on the Surplus Hamiltonian. Section 3-6 discusses the application of control techniques from port-Hamiltonian systems on the derived models. Section 3-7 provides a conclusion and discussion of this chapter.

3-2 Background: Fractional Derivatives

Whereas integer-order calculus only considers $\frac{d^\alpha f(t)}{dt^\alpha}$ for integer values of α, fractional calculus considers the differential for every real or complex valued α.

In this section I omit some mathematical formalities of the fractional derivative and approach it from an engineering perspective, using operator calculus. A more formal formulation of fractional derivatives is presented hereafter.

3-2-1 Basic Concept

Let us first explore the fractional derivative through an example using the exponent function $f(t) = e^{at}$. It is known from integer-order calculus that the first order derivative of this function is

$$\mathsf{D}^1 f(t) = ae^{at},$$

Here, the operator D^n represents time derivation of order n: $D^n = d^n/dt^n$. The second order derivative of $f(t)$ is

$$D^2 f(t) = a \cdot ae^{at} = a^2 e^{at}$$

For negative valued n, we get the indefinite integral, or anti-derivative, e.g. in case $n = -1$ we have

$$D^{-1} = I^1 = a^{-1} e^{at}$$

In general, the $n^{th}-$order derivative is given by

$$D^n f(t) = a^n e^{at}$$

where n is any integer. Fractional calculus extends this generalization by letting n be any rational number.

Setting $n = \frac{1}{2}$ we get

$$D^{\frac{1}{2}} f(t) = \sqrt{a} e^{at}$$

Taking this derivative twice result in the first order derivative of $f(t)$:

$$D^{\frac{1}{2}} D^{\frac{1}{2}} f(t) = D^1 f(t) = ae^{at}$$

The above two equations play an important role in the fractional calculus of variations of Lagrangians as we will see in the next section.

The above example is only valid for exponential functions in case the interval of evaluation runs from zero to infinity. Fractional-order operators, fractional derivatives in particular, have some subtleties that are not present in integer-order operators. Contrary to integer-order derivatives, the fractional derivatives are non-local operator; the values of fractional derivatives are dependent of past or future states during a time interval. Furthermore, there exists more than one definition, each with its own calculus. In the next section, a more formal introduction to fractional-order operators is provided. An historic overview of the fractional calculus can be found in [8].

3-2-2 Fractional Calculus

There are several definitions for a fractional derivative. In this work I use the definitions from Riemann-Liouville and that of Caputo, see [6]. Both definitions are based on the **Riemann-Liouville fractional integral**:

$$_aI_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{3-1}$$

and

$$_tI_b^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{x(\tau)}{(\tau-t)^{1-\alpha}} d\tau \tag{3-2}$$

where $\Gamma(\alpha)$ denotes the Gamma function. The fractional integral operators $_aI_t^\alpha$ and $_tI_b^\alpha$ are called the left and right fractional integrals, respectively.

On an interval $[a, b]$, the left integral is done from a up to the point of evaluation, t, and the right integral is done from t to b.

The **Riemann-Liouville fractional derivatives** are defined by

$$ {}_aD_t^\alpha x(t) \equiv D_t^n {}_aI_t^{n-a} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{x(\tau)}{(t-\tau)^{1+\alpha-n}} d\tau \tag{3-3} $$

and

$$ {}_tD_b^\alpha x(t) \equiv D_t^n {}_tI_b^{n-a} x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b \frac{x(\tau)}{(\tau-t)^{1+\alpha-n}} d\tau \tag{3-4} $$

where $n = [\alpha] + 1$.

The Caputo definition of the fractional derivatives changes the order of derivation and integration with respect to the Riemann-Liouville derivative. The left and right **Caputo fractional derivatives** are defined as

$$ {}_a^C D_t^\alpha x(t) \equiv {}_aI_t^{n-\alpha} D_t^n x(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{x^{(n)}(\tau)}{(t-\tau)^{1+\alpha-m}} d\tau \tag{3-5} $$

and

$$ {}_t^C D_b^\alpha x(t) \equiv (-1)^n {}_tI_b^{n-\alpha} D_t^n x(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{x^{(n)}(\tau)}{(\tau-t)^{1+\alpha-m}} d\tau \tag{3-6} $$

The difference between the Riemann-Liouville and the Caputo definitions thus is the order in which integration and differentiation are performed.

From the above definitions it is clear that contrary to ordinary derivatives, fractional derivatives are non-local operators. The left fractional derivatives depend on all past and present values and the right fractional derivatives depend on the present and future values on a specified interval. Therefore, the interpretation is that the left fractional derivatives as causal operators with memory where the size of the memory depends on the length of the evaluated interval [8]. The right fractional derivative is anticausal[1] since it depends not only on present values, but also all future values on the evaluated interval.

In this chapter the following properties of fractional derivatives are needed in the derivation of the price equation from the principle of maximum utility.

Property I Integration by parts, see Klimek [9]

$$ \int_a^b f(t) {}_aD_t^\alpha g(t) dt = \int_a^b g(t) {}_t^C D_b^\alpha f(t) dt + \Big[f(t) {}_aI_t^{1-\alpha} g(t) \Big]_a^b \tag{3-7} $$

Property II On a short interval, the following approximation holds, see Lazo [10]

$$ {}_t^C D_b^{\frac{1}{2}} \approx -{}_a^C D_t^{\frac{1}{2}}, \qquad \text{for} a - b << 1, t = a + \frac{1}{2}(b-a). \tag{3-8} $$

Property III From the general property ${}_aI_t^\alpha {}_aI_t^\beta = {}_aI_t^{\alpha+\beta}$, see Kilbas [11], it follows that, see Lazo [10]

$$ {}_aD_t^{\frac{1}{2}} {}_a^C D_t^{\frac{1}{2}} f(t) = \frac{d}{dt} {}_aI_t^{\frac{1}{2}} {}_aI_t^{\frac{1}{2}} \frac{d}{dt} = \frac{d}{dt} f(t) \tag{3-9} $$

[1] Sometimes only operators that depend exclusively on future values are called anticausal.

3-3 Principle of Maximum Utility with Fractional Derivatives

Following Riewe's [4, 5, 10] method of including dissipative elements in the Lagrangian, I include consumption in the Utility Lagrangian. In particular, I include consumption in the Utility Lagrangian so that the variational principle results in a price equation (Euler-Lagrange) that includes the effect of consumption on the price. The effect of consumption on the price is an extension of the price equation (2-11) in economic engineering.

3-3-1 Fractional Derivative as Consumption

Let me introduce the right-sided Caputo fractional derivative with order $n = 1/2$ of stock as consumption variable, see the definition in (3-6)

$$\breve{q} \equiv {}_t^C\mathsf{D}_b^{\frac{1}{2}} q \tag{3-10}$$

Here, I introduce the breve notation to represent half a fluxion, Newton's notation for a flow. That is,

$$\begin{aligned} \mathsf{D}^0 &= f \\ \mathsf{D}^{1/2} &= \breve{f} \\ \mathsf{D}^1 &= \dot{f} \end{aligned}$$

To include the effect of consumption on the utility, I introduce the consumption as an independent variable in the Utility Lagrangian, i.e. $L = L(q, \breve{q}, \dot{q})$. The intertemporal utility is now a function of asset stock, q, asset flow, $\dot{q}$, and asset consumption, $\breve{q}$. For the sake of clarity, I will refer to the Lagrangian with the added fractional derivative, $L(q, \breve{q}, \dot{q})$, as the *Fractional Utility Lagrangian*.

Defining the consumption variable as the right-sided Caputo derivative, implies that the utility from consumption, $L(\breve{q})$, depends on future values, see the previous section. In economics, a key hypothesis is that agents base their decisions on the utility they expect to receive from making those particular decisions. This is known as the expected utility hypothesis, see e.g. [?]. Therefore, defining the consumption variable as the right-sided Caputo derivative results in a utility from consumption that the agent expects to receive over the interval $[t, b]$.

3-3-2 Fractional Principle of Maximum Utility

The total utility over a time interval $[t_1, t_2]$ is

$$S = \int_{t_1}^{t_2} L(q, \breve{q}, \dot{q}) dt \tag{3-11}$$

The *principle of maximum utility* from economic engineering states (see Section 2-3) that an economic agent adjusts its asset stock from $q(t_1)$ to $q(t_2)$ so that the integral in (3-11) takes the highest possible value. Assuming that the Fractional Utility Function is continuous and concave and the variables $(q, \dot{q}, \breve{q})$ are continuously differentiable, the total utility, S, is maximum if $\delta S = 0$, see e.g. [12].

3-3-3 Obtaining the Price Equation with Fractional Calculus of Variations

Applying the fractional calculus of variations [6], the principle of maximum utility will result in a differential equation that reveals the dynamic relation between the marginal utility from trade, $\dot{q}$, consumption, $\breve{q}$, and holding a stock, q, see Section 2-3.

Treating q, $\breve{q}$ and $\dot{q}$ as independent variables, the variation of the total utility can be written as

$$\delta S = \int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial \dot{q}} \right) \delta \dot{q} + \left(\frac{\partial L}{\partial \breve{q}} \right) \delta \breve{q} + \left(\frac{\partial L}{\partial q} \right) \delta q \right] dt \tag{3-12}$$

Using that $\delta \dot{q} = \frac{d\delta q}{dt}$ and $\delta \breve{q} = {}_t^C\mathsf{D}_b^{\frac{1}{2}} \delta q$, integration by parts can be performed such that the total utility is varied only over the asset stock, δq. Performing integration by parts (see (3-7) for the fractional case), the variation of the total utility can be rewritten as

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \left[{}_aI_t^{\frac{1}{2}} \frac{\partial L}{\partial \breve{q}} \delta q \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + {}_a\mathsf{D}_t^{\frac{1}{2}} \left(\frac{\partial L}{\partial \breve{q}} \right) + \left(\frac{\partial L}{\partial q} \right) \right] \delta q dt \tag{3-13}$$

Since the variation is done over the interval from t_1 to t_2 and not at the begin and end points, it holds that

$$\delta q(t_1) = \delta q(t_2) = 0; \tag{3-14}$$

the integrated terms in (3-13) vanish.

For the total utility to be maximized, the integral in (3-13) has to be zero for every variation δq. This implies that the total utility is maximum if the following condition is satisfied

$$-\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) + {}_a\mathsf{D}_t^{\frac{1}{2}} \left(\frac{\partial L}{\partial \breve{q}} \right) + \left(\frac{\partial L}{\partial q} \right) = 0 \tag{3-15}$$

Equation (3-15) is the fractional Euler-Lagrange (fEL) equation, see [10].

In the context of the principle of maximum utility, the fEL equation implies that there exists a differential equation, relating the marginal utilities —$\partial L/\partial \dot{q}$, $\partial L/\partial \breve{q}$, and $\partial L/\partial q$— when the utility is maximized. Comparing the fEL equation to the price equation in economic engineering (2-11), we see that the marginal utility from consumption is included in the price equation.

3-4 Marginal Utility from Consumption: Expense

The fEL equation (3-15) relates the price movement, $\dot{p}$, to the benefit of holding the stock[2], B, and the left Riemann-Liouville derivative of the effect on utility from consumption

$$\dot{p} = B + {}_a\mathsf{D}_t^{\frac{1}{2}} \left(\frac{\partial L}{\partial \breve{q}} \right) \tag{3-16}$$

That is, the price is not only influenced by the benefit, B, from holding the asset, but also by consumption of the asset.

[2] See (2-11)

To get an economic meaningful interpretation of the cost from consumption in the above equation, I interpret the marginal utility from consumption as the expense, γ

$$\frac{\partial L}{\partial \breve{q}} = \gamma \tag{3-17}$$

An expense is different from a cost [13]. A cost is related to the the expenditure to increase value of an asset, e.g. acquire material, allocation, production. An expense refers to the consumption of an asset. A key difference between a cost and an expense is that a cost refers to an underlying asset, q, whereas an expense does not.

In the fEL equation (3-15), the expense results in a cost (that causes a price movement, $\dot{p}$) by taking its half-order derivative. The resulting cost is the depreciation per unit, D [13].

$$D = {}_a\mathbf{D}_t^{\frac{1}{2}} \left(\frac{\partial L}{\partial \breve{q}} \right) = {}_a\mathbf{D}_t^{\frac{1}{2}} \gamma \tag{3-18}$$

Before comparing the depreciation per unit, D, to the price movement, $\dot{p}$, and the benefit, B, in the fEL equation, I will evaluate an important feature of the Caputo derivative in the variational principle, as introduced by Lazo [10]. In (3-10) I have introduced the consumption variable as the right-sided Caputo derivative of stock. This implies that the expense, γ, is a function of the right-sided Caputo derivative: $\gamma = \gamma({}_t^C\mathbf{D}_b^{\frac{1}{2}} q)$. In particular, if the expense is a linear function we have that

$$\gamma = c\,{}_t^C\mathbf{D}_b^{\frac{1}{2}} q \tag{3-19}$$

where c is a depreciation constant. Substituting the above expression of expense in (3-17) and subsequently in the (3-16) results in

$$\dot{p} = B + c\,{}_a\mathbf{D}_t^{\frac{1}{2}}\,{}_t^C\mathbf{D}_b^{\frac{1}{2}} q \tag{3-20}$$

Setting time, t, as the midpoint of a small interval $[a, b]$ and using property II (3-8) and property III (3-9), we get the following [10]

$$\dot{p} = B - c\dot{q} \tag{3-21}$$

This implies that the (linear) depreciation, $D = c\dot{q}$, has a negative effect on the price-movement:

$$\dot{p} = B - D \tag{3-22}$$

Finally, integrating both sides of this equation we get that the price at time t is equal to the spot-price plus the accrued benefit minus the accrued depreciation

$$p(t) = p(0) + \int^t B dt - \int^t D dt \tag{3-23}$$

To conclude, applying the principle of maximum utility to the Fractional Utility Lagrangian, $L(q, \breve{q}, \dot{q})$, includes the effect consumption in the price equation. By interpreting the marginal utility of consumption as an expense that results in a depreciation in the fEL equation, we get a clear distinction between the benefit (negative cost) and the depreciation. This distinction is also made in economics where benefits are linked to an asset, whereas depreciation is linked to the consumption of an asset [13].

Example: Buying a car
To illustrate the forward price equation in (3-23), let us consider the following situation. We want to buy a car that is manufactured in Japan. The price of the car itself is $p(0)$; this is the price at which the car is sold in Japan and is determined by the market

Since we want to use the car in Europe, we have two options: buy the car in Japan and ship it to Europe, or buy the car at a European dealership. Taking the first option would clearly induce costs; we would have to pay for the shipment of the car, ignoring any governmental fees. Taking the second option, it is likely that we are willing to pay the additional costs that we would have make to ship the car ourselves. Our reservation price $p(t)$ is

$$p(t) = p(0) + \int^{t} Bdt \tag{3-24}$$

where $p(0)$ is the market price of the car itself, $\int^{t} Bdt$ is the accrued benefit that the dealership added to the car by re-allocating it to Europe, and the benefit is equal to the costs that would have to be made by ourselves to ship the car.

We take the section option, we buy the car at the dealership for the price defined in (3-24). However, before driving the car we decide that we don't want the car. Apart from some administration costs (that we ignore in this example), this will not be a problem; the car is worth the same to another European buyer, because their price will also be the market value plus the accrued benefit.
This will not hold in we decide to not take the car after we have driven it. By driving (consuming) the car, we have extracted some of its utility. Another European buyer is now willing to pay the price in (3-24) less the accrued depreciation:

$$p(t) = p(0) + \int^{t} Bdt - \int^{t} Ddt \tag{3-25}$$

This shows that an asset (in this case the car) stores added utility as a benefit, but loses utility as depreciation.

3-5 Legendre Transform from Utility Lagrangian to Surplus Hamiltonian

In this section I will show the effect of consumption on surplus. At the end of Section 2-3, it was shown how utility and surplus are related by a Legendre transform. As the Utility Lagrangian, $L(\dot{q}, q)$, the Surplus Hamiltonian was not a function of the consumption and depreciation of the assets.

In the previous section, I have followed the works of Riewe and Lazo [4, 5, 10] to include consumption as analog of dissipation in the Utility Lagrangian. The works of Riewe and Lazo do also derive a Hamiltonian of a frictional mechanical system. This Hamiltonian is obtained by performing a double Legendre transform; the Legendre transform is performed between the momentum, p, and the velocity, $\dot{q}$, and between the fractional momentum, $\gamma = \partial L/\partial \breve{q}$, and the fractional derivative $\breve{q}$.

However, using this Legendre transform to derive a Fractional Surplus Hamiltonian from the Fractional Surplus Lagrangian leads to the incorrect result that an increase in depreciation has a positive effect on the surplus, see Appendix A-2. Therefore, I propose that the Fractional Surplus Hamiltonian should be obtained by performing the Legendre transform only over the price (generalized momentum), p, and the flow variable (velocity), $\dot{q}$. I motivate this proposition by the observation that whereas the price (generalized momentum) is a conserved property, there is no conservation law for the expense (fractional momentum).

