Stability analysis for two-phase pipe flow
Master Thesis
with liquid backflow along the wall

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Preface

In front of you lies my Master thesis, which presents the findings of my research into mechanistic modeling of two-phase pipe flow. This thesis is the product of my graduation project. Being unfamiliar with multiphase flow at the start, this research was challenging but it became a great opportunity to learn. I enjoyed to this research and proudly present this thesis.

I would like to thank Ruud Henkes and Alexander van Zuijlen for their excellent guidance during this research. Their ideas and the discussions during the meetings and their advise in the writing of this thesis truly contributed to the success of this research. I also like to thank Mick Bouman and Frank Bos from DRG for their support and offering their facilities during this research. Finally I would like to thank my friends and family for giving me a ’kick in the butt’ at tough times, especially my parents who are always there when I need them.

I hope that you enjoy reading this thesis!

Michiel Valentijn Bakker
Summary

Accurate modeling of gas-liquid pipe flow is essential for the flow assessment in the oil and gas industry. This can be for both onshore and offshore pipeline systems. The distance from the subsea gas and oil wells to the production facilities can be large, such as for the Ormen Lange field which lies more than 100 kilometers off shore. The pipe systems are so large (often over a hundred kilometers) that multi-dimensional modeling would require an excessive computational effort. Therefore the models are kept as simple as possible and are one-dimensional. For the largest part of the Ormen Lange pipeline system the pipelines are at a depth of roughly 95 meter. The seabed is not perfectly flat, which creates small inclination and declination angles, resulting in many lower elbow sections where liquids can accumulate, particularly for relatively low gas production rates at turndown and in later field life.

Especially those circumstances, where liquid is accumulating in the low spots and remains stagnant for some time, give modeling problems. This two-phase flow has a low liquid holdup and a zero net liquid upward inclined flow (since the liquid is dragged upward by the gas). At a certain gas velocity the liquid will be blown out of the lower elbow section and the moment just before this happens is called the critical condition. The accuracy of the prediction when and how this critical condition occurs is an important performance indicator of the model, i.e., whether it is suitable for flow assurance calculations.

The start of this research consists of an evaluation of existing liquid wall shear stress models and their properties. Often the effect of liquid backflow along the wall is neglected. This backflow certainly plays a role in the force equilibrium at and around the critical condition. Two mechanistic liquid wall shear stress models, also referred to as closure relations, do allow for liquid backflow. Their derivation and their performance is presented in this thesis.

Also a steady state analysis on the critical condition is performed. First, for all gas velocities there is the trivial solution, in which case there would be no liquid in the pipe. A steady state analysis shows that there is indeed a certain gas velocity for which a hanging liquid layer could just still exist in an upward inclined pipeline. A gas velocity above this critical gas velocity would blow the liquid out of the pipe. Below the critical condition the steady state analysis gives three solutions, namely zero liquid holdup, a lower liquid holdup and a higher liquid holdup. Moreover, the critical superficial gas velocities predicted by the laminar and turbulent liquid models were compared to the experimental results obtained by Milos Birvalski[1]. This showed that the laminar liquid model was capable of predicting the superficial gas velocity with an error below 1%. This is significantly better than the turbulent liquid model which gave an error of ±5%. Therefore the laminar liquid model is superior to the turbulent liquid model when it comes to predicting the critical condition.

Note that the critical condition is an exact point, in which two nonzero solutions are joined together. In practice it is interesting how the flow behaves right before or around the critical condition. Therefore the behaviour of the averaged liquid holdup around the critical condition with respect to time has also been simulated. Thereby a nonlinear system of ordinary differential equations is derived. In order to do this the pipeline is discretized for a certain pipe length along which the liquid holdup remains constant. The results show that there is indeed a domain of initial conditions that would result
in the highest liquid holdup found in the steady state analysis. The lower boundary of this domain is determined by the lower liquid holdup solution found in the steady state analysis, which behaves linearly unstable. The higher boundary of this stability domain of higher liquid holdup is determined by the length of the pipe set for the simulation. Therefore this analysis shows that there is a domain for which a higher liquid holdup is stable close to the critical condition.

It is, however, still unknown how interfacial waves influence the stability of the higher liquid holdup solution since interfacial waves were not modeled in the previous stability analysis. Therefore a wave stability analysis has been performed. This simulation focuses on the propagation of interfacial waves with respect to time and space. The main goal is to see whether waves would grow or decay in amplitude when the conditions are close to the critical one. By neglecting surface tension at the interface it becomes possible to rewrite the system of partial differential equations such that it has a hyperbolic character. This allows to obtain a set of characteristic velocities along which ordinary differential equations describe the complete system.

This wave equation is derived and implemented in a Matlab program and a perturbation wave is set as the initial condition. The results show that for small amplitude waves the wave amplitude will decay over time. For large wave amplitudes the wave amplitude will grow with respect to time. When the gas velocity is increased above the critical condition rolling waves occur. Therefore it is concluded that close to the critical condition the system is stable for small waves (linearly stable), but unstable for larger waves (nonlinearly unstable).
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<td>$\bar{\tau}_i$</td>
<td>Average interfacial shear stress [N/m$^2$]</td>
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<tr>
<td>$\bar{u}_g$</td>
<td>Average gas velocity [m/s]</td>
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<td>$\bar{u}_l$</td>
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</tr>
<tr>
<td>$\varepsilon_l$</td>
<td>Liquid holdup, $\frac{A_l}{A}$ [-]</td>
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<tr>
<td>$\eta$</td>
<td>Bipolar transformation coordinate</td>
</tr>
<tr>
<td>$\mu_l$</td>
<td>Viscosity of the liquid [kg/m/s]</td>
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<tr>
<td>$\nu_g$</td>
<td>Dynamic viscosity of the gas [m$^2$/s]</td>
</tr>
<tr>
<td>$\nu_l$</td>
<td>Dynamic viscosity of the liquid [m$^2$/s]</td>
</tr>
<tr>
<td>$\nu_{water}$</td>
<td>Dynamic viscosity of the gas [m$^2$/s]</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>Density of the gas phase [kg/m$^3$]</td>
</tr>
<tr>
<td>$\rho_l$</td>
<td>Density of the liquid phase [kg/m$^3$]</td>
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<td>$\sigma$</td>
<td>Interfacial tension between gas and liquid</td>
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<td>Wall shear stress of the gas phase [N/m$^2$]</td>
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<td>$\tau_i$</td>
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<td>$\tau_l$</td>
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\( f_g \) Gas wall friction coefficient [-]
\( f_i \) Interfacial friction coefficient [-]
\( f_l \) Liquid wall friction coefficient [-]
\( Fr_g \) Froude number of the gas phase [-]
\( g \) Gravitational acceleration [m/s\(^2\)]
\( k \) Average (pipe) wall roughness parameter
\( L \) Pipe length [m]
\( P \) Pressure [N/m\(^2\)]
\( R \) Pipe radius [m]
\( S \) Pipe perimeter [m]
\( S_g \) Perimeter of the gas phase [m]
\( S_i \) Perimeter of the gas-liquid interface [m]
\( s_i \) Interfacial surface parameter in MARS model
\( S_l \) Perimeter of the liquid phase [m]
\( u_g \) Gas velocity [m/s]
\( u_i \) Interface velocity [m/s]
\( u_l \) Liquid velocity [m/s]
\( u_{g,S} \) Superficial gas velocity [m/s]
\( u_{l,S} \) Superficial liquid velocity [m/s]
\( We_{l,S} \) Superficial liquid Weber number [-]
\( X^2 \) Lockhart-Martinelli parameter [-]
\( Y \) Normalized inclination parameter
\( Re_D \) Reynolds number with the pipe diameter as characteristic length [-]
\( Re_{g\text{MARS}} \) The Reynolds number of the gas phase in the MARS model [-]
\( Re_{l,S\text{MARS}} \) [-]
1. Introduction

With an increasing global demand for energy, obtaining fossil fuels remains essential. Over the past decade more offshore deep sea resources were found, which poses new challenges to pipeline engineers. One such example is described in the following section.

1.1 Ormen Lange gas field

In 1997 a large gas field was found about 100 kilometers from the coast of Norway. Ten years later the production started using two parallel pipelines which transported mostly gas, but also light oil (condensate) and water from the well to the processing facility on shore.

The pipelines lay on the seabed of which the surface is rough. This results in many ups and downs with small pipe inclination angles. This is clearly shown in figure 1.1. After a steep upward section for the first 25 kilometers the pipe system lays for approximately 95 kilometers at a depth of 100 meters. The pipe angles along the pipeline are small, ranging for the largest part from $-3^\circ$ to $3^\circ$.

One of the major challenges is the accumulation of liquid in the V-sections or lower elbow sections of the pipelines. The correct prediction of the minimum required velocity of the gas that will sweep out the liquid is essential for the flow assurance assessment.

