

# Railway Timetable Stability Analysis Using Stochastic Max-Plus Linear Systems

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## Abstract

Stability and robustness of a railway timetable are essential properties for punctual and reliable operations. Timetable performance evaluation is therefore an important aspect in the timetable design process. In particular, the stability and recoverability properties of a timetable with respect to daily process time variations must be well analysed. The timetable must be able to recover from primary delays due to stochastic process times and it must be robust against secondary delays due to train interactions. This paper presents a stability analysis approach based on stochastic max-plus linear system theory. Stochastic counterparts of well-established concepts from the deterministic max-plus stability analysis are proposed, like timetable stability and realizability. General probability distributions can be used to model the primary stochastic behaviour of process times, while delay propagation due to timetable and infrastructure constraints are computed from the stochastic recursive system equations. Recently developed powerful algorithms can be utilized to analyse and improve large-scale stochastic systems, and to establish the amount of stochastic variations that a timetable can absorb without external control.

## Keywords

Railway timetable, Periodic timetables, Stability analysis, Max-plus algebra, Stochastic event graph

## 1 Introduction

Timetable stability is rapidly gaining attention due to the increasingly saturated European railway infrastructure, where a slightly delayed train may cause a domino effect of consecutive delays over the entire network. From a stability point of view, a railway timetable must be insensitive with respect to small disturbances so that it can recover from small disruptions without external control. This self-regulating timetable behaviour after disruptions requires a careful distribution of recovery times and buffer times to reduce delays and prevent delay propagation, respectively. Railway timetabling models and methods are usually based on deterministic process times (running times, dwell times, transfer times, headway times, etc.). Moreover, running times are rounded and train paths are bended to fit timetable or infrastructure constraints. The validity of these timetabling decisions and simplifications

must be evaluated to guarantee feasibility, stability, and robustness, with respect to network interrelations and variations in process times.

The essential structure of railway traffic operating under a periodic railway timetable can be modelled as a (stochastic) linear system in max-plus algebra, where the process times are stochastic variables with specified distributions. Stability can be assessed by computing the so-called Lyapunov exponent of the associated stochastic state matrix containing all process times. This Lyapunov exponent is equivalent to the expected cycle time of all events in the system. Thus, a stable timetable requires that the Lyapunov exponent is smaller than the timetable period length.

In a deterministic setting, stability can be assessed by computing the eigenvalue of the state matrix in max-plus algebra [7, 8, 12]. This eigenvalue is the minimum cycle time required to satisfy all timetable and headway constraints, where a timetable operating with this minimum cycle time is given by the associated eigenvector. Thus, if the eigenvalue exceeds the intended timetable period length (e.g. an hour) then the timetable is unstable. When the process times (running times, dwell times, etc.) are stochastic the model becomes a stochastic max-plus linear system. The stochastic counterpart of the deterministic max-plus eigenvalue is the Lyapunov exponent which equals the expected cycle time of the system. If all process times are deterministic then the Lyapunov exponent is equal to the deterministic eigenvalue. The Lyapunov exponent can be used to assess stability taking into account the stochastic variation of the process times. Efficient algorithms have been developed for the computation of the Lyapunov exponent for deterministic matrices [5] and also existence theorems of Lyapunov exponents for random matrices are well-known [1, 6, 13, 14]. However, computing the Lyapunov exponent for stochastic matrices has been difficult due to stochastic dependencies between the events in the network which prohibit analytical solutions. Straightforward Monte Carlo simulation is also hampered by large computation times because the coupling time of a sample path before reaching the stationary regime can be extremely large. Recently, Goverde *et al.* [9] developed an effective perfect simulation algorithm to compute an unbiased estimate of the Lyapunov exponent for random matrices. Together with sparse matrix computations the algorithm is able to approximate the Lyapunov exponent for large-scale networks very fast. Hence, it is now feasible to extend the deterministic max-plus stability analysis approach as implemented in PETER [10] to stochastic systems as explained in this paper.

The next section introduces the macroscopic modelling approach of railway traffic systems as stochastic max-plus linear systems. Section 3 presents the ergodic theory of stochastic max-plus linear systems, which is applied to a stability analysis approach in Section 4. An example network illustrates the presented methodology in Section 5, after which the paper ends with some conclusions.

## 2 Stochastic Max-Plus Linear Systems

Max-plus timetable stability analysis is based on a macroscopic model of the scheduled railway traffic system with an emphasis on the synchronization of trains. Stability depends on timetable constraints, interconnection structure, infrastructure usage, rolling stock circulations, and the distribution of slack times and buffer times over the network. The variables of interest are the train event times at stations, which are connected by activities (train runs, stops, transfers, etc.). We assume a periodic timetable where each event repeats at a regular interval called the cycle time  $T$ , which is usually 60 minutes. Train lines with a frequency

of more than once per  $T$  minutes are modelled by separate train lines with equal characteristics but schedules that are shifted by a multiple of the line cycle time. A periodic railway timetable defines the scheduled arrival and departure times within a basic period of length  $T$  for each periodic train line at all served stations. Also the through times at stations where the trains do not stop are given, as well as the scheduled through times at e.g. junctions of merging railway lines, movable bridges, and any other ‘timetable points’ in the network where a minimum headway time has to be satisfied corresponding to a safe separation distance on conflicting train routes.

## 2.1 A Dynamic Railway Traffic Model

A max-plus linear system is a discrete-event dynamic system where the dynamics are driven by the occurrences of events and the state variables are the event times. In particular, max-plus linear systems describe the evolution of sequential and synchronized processes, like train runs and stops of individual train lines and transfer connections at stations where train lines meet, respectively. We consider periodic railway timetables corresponding to train circulations over train lines operating at regular intervals. Denote by  $x_i(k)$  the event time of the  $k$ th occurrence of event  $i$ . The vector  $x(k) = (x_1(k), \dots, x_n(k))'$  denotes the  $k$ th occurrence times of all events in the system. In railway traffic the events are departures from stations, arrivals at stations, and passages of stations and other ‘timetable points’ in the network where trains have no scheduled stop but must keep a safe separation distance on conflicting routes.

The event times are connected to each other by activities or processes. An event can only occur if all its preceding processes have completed. For instance, a train departure can occur only if the train has arrived and the dwell process has been completed, a feeder train has arrived and transferring passengers have boarded the train, and conflicting routes of preceding trains have been released. Since an event  $i$  must wait for the completion of all preceding processes, the earliest event time is given by

$$x_i(k) = \max_j (a_{ij}(k) + x_j(k - \mu_{ij})),$$

where the maximum is taken over all predecessors  $j$  of  $i$ ,  $a_{ij}(k)$  is the (stochastic) process time from event  $j$  to  $i$ , and  $\mu_{ij}$  is the *period shift* denoting whether both events  $i$  and its predecessor  $j$  are scheduled in the same period ( $\mu_{ij} = 0$ ), or whether the preceding event  $j$  is scheduled in a previous period ( $\mu_{ij} \geq 1$ ). Thus, if  $\mu_{ij} = 0$  then the scheduled process from  $j$  to  $i$  falls entirely within the same period, while for  $\mu_{ij} = 1$  the process crosses a period boundary. In general, a process may also cross two or more periods if the scheduled process time exceeds the basic period length. However, such a process can easily be split into one or more auxiliary events with unit period shift. Thus, without loss of generality, we assume that  $\mu_{ij} \in \{0, 1\}$  for all processes from  $j$  to  $i$ . The process times may vary over the different periods and therefore depend on the occurrence period  $k$ .