The Fractional Surplus Hamiltonian is derived by performing a Legendre transform on the Lagrangian over the velocity.

$$H = \frac{\partial L}{\partial \dot{q}}\dot{q} - L \tag{3-26}$$

Substituting the definition of reservation price ($p \equiv \partial L/\partial \dot{q}$), the surplus can be written as

$$H = p\dot{q} - L \tag{3-27}$$

The total derivative of this Surplus Hamiltonian, which I name the Surplus Statement[3], dH, is

$$dH = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial \breve{q}}d\breve{q} + \frac{\partial H}{\partial q}dq \tag{3-28}$$

This implies that the surplus changes with the change in price, expense, and stock multiplied by the factors $\frac{\partial H}{\partial p}$, $\frac{\partial H}{\partial \gamma}$, and $\frac{\partial H}{\partial q}$, respectively. These three factors can be identified by evaluating the total derivative of the Fractional Utility Lagrangian

$$dL = \frac{\partial L}{\partial \dot{q}}d\dot{q} + \frac{\partial L}{\partial \breve{q}}d\breve{q} + \frac{\partial L}{\partial q}dq \tag{3-29}$$

Substituting the price, p, yields

$$dL = pd\dot{q} + \frac{\partial L}{\partial \breve{q}}d\breve{q} + \frac{\partial L}{\partial q}dq \tag{3-30}$$

Using the product rule and subsequently moving the term $d(p\dot{q})$ to the left-hand side results in

$$d(p\dot{q} - L) = \dot{q}dp - \frac{\partial L}{\partial \breve{q}}d\breve{q} - \frac{\partial L}{\partial q}dq \tag{3-31}$$

Observing that the left-hand side is equal to the Surplus Hamiltonian (3-27) we find the following Hamilton equations

$$\frac{\partial H}{\partial p} = \dot{q}, \qquad \frac{\partial H}{\partial \breve{q}} = -\frac{\partial L}{\partial \breve{q}}, \qquad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}, \tag{3-32}$$

Substituting the equalities $\frac{\partial L}{\partial \breve{q}} = \gamma$ and $\frac{\partial L}{\partial q} = B$ we get that the Surplus Statement, dH, is

$$dH = \dot{q}dp - \gamma d\breve{q} - Bdq \tag{3-33}$$

That is, the change in surplus is a result of

[3] Using terminology from accounting; e.g. Income Statement, Cash Flow Statement [13]

1. the change in price multiplied by the flow,
2. the change in consumption multiplied by the expense, and
3. the change in stock multiplied by the cost

The Fractional Surplus Hamiltonian $H(p, \breve{q}, q)$ can be linked to the earnings before interest and tax (EBIT [4]) in accounting [13]. Comparing the Surplus Statement, dH, to a change in EBIT, I link the first effect to the change in revenue, the second effect to the depreciation, and the third effect to the cost of goods sold.

By deriving the Hamiltonian as done in (3-27), the depreciation has a negative effect on the Surplus Statement, dH, corresponding to the negative effect of depreciation on the EBIT. Riewe and Lazo argued that deriving the Hamiltonian as done in (A-1) results in a Hamiltonian as the total energy in the system [4, 5, 10]. Assuming their Hamiltonian is indeed the total energy, this implies that surplus is not analogous to energy in general, but to availability; whereas availability is the energy available to perform work, surplus is the cash flow available to perform economic activity.

3-6 Surplus Hamiltonian in Port-Hamiltonian Control

So far, this chapter concerned the first objective of this thesis: modeling consumption as economic analog of dissipation in the Lagrangian and Hamiltonian formalism. The second objective of this thesis is to use control methods from port-Hamiltonian systems theory to control an economic system in the presence of consumption. In the previous section, the Legendre transform was used to relate the economic fundamental concept of utility to the Hamiltonian description that is used in port-Hamiltonian systems theory.[5] Now, the goal is to include the Fractional Surplus Hamiltonian (3-27) in the port-Hamiltonian formalism.

However, there is a problem with including the Fractional Surplus Hamiltonian (3-27) in the port-Hamiltonian formalism. Whereas the Hamilton's equations of a traditional Hamiltonian system can be expressed with Poisson brackets (see Paragraph 42 of [12]), this is not the case for the Fractional Surplus Hamiltonian since the coordinate system $(p, \breve{q}, q)$ is not canonical. Naively implementing the Surplus Hamiltonian $H(p, \breve{q}, q)$ in the port-Hamiltonian formalism, we would write

$$\begin{pmatrix} \dot{q} \\ \dot{\breve{q}} \\ \dot{p} \end{pmatrix} = S \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial \breve{q}} \\ \frac{\partial H}{\partial p} \end{pmatrix}, \tag{3-34}$$

where S is some structure matrix. However, such a description cannot be obtained since $\dot{\breve{q}}$ is not defined. This implies that the Fractional Surplus Hamiltonian is inadmissible for the port-Hamiltonian formalism.

[4] EBIT= Revenue - costs and expenses (except interest and tax).

[5] The fractional Lagrangian can be used in the context of optimal control. As mentioned in the introduction, economists model utility as a Lagrangian in a similar fashion as done in optimal control techniques. The Fractional Utility Lagrangian introduced in this chapter is suitable to be included in optimization. Since this is not inside the scope of this thesis, I will not discuss this in further detail. Interested readers are referred to Section 5 of [14], where a similar fractional Lagrangian is used in optimal control in the context of a mechanical system with friction.

For this reason, I leave the port-Hamiltonian control of the Surplus Hamiltonian with fractional derivatives as an open problem. In the next chapter, I derive an alternative method to include consumption as analog of dissipation in the Hamiltonian, using complex-valued coordinates. The resulting Complex Hamiltonian is a function of a coordinate pair that is canonical.

3-7 Conclusions

This chapter addressed the problem that the Utility Lagrangian in economic engineering is not a function of consumption.

I have shown that with fractional calculus [4, 5, 10], consumption can be included as an independent variable in the Fractional Utility Lagrangian. Applying the fractional calculus of variations to the principle of maximum utility — stating that economic agents adjust their asset stock so as to maximize their utility — results in a fractional Euler-Lagrange equation that represents the price dynamics. Whereas the classical Euler-Lagrange equation used in economic engineering relates a change in price to an accrued benefit, the fractional Euler-Lagrange equation relates a change in price to the accrued benefit less the accrued depreciation from consuming the asset. The classical example of this phenomenon is that the value of a car decreases proportional to the distance it has covered over its lifetime.

Performing a Legendre transform on the Fractional Lagrangian results in the Fractional Surplus Hamiltonian, which is an extension to the Fractional Surplus Hamiltonian in economic engineering. I showed that the Legendre transform should be performed over the flow of assets only, rather than over the flow of assets and the consumption, as done in [4, 5, 10]. The former results in a Fractional Surplus Hamiltonian that decreases when depreciation increases, whereas the latter would result in a Fractional Surplus Hamiltonian that increases when depreciation increases.

In this thesis I aim to apply control formalisms from port-Hamiltonian systems theory to economic systems. The Fractional Surplus Hamiltonian is, however, not a function of a set of canonical coordinates. As a result, control formalisms of port-Hamiltonian systems theory cannot be applied to the Fractional Surplus Hamiltonian.

Contributions

1. Consumption is included as an independent variable in the Utility Lagrangian
2. The fractional Euler-Lagrangian equation includes depreciation from consumption in the price dynamics
3. It was shown that economic surplus is analogous to the availability rather than the energy in general

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Chapter 4

Complex-Hamiltonian Systems and Control

4-1 Introduction

Although port-Hamiltonian systems theory does account for dissipative systems, it does not include the dynamics of dissipation in the Hamiltonian itself, see Section 2-4. Conceptually, this is not in line with the Hamiltonian description of economic engineering where consumption — the economic analog of dissipation — is endogenous to the economic agent, as shown in Section 2-3. Furthermore, the passivity-based control and control by interconnection formalisms from port-Hamiltonian systems are constrained by the dissipative elements, see Section 2-4-3. The latter is known as the dissipation obstacle [1, 2, 3, 4].

Problem

1. In port-Hamiltonian systems theory the dynamics of dissipation are handled separately from the Hamiltonian
2. Control formalisms in port-Hamiltonian systems theory are constrained by the dissipation obstacle

In this chapter, I develop a theory that both models dissipation as an endogenous effect of the Hamiltonian and bypasses the dissipation obstacle. To achieve this, I propose the following solution.

Solution

Include the dynamics of dissipative elements in the Hamiltonian by using complex state variables.

This chapter is structured as follows. Section 4-2 introduces the Hamiltonian description of (damped) harmonic oscillators with complex state variables. Section 4-4 joins the Complex-Hamiltonian description and the port-based description of port-Hamiltonian systems, resulting in complex-port-Hamiltonian systems. Section 4-5 applies the control formalisms from port-Hamiltonian systems theory to complex-port-Hamiltonian systems. Section 4-6 concludes and discusses the chapter.

In order to relate the developments I make in this chapter to port-Hamiltonian theory, I will consider mechanical systems only. Chapter 5 applies the developed formalisms to the economic agent.

4-2 From Modeling to Complex-Hamiltonian Systems

In this section I show how physical systems can be described as *Complex-Hamiltonian systems*. This will be done in a constructive manner using the mechanical harmonic and damped harmonic oscillator as fundamental systems. The purpose of this section is to introduce the similarity between the Complex-Hamiltonian and the port-Hamiltonian description. The Complex-Hamiltonian and the complex equations of motion are formalized in Section 4-3.

4-2-1 Harmonic Oscillator

Complex Equation of Motion

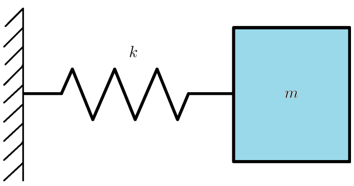

Figure 4-1: Mass-Spring system as harmonic oscillator

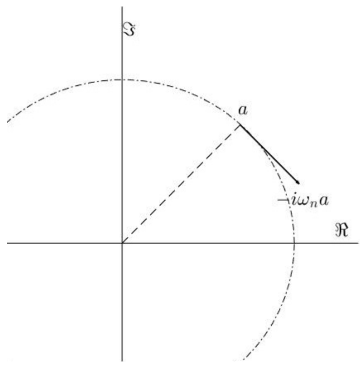

Figure 4-2: Geometric representation of (4-1). The equation of motion can simply be read off using the structure of the complex plane. The dash-dotted circle represents the Hamiltonian $H = a\bar{a}$.

Consider the mass-spring system in Figure 4-1. It is well-known that the systems time evolution can be written either as a set of two coupled first-order equations or one second-order equation of motion. Less known is the fact that it can be written as a single first-order equation in a complex state variable a as follows, see e.g. [5]

$$\dot{a} = -i\omega_n a \tag{4-1}$$

Here, ω_n denotes the natural frequency of the system and the complex-valued state is defined as

$$a = \sqrt{V} + i\sqrt{T} = \sqrt{\frac{k}{2}}q + i\sqrt{\frac{1}{2m}}p \tag{4-2}$$

with V the potential and T the kinetic energy. The complex equation of motion (4-1) is easily verified geometrically, see Figure 4-2.

Hamilton's Equation

Alternatively, the system can be described using Hamilton's equations. In port-Hamiltonian systems this is done in matrix form, see [3]

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}, \qquad H = \frac{1}{2}kq^2 + \frac{p^2}{2m} \tag{4-3}$$

where J is a skew-symmetric structure matrix such that $J^2 = -I$.

In complex state variables, Hamilton's equations reduces to a single equation. First notice that the Hamiltonian for the undamped oscillator becomes:

$$H = a\bar{a} \tag{4-4}$$

as can be verified by substituting (4-2) into this expression. This allows the complex equation of motion to be written as:

$$\dot{a} = -i\omega_n \frac{\partial H}{\partial \bar{a}}, \qquad H = a\bar{a} \tag{4-5}$$

Compare this to the port-Hamiltonian formulation. It can be seen that the complex structure J-matrix is replaced by the factor $-i$. However, whereas the J-matrix is an abstract matrix with $J^2 = -I$, the factor $-i$ directly corresponds to the -90° rotation required to go from the gradient of H to the direction of $\dot{a}$.

4-2-2 Damped Harmonic Oscillator

Complex Equation of Motion

I now extend this analysis to the damped harmonic oscillator.

Expressing the equation of motion of a mass-spring damper system in complex state variables, the complex equation of motion becomes

$$\dot{a} = -(\beta + i\omega_n)a + \beta\bar{a} \tag{4-6}$$

where $\beta = \frac{b}{2m}$ is the damping coefficient. Figure 4-4 shows a geometric representation of the complex equation of motion. This is a single first-order equation that replaces the usual system given by the A-matrix formalism. The derivation of the complex equation of motion (4-6) is done step-by-step in Appendix D-1.

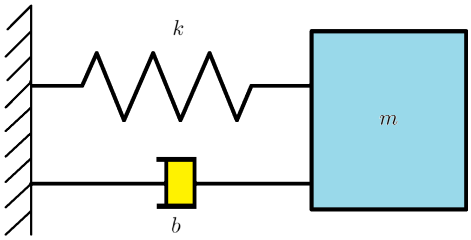

Figure 4-3: Mass-Spring-Damper System as damped harmonic oscillator

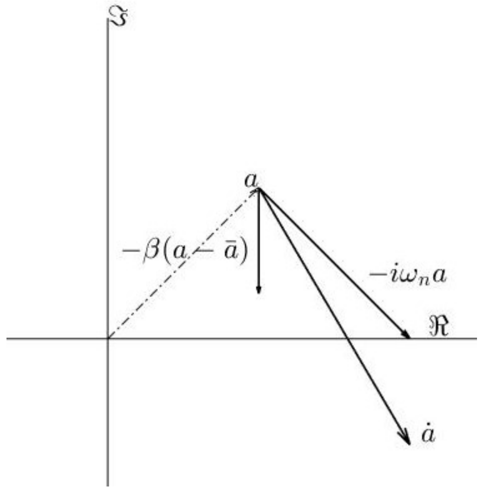

Figure 4-4: Complex equation of motion of a mass-spring-damper system

Hamilton's Equation

Non-conservative elements are typically not included in the Hamiltonian, see e.g. [6]. In Section 2-4 it was shown that in port-Hamiltonian systems this issue is solved by introducing a dissipation matrix R and representing dissipative systems as

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = (J - R) \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}, \qquad H = \frac{1}{2}kq^2 + \frac{p^2}{2m} \tag{4-7}$$

However, as was reviewed in Section 2-4-3, the R matrix causes the dissipation obstacle in several control techniques of port-Hamiltonian systems theory.

I now show that damping can be included in the Hamiltonian with complex numbers so that the equation of motion of a damped system can be written as a Hamilton's equation.

I introduce the complex-valued Hamiltonian of a damped system (see Section 4-3):

$$H(a) = (1 - i\zeta)a\bar{a} - \frac{1}{2}i\zeta a^2 + \frac{1}{2}i\zeta\bar{a}^2 \tag{4-8}$$

where $\zeta = \frac{b}{2\sqrt{km}}$ is the damping ratio. Following the same procedure as in (4-5), I find the expression for the equation of motion as

$$\dot{a} = -i\omega_n \frac{\partial H}{\partial \bar{a}}, \quad H = (1 - i\zeta)a\bar{a} - \frac{1}{2}i\zeta a^2 + \frac{1}{2}i\zeta\bar{a}^2 \tag{4-9}$$

This expression indeed coincides with the expression found in (4-6). Again, compare this result to the port-Hamiltonian formulation in (4-3-5).

By describing a system with complex state variables, the equation of motion can be defined as a single Hamilton's equation even in the presence of damping. This allows the structure of the Hamilton's equation, $-i$, to be maintained in the presence of damping. The dissipation matrix R used in port-Hamiltonian systems (4-3-5) thus becomes obsolete. In Section 4-5 I will show that this is an important result for solving the dissipation obstacle.

4-3 The Complex Hamiltonian

In this section, I show that the Complex-Hamiltonian of a damped harmonic oscillator is defined on a complex symplectic structure. This will show to be an important result regarding the dissipation obstacle.

First I define the complex state, a. Then I define the complex equation of motion, $\dot{a}$, as the Poisson bracket Finally, I derive the Complex-Hamiltonian, H, of a damped harmonic oscillator.

4-3-1 The Complex State

For a set of symplectic state variables $(q_j, p_j),\ j = 1, \ldots, n$, the corresponding complex states are (see e.g. [5, 7])

$$a_j = \alpha_{1,j} q_j + i \alpha_{2,j} p_j, \tag{4-10}$$

Here, the α's are some complex constants.

For the mechanical oscillator, I choose these complex coordinate so that iso-energetic curves are simply represented by circles on the complex plane. This is achieved by choosing a as

$$a_j = \sqrt{V_j(q_j)} + i\sqrt{T_j(p_j)} = \sqrt{\frac{k_j}{2}} q_j + i\sqrt{\frac{1}{2m_j}} p_j, \tag{4-11}$$

with $V_j(q_j)$ and $T_j(p_j)$ the potential and kinetic energy, respectively. A circle, $a\bar{a}$, on the complex plane coincides with the Hamiltonian of a conservative system on the $(q,p)-$plane

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2} k q^2,$$

implying that a circle on the complex plane indeed represents constant energy.

4-3-2 The Complex Poisson Bracket

Following the lecture notes of Svistunov [7], I define the time evolution of a system as the *canonical* Poisson bracket between the complex state, a, and the Complex-Hamiltonian, H

$$\dot{a} \equiv \{a, \mathsf{H}\} \tag{4-12}$$

In matrix form, the complex Poisson bracket can be written as

$$\{A, B\} = \Omega \frac{\partial^{|} A}{\partial \mathbf{a}} \mathsf{J} \frac{\partial B}{\partial \mathbf{a}}, \tag{4-13}$$

where, the complex state vector, $\mathbf{a}$ C^{2n}, is constructed as n pairs of complex states, a_j, and their complex conjugate, $\bar{a}_j$, for $j = 1, \ldots n$. Matrix Ω is constructed as

$$\Omega = \begin{pmatrix} \rho_1 & 0 & \ldots & 0 & 0 \\ 0 & \rho_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \rho_n & 0 \\ 0 & 0 & \ldots & 0 & \rho_n \end{pmatrix}, \tag{4-14}$$

with $\rho_j = (\alpha_{1,j}\bar{\alpha}_{2,j} + \bar{\alpha}_{1,j}\alpha_{2,j})$, for $j = 1, \dots n$ [7]. For the complex state in (4-11) this implies $\rho_j = \sqrt{\frac{k_j}{m_j}} = \omega_n$, is the natural frequency.