![Characteristics of the seabed of the Orman Lange pipeline system. [1]](image)
1.2 Single-phase pipe flow modeling

Flows through pipes have been around for centuries. It was therefore no surprise when engineers tried to come up with calculations in order to be able to assure that a fluid would go from point A to point B. Soon it was found that the effect of the wall friction was hard to model. The so-called head loss $H_l$ is a measure of wall friction in pipe flow defined from the Bernoulli principle and is shown in equation 1.1,

$$H_l = \left( \frac{U_1^2}{2g} + \frac{p_1}{\rho g} + z_1 \right) - \left( \frac{U_2^2}{2g} + \frac{p_2}{\rho g} + z_2 \right)$$  (1.1)

Where $U_l$ is the average liquid velocity, $\rho$ the liquid density, $g$ the gravitational acceleration, $p$ the static pressure and $z$ the height at the specific location 1 or 2.

Julius Weisbach\[5\] was among the first to present a model. He determined that the head loss was a function of the fluid velocity, fluid properties, pipe length and diameter and other pipe (wall) properties. Therefore he proposed equation 1.2,

$$H_l = f \frac{L}{D} \frac{U_1^2}{2g}$$  (1.2)

Where $D$ is the pipe diameter, $L$ is the pipe length and $f$ is the pipe (wall) friction factor. The friction factor was then found using equation 1.3,

$$f = \alpha_{\text{Weisbach}} + \frac{\beta_{\text{Weisbach}}}{U_1^2}$$  (1.3)

and in this equation $\alpha_{\text{Weisbach}}$ and $\beta_{\text{Weisbach}}$ are friction coefficients that would vary for different pipe diameter and pipe material. Experiments were performed in order to acquire data on the friction coefficients. Weisbach made a great start in the modeling of pipe flow with one fluid. Many experiments were performed in order to achieve data on the friction coefficients. However, it was Johann Nikuradse who took a different approach to determine the friction factor $f$.

Johann Nikuradse was a student of Ludwig Prandtl. Nikuradse has performed numerous experiments to see how the surface roughness of the wall in pipe flows would affect the friction (and therefore the head loss) \[2\]. In his experiments Nikuradse glued sand grains to the pipe wall to create a specific and uniform roughness. One of his results is shown in figure 1.2

Nikuradse’s results may look familiar. A more common plot on friction factors versus Reynolds number is the Moody diagram, but more on that later. For the laminar region (low Reynolds numbers) it were Jean Poiseuille (1841) and Gotthilf Hagen (1839)\[5\], who independently found the relation for the friction factor that is now known as,

$$f = \frac{64}{\text{Re}_D}$$  (1.4)

Where the Reynolds number is based on the pipe diameter.

1.2.1 Colebrook-White equation

For the transitional and turbulent region (where the Reynolds numbers are larger than 2300) there is not such a straight forward relation between the Reynolds number and the friction coefficient. Colebrook and White\[3\] proposed relations based on multiple experiments. In figure 1.3 multiple lines from experiments (also those of Nikuradse) are shown together with their proposed relation. Note that Nikuradse’s results are different than the results of other experiments and to the proposed relation by Colebrook and White. For his experiments Nikuradse used a uniform sand grain-size surface, which seemed more ‘perfect’ and therefore showed different results than for instance a galvanized or wrought iron surface.
Figure 1.2: One of Nikuradse’s measurement results for water pipe flow[2].

Figure 1.3: Colebrook’s proposed relation compared to various experiments. [3]
Colebrook and White[3] choose to approximate the results of the measurements by using more ‘realistic’ surfaces instead of Nikuradse’s ‘idealistic’ surface and this led to equation 1.5.

\[
\frac{1}{\sqrt{f}} = -2\log_{10} \left( \frac{\varepsilon}{3.7D_h} + \frac{2.51}{	ext{Re} \sqrt{f}} \right)
\]  

(1.5)

Moody[6] created a diagram in 1944 combining most of the available knowledge on friction factors, i.e. Poiseuille’s relation, Colebrook-White equation. Actually, Rouse[5] had proposed such a diagram already, only using a less practical grid. It is therefore that his diagram did not get much attention.

Later the American Gas Association[7] decided to use a different method to calculate the friction factor. Panhandle Eastern Pipe Line Company had acquired measurements from one of their pipelines and used those measurements to set up a model. This model was more practical in the sense that it could also correct for weldings or small amounts of solids which could be in the pipelines. Also the use of the Panhandle A, Modified Panhandle and Weymouth equations is very limited. Moreover, the way that they take into account these weldings or other imperfections is by using a so-called efficiency factor which has to be determined per specific calculation. This makes it only possible to calculate very rough indications of the head loss. From this point there were two different paths followed in order to optimize the friction factor calculation.

### 1.2.2 Explicit approximations of Colebrook-White equation

One was to try to approach the implicit Colebrook-White equation by an explicit equation so that it would be much faster to calculate the friction factor. Moody was one of the first to propose an explicit approximation to the implicit Darcy-Weisbach equation. Therefore he was able to create his famous diagram at that time. Later on Haaland[8] noted that Moody’s approximation had an error from -16% to 13%. He also had the opinion that the Colebrook-White equation was not that accurate in itself being made from experimental results. Therefore it seemed unnecessary to try and approximate those implicit relations within 1% error for the overall range. His criticism was not appreciated by everyone in the field[9]. Still, Haaland’s requirement was already achieved by Swamee and Jain[10] or Zigrang and Sylvester[11] but also by Haaland’s own proposed formula (which looked a lot like Swamee and Jain’s formula [12]).

Even though, according to Haaland, it seems unnecessary to further improve the accuracy of explicit approximations to the Colebrook-White equation, researchers kept coming with improvements. Finally Buzzelli[13] proposed the best performing equation, being accurate and relatively simple at the same time according to Heydari, Namiri and Pakniya[14].

### 1.2.3 Proposed improved physical models

The second way was to look at the measurement data and modify or extend the method of modeling the friction factor. The goal is to add more physical behaviour in the model, to see if it would give more knowledge on the behaviour of the friction factor. Bratland[15] tried adding a surface uniformity parameter and rewrote the Colebrook-White equations, also using the results of Nikuradse. Finally he presented a modified Moody diagram.

Herwig[16] proposed a different approach in order to obtain the friction factor. Using the thermodynamic laws the mechanical dissipation was modeled using CFD for several pipes with periodic wall surfaces. These results were similar to experimental results, therefore they were able to calculate friction factors based on the second thermodynamic law based on mechanical dissipation.
1.3 Two-phase pipe flow modeling

As shown in the previous section, single-phase pipe flow modeling is challenging. Especially accurate wall friction factors predictions. Pipe flows with multiple phases increase the complexity.

Multiphase flows are very common in practice. As soon as there is condensation in a gas pipe flow it becomes a multiphase flow. Or also when there is debris in a water flow (or gas flow) it becomes a multiphase flow. Simply stated, when there are multiple phases or different fluids in a flow it becomes multiphase. Multiphase flows behave differently compared to single-phase flows. For instance, the presence of a small amount of gas in a liquid flow makes the speed of sound in that flow drop dramatically. There are also fluid flows where more than two phases occur. Realistically in oil-pipe systems gas, oil and water are present for instance. This complicates the analysis of such systems drastically. This paper will restrict itself to two-phase flows, more specifically to gas-liquid systems. What happens in such gas-liquid systems? What are the the biggest challenges in gas-liquid systems?

1.3.1 Flow regimes

Bratland [4] describes in his book on multiphase flows the different flow regimes which occur in a two-phase flow. A flow regime is the ‘type’ of flow that occurs in a pipe flow. The regime is affected by the pipe diameter, the pipe inclination, the volumetric throughput of gas and liquid, and of the fluid properties. Accurate flow regime analysis remains a challenge and Taitel and Dukler[17] proposed a relation which is still used.

There are multiple flow regimes defined, see figure 1.4. The two major types are dispersed flow and separated flow. In a dispersed flow one of the two phases is prevalent, for instance a liquid flow with small gas bubbles or a gas flow with small droplets of liquid. Note that the speed of sound is drastically adjusted for a liquid flow with tiny gas bubbles. Stratified flow (which is an example of separated flow) contains of two layers of the different fluids flowing above and below the other. The liquid flows on the bottom of the pipe and a gas flows above it in case of a gas-liquid stratified flow. Within the stratified flow regime there are different regimes as well. At higher
gas flow velocities for instance a wavy liquid surface starts to form, this is called the wavy-stratified flow regime. Under certain conditions, especially with higher liquid holdup, slug flow will occur, in some regions of the pipe flow there will be only liquid flow and other regions there will be stratified or even only gas flow. These slug flows create problems for example for pumps, where suddenly the inflow changes from only liquid to a little liquid with a lot of gas [4][17]. This research will continue to focus on stratified gas-liquid flows without slugs. How are such flows modeled? What are the modeling challenges? What are the accuracies achieved by different methods? Modeling multiphase flows requires much more complex models compared to single-phase flow. The interaction between the flows greatly changes the behaviour of the systems as a whole. Different flow regimes act very differently and therefore require different modeling. In this study the focus is on the stratified two-phase flow models. In the beginning the models were based upon empirics but later on more sophisticated models based on the physical behaviour of multiphase flow were created. In the following chapters two of these models will be discussed and explained in detail.