Defining  $a_{ij}(k) = -\infty$  for all event pairs  $(j, i)$  that are not directly connected by a process from  $j$  to  $i$ , we can write

$$x_i(k) = \max (a_{i1}(k) + x_1(k - \mu_{i1}), \dots, a_{in}(k) + x_n(k - \mu_{in})) \quad (1)$$

for all  $1 \leq i \leq n$ . In general, multiple parallel processes may be defined between two events with possibly different period shifts. We will assume that only single processes occur for notational convenience but this can be relaxed easily.

For a given initial condition  $x(0) = x_0$ , (1) defines a stochastic recursive sequence  $\{x(k)\}_{k \in \mathbb{N}}$  depending on the random parameters  $a_{ij}(k)$ . This dynamic equation has a special form that fits the framework of *max-plus algebra*.

## 2.2 Max-Plus Algebra

The recursive equation (1) of the successive event times is nonlinear but becomes linear in the so-called *max-plus algebra* [2, 12]. Note that in (1) only the operations  $\max$  and  $+$  appear. Max-plus algebra is defined on the real numbers extended with  $\varepsilon := -\infty$ , with the two operations defined as

$$\begin{aligned} a \oplus b &:= \max(a, b) \\ a \otimes b &:= a + b \end{aligned}$$

for all  $a, b \in \mathbb{R} \cup \{\varepsilon\}$ . This algebraic structure is an idempotent semiring<sup>4</sup>, or dioid, denoted as  $\mathbb{R}_{\max} = (\mathbb{R} \cup \{\varepsilon\}, \oplus, \otimes)$ . Note that  $\varepsilon$  is the ‘zero’ element for addition, since  $a \oplus \varepsilon = \varepsilon \oplus a = a$  for all  $a \in \mathbb{R}_{\max}$ . The ‘unit’ element is  $e = 0$ , because  $a \otimes e = e \otimes a = a$ . A power in max-plus algebra is defined as  $a^{\otimes l} := a \otimes \dots \otimes a = l \cdot a$  for any nonnegative integer, with  $a^{\otimes 0} = e$ . The superscript  $\otimes$  in a power indicates that the power must be understood with respect to max-plus algebra. The relevance of these definitions becomes clear when extending the operations to matrices in a straightforward way as follows. For matrices  $A = (a_{ij})$  and  $B = (b_{ij})$  of appropriate size over  $\mathbb{R}_{\max}$ , max-plus matrix addition and multiplication is defined as

$$\begin{aligned} (A \oplus B)_{ij} &:= a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij}) \\ (A \otimes B)_{ij} &:= \bigoplus_{l=1}^n (a_{il} \otimes b_{lj}) = \max_{l=1, \dots, n} (a_{il} + b_{lj}). \end{aligned}$$

Note that  $\bigoplus_{l=1}^n c_l = c_1 \oplus \dots \oplus c_n = \max_l c_l$  denotes repeated application of the ‘addition’ operation  $\oplus$ . The set of  $m \times n$  matrices over  $\mathbb{R}_{\max}$  is denoted by  $\mathbb{R}_{\max}^{m \times n}$ . The set of  $n$ -dimensional vectors over  $\mathbb{R}_{\max}$  is denoted by  $\mathbb{R}_{\max}^n$ . The unit vector is also denoted by  $e = (0, \dots, 0)' \in \mathbb{R}_{\max}^n$ , where the appropriate dimension is understood from the context.

Now define the matrices  $A_0(k)$  and  $A_1(k)$  in  $\mathbb{R}_{\max}^{n \times n}$  by

$$(A_l(k))_{ij} = \begin{cases} a_{ij}(k) & \text{if } \mu_{ij} = l \\ \varepsilon & \text{otherwise.} \end{cases}$$

Then we can write the nonlinear recursive equations (1) in max-plus algebra as

$$x(k) = A_0(k) \otimes x(k) \oplus A_1(k) \otimes x(k-1),$$

where  $\{A_0(k), A_1(k)\}$  is a sequence of matrices in  $\mathbb{R}_{\max}^{n \times n}$ . If an initial condition  $x(0) = x_0 \in \mathbb{R}^n$  is given, we obtain the *stochastic max-plus linear system*

$$x(k) = A_0(k) \otimes x(k) \oplus A_1(k) \otimes x(k-1), \quad x(0) = x_0. \quad (2)$$

Note that  $A_1(k)$  contains the process times of all processes starting in period  $k-1$  and crossing the period border into period  $k$ , and  $A_0(k)$  contains all process times of processes

scheduled entirely within period  $k$ . A max-plus linear system (2) is equivalent to a subclass of stochastic Petri nets called *stochastic event graphs*, see Baccelli *et al.* [2] for more details.

In the sequel, we need some terminology from probability theory applied to max-plus algebra. A property of a random variable holds *almost surely (a.s.)* if it holds with probability 1. A random matrix  $A \in \mathbb{R}_{\max}^{n \times n}$  is said to have a *fixed support* if each entry is either a.s. equal to  $\varepsilon$  or a.s. finite. A random matrix sequence  $\{A(k)\}$  is said to have fixed support if each  $A(k)$  has fixed support and the position of a.s. finite elements is independent of  $k$ . A random matrix  $A \in \mathbb{R}_{\max}^{n \times n}$  with finite support is called *integrable* if  $\mathbb{E}[|a_{ij}|]$  is finite for all elements in the support of  $A$ ,  $\text{supp}(A) := \{(i, j) | a_{ij} \neq \varepsilon \text{ a.s.}\}$ . A random matrix  $A \in \mathbb{R}_{\max}^{n \times n}$  is a.s. *irreducible* if it has fixed support and the associated precedence graph  $G(A) = (V, E)$  is strongly connected, with  $V = \{1, \dots, n\}$  and  $E = \{(j, i) | a_{ij} \neq \varepsilon \text{ a.s.}\}$ . Likewise, a sequence  $\{A(k)\}$  is irreducible if it has fixed support and the precedence graph  $G(A(1))$  is strongly connected. A sequence  $\{A(k)\}$  is *independent and identically distributed (i.i.d.)* if all  $A(k)$  have the same probability distribution and all are mutually independent. Note that in an i.i.d. sequence of random matrices the entries within each matrix can be differently distributed and jointly dependent.

The interconnection structure (or topology) of a system (2) can be visualized by a (marked) digraph with nodes 1 to  $n$  and an arc  $(j, i)$  if  $(A_l(1))_{ij} \neq \varepsilon$  almost surely. The period shifts can be depicted in this graph by drawing a token next to arc  $(j, i)$  if  $\mu_{ij} = 1$  and no token if  $\mu_{ij} = 0$ , see Figure 1 in Section 5. The token distribution over the arcs is also called the initial marking of the marked graph. The precedence graphs  $G(A_0(1))$  and  $G(A_1(1))$  are subgraphs of this marked graph corresponding to the arcs with zero or one token, respectively.