Matrix $\boldsymbol{J}$ is constructed as

$$\boldsymbol{J} = \begin{pmatrix} 0 & -i & \dots & 0 & 0 \\ i & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -i \\ 0 & 0 & \dots & i & 0 \end{pmatrix} \tag{4-15}$$

Notice that we have that $\boldsymbol{J}^2 = -I$. Compare this to the symplectic matrix of the Hamilton's equations on the $(q,p)-$plane

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4-16}$$

for which it holds that $J^2 = -I$.

Since the Poisson bracket (4-13) is canonical, it obeys the rules of anticommutivity, bilinearity, Leibniz's rule and Jacobi's identity. More importantly, the Poisson bracket satisfies $\dot{\mathsf{H}} = \{\mathsf{H}, \mathsf{H}\} = 0$. The latter observation means that the Complex Hamiltonian is a constant of the motion, i.e. it is a conserved property.

$$\dot{\mathsf{H}} = \{\mathsf{H}, \mathsf{H}\} = 0 \tag{4-17}$$

4-3-3 The Complex Hamilton's Equations

For a simple (damped) harmonic oscillator with complex state $\mathbf{a}$ $\quad\mathsf{C}^2$, (4-12) and (4-13) imply that the complex equation of motion is

$$\dot{a} = \{a, \mathsf{H}\} = \omega_n \begin{pmatrix} \frac{\partial a}{\partial a} & \frac{\partial a}{\partial \bar{a}} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathsf{H}}{\partial a} \\ \frac{\partial \mathsf{H}}{\partial \bar{a}} \end{pmatrix} = -i\omega_n \frac{\partial \mathsf{H}}{\partial \bar{a}} \tag{4-18}$$

By complex conjugation it immediately follows that

$$\dot{\bar{a}} = \overline{\{a, \mathsf{H}\}} = \omega_n \begin{pmatrix} \frac{\partial \bar{a}}{\partial \bar{a}} & \frac{\partial \bar{a}}{\partial a} \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{\mathsf{H}}}{\partial \bar{a}} \\ \frac{\partial \bar{\mathsf{H}}}{\partial a} \end{pmatrix} = i\omega_n \frac{\partial \bar{\mathsf{H}}}{\partial a} \tag{4-19}$$

I will refer to (4-18) and (4-19) as the Complex Hamilton's equations. This implies that the Complex-Hamiltonian is— as is the classical Hamiltonian— the generator of the time-evolution of the system. Also similar to classical Hamiltonian mechanics, the Complex Hamilton's equations have an equivalent expression in terms of the canonical Poisson bracket. The latter implies that the complex coordinate space $(a, \bar{a})$ is canonical, see e.g. [8].

4-3-4 The Complex Hamiltonian of a Damped Harmonic Oscillator

In this section I derive the Complex Hamiltonian of a damped harmonic oscillator. The Complex Hamiltonian is derived by comparing the complex equation of motion derived from the definition of the complex state (4-11) and the Complex Hamilton's equations.

For a damped harmonic oscillator, the complex equation of motion and its conjugate are (see Appendix D-1 for the derivation)

$$\dot{a} = -(\beta + i\omega_n)a + \beta\bar{a} \tag{4-20}$$

$$\dot{\bar{a}} = \beta a - (\beta - i\omega_n)\bar{a} \tag{4-21}$$

From the Complex Hamilton's equations, (4-18) and (4-19), we also have that

$$\dot{a} = -i\omega_n \frac{\partial H}{\partial \bar{a}} \tag{4-22}$$

$$\dot{\bar{a}} = i\omega_n \frac{\partial \bar{H}}{\partial a} \tag{4-23}$$

Substituting (4-20) and (4-21) in the Complex Hamilton's equation yields

$$-(\beta + i\omega_n)a + \beta\bar{a} = -i\omega_n \frac{\partial H}{\partial \bar{a}} \tag{4-24}$$

$$\beta a - (\beta - i\omega_n)\bar{a} = i\omega_n \frac{\partial \bar{H}}{\partial a} \tag{4-25}$$

Using $\beta = \zeta\omega_n$, the above equations can be rewritten as

$$-(i\zeta + 1)a + i\zeta\bar{a} = \frac{\partial H}{\partial \bar{a}} \tag{4-26}$$

$$-i\zeta a + (i\zeta + 1)\bar{a} = \frac{\partial \bar{H}}{\partial a} \tag{4-27}$$

Integrating these equalities with respect to a and $\bar{a}$, respectively, we find the Complex Hamiltonian (up to a constant) as

$$H = (1 - i\zeta)a\bar{a} - \frac{1}{2}i\zeta a^2 + \frac{1}{2}i\zeta\bar{a}^2 \tag{4-28}$$

Alternatively, the Complex Hamiltonian, H, can be written as

$$H = E - iD, \tag{4-29}$$

with E and D the mechanical energy and the dissipation function, respectively. The mechanical energy, E, was already evaluated in Section 4-3-1 as

$$E = a\bar{a} \tag{4-30}$$

This leaves the dissipation function, D, as

$$D = \frac{1}{2}\zeta(a^2 + 2a\bar{a} - \bar{a}^2) \tag{4-31}$$

The alternative description of the Complex Hamiltonian can be used to illustrate the geometry of the complex equation of motion, $\dot{a}$. The complex equation of motion is generated by the Complex Hamiltonian as the Complex Hamilton's equation, see (4-18). Using the alternative expression for the Complex Hamiltonian (4-29), $\dot{a}$ can be written as the combination of the motion generated by the energy function, E, and by the dissipation function, D

$$\dot{a} = \dot{a}_E - i\dot{a}_D - i\omega_n \frac{\partial E}{\partial \bar{a}} - \omega_n \frac{\partial D}{\partial \bar{a}} = -i\omega_n + \beta(\bar{a} - a) \tag{4-32}$$

This construction of the complex equation of motion is visualized in Figure 4-5. Notice that the motion generated by the dissipation function, $\beta(\bar{a} - a)$, is always directed at the real axis and cancels when $Im(a) = 0$. This implies that the dissipation is negative proportional to the momentum.

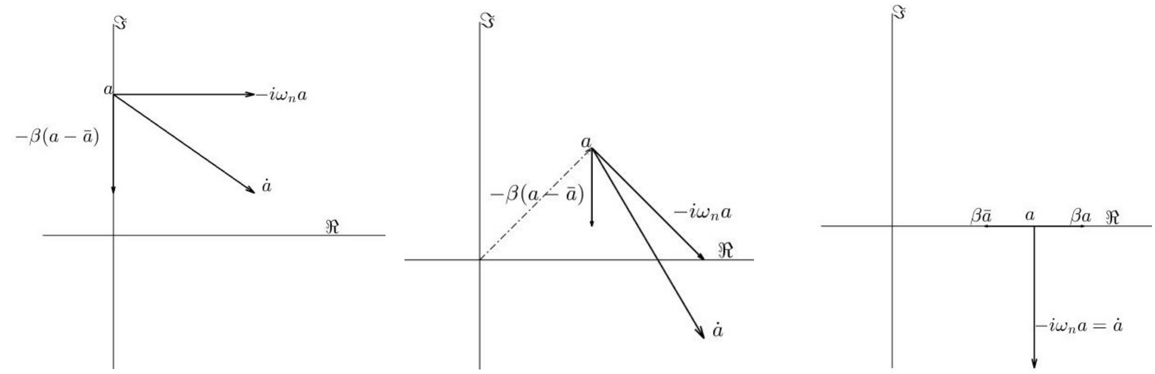

Figure 4-5: Complex equation of motion, $\dot{a}$, generated by the mechanical energy, E, and the dissipation function, D, for different states. The motion generated by the dissipation function is always directed at the real axis and cancels when $\mathrm{Im}(a) = 0$. This implies that the dissipation is negative proportional to the momentum.

4-3-5 Concluding Remarks on the Complex Hamiltonian

In contrast to the classical Hamiltonian, the Complex Hamiltonian can describe damped harmonic oscillators. In fact, the Complex Hamiltonian is a generalization of the classical Hamiltonian on the complex plane. Indeed, setting $\zeta = 0$ the Complex Hamiltonian (4-28) reduces to the classical Hamiltonian. This is easily verified with the alternative formulation $H = E + iD$ with E and D defined in (4-30) and (4-31), respectively. The Complex Hamiltonian is indeed a Hamiltonian, since it is the generator of the time-evolution of the system, obeying the algebra of the canonical Complex Poisson bracket.

The statement that the Complex Hamiltonian is the generalization of the classical Hamiltonian implies that the Complex Hamilton's equation is the generalization of the Hamilton's equation. This is an important result regarding the dissipation obstacle in pH systems theory.

In pH systems, dissipation is accounted for by subtracting a damping matrix, R, from the symplectic matrix, J.

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = (J - R) \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}, \qquad H = \frac{1}{2}kq^2 + \frac{p^2}{2m}$$

However, if $R = 0$ the damping matrix, R, clearly violates the symplectic structure of the Hamiltonian vector field. In comparison, the Complex-Hamiltonian description of the same system is

$$\dot{a} = -i\omega_n \frac{\partial H}{\partial \bar{a}} \tag{4-33}$$

This description is the same in the absence and the presence of dissipation; the structure of the Complex Hamiltonian vector field is not violated by dissipation.

The algebraic link between the classical Hamiltonian and the Complex Hamiltonian is due to the link between symplectic and complex geometry [9]. I argue that the complex geometry on which the Complex Hamiltonian is defined facilitates a structure that allows damped harmonic oscillators to be described by Hamiltonian mechanics. However, finding the corresponding manifolds and algebraic groups is beyond the scope of this thesis.

4-4 Complex-Port-Hamiltonian Systems

In this section I employ the Complex Hamiltonian in pH systems theory. The combination of the Complex-Hamiltonian and pH systems theory will be referred to as the complex-port-Hamiltonian (cpH) theory. The structure of the Complex Hamilton's equations is not affected by dissipation. In the cpH description, control formalisms from pH systems are not affected by the dissipation obstacle. Furthermore, since Complex-Hamiltonians can include damping, the Energy Shaping and Damping Injection formalisms reduce to a single formalism.

4-4-1 Stateful and Stateless Elements

The first implication of the Complex Hamiltonian regarding the port-based approach is that energy-storing and energy-dissipating do not have to be treated separately. Whereas the traditional port-Hamiltonian formalism divides system elements into energy-storing, energy-dissipating and energy-routing elements, the cpH formalism divides elements into two classes. These two classes will be referred to as *Stateful*, for elements that store a state, and *Stateless*, for elements that do not store any state, borrowing terminology from computer science. The comparison between element classes of cpH and pH systems is summarized in Table 4-1.

The difference in element classes illustrates the difference in pH and cpH systems. All elements that can be specified by a Complex-Hamiltonian are stateful elements. In contrast to in pH systems, this includes dampers. As a result, there is no dissipation matrix in the input-state-output description of a cpH system, as will be shown in the next section.

Complex-Port-Hamiltonian	Port-Hamiltonian	Physical components
Stateful	Energy-Storing	Masses, inductors, ... Spring, capacitors, ...
	Energy-Dissipating	Damper, resistors, ...
Stateless	Energy-Routing	Gears, wires, transformers, ...

Table 4-1: Classes of elements in complex-port-Hamiltonian, port-Hamiltonian, and the relationship to the physical components they describe.

4-4-2 Input-State-Output Description

In this section, I introduce physical inputs and outputs to the Complex Hamiltonian description. Although complex state variables are useful for incorporating damping in the Hamiltonian description of mechanics, they are inconvenient to use as physical inputs and outputs, such as forces and velocities. I use the definition of the complex state (4-10) to convert real-valued physical input and output signals to complex-valued state variables. This way, physical inputs and outputs can be used to measure and actuate the system, while the dynamics of the systems are computed with Complex-Hamiltonian mechanics.

The input-state-output description of a cpH system is

$$\begin{aligned} \dot{\mathbf{a}} &= \mathsf{J} \quad \mathsf{H} + \mathsf{G}u \\ y &= \mathsf{G}^{\intercal} \quad \mathsf{H} \end{aligned} \tag{4-34}$$

with state vector $\mathbf{a} \quad \mathsf{C}^{2n}$, input vector $u \quad \mathsf{R}^m, m \leq 2n$, output vector $y \quad R^m$, complex matrix $\mathsf{J} : \mathsf{C}^{2n} \Rightarrow \mathsf{C}^{2n\times 2n}$ (defined in (4-15)), input matrix $\mathsf{G} : \mathsf{C}^{2n\times m}$ —relating pairs of real state variables to complex state variables — and the operator $\quad \mathsf{H} \quad \mathsf{C}^{2n}$ constructed as[1]

$$\mathsf{H} = \begin{pmatrix} \frac{\partial \bar{\mathsf{H}}}{\partial a_1} & \frac{\partial \mathsf{H}}{\partial \bar{a}_1} & \dots & \frac{\partial \bar{\mathsf{H}}}{\partial a_n} & \frac{\partial \mathsf{H}}{\partial \bar{a}_n} \end{pmatrix}^{\intercal} \tag{4-35}$$

Comparing the description in (4-34) to the input-state-output description of a pH system (2-14):

$$\begin{aligned} \dot{x} &= [J - R] \quad H + g(x)u \\ y &= g^{\intercal}(x) \quad H, \end{aligned} \tag{PH-1}$$

it can immediately be seen that the descriptions are similar However, whereas in the pH description a dissipation matrix, $R(x)$, is needed to account for damping, the cpH description has no need for the dissipation matrix. See (2-14) for the definitions of the elements in (PH-1).

Note that including physical real-valued inputs and outputs requires the complex conjugates a_j, for $j = 1, \dots, n$ to be incorporated in the input-state-output description. This is a result of the inverse definitions of the complex state,

$$q = \sqrt{\frac{1}{2k}}(a + \bar{a}), \qquad p = -i\sqrt{\frac{m}{2}}(a - \bar{a}), \tag{4-36}$$

Example: Mass-Spring-Damper system with Force input

Consider a linear Mass-Spring-Damper system with mass m, spring constant k, damping constant b, natural frequency $\omega_n = \sqrt{\frac{k}{m}}$, damping ratio $\zeta = \frac{b}{2\sqrt{km}}$ and Complex-

[1] H is constructed so that $\mathsf{J} \quad \mathsf{H}$ is a shorthand notation for $\left(\{a_1, \mathsf{H}\} \quad \overline{\{a_1, \mathsf{H}\}} \quad \dots \{a_n, \mathsf{H}\} \quad \overline{\{a_n, \mathsf{H}\}}\right)^{\intercal}$

Hamiltonian

$$H = (1 - i\zeta)a\bar{a} - \frac{1}{2}i\zeta a^2 + \frac{1}{2}i\zeta\bar{a}^2 \tag{4-37}$$

The system is driven by a force acting on the mass, $u = F$. From the port-based approached it follows that the velocity of the system is the output of the system: $y = \dot{q}$. From (4-34), the input-state-output description of this system is

$$\begin{aligned}
\begin{pmatrix}\dot{a}\\ \dot{\bar{a}}\end{pmatrix} &= \begin{pmatrix}0 & -i\omega_n\\ i\omega_n & 0\end{pmatrix}\begin{pmatrix}\frac{\partial \bar{H}}{\partial a}\\ \frac{\partial H}{\partial \bar{a}}\end{pmatrix} + \begin{pmatrix}i\sqrt{\frac{1}{2m}}\\ -i\sqrt{\frac{1}{2m}}\end{pmatrix}u\\
y &= \begin{pmatrix}i\sqrt{\frac{1}{2m}} & -i\sqrt{\frac{1}{2m}}\end{pmatrix}\begin{pmatrix}\frac{\partial \bar{H}}{\partial a}\\ \frac{\partial H}{\partial \bar{a}}\end{pmatrix}
\end{aligned} \tag{4-38}$$

Notice that the output y can be written as

$$\begin{aligned}
y &= \begin{pmatrix}-i\sqrt{\frac{1}{2m}} & i\sqrt{\frac{1}{2m}}\end{pmatrix}\begin{pmatrix}\frac{\partial \bar{H}}{\partial a}\\ \frac{\partial H}{\partial \bar{a}}\end{pmatrix}\\
&= \begin{pmatrix}-i\sqrt{\frac{1}{2m}} & i\sqrt{\frac{1}{2m}}\end{pmatrix}\begin{pmatrix}-i\omega_n^{-1}\dot{\bar{a}}\\ i\omega_n^{-1}\dot{a}\end{pmatrix} = \sqrt{\frac{1}{2k}}(\dot{a}+\dot{\bar{a}}) = \dot{q}
\end{aligned} \tag{4-39}$$

From the first equation in (4-36) we have that this indeed is the velocity.

4-5 Complex-Port-Hamiltonian Control

In this section, I apply the control formalisms from pH systems theory (see Appendix C) to cpH systems. First I show that the energy flow at the port of the system is the product of the input and the output signals. Then, I apply the passivity based control and control by interconnection formalisms to the cpH systems description. It will appear that in both formalisms cpH systems are not affected by the dissipation obstacle.