1.4 Empirical versus mechanistic modeling approaches

When looking at models that simulate pipe flow one comes across two different approaches namely, empirical and mechanistic. All models are both empirical and mechanistic, however some are more mechanistic and some more empirical. The main difference between a mechanistic and empirical model is that mechanistic models try to include relations based on physical principles, whereas empirical models try to find the best fit to experimentally obtained results. Often mechanistic models are more generally applicable while empirical models have a restricted domain.

The first widely accepted more mechanistic model was proposed by Taitel and Dukler[18]. In short the TD model uses two sets of mass and momentum conservation laws, one for each phase. Thereafter the sets of equations are coupled through the boundary conditions in which both have a wall boundary condition and an interfacial boundary condition. The relations for these boundary conditions are called the closure relations. The closure relations proposed by Taitel and Dukler were based mostly on empirical results from experiments.

In 2000 it were Petalas and Aziz[19] who proposed another mechanistic model. This model was one of the first that tried to model the interfacial friction by finding closure relations for the interfacial waves. There results were good but not great as they still could not represent the detailed properties of the average wave accurately. However this model was a step towards more physical models.

Dag Biberg proposed a mechanistic model in 1999[20]. This model will also later be discussed in more detail. In essence a turbulence model is used as a basis to model the interfacial friction. Moreover this turbulence model describes the velocity distribution using pre-integrated equations. This velocity distribution is then used to find the shear stresses, closing the model.

1.5 The critical condition

As mentioned in the case study of the Ormen Lange gas field, the modeling of the sweep out velocity of the liquid in lower elbow sections is challenging but essential for flow assurance assessments. When this would not be correctly modeled the residue could accumulate in the pipes and more complex flows would arise including slugs. This would endanger the flow assurance.

The ’sweep-out’ condition is defined as the critical condition. When there is no gas flow, the liquid, in this case the liquid, is at rest in the bottom of the lower elbow section; see figure 1.5. As soon as the gas starts to flow, interfacial friction will drag the liquid upwards along with the direction of the gas flow. Due to the fact that the
density of the liquid is much higher the gravitational forces are dominant in the liquid flow. Therefore the liquid will flow downwards. At some point, right before the liquid would be blown out/upwards in the pipe, a steady state occurs and all the liquid 'stands' or 'hangs' in the upward inclined pipe. This situation is depicted in the lowest pipe shown in figure 1.5. This condition is called the critical condition. It is critical because as soon as the gas velocity would slightly increase the liquid would start to accelerate upwards in the pipe. Before this critical velocity is reached, all the liquid would simply remain in the V-section of the pipe. Birvalski[1] performed research into this critical condition, and published his findings in a PhD thesis. He concludes that this phenomenon is hard to model and that current closure relations will have to be improved.

1.6 Scope

This study focuses on the critical condition for upward inclined gas-liquid pipe flow for small inclination angles. This means that the liquid will have a zero net flow rate. Strictly stratified flow with low liquid holdup will be considered. This is the kind of flow that occurs when liquid will have to be swept out of V-sections in pipeline systems that lay on the bottom of the sea. It is this specific application where current models struggle to correctly predict the flow. The focus of the research is on the theoretical and analytical aspects of the flow. Throughout this thesis the models will be compared to experimental results if possible. Performing experiments lies beyond the scope of this research.

1.7 Research question and subquestions

With respect to the scope presented in section 1.6, the following research question is formulated:

To what extent are mechanistic models able to describe the critical gas velocity to sweep out liquids in the low elbows of pipelines?
This is the main research question. In order to be able to answer this main research question, three subquestions are defined below.

1. **What mechanistic two-phase stratified pipe flow models that allow for liquid backflow along the wall do currently exist?**

An investigation is performed into different closure relations published in literature. The result should be a selection of closure relations that will be used for further analysis. These models will be used and compared to answer the next subquestions and finally answer the main research question.

2. **How well is the critical condition predicted with mechanistic closure relations that allow for liquid backflow along the wall?**

Do the selected mechanistic closure relations model the critical condition? Are there differences between different closure relations and can they be explained? How are the shear stresses distributed? All these points should be addressed when answering the second subquestion.

3. **How does the liquid holdup behave in transient simulations close to the critical condition, using mechanistic closure relations that allow for liquid backflow along the wall?**

A pure theoretical study into the stability of the selected closure relations can be carried out. How do the closure relations behave with respect to time? Is the critical condition obtainable according to these closure relations? How do the closure relations respond to perturbations close to the critical condition?

### 1.7.1 Approach

In this thesis these questions will be answered. The first chapter will focus on different closure relations and it will present the derivations. These closure relations all have to meet the requirements set in the research scope and research questions. Once the closure relations are selected the second chapter will continue on how each set predicts the critical condition. These results will be compared to experiments performed by Milos Birvalski[1]. This will determine the capability in critical condition prediction of each model. The last two chapters, chapter four and five, present two different stability analyses. In chapter four a linear stability analysis is performed. This analysis shows the linear behaviour of a two-phase pipeflow with the selected closure relations. The interface is assumed flat during this analysis. In order to analyze the effect of waves on the surface a nonlinear wave analysis is performed in chapter five. This analysis studies the effect of waves and how they propagate through the pipe. Finally the research and subquestions will be answered in the overall conclusion, which is the last chapter of this thesis.
2. The holdup equation and closure relations

As shown in the introduction chapter there is a need for fast, transient multiphase pipeflow models that can be used for long pipeline systems. One-dimensional models are used to fulfill the need to predict the multiphase flow velocity, pressure drop and holdup values for these pipeline systems. In stratified low-liquid loading two-phase flow accurate friction modeling is essential for flow assurance calculations. The methods used to calculate the friction forces are often called closure relations.

This chapter presents several common models that are used in one-dimensional steady-state two-phase flow analysis. These are all based on the general Taitel-Dukler approach, where multiple different closure models have been proposed through the years. The first closure relations were largely based on empirical findings. These experiments do not cover the wide range of conditions of the actual deep sea pipe systems. The inclusion of more actual physical behavior of the two-phase flow instead of empirics should improve the applicability of the models.

The outline of this chapter is as follows. First the general Taitel and Dukler approach[18] will be reviewed and discussed. Thereafter the closure relations by Fortuin and Grolman[21] are presented followed by two different closure relations by Dag Biberg for the wall shear stress are described. Finally the models are compared to each other and discussed in order to justify the application of the models used further for upward inclined stratified two-phase flow.

2.1 Taitel and Dukler approach: the holdup equation

Taitel and Dukler[18] proposed a one-dimensional model that became a general model for stratified pipe flows. Originally it was a model designed to predict the correct flow regime of a two-phase pipeflow. However it became widely used for calculating holdup and pressure drop properties for two-phase stratified flow. Their full model will not be discussed here, however they were the first to write down the holdup equation. This is done by setting up the conservation law for the momentum for each phase individually. The subscripts \( l \) and \( g \) indicate the liquid phase and the gas phase, respectively.

The following assumptions are made:

- two-phase, stratified, steady-state, incompressible flow through a constant area pipe
- equal pressure gradients \( [dP/dL]_l = [dP/dL]_g = [dP/dL] \)

The model starts with two momentum balance equations, one for the liquid phase and one for the gas phase. These are shown in equations 2.1 and 2.2. See figure 2.2 for a physical interpretation of the momentum equations.
A \left( -\frac{dP}{dx} \right)_g - \tau_g S_g - \tau_i S_i + A_g \rho_g g \sin \alpha = 0 \quad (2.1)

A_l \left( -\frac{dP}{dx} \right)_l - \tau_l S_l - \tau_i S_i + A_l \rho_l g \sin \alpha = 0 \quad (2.2)

Here the $A$ represents the area for each phase, $dP/dx$ is the pressure gradient, $S$ is the wetted wall or interface surface, $\rho$ the density, $g$ the gravitational acceleration and $\alpha$ is the inclination angle with respect to the horizontal (positive angle for an upward inclination). The $\tau$ is the shear stress, which values will be calculated by the closure relations.

