Active Directional Wave Absorption Theory

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Abstract

Panel segment type wave-makers are being used to realize multi-directional irregular waves in laboratory wave basins. In order to reproduce the real sea condition, the wave-maker should generate the incident waves, at the same time absorb reflected waves from model structures. The present study was conducted to give a multi-directional wave absorption theory as the first step of developing multi-directional absorbing wave making system. First a multi-directional wave absorption theory was presented for waves in a rectangular basin with reflective side walls. The theory relates the reflected wave profile to the motion of the wave board. Second practical approximation methods were given to realize the real time operation. Last the performance of the methods were analysed theoretically for a typical wave experiment condition. The result of the analysis showed that these methods have sufficient capability of absorbing reflected multi-directional irregular waves.
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1 Introduction

Panel segment type wave-makers are being used to generate multi-directional random waves within laboratory wave basins. The directional wave making theory which relates the motions of wave boards to the wave fields in wave basins has been discussed by several authors, such as Takayama[1], Sand and Mynett[2], Funke and Miles[3], Dalrymple[4], Isacson[5]. This provides the capability of using multi-directional random waves for coastal and offshore hydraulic model tests.

However, in wave basins with the installation of model structures, waves reflected by the structures travel back to the wave-maker and are re-reflected there. Thus the incident wave field is disturbed. Therefore, in order to reproduce the real sea situation, the wave-maker should be controlled in such a way that it makes the correct incident waves and at the same time absorbs the reflected waves.

In case of wave flume tests, where incident and reflected waves propagate perpendicular to the wave board, the wave making system with reflected wave compensation is widely used(Salter[6], Kawaguchi[7]). However, no absorbing multi-directional wave-maker is reported to be available at this moment. This report describes the multi-directional wave absorbing theory as the first step of developing multi-directional absorbing wave making system. First, the relationship between the motion of the wave-maker and the wave motion in the wave basin with reflective sidewalls is considered. Second, the mathematical expression for multi-directional wave absorption is derived. Last, practical approximation methods are presented and the performance of them are discussed.
2 Theoretical model

2.1 Schematization

The schematized situation in the wave basin is sketched in Figure 1. The wave basin is rectangular, with width $2b$. Mean water depth $h$ is constant in the basin. Side walls of the basin are vertical and impermeable. The wave-maker is mounted against one end of walls. The distribution of water surface elevation along the wave-maker is measured.

A Cartesian coordinate system $O-xyz$ is used for expressing the waves in the wave basin. The origin of the coordinate, $O$, is located at the center of the wave-maker on the mean water level. $x$ is a horizontal axis perpendicular to the wave-maker, positive inward. $y$ is another horizontal axis along the wave-maker. $z$ is a vertical axis, positive upward.

![Figure 1: Definition sketch](image-url)
2.2 Waves in a basin with reflective side walls

Fluid, fluid motion and boundary conditions are assumed as follows:

1. The fluid is non-viscous and incompressible.

2. The fluid motion is irrotational.

3. The displacement of the wave board is sufficiently small compared with depth and wave length. This means that the small wave amplitude theory can be used.

4. The displacement of the wave board is uniform in z direction, i.e. piston-type motion, and continuous in y direction.

5. Bottom, sidewalls, and wave board are impermeable. This means that any wave motion directing to the sidewalls or to the wave board is perfectly reflected.

Under the above assumptions, the fluid motion can be expressed by a velocity potential \( \phi \), which satisfies the following Laplace equation.

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0
\]

The boundary conditions for waves traveling in a wave basin with side walls are expressed by the following formulas.

\[
z = 0 : \quad \frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0
\]

\[
z = -h : \quad \frac{\partial \phi}{\partial z} = 0
\]

\[
y = \pm b : \quad \frac{\partial \phi}{\partial y} = 0
\]

where, \( t \) and \( g \) denote time and acceleration of gravity respectively.