4-5-1 Passivity-Based Control

A key concept in pH systems theory is passivity [3]. A system is said to be passive if it is not able to autonomously generate energy along its trajectories. An important equation in this technique is the energy-balance equation [4]

$$\underbrace{H[x(t)] - H[x(0)]}_{\text{stored energy}} = \underbrace{\int_0^t u^\intercal(s)y(s)ds}_{\text{supplied energy}} - \underbrace{d(t)}_{\text{dissipated energy}}$$

In the passivity based control (PBC) formalism , it is assumed that a perfect model of a plant is provided and no noise is acting on the system or the measurement, i.e. static control. The PBC formalism exploits the physical structure and the energy balance of physical systems [10, 11, 12]. For an actuated system, the energy balance states that the stored energy is equal to the supplied energy minus the dissipated energy. With controls, a desired amount of stored energy can be realized by supplying additional energy and increasing the dissipation to render a system closed-loop passive [3].

The first occurrence of the dissipation obstacle in pH systems arises from pervasive dissipation [4]. The power balance of a passive system with input-output description (PH-1) is

$$\begin{aligned} \frac{dH}{dt} &= H^{\intercal}\dot{x} \\ &= H^{\intercal}([J - R] \quad H + g(x)u) \\ &= -\underbrace{H^{\intercal}R \quad H}_{d(x)} + u^{\intercal}y \end{aligned} \tag{PH-2}$$

Assume that the control objective is to steer the system to a desired equilibrium $x \quad \mathsf{R}^n$. Since the energy flow, dH/dt, should be zero at x , we have that for a dissipative system, $R > 0$, the dissipated power has to be compensated by the controller:

$$u \; ^{\intercal}y \;=\; ^{\intercal}H(x\;)R(x\;) \quad H(x\;) \tag{PH-3}$$

where u and y are the input and output at the equilibrium, respectively. However, the power that can be extracted from a passive system is bounded [16]. This implies that if the system dissipates power at the equilibrium —a phenomenon known as pervasive dissipation— it cannot be stabilized by a passive controller. Therefore, only systems for which it holds that

$$R(x\;) \quad H(x\;) = 0 \tag{PH-4}$$

can be stabilized with passive controllers. Systems for which (C-25) does not hold are constrained by the dissipation obstacle [4]

Now compare this to the same situation for a cpH description. From the cpH input-state-output description in (4-34) it can be derived that the energy flow at the port of a cpH system is

$$\begin{aligned} \frac{d\mathsf{H}}{dt} &= \mathsf{H}^{\intercal}\dot{\mathbf{a}} \\ &= \mathsf{H}^{\intercal}\mathsf{J} \quad \mathsf{H} + \quad \mathsf{H}^{\intercal}\mathsf{G}u \\ &= y^{\intercal}u \end{aligned} \tag{4-40}$$

Since the Complex Hamilton's equation is embedded in y, (4-40) returns the equation of motion with the dissipation accounted for. If we steer the cpH system to a desired equilibrium $a \quad \mathsf{C}^n$, the complex Hamiltonian flow, $d\mathsf{H}/dt$ should be zero at a . This results in

$$y \; ^{\intercal}u \;\; = 0 \tag{4-41}$$

Compare (4-41) to (PH-3). In (PH-3), the system can only be stabilized with a passive controller if condition (PH-4) holds. In (4-41) this is clearly not the case. The dissipation obstacle is thus bypassed in the cpH description.

Rather than describing the energy flow at the port of the storage element, as done in pH systems, the energy flow of a cpH system is described at the port of the stateful element. For a damped harmonic oscillator, the stateful element represents the entire physical system.

From the energy flow at the port of a cpH system (4-40), we can obtain a control law, u, that renders the closed-loop system passive. The most straightforward choice for this control law is $u = -Ky$ where K is a positive constant. Indeed, this control law results in the following energy flow at the port of the system

$$\frac{d\mathsf{H}}{dt} = -y^{\top} K y \leq 0 \tag{4-42}$$

Since the energy of this system is always decreasing, the stability of the system is guaranteed.

4-5-2 Energy-Shaping and Damping Injection

To emphasize the advantage of using the cpH systems description for the design of controllers with respect to using the pH description, I first summarize the energy-shaping and damping-injection (ES-DI) formalism for a pH system and then show the simplicity of the same method for a cpH system. Moreover, I show how the dissipation obstacle that affects the (ES-DI) formalism for a pH system is bypassed in the cpH description.

ES-DI in PH Systems

Stabilization via energy balancing [4] is a control method that uses the additive property of the Hamiltonian. A controller with added energy function H_a can be designed so that a desired closed-loop energy function H_d is obtained

$$H_d(x) = H(x) + H_a(x) \tag{PH-5}$$

If H_d has a minimum at the desired equilibrium x , the system will be stable.

In [7] this technique is further developed by introducing a ***damping injection*** input in order to ensure convergence of the state trajectories to the desired equilibrium. This procedure is known as Energy-shaping and damping injection (ES-DI). Note that since the classical Hamiltonian does not represent damping, damping injection has to be implemented separately from energy-shaping.

For a pH plant such as in (2-14) the ES-DI method attempts to obtain a desired closed-loop system

$$\dot{x} = (J(x) - R_d(x)) \quad H_d(x) \tag{PH-6}$$

where R_d is the desired damping matrix

$$R_d(x) = R(x) + g(x)K_d(x)g^{\top}(x)$$

with damping injection matrix K_d. The desired Hamiltonian H_d is defined in (C-5).

The desired closed-loop dynamics can be realized with control input

$$u(x) = u_{ES}(x) + u_{DI}$$

where the energy shaping input is generated by the added energy Hamiltonian H_a and the damping injecting input generated by desired Hamiltonian H_d

$$\begin{aligned} u_{ES} &= (g^{\top}(x)g(x))^{-1}g^{\top}(x)[J(x) - R(x)]\frac{\partial H_a}{\partial x}(x) \\ u_{DI} &= -K_d(x)g^{\top}(x)\frac{\partial H_d}{\partial x}(x) \end{aligned}$$

The added energy function $H_a(x)$ can then be found solving the following set, see equation (15.18) of [3]

$$\begin{bmatrix} g \ \ (x)[J(x) - R(x)] \\ g^{\top}(x) \end{bmatrix} \frac{\partial H_a}{\partial x}(x) = 0 \qquad \text{(PH-7)}$$

The solution for H_a that satisfies $x \ = \text{argmin}(H_d = H + H_a)$ is then selected.

However, the PDE's in (PH-7) imply that the controlled state cannot be affected by damping. Since in physical systems states are in general affected by damping, this implication reduces the applicability of the ES-DI method. This is known as the dissipation obstacle shown in e.g. Section 15.7 of [3].

ES-DI in CPH Systems

The Complex-Hamiltonian acts as both the generator of the equations of motion of the system and as an energy function. In (C-5) it was shown how a synthesized additive energy function $H_a(x)$ can be added to the energy function of the plant in order to obtain a desired energy function H_d. For cpH systems, this results in a Hamiltonian balance equation:

$$H_d(a) = H_p(a) + H_c(a) \qquad (4\text{-}43)$$

Since the controller Hamiltonian[2] $H_c(a)$ uses the same state as the plant $H_p(a)$, the closed-loop dynamics are obtained as the Poisson bracket (4-13) between the state and the desired Hamiltonian H_d. From the sum rule in differentiation we have that the desired closed-loop equation of motion is

$$\dot{a}_d = \{a, H_d\} = \{a, H_p\} + \{a, H_c\} \qquad (4\text{-}44)$$

Compare this to (PH-6) in a pH system. Since the Complex Hamiltonian can include damping, the damping injection is done simultaneously with the energy-shaping in the Hamiltonian balance equation $(4 - 43)$. As I will now show, the ES-DI formalism,— from (C-5) up to (PH-7)— is reduced to (4-43) and (4-44).

[2] I use the term controller Hamiltonian H_c rather than added energy H_a, since in contrast to the added energy Hamiltonian in pH systems, the complex controller Hamiltonian, H_c, can also inject damping.

For a CpH system with input-state-output description

$$\begin{aligned} \dot{\mathbf{a}} &= \mathsf{J} \quad \mathsf{H}(a) + \mathsf{G}u, \\ y &= \mathsf{G}^{|} \quad \mathsf{H}(a), \end{aligned} \tag{4-45}$$

Energy-Shaping and Damping Injection (ES-DI) —see Section C-0-2— can be used to obtain the desired closed-loop system

$$\dot{\mathbf{a}}_d = \mathsf{J} \quad \mathsf{H}_d(a) \tag{4-46}$$

The desired Hamiltonian H_d is found using (4-43) so that it has a minimum at some desired equilibrium a :

$$a \quad = \arg\min(\mathsf{H}_d(a)) \tag{4-47}$$

The desired closed-loop dynamics (4-46) can be achieved by introducing the feedback law

$$u = (\mathsf{G}^{|}\mathsf{G})^{-1}\mathsf{G}^{|}\mathsf{J} \quad \mathsf{H}_c(a) = \{a, \mathsf{H}_c\} \tag{4-48}$$

as control input u of the system in (4-45). This indeed corresponds to the closed-loop equation of motion described by the Poisson bracket (4-44):

$$\dot{\mathbf{a}}_d = \mathsf{J} \quad \mathsf{H}(a) + \mathsf{J} \quad \mathsf{H}_c(a) = \{a, \mathsf{H}\} + \{a, \mathsf{H}_c\} \tag{4-49}$$

Substituting (4-46) and (4-48), this can be rewritten as

$$u = \dot{\mathbf{a}}_d - \dot{\mathbf{a}} = \{\mathbf{a}, \mathsf{H}_d\} - \{\mathbf{a}, \mathsf{H}\} \tag{4-50}$$

The simplicity of the above expression for the control law, u, is due to ability of Complex-Hamiltonians to include damping. Whereas the Hamiltonian dynamics of a controller in pH systems can only generate the energy shaping input, the dynamics of a CpH controller generates both the energy shaping and dissipation injection inputs. For a CpH system, the Energy-Shaping and Damping-Injection formalisms are thus combined in a single formalism.

More importantly, the controller Hamiltonian is a solution to the set

$$\begin{pmatrix} \mathsf{G} \quad (a)\mathsf{J}\,(a) \\ \mathsf{G}^{|} \end{pmatrix} \quad \mathsf{H}_c(a) = 0 \tag{4-51}$$

where $\mathsf{G} \quad (a)$ is the left-annihilator of $\mathsf{G}(a)$.

In the previous Section in was shown that the PDE's in (PH-7) imply that the controlled state cannot be affected by damping: the dissipation obstacle. The set of PDE's in (4-51) clearly does not imply that the to be shaped state cannot be affected by damping. Therefore, the dissipation obstacle is bypassed in the cpH description.

Bypassing the dissipation obstacle is the result of the structure of the Complex Hamilton's equation (or the Complex Hamiltonian vector field). Since the structure is not violated by damping, the set of PDE's in (4-51) does not require the to be shaped state to be unaffected by damping.

It should be kept in mind that the cpH theory introduced in the present chapter has so far only been developed for linear damped harmonic oscillators. Therefore, it can not be concluded that using Complex-Hamiltonians will solve the dissipation obstacle in port-Hamiltonian systems in general. This needs to be further researched.

Example: Energy-Shaping and Damping Injection

Consider a linear Mass-Spring-Damper system with mass m, spring constant k, damping constant b, natural frequency $\omega_n = \sqrt{\frac{k}{m}}$, damping ratio $\zeta = \frac{b}{2\sqrt{km}}$, input matrix $G = \begin{pmatrix} i\sqrt{\frac{1}{2m}} & -i\sqrt{\frac{1}{2m}} \end{pmatrix}^{\mathsf{T}}$. The Complex-Hamiltonian of the system is

$$H = \frac{1}{2}\begin{pmatrix} a & \bar{a} \end{pmatrix} \begin{pmatrix} -i\zeta & (1-i\zeta) \\ (1-i\zeta) & i\zeta \end{pmatrix} \begin{pmatrix} a \\ \bar{a} \end{pmatrix} = \frac{1}{2}\mathbf{a}^{\mathsf{T}}\mathbf{H}\mathbf{a} \tag{4-52}$$

The input-state-output description of this system is

$$\begin{aligned} \begin{pmatrix} \dot{a} \\ \dot{\bar{a}} \end{pmatrix} &= \begin{pmatrix} 0 & -i\omega_n \\ i\omega_n & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{H}}{\partial a} \\ \frac{\partial H}{\partial \bar{a}} \end{pmatrix} + \begin{pmatrix} i\sqrt{\frac{1}{2m}} \\ -i\sqrt{\frac{1}{2m}} \end{pmatrix} u \\ y &= \begin{pmatrix} i\sqrt{\frac{1}{2m}} & -i\sqrt{\frac{1}{2m}} \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{H}}{\partial a} \\ \frac{\partial H}{\partial \bar{a}} \end{pmatrix} \end{aligned} \tag{4-53}$$

Suppose we want to stabilize the system around a desired equilibrium $(q, p) = (q\ , 0)$ or, in complex state variables, $a\ = \bar{a}\ = \sqrt{\frac{k}{2}}q\ $.

The desired Hamiltonian, H_d, thus is

$$H_d = \frac{1}{2}\left(\mathbf{a} - \mathbf{a}\ \right)^{\mathsf{T}} \begin{pmatrix} -i\zeta & (1-i\zeta) \\ (1-i\zeta) & i\zeta \end{pmatrix} \left(\mathbf{a} - \mathbf{a}\ \right) \tag{4-54}$$

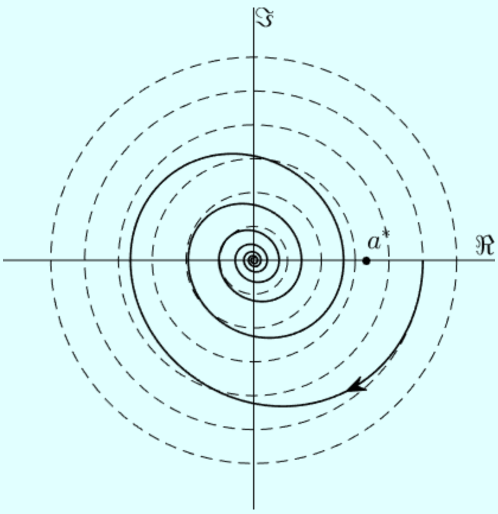

Figure 4-6: Trajectory of autonomous system. The dashed circles represent iso-energetic lines. $a^{\square}$ is the desired forced equilibrium

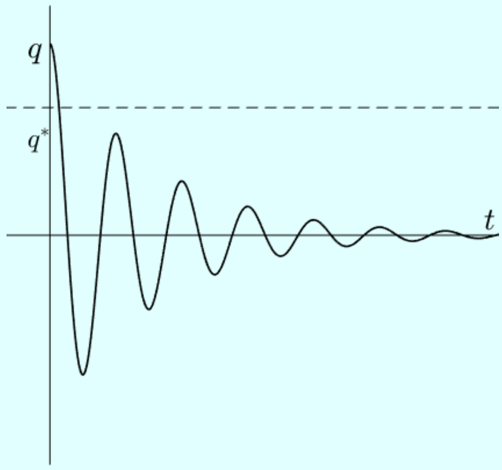

Figure 4-7: Signal trajectory of position q for the autonomous system.

From (4-50) we can immediately find the control law that results in the desired closed-loop dynamics as

$$u = \{a, H_d\} - \{a, H\} \tag{4-55}$$

The controller that achieves the desired closed-loop dynamics is specified by the Hamiltonian

$$H_c = i\zeta\left(a(a^* - \bar{a}^*) + \bar{a}(a^* - \bar{a}^*)\right) - a\bar{a}^* - \bar{a}a^* - a^*\bar{a} \tag{4-56}$$

Verifying that H_c is indeed a solution for (4-51)

$$\begin{pmatrix} G^\perp(a)J(a) \\ G^\top \end{pmatrix} H_c(a) = \begin{pmatrix} i\omega_n & -i\omega_n \\ i\sqrt{\frac{1}{2m}} & -i\sqrt{\frac{1}{2m}} \end{pmatrix} \begin{pmatrix} -i\zeta(\bar{a}^* - a^*) - \bar{a}^* \\ i\zeta(a^* - \bar{a}^*) - a^* \end{pmatrix} = 0, \tag{4-57}$$

with left-annihilator $G^\perp = \begin{pmatrix} 1 & 1 \end{pmatrix}$ and using that $a^* = \bar{a}^*$.

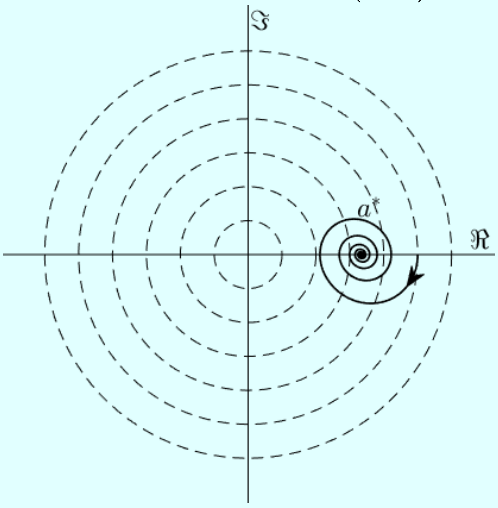

Figure 4-8: Trajectory of system stabilized around forced equilibrium a^*

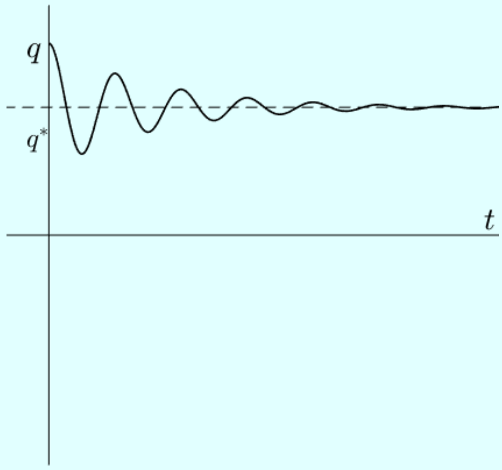

Figure 4-9: Position signal of system stabilized around forced equilibrium.