The random matrices  $A_0(k)$ ,  $k \geq 1$ , must be acyclic because otherwise the system would be deadlocked. Acyclic  $A_0(k)$  means that the precedence graph  $G(A_0(k)) = (V, E_0)$  with  $V = \{1, \dots, n\}$  and  $E_0 = \{(j, i) | a_{ij}(k) \neq \varepsilon, \mu_{ij} = 0\}$  is acyclic. Note that if there would be a circuit in this graph then the system is deadlocked: let  $i$  and  $j$  be two events on the circuit. Then there is a sequence of processes from  $j$  to  $i$  within a basic period that must be completed before event  $i$  can occur, but likewise event  $j$  waits for the completion of a process sequence starting with event  $i$  in the same period. So  $x_i(k)$  can occur only after the occurrence of  $x_j(k)$  and vice versa, which is a deadlock. A process sequence from an event  $i$  to itself can thus only occur if the sequence crosses one or more period boundaries.

A basic result in max-plus algebra is that a system of the form (2), with acyclic  $A_0(k)$ , can be formulated in its (pure) 1st-order representation

$$x(k) = A(k) \otimes x(k-1), \quad x(0) = x_0, \quad (3)$$

with  $A(k) := A_0^*(k) \otimes A_1(k)$ . Here,  $A_0^*(k)$  is the *Kleene star* of  $A_0(k)$  defined for acyclic  $A_0(k)$  as

$$A_0^*(k) := \bigoplus_{l=0}^{n-1} A_0^{\otimes l}(k), \quad (4)$$

with  $A_0^0(k) = E$  the unit matrix defined as  $E_{ii} = 0$  and  $E_{ij} = \varepsilon$  for  $j \neq i$ ,  $1 \leq i, j \leq n$ . The diagonal entries of  $A_0^*(k)$  are all 0, and for  $i \neq j$ ,  $(A_0^*(k))_{ij}$  is the weight of the critical path from  $j$  to  $i$  in the precedence graph  $G(A_0(k))$ . Note that a power  $A_0^{\otimes l} = A_0 \otimes \dots \otimes A_0$  (with  $l$  factors) is the matrix whose entries are the maximum weights of paths with length  $l$ . A path in an acyclic graph with  $n$  nodes has at most  $n - 1$  arcs (otherwise there is a cycle), and so (4) is the maximum of the path weights over paths with lengths from 0 to

$n - 1$ . In general, if  $A_1(k)$  has columns with all  $\varepsilon$ 's then so has  $A(k)$ . These columns correspond to nodes in the original marked graph that have only outgoing arcs with zero period shift, i.e., processes to events scheduled in the same period. If the marked graph associated to (2) is strongly connected then the all- $\varepsilon$  columns and associated rows can be deleted from  $A(k) = A_0^*(k) \otimes A_1(k)$  to obtain the irreducible matrix corresponding to the strongly-connected component of the precedence graph  $G(A(1))$ . In the sequel, we denote by  $A(k)$  this reduced irreducible matrix.

The successive event times  $x(k)$  are determined by the initial condition  $x_0$  and the matrices  $A(k)$  for  $k \geq 1$ . Mathematically,  $\{x(k)\}_{k \geq 0}$  is a stochastic recursive sequence determined by the matrix sequence  $\{A(k)\}_{k \geq 1}$  and  $x_0$ . We also write the solution of (2) or (3) as  $x(k; x_0)$  to emphasize the initial condition  $x(0) = x_0$ .

The system equations (2) and (3) give the *earliest* possible event times that satisfy all precedence relations. In practice, departures may not occur early, i.e., before their scheduled departure times. A timetable can be incorporated into the system equations by adding an inhomogeneous term. Define the timetable vector  $d(k)$  of the  $k$ th scheduled event times. Then the *scheduled max-plus linear system* is

$$x(k) = A_0(k) \otimes x(k) \oplus A_1(k) \otimes x(k-1) \oplus d(k), \quad x(0) = x_0, \quad (5)$$

or in the pure 1st-order representation

$$x(k) = A(k) \otimes x(k-1) \oplus d(k), \quad x(0) = x_0. \quad (6)$$

If  $d(k)$  is a periodic timetable with cycle time  $T$  and initial basic timetable  $d(0) = d_0$ , with  $d_i(0) \in [0, T) \cup \{\varepsilon\}$  for all  $1 \leq i \leq n$ , then the successive scheduled event times are given by the *deterministic* max-plus linear system  $d(k) = T \otimes d(k-1)$ , for  $k \geq 1$ , or its solution

$$d(k) = d_0 \otimes T^{\otimes k} = (d_i(0) + k \cdot T).$$

If  $d_i(0) = \varepsilon$  then event  $i$  may occur early as is common for arrivals and passages. Note that the event times  $x(k)$  generated by (5), or (6), satisfy  $x(k) \geq d(k)$ . We also denote the solution of (5), or (6), as  $x(k; x_0, d_0, T)$  to emphasize the initial condition and the periodic timetable.

### 2.3 Random Matrices and Delay Propagation

The stochasticity can be defined in various ways depending on the characteristics of the matrix sequence  $\{A(k)\}$ . Typical random matrices are:

1. **Continuous random matrices.** The matrices have fixed support and the finite elements are continuous stochastic variables with given (joint or independent) distribution functions. In this case the stochastic process times model the primary variations in the process times. The secondary delays caused by waiting on other processes are *not* included in these distributions, but are computed by the dynamic max-plus recursions.
2. **Discrete random matrices.** The matrices belong to a finite discrete set of matrices from which each matrix occurs with some given probability. This case models a system with a nominal state matrix  $A$  which is disrupted with some probability by one or more perturbation matrices. This models for instance random breakdowns in an otherwise regular system.

3. **Switching matrices.** The state matrix can switch between different modes in each period according to alternative dispatching decisions, possibly depending on large delays. This case differs from the former in that the state matrices  $A(k)$  are not selected randomly but by supervisory control actions. This is useful in model-predictive control of essentially regular max-plus linear systems. The ‘randomness’ is here the result of choices to be made by a controller.

Also a mixture of the above stochastics can be applied. For instance, a mixture of all three types of randomness defines a stochastic max-plus linear system where the process times are all continuous stochastic variables, random incidents may occur that change the interconnection structure or the distributions of the individual process times, and emergency timetables can be imposed by a controller to deal with these unexpected incidents until the nominal system behaviour restores.

The continuous stochastic systems of type one are considered in Goverde *et al.* [9]. Heidergott [11] gives an in-depth treatment of discrete stochastic systems of the second type. Van den Boom & De Schutter [15] consider the switching max-plus linear systems of type three. In the sequel of this paper, the focus will be on max-plus linear systems with continuous random state matrices with fixed support (type one).

Assume that  $\{A(k)\}_{k \in \mathbb{N}}$  is a sequence of integrable random matrices in  $\mathbb{R}_{\max}^{n \times n}$  with fixed support and  $d_0 \in [0, T]^n$  a finite vector. Then the scheduled stochastic max-plus linear system (6), or (5), models the delay propagation over time and space due to stochastic initial delays and delayed process times, including the recovery of train delays by using time reserves in the timetable. There are two types of delays: primary (or original) and secondary (or knock-on) delays. A primary delay is caused by a process that exceeds its scheduled process time. A secondary delay is caused by interaction with another train, such as a route conflict or a secured transfer. The delay of an event  $i$  scheduled in period  $k$  is denoted as

$$z_i(k) := x_i(k) - d_i(k), \quad (7)$$

with  $x(k) = x(k; x_0, d_0, T)$ . We also write  $z(k; x_0)$  to stress the initial condition. Note that  $z(k) \geq 0$  for finite  $d_0$ . We may relax the assumption of a finite  $d_0$  by defining  $z_i(k)$  as (7) for  $i \in \text{supp}(d_0)$  and  $z_i(k) = \varepsilon$  otherwise. Hence, delays are only defined for events with a scheduled event time. Delays are identified by a triple  $(i, k, z_i(k))$  of the event number, the occurrence period, and the amount of delay.

