Probabilistic analysis of the Stüssi-Kollbrunner paradox

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In most text books on Plasticity Theory a reference is made to the famous tests performed by Stüssi and Kollbrunner in 1935. The test object was a continuous beam on four supports, loaded by a concentrated point load halfway the central span. According to a straight forward application of the theory of plasticity the load carrying capacity of this beam is independent of the length of the outer spans. However, if the length of the outer span approaches infinity, the stiffness of the outer beam parts goes to zero. The centre span is then to be considered as a simply supported beam, having only half the load carrying capacity of the continuous beam. As usual, such a paradox can only be solved by using a more advanced theory, including both elastic and plastic deformations, as well as the real stress strain relationship. This of course has been done in the past successfully and the theoretical results were confirmed by tests.
The present paper intends to reconsider this problem, however including also the random nature of the material properties. This way an adequate approach to the reliability aspects of the problem can be achieved. Such an approach is considered especially useful for high strength steel and concrete materials that are used nowadays. For these materials aspects of deformation capacity become more and more important
Apart from some simplified calculations, the numerical results in this paper have been attained using a special purpose computer program that combines the standard Finite Element Method and First Order Reliability Methods. Some attendance will be given to the possibilities and limitations of the present program and the techniques that are used to reduce the computation time.

Key words: **reliability, rotation capacity, plasticity, steel structure**

1 Introduction

As computers become faster and faster, it becomes feasible to carry out probabilistic calculations using standard Finite Element Codes like the TNO program DIANA for the modelling of the structural behaviour. Especially when the physically and geometrically non-linear behaviour of the structure is taken into account, this combination offers the opportunity to make numerical estimates of the structural systems reliability behaviour. But also on a more modest level (local failure, serviceability limit states) interesting results can be obtained.

For a probabilistic analysis to be useful, all variables should be treated as stochastic variables, in particular:

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– Material properties	yield stress modulus of elasticity ultimate stress ultimate strain
– Geometrical properties	cross-sectional dimensions nodal co-ordinates (eccentricities)
– Loads	nodal forces settlements of supports temperature loads

The input option should allow for a wide variety of distribution types and correlation patterns.

In real structures many stochastic variables may be present. Consider by way of example a structure that is modelled by a FEM model consisting of 50 elements. If the description of the σ-ε diagram for each element requires four variables, as indicated above, the material properties amount already to $50 \times 4 = 200$ stochastic variables. If every element is loaded by two forces, an additional 100 stochastic load variables have to be used. When the geometrical properties are stochastic as well, the total number of stochastic variables will easily go up to 1000. Of course, when some variables are completely correlated, this number can be reduced, but that is not the general case.

Additionally, a real structure may have many failure path's and limit states. Therefore, the calculation of the reliability of such a structure with standard reliability analysis tools like FORM [8], Monte Carlo [12] or Numerical Integration is very time consuming. Having a 1000 stochastic variables, the numerical integration methods are simply out of the question. The standard Monte Carlo simulation (assuming a failure probability of $P_f = 10^{-4}$) needs approximately $5\ 10^6$ FEM computations. This number can be decreased using more advanced techniques like importance sampling or directional sampling but still an enormous amount of FEM computations will remain. The standard FORM procedure is also very time consuming as well. The number of FEM computations is approximately $5\ 10^4$ limit state function evaluations.

In order to reduce the number of FEM calculations a Response Surface Technique can be used [5]. The standard option is to make a number of FEM calculations, then to construct a Response Surface and then to perform a Reliability Analysis using the Response Surface for the limit state function evaluations.

In this paper a strategy will be used where the Response Surface technique will be integrated with FORM. The point is that in every FORM iteration (except the first one) only one FEM calculation is being performed and partial derivations are based on the Response Surface. The Response Surface, on its turn, is adapted on the basis of the FEM calculation during each FORM iteration. Chapter 2 will provide further details.