The assumption that the pressure gradients are equal makes it possible to write the momentum balances in one equation. This equation, also known as the holdup equation, is shown in equation 2.3. See figure 2.2 for a physical interpretation of the shear stresses of the pipe flow.

$$
\tau_g \frac{S_g}{A_g} - \tau_l \frac{S_l}{A_l} + \tau_i \left[ \frac{1}{A_l} + \frac{1}{A_g} \right] + (\rho_l - \rho_g) g \sin \alpha = 0
$$

(2.3)

Taitel and Dukler then used the same approach as was always done for single-phase pipe flow, using friction factors. See equation 2.4.

$$
\tau_g = \frac{1}{2} \rho_g \frac{u_{g,S}^2}{(1 - \varepsilon_l)^2}, \quad \tau_l = \frac{1}{2} \rho_l \frac{u_{l,S}^2}{\varepsilon_l^2}, \quad \tau_i = \frac{1}{2} \rho_g \left[ \frac{u_{g,S}}{(1 - \varepsilon_l)} - u_i \right]^2.
$$

(2.4)

Where $\varepsilon_l = A_l/A$ is the liquid holdup. Also $u_{g,S}$ and $u_{l,S}$ are the superficial velocities, meaning the average velocity that the fluid would have if it would be a single phase flow in the pipe. The modeling the shear stresses requires additional closure relations for the friction factors. These factors were usually found in tables which were created using a lot of experiments.

The closure relations complete the multiphase pipe flow model and they can be separated into two categories: dimensional and frictional relations. The dimensional closure relations describe how the fluid areas and perimeters are defined and calculated. They always need to be physically consistent, which means that,

$$
A_g + A_l = A \quad \text{and} \quad S_g + S_l = S
$$

(2.5)

since the sum of the areas of the two phases in two-phase flow equals the total pipe area, similarly for the wall surface. The interface shape (and therefore the interface perimeter) and fluid areas and perimeters are defined as functions of only the pipe diameter and a variable describing the liquid holdup $\varepsilon_l$ or gas holdup ($\varepsilon_g = A_g/A$).

The frictional closure models describe the fluid wall friction per phase and the interfacial friction between both phases. Wall friction modeling is a challenging task and for the friction calculation in single phase flow there exist no exact models. Therefore in single phase pipe flow, empirical results are used to estimate the wall friction. Moreover, applying the single phase wall friction models as multiphase friction models might introduce (large) errors. The phase interaction affect the force balance significantly.

Therefore finally there are the closure relations for the interfacial friction, which link both phases to each other. This connecting closure relation results in the interfacial shear stress. Once all the closure relations are defined the holdup equation can be solved. A closure relation is usually a set of equations which eventually determines a value as a function of the liquid holdup. The liquid holdup fraction is defined as the ratio of the liquid area and the total area ratio. Some closure relations use different variables like the wetted wall fraction to describe the liquid holdup as we will see later in this chapter.
2.2 Modified Apparent Rough Surface closure model

One of the possible set of closure relations is the Modified Apparent Rough Surface (MARS) model presented by Grolman and Fortuin\[21\]. This model was developed for two-phase pipe flow with a low liquid-gas ratio. The model has been verified against 2400 laboratory experiments and is claimed to be valid for a liquid holdup $\varepsilon_l$ in the range from 0 to 0.42, a superficial gas velocity $u_{g,S}$ in the range of 1.8 m s\(^{-1}\) to 34 m s\(^{-1}\) and a superficial liquid velocity $u_{l,S}$ in the range of 0 m s\(^{-1}\) to 0.06 m s\(^{-1}\). This makes this model suitable for the modeling of the cases measured by Birvalski [1] for comparison. Moreover, Birvalski also used this model in his simulations.

Due to its properties, the MARS model presented in a paper by Grolman and Fortuin\[21\], will be summarized in this section.

2.2.1 Friction factor relations

In the model also the relations for the friction factors are needed as a closure. Grolman and Fortuin based these relations mostly on the superficial Reynolds number of the phases. For instance, see the relation for the factor of the gas wall friction in equation 2.6,

$$f_{g,MARS} = 0.07725 \left( \log_{10} \left( \frac{Re_{g,MARS}}{7} \right) \right)^2 \quad \text{where,} \quad Re_{g,MARS} = \frac{\rho_g u_{g,S} D}{\nu_g (1 - \theta + s_i)}$$

For now ignore the $\theta$ and $s_i$, as these will be discussed later on. Note that this is an empirical relation based on laboratory experiments.

Now, the liquid wall friction factor is defined a little differently, see equation 2.7.

$$f_{l,MARS} = \begin{cases} f_i 202 \left( \frac{\nu_{water}}{\nu_l} \right)^{0.274} \theta_0 Re_{l,MARS}^{-1} Re_{g}^{0.25} & \text{whichever is the biggest} \end{cases} \quad (2.7)$$

where, $Re_{l,MARS} = \frac{\rho_l u_{l,S} D}{\nu_l}$

The interfacial friction factor has a slightly more complicated calculation method. Grolman and Fortuin found that the relative (sand) roughness can be related to the interfacial shear stress. This is not surprising since the interfacial shear stress influences the interfacial waves and vice versa. These waves create a surface roughness which can be modeled as a sand surface roughness. An iterative method was developed that converges towards an interfacial friction factor, which was fine-tuned using experimental results.

The iterative method consists of several steps, described below. Grolman and Fortuin defined a so-called friction number $Fn$, see equation 2.9,

$$Fn = \frac{f_i (0.05 + f_i) (1 - \varepsilon_l)^{3/2}}{(\sigma / \sqrt{gD}) \left( \frac{\sigma}{\sqrt{gD}} \right)^{0.04} \left( \frac{\rho_g D^2}{\sigma} \right)^{0.22}}$$

where $\sigma$ is the surface tension between the gas and liquid. As an initial value for $f_i$ the gas wall friction factor $f_g$ is used. The second step determines the relative (sand) roughness, see equation 2.10,

$$k = 0.5145 \varepsilon_l \left( \frac{S_i}{\pi D} \right)^{-3/2} (\tanh (0.05762 (Fn - 33.74)) + 0.9450) \quad (2.10)$$

The final step calculates the interfacial friction factor. This is the basic friction factor relation for rough pipes by Eck, see equation 2.11,
Figure 2.1: The cross section showing the curved interface between the gas and liquid for the MARS model. Note that the $\theta_{\text{MARS}}$ value does not have a geometrical meaning.

\[ f_i = \frac{0.0625}{(\log_{10}(\frac{15}{Re_g} + \frac{4}{3.713D}))^2}. \tag{2.11} \]

The resulting interfacial friction factor is then used as an input for the first step. This iterative process is continued until the interfacial friction factor is converged and does not change more than 0.1% between iterations.

To find the interfacial shear stress with equation 2.4 an interfacial velocity needed. This velocity is found using equation 2.12,

\[ u_i = \begin{cases} 
1.8 \frac{u_l S}{\varepsilon_l}, & \text{Re}_l < 2100 \\
\frac{u_l S}{\varepsilon_l}, & \text{Re}_l \geq 2100.
\end{cases} \tag{2.12} \]

### 2.2.2 Geometrical closure relations (MARS)

There is a final set of equations needed to close the model by Grolman and Fortuin. These relations describe the stratified flow domain, in other words the perimeter and area for the liquid, gas and interface. The MARS model does not use a flat interface between the gas and liquid phase. Instead a correction factor $s_i$ is defined to represent the curvature. The geometrical closure relations are shown below in equations 2.13 and 2.14 and figure 2.1

\[ S_i = s_i \pi D, \quad S_l = \theta \pi D, \quad S_g = (1 - \theta) \pi D, \tag{2.13} \]

and,

\[ A = \frac{\pi D^2}{4}, \quad A_l = \varepsilon_l A, \quad A_g = (1 - \varepsilon_l) A. \tag{2.14} \]

A couple of new terms enter the previous equations ($\theta_0$, $\theta$ and $s_i$). Here $\theta$ is the wetted wall fraction, defined as $\theta = S_l / (\pi D)$. The interfacial surface is not assumed to be flat but it is modeled such that the liquid will creep upwards along the pipe. Grolman and Fortuin adjusted the original relation by Hart et al. to make it work in upward and downward flow situations. They proposed equation 2.15,

\[ \theta_{\text{MARS}} = \theta_0 \left( \frac{\sigma_{\text{water}}}{\sigma} \right)^{0.15} + \frac{W_{lS}^{0.25} F_{lS}^{0.8} \rho_g}{(\rho_l - \rho_g) \cos \alpha}. \tag{2.15} \]

where the liquid Weber number and gas Froude number are defined as,
We \( \text{We}_{l,s} = \frac{\rho_l u_{l,s}^2 D}{\sigma} \), and, \( \text{Fr}_g = \frac{u_{g,s}^2}{(1 - \varepsilon_l)^2 g D} \).

Here \( \theta_0 \) is the minimal wetted wall fraction from a geometrical point of view. Therefore \( \theta_0 \) is described as implicitly as,

\[
\varepsilon_l = \theta_0 + \frac{\sin(2\pi \theta_0)}{2\pi}, \quad \text{or by explicit approximation,} \quad \theta_0 = 0.624\varepsilon_l^{0.374} \tag{2.16}
\]

Finally the normalized interfacial perimeter is found by an interpolation between the minimum value \( \theta_0 \) and the maximum value. This maximum value is found by letting \( \theta = 1 \), which physically means that the flow is annular. This results in the following equation 2.17,

\[
\frac{S_i}{\pi D} \equiv s_i = \frac{\theta - \theta_0}{1 - \theta_0} \sqrt{1 - \varepsilon_l} + \frac{1 - \theta \sin(\pi \theta_0)}{1 - \theta_0} \pi \tag{2.17}
\]

These relations can be substituted into the holdup equation 2.3. Notice that once doing so only the liquid holdup \( \varepsilon_l \) remains as a variable. Therefore one equation with one unknown remains which can be solved using a computer program to find the holdup value for which the holdup equation is valid. This concludes the MARS model description.

