The interaction between bed-load transport and dune orientation

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Preface

This report is my additional graduation work, and is part of my studies at the TU Delft, at the faculty of Civil Engineering and Geosciences. The work has been done at Deltares at the department of River Dynamics and Inland Water Transport during a three-month span.

I would hereby like to thank my committee, consisting of Kees Sloff, Erik Mosselman, Wim Uijttewaal and Arno Talmon for their support and guidance during my work. I would also like to thank Sanjay Giri for help with this project. I would like to thank Deltares for giving me the chance to work on this project and for providing me with the necessary tools. Finally I would like to thank my fellow students at Deltares for their help, and creating a good atmosphere to work in.

Abstract

Oblique dunes have an important influence on the direction of the bed-load sediment in rivers. However, not much is known about these effects, and about what affects the angle of oblique dunes. Therefore tests have been carried out to look into this effect of oblique dunes on sediment transport.

Based on these tests, formulae for bed-load transport direction which take the dune angle into account have been derived. Application of these formulae shows a large effect of the dune angle on the transport direction, but the formulae have not been validated yet.

Based on the hypothesis that dune angles are related to the dune migration rate along the crest, a formula to calculate the dune angle has been derived. The formula has not been compared to laboratory tests or field measurements, but the validity of the formula is questionable as in the test cases very large angles are found.

Formulae for the dune angle and the bed-load transport have been implemented in a numerical model. For the dune angle formula an iterative scheme is needed to solve it. Three methods have been tested. Out of these three the bisection method is the most promising scheme.
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<td>$\bar{z}$</td>
<td>Bed level without bedforms</td>
<td>$m$</td>
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\(\alpha\) Bed-form orientation angle rad
\(\alpha_{MPM}\) Calibration factor for Meyer-Peter-Müller formula
\(\Delta U\) Difference between the near bed flow velocity \(U\) and the sediment velocity \(v\) m/s
\(\delta_{fs}\) Direction of the near-bed flow on the stoss side of the dune rad
\(\delta_{pl}\) Direction of the sediment particle on the lee side of the dune rad
\(\delta_{ps}\) Direction of bed-load transport at stoss side of the dune rad
\(\delta_{pt}\) Direction of the sediment particle in the transition zone of the dune rad
\(\delta_p\) Bedform averaged direction of the sediment particle rad
\(\mu\) Particle friction coefficient -
\(\mu_H\) Efficiency factor -
\(\rho\) Specific density of the water kg/m\(^3\)
\(\rho_s\) Specific density of the sediment kg/m\(^3\)
\(\theta\) The shields parameter as defined by Shields [1936] -
\(\theta_c\) The critical shields parameter -
\(\varepsilon\) Calibration parameter -
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1 Introduction

1.1 Problem Definition

Dunes are one of the main bedforms found in rivers. Dunes are large sand waves with a mostly triangular shape. They have a mild and slightly curved upstream slope and a downstream slope which is roughly equal to the angle of repose of the bed material [Engelund & Fredsoe, 1982]. At the downstream slope flow separation occurs. For a schematised drawing of a dune, see figure 1.1. After the dune crest the flow cannot follow the quick drop in bed level. This causes the flow to separate, causing a circulating flow at the lee side of the dune. The dunes migrate by sediment transported over the crest and settling on the lee side of the dune.

These dunes are important because they play a large role in the sediment transport. This is important, for example, in the maintenance of navigation channels. Dunes also influence the roughness off the bed. This is important in calculating water levels in case of a flood.

Because of their importance many studies have been carried out to investigate dunes aligned perpendicular to the flow, called transverse dunes [e.g. Müller & Gyr, 1986; Nelson et al., 1993]. A good overview of research on dunes, focused mostly on the fluid dynamics, can be found in Best [2005]. While most research is on transverse dunes, oblique dunes do occur regularly in rivers. An example of this is seen in figure 1.2.

Talmon [2009] carried out experiments to determine the effect of oblique dunes on sediment transport. He found a significant effect of the dune angle on the direction of sediment transport. This is caused by (a) The change of the local bed slope caused by the dunes, and (b) by helical vortex generated on the lee side of the dune [Walker & Nickling, 2002], see figure 1.3. This helical vortex causes sediment transport in the cross stream direction. This cross stream sediment transport was also found by Dietrich & Smith [1984] in their field study in the Muddy Creek.

For large-scale modelling it is not feasible to use a grid size small enough to model the dunes directly. To include the effect of oblique dunes on the sediment transport, the dune angle has to be parameterised. Sieben & Talmon [2011] derived a formula for the sediment transport direction from the experiments by Talmon [2009]. Their formula includes the dune angle. To apply the formula the dune angle has to be known or predicted. Predicting the dune angle is not so simple. Experiments have been carried out to predict this dune angle [Rubin & Hunter, 1987; Rubin & Ikeda, 1990; Werner & Kocurek, 1997], but these were based on directionally varying flow. Which is most likely not the cause for oblique dunes in rivers, because the flow is mostly in one direction (except for tidal rivers). Sieben & Talmon [2011] proposed to derive the dune angle by relating it to the relative dune migration rate along the dune crest. This migration rate depends on the sediment transport direction. This means there is a feedback between the dune angle and sediment transport direction. This relation for the dune angle still has to be tested to see if it can be applied in large-scale modelling.

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Figure 1.1: Schematisation of longitudinal cross section of dunes

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Figure 1.3: Helical vortex generated on the lee side of the dune
Figure 1.2: Bathymetry of the Waal river between kilometre 947 and 949

Figure 1.3: Schematisation of the helical vortex on the lee side [Walker & Nickling, 2002]
1.2 Objectives

The main objective of this research is to test the formulae given by Sieben & Talmon [2011]. This has been formulated as the following main question.

*Can the dune angle and bed-load transport direction be computed well with the formulae derived by Sieben & Talmon [2011]?*

This main objective has been divided into several subobjectives.

1. *How can the dune angle formula best be implemented in a numerical model?*

   The formula derived for the dune angle cannot be solved directly, but requires an iterative scheme. Several schemes are tested. The aim is to find a scheme that is stable and finds a solution within a limited amount of computations.

2. *How does the dune angle formula perform?*

   To test if the formula gives realistic results, it is applied to several different cases. The results are compared to the expected results.

3. *How does the bed-load transport direction formula perform?*

   Similar to the dune angle formula, the formula for bed-load transport direction will be tested as well.
2 Dune angle

In this chapter the formula for the dune angle is derived in section 2.1. A qualitative analysis for a simplified case will be carried out in section 2.2, and some field measurements of Dutch rivers are analysed in section 2.3 to see if the observed bed form orientations can be explained by the derived formula.

2.1 Derivation of the Formula

It has been derived from the Exner equation [Exner, 1931] that for dunes migrating without changing in size or shape, the migration rate of dunes is proportional to the bed-load transport perpendicular to the dune crest as follows [e.g. Niño et al., 2002],

\[ S_s = S \cos(\delta_{ps} - \alpha) \propto wH \]  

(2.1)

where:

- \( S_s \) Bed-load transport over the dune crest per unit width [m\(^2\)/s]
- \( S \) Bed-load transport magnitude per unit width [m\(^2\)/s]
- \( \delta_{ps} \) Direction of bed-load transport at stoss side of the dune [rad]
- \( \alpha \) Bed-form orientation angle [rad]
- \( H \) Dune height [m]
- \( w \) Dune migration rate [m/s]

This relation between sediment transport \( S \), dune height \( H \), and dune migration rate \( w \) has been confirmed by experimental results of Coleman & Melville [1994] and numerical experiments by Niño et al. [2002]. However, Friedrich et al. [2004] concluded that the assumption of a constant shape can have large effects on the calculated dune migration rate. It should also be noted that the experiments mentioned were carried out with transverse dunes, not oblique dunes.

When the migration rate is not constant over the entire dune crest, the angle of this dune in relation to the flow will change. It is therefore possible that dune crests are not perpendicular to the direction of flow. A relation between the dune migration, \( w \), and the bed-form orientation angle, \( \alpha \), is derived with figure 2.1.

\[ \text{dune rest at } t = t_1 + \Delta t \]
\[ \text{dune crest at } t = t_1 \]
\[ \Delta \alpha \]
\[ w \times \Delta t \]
\[ \Delta n \]
\[ \Delta m \]

Figure 2.1: Schematisation of dune angle

Here the \( n \)-direction is perpendicular to the dune crest and the \( m \)-direction parallel to the dune-crest. The change in bed-form orientation angle \( \Delta \alpha \) can be calculated as follows:
\[
\tan \Delta \alpha = \frac{w_1 \Delta t - w_2 \Delta t}{\Delta m} = \Delta t \frac{w(m) - w(m + \Delta m)}{\Delta m}
\]  
(2.2)

\[
\Delta t = \frac{\Delta n}{w}
\]  
(2.3)

Making \(\Delta n\) and \(\Delta m\) infinitesimally small gives the following partial differential equation:

\[
\frac{\partial \alpha}{\partial n} - \tan (\delta_{ps} - \alpha) \frac{\partial \alpha}{\partial m} = \frac{1}{H} \frac{\partial H}{\partial m} - \frac{1}{S} \frac{\partial S}{\partial m} + \tan (\delta_{ps} - \alpha) \frac{\partial \delta_{ps}}{\partial m}
\]  
(2.4)

Equation (2.4) is similar to the refraction formula used for waves in shallow water [e.g. Holthuijsen, 2007]. The dune angle calculated with this equation is the equilibrium dune angle. The time needed to reach this equilibrium dune field is not taken into account in this study. Combining equations (2.1) and (2.4), we get the following partial differential equation:

\[
\frac{\partial \alpha}{\partial n} = \frac{1}{w \Delta m} \frac{\partial w}{\partial m}
\]  
(2.5)

If we define \(\alpha\) as zero when the dune crest is perpendicular to the x-axis, then we can use the following to transform the formula to x- and y-coordinates:

\[
\frac{\partial}{\partial n} = \cos \alpha \frac{\partial}{\partial x} + \sin \alpha \frac{\partial}{\partial y}
\]  
(2.6a)

\[
\frac{\partial}{\partial m} = -\sin \alpha \frac{\partial}{\partial x} + \cos \alpha \frac{\partial}{\partial y}
\]  
(2.6b)

Applying this gives us the following equation:

\[
u_{\alpha} \frac{\partial \alpha}{\partial x} + v_{\alpha} \frac{\partial \alpha}{\partial y} = RHS_{H} + RHS_{S} + RHS_{\delta}
\]  
(2.7)

where:

\[
u_{\alpha} = (\cos \alpha - \sin \alpha \tan (\delta_{ps} - \alpha))
\]  
(2.8a)

\[
v_{\alpha} = (\sin \alpha + \cos \alpha \tan (\delta_{ps} - \alpha))
\]  
(2.8b)

and:

\[
RHS_{H} = -\frac{1}{H} \left( \sin \alpha \frac{\partial H}{\partial x} - \cos \alpha \frac{\partial H}{\partial y} \right)
\]  
(2.9a)

\[
RHS_{S} = \frac{1}{S} \left( \sin \alpha \frac{\partial S}{\partial x} - \cos \alpha \frac{\partial S}{\partial y} \right)
\]  
(2.9b)

\[
RHS_{\delta} = -\tan (\delta_{ps} - \alpha_{i,j}) \left( \sin \alpha \frac{\partial \delta_{ps}}{\partial x} - \cos \alpha \frac{\partial \delta_{ps}}{\partial y} \right)
\]  
(2.9c)

This equation cannot be solved directly, but has to be solved iteratively. The iterative scheme used to solve the equation is given in chapter 4.
2.2 Analysis of the Formula

Let us analyse the proposed formula for a strongly simplified case: A straight channel with a distribution of $S/H$ along the $y$-axis, and no change in $S/H$ in the $x$-direction. This $S/H$ is the relative dune migration rate in the case of flow and bed-load transport perpendicular to the dune. $S/H$ is highest in the middle of the channel, and decreases towards the banks. The bed-load transport is in the $x$-direction, with no transport in the $x$-direction. See also figure 2.2.