Now suppose we want to achieve the closed-loop dynamics

$$\dot{a}_d = -(\delta + i\psi)(a - a^*) + \delta(\bar{a} - \bar{a}^*) \tag{4-58}$$

Here, δ is the closed-loop damping coefficient and ψ is the closed-loop frequency. This implies that the controller law is

$$u = \{a, H_c\} = \underbrace{(\delta - \beta)}_{\text{DI}}(\bar{a} - a) - i\underbrace{(\psi - \omega_n)}_{\text{ES}}a + \underbrace{(\delta + i\psi)a^* - \delta\bar{a}^*}_{\text{Forced equilibrium}} \tag{4-59}$$

From (4-43), the controller Hamiltonian is simply found as

$$H_c = \frac{1}{2}\left(\mathbf{a} - \mathbf{a}^*\right)^\top \begin{pmatrix} -iD_d & E_d - iD_d \\ E_d - iD_d & iD_d \end{pmatrix} \left(\mathbf{a} - \mathbf{a}^*\right) - H \tag{4-60}$$

with $D_d = \delta/\omega_n$ and $E_d = \psi/\omega_n$.

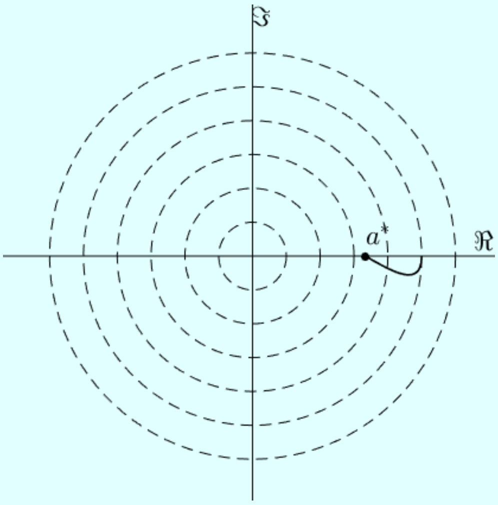

Figure 4-10: Trajectory of the closed-loop system stabilized with ES-DI around forced equilibrium $a^\star$

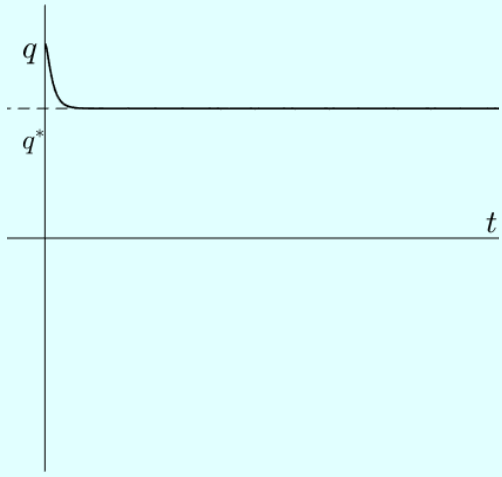

Figure 4-11: Position signal of closed-loop system stabilized with ES-DI around forced equilibrium

In Example 15.5 of [3], it is shown that ES-DI controllers (such as the one we obtained in the previous example) are classical PD controllers. However, designing such controllers with energy-shaping and damping-injection from pH systems theory results in a closed-loop energy balance. In pH systems theory, this balance is (see the end of Section 15.6 of [3])

$$H_d(x) = H(x) - \int_0^t u^\top(\tau)y(\tau)d\tau \tag{4-61}$$

For cpH systems, this energy balance is

$$\mathsf{H}_d = \mathsf{H} + \mathsf{H}_c \tag{4-62}$$

Again, in contrast to in pH systems theory the damping-injecting (DI) controller can be described by a Hamiltonian, H_c, in cpH systems theory. Hence, in cpH systems the energy balance (4-62) is simply the balance of Hamiltonians whereas in the pH (4-61), the power at the port of the system needs to be integrated in order to obtain the energy.

4-5-3 Control by Interconnection

The PBC formalism is used to obtain passive controllers. In pH systems theory, dynamic controllers can be obtained with the control by interconnection (CbI) formalism. The CbI formalism is summarized in Appendix C of this thesis. In the CbI formalism, the controller is modeled as a pH system itself. The output of the plant is the input of the controller and, up to a sign, vice versa. In contrast to the PBC control formalism, CbI considers dynamic controllers with controller state $\mathbf{x}_c \in \mathsf{X}_c$.

A key open topic in controller synthesis in the CbI formalism is how to efficiently shape the energy of the closed loop between the plant and the controller. A widely proposed formalism is finding dynamical invariants that relate states of the controller to states of the plant, decreasing the degrees of freedom in the choice of possible controllers. This formalism, known as the Energy-Casimir formalism, can however not be applied to states that are subject to damping; as the PBC formalism, the CbI formalism is affected by the dissipation obstacle. For a summary of the CbI formalism, the Energy-Casimir formalism, the dissipation obstacle, as well as for references to the literature, the reader is referred to Appendix C-0-3.

In this Section, I integrate the CbI formalism in cpH systems theory. The starting point of the CbI formalism is that a controller is modeled as a cpH system itself. That is, the controller has a Hamiltonian $H_c(a_c)$, with controller state a_c, and a port connecting the controller to its environment. The input-state-output description of the controller is

$$\begin{aligned} \dot{\mathbf{a}}_c &= \mathsf{J}_c \quad \mathsf{H}_c + \mathsf{G}_c u_c \\ y_c &= \mathsf{G}_c^{\dagger} \quad \mathsf{H}_c \end{aligned} \tag{4-63}$$

with natural frequency of the controller ω_c, controller-state vector $\mathbf{a}_c \quad \mathsf{C}^{2v}$, input vector $u_c \quad \mathsf{R}^w, w \leq 2v$, output vector $y_c \quad R^w$, complex structure matrix[3] $\mathsf{J}_c : \mathsf{C}^{2v} \Rightarrow \mathsf{C}^{2v\times 2v}$, input matrix $\mathsf{G}_c : \mathsf{C}^{2v\times w}$, and the operator $\quad \mathsf{H}_c \quad \mathsf{C}^{2v}$ constructed as

$$\mathsf{H}_c = \begin{pmatrix} \frac{\partial \bar{H}_c}{\partial a_{c,1}} & \frac{\partial H_c}{\partial \bar{a}_{c,1}} & \cdots & \frac{\partial \bar{H}_c}{\partial a_{c,n}} & \frac{\partial H_c}{\partial \bar{a}_{c,n}} \end{pmatrix}^{\dagger} \tag{4-64}$$

A closed-loop system can be achieved by interconnecting a plant with input-output description (4-34) and a controller (4-63). Using a negative feedback interconnection implies

$$u_c = y, \qquad\qquad u = -y_c \tag{4-65}$$

This results in a closed-loop system with input-state-output description

$$\begin{aligned} \begin{pmatrix} \dot{\mathbf{a}} \\ \dot{\mathbf{a}}_c \end{pmatrix} &= \begin{pmatrix} \mathsf{J}_p & -\mathsf{G}_p\mathsf{G}_c^{\dagger} \\ \mathsf{G}_c\mathsf{G}_p^{\dagger} & \mathsf{J}_c \end{pmatrix} \begin{pmatrix} \mathsf{H}_p \\ \mathsf{H}_c \end{pmatrix} \\ \begin{pmatrix} y \\ y_c \end{pmatrix} &= \begin{pmatrix} \mathsf{G}_p^{\dagger} & 0 \\ 0 & \mathsf{G}_c^{\dagger} \end{pmatrix} \begin{pmatrix} \mathsf{H}_p \\ \mathsf{H}_c \end{pmatrix} \end{aligned} \tag{4-66}$$

with state vector $\mathbf{a} \quad \mathsf{C}^{2n}$, input vector $u \quad \mathsf{R}^m, m \leq 2n$, output vector $y \quad R^m$, complex matrix $\mathsf{J} : \mathsf{C}^{2n} \Rightarrow \mathsf{C}^{2n\times 2n}$ (defined in (4-15)), input matrix $\mathsf{G} : \mathsf{C}^{2n\times m}$, and the operator $\quad \mathsf{H}_p \quad \mathsf{C}^{2n}$ as defined in (4-35).

Comparing the closed-loop system in (4-66) to a closed-loop system obtained with CbI in pH systems theory (see (C-0-3)), we notice that the cpH description does not contain a dissipation matrix. This is again an important result.

In the Energy-Casimir formalism for pH systems, a Casimir function is found as a solution $F(x)$ for

$$\begin{pmatrix} -\frac{\partial^{\dagger} F(x)}{x} & I_{nc} \end{pmatrix} \begin{pmatrix} J(x) - R(x) & -g(x)g_c^{\dagger}(\xi) \\ g_c(\xi)g^{\dagger}(x) & J_c(\xi) - R_c(\xi) \end{pmatrix} = 0 \tag{PH-8}$$

[3] The structure matrix is constructed as (4-15). Here, the natural frequency is that of the controller.

However, the equality $R(x)\partial F/\partial x = 0$ implies that the to be shaped states cannot be affected by damping [16]. This is known as the dissipation obstacle for the CbI formalism.

As for pH systems, Casimir functions for cpH systems (4-66) can be found by relating the states of the plant to the states of the controller. This can be done by finding dynamical invariants (Casimirs) for the closed-loop system [3, 13, 14, 15, 16]. In general, this implies finding functions $\mathsf{C}(\mathbf{a}, \mathbf{a}_c)$ that satisfy[4]

$$\begin{pmatrix} \frac{\partial^{\dagger} \mathsf{C}}{\partial \mathbf{a}} & \frac{\partial^{\dagger} \mathsf{C}}{\partial \mathbf{a}_c} \end{pmatrix} \times \begin{pmatrix} J_p & -\mathsf{G}_p \mathsf{G}_c^{\dagger} \\ \mathsf{G}_c \mathsf{G}_p^{\dagger} & J_c \end{pmatrix} = 0 \tag{4-67}$$

and present a Lyapunov candidate function

$$V = H_p + H_c + \mathsf{C} \tag{4-68}$$

that has a minimum at $(\mathbf{a}\ \ , \mathbf{a}_c)$.

The typical candidate function used in pH systems, expressed in complex state variables becomes

$$\mathsf{C}(a, a_c) = a_c - F(a) = \kappa \tag{4-69}$$

where $F(a)$ is some differentiable function and κ is some constant. With the Casimir candidate, $\mathsf{C}(a, a_c)$, the closed-loop energy can be written as

$$H_{cl} = H(a) + H_c(C(F(a) + \kappa)) \tag{4-70}$$

The closed-loop energy function is conserved (invariant) if [13, 14, 15]

$$\frac{d}{dt}\mathsf{C}(a, a_c) = 0 \tag{4-71}$$

If this is the case, C is indeed a Casimir function and is independent of the Hamiltonian.

A Casimir function is found as a solution $F(x)$ for the following PDE's

$$\begin{pmatrix} -\frac{\partial^{\dagger} F(\mathbf{a})}{\partial^{\dagger} \mathbf{a}} & I \end{pmatrix} \begin{pmatrix} J_p & -\mathsf{G}_p \mathsf{G}_c^{\dagger} \\ \mathsf{G}_c \mathsf{G}_p^{\dagger} & J_c \end{pmatrix} = 0 \tag{4-72}$$

Compare (4-72) to (PH-8). Clearly, (4-72) —nor (4-67))— does not imply that damping cannot act on the complex states that need to be shaped. This implies that the dissipation obstacle in the Energy-Casimir formalism is bypassed by the cpH description

4-6 Conclusions

A Complex-Hamiltonian theory can be developed by defining the equations of motion of a damped harmonic oscillator as the complex Poisson bracket between the state and the Complex Hamiltonian. Using this definition, the Complex Hamiltonian can be derived by substituting the known equations of motion of a damped harmonic oscillator. Including the Complex Hamiltonian in the port-Hamiltonian formalism generalizes energy-storing and energy-dissipating elements as stateful elements. Contrary to port-Hamiltonian systems, Complex-Port-Hamiltonian systems include the effect of dissipation in the Hamilton's equation. This

[4] Notice that this set of PDE's is the complex equivalent of the set of PDE's in the lossless case of pH systems. See equation (15.9) of Van Der Schaft [3]

has two main results. The first result is that the Energy-Shaping and Damping-Injection control formalism reduces to a single Complex-Hamiltonian balance function. The second, and most important, result is that the control formalisms for a Complex-Port-Hamiltonian systems is not affected by the dissipation obstacle.

I conclude that defining the complex equation of motion as the Poisson bracket between the state and the Hamiltonian has a crucial step in the derivation of the Complex Hamiltonian. Resulting from this step, the Complex Hamiltonian always fulfills the requirements that it should generate the time-evolution of the system and is a constant of the motion. Furthermore, I argue that this is the reason why the Complex-Hamiltonian is not affected by the dissipation obstacle. The control formalisms from port-Hamiltonian systems theory are defined for systems with a Poisson algebra. The dissipation matrix in port-Hamiltonian systems, however, violates the Poisson bracket of the classical Hamiltonian. As a result, the control methods from port-Hamiltonian systems theory fail for dissipative systems. Since the Complex Hamiltonian is defined so that the Poisson bracket defines the equation of motion of a damped harmonic oscillator, the Complex Hamiltonian bypasses the dissipation obstacle. Proof of these notions is, however, left for further research.

Future work is also needed to relate the Complex Hamiltonian to a Lagrangian, e.g. the fractional Lagrangian in Chapter 3. This would imply that dissipative systems follow a fundamental principle. Furthermore, the theory developed in this chapter has only considered linear second-order damped harmonic oscillators. Conceptually, it is possible to model higher order systems as interconnected second-order systems, but a structured formalism to match the natural frequencies of the individual subsystems has to be developed. Finally, the Complex-Hamiltonian theory may contribute in modeling and controlling dissipative nonlinear systems from an energy perspective.

Contributions

1. Damped harmonic oscillators are included in the Hamiltonian formalism
2. The equation of motion of a damped harmonic oscillator is expressed as a canonical Hamilton's equation
3. Dissipative cpH systems do not violate the symplectic structure, in contrast to dissipative pH systems
4. For cpH systems, the ES-DI method reduces to a single complex equation of motion
5. The dissipation obstacle that affects the control of pH systems is bypassed in cpH systems

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Chapter 5

The Mechanics of Consumption with Complex-Hamiltonian Systems

5-1 Introduction

In economic engineering, available surplus is the economic analog of the Hamiltonian from mechanics, see Section 2-3. Modeling an economic agent as a Surplus Hamiltonian system results in a description of a trader that is analogous to a mechanical harmonic oscillator. However, the mechanical Hamiltonian only describes conservative systems, see e.g. the introduction of [1] and references therein. For the description of an economic agent, this implies that consumption (and profits) are not included in the Surplus Hamiltonian.

Problem
The Surplus Hamiltonian from Economic Engineering is not a function of consumption

In this Chapter, I use the Complex Hamiltonian, developed in Chapter 4, to include consumption (and profit) in the Surplus Hamiltonian.

Solution
Include Consumption in the Surplus Hamiltonian following the Complex Hamiltonian theory

In Chapter 4, the Complex Hamiltonian was introduced to include dissipative systems in the Hamiltonian formalism. In the present chapter, the Complex Hamiltonian will be used to include consumption in the Hamiltonian formulation of economic agents [2]. The economic theory of the saving-investment cycle [3, 4] is interpreted in the framework of economic engineering in Section 5-2. The Complex Hamiltonian represents the saving function, constructed as the sum of the disposable income function and the consumption function. In Section 5-3, the saving function[4] as Complex Hamiltonian system is reformulated as a cpH system with physical inputs and outputs. In Section 5-4, control methods of cpH systems theory are applied to the cpH formulation of the saving function.

5-2 The Saving-Investment Cycle

In the financial system, surplus from one economic agent, e.g. a household, is used to finance the investment of another agent, e.g. a firm. This is known as the saving-investment cycle [3]. The saving, S, of an agent equals their disposable income, Y_d, minus their consumption, C, see e.g. [4]

$$S = Y_d - C \tag{ECON-1}$$

Here, disposable income, Y_d, is the income after taxes, social security fees and necessities [5]. The agent is thus free to choose whether to save or consume its disposable income. The consumption function, C, does not include the consumption necessary for survival, i.e. food and shelter; only the consumption of disposable income, $C = C(Y_d)$.

5-2-1 Some Preliminaries on the Hamiltonian Surplus Function

The goal of this chapter is to present a prove of concept for the use of the Complex Hamiltonian to model the saving function (ECON-1). As is done in economic engineering theory in general, economic agent is modeled as a simple (damped) mechanical oscillator [?]. The agent is defined by a single capital asset stock, q, with price p. The disposable income, Y_d, is the capital income obtained by trading the capital asset, q, see Figure 5-1. Here, it is assumed that the agent has a fixed Trade Frequency, ω_T.