In the max-plus model delays are initiated from two sources:

1. **Initial delays.** Delays in the initial period,

$$Z_{\mathcal{I}} := \{(i, 0, z_i(0)) \mid z_i(0) = (x_0)_i - d_i(0) > 0\}.$$

2. **Primary delays.** Delays caused by exceeding scheduled process times,

$$Z_{\mathcal{P}} := \{(i, k, z_i(k)) \mid z_i(k) = d_j(k - \mu_{ij}) + a_{ij}(k) - d_i(k) > 0, k \geq 1\}.$$

Initially and primary delayed trains may cause secondary delays, which are computed from the max-plus system equations. Moreover, secondary delays may further generate more secondary delays. We distinguish between two types of secondary delays:

3. **Consecutive delays.** Existing, possibly partially recovered, train delays,

$$Z_{\mathcal{C}} := \{(i, k, z_i(k)) \mid z_i(k) > 0, z_j(k - \mu_{ij}) > 0, a_{ij}(k) \text{ running or dwell time}, k \geq 1\}.$$

4. **Knock-on delays.** Delays caused by interaction with other trains,

$$Z_{\mathcal{K}} := \{(i, k, z_i(k)) \mid z_i(k) > 0, k \geq 1\} \setminus (Z_{\mathcal{P}} \cup Z_{\mathcal{C}}).$$

A delay may originate from different sources. For instance, a consecutive delay may increase by an additional primary delay. In this case, the delay is partially primary and partially consecutive. Also a primary delay or consecutive delay may be superseded by a larger knock-on delay from another train.

The dynamic system behaviour can be analysed by simulating the max-plus linear systems. However, the essential system behaviour can also be characterized by application of the ergodic theory of stochastic max-plus linear systems as considered in the next two sections. This is the main advantage of the max-plus modelling.

### 3 Ergodic Theory

In this section we consider stochastic systems (3) of the successive earliest event times, so without timetable. We show that the mean behaviour of these systems is generally eventually periodic. This property is useful in the design of periodic timetables and for stability analysis of existing periodic timetables. The connection to the stability of scheduled systems will be considered in the next section.

The sequence  $\{x(k)\}$  defined by the max-plus linear system (3) is monotonously non-decreasing but reaches a regular behaviour when we look at the (mean) interval between event occurrences,  $\mathbb{E}[x_i(k)] - \mathbb{E}[x_i(k-c)]$  for some integer  $c \geq 1$ . Since  $x(k-c)$  has the same law as the (shifted) sequence  $x^c(k)$  defined by the recursive equation (3) for  $k \geq c$  and initial condition  $x^c(c) = x_0$ , we will investigate

$$\lim_{k \rightarrow \infty} \mathbb{E}[x_i(k)] - \mathbb{E}[x_i(k-c)] = \lim_{k \rightarrow \infty} \mathbb{E}[x_i(k) - x_i^c(k)] \quad (8)$$

for some positive integer  $c$  and all  $1 \leq i \leq n$ . Note that if the limit converges for some  $c$  then it also converges for any integral multiple of  $c$ . Whenever it exists, the limit (8) is equal to  $c$  times the (asymptotic) *cycle time*, defined as

$$\lim_{k \rightarrow \infty} \frac{x_i(k)}{k}$$

for each event  $i$ . If the latter limits exist for all  $1 \leq i \leq n$  and all have the same value then this value is called the *Lyapunov exponent* of the sequence  $\{A(k)\}$ , denoted by  $\lambda$ . It is independent of the initial condition as will be seen in the theorems below.

The following theorem is a fundamental result of max-plus ergodic theory [6, 11, 14] and fundamental to our stability theory.

**Theorem 1** *Let  $\{A(k)\}_{k \in \mathbb{N}}$  be an i.i.d. sequence of integrable and irreducible random matrices in  $\mathbb{R}_{\max}^{n \times n}$ , and  $\{x(k; x_0)\}$  the stochastic recursive sequence defined by (3). Then there is a real constant  $\lambda \in \mathbb{R}$  such that for any (finite) initial condition  $x_0 \in \mathbb{R}^n$ ,*

$$\lim_{k \rightarrow \infty} \frac{x_i(k; x_0)}{k} = \lambda \quad \text{a.s.}$$

for all  $1 \leq i \leq n$ . If the initial condition  $x_0$  is integrable then also for all  $1 \leq i \leq n$ ,

$$\lim_{k \rightarrow \infty} \frac{\mathbb{E}[x_i(k; x_0)]}{k} = \lambda.$$



Note that the second limit holds for random initial conditions corresponding to random delays present at some initial reference time.

In the special case of deterministic max-plus linear systems with irreducible  $A(k) = A \in \mathbb{R}_{\max}^{n \times n}$  for all  $k \geq 1$ ,

$$x(k) = A \otimes x(k-1), \quad x(0) = x_0,$$

it is well-known that the event times reach a periodic regime independent of the initial condition, i.e.,  $x(k) = \lambda^{\otimes c} \otimes x(k-1)$  for some  $c$  and sufficiently large  $k$ , where  $\lambda$  is the unique *eigenvalue* defined by the eigenvalue problem

$$A \otimes v = \lambda \otimes v.$$

Here,  $v$  is an eigenvector associated with the eigenvalue  $\lambda$ . If the initial condition is given by the eigenvector,  $x_0 = v$ , then the system is immediately periodic with cycle time  $\lambda$  since  $x(k) = A^{\otimes k} \otimes v = \lambda^{\otimes k} \otimes v$  for all  $k \geq 1$ . Thus, the eigenvector is an optimal steady-state timetable with (minimal) cycle time  $\lambda$ . Moreover, the eigenvalue of a deterministic irreducible matrix  $A \in \mathbb{R}_{\max}^{n \times n}$  equals the maximum cycle mean

$$\lambda = \max_{\xi \in C} \frac{w(\xi)}{\mu(\xi)}, \quad (9)$$

where  $C$  is the set of all circuits in the precedence graph  $G(A)$ ,  $w(\xi)$  is the sum of all process times in circuit  $\xi$  (the circuit weight), and  $\mu(\xi)$  is the number of periods covered by circuit  $\xi$  (the sum of period shifts). Also note that for  $x_0 = v$ , with  $v$  an eigenvector associated with  $\lambda$ ,

$$\lim_{k \rightarrow \infty} \frac{x_i(k; v)}{k} = \lim_{k \rightarrow \infty} \frac{\lambda^{\otimes k} \otimes v_i}{k} = \lim_{k \rightarrow \infty} \frac{k \cdot \lambda + v_i}{k} = \lim_{k \rightarrow \infty} \lambda + \frac{v_i}{k} = \lambda.$$

Thus, Theorem 1 generalizes the spectral theory of deterministic max-plus linear systems to stochastic systems. For more details on deterministic max-plus linear systems, see e.g. Baccelli *et al.* [2] or Heidergott *et al.* [12].