![Figure 2.2: A straight channel, with the given distribution of $S/H$.](image)

We start with a dune crest perpendicular to the flow direction at $t = t_0$. The theory now says that the transverse gradient in $S/H$ will cause the crest to turn, as the middle part travels faster than the outer parts of the dune. At $t = t_1$ the dune crest has the shape as in figure 2.2. With the angle of the dune crest increasing from the middle of the channel towards the banks.

The angle of the dune crest influences the relative dune migration rate. When the bed-load transport is perpendicular to the dune, all the sediment crosses the crest. But when the sediment transport direction is at an angle with the dune, part of the sediment is transported parallel to dune and less sediment crosses the dune crest, see figure 2.3. This means the dune angle causes an even greater gradient in the relative dune migration rate, which causes the dune crests to turn even faster until the dunes are eventually parallel to the bed load transport direction.

![Figure 2.3: The effect of dune angle on sediment passing the crest](image)

However, as indicated in the introduction, the dune angle influences the sediment transport direction. The change in direction of the transport is towards the more advanced part of the dune [Talmon, 2009], see figure 2.4. This actually causes an even more decreased relative dune migration rate, as the angle between dune crest and sediment transport increases further. This would mean that with the suggested formula, the dune crests in a straight channel will eventually become parallel to the sediment transport.
direction, as in this case there is no more gradient in dune migration rate along the dune crest. This is not a realistic result.

Without effect of dune angle

With effect of dune angle

Figure 2.4: The effect of dune angle on the sediment transport direction

However, in the proposed theory it is assumed that the dune crest migrates in a direction perpendicular to the crest. This means an oblique dune crest also migrates in the y-direction, as well as in the x-direction. Therefore it could be possible to reach an equilibrium dune angle, where the dune angle would stay constant in the x-direction. However, results of tests in section 5 do show angles almost perpendicular to the flow in straight channels.

2.3 Observations

In this section bedforms in Dutch rivers are analysed to see if the observed bedform orientations can be explained qualitatively by the equations derived in section 2.1.

2.3.1 The Waal

Figures 2.5 and 2.6 show the bathymetry of the Waal river between kilometres 946 and 949. The river flows from east to west. On the northern part of the river oblique dunes can clearly be observed. These dunes lie under an angle of about 50 degrees with the river axis. There is a mild bend in the river here. Near the outside bank of the bend the sediment transport rates are assumed to be higher than in the inner bank.

From equation (2.7) it would then be expected that the dunes would turn counter clockwise along the bend, unless the gradient in dune height is larger than the gradient in the sediment transport. However, the angle of the dunes relative to the river axis does not seem to change. At the southern bank disturbances in the bed caused by the groins can be seen. These disturbances have an angle similar to the dunes, perhaps this can give a clue to the possible mechanisms causing the oblique dunes. Although in figure 2.7 (The Waal river between kilometres 890 and 892) the dune crests are aligned perpendicular to the flow direction while there are groynes present. The size of these dunes is smaller than the ones in figures 2.5 and 2.6.

2.3.2 The IJssel

In figure 2.8a the bathymetry of the IJssel river between kilometres 959 and 962 is shown. Here we can clearly see the dune crests turning in clockwise direction along the bend as expected. This is also seen in figure 2.8b (The IJssel river between kilometres 971 and 973) for a gentler bend.

Figure 2.9 shows the IJssel river bathymetry between kilometres 984 and 985. In this figure we can see the dune crest turning counter clockwise along the bend. Just after kilometre 984 a quick change in dune angle can be seen from perpendicular to an angle of 50 degrees as seen in figures 2.5 and 2.6. This quick change cannot easily be explained with equation (2.7).
Figure 2.5: Bathymetry of the Waal river between kilometre 947 and 949

Figure 2.6: Bathymetry of the Waal river between kilometre 946 and 948
Figure 2.7: Bathymetry of the Waal river between kilometre 890 and 892
Figure 2.8: Bathymetry of the IJssel river between kilometre 959 and 962 (a), and 971 and 973 (b)
Figure 2.9: Bathymetry of the IJssel river between kilometre 984 and 985
3 Bed-load Transport Direction

The dune angle has an effect on the bed-load transport direction over the dune. One cause of this, is the gravity. A dune with a crest rotated clockwise will give a transport direction to the left at the lee side of the dune, and to the right at the stoss side of the dune, due to the change in the bed slope.

Another cause is the changed flow field in the wake of the dune. For oblique bed forms, a helical vortex develops in the separation zone (see figure 1.3). This causes a bed-load transport to the left for a crest rotated clockwise. The effect on the flow direction of this helical vortex can be seen in figure 3.1, this picture was taken during tests by Talmon [2009]. To visualize the near bed flow pattern, wool threads were attached to a dune. The mean flow direction is from right to left. The wood panel on the right side of the image is the lee side of the upstream dune. On the lee side of the dune, the current is almost completely in the transverse direction toward the left bend. Also behind the lee side a strong transverse component is seen. After the flow reattaches, the current starts deflecting toward the depth averaged flow again. The effects of the dune angle on the sediment transport direction are summarised in figure 3.2.

As seen in figure 3.1 the dune can roughly be divided into 3 zones:

1. The lee zone;
2. The transition zone behind the lee zone, with recirculating flow;
3. The stoss zone minus the transition zone.

For each zone a bed-load transport direction formula is derived. $\delta_{pl}$ is the transport direction on the lee side of the dune, $\delta_{pt}$ is the transport direction in the transition zone, and $\delta_{ps}$ is the transport direction on the stoss side of the dune minus the transition zone. The bedform averaged bed-load transport direction is $\delta_{p}$. All the bed-load transport directions are defined relative to the depth averaged flow direction.

The effects of the oblique dune on the bed-load transport direction found by Talmon [2009] can be summarised with figure 3.3. This figure shows the direction of the bed-load transport for different locations along the dune, and for different values of $u/u_{cr}$. The thick black line is the simplified dune profile. At the lee of the dune there is a large difference between the measured transport directions. For higher values of $u/u_{cr}$, the influence of the helical flow becomes larger and a transport direction opposite the dune angle is observed. For smaller values of $u/u_{cr}$ the gravity effect of the lee side becomes more important and the transport direction goes towards the dune angle. In the wake zone after the dune there is little effect of $u/u_{cr}$ on the transport direction observed. The direction here stays at around 20° in the direction opposite to the dune angle. On the stoss side there is a small angle of sediment transport in the same direction as the dune angle. This is not what would be expected from the change in slope at oblique dunes.

![Figure 3.3: Direction of sand particles for $\alpha = -20^\circ$, $H/L_s$ between 0.53 and 0.64 (Higher angles for steeper lee side slopes), relative to flume axis versus distance from crest. The thick black line is the simplified dune profile. [Sieben & Talmon, 2011]](image)

### 3.1 Bed-load Transport at Stoss Side

The bed-load transport direction on the stoss side of the dune is estimated with a momentum balance for a spherical particle. The forces acting on a sediment particle are (See also figure 3.4):

1. The drag force $F_d$;
2. The lift force $F_l$;
3. The friction force between the particle and the bed $F_f$;
4. The gravity force $F_g$, which has components $F_{g \perp}$ perpendicular to the riverbed, and $F_{g \parallel}$ parallel to the bed, see figure 3.4b.
Figure 3.4: Forces acting on a sediment particle. The flow direction is from left to right.

The drag force $F_d$ on the particle is defined as follows for spherical particles [e.g. Seminara et al., 2002]:

$$F_d = \rho C_d \pi \frac{D^2}{4} \Delta U^2$$

(3.1)

where:
- $\rho$ Specific density of the water [kg/m$^3$]
- $C_d$ Drag coefficient [-]
- $A$ Area of the cross section [m$^2$]
- $\Delta U$ Difference between the near bed flow velocity $U$ and the sediment velocity $v$ [m/s]

The difference between the near bed flow velocity $U$ and the sediment velocity $v$ is defined as follows:

$$\Delta U = \sqrt{(U_x - v_x)^2 + (U_y - v_y)^2}$$

(3.2)

where:
- $U_i$ Near-bed flow velocity in i direction [m/s]
- $v_i$ Sediment velocity in i direction [m/s]

The drag force $F_d$ can be split up into a drag force in x direction $F_{dx}$ and a drag force in y direction $F_{dy}$:

$$F_{dx} = F_d \frac{U_x - v_x}{\Delta U}$$

(3.3a)

$$F_{dy} = F_d \frac{U_y - v_y}{\Delta U}$$

(3.3b)

The x- and y-components of the parallel gravity forces $F_{g//x}$ and $F_{g//y}$, on the particle are defined as follows:

$$F_{g//x} = \left(\rho_s - \rho\right) \frac{\pi}{6} g D^3 \frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}$$

(3.4a)

$$F_{g//y} = \left(\rho_s - \rho\right) \frac{\pi}{6} g D^3 \frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}$$

(3.4b)
The bed level gradients $z_x$ and $z_y$ are defined as follows:

\begin{align}
   z_x &= \frac{\partial \bar{z}}{\partial x} + \cos(\alpha) \left( \frac{H}{L_s} \right) \\
   z_y &= \frac{\partial \bar{z}}{\partial y} + \sin(\alpha) \left( \frac{H}{L_s} \right)
\end{align}

where:

$L_s$ Length of the dune stoss side [m]
\bar{z} Bed level without bedforms [m]

A Coulomb approach is used for the calculation of the dynamic friction force $F_b$ between the moving particle and the river bed. In a Coulomb approach the friction force $F_b$ is proportional to the total downward force. The bed friction force $F_b$ is then defined as follows:

$$F_b = \mu \left( F_{g\perp} - F_{li} \right) = \mu \left( (\rho_s - \rho) \frac{\pi}{6} gD^3 \sqrt{1 + \bar{z}_x^2 + \bar{z}_y^2} \frac{1}{\sqrt{1 + \bar{z}_x^2 + \bar{z}_y^2}} - \xi F_d \right)$$

where:

$\xi$ Lift to drag ratio [-]
$\mu$ Particle friction coefficient [-]

Here it is assumed that the lift force $F_{li}$ is proportional to the drag force $F_d$ by $F_{li} = \xi F_d$.