Equations (1)–(4) are solved by applying the separation of variables. The solution which corresponds to the waves traveling \( x \) positive direction can be written by,

\[
\phi(x, y, z, t) = \sum_{n=1}^{\infty} \left\{ \Phi_n^{(1)} + \Phi_n^{(2)} + \Phi_n^{(3)} \right\} e^{-i(\omega_n t + \xi_n)}
\]
\[ \Phi_{n}^{(1)}(x, y, z) = \sum_{m=0}^{\alpha_n} A_{n,m,0} e^{i\sqrt{k_n^2 - \lambda_n^2} x} \cos \lambda_m y \cosh k_n (h + z) \\
+ \sum_{m=0}^{\beta_n} B_{n,m,0} e^{i\sqrt{k_n^2 - \gamma_n^2} x} \sin \gamma_m y \cosh k_n (h + z) \]

\[ \Phi_{n}^{(2)}(x, y, z) = \sum_{m=\alpha_n+1}^{\infty} A_{n,m,0} e^{-\sqrt{\lambda_n^2 - k_n^2} x} \cos \lambda_m y \cosh k_n (h + z) \\
+ \sum_{m=\beta_n+1}^{\infty} B_{n,m,0} e^{-\sqrt{\gamma_n^2 - k_n^2} x} \sin \gamma_m y \cosh k_n (h + z) \]

\[ \Phi_{n}^{(3)}(x, y, z) = \sum_{m=0}^{\alpha_n} \sum_{s=1}^{\infty} A_{n,m,s} e^{-\sqrt{\nu_n^2 + \lambda_n^2} x} \cos \nu_m y \cos \nu_{n,s} (h + z) \\
+ \sum_{m=0}^{\beta_n} \sum_{s=1}^{\infty} B_{n,m,s} e^{-\sqrt{\nu_n^2 + \gamma_n^2} x} \sin \gamma_m y \cos \nu_{n,s} (h + z) \]

where, \( \omega_n, \epsilon_n, k_n \) are the angular frequency, the phase-lag and the wave number of the nth component wave, \( k_n \) satisfies the following dispersion relationship.

\[ \omega_n^2 = g k_n \tanh k_n h \]

\( \nu_{n,s} \) is the sth root of the following equation.

\[ \omega_n^2 = -g k_{\nu_n,s} \tan \nu_{n,s} h \]

Furthermore, \( \lambda_m = m \pi / b \) and \( \gamma_m = (m + 1/2) \pi / b \) for \( m = 0, 1, \ldots, \infty \). \( \alpha_n \) and \( \beta_n \) are the maximum values of \( m \) which satisfy \( k_n > \lambda_m \) and \( k_n > \gamma_m \). \( A \) and \( B \) are unknown complex coefficients.

The velocity potential of the waves traveling \( x \) negative direction is given by replacing \( x \) with \(-x\) in equations (6)-(8).

### 2.3 Relationship between wave field and wave board motion

The relationship between the wave motion and the wave board motion is given by the following wave board boundary condition.

\[ x = 0 : \frac{\partial \phi}{\partial x} = u(y, t) \]
where $u(y,t)$ is velocity of wave board motion.

In equation (5) the first term represents the "propagating wave mode", and other terms express "evanescent wave mode", i.e. wave decaying with traveling. Since we discuss active wave absorption, we will focus on the behavior of propagating wave mode, the velocity potential of which can be written as

$$\phi^{(1)}(x,y,z,t) = \sum_{n=1}^{\infty} \Phi_{n}^{(1)} e^{-i(\omega_{n}t + \epsilon_{n})}$$

(12)

Considering the orthogonality of functions $\cosh k_{n}(h + z)$ and $\cos k_{n}z$ for $j = 1, 2, \ldots, \infty$ (Takayama[1]), we get the following relationship between $u$ and $\phi^{(1)}$.