We now return to the 2nd order limit (8) of event time differences. First, we give a formal definition of the cyclicity  $c$ . The *cyclicity*  $c$  of an irreducible matrix  $A \in \mathbb{R}_{\max}^{n \times n}$  is the least integer  $c$  such that  $A^{\otimes l+c} = \lambda^{\otimes c} \otimes A^{\otimes l}$  for all  $l \geq \eta$ , where  $\lambda$  is the eigenvalue and  $\eta$  the coupling time of  $A$ . Moreover,  $c$  equals the least common divisor (l.c.d.) of the lengths of all (critical) circuits in the critical graph  $G^c(A)$  consisting of all circuits having maximum cycle mean  $\lambda$ . Goverde *et al.* [9] proved the following theorem.

**Theorem 2** *Let  $\{A(k)\}_{k \in \mathbb{N}}$  be a generic i.i.d. sequence of integrable and irreducible random matrices in  $\mathbb{R}_{\max}^{n \times n}$  with fixed support,  $x_0 \in \mathbb{R}^n$  integrable, and  $\{x(k; x_0)\}$  the stochastic recursive sequence defined by (3). Then there is a positive integer  $c$  and a random integer  $k_0 \in \mathbb{N}$  such that for all  $k \geq k_0$  and all  $1 \leq i \leq n$ ,*

$$x_i(k) - x_i^c(k) = x_i(k_0) - x_i^c(k_0)$$

and moreover

$$\frac{1}{c} \mathbb{E}[x_i(k_0) - x_i^c(k_0)] = \lambda.$$

For deterministic systems with  $A(k) = A$  for all  $k \geq 1$ , the least  $c$  satisfying Theorem 2 is the cyclicity of  $A$ . For random matrices  $A(k)$  that are close or proportional to a deterministic matrix  $A$  a good choice for  $c$  is the cyclicity of this  $A$ .

A general lower bound on the cycle time of a random matrix  $A \in \mathbb{R}_{\max}^{n \times n}$  can be given by the cycle time of its expectation  $\mathbb{E}[A]$ , which is the deterministic matrix obtained by replacing the process times by their mean values. For a proof of the following theorem, see Baccelli *et al.* [2, Cor. 8.24].

**Theorem 3** *Let  $\{A(k)\}_{k \in \mathbb{N}}$  be an i.i.d. sequence of integrable and irreducible random matrices in  $\mathbb{R}_{\max}^{n \times n}$ . Then the cycle time of the stochastic max-plus linear system (3) satisfies the bound*

$$\lambda \geq \max_{\xi \in C} \frac{\mathbb{E}[w(\xi)]}{\mu(\xi)},$$

where  $C$  is the set of all circuits in  $G(\mathbb{E}[A])$  and  $\mathbb{E}[w(\xi)]$  is the expectation of the sum of process times in circuit  $\xi$ .

Hence, the variances of the process times in a stochastic system lead to a higher cycle time than the deterministic system obtained by simply replacing the random process times by their means.

## 4 Timetable Stability Analysis

From now on, we consider scheduled stochastic max-plus linear systems

$$x(k) = A(k) \otimes x(k-1) \oplus d(k), \quad k \geq 1, \quad (10)$$

with  $x(0) = x_0$  and  $d(k) = d_0 \otimes T^{\otimes k}$ , under the following assumptions:

- A1.  $\{A(k)\}_{k \in \mathbb{N}}$  is a sequence of i.i.d. integrable random matrices,
- A2.  $\{A(k)\}_{k \in \mathbb{N}}$  has fixed support,
- A3.  $A(1)$  is irreducible.

The second assumption implies that we consider a fixed timetable structure, so we will not allow dispatching actions that change e.g. the order of trains, routings, or rolling stock circulations. Indeed, we will be concerned with the properties of the given timetable structure and study its behaviour with respect to stochastic disturbances. For this, we assume that the process times of each period are stochastic variables with finite absolute mean as in assumption A1. Assumptions A2 and A3 together imply that the sequence  $\{A(k)\}$  is irreducible. In addition, we naturally assume that process times are nonnegative,  $a_{ij}(k) \geq 0$ . Therefore we may assume without loss of generality that  $A_{ii}(k) \geq e$ , i.e.,  $A(k) := A(k) \oplus E$ . This assumption is required to apply some theorems from the literature [1, 2].

The stochastic system behaviour depends on two system properties: stability and realizability, which will be discussed next. A scheduled max-plus linear system (10) is *stable* if it is able to operate with cycle time  $T$  demanded by the periodic timetable, i.e.,

$$\lim_{k \rightarrow \infty} \frac{x(k; x_0, d_0, T)}{k} = T \quad \text{a.s.} \quad (11)$$

The timetable system  $d(k) = d(k-1) \otimes T$ ,  $d(0) = d_0$  can be interpreted as a deterministic input sequence with cycle time  $T$  and basic timetable structure  $d_0$ . Hence, the periodic timetable  $d(k) = d_0 \otimes T^{\otimes k}$  is a stationary input sequence to the internal autonomous system  $x(k) = A(k) \otimes x(k-1)$ ,  $x(0) = x_0$ . The overall cycle time of the scheduled system is given in the following theorem, adapted from [1, 14].

**Theorem 4** *Let  $\{A(k)\}_{k \in \mathbb{N}}$  be an i.i.d. sequence of integrable and irreducible random matrices in  $\mathbb{R}_{\max}^{n \times n}$  with fixed support,  $x_0 \in \mathbb{R}^n$  integrable,  $d(k) = d_0 \otimes T^{\otimes k}$  with  $d_0 \in [0, T]^n$  a finite initial timetable, and  $\{x(k; x_0, d_0, T)\}$  the stochastic recursive sequence defined by the scheduled max-plus linear system (10). If  $\lambda$  is the cycle time of the autonomous system (3) then the cycle time of the scheduled system (10) is*

$$\lim_{k \rightarrow \infty} \frac{x(k; x_0, d_0, T)}{k} = \lim_{k \rightarrow \infty} \frac{\mathbb{E}[x(k; x_0, d_0, T)]}{k} = \lambda \oplus T.$$

**Proof.** Rewrite the scheduled system (10) as the autonomous max-plus linear system with an internal clock event  $u(k)$  giving the starting time of period  $k+1$ ,

$$\begin{cases} u(k) = T \otimes u(k-1) \\ x(k) = A(k) \otimes x(k-1) \oplus d_0 \otimes u(k-1) \end{cases}$$

with  $u(0) = T$  and  $x(0) = x_0$ , or equivalently as

$$\begin{pmatrix} u(k) \\ x(k) \end{pmatrix} = \begin{pmatrix} T & \varepsilon \\ d_0 & A(k) \end{pmatrix} \otimes \begin{pmatrix} u(k-1) \\ x(k-1) \end{pmatrix}, \quad \begin{pmatrix} u(0) \\ x(0) \end{pmatrix} = \begin{pmatrix} T \\ x_0 \end{pmatrix}.$$

Note that  $u(k) = T^{\otimes k+1}$  is the starting time of period  $k+1$ . This system is reducible with two strongly connected components: a first component consisting of a loop around  $u(k)$  which recycles the input clock event with cycle time  $T$ , and a second component consisting of the stochastic autonomous system given by  $x(k) = A(k) \otimes x(k-1)$  with cycle time  $\lambda$  due to Theorem 1. The clock event has access to event  $x_i$  via an arc with deterministic weight  $d_i(0)$  for all  $1 \leq i \leq n$ . The cycle time of a component in a reducible system is the maximum of the cycle times over all components having access to it [1, 14]. Hence, the first component has cycle time  $T$  and the second component has cycle time  $\lambda \oplus T$ .  $\square$

Note that stability is independent on the initial condition due to Theorem 1. The assumption of a finite initial timetable  $d_0 \in [0, T]^n$  can be relaxed to a vector  $d_0 \neq \varepsilon$ . In particular if  $d_0$  has only one finite entry, then the theorem is already valid: in the proof of the theorem there is now one arc from the clock event to some event  $x_i$  in the second (strongly-connected) component and therefore this component still has an upstream component with cycle time  $T$ . Theorem 4 leads to the following stability condition.