### 2.3 Turbulent liquid wall shear stress model for stratified flow by Biberg

One year after the MARS model was presented, Biberg\[22\] published a model that used a more analytical approach. This model used pre-integrated velocity distributions in order to determine the wall and interfacial shear stresses. Birvalski also used this model to simulate inclined upward zero-net-liquid two-phase pipe flow. The model for the calculation of the wall shear stresses consists of two components. One component describes the wall shear stress as if the phase was a single-phase free surface flow where the interface is frictionless. The second component takes the effect of the interface on the wall shear stress into account. According to Biberg this improves the accuracy of the wall shear modeling. Also the interfacial shear stress is calculated accordingly. The two-phases are both assumed turbulent. First the velocity distributions and shear stresses are derived for channel flow. Thereafter the channel flow equations are transformed to hold for pipe flow.

Note that in this report various types of velocities are used. Average velocities are indicated by a capital \( U \) while local or velocity distributions are indicated by a small \( u \). Superficial velocities are a time independent phase-area-scaled average velocity.

The model starts with an assumption for the shape of the gas velocity distribution, as shown in equation 2.18,

\[
u_g = a \ln \left( 1 - \frac{y}{h_g} \right) + b \ln \left( \frac{y}{h_g} \right) + c \tag{2.18}
\]

Two friction velocities are defined in order to find the coefficients \( a, b \) and \( c \). These are the friction velocities for the gas wall and interface, respectively shown in equation 2.19,

\[
u_g^* = \text{sign}(\tau_g) \sqrt{\frac{\varepsilon_g}{\rho_g}} \quad \text{and,} \quad u_i^* = \text{sign}(\tau_i) \sqrt{\frac{\varepsilon_i}{\rho_i}} \tag{2.19}
\]

Eventually the coefficients \( a, b \) and \( c \) are determined to be,
\[ a = \frac{u_g^*}{\kappa} \quad b = \frac{u_i^*}{\kappa} \quad c = \frac{u_g^*}{\kappa} \left\{ \ln \left( \frac{|u_g^*| \nu}{V_g} \right) + \kappa B_{Biberg} \right\} \text{ smooth} \]

Where \( \kappa \approx 0.41, B_{Biberg} \approx 5 \) and \( A_{Biberg} \approx 8.5 \). The upper case of equation 2.20 is used for smooth flow, the lower case for rough flow. This results in the following expression for the turbulent gas velocity distribution,

\[ u_g = \frac{u_g^*}{\kappa} \ln \left( \frac{y}{h_g} \right) + \frac{u_g^*}{\kappa} \left\{ \ln \left( \frac{|u_g^*| \nu}{V_g} \right) + \kappa B_{Biberg} \right\} \text{ smooth} \]

Note in equation 2.21 that the first term represents the effect of the interfacial shear flow while the second term represents the effect of the free surface flow (as if the gas flow was single-phase). Now the average gas velocity is determined by integrating the gas velocity distribution presented in equation 2.21. This results in,

\[ U_g = \frac{u_g^*}{\kappa} + \frac{u_g^*}{\kappa} \left\{ \ln \left( \frac{|u_g^*| \nu}{V_g} \right) + \kappa B_{Biberg} - 1 \right\} \text{ smooth} \]

A wall friction factor \( \lambda_g \) is introduced as,

\[ \tau_g = \frac{\lambda_g}{4} \rho_g \left| \frac{U_g}{U_g} \right| \left| U_g \right|^2 \]

The gas and interfacial friction velocities can be rewritten using 2.23 to be,

\[ u_g^* = \frac{U_g}{2} \sqrt{\frac{\lambda_g}{2}} \quad \text{and} \quad u_i^* = \text{sign} \left( \frac{\tau_i}{\tau_g} \right) \sqrt{\left| \frac{\tau_i}{\tau_g} \right| U_g} \left| U_g \right|^2 \]

By substituting these friction velocities back into equation 2.22 one can derive a definition of the gas wall friction factor. Note that the logarithms are switched to regular 10-based logarithms, this all results into,

\[ \frac{1}{\sqrt{\lambda_g}} = \left[ 1.986 \log Re_{D_k} \sqrt{\lambda_g} - 0.837 \right] + \left[ -C \text{ sign} \left( \frac{\tau_i}{\tau_g} \right) \sqrt{\left| \frac{\tau_i}{\tau_g} \right| U_g} \left| U_g \right|^2 \right] \]

where,

\[ Re_{D_k} = \frac{|U_g| D_g^*}{V_g} \quad \text{and} \quad D_g^* = \frac{8 h_g}{3} \]

Still, when looking at equation 2.25 note that the first term covers the effect of the free surface flow and the second term accounts for the shear flow. The diameter in equation 2.26 is the effective free surface channel flow diameter. Equation 2.25 can be changed such that one term describes the closed-duct (single-phase) contribution and a second term describes the two-phase flow contribution.

Now that the equations have been derived for one phase of a multiphase channel flow system the equations will be transformed to pipe flow. First we will take a closer look at the difference between the effective diameters earlier defined. To start the effective diameters defined here are effective diameters for single-phase laminar flow, not hydraulic diameters. Then finally there are two groups of effective diameters, namely ‘free surface flow’ (with the asterisk) and ‘closed duct flow’ (without the asterisk). For closed duct flow the interface between both fluids is seen as a solid wall, also adding to the shear stress. The free surface flow does not include the interface as a wall. This finally brings us to the following definitions, first for the channel flow,
\[ D_{g,\text{channel}}^* = \frac{8h_g}{3} \] free surface flow, \[ D_{g,\text{channel}} = \frac{4h_g}{3} \] closed duct flow. (2.27)

and for pipe flow,

\[ D_{g,\text{pipe}}^* = \frac{4A_g}{S_g} \] free surface flow, \[ D_{g,\text{pipe}} = \frac{4A_g}{S_g + S_i} \] closed duct flow. (2.28)

One can rewrite equation 2.25 such that instead of the 'shear flow' and 'free surface flow' terms the equations gets a 'closed duct flow' term and a 'two-phase effect' term. This means that the friction coefficient is build up of a laminar single-phase closed duct flow term and a term that describes the effect on the friction coefficient of the second phase in the pipe. The first term in equation 2.29 is the laminar single-phase closed duct flow, whereas the second term describes the second phase effects.

\[
\frac{1}{\sqrt{\lambda_g}} = \left[ \frac{1.986 \log Re_{D_g} \sqrt{\lambda_g} - 0.837}{1.986 \log \frac{D_{g,\text{pipe}}^*}{S_g} + 1.895} \right] + \left[ 1.986 \log \left( \frac{D_{g,\text{pipe}}^*}{D_{g,\text{pipe}}} \right) - C \text{sign} \left( \frac{\tau_i}{\tau_g} \right) \sqrt{\left| \frac{\tau_i}{\tau_g} \right|} \right].
\] (2.29)

Still there remains an unknown \( C \) in the equation. Now in one of Biberg’s papers [22] the \( C \) value is motivated by the statement that in a laminar and turbulent closed duct pipe or duct flow the ratio \( \sqrt{\tau_i/\tau_g} \approx 1 \). The paper describes that when a pure closed duct flow is present the 'two-phase effects' term should cancel out and become zero. This results in equation 2.30 for the value of \( C \).

\[
C \approx 2 \log \left( \frac{D_{g,\text{pipe}}^*}{D_{g,\text{pipe}}} \right) = 2 \log \left( 1 + \frac{S_i}{S_g} \right).
\] (2.30)

In a different paper[23] Biberg continues this derivation. Biberg starts by noting that the equation is equally valid for the liquid shear stress once the sign of the 'two-phase effect' term is changed. Since the value for the \( C \) term has already been evaluated the equation is rewritten to incorporate the definitions of the effective diameters for free surface flow. Moreover for the actual 'free surface flow' term (the first one) Biberg found an improved fit using the experimental results by Espedal[24]. This improved fit is shown in equation 2.31

\[
\frac{1}{\sqrt{\lambda_l}} \bigg|_{\text{free surface flow}} = 1.47 \log \left( \frac{Re_{D_l}}{1.44} \right).
\] (2.31)

This improved fit is therefore used by Biberg in the definition of the liquid wall friction factor. Finally Biberg proposes to change the second term as well to make it consistent with the findings from Espedal’s experiments. Due to the effects of interfacial waves, which are not covered in the theoretical models a rather drastic change is presented. The so-called 'wave term' is shown in equation 2.32 and is a complete replacement of the 'two-phase effect' term. The 'wave term' is based on the empirical results of Espedal and originates from the trail and error method.