$$u(y, t) = \int_{-h}^{0} \frac{\partial \phi^{(1)}}{\partial x} \bigg|_{x=0} \cosh k_{n}(h + z)dz / \int_{-h}^{0} \cosh k_{n}(h + z)dz$$

$$= \sum_{n=1}^{\infty} \frac{\partial \Phi_{n}^{(1)}}{\partial x} \bigg|_{x=0} \frac{2k_{n}h + \sinh 2k_{n}h}{4 \sinh k_{n}h} e^{-i(\omega_{n}t + \epsilon_{n})}$$

(13)

$$= \sum_{n=1}^{\infty} \left( \sum_{m=0}^{\alpha_{n}} A_{n,m,0} \sqrt{k_{n}^{2} - \lambda_{m}^{2}} \cos \lambda_{m}y$$

$$+ \sum_{m=0}^{\beta_{n}} B_{n,m,0} \sqrt{k_{n}^{2} - \gamma_{m}^{2}} \sin \gamma_{m}y \right)$$

$$\times \frac{2k_{n}h + \sinh 2k_{n}h}{4 \sinh k_{n}h} e^{-i(\omega_{n}t + \epsilon_{n})}$$

Since the propagating wave profile $\eta(x,y,t)$ is expressed as

$$\eta(x,y,t) = -\frac{1}{g} \frac{\partial \phi^{(1)}}{\partial t} \bigg|_{z=0}$$

$$= \sum_{n=1}^{\infty} \frac{i\omega_{n}}{g} \Phi_{n}^{(1)} \bigg|_{z=0} \cosh k_{n}he^{-i(\omega_{n}t + \epsilon_{n})}$$

(14)

from equations (6), (9) and (14), wave profile at the wave board denoted by $\hat{\eta}(y,t)$ is given by the following equation.

$$\hat{\eta}(y,t) = \eta|_{x=0}$$

$$= \sum_{n=1}^{\infty} \left( \sum_{m=0}^{\alpha_{n}} A_{n,m,0} \cos \lambda_{m}y + \sum_{m=0}^{\beta_{n}} B_{n,m,0} \sin \gamma_{m}y \right)$$

$$\times \frac{k_{n} \sinh k_{n}h}{\omega_{n}} e^{-i(\omega_{n}t + \epsilon_{n})}$$

(15)

Comparing equations (13) and (15), it is found out that $u$ and $\eta$ are in phase. This gives the capability of the absorbing wave-maker with the water surface profile sensor and the wave board velocity control to be real time operated.
2.4 Wave absorption

Let's consider the situation schematized in Figure 2. The wave reflected by the model structure ($\eta (r)$) propagates toward the wave-maker obliquely and re-reflected wave ($\eta (rr)$) travels back. If the wave-maker generates the wave ($\eta (c)$) which has the same amplitude and propagating angle, but has phase 180 degree out of phase, as the re-reflected wave, the re-reflected wave will be compensated. In other words reflected wave will be absorbed by the wave-maker.

![Figure 2: Principle of wave absorption](image)

From equations (6) and (14), in the wave basin with reflective sidewalls, the reflected waves $\eta (r)$ can be written in the form

$$\eta (r) = \sum_{n=1}^{\infty} \left\{ \sum_{m=0}^{\infty} P_{n,m} e^{-i\sqrt{k_n^2 - \lambda_n^2} x} \cos \lambda_m y \\
+ \sum_{m=0}^{\infty} Q_{n,m} e^{-i\sqrt{k_n^2 - \gamma_m^2} x} \sin \gamma_m y \right\} x e^{-i(\omega_n t + \epsilon_n)}$$

(16)

Since waves are assumed to be perfectly reflected at the wave board, the re-reflected wave profile is written as
Based on the above discussion, the compensation wave has the following wave profile.