The forces mentioned can be used in a momentum balance in x- and y-direction:

\begin{align}
   F_d \frac{U_x - v_x}{\Delta U} &- F_{g/x} - F_b \frac{v_x}{v} = 0 \quad (3.7a) \\
   F_d \frac{U_y - v_y}{\Delta U} &- F_{g/y} - F_b \frac{v_y}{v} = 0 \quad (3.7b)
\end{align}

If we assume the flow and particle velocity in y-direction to be small compared to the flow and particle velocity in x-direction we can simplify equation (3.7a) to

$$F_d \approx F_{g/x} + F_b = (\rho_s - \rho) \frac{\pi}{6} gD^3 \frac{1}{\sqrt{1 + \bar{z}_x^2 + \bar{z}_y^2}} \left( z_x + \mu \right) - \xi F_d$$

This can be rewritten to

$$F_d \approx (\rho_s - \rho) \frac{\pi}{6} gD^3 \frac{1}{\sqrt{1 + \bar{z}_x^2 + \bar{z}_y^2}} \frac{z_x + \mu}{1 + \xi \mu}$$

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Combining equation (3.1) and equation (3.9) gives us the following,

$$
\Delta U = U - v = \sqrt{\frac{4 \Delta}{3 C_d}} gD \sqrt{\frac{z_x + \mu}{1 + \xi \mu} (1 + z_x^2 + z_y^2)^{-1/4}}
$$

(3.10)

In Sieben & Talmon [2011] $U_{cr}$ is taken equal to

$$
U_{cr} = \sqrt{\frac{4 \Delta}{3 C_d}} gD
$$

(3.11)

The reason for this is not mentioned in the article. For small values of $z_x$ and $z_y$ the term $(1 + z_x^2 + z_y^2)^{-1/4} \approx 1$. This gives for $\Delta U$,

$$
\Delta U = U_{cr} \sqrt{\frac{z_x + \mu}{1 + \xi \mu}}
$$

(3.12)

With equation (3.7), a formula for the particle direction on the stoss side of the dune can be derived:

$$
\tan(\delta_{ps}) = \frac{U_y p^2 - \Delta U z_y}{U_x p^2 - \Delta U z_x} = \frac{\sin(\delta_{fs}) - \frac{1}{f(\theta)}((\partial \bar{z}/\partial y) + \sin \alpha (H/L_s))}{\cos(\delta_{fs}) - \frac{1}{f(\theta)}((\partial \bar{z}/\partial x) + \cos \alpha (H/L_s))}
$$

(3.13)

where:

- $\delta_{ps}$: Direction of the sediment particle on the stoss side of the dune [rad]
- $\delta_{fs}$: Direction of the near-bed flow on the stoss side of the dune [rad]

and

$$
f(\theta) = p \sqrt{\frac{U}{U_{cr}}} = \sqrt{\frac{z_x + \mu}{1 + \xi \mu} \sqrt{\frac{\theta}{\theta_{cr}}}}
$$

(3.14)

and

$$
p = \sqrt{\frac{z_x + \mu}{1 + \xi \mu}} \approx \sqrt{\frac{H/L_{ss} + \mu}{1 + \xi \mu}}
$$

(3.15)

In Sieben & Talmon [2011] the terms $U_x$ and $U_y$ in equation (3.13) are not multiplied by $p^2$ (For a more detailed derivation of equation (3.13) see appendix A). This gives a different formula for $f(\theta)$,

$$
f(\theta) = \frac{1}{p} \sqrt{\frac{U}{U_{cr}}} = \sqrt{\frac{1 + \xi \mu}{z_x + \mu} \sqrt{\frac{\theta}{\theta_{cr}}}}
$$

(3.16)

From equation (3.13) it can be seen that the direction of the bed-load transport $\delta_{ps}$ depends on the direction of the near-bed flow $\delta_{fs}$ and the slope of the bed. The slope of the bed has a bed-form averaged bed slope and a slope caused by the bed-form itself. The slope direction of the dune is dependent on the dune angle $\alpha$. 

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Equation (3.13) is equal to the formulae by Parker & Andrews [1985] and Talmon et al. [1995] as presented in Deltares [2011], but with different values for \( f(\theta) \). Also, the bed level gradients used now include both the bed-form averaged gradient as well as the gradient of the bed form, \( H/L_s \). The different formula for \( f(\theta) \) are given in table 3.1.

<table>
<thead>
<tr>
<th>Formula</th>
<th>( f(\theta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation (3.13)</td>
<td>( \sqrt{\frac{z_x + \mu}{1 + \xi \mu}} \sqrt{\frac{\theta}{\theta_{cr}}} )</td>
</tr>
<tr>
<td>Sieben &amp; Talmon [2011]</td>
<td>( \sqrt{\frac{1 + \xi \mu}{H/L_s + \mu}} \sqrt{\frac{\theta}{\theta_{cr}}} )</td>
</tr>
<tr>
<td>Parker &amp; Andrews [1985]</td>
<td>( \frac{\mu}{1 + \xi \mu} \sqrt{\max\left(\frac{1}{\pi \theta_{cr}}, \frac{1}{\theta_{cr}}\right)} )</td>
</tr>
<tr>
<td>Talmon et al. [1995]</td>
<td>( A_{sh} \theta B_{sh} \left( \frac{D_{sh}}{d} \right) C_{sh} \left( \frac{D_{sh}}{D_{m}} \right) D_{sh} )</td>
</tr>
</tbody>
</table>

Table 3.1: \( f(\theta) \) in different formulae

where:

- \( D_{50} \) Median grain diameter [m]
- \( D_m \) Mean grain diameter [m]
- \( A_{sh}, B_{sh}, C_{sh}, D_{sh} \) Tuning coefficients [-]

If \( z_x \) is small, and we use \( v = 0 \) to determine \( U_{cr} \), equation (3.13) is equal to the formula of Parker & Andrews [1985]. The formula of Talmon et al. [1995] is equal to that of Sieben & Talmon [2011] for

\[
\begin{align*}
A_{sh} &= \frac{z_x + \mu}{1 + \xi \mu} \frac{1}{\theta_{cr}} \\
B_{sh} &= 0.5 \\
C_{sh} &= D_{sh} = 0
\end{align*}
\]

It is possible to apply the formulae of Talmon et al. [1995] for the transport direction on the stoss slope if we include the slope of the bed-form in the total bed gradient. These formulae do not reproduce the small angle in the direction of the dune angle seen in figure 3.3.

### 3.2 Bed-load Transport at Lee Side

The formula for the bed-load transport direction at the lee of the dune is not derived from a momentum balance. It is based on the observation that the direction of the bed-load transport will be equal to \( \alpha \) when \( u \) is equal to \( u_{cr} \), because here the effect of the helical flow on the particle is not felt yet. This means the transport direction is entirely influenced by the slope direction of the lee. Then as the term \( u/u_{cr} \) increases the effect of the helical flow becomes larger and the angle between \( \alpha \) and the transport direction becomes larger. With this the following formula is proposed.

\[
\tan (\delta_{pl} - \alpha) \approx \left( \sqrt{\frac{\theta}{\theta_{cr}}} - 1 \right) \sin (\delta_{fs} - \alpha)
\]
(δ_{pl} - \alpha) is the bed-load transport direction at the lee, δ_{pl}, relative to the dune angle, \alpha. With this formula the transport direction does indeed equal \alpha when u = u_{cr}. For large values of u/u_{cr} the angle between the transport direction and \alpha can become larger than 90°. This is not realistic, the formula is not valid for large values of u/u_{cr}.

It should also be noted that this formula has not been validated with the data of Talmon [2009], due to missing data for the near bed flow direction. There is also no influence of the bed slope in the formula. For large values of \alpha, large values of δ_{pl} occur, and the effect of the bed slope is negligible. However, for small values of \alpha the effect of the bed slope should not be neglected.

### 3.3 Bed-load Transport in the Transition Zone

The transition zone after the lee side lies on the flatter part of the upstream stoss. Therefore the effect of slopes is considered negligible. This means the particle direction will be equal to the near-bed flow direction. The near-bed flow component parallel to the dune crest is considered to be equal to the parallel component of the near-bed flow at the crest of the dune. This gives us the following relation.

\[ U_{lm} \approx U_{crest} \sin (\delta_{fs} - \alpha) \] (3.19)

where:
- \( U_{lm} \) Near bed flow component parallel to the crest [m/s]
- \( U_{crest} \) Near-bed flow magnitude at the dune crest [m/s]

If we apply conservation of depth averaged mass flux in the direction perpendicular to the crest we get the following [Chow, 1985],

\[ \tan (\delta_{pt} - \alpha) \approx \left(1 + r \frac{H}{d}\right) \tan (\delta_{fs} - \alpha) \] (3.20)

Here \( r \) is an empirical coefficient. If we apply the data of Talmon [2009] (\( \delta_{fs} \approx 0, \alpha = -0.35, \delta_{pt} = 0.35 \) and \( H/d = 0.23 \)) a value for \( r \) in the order of 4-5 is found. Equation (3.20) is the formula derived in Sieben & Talmon [2011].

As for the formula in the lee zone, this formula has not been validated with the data of Talmon [2009]. There are also no effects of bed slope taken into account in the formula.