**Theorem 5** *The scheduled system (10) is stable iff  $\lambda \leq T$ . It is called strictly stable if  $\lambda < T$ , critical if  $\lambda = T$ , and unstable if  $\lambda > T$ .*

**Proof.** The theorem directly follows from Theorem 4 and definition (11) of stability.  $\square$

Obviously, a system with autonomous cycle time exceeding  $T$  can not realize its processes within one timetable period and thus will lead to increasing delays. On the other hand, if the cycle time of the autonomous system is smaller than the timetable period length then there

are positive mean recovery times and buffer times so that delays can settle and the system can return to a steady-state timetable with cycle time  $T$ . This steady-state timetable equals  $d(k)$  only if it is realizable as discussed next.

In deterministic systems a realizable process time is one for which the (deterministic) process time does not exceed the scheduled process time, and a realizable timetable is one in which all process times are realizable [7]. In stochastic systems the process times are random variables and therefore may exceed the scheduled process times with some probability. We therefore generalize the notion of realizability for stochastic systems as follows. A periodic timetable is *realizable* if the expected delays of the system with initial condition  $x_0 = d_0$  reach zero,

$$\lim_{k \rightarrow \infty} \mathbb{E}[z(k; d_0)] = 0, \quad (12)$$

with  $z(k; d_0) = x(k; d_0, d_0, T) - d(k)$  determined by the scheduled linear system (10) with  $x_0 = d_0$ . This asymptotic condition can also be substituted by the stronger condition that there exists a finite  $K \in \mathbb{N}$ , such that  $\mathbb{E}[z(k; d_0)] = 0$  for all  $k \geq K$ . A necessary condition for realizability is that the scheduled process times must exceed the mean process times,  $\mathbb{E}[a_{ij}(k)] < d_i(0) - d_j(0) - \mu_{ij}T$ . If  $d_0$  has some entries equal to  $\varepsilon$  then the limit in (12) must be understood to hold for the finite entries in  $d_0$ , i.e.,  $\lim_{k \rightarrow \infty} \mathbb{E}[z_i(k; d_0)] = 0$  for all  $i \in \text{supp}(d_0)$ .

It could however occur that a process time is scheduled too tight so that there will be a positive mean delay whatever the cycle time of the autonomous system. The system still could be called stable if downstream processes have enough recovery time to compensate for the unrealizable process times. Nevertheless, the limit (12) is then no longer satisfied, but instead a periodic timetable  $\bar{d}(k) = \bar{d}_0 \otimes T^{\otimes k}$  will be reached with a stationary delay regime  $\bar{z}(k) = \bar{d}(k) - d(k)$ . This is formalized in the next theorem.

**Theorem 6** *If the scheduled system (10) is strictly stable,  $\lambda < T$ , then there exists a uniquely defined periodic timetable  $\bar{d}(k) = \bar{d}_0 \otimes T^{\otimes k}$  with finite  $\bar{d}_0 \in [0, T]^n$ , such that the scheduled max-plus linear system  $x(k) = A(k) \otimes x(k-1) \oplus \bar{d}(k)$  is strictly stable and realizable, and for all integrable initial conditions  $x_0 \geq e \in \mathbb{R}^n$ ,*

$$\lim_{k \rightarrow \infty} \mathbb{E}[z(k; x_0, \bar{d}_0, T)] = 0,$$

with  $z(k; x_0, \bar{d}_0, T) = x(k; x_0, \bar{d}_0, T) - \bar{d}(k)$ . Moreover, with this timetable any delay  $z(k) \geq e$  settles with a mean rate of  $T - \lambda$  per period.

**Proof.** Consider the nonautonomous max-plus linear system with a recycled clock event with cycle time  $T$ ,

$$x(k) = A(k) \otimes x(k-1) \oplus e \otimes u(k-1), \quad k \geq 1, \quad (13)$$

with  $e \in \mathbb{R}_{\max}^n$ ,  $x(0) = x_0 \geq e$ , and  $u(k) = T \otimes u(k-1)$ ,  $u(0) = T$ . We now focus on the increment process  $y_i(k) := x_i(k) - u(k-1)$  for  $1 \leq i \leq n$ ,  $k \geq 0$ , i.e., the offset times with respect to the start of the  $k$ th period. We will see that these offset times couple to a uniquely defined stationary and ergodic sequence and define  $\bar{d}_0$  as the (mean of) this stationary solution.

Subtracting the scalar  $u(k-1)$  from each row on both sides of (13) gives the stochastic recursive sequence

$$\begin{aligned} y_i(k) &= \bigoplus_{j=1}^n A_{ij}(k) \otimes (x_j(k-1) - u(k-1)) \oplus e \\ &= \bigoplus_{j=1}^n A_{ij}(k) \otimes (x_j(k-1) - u(k-2) - T) \oplus e \\ &= \bigoplus_{j=1}^n (A_{ij}(k) - T) \otimes y_j(k-1) \oplus e. \end{aligned}$$

Define the matrix  $A_T(k) := T^{\otimes -1} \otimes A(k) = (A_{ij}(k) - T)$ , i.e., the matrix obtained by subtracting  $T$  from each finite entry of  $A$ . Then we obtain in matrix notation

$$y(k) = A_T(k) \otimes y(k-1) \oplus e, \quad k \geq 1,$$

with initial condition  $y(0) = y_0 := x_0 \geq e$ . This initial condition is obtained using the backward continuation of the periodic sequence  $u(k) = T \otimes u(k-1)$ ,  $k \in \mathbb{Z}$ , which gives  $u(-1) = u(0) - T = 0$  and therefore  $y_i(0) = x_i(0) - u(-1) = x_i(0) - 0 = x_i(0)$  for all  $1 \leq i \leq n$ . This system can be written as the autonomous system

$$\begin{pmatrix} r(k) \\ y(k) \end{pmatrix} = \begin{pmatrix} e & \varepsilon \\ e & A_T(k) \end{pmatrix} \otimes \begin{pmatrix} r(k-1) \\ y(k-1) \end{pmatrix}, \quad \begin{pmatrix} r(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} e \\ x_0 \end{pmatrix}. \quad (14)$$