Notice that the model only regards disposable income. Zero disposable income does not mean that the agent is broke; the agent still finances its necessities with its non-disposable income. The capital itself cannot be consumed; the agent can only consume the monetary value of the capital after liquidating the capital stock. Depreciation of capital is not considered in the model.

5-2-2 The Complex Hamiltonian as the Complex Surplus Function

Whereas the Surplus Hamiltonian from economic engineering can only model the surplus from disposable income (see Section 2-3), the Complex Hamiltonian can conceptually also include the surplus from consumption as economic analog of dissipation, see Section 4-3. Modeling the agent as a Complex-Hamiltonian results in a Complex Surplus Hamiltonian, Σ, as the generator of a saving equation corresponding to (ECON-1).

The state of the economic agent is defined by the stock of capital assets, q, and the corresponding price, p. Integrating this state in the Complex-Hamiltonian theory, the complex state is defined as

$$z \equiv \alpha_1 q + i\alpha_2 p \tag{5-1}$$

Here, α_1 and α_2 depend on the illiquidity, κ, and the slope of the demand curve (price rigidity), μ, respectively, see section 2-3. Assuming that both the illiquidity and the price rigidity are

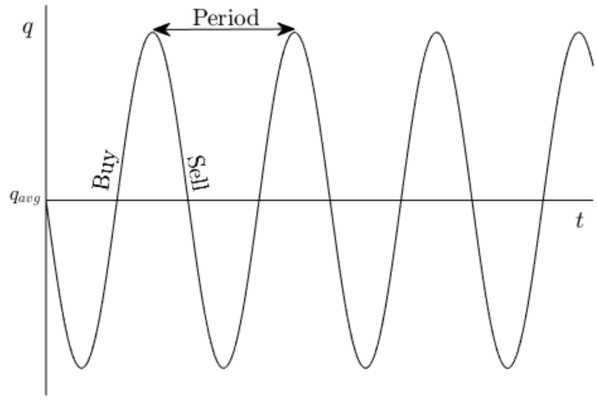

Figure 5-1: Periodic fluctuation in capital stock due to trade.

linear, the complex economic state is

$$z = \sqrt{\frac{\kappa}{2}}q + i\sqrt{\frac{1}{2\mu}}p, \tag{5-2}$$

Analogous to the Complex Hamiltonian (see Section (4-3)), the Complex Surplus Hamiltonian, Σ, is composed by a real part representing the Disposable Income function, Y_d, and an imaginary part representing the Consumption function, C.

$$\Sigma = Y_d - iC \tag{5-3}$$

The Disposable Income function, Y_d, is analogous to the mechanical energy (E in (4-29))[1] and the Consumption function is analogous to the Dissipation function (D in (4-29)). Following this analogy, the Disposable Income function of an economic agent with state (5-2) is

$$Y_d = z\bar{z} \tag{5-4}$$

When the Consumption, C, is zero and no profits are made by trading the capital, the surplus, Σ, is constant. In that case, the Complex Surplus Hamiltonian reduces to the Surplus Hamiltonian introduced by economic engineering in Section 2-3.

5-2-3 Damping Ratio as the Marginal Propensity to Consume

Now we will evaluate the analogy between dissipation and consumption. In economics, the consumption function, C, introduced by Keynes [6], relates consumption to Disposable Income, Y_d [7]. In its simplest form, the Consumption function is a linear function of the disposable income, see e.g. Chapter 4 of [4]

$$C = a + \lambda Y_d \tag{ECON-2}$$

[1] For your convenience (4-29): $H = E - iD$

Here, a is the (constant) autonomous consumption and λ is the ***marginal propensity to consume***. The autonomous consumption is the consumption that is independent of the income. Since it is a constant, the autonomous income will not have an influence on the dynamics of the economic agent and can therefore be omitted.

In economics, the marginal propensity to consume (MPC) is the change in consumption due to a change in disposable income [4]

$$\lambda = \frac{\Delta C}{\Delta Y} \tag{ECON-3}$$

Substituting the complex Disposable Income function, Y_d in (5-4), it follows that the complex Consumption function, C, should be proportional to the Disposable Income function multiplied by the MPC, λ:

$$C \quad \lambda Y_d$$

The economic agent generates disposable income by trading capital. However, capital itself cannot be consumed; the agent first needs to liquidate the capital in order to consume. This is the same effect as was found for the damped harmonic oscillator in Chapter 4. The damped harmonic oscillator can only dissipate kinetic energy. To capture this effect, I will use the dissipative function, D, of the Complex Hamiltonian of a damped harmonic oscillator (4-29) to model the Consumption function:

$$C := \lambda \left(z\bar{z} + \frac{z^2 - \bar{z}^2}{2} \right) \tag{5-5}$$

This implies that the marginal propensity to consume, λ, is analogous to the mechanical damping ratio, ζ. In economics, the MPC is the ratio of the differences in consumption and disposable income over a given period [4]

$$MPC \equiv \frac{\Delta C}{\Delta Y} \tag{ECON-4}$$

Substituting the (5-4) and (5-5) this becomes

$$MPC = \frac{\Delta\lambda \left(z\bar{z} + \frac{z^2 - \bar{z}^2}{2} \right)}{\Delta z\bar{z}} = \lambda \left(\frac{\Delta z\bar{z}}{\Delta z\bar{z}} + \frac{\Delta(z^2 - \bar{z}^2)}{2\Delta z\bar{z}} \right) \tag{5-6}$$

The first fraction on the right-hand side clearly reduces to identity. The second fraction takes values between $-i$ and i and vanishes when the interval over which the difference is measured is set to the economic period[2] of the agent. Therefore, it follows that

$$MPC := \lambda \tag{5-7}$$

Comparing the Consumption function, C, in (5-5) with the Complex Hamiltonian of a damped harmonic oscillator in (4-8), it can be seen that the MPC in the Consumption function is analogous to the damping ratio, ζ in the Complex Hamiltonian of a mechanical system, see Section 4-3.

[2] See Figure 5-1

Mathematically, we can also evaluate the instantaneous change in consumption with respect to the disposable income:

$$\Lambda = \frac{dC}{dY} = \lambda \left(1 + \frac{zdz - \bar{z}d\bar{z}}{zd\bar{z} + \bar{z}dz}\right) \tag{5-8}$$

Here, Λ is the instantaneous marginal propensity to consume (iMPC). In contrast to the MPC, the iMPC is not a constant but a complex function. The real part of the iMPC is equal to the MPC (λ) and the imaginary part, generated by $\frac{zdz-\bar{z}d\bar{z}}{zd\bar{z}+\bar{z}dz}$, takes values between $-i\lambda$ and $i\lambda$.

Modeling an economic agent as a Complex Hamiltonian Surplus function thus implies that the damping ratio, ζ, in mechanical systems is analogous to the MPC in economics. Furthermore, the model implies that the instantaneous MPC is not constant, but a complex function with constant real part (the MPC) and an imaginary oscillation. This fluctuation in the instantaneous MPC, Λ, is due to the fact that the economic agents first has to liquidate its capital before he can consume its value. This is a similar effect as observed in the Complex Hamiltonian of a mechanical system, where the dissipation function in the Hamiltonian accounts for the fact that damping can only act on the kinetic energy.

5-2-4 Complex Hamilton's Equation as Saving Equation

Now we evaluate the time-evolution of the complex state, z, following the Complex Hamilton's equations, see Section 4-3-3. From the theory on Complex Hamiltonians developed in Chapter 4, we have that the equations of motion are defined as the Poisson bracket between the state and the Complex Hamiltonian, see Section 4-3-2. This implies that the flow of the economic state is defined by the Poisson bracket between the economic state, z, and the complex surplus function, Σ.

$$\dot{z} = \{z, \Sigma\} \tag{5-9}$$

With the definition of the Poisson bracket [8], this can be written as

$$\dot{z} = \frac{i}{\rho}\left(\frac{\partial\Sigma}{\partial z}\frac{\partial z}{\partial\bar{z}} - \frac{\partial z}{\partial z}\frac{\partial\Sigma}{\partial\bar{z}}\right) \tag{5-10}$$

Substituting $\rho = 1/\alpha_1\bar{\alpha_2} + \bar{\alpha_1}\alpha_2 = \sqrt{\frac{\mu}{\kappa}}$, results in the economic interpretation of the Complex Hamilton's equation

$$\dot{z} = -i\sqrt{\frac{\kappa}{\mu}}\frac{\partial\Sigma}{\partial\bar{z}} \tag{5-11}$$

Here, $\sqrt{\frac{\kappa}{\mu}}$ is defined as the rate of the economic cycle [3], the trade frequency, of the agent [?]

$$\omega_T \equiv \sqrt{\frac{\kappa}{\mu}} \tag{5-12}$$

From the sum rule in differentiation, the flow of the economic state is the combined effect of the flow generated by the disposable income function, Y_d, and the consumption function, C,

$$\dot{z} = \{z, \Sigma\} = \{z, Y_d\} - \{z, iC\} \tag{5-13}$$

[3]Evaluating the units of κ and μ, we have that since μ provides a relation between a flow of capital $\left[\frac{\#}{yr}\right]$ and price $\left[\frac{\$}{\#}\right]$ and κ provides a relation between a stock of capital [#] and a cost $\left[\frac{\$}{\#yr}\right]$. The units of ω_T thus are $\left[\frac{1}{yr}\right]$.

Defining the flow generated by the Disposable Income function, Y_d, as the disposable income, y_d,

$$y \equiv \{z, \mathsf{Y}_d\} \tag{5-14}$$

and the flow generated by the Consumption function, C, as the consumption, c,

$$c \equiv \{z, iC\}, \tag{5-15}$$

it follows that the flow of the economic state, $\dot{z}$, represents the saving (ECON-1)

$$\dot{z} = y_d - c = s \tag{5-16}$$

Note that the saving, s, is the amount of money saved per unit of time —thus, a flow variable — and not the amount of money saved [6].

Example: Income and Consumption from an Endowment

Consider an economic agent that receives an endowment $q_0 = q(t_0)$ at time $t = t_0$. The price of the endowment asset at time $t = t_0$ is $p(t_0) = p_0$. The agent has a price rigidity μ, an illiquidity κ, a marginal propensity to consume λ, and its complex state z is

$$z = \sqrt{\frac{\kappa}{2}}q + i\sqrt{\frac{1}{2m}}p \tag{5-17}$$

The agent can be described by the Complex Hamiltonian

$$\Sigma = \mathsf{Y}_d - i\mathsf{C} \tag{5-18}$$

Here Y_d is the disposable income function and C is the consumption function, respectively given as

$$\mathsf{Y}_d = z\bar{z} \tag{5-19}$$

$$\mathsf{C} = \lambda\left(z\bar{z} + \frac{z^2 - \bar{z}^2}{2}\right) \tag{5-20}$$

If the MPC is zero, $\lambda = 0$, the saving of the agent is equal to the income of the agent, generated by the initial endowment

$$s = \{z, \Sigma\} = \{z, \mathsf{Y}_d\} = y_d = -i\omega_T z \tag{5-21}$$

Clearly, in this case the surplus of the agent is conserved.
If the MPC is greater than zero, $\lambda > 0$, the agent will consume a part of the Disposable Income, Y_d. The saving of the agent then becomes

$$s = \{z, \Sigma\} = \{z, \mathsf{Y}_d\} - \{z, iC\} = y_d - c = -i\omega_T z - \lambda\omega_T(\bar{z} - z) \tag{5-22}$$

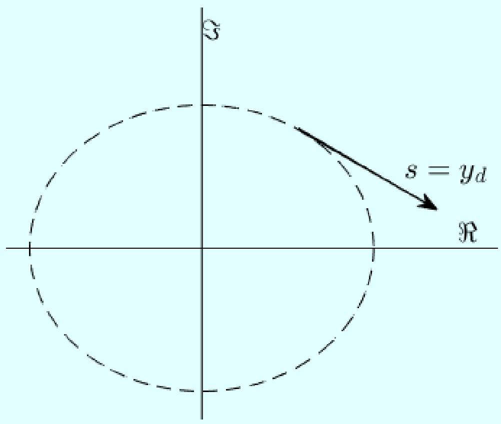

Figure 5-2: Trajectory of the complex state, z, of the economic agent. Since the agent does not consume, the saving, s, is equal to the disposable income, y_d.

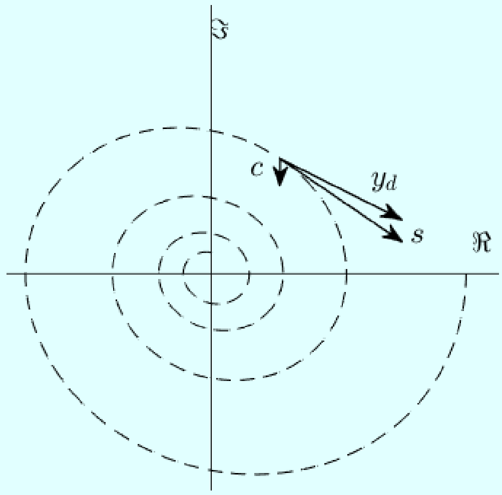

Figure 5-3: Trajectory of the complex state, z, with consumption, c. Since the agent consumes a part of its disposable income, y_d, its saving, s, decreases.

Applying the Complex-Hamiltonian of a damped harmonic oscillator, introduced in Chapter 4, allows us to model the saving, s of an economic agent that obtains disposable income,y_d, from trading capital and consumes, c, a part of the disposable income. In general, this implies that the Complex-Hamiltonian theory can include consumption in the Hamiltonian formulation of an economic agent, introduced by [2].

5-3 Evaluating the Economic Agent as a Complex-Port-Hamiltonian System

In the next two section, I show how the economic agent with a Complex Surplus Hamiltonian can be described as Complex-Port-Hamiltonian systems and introduce the interpretation of the cpH controller for the economic agent. Since these topics are beyond the scope of this thesis, they are only discussed briefly to illustrate the potential of the Complex-Hamiltonian theory for economic systems.

The description of an economic agent as a Complex-Hamiltonian system includes the marginal propensity to consume, λ, as an endogenous variable in the Complex Surplus Hamiltonian, Σ. However, the complex state, z, itself is an abstract description of the economic agent. In Section 4-4, physically meaningful input and output signals are connected to the Complex Hamiltonian system by describing the system as a complex-port-Hamiltonian system (cpH). By describing the economic agent as a cpH system, ”physical”economic input and output signals can be connected to the Complex Surplus Hamiltonian. From economic engineering [2], we have that pairs of price movement, $\dot{p}$, and flow of assets, $\dot{q}$, serve as pairs of inputs and outputs.

The input-state-output description of the economic agent with Complex Surplus Hamiltonian,

Σ, as a cpH system is

$$\begin{aligned} \dot{\mathbf{z}} &= \mathsf{J} \quad \Sigma + \mathsf{G}u \\ y &= \mathsf{G}^{\top} \quad \Sigma \end{aligned} \tag{5-23}$$

with state vector $\mathbf{z}$ $\quad \mathsf{C}^{2n}$, input vector u $\quad \mathsf{R}^m, m \leq 2n$, output vector y $\quad R^m$, complex matrix $\mathsf{J} : \mathsf{C}^{2n} \Rightarrow \mathsf{C}^{2n \times 2n}$ (defined in (4-15)), input matrix $\mathsf{G} : \mathsf{C}^{2n \times m}$ —relating pairs of real coordinates to complex coordinates — and the operator $\quad \Sigma \quad \mathsf{C}^{2n}$ constructed as[4]

$$\Sigma = \begin{pmatrix} \frac{\partial \bar{\Sigma}}{\partial z_1} & \frac{\partial \Sigma}{\partial \bar{z}_1} & \dots & \frac{\partial \bar{\Sigma}}{\partial z_n} & \frac{\partial \Sigma}{\partial \bar{z}_n} \end{pmatrix}^{\top} \tag{5-24}$$

From the input-state-output description in (5-23) it can be derived that the Complex Surplus flow at the port of the system is

$$\begin{aligned} \frac{d\Sigma}{dt} &= \quad \Sigma^{\top} \dot{\mathbf{z}} \\ &= \quad \Sigma^{\top} \mathsf{J} \quad \Sigma + \quad \Sigma^{\top} \mathsf{G} u \\ &= y^{\top} u \end{aligned} \tag{5-25}$$

This implies that the loss or gain in surplus is a result of the interaction of the agent with its environment. Conceptually, this implies that the agent cannot loose or gain surplus if there is no counterparty to interact with.

5-4 Evaluating Complex Surplus Hamiltonian Control

The cpH description of the economic agent enables the use of control formalisms from cpH systems theory. In this section, I briefly discuss the interpretation of a cpH controller for economic systems. Again, note that this is beyond the scope of this thesis,

For mechanical systems, cpH controllers are defined as a Complex Hamiltonian, see Section 4-5. For an economic system, this implies that a cpH controller is defined as a Complex Surplus Hamiltonian. That is, the controller will be modeled as an economic agent that has an income and consumes. The total Complex Surplus Hamiltonian, Σ_T, is then the sum the Complex Surplus Hamiltonian of the agent, Σ, and the external Complex Surplus Hamiltonian from the controller, Σ_E:

$$\Sigma_T = \Sigma + \Sigma_E \tag{5-26}$$

The Complex Surplus Controller will, by analogy [2], control the economic agent by adding or subtracting surplus. A passive Complex Surplus Controller will subtract economic surplus from the economic agent and an active Complex Surplus Controller will add surplus to the agent. A possible economic interpretation of the passive Complex Surplus Controller is a firm that drains surplus from its clients. A possible interpretation of the active Complex Surplus Controller is an investor that provides surplus to a firm.