### 3.4 Bedform Averaged Bed-load Transport

The three separate bed-load transport directions can be used to compute the bed form averaged transport direction \( \delta_p \) as follows,

\[ \tan (\delta_p - \alpha) = \frac{L}{L} \tan (\delta_{pl} - \alpha) + \frac{L_{wake} - L_i}{L} \tan (\delta_{pt} - \alpha) + \frac{L - L_{wake}}{L} \tan (\delta_{ps} - \alpha) \] (3.21)

This formula includes the effect of dune angle, bed slope and near-bed flow direction.
4 Discretisation

The dune angle formula derived in chapter 2 will be discretised so it can be used in a numerical simulation. For convenience the final equation of chapter 2 is repeated here:

\[
\frac{1}{S} \left( \sin \alpha \frac{\partial S}{\partial x} - \cos \alpha \frac{\partial S}{\partial y} \right) \tan (\delta_{ps} - \alpha) - \frac{1}{H} \left( \sin \alpha \frac{\partial H}{\partial x} - \cos \alpha \frac{\partial H}{\partial y} \right) = \frac{1}{S} \left( \sin \alpha \frac{\partial \alpha}{\partial x} - \cos \alpha \frac{\partial \alpha}{\partial y} \right) + \frac{1}{S} \left( \sin \alpha \frac{\partial \delta_{ps}}{\partial x} - \cos \alpha \frac{\partial \delta_{ps}}{\partial y} \right) - \tan (\delta_{ps} - \alpha) \left( \sin \alpha \frac{\partial \delta_{ps}}{\partial x} - \cos \alpha \frac{\partial \delta_{ps}}{\partial y} \right) \tag{4.1}
\]

This is discretised as follows:

\[
u_\alpha \frac{\Delta \alpha}{\Delta x} + v_\alpha \frac{\Delta \alpha}{\Delta y} = RHS_S + RHS_\delta + RHS_H \tag{4.2}
\]

where:

\[
u_\alpha = (\cos \alpha_i,j - \sin \alpha_i,j \tan (\delta_{ps,i,j} - \alpha_i,j)) \tag{4.3a}
\]

\[
v_\alpha = (\sin \alpha_i,j + \cos \alpha_i,j \tan (\delta_{ps,i,j} - \alpha_i,j)) \tag{4.3b}
\]

And:

\[
RHS_S = \frac{1}{S} \left( \sin \alpha_i,j \frac{\Delta S_x}{\Delta x} - \cos \alpha_i,j \frac{\Delta S_y}{\Delta y} \right) \tag{4.4a}
\]

\[
RHS_\delta = - \tan (\delta_{ps,i,j} - \alpha_i,j) \left( \sin \alpha_i,j \frac{\Delta \delta_{ps,x}}{\Delta x} - \cos \alpha_i,j \frac{\Delta \delta_{ps,y}}{\Delta y} \right) \tag{4.4b}
\]

\[
RHS_H = - \frac{1}{H} \left( \sin \alpha_i,j \frac{\Delta H_x}{\Delta x} - \cos \alpha_i,j \frac{\Delta H_y}{\Delta y} \right) \tag{4.4c}
\]

Here \(i\) denotes the x-coordinate and \(j\) the y-coordinate. For the discretisation upwinding is used. This means that:

\[
\Delta \alpha_x = \begin{cases} 
\alpha_{i,j} - \alpha_{i-1,j} & \text{for } u \geq 0 \\
\alpha_{i+1,j} - \alpha_{i,j} & \text{for } u < 0 
\end{cases} \tag{4.5a}
\]

\[
\Delta \alpha_y = \begin{cases} 
\alpha_{i,j} - \alpha_{i,j-1} & \text{for } v \geq 0 \\
\alpha_{i,j+1} - \alpha_{i,j} & \text{for } v < 0 
\end{cases} \tag{4.5b}
\]

\(\Delta S, \Delta H\) and \(\Delta \delta_{ps}\) are discretised in a similar manner. Equation (4.2) cannot be solved directly, an iterative scheme is needed. To find the solution for \(\alpha\) we must find the solution to:

\[
F(\alpha) = RHS - LHS = 0 \tag{4.6}
\]
where:

\[
\begin{align*}
RHS &= RHS_S + RHS_\delta + RHS_H \\
LHS &= \frac{\Delta \alpha_x}{\Delta x} + \frac{\Delta \alpha_y}{\Delta y}
\end{align*}
\] (4.7a) (4.7b)

The right hand side minus the left hand side of the equation, \(F(\alpha)\), usually has a form similar to figure 4.1. The formula is discontinuous: \(F(\delta_{ps} - \pi/2) = \infty\) and \(F(\delta_{ps} + \pi/2) = -\infty\). The solution of \(F(\alpha) = 0\) can be found between \((\delta_{ps} - \pi/2)\) and \((\delta_{ps} + \pi/2)\).

![Figure 4.1: Standard shape of equation (4.2)](image)

The discontinuous nature of equation (4.2) limits the possible schemes to be used. A pseudo time iteration has been proposed, as well as a predictor-corrector method. Because the limits of the solution are well known \((\delta_{ps} - \pi/2 < \alpha < \delta_{ps} + \pi/2)\). The bisection method is easy to apply in this case. But before looking at the possible iteration schemes, there are a number of problems that can arise when using equation (4.2).

### 4.1 General Problems

Two problems will be mentioned here, these are:

1. \(F(\alpha)\) has multiple solutions on the interval \([\delta_{ps} - \pi/2, \delta_{ps} + \pi/2]\), or
2. \(F(\alpha)\) has no solutions on the interval \([\delta_{ps} - \pi/2, \delta_{ps} + \pi/2]\).

#### 4.1.1 Multiple Solutions

One problem is the possibility to have multiple solutions on the interval \([\delta_{ps} - \pi/2, \delta_{ps} + \pi/2]\), see figure 4.2. Right now it is unknown which of these solutions is the right one. This is also problematic in all three iterative schemes, because one cannot control to which solution the methods converge. The likelihood of this occurring depends mostly on the following factor:
\[ f_{HS} = f_H + f_S = \frac{1}{H} \left( \frac{\Delta H_x}{\Delta x} + \frac{\Delta H_y}{\Delta y} \right) + \frac{1}{S} \left( \frac{\Delta S_x}{\Delta x} + \frac{\Delta S_y}{\Delta y} \right) \] (4.8)

The likelihood of occurrence is larger when \(|f_{HS}|\) is larger. This can occur when the values of \(S\) or \(H\) are very small, and thus the values of \(1/S\) or \(1/H\) are very large. This can also occur for very large gradients in \(S\) or \(H\), which is less likely because the gradients needed are not realistic. For small values of \(|f_{HS}|\) it is unlikely that multiple solutions occur, see figure 4.3. Of course it also depends on \(LHS\) and \(RHS_S\), but their influence is much smaller.

![Figure 4.2: The effect on \(F(\alpha)\) of large values of \(f_H\) and \(f_S\)](image)

![Figure 4.3: \(F(\alpha)\) for the same values as in figure 4.2, but with smaller values for \(f_H\) and \(f_S\)](image)

### 4.1.2 No Solution

It is also possible that there are no solutions on the interval \([\delta_{ps} - \pi/2, \delta_{ps} + \pi/2]\). This can happen when the values of \(\alpha\) in the surrounding grid-cells do not fall within the interval \([\delta_{ps} - \pi/2, \delta_{ps} + \pi/2]\). When the values of \(\alpha\) in the surrounding grid-cells do fall within the interval, \(F(\alpha)\) goes to \(-\infty\) at \(\delta_{ps} - \pi/2\) because the values of \(\partial\alpha/\partial x\) and \(\partial\alpha/\partial y\) will be negative, and \(F(\alpha)\) goes to \(+\infty\) at \(\delta_{ps} + \pi/2\) because the values of \(\partial\alpha/\partial x\) and \(\partial\alpha/\partial y\) are positive here. This means that \(F(\alpha)\) will cross the x-axis, since it is continuous on the interval \([\delta_{ps} - \pi/2, \delta_{ps} + \pi/2]\).

when the values of \(\alpha\) in the surrounding grid-cells do not fall within the interval \([\delta_{ps} - \pi/2, \delta_{ps} + \pi/2]\), The values of \(\partial\alpha/\partial x\) and \(\partial\alpha/\partial y\) do not change sign. When this happens, the equation has no guaranteed
solution in this interval (See figure 4.4). This is a discretisation problem, and not a problem with the
equation. This problem can come into play, for example, when dealing with reversing flows.

Figure 4.4: Shape of equation (4.2), when the term \( \alpha \)'s in the surrounding grid cells does not lie on the
interval \([\delta_{ps} - \pi/2, \delta_{ps} + \pi/2]\). In this case \( \delta_{ps} = 0.1 \) and \( \alpha_{i,j} = -\pi/2 \) are applied.

No solution can also happen when the gradient of \( \delta_{ps} \) is larger than the gradient of \( \alpha \) at \( \delta_{ps} \pm \pi/2 \). In
this case RHS is larger than LHS. Both contain the term \( \tan(\delta_{ps} - \alpha) \), therefore both go to \( \pm\infty \) at
this location.

4.2 Iterative Scheme

In this section three different iteration schemes to solve 4.2 are described and analysed. These schemes are:

1. Pseudo-time iteration
2. Predictor-corrector
3. Bisection method

It is also possible to calculate the solution of \( F(\alpha) \) for every possible point. In fact, this is how the plots
in this chapter are generated. This will take thousands of calculations per grid-cell. The goal of this
section is to find a solution that limits the needed calculations as much as possible while still finding
solutions in most cases.

4.2.1 Pseudo-time Iteration

The first method is the pseudo-time iteration. In this scheme a ‘time derivative’ is added to equation
(4.2). This gives us the following equation:

\[
\frac{\alpha_{i,j}^{k+1} - \alpha_{i,j}^k}{\Delta \tau} + u_a \frac{\Delta \alpha_x}{\Delta x} + v_a \frac{\Delta \alpha_y}{\Delta y} = RHS_S + RHS_\delta + RHS_H
\]  

(4.9)

Here \( k \) is the iteration step. \( \Delta \tau \) is not the same as the time step for the flow, but is used for an internal
loop. \( \alpha_{i,j}^{k+1} \) is solved until \( |\alpha_{i,j}^{k+1} - \alpha_{i,j}^k| \) is small enough. The following stability criterion is suggested to
determine \( \Delta \tau \):
\[ \Delta \tau \left( \left| \frac{u}{\Delta x} \right| + \left| \frac{v}{\Delta y} \right| \right) \leq 1 \]  

(4.10)

Using equation (4.10) we can calculate \( \alpha_{i,j}^{k+1} \) as follows:

\[ \alpha_{i,j}^{k+1} = (RHS - LHS) \Delta \tau = (RHS - LHS) \sigma \left( \frac{\Delta x}{|u|} + \frac{\Delta y}{|v|} \right) \]  

(4.11)

Where \( \sigma \) is a number smaller than 1. If equation (4.10) is a stability criterion, any value under 1 would suffice, and the highest number should be chosen. This is however not the case. This can be visualised if equation (4.11) is interpreted as a fixed point iteration scheme [Vuik et al., 2006].

In figure 4.5 The left hand side of equation (4.11) is plotted as the dashed green line, and the right hand side is plotted for a \( \sigma \) of 0.3 (the blue line) and 0.05 (the red line). The arrows show the first few iterations of the scheme. The solution is where where the lines cross the green dotted line which represents \( \alpha \).

Figure 4.5: Pseudo time iteration visualised as a fixed point iteration, for \( \sigma \) is 0.3 and 0.05. For each \( \sigma \) three iterations have been done.

It can be seen that with a \( \sigma \) of 0.3 this method will not be stable, the arrows only go farther away from the solution. With a \( \sigma \) of 0.05 the scheme does converge toward the solution. The fixed point iteration scheme requires slopes between 1 and -1 for convergence. With a \( \sigma \) of 0.3 the slope is lower than -1. The fastest convergence happens for slopes near 0. When the slope is near 1 or -1, the convergence is very slow.

The upper limit of \( \sigma \) for convergence differs per gridcell. If we chose a fixed number for \( \sigma \) we get too slow convergence for some cells, while we get no convergence for other cells. If there is a solution, it is always possible to have convergence by picking a small enough \( \sigma \). This is because the slope of equation (4.11)
at any point is -1 for a \( \sigma \) of 0, and the slope increases linearly with \( \sigma \). Therefore it is always possible to attain a slope between -1 and 1.