\[ \eta^{(rr)} = \sum_{n=1}^{\infty} \left\{ \sum_{m=0}^{\infty} P_{n,m} e^{i\sqrt{k_n^2 - \lambda_n^2} \cos \lambda_m y} + \sum_{m=0}^{\beta_n} Q_{n,m} e^{i\sqrt{k_n^2 - \gamma_m^2} \sin \gamma_m y} \right\} \times e^{-i(\omega_n t + \epsilon_n)} \]  

(17)

By using the relationship between wave board motion and the wave field expressed by the equations (13) and (15), the wave board motion which generates the waves \( \eta^{(e)} \) can be written as follows.

\[ \eta^{(e)} = \sum_{n=1}^{\infty} \left\{ \sum_{m=0}^{\infty} -P_{n,m} X_{n,m} \cos \lambda_m y \right\} \times e^{-i(\omega_n t + \epsilon_n)} \] 

(18)

where, \( X_{n,m} \) and \( Y_{n,m} \), called as the wave absorption transfer function, are expressed by the following functions.

\[ X_{n,m} = \frac{\omega_n \sqrt{k_n^2 - \lambda_n^2}}{F_n k_n} \]  

(20)

\[ Y_{n,m} = \frac{\omega_n \sqrt{k_n^2 - \gamma_m^2}}{F_n k_n} \]  

(21)

\[ F_n = \frac{4 \sinh^2 k_n h}{2 k_n h + \sinh 2 k_n h} \]  

(22)
In case of absorbing wave making, the water surface profile measured at the wave board is the summation of incident and reflected waves. Because incident wave profile at the wave board can be estimated by using the wave making theory, the reflected wave profile can be calculated by simply subtracting incident wave profile from the observed water surface profile. Once the reflected wave profile is estimated, the wave board motion which should be superimposed in order to absorb the reflected wave can be analysed by equations (19)–(22).
3 Practical aspects

3.1 Wave-maker configuration

In the previous section, the wave absorption theory was derived based on the eigenvalue analysis. The theory relates the distribution of water surface elevation along the wave board to the velocity distribution of the wave board. In practice, however, water surface elevations can be measured at some discrete points, and the continuous motion of wave board is approximated by the motions of numbers of narrow wave paddles. So, before we apply the theory to the practical situation, we need to specify the wave puddle configuration and water surface measurement equipment.

As for the configuration of wave paddles, we consider two types of horizontal structure. They are, separate type, the midpoints of separate paddles are driven, and linked type, the hinges between adjacent paddles are driven (see Fig. 3). For simplicity, we assume that water surface elevations are measured at the driving points, $y$ coordinates of which are denoted by $y_j$, $j = 1, \ldots, N$. This means that the number of measurement points is equal to that of driving points. Thus, in practice, we need to relate the reflected wave profiles measured at driving points $(y_j^{(r)}, j = 1, \ldots, N)$ to the absorbing driving velocity signals $(u_j^{(s)}, j = 1, \ldots, N)$.

![Wave-maker configurations](image)

Figure 3: Wave-maker configurations. a. linked type, b. separate type
3.2 Practical approximations

Because the wave absorption transfer functions $X_{n,m}$ and $Y_{n,m}$ depend on both frequencies and eigenvalues, it is not feasible to make the numerical filter which has characteristics expressed by equations (20) and (21). So, in order to make the real time operation possible, practical approximations are needed.

3.2.1 Narrow frequency band approximation (Method 1)

When the frequency spectra of reflected waves are narrow band, the wave absorption transfer functions, $X_{n,m}$ and $Y_{n,m}$ are well approximated by the values at peak frequency of the spectra, $X_{m}^{(1)}$ and $Y_{m}^{(1)}$

$$X_{m}^{(1)} = \frac{\omega_{p}\sqrt{k_{p}^{2} - \lambda_{m}^{2}}}{k_{p}}$$

$$Y_{m}^{(1)} = \frac{\omega_{p}\sqrt{k_{p}^{2} - \gamma_{m}^{2}}}{k_{p}}$$

where, subscript $p$ means the values at the peak frequency. This approximation makes it possible to rewrite the reflected wave profile at the wave paddles