\[
\frac{1}{\sqrt{\lambda_l}} \bigg|_{\text{two-phase effect}} = 1.47 \log \left( 1 - \tanh \left( 2.2 \frac{\tau_i}{\tau_l} \right) \frac{S_i}{S_l} \right) \sqrt{\left| \frac{\tau_i}{\tau_l} \right|}.
\] (2.32)

The final liquid wall shear stress model then becomes,

\[
\tau_l = \frac{\lambda_l \rho_l |U_l| U_l}{4}.
\] (2.33)
\[ \frac{1}{\sqrt{\lambda_l}} = \frac{1}{\sqrt{\lambda_l}} \bigg|_{\text{free surface flow}} + \frac{1}{\sqrt{\lambda_l}} \bigg|_{\text{two-phase effect}} \]
\[ \frac{1}{\sqrt{\lambda_l}} = 1.47 \log \left( \frac{Re_D^*}{1.44} \right) + 1.47 \log \left( 1 - \tanh \left( \frac{2.2}{\tau_i} \frac{S_i}{S_l} \right) \right) \sqrt{\frac{\tau_i}{\tau_l}} \quad (2.34) \]

So this model still uses empirics but is also based on more physical principles than the MARS model previously discussed. Note that still this model for the liquid wall shear stress makes use of an friction factor. This friction factor is however calculated using the free surface flow effect and the two-phase effect. The effect of backflow along the wall is covered by this two-phase effect in the friction factor. However, since the liquid wall shear stress is calculated using a friction factor, the liquid wall shear stress will be zero when the average liquid velocity is zero. Even though the friction factor will not be zero for a zero net liquid flow, the liquid wall shear stress will be. At the critical condition the average liquid velocity is zero.

### 2.4 Laminar liquid wall shear stress model for stratified flow by Biberg

One year later Biberg[25] published another paper where he proposed a new closure model for the liquid wall shear stress. This model is fully analytical as it assumes an exact laminar flow for the liquid which can be solved for pipe flow domains. For the low liquid velocities this is a reasonable assumption.

This section summarizes a paper published by Biberg on the liquid wall friction in turbulent gas and laminar liquid stratified pipe flow.

Assumptions for the liquid:
- homogeneous
- incompressible
- Newtonian
- flow is only in x-direction, \( \vec{U} = u_l(y, z) \vec{i}_x \)
- constant shear at the interfacial boundary, \( \bar{\tau}_i = \text{constant} \)
- constant viscosity

The Navier-Stokes equation in the x-direction is given by,
\[ \nabla^2 u_l = \frac{1}{\mu_l} \left( \frac{\partial P}{\partial x} + \rho_l g \sin \alpha \right) \quad \text{where,} \quad u_l|_{S_{lw}} = 0, \quad \mu_l \frac{\partial u_l}{\partial z} |_{S_i} = \bar{\tau}_i \quad (2.35) \]

See figure 2.2 for the definition of the domain. The boundary conditions mean that the velocity at the pipe wall is zero and the shear flow at the interfacial boundary equals the interfacial shear stress.

Equation 2.35 can be solved and results in the Hagen-Poiseuille solution. Which will give,
\[ u_l = -\frac{B_l}{4\mu_l} \left( R^2 - r^2 \right) + \phi \quad \text{where,} \quad r^2 = y^2 + (z - R \cos \delta_l)^2 \quad (2.36) \]
\[ \text{and,} \quad B_l = \frac{\partial P}{\partial x} + \rho_l g \sin \alpha \]
In this solution the first product is the particular solution to the partial differential equation and the \( \phi \) is the homogeneous solution. In order to find the homogeneous solution the equation is transformed to a bipolar coordinate system. This transformation is defined as:

\[
y = R \sin \delta_l \sinh \xi \cosh \xi + \cos \eta \\
z = R \sin \delta_l \sin \eta \cosh \xi + \cos \eta
\]

The transformation shown in equation 2.37 is applied to the particular solution part of equation 2.36. Note that the boundaries will have a constant value for \( \eta \), for the wall boundary \( \eta = -\delta_l \) and for the interface boundary \( \eta = 0 \). The transformation is starting with,

\[
R^2 - r^2 = R^2 - \left( x^2 + (y - R \cos \delta_l)^2 \right)
\]

\[
= R^2 - R^2 \sin^2 \delta_l \sinh^2 \xi \cosh \xi + \cos \eta \cosh \xi + \cos \eta - R^2 \sin^2 \delta_l \sin^2 \eta \cosh \xi + \cos \eta \cosh \xi + \cos \eta + 2R \sin \delta_l \sin \eta \cosh \xi + \cos \eta \cos \delta_l - R^2 \cos^2 \delta_l
\]

The product within the straight brackets can be simplified to,

\[
\sin^2 \delta_l - \sin^2 \delta_l \sinh^2 \xi + \sin^2 \eta \cosh \xi + \cos \eta \cosh \xi + \cos \eta
\]

\[
= \sin^2 \delta_l \left( 1 - \sin^2 \xi \cosh \xi + \cos \eta \right)
\]

\[
= \sin^2 \delta_l \left( \cosh \xi + \cos \eta \cosh \xi + \cos \eta \right)
\]

\[
= \sin^2 \delta_l \left( \cos \eta \cosh \xi + \cos \eta \cosh \xi + \cos \eta \right)
\]

\[
= \sin^2 \delta_l \left( \cos \eta \cosh \xi + \cos \eta \cosh \xi + \cos \eta \right)
\]

\[
= \sin^2 \delta_l \left( \frac{2 \cos \eta \cosh \xi + \cos \eta}{\cosh \xi + \cos \eta} \right)
\]

\[
= \sin^2 \delta_l \left( \frac{2 \cos \eta \cosh \xi + \cos \eta}{\cosh \xi + \cos \eta} \right)
\]
Now continuing the derivation from equation 2.39 and implementing the result, equation 2.40,

\[ R^2 - r^2 = R^2 \left( \sin^2 \delta_l \left( \frac{2 \cos \eta}{\cosh \xi + \cos \eta} \right) + 2 \sin \delta_l \sin \eta \cos \delta_l \right) \]

\[ = R^2 \left( \frac{2 \sin^2 \delta_l \cos \eta + 2 \sin \delta_l \sin \eta \cos \delta_l}{\cosh \xi + \cos \eta} \right) \]

\[ = 2R^2 \sin \delta_l \left( \frac{\sin \delta_l \cos \eta + \sin \eta \cos \delta_l}{\cosh \xi + \cos \eta} \right) \]

\[ = 2R^2 \sin \delta_l \left( \frac{\sin (\delta_l + \eta)}{\cosh \xi + \cos \eta} \right) \]

Therefore after the transformation the solution for the liquid velocity in terms of \( \xi \) and \( \eta \) becomes,

\[ u_l = -\frac{B_l R^2 \sin \delta_l \sin (\delta_l + \eta)}{2\mu_l (\cosh \xi + \cos \eta)} + \phi \] (2.42)

The Laplace operator for bipolar coordinates is defined as,

\[ \nabla^2 u_l = \frac{(\cosh \xi + \cos \eta)^2}{R^2 \sin^2 \delta_l} \left( \frac{\partial^2 u_l}{\partial \xi^2} + \frac{\partial^2 u_l}{\partial \eta^2} \right) = \frac{B_l}{\mu_l} \] (2.43)

The substitution of \( u_l \) is as follows,

\[ \frac{\partial^2 u_l}{\partial \xi^2} = -\frac{B_l R^2 \sin \delta_l \sin (\delta_l + \eta)}{2\mu_l} \left( \frac{2 \sinh^2 \xi}{(\cosh \xi + \cos \eta)^3} - \frac{\cosh \xi}{(\cosh \xi + \cos \eta)^2} \right) + \frac{\partial^2 \phi}{\partial \xi^2} \] (2.44)

\[ \frac{\partial^2 u_l}{\partial \eta^2} = -\frac{B_l R^2 \sin \delta_l}{2\mu_l} \left( \frac{2 \sin \eta (\cosh \xi \cos (\delta_l + \eta) + \cos \delta_l)}{(\cosh \xi + \cos \eta)^3} - \frac{\cosh \xi \sin (\delta_l + \eta)}{(\cosh \xi + \cos \eta)^2} \right) + \frac{\partial^2 \phi}{\partial \eta^2} \] (2.45)

Substitution of equations 2.44 and 2.45 into equation 2.43 and simplifying gives,

\[ \nabla^2 u_l = -\frac{B_l}{2\mu_l \sin \delta_l} \left( 2 \sinh^2 \xi \sin (\delta_l + \eta) + 2 \sin \eta (\cosh \xi \cos (\delta_l + \eta) + \cos \delta_l) \right) \]

\[ \text{...} \]

\[ \text{...} - 2 \cosh \xi \sin (\delta_l + \eta) \right) + \frac{(\cosh \xi + \cos \eta)^2}{R^2 \sin^2 \delta_l} \left( \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} \right) = \frac{B_l}{\mu_l} \] (2.46)