The relation between several variables and this maximum \( \sigma \) has been investigated in appendix B. The bed-load transport direction \( \delta_{ps} \) has a large effect on the maximum \( \sigma \). This is because the values of \( u_\alpha \) and \( v_\alpha \) in equation (4.11) are related to \( \delta_{ps} \) as follows

\[
\frac{v_\alpha}{u_\alpha} = \tan(\delta_{ps})
\]  

(4.12)

When \( \delta_{ps} \) is 0, \( v_\alpha \) is 0, and \( \Delta \tau \) will become infinity. When \( \delta_{ps} \) is \( \pm \pi/4 \), \( u_\alpha \) is 0, and \( \Delta \tau \) will again become infinity. This means that for these values of \( \delta_{ps} \) a very low value of \( \sigma \) is required.

Besides \( \delta_{ps} \), also the gradients in \( \delta_{ps} \), sediment transport rate \( S \) and dune height \( H \) influence the maximum \( \sigma \), as well as the values of \( \alpha \) used in the surrounding grid cells. Because many variables influence the maximum \( \sigma \), it is hard to define clear relations. This makes it difficult to pick a right value of \( \sigma \).

### 4.2.2 Predictor Corrector

The second method is the predictor corrector method. In this method we use a predictor to calculate \( \alpha^* \) and then use a corrector to calculate a new \( \alpha \). For the predictor it is assumed that \( \alpha \approx \delta_{ps} \). This gives us the following equation:

\[
\cos \delta_{ps,i,j} \frac{\Delta \alpha_x}{\Delta x} + \sin \delta_{ps,i,j} \frac{\Delta \alpha_y}{\Delta y} = \frac{1}{S} \left( \sin \delta_{ps,i,j} \frac{\Delta S_y}{\Delta x} - \cos \delta_{ps,i,j} \frac{\Delta S_x}{\Delta y} \right) - \frac{1}{H} \left( \sin \delta_{ps,i,j} \frac{\Delta H_y}{\Delta x} - \cos \delta_{ps,i,j} \frac{\Delta H_x}{\Delta y} \right)
\]  

(4.13)

with

\[
\Delta \alpha_x = \begin{cases} 
\alpha^* - \alpha_{i-1,j} & \text{for } u^* \geq 0 \\
\alpha^* - \alpha_{i,j} & \text{for } u^* < 0 
\end{cases}
\]  

(4.14a)

\[
\Delta \alpha_y = \begin{cases} 
\alpha^* - \alpha_{i,j-1} & \text{for } v^* \geq 0 \\
\alpha^* - \alpha_{i,j} & \text{for } v^* < 0 
\end{cases}
\]  

(4.14b)

where:

\[
u^* = \cos \delta_{ps}
\]  

(4.15a)

\[v^* = \sin \delta_{ps}
\]  

(4.15b)

The \( \alpha^* \) calculated with this formula is then used in equation (4.2) as follows.

\[
u_\alpha^* \frac{\Delta \alpha_x}{\Delta x} + v_\alpha^* \frac{\Delta \alpha_y}{\Delta y} = RHS_S^* + RHS_S^* + RHS_H^*
\]  

(4.16)
Where the * indicates that $\alpha^*$ is used instead of $\alpha$ in these terms. Much better precision can be reached when we repeat the corrector phase. We then use the last calculated alpha as $\alpha^*$ in equation (4.16). Similar to the pseudo-time iteration, this can be interpreted as a fixed point iteration as follows,

$$
\alpha_{i,j}^{k+1} = \begin{cases}
(RHS_S^k + RHS_S^k + RHS_H^k + u^k_\alpha \frac{\alpha_{i-1,j}}{\Delta x} + v^k_\alpha \frac{\alpha_{i,j-1}}{\Delta y}) / (u^k_\alpha + \frac{v^k_\alpha}{\Delta y}) & \text{for } u^k_\alpha \geq 0, v^k_\alpha \geq 0 \\
(RHS_S^k + RHS_S^k + RHS_H^k - u^k_\alpha \frac{\alpha_{i+1,j}}{\Delta x} + v^k_\alpha \frac{\alpha_{i,j+1}}{\Delta y}) / (-u^k_\alpha + \frac{v^k_\alpha}{\Delta y}) & \text{for } u^k_\alpha < 0, v^k_\alpha \geq 0 \\
(RHS_S^k + RHS_S^k + RHS_H^k + u^k_\alpha \frac{\alpha_{i+1,j}}{\Delta x} - v^k_\alpha \frac{\alpha_{i,j+1}}{\Delta y}) / (u^k_\alpha - \frac{v^k_\alpha}{\Delta y}) & \text{for } u^k_\alpha \geq 0, v^k_\alpha < 0 \\
(RHS_S^k + RHS_S^k + RHS_H^k - u^k_\alpha \frac{\alpha_{i-1,j}}{\Delta x} - v^k_\alpha \frac{\alpha_{i,j-1}}{\Delta y}) / (-u^k_\alpha - \frac{v^k_\alpha}{\Delta y}) & \text{for } u^k_\alpha < 0, v^k_\alpha < 0
\end{cases}
$$

(4.17)

$k$ denotes the current iterations step.

The fixed point iteration scheme can be visualised, this has been done in figure 4.6a. $\alpha$ is plotted as a dashed green line. The right hand side of equation (4.17) is plotted as the blue line. We start with the initial guess of $\alpha_{i,j}^0 = \delta_{ps}$. Using this value in equation (4.17) gives us $\alpha_{i,j}^1$, this is visualised in the figure as the first horizontal arrow. This $\alpha_{i,j}^1$ is then used in equation (4.17), this is visualised as the vertical arrow. This gives us $\alpha_{i,j}^2$, which is visualised as the second horizontal arrow. This continues until we get an value for $\alpha_{i,j}$ close enough to the real solution.

In many cases this method will give fast convergence. The speed of convergence depends on the slope of the blue line. For a slope lower than -1 this method does not converge. A slope close to 0 gives the fastest convergence. A calculation with the predictor corrector method is shown which fails to converge in figure 4.6b.

![Figure 4.6: Calculated $\alpha$ with the predictor corrector method, the method fails to converge.](image)

The dependence of the convergence on several variables has been investigated in appendix C. Whether the scheme converges or not seems to depend mostly on the right hand side terms. $RHS_S$ and $RHS_H$ have the largest effect. There is also an effect of the $\alpha$’s in the surrounding grid cells and the $\delta_{ps}$, but this effect is limited. In the cases tested in chapter 5 the predictor-corrector did converge.

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4.2.3 Bisection Method

The bisection method is a simple and robust method for finding the root of a continuous function. It starts with picking two values at each side of the root. These values should have opposite signs and the formula should be continuous as well. In this case this is easy, we can take $\delta_{ps} - \pi/2$ and $\delta_{ps} + \pi/2$.

The algorithm for finding the root is as follows:

1. Pick two values $x_{a,1}$ and $x_{b,1}$ at either side of the root. In this case $x_{a,1} = \delta_{ps} - \pi/2$ and $x_{b,1} = \delta_{ps} + \pi/2$.

2. Calculate $x_{c,n} = 0.5 \left( x_{a,n} + x_{b,n} \right)$

3. If the sign of $f(x_{c,n})$ is equal to the sign of $f(x_{a,n})$, set $x_{a,n+1}$ to $x_{c,n}$. Else set $x_{b,n+1}$ to $x_{c,n}$. If $x_{c,n} = 0$ we can stop.

4. Go back to step 2.

At the start of the algorithm, the maximum error is $(x_{b,1} - x_{a,1})$, which is $\pi$ in this case. With each step the maximum error is halved. Thus after $n$ iterations, the error is:

$$|f(x_{c,n}) - p| \leq \frac{x_{b,1} - x_{a,1}}{2^n} = \frac{\pi}{2^n} \quad (4.18)$$

The number of iterations needed for a certain maximum allowed error can be calculated as follows:

$$n = \frac{\ln \pi - \ln \varepsilon}{\ln 2} \quad (4.19)$$

This means that the number of necessary iterations for different precisions are given in table 4.1.

<table>
<thead>
<tr>
<th>Maximum allowed error</th>
<th>Iterations needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>9</td>
</tr>
<tr>
<td>0.001</td>
<td>12</td>
</tr>
<tr>
<td>0.0001</td>
<td>15</td>
</tr>
<tr>
<td>0.00001</td>
<td>18</td>
</tr>
</tbody>
</table>

Table 4.1: Number of iterations needed for different maximum allowable errors

The advantage of the bisection method is that it always converges when there is a single solution on the interval $[\delta_{ps} - \pi/2, \delta_{ps} + \pi/2]$. It can also be seen as an advantage that the number of necessary calculations is known beforehand. It does need more iterations than the predictor corrector in most cases.
5 Application

In this chapter the formulae derived earlier for the dune angle and bed-load direction are applied to two test cases. First the methods used are discussed, then the results of the application are given.

5.1 Methods

The formulae derived in chapters 2 and 3 have been added to the source code of Delft3D. At the start of the project, code was supplied for calculation of the dune angle, this code was very incomplete. Adjusting the code took most of the time in this project and there are likely still mistakes present in the code. The method used is the bisection method.

5.1.1 Description of Delft3D

Delft3D is a 2DH numerical modelling package. It can simulate the hydrodynamics and transport phenomena such as sediment transport. The hydrodynamics are calculated based on the Navier-Stokes equations for incompressible free surface flow. These are a set of equations containing the mass and momentum balance for fluids [Temam, 2001]. Delft3D also uses the shallow water assumption. With this assumption it is assumed that the length scale is much larger than the depth scale. In this case the vertical momentum equation is reduced to the hydrostatic pressure relation.

5.1.2 Settings in Delft3D

For the prediction of dune heights the predictor of Fredsoe [1982] combined with the sediment transport formula of Meyer-Peter & Müller [1948]. According to Fredsoe [1982] the dune height $H$ relates to the sediment transport $S$ as follows,

$$H = \varepsilon Sd \frac{\partial S}{\partial U} - d \frac{\partial S}{\partial d} - k \frac{\partial S}{\partial k}$$

where:

- $d$: Water depth [m]
- $U$: Flow velocity magnitude [m/s]
- $k$: The Nikauradse bed roughness [m]
- $\varepsilon$: Calibration parameter [-]

If we use the transport formula of Meyer-Peter & Müller [1948] as the sediment transport formula in equation (5.1), we get the following formula for the dune height,

$$H = \frac{24}{63} \varepsilon d \max \left(1 - \frac{\theta_{cr}}{\mu H}, 0\right)$$

where:
\( \theta \) The shields parameter as defined by Shields [1936] [-]

\( \theta_{cr} \) The critical shields parameter [-]

\( \mu H \) Efficiency factor [-]

For the dune length the predictor of van Rijn [1984] is used. In this the dune length \( L \) relates to the dune height \( H \) as follows,

\[
L = 7.3d
\] (5.3)

The sediment transport is calculated with the formula of Meyer-Peter & Müller [1948].

\[
S = 8\alpha_{MPM}D_{50}\Delta gD_{50}^{1/2}(\mu\theta - \theta_{cr})^{3/2}
\] (5.4)

Here \( \alpha_{MPM} \) is a calibration factor specified by the user.