$$\dot{\eta}_{j}^{(r)}(t) = \sum_{m=0}^{\alpha_{p}} -\bar{P}_{m}(t) \cos \lambda_{m}y_{j} + \sum_{m=0}^{\beta_{p}} -\bar{Q}_{m}(t) \sin \gamma_{m}y_{j}$$  

$$\bar{P}_{m}(t) = \sum_{n=1}^{\infty} P_{m,n} e^{i(\omega_{n} t + \xi_{n})}$$

$$\bar{Q}_{m}(t) = \sum_{n=1}^{\infty} Q_{m,n} e^{i(\omega_{n} t + \xi_{n})}$$

The compensation paddle driving velocity can be approximated as

$$u_{j}^{(e)}(t) = \sum_{m=0}^{\alpha_{p}} -X_{m}^{(1)} \bar{P}_{m}(t) \cos \lambda_{m}y_{j} + \sum_{m=0}^{\beta_{p}} -Y_{m}^{(1)} \bar{Q}_{m}(t) \sin \gamma_{m}y_{j}$$

By eliminating $\bar{P}_{m}$ and $\bar{Q}_{m}$ in equations (27) and (30), we can directly relate $u_{j}^{(e)}$ to $\dot{\eta}_{j}^{(r)}$ in the following form.

$$u_{j}^{(e)} = \sum_{s=0}^{N} Z_{j,s} \dot{\eta}_{s}^{(r)}$$
in which \( Z \) is a \( N \times N \) matrix which can be approximated by the following matrix when \( N \) is large.

\[
[z] = -
\begin{bmatrix}
X_0^{(1)} \cos \lambda_0 y_1 & \cdots & X_{\alpha_p}^{(1)} \cos \lambda_{\alpha_p} y_1 & Y_0^{(1)} \sin \gamma_0 y_1 & \cdots & Y_{\beta_p}^{(1)} \sin \gamma_{\beta_p} y_1 \\
\vdots & \ddots & \vdots & \vdots & & \vdots \\
X_0^{(1)} \cos \lambda_0 y_N & \cdots & X_{\alpha_p}^{(1)} \cos \lambda_{\alpha_p} y_N & Y_0^{(1)} \sin \gamma_0 y_N & \cdots & Y_{\beta_p}^{(1)} \sin \gamma_{\beta_p} y_N \\
\frac{2}{N} \cos \lambda_1 y_1 & \cdots & \frac{2}{N} \cos \lambda_1 y_N \\
\vdots & \ddots & \vdots & \vdots & & \vdots \\
\frac{2}{N} \cos \lambda_{\alpha_p} y_1 & \cdots & \frac{2}{N} \cos \lambda_{\alpha_p} y_N \\
\frac{2}{N} \sin \gamma_0 y_1 & \cdots & \frac{2}{N} \sin \gamma_0 y_N \\
\vdots & \ddots & \vdots & \vdots & & \vdots \\
\frac{2}{N} \sin \gamma_{\beta_p} y_1 & \cdots & \frac{2}{N} \sin \gamma_{\beta_p} y_N
\end{bmatrix}
\]

(30)

3.2.2 Narrow directional spreading approximation (Method 2)

From the system engineering point of view, equation (29) is still complicated, because all of the wave gauge signals are needed to determine one driving velocity signal. So, second approximation is made. When the reflected waves travel meanly normal to the wave board and the directional spreading is relatively small, \( \lambda_m \) and \( \gamma_m \) are smaller than \( k_p \). Thus, the wave absorption transfer function \( X_m^{(1)} \) and \( Y_m^{(1)} \) in equations (23) and (24) can be approximated by the following functions.

\[
X_m^{(2)} = \frac{\omega_p}{F_p} \left\{ 1 - \frac{1}{2} \left( \frac{\lambda_m}{k_p} \right)^2 \right\}
\]