The terms within the brackets can be simplified, which is shown below,
\[
\left(2\sinh^2\xi \sin(\delta_i + \eta) + 2\sin\eta (\cosh\xi \cos(\delta_i + \eta) + \cos\delta_i) - 2\cosh\xi \sin(\delta_i + \eta)\right) \\
= 2\left(\left(\cosh^2\xi - 1\right) \sin(\delta_i + \eta) + \sin\eta \cosh\xi \cos(\delta_i + \eta) + \sin\eta \cos\delta_i\right) \\
\cdots \\
= 2\left(\sin(\delta_i + \eta) + \cosh\xi (\sin\eta \cos(\delta_i + \eta) - \sin(\delta_i + \eta) \cos\eta) + \sin\eta \cos\delta_i\right) \\
\cdots \\
= 2\left(\cosh\xi (\sin\eta (\cos\delta_i \cos\eta - \sin\delta_i \sin\eta) - \cos\eta (\sin\delta_i \cos\eta + \sin\eta \cos\delta_i)) \right) \\
\cdots \\
= 2\left(- \cosh\xi (\sin^2\eta \sin\delta_i + \cos^2\eta \sin\delta_i) - \sin\delta \cos\eta\right) \\
= -2 \left(\frac{\cosh\xi \sin\delta_i + \cos\eta \sin\delta_i}{\cosh\xi + \cos\eta}\right) = -2 \sin\delta_i \\
\text{(2.47)}
\]

Now when this result of the derivation shown in equation 2.47 is substituted into equation 2.46 one will get,

\[
\nabla^2 u_l = \frac{B_1}{\mu_i} + \frac{(\cosh\xi + \cos\eta)^2}{R^2 \sin^2\delta_i} \left(\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2}\right) = \frac{B_1}{\mu_i} \\
\text{(2.48)}
\]

Or simply,

\[
\nabla^2 u_l = \frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0 \\
\text{(2.49)}
\]

So the transformation to bipolar coordinates results in a Poisson equation which can then be solved in the conventional way using a Fourier transformation. More specifically, due to the boundary conditions, a Fourier cosine transform can be used.

This results in the following velocity distribution for the liquid,

\[
u_l = -\frac{B_1 R^2 \sin\delta_i}{2\mu_i} \times \left[\frac{\sin(\delta_i + \eta)}{\cosh\xi + \cos\eta} - 2\cos\delta_i \int_0^\infty \frac{\sinh (\omega (\delta_i + \eta))}{\sinh (\omega \pi) \cosh (\omega \delta_i)} \cos (\omega \xi) \ d\omega\right] \cdots \\
\cdots + \tau \frac{2R \sin\delta_i}{\mu_i} \int_0^\infty \frac{\sinh (\omega (\delta_i + \eta))}{\sinh (\omega \pi) \cosh (\omega \delta_i)} \cos (\omega \xi) \ d\omega \\
\text{(2.50)}
\]

Now the velocity distribution is derived. To obtain the wall shear distribution note that the shear stress distribution is defined in \( z \)-direction for constant \( \eta \) as,

\[
\tau_{l,\eta,z} = \mu_i \left(\frac{\cosh\xi + \cos\eta}{R \sin\delta_i}\right) \frac{\partial u_l}{\partial \eta} \\
\text{(2.51)}
\]

Now by differentiating the velocity distribution given by equation 2.50, combining this with equation 2.51 and substituting the value \( \eta = -\delta_i \) will give the wall shear stress distribution. Note that this is independent of the viscosity. The wall shear stress distribution is only dependent on \( \xi \) and is given in equation 2.52 below,

\[
\tau = -\frac{B_1 R}{2} \times \left[1 - 2 (\cosh\xi + \cos\delta_i) \cos\delta_i \int_0^\infty \frac{\omega \cos (\omega \xi)}{\sinh (\omega \pi) \cosh (\omega \delta_i)} \ d\omega\right] \cdots \\
\cdots + 2\tau \left(\cosh\xi + \cos\delta_i\right) \int_0^\infty \frac{\omega \cos (\omega \xi)}{\sinh (\omega \pi) \cosh (\omega \delta_i)} \ d\omega \\
\text{(2.52)}
\]
Note that when the value \( \eta = 0 \) is substituted the result of the shear stress will be \( \bar{\tau} \) which is consistent with the boundary conditions set in equation 2.35. There are however two points where the interface and wall meet, which is equivalent to \( \xi \rightarrow \infty \). Biberg applied the residue theorem to these points and concluded that as long as the liquid holdup is in the lower half of the pipe the shear stress will have a constant limit,

\[
\tau_i (\xi \rightarrow \infty) = \frac{\bar{\tau}_i}{\cos \delta_i} \quad \text{for,} \quad \delta_i < \frac{\pi}{2} \tag{2.53}
\]

The goal of these derivations is to end up with an average wall shear stress depending on the average fluid velocity. The mean wall shear stress can be found by,

\[
\bar{\tau}_i = \frac{1}{S_i} \int_{S_i} \tau_i dS \tag{2.54}
\]

Using equation 2.52 with equation 2.54 will result in,

\[
\bar{\tau}_i = -\frac{B_l}{S_i} \left( \delta_i - \frac{1}{2} \sin 2\delta_i \right) R^2 + \frac{S_i}{S_i} \bar{\tau}_i \tag{2.55}
\]

To get the average velocity in the equation the volumetric flow rate is derived. It is simply defined as,

\[
Q_i = \int_{A_i} u_i dA = -\frac{B_l R^2 \sin \delta_i}{2 \mu_l} \int_{A_i} \sin \left( \delta_i + \eta \right) \cos \left( \delta_i \right) dA \ldots \\
\ldots + \int_0^{\omega_\pi} \frac{2 R \sin \delta_i}{\mu_l} \left( \frac{B_l R \cos \delta_i}{2} + \bar{\tau}_i \right) \frac{1}{\sinh \left( \omega \pi \right) \cosh \left( \omega \delta_i \right)} d\omega \ldots \\
\ldots \times \int_{A_i} \sinh \left( \omega \left( \delta_i + \eta \right) \right) \cos \left( \omega \delta_i \right) dA \tag{2.56}
\]

The details of the simplification are not given here, but the end result is,

\[
Q_i = -\frac{B_l \pi R^4}{8 \mu_l} \phi_B (\delta_i) + \frac{\pi R^3}{4 \mu_l} \phi (\delta_i)
\]

where, \( \phi_B (\delta_i) = \frac{1}{\pi} \left( \delta_i - \sin 2\delta_i + \frac{1}{4} \sin 4\delta_i \right) + 2 \sin^2 \left( 2\delta_i \right) F (\delta_i) \)

and, \( \phi (\delta_i) = \frac{4}{3} \sin^2 \delta_i \left( \frac{\delta_i}{\pi} - 6 \cos \delta_i F (\delta_i) \right) \)

with, \( F (\delta_i) = \int_0^{\omega_\pi} \frac{\omega \tanh \left( \omega \delta_i \right)}{\sinh^2 \left( \omega \pi \right)} d\omega \tag{2.57} \)

Note that the volumetric flow rate is, apart from several constants and conditions, a function of the liquid holdup angle \( \delta_i \). This last integral is the only integral in this model for the mean wall shear stress that has no exact solution. This integral will have to be numerically evaluated, but it only depends on one variable and should not be computational expensive.

After rewriting equation 2.57 as a function for \( B_l \) it can be implemented into equation 2.55. In this way the average velocity \( U_i = Q_i/A_i \) shows up in the equation,

\[
\bar{\tau}_i = \frac{8 \mu_l U_i}{D_i (\delta_i)} - C_i (\delta_i) \bar{\tau}_i \tag{2.58}
\]

with, \( D_i (\delta_i) = \left[ \frac{\delta_i}{\pi} \left( \frac{\delta_i}{\pi} - \frac{1}{2} \sin 2\delta_i \right) \right] \\
\text{and,} \quad C_i (\delta_i) = \frac{\delta_i - \frac{1}{2} \sin 2\delta_i}{\delta_i} \frac{\phi (\delta_i)}{\phi_B (\delta_i)} - \frac{\sin \delta_i}{\delta_i} \)
Equation 2.58 shows that the average liquid wall shear stress is depending on two terms namely, on the free surface flow contribution and the (interfacial) shear flow contribution, similar to the liquid wall shear stress model presented in section 2.3. This equation states that the total wall shear stress equals the wall shear stress that would occur in a free surface flow, plus the shear stress contribution due to the interfacial shear stress. Note that the free surface term looks similar to the wall shear stress of a one-phase pipeflow. Usually the hydraulic diameter is defined differently, however Biberg showed that this definition of the diameter only differs about ±3% as long as δ_l/π < 0.7 (which is true for most applications, especially for low liquid loading). Still this is only true when τ_i = 0.

\[ D_h = D \left[ \frac{\delta_l - \frac{1}{2} \sin 2\delta_l}{\delta_l} \right] \approx D \left[ \frac{\delta_l}{\pi} \left( \frac{\varphi_B(\delta_l)}{\delta_l - \frac{1}{2} \sin 2\delta_l} \right)^2 \right] = D_l \quad \text{as long as,} \quad \tau_i = 0 \]  

(2.59)

Obviously, two-phase effects are not included in the one-phase hydraulic pipe diameter definition. Therefore for a two-phase flow, the one-phase hydraulic pipe diameter definition should be adapted to include two-phase effects.