As a second strategy chapter 2 also shows a combined Directional-Importance Sampling approach. This Directional Importance Sampling gets relatively faster if more limit state functions in the same analysis are involved. For 1000 stochastic variables, approximately 50 samples are needed (the number of limit state evaluations is approximately 5 times higher). The only condition is that a minimum of 20 out of the 1000 variables are of a typical "load" or "resistance" type, that is, for 20 variables the sign of the partial derivative should be known.

The number of function evaluations needed is summarised in Table 1. A disadvantage of the standard FORM method is that the number of computations is influenced by the number of limit state functions and failure paths. The other methods don't have this disadvantage.

Table 1. Number of FEM computations for the various probabilistic calculation techniques in the case of 1000 random variables.

Method	typical number of FEM computations
Numerical Integration	too many
Crude Monte Carlo	5000000
Standard FORM	20000
Directional Sampling with adapted Response Surface	20000
FORM using an adapted Response Surface	1000
Importance Directional Sampling[1]	250
Same with adapted Surface Technique[1]	50

[1] assuming that 20 variables have either an outspoken load or an outspoken resistance character.

The methods, discussed in section 2, will be demonstrated in chapter 3. The example structure is the famous Stüssi-Kollbrunner beam. In this beam the load bearing capacity depending on the span ratio's and the σ-ε diagram may be governed by the rotation capacity.

2 Methods for the calculation of failure probabilities

In this chapter two methods for the reliability analysis of structures using the FE-analysis are discussed. The first method starts from the FORM technique, the second one form Directional Sampling. In both cases an adapted Response Surface Method is used.

FORM – *integrated adapted response surface technique*

A method for calculating the reliability using the FORM method combined with FE-analysis is described in [16][17]. The FORM method is interacting with the FE-code through a Response Surface. The Response Surface is built up using the FORM results and updated every time a FORM iteration has taken place. The Response Surface technique has been used earlier by [5] and [2]. The improvement of the present method lies in an enhanced accuracy of the response surface [18] in the neighbourhood of the latest design point and the simultaneous treatment of a number of limit states.

In short the method can be described as follows:

1. A starting point X_0 is defined, X being the vector with random variables.
2. The Z-functions and gradients for all limit states are determined; this requires $(1 + n)$ FEM computations, where n is the number of random variables.
3. For every individual limit state a design point is estimated using the standard FORM procedure and the corresponding Z-values are evaluated using FEM. This brings the total number of FEM computations at $(n + 1 + k)$, where k is the number of limit states.
4. A non-linear Response Surface is constructed on the basis of the $(1 + n + k)$ results
5. An iteration procedure is started, each iteration consisting out of three substeps:
   (i) a standard FORM calculation on the basis of the current Response Surface, leading to a new set of k design point estimations
   (ii) a FEM calculation for each new design point
   (iii) the construction of a new Response Surface based on the extended data base of design point estimations

After N iterations the number of FEM calculations is $(1 + n + Nk)$.

6. The iteration process is terminated if for all limit states the value of the limit state function is sufficiently close to zero.

The advantage of the above scheme is that number of the time consuming (non-linear) FEM-computations is reduced. Some remarks should be made:

(1) The Response Surface is constructed in such a way that it gives a good approximati-on of the limit state function in the neighbourhood of the last design point X_j in the FORM iteration procedure. When sufficient data is available a Response Surface function with a number of quadratic terms fitted to this data. The most important variables are weighting more then other ones [18].
(2) Standard non-linear FEM computations are normally carried out by applying load controlled increments. In the present method not only the final result (final increment) is stored in the data base. The results at the intermediate increments are stored as well and are used in the construction of the Response Surface.
(3) In some cases there will be no quantitative final result of the FEM computations, for instance due to geometrical instability. In that case the value of the Z-function is negative but unknown and is not used in the procedure. Since there are results of the intermediate load increment steps (see (2)), the Response Surface can still be updated.

After the FORM procedure has been completed a combination procedure has to be applied to compute the system reliability. Furthermore the procedure has to be repeated until all failure paths have been investigated.