The morphology is not updated during the simulations. The parameters used in the bed-load transport equations are given in table 5.1. The parameters suggested in Sieben & Talmon [2011] are used. These are given in table 5.1. They are based on Seminara et al. [2002], Franchalanci & Solari [2007], and Carling et al. [2000].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu )</td>
<td>0.3</td>
</tr>
<tr>
<td>( \xi )</td>
<td>1.25</td>
</tr>
<tr>
<td>( r )</td>
<td>5</td>
</tr>
<tr>
<td>( L_s/H )</td>
<td>0.5</td>
</tr>
<tr>
<td>( L_l/H )</td>
<td>2.7</td>
</tr>
<tr>
<td>( L_{wake}/H )</td>
<td>6.0</td>
</tr>
</tbody>
</table>

Table 5.1: Parameters used in the bed-load direction formulae

5.1.3 Test Cases

The first case tested is a straight channel with a width of 60 metres and a length of 500 metres. The bed is sloped in the transverse direction. The grid size is 6 metres in transverse direction and 10 metres in the flow direction. The bed level relative to the reference level varies from 0.4 metres near the bank, to 2 metres in the centre of the channel, see figure 5.1. These slopes should induce a gradient in the relative bedform migration rate, and therefore cause oblique dunes. The transverse slopes also serve as a test for the transport direction formula in the case of sloped beds.

The second case is a curved channel with a width of 6 metres and a radius of 25 metres. The bed level is constant over the entire bend. The average grid size in the flow direction is 0.5 metres, the average grid size in transverse direction is 1 metre.
Figure 5.1: The cross section of the first test case

Figure 5.2: The grid of the first test case

Figure 5.3: The grid of the second test case
5.2 Bedform Orientation

5.2.1 Straight Channel

The first case is the straight channel with transverse slopes. These transverse slopes cause a gradient in relative dune migration speed (see figure 5.4b), and bed-load transport direction. This relative dune migration forces the change in dune angle. The highest relative dune migration speed occurs at the centreline of the channel. This causes the dunes to turn towards the banks and almost turn into longitudinal dunes, see figure 5.4a. This is in accordance with equation (2.7), but it is not a realistic outcome.

It could be possible that the transverse slopes used give an unrealistically high gradient in the relative dune migration rate. To test this another run has been made with a flat bottom and a partial slip condition at the walls. The roughness length used was 0.01 metres. The resulting relative dune migration rate is shown in figure 5.6. The gradient is much smaller than in figure 5.4b. When we look at the resulting dune angle in figure 5.5 we see that the equilibrium dune crest has not been reached yet. However, it does look like the dune angles go towards the almost longitudinal alignment seen in the case with longitudinal slopes. If we look at the relative dune migration rate with the effect of dunes included in figure 5.7 we can see why this is. As the dune crests turn, they lie at a greater angle with the bed-load transport direction. This decreases their relative migration rate. Because the dune angle increases from the centreline towards the banks, the relative dune migration rate now has an increased gradient with a maximum value at the centreline.

5.2.2 Curved Channel

The second case is a curved channel with a flat bottom, see figure 5.8a. Because of the flat bottom, the water takes the path of least resistance through the inner bend. This causes the relative dune migration rate to be higher in the inner bank, see figure 5.8b. This causes the dune crests to skew so that the
Figure 5.5: The calculated bedform orientation of the first case, with a flat bed.

Figure 5.6: The calculated relative dune migration of the first case, with a flat bed.

Figure 5.7: The calculated bedform orientation of the first case, with a flat bed. The dune angle is taken into account.
Figure 5.8: The calculated bedform orientation (a), and relative dune migration rate for transverse dunes (b) in a curved channel with a flat bed.

The lee side of the dune faces the outer bank. After a while the dune orientations seem to have reached an equilibrium. This equilibrium can be reached because in this case the dune angles are highest at the point of highest relative dune migration rate, contrary to the case of a straight channel. This causes the gradient in relative migration rate to decrease, as seen in figure 5.9. The migration rate is now highest in the outer bend. Because the crest in the outer bend need to travel farther, this gives an equilibrium dune crest.

5.3 Bed-load Transport Direction

5.3.1 Straight Channel

For the straight channel the resulting bed-load transport directions are given in figure 5.10a and 5.10b. The dune angle along the cross section is given in figure 5.11. The transverse slopes cause the bed-load direction to be toward the channel centre. The pink line shows the direction calculated without the effect of dunes. It predicts the direction of the bed-load transport due to the transverse slopes well. The blue line gives the bedform averaged bed-load transport direction. There is a large difference between the two lines, the angle of bed-load transport increases 4 to 2.5 times. This indicates a large effect of the dune angle on the bed-load transport equation.

The total transport consists of the different angles at the stoss side (green line), lee side (red line) and the wake zone. The transport direction on the stoss side is very similar to the direction without the effect of dunes, but with a slight increase in angle. Talmon [2009] also found a small difference at the stoss side, but he found this change in the other direction. At the lee and in the wake zone large angles are calculated. Because these angles are large they have a strong influence on the bedform averaged transport direction, although it should be noted that very high values for the dune angle are used in this case. The transport angles at the lee side and in the wake zone do not increase with increasing dune angle along the cross section. This is due to the influence of the velocity on these angles. Lower velocities give a lower influence of the helical wake flow on the sediment transport. This means smaller angles with the dune crest for lower velocities. The distribution of the velocity along the cross section is given in figure 5.12.
Figure 5.9: The dune migration rate of case 2, with the dune angle taken into account.

Figure 5.10: The calculated bed-load transport directions (a), and the effect of dunes on the transport direction (b) in case 1.

Figure 5.11: Calculated dune angle $\alpha$ along the cross section for case 1.
Figure 5.12: Depth averaged velocity $\alpha$ along the cross section for case 1

Figure 5.13: The calculated bed-load transport directions (a), and the effect of dunes on the transport direction (b) in case 2.

5.3.2 Curved Channel

For the curved channel the resulting bed-load transport directions are given in figure 5.13a and 5.13b. The dune angle along the cross section is given in figure 5.14. The spiral flow causes the bed-load direction to be toward the inner bend. The pink line shows the direction calculated without the effect of dunes. It predicts the direction of the bed-load transport due to spiral flow well.

The blue line gives the bedform averaged bed-load transport direction. There is again a large effect of the dune angle on the bed-load transport direction. The total transport consists of the different angles at the stoss side (green line), lee side (red line) and the wake zone. The transport direction on the stoss side is again very similar to the direction without the effect of dunes, and the change in angle is again in the ‘wrong’ direction. The angle of the bed-load transport at the lee and in the wake zone are again much larger than the angle at the stoss side.

The angle of the bed-load transport at the lee and in the wake zone are smallest at the inner bend, even though the dune angle here is largest, as well as the depth averaged velocity (See figure 5.15). The reason for this is that the angle relative to the dune crest increases for increasing dune angle. However, it is possible that this increase in bed-load transport angle is smaller than the increase in dune angle.
Figure 5.14: Calculated dune angle $\alpha$ along the cross section for case 2

Figure 5.15: Depth averaged velocity $\alpha$ along the cross section for case 2
6 Discussion

In the bisection method a numerical scheme has been found which performs reasonably well in most circumstances. There is likely room for improvement, but the focus of future research should first be on the starting assumptions used, because the resulting dune angles from calculations are not realistic. The model has not been compared to laboratory results or field measurements, therefore little can be said at the moment about its performance, but there are strong indications that there are some errors in the starting assumptions.

An application of the formula to the simple case of a straight channel in section 2.2 showed that dune crests are likely to become parallel to the flow direction. The results of the numerical calculations in chapter 5 seem to confirm this. The dune crests do not become completely parallel but find a balance very close to that. In the curved channel the dunes did not become parallel to the flow, but there were still quite large dune angles calculated, although this might be because of the flat bed that was used. When we look at field measurements in Dutch rivers in section 2.3, there are also some phenomena that cannot be easily explained with the formula derived in chapter 2.

It is likely that there is a mechanism that is not accounted for. One thing not taken into account is the effect of dunes changing in shape. Equation (2.1) only applies in the case of similarly shaped cross sections. It is also possible that a diffusion term needs to be added in the formula. The results show quick changes in dune angle along the channel. A diffusion term could dampen these results, and give more realistic results with smaller dune angles.

Another possibility is a link between characteristic directions of morphology. The similarities between the direction of the erosion caused by groynes and the dune angle indicate there might be a relation between the two. A characteristics analysis of a bottom disturbance shows that this disturbance eventually develops into a triangular shape as in figure 6.1. [De Vriend, 1987]. Perhaps the riverbank acts as a disturbance, and the dunes are one point of this triangle.

![Figure 6.1: Evolution of a bottom disturbance according to the characteristics analysis [De Vriend, 1987]](image)

When we look at the results of the bed-load transport formulae they seem to perform well, but it is uncertain whether the formulae replicate reality. Many assumptions have been made in the derivation, and the formulae have not been validated with test results. The change in direction on the stoss side of the dune is opposite of that observed by Talmon [2009] as seen in figure 3.3.
7 Conclusions and Recommendations

7.1 Conclusions

In this work a dune angle formula was discretised so it could be implemented in a numerical simulation. The formula required iteration to be solved. Three different schemes were tested to solve the dune angle equation: Pseudo time iteration scheme, the Predictor-corrector scheme, and the bisection method. It is concluded that the bisection method is the most reliable of the three. It always finds the solution if there is one. Its computation time is also predictable as the number of iterations is fixed. The pseudo time iteration also converges in all cases, but for this a variable $\sigma$ has to be chosen correctly. This is not always possible. When the wrong value is chosen, slow convergence or instability can occur. The predictor correcter is fast, but has problems with instability in some cases.

The implementation of the dune angle formula in a numerical simulation was successful, but the resulting solutions are likely not realistic. Very large dune angles are calculated. Results have not been compared with laboratory tests or field measurements. Bathymetry data of Dutch rivers have been analysed, and some of the observed phenomena were not easily explained by the dune angle formula, indicating that the derived formula does not fully describe the behaviour of oblique dunes.

The formulae for bed load transport direction give expected results, except for the transport direction on the stoss side of dunes. Here the formula predicts a change in direction opposite that observed by Talmon [2009]. Because the results have not been compared to actual measurements, it is hard to draw conclusions about its performance.

7.2 Recommendations

It is recommended to first look at the starting assumptions for the dune angle formula, before improvements on the numerical scheme are made. Because without a working dune angle formula the numerical scheme would not be useful.

The results of the dune angle formula, as well as the bed-load transport formulae should be validated with lab results or field measurements. If no suitable laboratory results or field measurements are available, these should be carried out first.