[4] See (4-35)

5-5 Conclusions

Consumption has been included in the Surplus Hamiltonian, using the Complex-Hamiltonian theory that was developed in Chapter 4 of this thesis. Utilizing the analogies between mechanical and economics systems, the Complex-Hamiltonian description of the agent resulted in a dynamical reformulation of the saving function from economics. Analyzing the Complex Surplus Hamiltonian, it was shown that the marginal propensity to consume of an economic agent is analogous to the damping ratio of a damped harmonic oscillator.

Describing economic systems with Complex Surplus Hamiltonians enables the use of control formalisms from Complex-Hamiltonian theory. This is an important result regarding economic engineering. Since the scope of this thesis was to include consumption in the Surplus Hamiltonian, control of the Complex Surplus Hamiltonian has been illustrated but has not been formally developed. However, combining the analogies from economic engineering, the interpretation of the economic agent as Complex-Hamiltonian system and the control formalisms of Complex-Port-Hamiltonian provides a strong template for further research.

Contributions

1. Consumption is included in the Surplus Hamiltonian
2. The Marginal Propensity to Consume is analogous to the Damping Ratio
3. Economic systems can be controlled with control formalisms from Complex-Port-Hamiltonian systems theory

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Chapter 6

Additional Results on Fractional Energy-Storage Variables for Dissipative Systems

6-1 Introduction

Resistors do not store energy. This makes resistors different from the other two basic dynamical elements, inertances and capacitors [1]. Inertances and capacitors store respectively kinetic energy as a function of momentum and potential energy as a function of position. Resistors dissipate mechanical (or electric) energy into heat. However, heat (or internal energy) is not expressed in mechanical properties but in the thermodynamic properties temperature and entropy, see e.g. [2]. The amount of heat is quantitatively equal to the amount of energy subtracted from the system and transferred to the environment [3].

Riewe [4, 5] showed that dissipative elements can be included in the mechanical Lagrangian with fractional derivatives. In particular, Riewe used the half-order derivative of position By performing a Legendre transform, he obtained a Hamiltonian that included the energy from dissipative elements. Improvements on Riewe's Hamiltonian have been proposed in [6], in [7] it was shown that Riewe's Hamiltonian is conserved by applying a fractional Noether's theorem.

In this additional chapter, I show how Riewe's fractional derivative can be used to obtain a mechanical storage function for internal energy generated by heat from dissipation. The bond graph approach for dynamical systems [1] is used as methodological framework for this derivation. This will result in a mechanical expression for internal energy, $U(\breve{q})$, analogous to the the well-know mechanical expressions for the variation of kinetic energy $T(p)$ and potential energy $V(q)$

$$T(p) = \int^{p} f(p)dp \qquad V(q) = \int^{q} e(q)dq \qquad U(\breve{q}) = \int^{\breve{q}} \breve{p}(\breve{q})d\breve{q} \tag{6-1}$$

Here, $\breve{q}$ is the half-order derivative of the position, and $\breve{p}$ is its conjugate.

This chapter is structured as follows.

6-2 Energy-Storage Variables

In this section, I summarize from Karnopp [1] the derivation of the expressions for kinetic and potential energy stored in an inertance and a capacitor, respectively. Furthermore, I show why such an expression cannot be obtained for a resistor using integer calculus.

Whereas in classical mechanics, power is derived from energy by time-derivation, the bond graph approach starts by defining power. The basic bond graph elements (inertances, capacitors, and resistors) have a common expression for power:

$$P(t) = e(t)f(t) \tag{6-2}$$

Here, $e(t)$ is the effort signal, e.g. a force, and $f(t)$ is the flow signal, e.g. a velocity.

The energy expressions for the basic elements are, however, distinct for each element. This distinction arises from the relations between the power and energy variables[1] and the constitutive laws of the basic elements[1]. The power variables are expressed as the time derivatives of the energy variables:

$$\frac{dp}{dt} = e(t) \tag{6-3}$$

$$\frac{dq}{dt} = f(t) \tag{6-4}$$

The general expression for energy is the integral of power over a time interval

$$E(t) = \int^{t} e(t)f(t)dt \tag{6-5}$$

By substituting $edt = dp$ (6-3) or $fdt = dq$ (6-4), the integral is varied over one of the energy variables (q or p) instead of over time. However, this is observation is only useful if the flow, $f(t)$, can be expressed as a function of the momentum, p, or the effort, $e(t)$, can be expressed as a function of the position, q, respectively. The energy can then be expressed as an explicit function of one of the energy variables.

6-2-1 Inertance

For an inertance, the constitutive law states that the flow variable is

$$f(p) = \frac{p}{m}, \tag{6-6}$$

with m the inertance constant. Thus, by substituting $edt = dp$ and the expression for $f(p)$, the energy stored in an inertance is an explicit function of the momentum

$$E(p) = \int^{p} \frac{p}{m}dp \tag{6-7}$$

Solving $E(p)$ as an indefinite integral results in the familiar expression for kinetic energy $T(p) = \frac{p^2}{2m}$.

[1] Power Variables: $e(t)$, $f(t)$
Energy Variables: $p(t)$, $q(t)$

6-2-2 Capacitor

The potential energy for a capacitor can be derived similarly. The constitutive law of a capacitor relates the effort signal to the generalized position

$$e(q) = kq, \tag{6-8}$$

with k the capacitor constant. Substituting this expression in the energy expression and replacing fdt by dq results in

$$E(q) = \int^q kqdq \tag{6-9}$$

Solving $E(q)$ as an indefinite integral yields the well-known expression for potential energy in capacitor elements $V(q) = \frac{1}{2}kq$.

6-2-3 Resistor

This approach shows why such an energy expression does not exist for resistors. Since the constitutive law of a resistor,

$$e(t) = Rf(t), \tag{6-10}$$

relates the two power variables by the resistor constant, the energy can not be written as an explicit function of one of the energy variables. Because neither efforts and flows can be stored, there is no storage variable for dissipated energy in the integer-order framework. Using fractional calculus we will show how internal energy, generated by dissipation, can be stored in an additional coordinate.

6-3 Half-Order Derivative of Position as Storage Variable of Heat

In this section I derive an expression for internal energy as a function of the half-order derivative of position. The definitions and properties from fractional calculus used in this section can be found in 3-2.

To express internal energy as a function of a coordinate, instead of time — similar as in (6-9) and (6-7)— a relation between a power and an energy variable is needed. For integer order variables, such a relation does not exist, see (6-10). However, using fractional order variables allows us to define an energy variable over which the energy integral can be varied.

I define the internal energy variable, $\breve{q}$, as the integral of the left Riemann-Liouville derivative of the power variable $f(t)$

$$\breve{q}(t) \equiv \int^t {}_a\mathsf{D}_t^\alpha f(t)dt \tag{6-11}$$

In differential form this is written as

$$\frac{d}{dt}\breve{q} = {}_a\mathsf{D}_t^\alpha f(t) \tag{6-12}$$

By using some properties of fractional calculus, it is possible to arrive at an expression for energy as a function of $\breve{q}$.

$$E(t) = \int^t e(t)f(t)dt \Rightarrow E(\breve{q}) = \int^{\breve{q}} \gamma(\breve{q})d\breve{q} \tag{6-13}$$

where $\gamma(\breve{q})$ is the power variable of $\breve{q}$.

The first property is, that for any function $x(t)$, continuous on the interval $[a, b]$, we have that [8]

$$\left({}^C_t\mathbf{D}^\alpha_b \circ {}_t\mathbf{I}^\alpha_b\right) x(t) = x(t) \tag{6-14}$$

This allows us to perform the right-sided Caputo derivative and the right-sided fractional integral on the effort signal, $e(t)$:

$$E(t) = \int^t e(t)f(t)dt = \int^t {}^C_t\mathbf{D}^\alpha_b {}_t\mathbf{I}^\alpha_b e(t)f(t)dt \tag{6-15}$$

This notation allows us to use the second property: fractional integration by parts. The fractional integration by parts is defined as, [9]

$$\int_a^b x(t) {}^C_t\mathbf{D}^\alpha_b y(t)dt = \int_a^b y(t) {}_a\mathbf{D}^\alpha_t x(t)dt + x(t) {}_aI^{1-\alpha}_t y(t)\big|_a^b, \tag{6-16}$$

In case $x(a) = x(b) = 0$, this reduces to

$$\int_a^b x(t) {}^C_t\mathbf{D}^\alpha_b y(t)dt = \int_a^b y(t) {}_a\mathbf{D}^\alpha_t x(t)dt \tag{6-17}$$

Substituting $x(t) = f(t)$ and $y(t) = {}_t\mathbf{I}^\alpha_b e(t)$ and assuming for the sake of simplicity[2] that the begin and end values of $f(t)$ are zero, (6-15) can be rewritten as

$$E(t) = \int^t \left[{}_t\mathbf{I}^\alpha_b e(t)\right] {}_a\mathbf{D}^\alpha_t f(t)dt \tag{6-18}$$

Using (6-12), to substitute $d\breve{q}$ for ${}_a\mathbf{D}^\alpha_t f dt$, the integral can be varied over the fractional energy variable $\breve{q}$

$$E(t) = \int^t {}_t\mathbf{I}^\alpha_b e(t)d\breve{q} \tag{6-19}$$

Now, the goal is to derive an expression for the fractional integral of the effort signal, ${}_t\mathbf{I}^\alpha_b e(t)$ as a function of $\breve{q}$.

For linear resistors, the effort signal is a function of the flow signals, $e(t) = e(f(t)) = Rf(t)$. Using the definition of the right-sided Caputo derivative (3-6) it can be derived that

$${}_t\mathbf{I}^{\frac{1}{2}}_b e(f(t)) = {}_t\mathbf{I}^{\frac{1}{2}}_b R\dot{q} = R {}_t\mathbf{I}^{\frac{1}{2}}_b \frac{d}{dt} q = R\, {}^C_t\mathbf{D}^{\frac{1}{2}}_b q \tag{6-20}$$

Then, from the definition of $\breve{q}$ (6-11) and the definitions of Riemann-Liouiville (3-3) and Caputo derivatives (3-5), $\breve{q}$ can be expressed as the left-sided Caputo derivative of q

$$\breve{q} = \int {}_a\mathbf{D}^{\frac{1}{2}}_t \frac{d}{dt} q dt = {}_aI^1_t \frac{d}{dt} {}_a\mathbf{I}^{\frac{1}{2}}_t \frac{d}{dt} q = {}_a\mathbf{I}^{\frac{1}{2}}_t \frac{d}{dt} q = {}^C_a\mathbf{D}^{\frac{1}{2}}_t q \tag{6-21}$$

[2] In case the begin and end values are nonzero, the integrated part of (6-16) will have to be added.

Finally, we use the relation between left- and right-sided Caputo derivatives. As shown in [6], taking the limit $a \Rightarrow b$ and setting t as the midpoint of the $[a, b]$ allows the approximation

$$ {}_{t}^{C}D_{b}^{\alpha} \approx -{}_{a}^{C}D_{t}^{\alpha} \quad \text{for} \quad a - b << 1 \tag{6-22}$$

so that 6-20 can be rewritten as

$$ {}_{t}I_{b}^{\frac{1}{2}} e(f(t)) = -R\breve{q} \tag{6-23}$$

Therefore, we can define a fractional effort variable, $\breve{p}$, that is a function of $\breve{q}$

$$\breve{p}(\breve{q}) \equiv {}_{t}I_{b}^{\frac{1}{2}} e(f(t)) = -R\breve{q} \tag{6-24}$$

Finally, we arrive at an expression for internal energy, U, from heat generated by dissipation as an explicit function of $\breve{q}$

$$U(\breve{q}) \equiv E(\breve{q}) = \int^{\breve{q}} \breve{p}(\breve{q}) d\breve{q} \tag{6-25}$$

This expression mathematically shows that it is possible to store internal energy by varying $\breve{q}$. Computing the integral without defining the begin and end points, results in the square energy function

$$U(\breve{q}) = -\frac{1}{2} R\breve{q}^2 \tag{6-26}$$

This is up to the sign, the energy function that is used in the Lagrangian and Hamiltonian in the work of Lazo [6]. This implies that in Lazo's work, the Lagrangian is constructed as

$$L = T - V - U \tag{6-27}$$

with T, V, and U, the kinetic, potential, and internal energy respectively. Lazo, however, does not specify this. His Lagrangian is constructed such that the action principle leads to the desired Euler-Lagrange equation, without remarking whether the internal energy should be positive or negative in the Lagrangian.

6-4 Conclusions

An expression for internal energy generated by heat from dissipation as a function of a mechanical variable was derived. The starting point of this derivation was the observation that the expression for power in a resistor is the product of the effort and the flow. The expression for energy dissipated by a resistor is the integral of this expression for power. Defining the internal energy variable as the integral of the fractional derivative of position, fractional calculus was used to vary the expression for energy over the internal energy variable instead of over time. The resulting expression for internal energy is analogous to the mechanical expressions for the variation of kinetic energy, $T(p)$, and potential energy, $V(q)$, see (6-1). Furthermore, for a linear damper, the expression coincided with the fractional energy introduced in the Lagrangian of Lazo [6] and of Riewe [4, 5].

The expression for internal energy derived in this chapter shows that it is mathematically possible to store and vary internal energy as a function of a mechanical variable. However, it was not shown in this chapter how the mechanical expression for internal energy relates to the thermodynamic expression for internal energy. This brings up the question: how do the fractional derivative of position and its corresponding fractional momentum relate thermodynamic properties. In particular, if we consider a closed, isochoric (constant volume) system: how does the fractional derivative of position relate to entropy and how does the fractional momentum relate to temperature? These are interesting topics for further research in the field of fractional calculus.

Contributions

1. Internal energy generated by heat from dissipation is expressed as a mechanical property
2. Mathematically, it is possible to vary internal energy over the half-order derivative of position

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Chapter 7

Conclusions

7-1 Conclusions

The purpose of this thesis was to include consumption in the analytical mechanics approach of economic engineering. For this purpose, two main contribution have been made in this thesis.

Following the method of including dissipation in the mechanical Lagrangian with fractional derivatives, consumption is included in the Utility Lagrangian from economic engineering. Assuming that an economic agent always allocates its assets so as to maximize its utility, the fractional action principle is reformulated as a fractional principle of maximum utility. The fractional calculus of variations can be used to find a fractional Euler-Lagrange equation that reveals the price dynamics that lead to maximum utility. This price dynamics equation is similar to the one already present in economic engineering, but it includes the effect of depreciation due to consumption. A Fractional Surplus Hamiltonian that includes consumption in the mechanics of surplus can be obtained by performing a Legendre transform on the Fractional Utility Lagrangian. However, this Fractional Surplus Hamiltonian cannot be used for control formalisms from port-Hamiltonian theory, since it is not a function of canonical state variables.

A second method to include consumption in the analytical mechanical approach formulates a Complex Surplus Hamiltonian with complex-valued state variables. This thesis has developed a Complex-Hamiltonian theory by defining the equations of motion of a damped harmonic oscillator as the complex Poisson bracket between the state and the Complex Hamiltonian. Using this definition, the Complex Hamiltonian has been derived by substituting the known equations of motion of a damped harmonic oscillator. Utilizing the analogies between mechanical and economics systems, the Complex-Hamiltonian results in an analytical description of surplus that includes the effect of consumption. Not only can the Complex Surplus Hamiltonian be used to model surplus and consumption with analytical mechanics, it can also be used to apply control formalisms to economic systems.

Besides the contributions to economic engineering, this thesis made two additional contributions to engineering.

The first, and most important, of these two additional contributions is the theory of Complex-Hamiltonians for damped harmonic oscillators. This theory resulted in a novel formalism that describes damped harmonic oscillator with Hamiltonian mechanics. Besides this novelty it was shown that when applied to port-Hamiltonian systems, the Complex-Hamiltonian theory solves a major problem: the dissipation obstacle.

The second additional contribution showed that with fractional calculus internal energy, generated by heat from dissipation, can be varied over a mechanical state variable. The resulting expression for internal energy coincides with the frictional energy used by Riewe and Lazo in their fractional Lagrangians. The derivation made in this thesis implied that the internal energy as a function should be subtracted from the kinetic energy in the Lagrangian.

7-2 Recommendations

This thesis developed two alternative theories to solve the same problem: include consumption in the analytical mechanics of utility and surplus. The first theory had the advantage of being based on a fundamental principle, the second had the advantage of possessing a Poisson structure.

However, it was not shown in this thesis how these two theories are related. Both the Fractional Utility Lagrangian and the Complex Surplus Hamiltonian can generate the equations of motion (the economic analog of) a damped harmonic oscillator. Furthermore, it is known that the Lagrangian and the Hamiltonian are equivalent by the Legendre transform. In fact, the Fractional Surplus Hamiltonian was derived from the Fractional Utility Lagrangian by performing a Legendre transform. This implies that the Fractional Utility Lagrangian and the Complex Surplus Hamiltonian are equivalent by some unknown transform. Further research on deriving this unknown transform will relate the fundamental principle of the Fractional Utility Lagrangian to the computational-friendly Complex Surplus Hamiltonian.

Besides further research on unifying the two theories developed in this thesis, the theories themselves will benefit from further developments.