Here,  $r(k) \equiv e \in \mathbb{R}_{\max}$  is the reference time at the start of each period, and  $\{y(k)\}$  is the sequence of offset times in each period  $k$ . Like in the proof of Theorem 4 this system is reducible and consists of two components: the deterministic reference subsystem and a downstream component corresponding to the autonomous system. It can be proven that the cycle time of the autonomous system determined by  $\{A_T(k)\}$  is  $\lambda - T < e$  a.s. for any initial condition  $x_0 \geq e$  [2, Lemma 7.52]. The deterministic reference system  $r(k) = e \otimes r(k-1)$ ,  $r(0) = e$  has a precedence graph consisting of a single loop with arc weight  $e$ . Hence, the maximum cycle mean is  $e$  and since by assumption  $\lambda - T < e$ , the critical graph consists of one circuit with length one, and therefore the cyclicity of the partitioned state matrix in (14) is  $c = 1$ . The sequence  $\{y(k) - y(k-1)\}$  couples in finite time with a uniquely defined stationary and ergodic sequence with cycle time  $\mathbb{E}[y(k) - y(k-1)] = e$ , regardless of the initial condition [3, Th. 3.4], see also [1, Th. 9], [2, Th. 7.67]. Define

$$\bar{d}_0 := \lim_{k \rightarrow \infty} \mathbb{E}[y(k)]. \quad (15)$$

Then we obtain the scheduled max-plus linear system

$$x(k) = A(k) \otimes x(k-1) \oplus \bar{d}_0 \otimes u(k-1), \quad k \geq 1,$$

with  $x(0) = x_0$  and  $u(k) = T \otimes u(k-1)$ ,  $u(0) = T$ . Similar to the increment process  $y(k)$  above, we can construct the delay process  $z(k) = x(k) - \bar{d}(k)$  with  $\bar{d}(k) = \bar{d}_0 \otimes u(k-1) = \bar{d}_0 \otimes T^{\otimes k}$  as

$$\begin{pmatrix} r(k) \\ z(k) \end{pmatrix} = \begin{pmatrix} e & \varepsilon \\ \bar{d}_0 & A_T(k) \end{pmatrix} \otimes \begin{pmatrix} r(k-1) \\ z(k-1) \end{pmatrix}, \quad \begin{pmatrix} r(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} e \\ x_0 - \bar{d}_0 \end{pmatrix}.$$

Again the autonomous system has cycle time  $\lambda - T < e$  indicating that any delay  $z_i(k) > e$  reduces with a mean rate  $T - \lambda$  per period. Indeed, this system is stable like the original system with (possibly unrealizable) initial timetable  $d_0$ . Moreover, by the construction of  $\bar{d}_0$ , we have for any initial condition  $x_0 \geq e$ ,

$$\begin{aligned} \lim_{k \rightarrow \infty} \mathbb{E}[z(k; x_0)] &= \lim_{k \rightarrow \infty} \mathbb{E}[x(k; x_0) - \bar{d}(k)] \\ &= \lim_{k \rightarrow \infty} \mathbb{E}[x(k; x_0) - \bar{d}_0 - u(k-1)] \\ &= \lim_{k \rightarrow \infty} \mathbb{E}[x(k; x_0) - u(k-1)] - \bar{d}_0 \\ &= \lim_{k \rightarrow \infty} \mathbb{E}[y(k; y_0)] - \bar{d}_0 = e. \end{aligned}$$

In particular, these limits hold for initial condition  $x_0 = \bar{d}_0$ , which proves that  $\bar{d} = \bar{d}_0 \otimes T^{\otimes k+1}$  is realizable.  $\square$

For a strictly stable system there are many realizable timetables depending on the distribution of the remaining slack time over the scheduled process times. The stationary timetable  $\bar{d}(k)$  of Theorem 6 is the optimal one in the sense that it could be applied with minimal cycle time  $\lambda$ , i.e.,  $d(k) = \bar{d}_0 \otimes \lambda^{\otimes k}$ . So  $\bar{d}(k)$  gives the tightest event times such that all processes are realizable. Theorem 6 does not only provide an existence theorem of a uniquely defined initial timetable vector  $\bar{d}_0$ , but its constructive proof also indicates how to actually compute it using (14) and (15). For this the perfect simulation algorithm of Bouillard & Gaujal [4] can be used, applied to the max-plus system (14). The stability margin  $T - \lambda$  can be used to add running time supplements or buffer times to the scheduled process times (e.g. rounding up to whole minutes). The resulting timetables are again realizable timetables as long as the cycle time of the deterministic system with process times equal to these scheduled process times does not exceed  $T$ . This approach can be used to compute robust timetables.

## 5 Example Application

This section illustrates the developed methodology of the previous sections with a small railway network consisting of two main stations and three train lines: line 1 serves the regional area around station  $S_1$ , line 2 circulates between the two stations, and line 3 serves the regional area around station  $S_2$ , see Figure 1. Both stations have two arrival and departure events. The departures are numbered  $x_1$  to  $x_4$  and the arrivals  $x_5$  to  $x_8$ , corresponding to the nodes 1 to 8 in Figure 1. The arc weights are the *minimum* running and dwell times (black arcs), minimum transfer times (dark gray arcs), and minimum departure headway times (light gray arcs).

The state vector  $x(k) = (x_1(k), \dots, x_8(k))'$  contains both departure and arrival events. The timetable is given by

$$d(k) = (31, 30, 0, 1, 21, 56, 26, 56)' \otimes 60^{\otimes k}.$$

Hence, the timetable  $d(k)$  is periodic with cycle time  $T = 60$  minutes. The arcs with a token in Figure 1 denote the processes that cross the hour with respect to this timetable. For example, line 3 arrives at xx:56 each hour in station  $S_2$  (node 8) and departs at xx:01 in the next hour (node 4). Consider first the deterministic max-plus linear system with respect to the minimum process times,

$$x(k) = A_0 \otimes x(k) \oplus A_1 \otimes x(k-1), \quad (16)$$

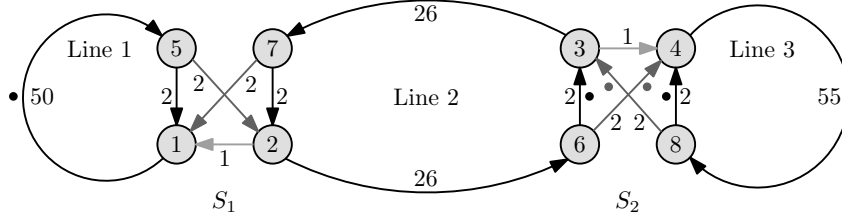


Figure 1: The example railway network

with

$$A_0 = \left( \begin{array}{cccc|cccc} \varepsilon & 1 & \varepsilon & \varepsilon & 2 & \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & 2 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \hline \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 26 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 26 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 55 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array} \right), \quad A_1 = \left( \begin{array}{cccc|cccc} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & 2 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 & \varepsilon & 2 \\ \hline 50 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{array} \right).$$

The matrices  $A_0$  and  $A_1$  correspond to the processes scheduled within an hour and crossing the hour, respectively, and are partitioned into four parts corresponding to the departure and arrival events. The lower-left part contains the running times, the upper-right part contains all dwell and transfer times from arrivals to departures, the upper-left part contains minimum headway times between departures, and the lower-right part contains minimum headway times between arrivals (none in this example). Minimum headway times between arrivals and departures can also be defined in the lower-left and upper-right parts.

The critical circuit of this deterministic network is the circuit over the nodes 3-4-8-3 with a total process time of 58 minutes and covering 1 period, since the only process in this circuit that crosses the hour is the transfer time from the basic scheduled event time  $d_8(0) = 56$  to  $d_3(0) = 0$ . Hence, by (9) the maximum cycle mean is  $\lambda = 58/1 = 58$  minutes. The deterministic system operating with minimum process times is therefore stable with a stability margin  $\Delta = 60 - 58 = 2$  minutes. For a detailed deterministic stability analysis of this network, see Goverde [7].