Possible mechanisms that could be important for the dune angle are:

- An extra term for diffusion to dampen the size of the dune angle
- A possible link with the characteristics of a bed disturbance
- The effects of a change in shape of the dunes

To increase understanding in the behaviour of oblique dunes, laboratory tests or detailed numerical simulations should be carried out. Numerical simulations need to be able to reproduce the helical flow pattern on the lee side of the dune.

It is also recommended to have a good look at the derived formulas for bed load transport direction, and to validate the formulas with laboratory results or field measurements.

For the numerical model, the dune angle and bed-load transport should be linked, so the interaction between the two can be modelled.
References


A Derivation of $\delta_{ps}$

This appendix contains the derivation of the direction of sediment particle on the stoss side of a dune, $\delta_{ps}$. It is based on Sieben & Talmon [2011]. Equations A.1 till A.10 are taken from this article. These are then used in the following equations to derive $\delta_{ps}$.

A.1 Used formulae

The total drag force $F_d$ is:

$$ F_d = \rho \frac{C_d \pi}{4} D^2 \Delta U^2 \quad (A.1) $$

With components

$$ F_{dx} = \frac{F_d}{\Delta U} \left( U_x - v_x \right) \quad \text{and} \quad F_{dy} = \frac{F_d}{\Delta U} \left( U_y - v_y \right) \quad (A.2) $$

The difference between the near bed flow velocity $U$ and the sediment velocity $v$, $\Delta U$, is

$$ \Delta U = \sqrt{(U_x - v_x)^2 + (U_y - v_y)^2} \quad (A.3) $$

The parallel components of the gravity are:

$$ F_{g/x} = (\rho_s - \rho) \frac{\pi}{6} g D^3 \frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}} \quad (A.4a) $$

$$ F_{g/y} = (\rho_s - \rho) \frac{\pi}{6} g D^3 \frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}} \quad (A.4b) $$

The bed friction force $F_b$ is

$$ F_b = \mu \left( (\rho_s - \rho) \frac{\pi}{6} g D^3 \frac{1}{\sqrt{1 + z_x^2 + z_y^2}} - F_{li} \right) \quad (A.5) $$

$z_x$ and $z_y$ are the gradients of the bed level in x- and y-direction and can be written as follows

$$ z_x = \frac{\partial \bar{z}}{\partial x} + \cos(\alpha) \left( \frac{H}{L_s} \right) \quad (A.6a) $$

$$ z_y = \frac{\partial \bar{z}}{\partial y} + \sin(\alpha) \left( \frac{H}{L_s} \right) \quad (A.6b) $$

In which $F_{li}$ is related to the drag force and is defined as:

$$ F_{li} = \xi F_d \quad (A.7) $$

The components of the friction force are:
\[ F_{bx} = \frac{v_x}{v} F_b \text{ and } F_{by} = \frac{v_y}{y} F_b \]  
\text{(A.8)}

These forces can be combined in the momentum balance equations in x- and y-direction as follows:

\[ F_d \frac{U_x - v_x}{\Delta U} - F_{g/x} - F_{bx} \frac{v_x}{v} = 0 \]  
\text{(A.9a)}

\[ F_d \frac{U_y - v_y}{\Delta U} - F_{g/y} - F_{by} \frac{v_y}{v} = 0 \]  
\text{(A.9b)}

And approximation of the drag force can be made by ignoring the transverse components:

\[ F_d = F_{g/x} + F_b \]  
\text{(A.10)}

\subsection{A.2 Derivation}

Equation A.10 can be rewritten as

\[ F_d = Qz_x + Q\mu - \mu \xi F_d \text{ or } F_d = \frac{Qz_x + Q\mu}{1 + \mu \xi} \]  
\text{(A.11)}

Where

\[ Q = (\rho_s - \rho) \frac{\pi}{6} g D^3 \frac{1}{\sqrt{1 + z_x^2 + z_y^2}} \]  
\text{(A.12)}

Equations A.9a and A.9b can now be written as

\[ \left( \frac{U_x - v_x}{\Delta U} \right) \frac{Qz_x + Q\mu}{1 + \mu \xi} - F_{g/x} = F_b \frac{v_x}{v} \]  
\text{(A.13a)}

\[ \left( \frac{U_y - v_y}{\Delta U} \right) \frac{Qz_x + Q\mu}{1 + \mu \xi} - F_{g/y} = F_b \frac{v_y}{v} \]  
\text{(A.13b)}

\[ \left( \frac{F_b}{v} \right) v_x = \left( \frac{U_x - v_x}{\Delta U} \right) \frac{Qz_x + Q\mu}{1 + \mu \xi} - Qz_x \]  
\text{(A.14a)}

\[ \left( \frac{F_b}{v} \right) v_y = \left( \frac{U_y - v_y}{\Delta U} \right) \frac{Qz_x + Q\mu}{1 + \mu \xi} - Qz_y \]  
\text{(A.14b)}

This can be further rewritten to

\[ \left( \frac{\Delta U F_b}{vQ} \right) v_x = (U_x - v_x) p^2 - \Delta U z_x \]  
\text{(A.15a)}

\[ \left( \frac{\Delta U F_b}{vQ} \right) v_y = (U_y - v_y) p^2 - \Delta U z_y \]  
\text{(A.15b)}
or

\[
\left( \frac{\Delta U_v}{F_b Q} + p^2 \right) v_x = U_x p^2 - \Delta U z_x \tag{A.16a}
\]

\[
\left( \frac{\Delta U_v}{F_b Q} + p^2 \right) v_y = U_y p^2 - \Delta U z_y \tag{A.16b}
\]

Where

\[ p = \sqrt{\frac{z_x + \mu}{1 + \mu \xi}} \approx \sqrt{\frac{H/L_s + \mu}{1 + \mu \xi}} \tag{A.17} \]

The direction of the sediment transport \( \delta_{ps} \) can now be calculated as follows

\[
\tan(\delta_{ps}) = \frac{U_y p^2 - \Delta U z_y}{U_x p^2 - \Delta U z_x} = \frac{(U_y/U)p^2 - (\Delta U/U)z_y}{(U_x/U)p^2 - (\Delta U/U)z_x} = \frac{p^2 \sin(\delta_f) - G [\partial \bar{z}/\partial y] - \sin \alpha (H/L_s)}{p^2 \cos(\delta_f) - G [\partial \bar{z}/\partial x] - \cos \alpha (H/L_s)} \tag{A.18}
\]

Where

\[
G = \frac{\Delta U}{U} = \frac{\Delta U}{\sqrt{U_x^2 + U_y^2}} = \sqrt{\frac{H/L_s + \mu}{1 + \xi \mu}} \frac{U_{cr}}{\sqrt{U_x^2 + U_y^2}} = p \sqrt{\frac{\theta_{cr}}{\theta}} \tag{A.19}
\]

The factor \( p^2 \) is not present in the formula given by Sieben & Talmon [2011].
B Convergence Pseudo-time Iteration

In this appendix the convergence of the pseudo time iteration scheme described in section 4.2.1 is investigated. In this scheme the value of \( \alpha \) is iteratively calculated as follows.

\[
\alpha_{i,j}^{k+1} = (RHS - LHS) \Delta \tau \\
\Delta \tau = \sigma \left( \frac{\Delta x}{|u|} + \frac{\Delta y}{|v|} \right)
\]

The convergence is influenced by the value of \( \sigma \). The maximum allowable \( \sigma \) for which convergence still occurs is called \( \sigma_{\text{max}} \). The depence of this \( \sigma_{\text{max}} \) on different variables is investigated in this appendix.

B.1 Method

The effects of 4 different variables on the maximum allowable \( \sigma \) have been tested. These variables are: The gradient in \( S/H \) as \( \Delta x, y \times RHS_{SH} \), the direction of the bed load transport \( \delta_{ps} \), the gradient in the bed load transport direction \( \partial \delta_{ps} / \partial x, y \), and the \( \alpha \)'s in the surrounding grid cells. \( RHS_{SH} \) is the sum of \( RHS_H \) and \( RHS_S \). The gradients in x- and y-direction have been set equal to each other, as well as the \( \alpha \)'s in different surrounding grid cells. This is done to keep the number of variables manageable.

It is assumed that the method converges if the slope of the term \( (RHS - LHS) \Delta \tau \) is between -1 and 1 at the final solution. This does not guarantee convergence, but it should be a good indication. \( \sigma_{\text{max}} \) is then defined as the \( \sigma \) for which the term has a slope of 1 at this point.

To calculate this \( \sigma_{\text{max}} \), first the solution is determined with the bisection method. Next the gradient is calculated with \( \sigma = 1 \). Because the gradient of \( (RHS - LHS) \Delta \tau \) depends linearly on \( \sigma \), the \( \sigma_{\text{max}} \) for which the gradient is equal to 1 can easily be computed via linear interpolation.

B.2 Results

Figure B.1 shows the maximum \( \sigma \) for different values of \( \Delta x, y \times RHS_{SH} \) and \( \delta_{ps} \). The values for \( \partial \delta_{ps} / \partial x, y \), and the \( \alpha \)'s in the surrounding grid cells have both been set to 0. It is clear that for values of \( \pm \pi/2 \) and 0 for \( \delta_{ps} \), the \( \sigma_{\text{max}} \) goes towards 0 and has maxima near \( \pm \pi/4 \). The values of these maxima are affected by the term \( \Delta x, y \times RHS_{SH} \).

In figure B.2, the effect of a non-zero value for \( \partial \delta_{ps} / \partial x, y \) and surrounding \( \alpha \)'s is shown. In B.2a, the values of surrounding \( \alpha \)'s is set to 0.5 rad. On the left side of the graph \( \sigma_{\text{max}} \) is equal to zero. This is because the formula has no solutions here. For negative values of \( \delta_{ps} \) the \( \sigma_{\text{max}} \) seems slightly higher, while for positive values \( \sigma_{\text{max}} \) seems slightly smaller. In B.2b the values for \( \partial \delta_{ps} / \partial x, y \) are set to 0.2, this results in slightly smaller values of \( \sigma_{\text{max}} \), and the maxima occur at a slightly higher value of \( \Delta x, y \times RHS_{SH} \).
Figure B.1: The maximum $\sigma$ for different values of $\Delta x, \delta x y \times RHS_{SH}$ and $\delta_{ps}$. The values for $\partial \delta_{ps}/\partial x, y$, and the $\alpha$’s in the surrounding grid cells have both been set to 0.

Figure B.2: The maximum $\sigma$ for different values of $\Delta x, \delta x y \times RHS_{SH}$ and $\delta_{ps}$.
Figure B.3 shows $\sigma_{\text{max}}$ for different values of surrounding $\alpha$’s and $\delta_{\text{ps}}$. The values for $\partial \delta_{\text{ps}}/\partial x, y$, and $\Delta x, y \times \text{RHS}_{SH}$ have both been set to 0. In the top left and bottom right of the plot, $\sigma_{\text{max}}$ is zero because the formula has no solution here. There is no effect of the surrounding $\alpha$’s.