The consumption variable in the Fractional Utility Lagrangian was defined as the Caputo fractional derivative of asset stock. This definition was made following the literature. Although the resulting fractional Euler-Lagrangian resulted in a price equation that describes the effect of consumption corresponding to the economic literature, the behavior of consumption as a fractional derivative has not been evaluated. Further research on describing consumption as a fractional derivative is therefore needed,

Regarding the Complex-Hamiltonian theory, further research is needed to include other systems than the (damped) harmonic oscillator. Conceptually, it is possible to model higher order systems as interconnected second-order systems, but a structured formalism to match the natural frequencies of the individual subsystems has to be developed. Furthermore, in this thesis it was shown that Complex-Port-Hamiltonian systems bypass the dissipation obstacle, but mathematical proof to generalize this observation is needed. Finally, future research on Complex-Hamiltonian theory can contribute to modeling and control of dissipative nonlinear systems from an energy perspective.

Since the scope of this thesis was to include consumption in the Surplus Hamiltonian, control of the Complex Surplus Hamiltonian has been illustrated but has not been formally developed. However, combining the analogies from economic engineering, the interpretation of the economic agent as Complex-Hamiltonian system and the control formalisms of Complex-Port-Hamiltonian provides a strong template for further research.

Appendix A

Fractional Calculus

A-1 Fractional Calculus of Variations

The fractional calculus of variations was introduced by Riewe [1] who proposed to use the halfth-order derivative of q to model frictional energy such that the velocity-dependent frictional force resulted from the Euler-Lagrange equation, using the fact that coordinates in the Lagrangian proportional to $(d^n/dt^n)^2$ result in coordinates proportional to $d^{2n}/dt2^n$ in the Euler-Lagrange equation.

Consider for example a mechanical mass-spring system. Its Lagrangian is

$$L(\dot{q}, q) = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2,$$

which from the variational calculus results in the following Euler-Lagrange equation:

$$-m\ddot{q} - kq = 0$$

Riewe, and enhanced versions by e.g. [2], showed that by introducing the right Caputo derivative of position $\breve{q} = {}_t^C\mathsf{D}_b^{\frac{1}{2}} q$ as frictional energy-variable and constructing the Lagrangian of a mass-spring-damper system as

$$L(\dot{q}, \breve{q}, q) = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}b\breve{q}^2 - \frac{1}{2}kq^2$$

results in the following Euler-Lagrange equation

$$-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + {}_a\mathsf{D}_t^{\frac{1}{2}}\frac{\partial L}{\partial \breve{q}} + \frac{\partial L}{\partial q} = 0$$

Using that ${}_a\mathsf{D}_t^{\frac{1}{2}}\,{}_t^C\mathsf{D}_b^{\frac{1}{2}} f(t) = -\frac{d}{dt}f(t)$ this results in the equation of motion of a mass-spring damper system

$$-m\ddot{q} - b\dot{q} - kq = 0$$

A-2 Legendre Transform: The Wrong Way

Following Riewe's and Lazo's procedure, the Legendre transform switches from the Lagrangian description of an agent with the coordinates $(\dot{q}, \breve{q}, q)$ to a Hamiltonian description with the coordinates (p, γ, q) as follows

$$H = \frac{\partial L}{\partial \dot{q}}\dot{q} + \frac{\partial L}{\partial \breve{q}}\breve{q} - L \tag{A-1}$$

Substituting the definition of reservation price ($p \equiv \partial L/\partial \dot{q}$) and the expression for expense as marginal utility from consumption ($\gamma = \partial L/\partial \breve{q}$), the surplus can be written as

$$H(p, \breve{\gamma}, q) = p\dot{q} + \gamma\breve{q} - L \tag{A-2}$$

Taking the total derivative of the Surplus Hamiltonian in (A-2) yields

$$dH = \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial \gamma}d\gamma + \frac{\partial H}{\partial q}dq, \tag{A-3}$$

The surplus changes with the change in price, expense, and stock multiplied by the factors $\frac{\partial H}{\partial p}$, $\frac{\partial H}{\partial \gamma}$, and $\frac{\partial H}{\partial q}$, respectively. These three factors can be identified by evaluating the total derivative of the Fractional Utility Lagrangian

$$dL = \frac{\partial L}{\partial \dot{q}}d\dot{q} + \frac{\partial L}{\partial \breve{q}}d\breve{q} + \frac{\partial L}{\partial q}dq \tag{A-4}$$

Substituting price, p, and expense, γ, we get

$$dL = p d\dot{q} + \gamma d\breve{q} + \frac{\partial L}{\partial q}dq \tag{A-5}$$

Using the product rule we arrive at a total derivative that changes with the price, p, the expense, γ, and the stock, q:

$$\begin{aligned} dL &= d(p\dot{q}) - \dot{q}dp + d(\gamma\breve{q}) - \breve{q}d\gamma + \frac{\partial L}{\partial q}dq \\ d(p\dot{q} + \gamma\breve{q} - L) &= \dot{q}dp + \breve{q}d\gamma - \frac{\partial L}{\partial q}dq \end{aligned} \tag{A-6}$$

Here, the left-hand side is equal to the total derivative of the Surplus Hamiltonian —see (A-2)— so we have that

$$dH = \dot{q}dp + \breve{q}d\breve{\gamma} - \frac{\partial L}{\partial q}dq \tag{A-7}$$

Comparing this expression of the total derivative of the Surplus Hamiltonian to the expression in (A-3) we find the unknown factors as the economic Hamilton's equations

$$\frac{\partial H}{\partial p} = \dot{q}, \qquad \frac{\partial H}{\partial \breve{\gamma}} = \breve{q}, \qquad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -B, \tag{A-8}$$

This implies that the change in surplus is a result of

1. the change in price multiplied by the flow,
2. the change in expense multiplied by the consumption, and
3. the change in stock multiplied by the cost

Appendix B

Economic Engineering

Overview of the economical analogs introduced by [3]

Generalized	**Mechanical**	**Electrical**	**Economical**
Momentum, p	Momentum, p $[\frac{kgm}{s}]$	Flux linkage, λ [Vs]	Price, p $[\frac{\$}{\#}]$
Position, q	Position, q $[m]$	Charge, q $[C]$	Stock, q $[\#]$
Effort, e	Force, F [N]	Voltage, U [V]	Cost/Benefit, B $[\frac{\$}{\#yr}]$
Flow, f	Velocity v $[\frac{m}{s}]$	Current, I [A]	Flow, $\dot{q}$ $[\frac{\#}{yr}]$
Energy, H	Hamiltonian, $\frac{p^2}{2m} + \frac{kq^2}{2}$, $[J]$	Hamiltonian, $\frac{\lambda^2}{2I} + \frac{Cq^2}{2}$, $[J]$	Surplus, Σ $[\frac{\$}{yr}]$
Power, P	Power, $P = Fv$ $[W]$	Power, $P = UI$ $[W]$	Growth, G $[\frac{\$}{yr^2}]$

Table B-1: Overview of dynamical analogs in mechanical, electrical and economic engineering

Generalized	**Mechanical**	**Electrical**	**Economical**
Compliance, C	Spring	Capacitor	Rigidity, μ
Inertance, I	Mass	Inductor	Illiquidity, λ
Resistance, R	Damper	Resistor	Depreciation Constant, [**?**]

Table B-2: Overview of dynamical analogs in mechanical, electrical and economic engineering

Appendix C

Port- Hamiltonian Systems

C-0-1 Passivity-Based Control

Passivity-based control (PBC) is a control method in which a system is rendered passive in order to stabilize a system, [4]. A system is said to be passive if it is not able to autonomously generate energy along its trajectories. An important equation in this technique is the energy-balance equation[5]

$$\underbrace{H[x(t)] - H[x(0)]}_{\text{stored energy}} = \underbrace{\int_0^t u^{\top}(s)y(s)ds}_{\text{supplied energy}} - \underbrace{d(t)}_{\text{dissipated}} \tag{C-1}$$

From the above equation we can derive the power balance of the system described in (2-14)

$$\begin{aligned} \frac{dH}{dt} &= \quad H^{\top}\dot{x} \\ &= \quad H^{\top}([J - R] \quad H + g(x)u) \\ &= -\underbrace{H^{\top}R \quad H}_{d(x)} + u^{\top}y \end{aligned} \tag{C-2}$$

with natural dissipation $d(x)$. From the fact that R is semi-definite, it follows that

$$\frac{dH}{dt} \leq u^{\top}y \tag{C-3}$$

The above equation is known as the *passivity inequality* [6]. An immediate result that can be obtained from this equality is that a simple control law

$$u = -Ky \tag{C-4}$$

with $K = K^{\top} > 0$ ensures that the energy in the system will decrease. If the Hamiltonian is bound from below this guarantees that the system is stable [5].

An interesting technique introduced in [5] is ***stabilization via energy balancing***. Using that energy is an additive property, it is possible to design a controller with added energy function H_a so that the desired energy function H_d is obtained

$$H_d(x) = H(x) + H_a(x) \tag{C-5}$$

If H_d has a minimum at the desired equilibrium x , the system will be stable.

C-0-2 Energy-Shaping and Damping Injection

In [7] this technique is further developed by introducing a ***damping injection*** input in order to ensure convergence of the state trajectories to the desired equilibrium. This procedure is known as Energy-shaping and damping injection (ES-DI).

For a pH plant such as in (2-14) the ES-DI method attempts to obtain a desired closed-loop system

$$\dot{x} = (J(x) - R_d(x)) \quad H_d(x) \tag{C-6}$$

where R_d is the desired damping matrix

$$R_d(x) = R(x) + g(x)K_d(x)g^{|}(x) \tag{C-7}$$

with damping injection matrix K_d. The desired Hamiltonian H_d is defined in (C-5).

The desired closed-loop dynamics can be realized with control input

$$u(x) = u_{ES}(x) + u_{DI} \tag{C-8}$$

where the energy shaping input is generated by the added energy Hamiltonian H_a and the damping injecting input generated by desired Hamiltonian H_d

$$u_{ES} = (g^{|}(x)g(x))^{-1}g^{|}(x)[J(x) - R(x)]\frac{\partial H_a}{\partial x}(x) \tag{C-9}$$

$$u_{DI} = -K_d(x)g^{|}(x)\frac{\partial H_d}{\partial x}(x) \tag{C-10}$$

The added energy function $H_a(x)$ can then be found solving the following set, see equation (15.18) of [7]

$$\begin{bmatrix} g \quad (x)[J(x) - R(x)] \\ g^{|}(x) \end{bmatrix} \frac{\partial H_a}{\partial x}(x) = 0 \tag{C-11}$$

The solution for H_a that satisfies $x \ = \text{argmin}(H_d = H + H_a)$ is then selected. I will come back to the above set of PDE's in Section 2-4-3, where I discuss the dissipation obstacle.

C-0-3 Control by Interconnection

In order to obtain dynamic feedback control, the controller can be modelled as another pH system connected to the plant. This method is known as Control by Interconnection (CbI) [5, 6, 8]. Whereas most PBC controllers are full-state feedback controllers, CbI controllers only use the output signal of the plant.

The controller is modelled as a pH system with its own states $\xi \in \mathbb{R}^{nc}$ and energy function $H_c(\xi)$. The input-state output description of the controller is

$$
\begin{aligned}
\dot{\xi} &= [J_c(\xi) - R_c(\xi)] \nabla H_c(\xi) + g_c(\xi)u_c \\
y_c &= g_c(\xi)^\top \nabla H(\xi)
\end{aligned}
\tag{C-12}
$$

The control loop can then be closed by using a negative feedback interconnection

$$
\begin{pmatrix} u \\ u_c \end{pmatrix} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} y \\ y_c \end{pmatrix}
\tag{C-13}
$$

Since this is a power preserving interconnection, the composed system is also a pH system [7]. The input-state-output system of the composed system is

$$
\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} J(x) - R(x) & -g(x)g_c^\top(\xi) \\ g_c(\xi)g^\top(x) & J_c(\xi) - R_c(\xi) \end{pmatrix} \begin{pmatrix} \frac{\partial H_{cl}}{\partial x}(x,\xi) \\ \frac{\partial H_{cl}}{\partial \xi}(x,\xi) \end{pmatrix}
\tag{C-14}
$$

where the closed-loop energy function H_{cl} is

$$
H_{cl}(x,\xi) = H(x) + H_c(\xi)
\tag{C-15}
$$

Thus, since the energy function of the plant $H(x)$ is given, $H_c(\xi)$ can be freely determined in order to obtain the desired closed-loop energy function. However, it is not clear how the closed-loop energy is shaped effectively [9].

C-0-4 Energy-Casimir method

A widely proposed method to solve this problem is to investigate so-called *Casimir functions* that are dynamical invariants (conserved quantities) relating the states of the plant to the states of the controller [8, 10, 11, 12].

A typical candidate function to relate the states of the plant to the states of the controller is

$$
C(x,\xi) = \xi - F(x) = \kappa
\tag{C-16}
$$

where F is some differentiable function of x and κ is some constant. The closed-loop energy function is then $H_{cl} = H(x) + H_c(F(x) + \kappa)$. This function is invariant if

$$
\frac{d}{dt}C(x,\xi) = 0
\tag{C-17}
$$

If this is the case, C is indeed a Casimir function and is independent of the Hamiltonian.

A Casimir function is found as a solution $F(x)$ for the following PDE's

$$
\begin{pmatrix} -\frac{\partial^\top F(x)}{x} & I_{nc} \end{pmatrix} \begin{pmatrix} J(x) - R(x) & -g(x)g_c^\top(\xi) \\ g_c(\xi)g^\top(x) & J_c(\xi) - R_c(\xi) \end{pmatrix} = 0
\tag{C-18}
$$

Assuming that $R(x) \geq 0$ and $R_c(\xi) \geq 0$ (positive damping) the above PDE's are characterized by the following chain of equalities [12]

$$\frac{\partial^{\top} F}{\partial x}(x) J(x) \frac{\partial F}{\partial x}(x) = J_c(\xi) \tag{C-19}$$

$$R(x) \frac{\partial F}{\partial x}(x) = 0 \tag{C-20}$$

$$R_c(\xi) = 0 \tag{C-21}$$

$$J(x) \frac{\partial F}{\partial x}(x) = -g(x) g_c^{\top}(\xi) \tag{C-22}$$

C-0-5 Dissipation Obstacle

Condition (C-20) however has a problematic effect on finding a Casimir function for systems with damping. This condition implies that the Casimir functions cannot depend on coordinates that are subject to damping [9]. This condition is called the ***dissipation obstacle*** [5]. As a result, the Energy-Casimir method can only be used when dissipation is only acting on the states of the closed-loop system that do not require shaping of the Hamiltonian. In [9] removing the passivity constraint on the controller ($R_c \geq 0$) was proposed to resolve the dissipation obstacle. Other methods of resolving the damping obstacle were proposed in [5] and [6].

Not only the Energy-Casimir method is affected by the dissipation obstacle, also the PDE's in (C-11) of ES-DI method cannot be solved when $R(x) = 0$. This has been shown in Section 15.7 of Van Der Schaft's overview.

The first occurrence of the dissipation obstacle arises from pervasive dissipation [20]. The power balance of a passive system with input-output description (2-14) is

$$\frac{dH}{dt} = - \quad {}^{\top} H(x) R(x) \quad H(x) + u^{\top} y \tag{C-23}$$

Assuming that the control objective is to steer the system to a desired equilibrium $x \quad \mathbb{R}^n$. Since the energy flow dH/dt should be zero at x , we have that for a dissipative system, $R > 0$, the dissipated power has to be compensated by the controller:

$$u \ {}^{\top} y \ = \quad {}^{\top} H(x \ ) R(x \ ) \quad H(x \ ) \tag{C-24}$$

where u and y are the input and output at the equilibrium, respectively. However, the power that can be extracted from a passive system is bounded [22]. This implies that if the system dissipates power at the equilibrium —a phenomenon known as pervasive dissipation— it cannot be stabilized by a passive controller. Therefore, only systems for which is holds that

$$R(x \ ) \quad H(x \ ) = 0 \tag{C-25}$$

can be stabilized with passive controllers. Systems for which (C-25) does not hold are constrained by the dissipation obstacle [5].

Appendix D

Complex-Port-Hamiltonian Systems

D-1 Derivation of the Complex Equation of Motion

The complex state is defined as

$$a = \sqrt{\frac{k}{2}}q + i\sqrt{\frac{1}{2m}}p, \tag{D-1}$$

This implies that the position, q, and the momentum, p, can be retrieved as

$$q = \sqrt{\frac{1}{2k}}(a + \bar{a}), \qquad p = -i\sqrt{\frac{m}{2}}(a - \bar{a}), \tag{D-2}$$

For a mechanical damped harmonic oscillator with mass m, spring constant k, and damping constant b, the system dynamics are

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \tag{D-3}$$

Substituting the expression in (D-2), the system dynamics can be rewritten as

$$\dot{q}(a, \bar{a}) = -i\sqrt{\frac{1}{2m}}(a - \bar{a}) \tag{D-4}$$

$$\dot{p}(a, \bar{a}) = -\sqrt{\frac{k}{2}}(a + \bar{a}) + ib\sqrt{\frac{1}{2m}}(a - \bar{a}) \tag{D-5}$$

Since a, as defined in (D-1), is a linear function of q and p, it holds that

$$\dot{a} = \frac{1}{\sqrt{2}}\left(\dot{q}\sqrt{k} + i\dot{p}\sqrt{\frac{1}{m}}\right) \tag{D-6}$$

Substituting (D-4) and (D-5) in the above equation results in the time evolution of a damped harmonic oscillator on the $a-$plane and its conjugate

$$\dot{a} = -(\beta + i\omega_n)a + \beta\bar{a} \tag{D-7}$$

$$\dot{\bar{a}} = \beta a - (\beta - i\omega_n)\bar{a} \tag{D-8}$$

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