Directional Sampling

The method of Directional Sampling has first been suggested by Deák [3]. The first step is to transform the set of variables X into a set of standardised independent Gaussian variables U.
An estimate of the probability of failure P_f can be obtained by performing N simulations of the U-vector. Using the FE-code, for each run j and for each limit state i, the value of (is determined for which:

$$Z_{ij} = Z(\lambda_{ij} U / |U|) = 0 \tag{2.1}$$

Every simulation results in a sample value

$$p_{ij} = \chi^2(\lambda_{ij}, n) \tag{2.2}$$

$\chi^2(..)$ = The CHI squared distribution
n = The number of random variables in the limit state function

An estimate for the probability of failure for mode i is calculated as the mean value of the sample values p_1:

$$E(P_{fi}) = \frac{1}{N} \sum_{j=1}^{N} p_{ij} \tag{2.3}$$

The procedure of Directional Sampling as such is already a fast Monte Carlo method However, it still requires quite a number of simulations and a number of FE-calculations per simulation.
The procedure can be speeded up in a similar way as has been discussed for the FORM procedure, that is by fitting a Response Surface to former results. When sufficient data is available, the estimate of λ_{ij} using the current Response Surface will already be very close to the correct one. This will limit the necessary number of FEM calculations.

Another improvement is to combine the ideas of Directional Sampling and Importance Sampling, for instance by adapting the variance of the variables. This has first been proposed by [1]. However, it proves to be very difficult to find the optimum multiplication factors for the variances. A more promising procedure is to truncate the probability distributions at the 50 percent fractile for those variables that have a clear positive or negative influence on the limit state function, that is, the typical load and resistance parameters. It turns out that even if only a limited number of variables is treated this way, the number of runs can be reduced considerably [18].

3 Probabilistic analysis of the Stüssi-Kollbrunner beam

A well-known phenomenon in structural plasticity theory is the influence of the rotational capacity on the plastic resistance of structures. Results of elementary plastic analyses are valid only if the structure has sufficient deformation capacity. In some cases this deformation capacity even has to be infinite. Fortunately insufficient deformation capacity can be compensated for by some hardening properties of the material.

One of the famous examples to show the necessity of deformation capacity is the Stüssi-Kollbrunner beam [13] [11]. This I-beam (see figure 1) consists out of three adjacent spans, the middle span loaded by a concentrated force in the centre of the span. According to the theory of plasticity, the simple upper and lower bound for the load bearing capacity is given by:

$$F_y = 8M_y/l \tag{3.1}$$

F = mid span force
M_y = fully plastic moment of the beam
l = central span length

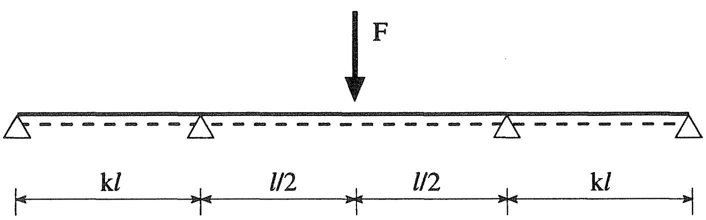

Fig. 1. Stüssi-Kollbrunner beam.

The point of discussion is that the load bearing capacity does not depend on the length of the outer spans. To investigate this point, the beam can be analysed assuming a perfect elastic plastic material behaviour with an infinite deformation capacity. This leads to the load deflection diagrams as seen in figure 2. Normally the load deflection line shows two changes of direction. The first change corresponds to the forming of a plastic hinge midspan, the second to the forming of plastic hinge at the supports. It turns out that for all values of k (= ratio of the span lengths) the elementary load bearing capacity of $8\,M_y/l$ can be reached, except for $k = \infty$. In that case the load bearing capacity is only $4\,M_y/l$, that means only 50 percent. This is called the Stüssi-Kollbrunner paradox.