We now consider the stochastic system. Assume that all process times are distributed according to a shifted Gamma distribution, with shift parameters equal to the minimum process times depicted in Figure 1. Alternatively, we say that the process times are given by a minimum process time plus a Gamma( $\alpha_{ij}, \lambda_{ij}$ ) distributed delay  $Z$ , with probability density

$$f_{ij}(z; \alpha_{ij}, \lambda_{ij}) = \frac{\lambda_{ij}^{\alpha_{ij}}}{\Gamma(\alpha_{ij})} z^{\alpha_{ij}-1} e^{-\lambda_{ij}z} \quad \text{for } z \geq 0$$

and  $f_{ij}(z) = 0$  for  $z < 0$ . Here,  $\alpha_{ij} > 0$  is the shape parameter and  $\lambda_{ij} > 0$  the scale parameter. The parameters of the Gamma distribution are determined by the method of matching moments. The mean and variance of a Gamma( $\alpha, \lambda$ ) distributed random variable  $Z$  are given by  $\mathbb{E}[Z] = \alpha/\lambda$  and  $\text{Var}[Z] = \alpha/\lambda^2$ . Hence, if we know the mean  $\mu_{ij}$  and variance  $\sigma_{ij}^2$  of a process time  $a_{ij}(k)$  then the associated parameters can be estimated as

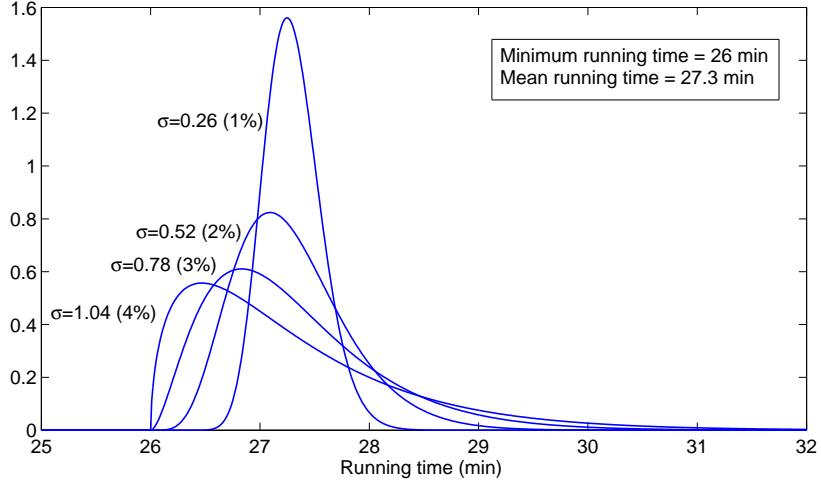


Figure 2: Shifted Gamma density  $f_{62}$  with minimum running time 26 min and 5% mean delay, and standard deviations ranging from 1% to 4%

$\alpha_{ij} = (\mu_{ij}/\sigma_{ij})^2$  and  $\lambda_{ij} = \mu_{ij}/\sigma_{ij}^2$ . For the stability analysis we now assume that the mean and standard deviation of the delays are given as percentage of the minimum process time, see Figure 2. Hence, we assume that the mean and standard deviation of a delay are increasing proportionally with increasing minimum process times.

Table 1 gives the cycle times computed by the Lyapunov exponent algorithm described in Goverde *et al.* [9] for Gamma distributed delays with mean  $\mu$  and standard deviation  $\sigma$  proportional to the minimum process times. The values are rounded to one decimal and thereby cover the 95%-confidence intervals which all have half-widths smaller than 0.05. The diagonal entries have a coefficient of variation  $CV = \sigma/\mu = 1$  and correspond to exponential delays. The lower triangular entries with  $CV < 1$  correspond to unimodal densities, while the upper triangular entries with  $CV > 1$  correspond to high-variance monotonously

Table 1: Cycle times in minutes depending on delay mean and standard deviation given in percentage of minimum process times. The cycle times are rounded to one decimal with the 95%-confidence intervals all within  $\pm 0.05$

Mean $\mu$	Standard deviation $\sigma$					
	0%	1%	2%	3%	4%	5%
0%	58.0	-	-	-	-	-
1%	58.6	58.6	58.6	58.7	58.9	59.0
2%	59.2	59.2	59.2	59.3	59.5	59.6
3%	59.7	59.7	59.8	59.9	60.0	60.2
4%	60.3	60.3	60.3	60.4	60.6	60.8
5%	60.9	60.9	61.0	61.0	61.2	61.4



decreasing densities. The first column with  $\sigma = 0$  corresponds to deterministic max-plus linear systems with mean process times, where the first entry  $\lambda = 58$  minute corresponds to the minimum process times.

Increasing all process times by a deterministic delay of  $3.45\% = 2/58$  of the minimum process times yields a deterministic max-plus linear system with cycle time  $\lambda = 60$ . The associated critical circuit is 3-4-8-3 with maximum cycle mean 60 minutes. According to Theorem 3 the stochastic systems with a mean delay over 3.45% of the minimum process times are all unstable. Indeed, Table 1 shows that the cycle times for mean delays of 4% and higher all exceed 60 minutes. For a mean delay of 3% the system becomes unstable when the standard deviation is 4% or higher. When the process times have a mean delay of 2% or lower the system is stable with standard deviations up to 5% (and higher).

Note that in this example the scheduled arrival times are chosen such that the scheduled running times equal the minimum running times, while the scheduled dwell and transfer times contain some positive dwell and transfer buffer times, respectively. Running time supplements may be added to the scheduled running times at the expense of the dwell and transfer buffer times. Moreover, the scheduled departure headways are here equal to the minimum departure headways, so that the departure times of train lines 1 and 3 will always have (small) primary delays due to the tight minimum departure headways after line 2. This timetable is therefore only realizable for the deterministic system with minimum process times, and unrealizable otherwise; even if we consider the scheduled departure times only, i.e.,  $d_0 = (31, 30, 0, 1, \varepsilon, \varepsilon, \varepsilon, \varepsilon)'$ . Nevertheless, it may be stable depending on the mean and variance of the delays.

## 6 Conclusions

This paper presented a stochastic framework to evaluate periodic railway timetables. Results known from deterministic max-plus stability analysis have been generalized to a stochastic setting where process times can have arbitrary stochastic distributions with finite mean. The developed methodology can be used to evaluate the influence of process time variabilities on stability and robustness of large-scale railway timetables. Because railway timetables are effectively modelled in the max-plus framework the computations required in the analyses are very fast, enabling a real-time interactive stability analysis approach as opposed to the alternative of network-wide simulation which requires very extensive computation times.

A current research subject is the sensitivity of the Lyapunov exponent (or cycle time) on the underlying process time distributions and parameters. Besides giving insight on the robustness of the Lyapunov exponent as stability performance indicator, this is also of interest to reduce the amount of input for large-scale networks as much as possible without sacrificing the quality of results. In this paper we assumed that the random state matrices have fixed support according to a fixed timetable structure. Further research will concentrate on switching max-plus linear systems which allows the modelling of order changes and other dispatching actions. Also the design of robust stationary timetables will be a topic of further research. Moreover, the algorithms will be implemented in the stability analysis tool PETER.

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