(31)

\[
Y_m^{(2)} = \frac{\omega_p}{F_p} \left\{ 1 - \frac{1}{2} \left( \frac{\gamma_m}{k_p} \right)^2 \right\}
\]

(32)

Under this approximation, the compensation driving velocity signals can be written as
\[ u_j^{(c)}(t) = \sum_{m=0}^{\alpha_p} -X_m^{(2)} \bar{P}_m(t) \cos \lambda_m y_s + \sum_{m=0}^{\beta_p} -Y_m^{(2)} \bar{Q}_m(t) \sin \gamma_m y_s \]

\[ = -\frac{\omega_p}{F_p} \left[ \sum_{m=0}^{\alpha_p} \bar{P}_m(t) \cos \lambda_m y_s + \sum_{m=0}^{\beta_p} \bar{Q}_m(t) \sin \gamma_m y_s \right] \]

\[ + \frac{\omega_p}{2F_p k_p^2} \left[ \sum_{m=0}^{\alpha_p} \lambda_m^2 \bar{P}_m(t) \cos \lambda_m y_s + \sum_{m=0}^{\beta_p} \gamma_m^2 \bar{Q}_m(t) \sin \gamma_m y_s \right] \]

\[ = -\frac{\omega_p}{F_p} \left( \frac{\partial^2 \eta^{(r)}}{\partial y^2} \right) \left( \eta_j + \frac{1}{2k_p^2} \frac{\partial^2 \eta^{(r)}}{\partial y^2} \right) \]  

(33)

Because the second term in equation (33) can be estimated by, for example,

\[ \frac{\partial^2 \eta}{\partial y^2} \approx \frac{1}{(\Delta y)^2} (\eta_{j+1} - 2\eta_j + \eta_{j-1}) \]  

(34)

where \( \Delta y = y_{j+1} - y_j = y_j - y_{j-1} \).

This means that \( j \)th compensate driving velocity signal can be estimated from \((j-1)\)th, \( j \)th and \((j+1)\)th wave gauges signals. Therefore, the second approximation reduces the great deal of complexity of the control system.

3.2.3 Unidirectional approximation (Method 3)

If the second term in equations (31) and (32) is neglected, the absorbing transfer functions become identical to that for uni-directional perpendicular waves.

\[ X_m^{(3)} = Y_m^{(3)} = \frac{\omega_p}{F_p} \]  

(35)

3.3 Performance comparison

In order to evaluate the performance of approximate methods presented in the previous sections, the absorption transfer functions \( X \)and \( Y \)are evaluated numerically for a typical experiment condition. The hydraulic conditions used are

\[ h = 0.5(m), \quad f_p = \omega_p/2\pi = 0.6(Hz), \quad b = 8(m) \]

Figure 3 shows the variation of \( X \)and \( Y \)with respect to frequency and for various values of \( m \). This figure shows that method 1 and method 2 fairly well approximate the exact solution over the frequency range \( 0.45-0.90(HZ) \), \( (0.75f_p-1.5f_p) \). Because the wave absorption transfer function of unidirectional wave is identical with the \( X_0^{(1)} \), It is obvious that the present methods 1 and 2 have better performance than the un-directional absorbing method. The difference between method 1 and 2 is small if \( m \) is small. Therefore, the present methods have the good performance of absorbing multi-directional waves, when their spectrum is narrow in frequency and directional spreading.
Figure 4: Comparison of wave absorption transfer functions
4  Conclusion

A multi-directional wave absorption theory is presented for waves in a rectangular basin with reflective side walls. The theory relates the reflected wave profiles measured at the wave paddles to velocities of wave paddle motion which should be superimposed to the wave making action. Practical approximation methods are given in order to provide the feasibly simple real time operation. The performance of these methods are analysed theoretically for a typical experiment condition. The result of the analysis shows that the use of the present methods provides better wave absorption performance than the use of uni-directional wave absorption algorithm for multi-directional waves.

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