Figure 2 learns that the deflection at which the load bearing capacity of $8\,M_y/l$ is reached, becomes larger and larger for increasing k. For $k = \infty$ the deflection simply becomes unlimited. For k up to 3 the theory was confirmed by tests, indicating that the rotation capacity of the midspan high was enough and that the plastic theory gave correct predictions for those values. For larger values of k the deflections were considered to be so large that practical applications were considered as being irrelevant. This solved the paradox at that time.

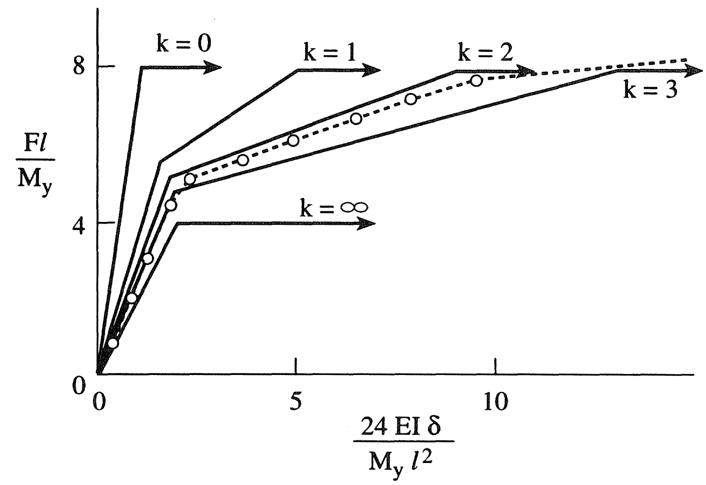

Fig. 2. Load deflection relations for beam of figure 1 (from [11]) (dotted line indicates test results).

In this paper we will investigate the beam using a more advanced model including the finite deformation capacity of the material. Also the uncertainties in the various parameters will be taken into account, using the theory of chapter 2. The following for two types of material have been considered (figure 3):

normal steel:	$r = f_u \,/\, f_y = 1.5$ and $\varepsilon_u = 0.5$
high strength steel:	$r = f_u \,/\, f_y = 1.1$ and $\varepsilon_u = 0.2$

where f_y and f_u represent the yield and rupture strength respectively and ε_u is the strain at rupture.

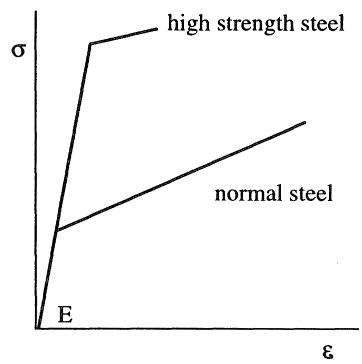

Fig. 3. σ-ε diagrams for various material properties.

The critical limit state in this problem is failure of the outer fibre at mid span. It is assumed that beyond this point no more load can be added. So, formally, the Limit State Function can be written as:

$$Z = \varepsilon_u - \varepsilon(f_y, r, F) \tag{3.2}$$

The computations do not include local buckling of flanges or similar mechanisms.

The mean value of the load on the beam in the probabilistic analysis has been chosen in such a way that normal design conditions are fulfilled:

$$\gamma_f \, \mu(F) = 8 W_y f_{yk} / (\gamma_m l) \tag{3.3}$$

where:

l	$= 10$ m	= span
W_y	$= 0.00126$ m^3	= plastic section modulus (HE 300 A)
f_{yk}	$= \mu(f) - 1.64\,\sigma(f)$	= characteristic value of the yield stress
γ_f	$= 1.5$	= partial load factor
γ_m	$= 1.1$	= partial resistance factor

The coefficients of variation (COV), needed for the probabilistic analysis, are presented in Table 2. The COV of the load is chosen as a weighted average for all loads (self-weight, wind, snow and live load). The mean values and COV's of the material behaviour are used from [15]. All variables are fully correlated over the length of the beam. Yield stress and the rupture yield ratio are assumed to