In figure B.4 the effect of non-zero $\partial \delta_{\text{ps}}/\partial x, y$, and $\Delta x, y \times \text{RHS}_{SH}$ on the relation between $\alpha$’s, $\delta_{\text{ps}}$ and $\sigma_{\text{max}}$ is shown. In figure B.4a the value for $\Delta x, y \times \text{RHS}_{SH}$ is set to 2. In figure B.4b the value for $\partial \delta_{\text{ps}}/\partial x, y$ is set to 0.2. With these values there is now a effect of the $\alpha$’s on $\sigma_{\text{max}}$. The maximum $\sigma_{\text{max}}$ changes for different $\alpha$’s. The $\delta_{\text{ps}}$ still has the largest effect.

Figure B.5 shows the $\sigma_{\text{max}}$ for different values of surrounding $\partial \delta_{\text{ps}}/\partial x, y$ and $\delta_{\text{ps}}$. The values for $\alpha$’s, and $\Delta x, y \times \text{RHS}_{SH}$ have both been set to 0. The effect of $\delta_{\text{ps}}$ on the $\sigma_{\text{max}}$ is again clear. There is also an effect of $\partial \delta_{\text{ps}}/\partial x, y$ on the $\sigma_{\text{max}}$. This effect is largest for positive values of $\delta_{\text{ps}}$. Here the $\sigma_{\text{max}}$ goes from 0.8 at $\partial \delta_{\text{ps}}/\partial x, y = -0.5$, to 0.3 for $\partial \delta_{\text{ps}}/\partial x, y = 0.5$. For a negative value of $\delta_{\text{ps}}$, the change of $\sigma_{\text{max}}$ is much smaller.
Figure B.5: The maximum $\sigma$ for different values of surrounding $\partial \delta ps/\partial x, y$ and $\delta ps$. The values for $\alpha$’s, and $\Delta x, y \times RHS_{SH}$ have both been set to 0.

In figure B.6 the effect of non-zero $\alpha$’s, and $\Delta x, y \times RHS_{SH}$ on the relation between $\partial \delta ps/\partial x, y$, $\delta ps$ and $\sigma_{\text{max}}$ is shown. In figure B.6a the value for surrounding $\alpha$’s is set to 0.5. This affects mostly the $\sigma_{\text{max}}$ for negative $\delta ps$. There is now a larger difference in $\sigma_{\text{max}}$ for different values of $\partial \delta ps/\partial x, y$, similar to the difference for positive $\delta ps$. In figure B.6b the value for $\Delta x, y \times RHS_{SH}$ is set to 2. This causes a larger difference in $\sigma_{\text{max}}$ for different $\partial \delta ps/\partial x, y$ as well as higher values overall. For positive $\delta ps$ the $\sigma_{\text{max}}$ is reduced.

Figure B.6: The maximum $\sigma$ for different values of $\partial \delta ps/\partial x, y$ and $\delta ps$.

B.3 Conclusion

The value of $\sigma_{\text{max}}$ depends on all variables. The clearest effect is that of the $\delta ps$. This is because $\Delta \tau$ goes to infinity when $\delta ps$ is a multiple of $\pi/2$. The gradients also have a large influence of the $\sigma_{\text{max}}$. The value of the $\alpha$’s of surrounding grid cells have an effect, but it is smaller than the effects of the other variables, and is only present for non-zero values of $\Delta x, y \times RHS_{SH}$ or $\partial \delta ps/\partial x, y$. Because all the
variables have an influence on the $\sigma_{\text{max}}$ it is difficult to give clear relations. This makes it hard to assign the right value to $\sigma$. 
C Convergence Predictor Corrector

In this appendix the convergence of the predictor corrector scheme, and its dependence on different variables is tested. The predictor corrector scheme is described in section 4.2.2. In this scheme the $\alpha$ is iteratively determined as follows.

$$
\alpha_{i,j}^{k+1} = \begin{cases} 
\left( RHS_S^k + RHS^k_S + RHS^k_H + u^{k}_{\alpha} \frac{\alpha_{i-1,j} - \alpha_{i,j}}{\Delta x} + v^{k}_{\alpha} \frac{\alpha_{i,j+1} - \alpha_{i,j}}{\Delta y} \right) / \left( \frac{u^k}{\Delta x} + \frac{v^k}{\Delta y} \right) & \text{for } u_{\alpha}^k \geq 0, v_{\alpha}^k \geq 0 \\
\left( RHS_S^k + RHS^k_S + RHS^k_H - u^{k}_{\alpha} \frac{\alpha_{i+1,j} - \alpha_{i,j}}{\Delta x} + v^{k}_{\alpha} \frac{\alpha_{i,j+1} - \alpha_{i,j}}{\Delta y} \right) / \left( \frac{u^k}{\Delta x} - \frac{v^k}{\Delta y} \right) & \text{for } u_{\alpha}^k < 0, v_{\alpha}^k \geq 0 \\
\left( RHS_S^k + RHS^k_S + RHS^k_H + u^{k}_{\alpha} \frac{\alpha_{i-1,j} - \alpha_{i,j}}{\Delta x} - v^{k}_{\alpha} \frac{\alpha_{i,j+1} - \alpha_{i,j}}{\Delta y} \right) / \left( -\frac{u^k}{\Delta x} - \frac{v^k}{\Delta y} \right) & \text{for } u_{\alpha}^k \geq 0, v_{\alpha}^k < 0 \\
\left( RHS_S^k + RHS^k_S + RHS^k_H - u^{k}_{\alpha} \frac{\alpha_{i+1,j} - \alpha_{i,j}}{\Delta x} - v^{k}_{\alpha} \frac{\alpha_{i,j+1} - \alpha_{i,j}}{\Delta y} \right) / \left( -\frac{u^k}{\Delta x} - \frac{v^k}{\Delta y} \right) & \text{for } u_{\alpha}^k < 0, v_{\alpha}^k < 0
\end{cases} \tag{C.1}
$$

C.1 Method

The effects of 4 different variables on convergence have been tested. These variables are the gradient in $S/H$ as $\Delta x,y \times RHS_{SH}$, the direction of the bed load transport $\delta_{ps}$, the gradient in the bed load transport direction $\partial\delta_{ps}/\partial x,y$, and the $\alpha$’s in the surrounding grid cells. $RHS_{SH}$ is the sum of $RHS_H$ and $RHS_S$. The gradients in x- and y-direction have been set equal to each other, as well as the $\alpha$ in different surrounding grid cells. This is done to keep the number of variables manageable.

It is assumed that the method converges if the slope of the right hand side of equation (C.1) is between -1 and 1 at the final solution. This does not guarantee convergence, but it should be a good indication. The final solution is determined with the bisection scheme.

C.2 Results

Figure C.1 shows the whether the predictor corrector converges for different combinations of $\Delta x,y \times RHS_{SH}$ with $\delta_{ps}$, $\partial\delta_{ps}/\partial x,y$, and surrounding $\alpha$’s. The green part indicates that the method converges. The red part indicates that the method does not converge. White parts indicate that no solution could be found.

There is a large influence of the term $\Delta x,y \times RHS_{SH}$ on the convergence of the predictor corrector method. The method seems to converge when $\Delta x,y \times RHS_{SH}$ lies between a certain upper and lower boundary. The upper and lower boundary are influenced by all the variables. The upper boundary lies between 1 and 2, while the lower boundary lies between -1 and -2.
Figure C.1: The influence of $\Delta x, y \times RHS_{SH}$ with $\delta_{ps}$ (a), $\partial \delta_{ps} / \partial x, y$ (b), and surrounding $\alpha$'s (c) on the convergence of the predictor corrector method. The green part indicates that the method converges. The red part indicates that the method does not converge. White parts indicate that no solution could be found.
Figure C.2 shows the convergence of the predictor corrector method for different combinations of $\delta_{ps}$ and $\partial \delta_{ps}/\partial x, y$. The values of the surrounding values of $\alpha$ are set to 0. $\Delta x, y \times RHS_{SH}$ is set to 0 in figure C.2a and to 1 in figure C.2b. It can be seen in figure C.2a that for $\Delta x, y \times RHS_{SH}$ equal to zero, the method converges for most combinations. Only small red parts can be seen for high values of $\partial \delta_{ps}/\partial x, y$. When we set $\Delta x, y \times RHS_{SH}$ to 1, the area where no convergence takes place increases for negative values of $\partial \delta_{ps}/\partial x, y$.

Figure C.3 shows the convergence of the predictor corrector method for different combinations of $\delta_{ps}$ and surrounding $\alpha$’s. The value of $\partial \delta_{ps}/\partial x, y$ is set to 0. $\Delta x, y \times RHS_{SH}$ is set to 0 in figure C.3a and to 2 in figure C.3b. It can be seen in figure C.3a that for $\Delta x, y \times RHS_{SH}$ equal to zero, the method converges for all the tested combinations. Even for a value of 1 this is the case. When we set $\Delta x, y \times RHS_{SH}$ to 2, in the upper part of the plot there is no more convergence.

Figure C.4 shows the convergence of the predictor corrector method for different combinations of $\partial \delta_{ps}/\partial x, y$ and surrounding $\alpha$’s. The value of $\delta_{ps}$ is set to 0. $\Delta x, y \times RHS_{SH}$ is set to 0 in figure C.4a and to 1 in figure C.4b. In figure C.4a it can be seen that for most combinations the method converges. Only for values of $\partial \delta_{ps}/\partial x, y$ higher than 0.7 does the method not converge for certain values of surrounding $\alpha$’s. When we set $\Delta x, y \times RHS_{SH}$ to 1 in figure C.4b, the area with no convergence increases for negative values of $\partial \delta_{ps}/\partial x, y$ and decreases for positive values of $\partial \delta_{ps}/\partial x, y$. 


Figure C.3: the convergence of the predictor corrector method for different combinations of $\delta_{ps}$ and $\partial \delta_{ps} / \partial x, y$. The values of the surrounding values of $\alpha$ are set to 0. The red part indicates that the method does not converge. White parts indicate that no solution could be found.

Figure C.4: the convergence of the predictor corrector method for different combinations of $\partial \delta_{ps} / \partial x, y$ and surrounding $\alpha$’s. The red part indicates that the method does not converge. White parts indicate that no solution could be found.
The convergence of the predictor corrector method depends mostly on the size of the term $\Delta x, y \times RHS_{SH}$. This term should be between a certain upper and lower boundary. The lower boundary usually lies between -1 and -2, while the upper boundary usually lies between 1 and 2. The direction of the bed load transport $\delta_{ps}$, The gradient in the bed load transport direction $\partial \delta_{ps}/\partial x, y$, and the $\alpha$’s in the surrounding grid cells influence the location of these boundaries.

While the convergence mostly depends on the term $\Delta x, y \times RHS_{SH}$. It can be seen in Figures C.2a and C.4a that when $\Delta x, y \times RHS_{SH}$ is 0, there can still be no convergence. This occurs for large values of $\partial \delta_{ps}/\partial x, y$. 

C.3 Conclusions