Table 2. Statistical Properties of random variables.

variable	unit	lower bound	mean	COV	distribution type
E	GPa	-	210	0	deterministic
F (normal steel)	kN	-	6.4	0.2	normal
f_y (normal steel)	MPa	-	240	0.08	normal
ε_u (normal steel)	-	-	0.5	0.25	normal
r (normal steel)	-	1.5	1.6	0.06	shifted lognormal
F (high steel)	kN	-	16.0	0.2	normal
f_y (high steel)	MPa	-	600	0.08	normal
ε_u (high steel)	-	-	0.2	0.25	normal
r (high steel)	-	1.1	1.2	0.08	shifted lognormal

be negatively correlated with $\rho(f_y,r) = -0.5$. This means that high yield stress does not automatically implies a high rupture strength. Note that due to the choice of a shifted normal for r the rupture stress is always greater than the yield stress.

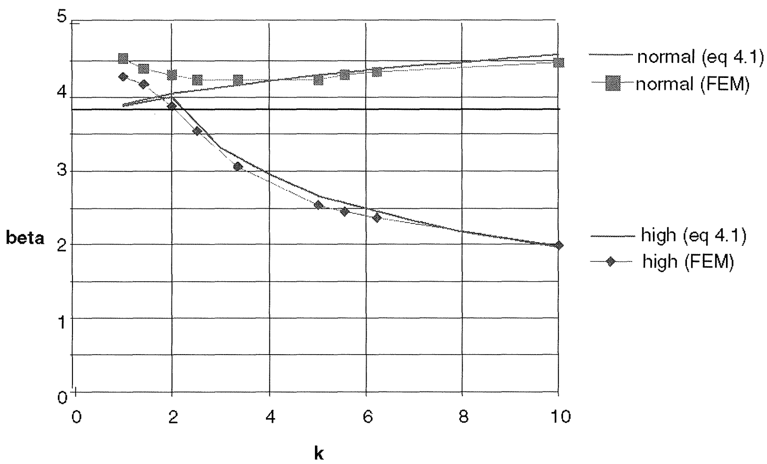

Fig. 4. *Reliability index β for normal and high strength steel as a function of k (kl = side span length); FEM and simplified calculation*

The results of the reliability analysis are shown in figure 4 (FEM lines) where the reliability index β is presented against the span ratio k for the two materials. The analysis is based on a FEM model with 20 beam elements for the midspan and 8 elements for the outer spans. The numerically integrated elements have 2 times 35 integration points. The reliability indices have been computed by both FORM and Directional sampling and give the same results. The reliability index does not decrease with increasing k for normal steel due to its large deformation capacity. High strength steel has a lower deformation capacity which is reflected in the results by a decrease of reliability with increasing k. Already for $k > 2$ it drops below the $\beta = 3.8$ limit that has been specified informally in the Eurocode as a target value.

For both high strength steel and normal steel, the force F has the maximum influence: $\alpha(F) = 0.7$; the other influences are evenly distributed over the material parameters, f_y, ε_u and r.

The computational time for the direction sampling with adapted surface technique without importance sampling is 20 minutes on a Pentium PC (120 samples using 1200 FEM computations). The FORM procedure with adapted response surface takes approximately 20 FEM computations. For this case, where only one limit state is important and where the limit state function near the design point is hardly nonlinear, the FORM is much faster than the Directional Sampling method. The efficiency of Directional Sampling is more pronounced when more limit states are involved and/or with highly nonlinear limit state functions.

4 Simplified model

It is interesting to see to compare the above results with those of a simplified model which is based on an M-κ-diagram (see figure 5). In this simplified model the Limit State Function can be presented as:

$$Z = R - F \tag{4.1}$$

The resistance R equals:

$$R = 4(M_f + M_s)/l \tag{4.2}$$

M_f = midspan moment
M_s = moment at the support

The mid span rotation corresponding to these moments is equal to (see figure 6):

$$\varphi_p = (0.67 M_s kl + 0.25 M_s l - 0.25 M_f l)/EI \tag{4.3}$$

Based on the M-κ-diagram, this rotation also equals:

$$\varphi_p = 0.5\,\kappa\Delta \tag{4.4}$$

κ = midspan curvature = $2\varepsilon/h$
h = beam height (= 300 mm)
Δ = half length of the plastic hinge

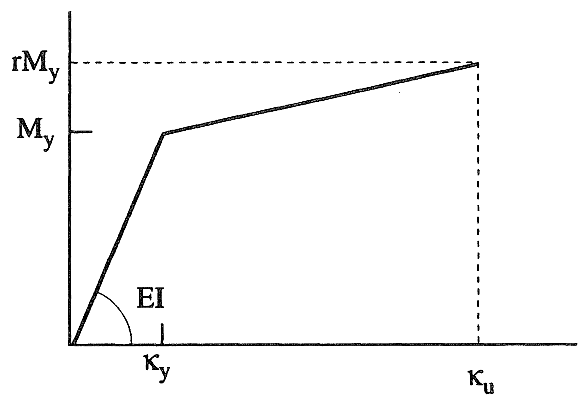

Fig. 5. M-κ-diagram.

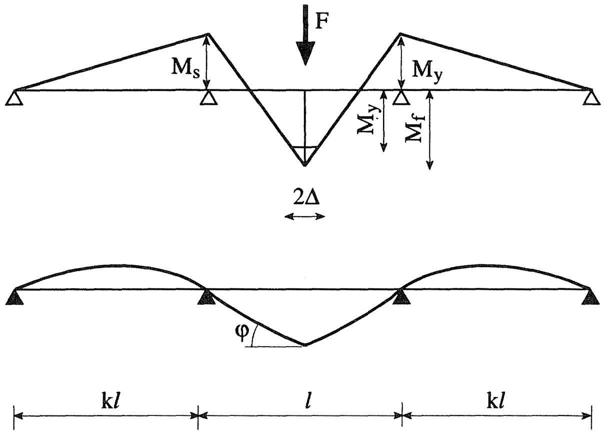

Fig. 6. Moments and displacements.

The "half length of the plastic hinge" Δ follows from (see figure 6):

$$M_y + R\Delta = M_f \tag{4.5}$$

where $M_y = W_y f_y$ and, according to the M-κ-diagram, the midspan moment can be expressed as:

$$M_f = M_y\left[1 + \frac{\kappa}{\kappa_u}(r-1)\right] \tag{4.6}$$

κ_u = curvature at rupture = $2\varepsilon_u/h$

Now, for each κ, using the formulas (4.2) - (4.6) the values of M_f, M_s, Δ and R can be found.
The resistance R corresponds to the lowest κ where either $M_s > M_y$ or $M_f > rM_y$. Applying the same data as before we find using standard FORM, the lines indicated in figure 4 as "eq. 4.1". The results are in good agreement with the FEM calculations. For $k < 2$ the limit $M_s = M_y$ leads to some difference with the FEM calculations where this limit was not introduced.

5 Conclusion

This paper has revisited the classical Stüssi-Kollbrunner beam, using a probabilistic approach, for both normal and high strength steel. The analyses confirm that normal steel easily fulfils the rotation capacity requirements for this case. On the other hand, high strength steel with a fracture over yield ratio of 1.1 and a reduced fracture strain value shows a clear shortage in the reliability index for span ratio k values over 3. It should be noted that in this analysis no attention has been given to unfavourable influences like eigen stresses, temperature loads, settlements, etc.

The paper has further attended the aspects of calculation technique and calculation efficiency.
The reliability analysis in this paper has been based on a Finite Element Program. In order to keep

the calculation time within suitable limits use has been made of FORM or Directional Sampling and an integrated form of Response Surface technique. The experiences in the problem at hand were very satisfactory.

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