### 2.5 Interfacial shear and gas wall shear models by Biberg

So far only the liquid wall friction models by Biberg were presented. However, in order to close the holdup equation 2.3 additional closure relations are needed. The closure relations for the gas wall shear stress and interfacial shear stress will be briefly described in this current section. They are based on Biberg [7]. Birvalski used these specific closure relations for his analysis as well.

#### 2.5.1 Interfacial shear stress

Biberg[26] describes an exact model for the interfacial shear stress of a laminar two-phase flow. Referring to that paper this derivation starts with the following exact expression of the interfacial shear stress for laminar two-phase flow. This exact definition for laminar flow is then combined with experimental results and a combined model is created. Basically the exact laminar case is corrected such that it will fit the experiments with a smooth interface, small waves and large waves. The presented model for the interfacial shear stress results in equation 2.60 according to Biberg, corrected for experiments by Espedal.

\[ \tau_i \approx KC_a \left( -\frac{Dg}{4} \left( \frac{\partial p}{\partial x} + \rho g \sin \alpha \right) \right) f(\delta_g) \]  

(2.60)

where,

\[ C_a = \frac{\mu_{ml} U_r}{\sigma}, \quad f(\delta_g) = \frac{(\delta_g + \sin \delta_g) (\sin \delta_g - \cos \delta_g \cos \delta_g)}{\delta_g (\delta_g - \frac{1}{2} \sin 2\delta_g)} \]  

(2.61)

and,

\[ U_r = \frac{U_g s}{1 - e_l} - \frac{u_1 s}{e_l} \]  

(2.62)

And K ≈ 750.

#### 2.5.2 Gas wall shear stress

Similar to the liquid wall shear stress the gas wall shear stress uses a friction coefficient defined by equation 2.23. Basically the gas wall friction model derived in the section
on the first liquid wall friction model is used as the gas wall friction model. Using equation 2.29 and equation 2.30 this model for the gas wall shear stress is obtained.

\[
\frac{1}{\sqrt{\lambda_g}} = \left[ 1.986 \log \text{Re}_{D_g} \sqrt{\lambda_g} - 0.837 \right] + \left[ 2 \log \left( 1 + \frac{S_i}{S_g} \right) \left( 1 - \text{sign} \left( \frac{\tau_i}{\tau_g} \right) \sqrt{\frac{\tau_i}{\tau_g}} \right) \right]
\]

Equation (2.63)

Note that the first term accounts for the ‘closed duct flow’ and the second term provides the ‘two-phase effect’. In Biberg’s paper \[7\] now the ‘closed duct flow’ term is replaced by Haaland’s formula \[8\] for closed duct flow, since this equation shows good compliance with both rough and smooth surface interfaces. Haaland’s formula is shown in equation 2.64,

\[
\frac{1}{\sqrt{\lambda_g}} = -1.8 \log \left( \frac{6.9}{\text{Re}_{D_g}} + \left( \frac{k_s}{3.7 \text{D}_g} \right)^{1.11} \right)
\]

Haaland’s formula for closed duct flow

Equation (2.64)

So the two-phase gas wall friction coefficient becomes,

\[
\frac{1}{\sqrt{\lambda_g}} = -1.8 \log \left( \frac{6.9}{\text{Re}_{D_g}} + \left( \frac{k_s}{3.7 \text{D}_g} \right)^{1.11} \right) + \left[ 2 \log \left( 1 + \frac{S_i}{S_g} \right) \left( 1 - \text{sign} \left( \frac{\tau_i}{\tau_g} \right) \sqrt{\frac{\tau_i}{\tau_g}} \right) \right]
\]

Equation (2.65)

Evaluation of this function requires an iterative process. However by setting up a momentum balance for only the gas phase,

\[
A_g \left( \frac{\partial p}{\partial x} + \rho_f g \sin \alpha \right) + \tau_g S_g + \tau_i S_i = 0
\]

Equation (2.66)

and by combining equation 2.60 with equation 2.66 one can rewrite those such that the ratio between \( \tau_i \) and \( \tau_g \) equals,

\[
\frac{\tau_i}{\tau_g} = \frac{S_i}{S_g + S_i} \frac{K C_a f(D_g)}{1 - \frac{\delta_l}{S_g + S_i} K C_a f(D_g)}
\]

Equation (2.67)

which is used directly into equation 2.65 in order to calculate the gas wall friction coefficient. In all the Biberg models a flat gas-liquid interface is assumed. Using equation 2.68 we obtain the gas wall shear stress.

\[
\tau_g = f_S \frac{1}{2} \frac{\Delta \rho_g u_g^2 S_g}{(1 - \epsilon_l)} \tau_g
\]

Equation (2.68)

where, \( f_S = \frac{\lambda_g}{4} \)

This concludes the interfacial and gas wall shear stress closure relations.

### 2.6 Discussion and Conclusion

This chapter presented two sets of closure relations, the MARS model and the mechanistic approach by Dag Biberg. Dag Biberg also presented two different methods to calculate the liquid wall shear stress. These sets of closure relations describe the geometry of the interface and how the perimeters and areas of the fluid-fluid phases are defined compared to the holdup ratios. These closure relations describe how the wall and interfacial shear stresses can be calculated as a function of the liquid holdup. This makes that once the closure relations are known, they can be implemented into the holdup equation and solved for the holdup (using a computer program).
All the closure relation models presented use a similar approach: all of them start with a single-phase approach and adjust that for the two-phase flow properties (such as the area). Then the effect of the other fluid present in the pipe flow is accounted for. The way how the models take these (geometrical) adjustments and multiphase effects into account is where the models differ.

Firstly the MARS model is considered. It mostly uses the geometrical closure relations to set up a two-phase model. The algorithm to estimate the interfacial shear stress uses a single-phase approach that tries to model the interfacial waves as if it was a sand surface roughness, which is a very familiar approach in single-phase flow. The MARS model alters the geometrical closure relations by an empirically based adjustment of the wetted wall angle. Unfortunately since the formula for the liquid wall shear stress contains the average liquid velocity, the liquid wall shear stress will always be modeled as zero for zero net liquid flow. This shows that the effect of backflow along the wall on the liquid wall shear stress is not modeled.

The closure relations by Dag Biberg use a very different approach. The geometrical closure relations assume a flat fluid-fluid interface, which is a simple approximation compared to MARS. Biberg has tried to include all the two-phase flow physics in the correlations for the shear stresses.

Both liquid wall shear stress models allow for backflow along the wall. This is done by assuming a velocity distribution based on a logarithmic function. This makes that for upward inclined flow the liquid could flow downward along the wall surface and upward along the interfacial surface. This allows for wall shear stresses to become opposite to the average liquid flow. This is not possible with the MARS model. However the turbulent liquid wall shear stress model by Biberg still uses a friction factor. The closure relation completely describes all the physical aspects of backflow in this friction factor. Unfortunately for zero-net-liquid flow the liquid wall shear stress becomes zero since the average liquid velocity becomes zero. Therefore at the critical condition defined by Birvalski for upward inclined two-phase pipe flow this turbulent-turbulent liquid wall shear stress model might not correctly model the liquid wall shear stress.

The laminar liquid-turbulent gas liquid wall shear stress model by Biberg, however, describes a fully analytical solution of the liquid wall shear stress. This model will not become zero for zero-net-liquid flow. Taking these properties of the models into account it is concluded to drop the MARS model for further analysis.

For the remainder of this thesis the closure relations for interfacial shear stress and gas wall shear stress by Biberg will be used. These were described in section 2.5. For the liquid wall shear stress two models are used and compared. One being the turbulent liquid model for liquid wall shear stress, described in section 2.3. The other is the laminar liquid model for liquid wall shear stress described in section 2.4. These models will from now on be referred to as the turbulent liquid model and the laminar liquid model, respectively.
3. Steady-state analysis for inclined zero net liquid pipe flow

A steady state analysis is the first and most simplistic method to investigate the behaviour of the models introduced in the previous chapter. It simulates a time and space independent situation, which allows for various simplifying assumptions. The mass and momentum conservations will be derived for the steady state analysis. This steady state analysis will give a first impression on how the mechanistic models by Biberg, described in chapter 2, differ.

The critical condition, earlier introduced in the chapter 1, is a steady state situation. Birvalski performed experiments to investigate this critical condition empirically. By performing this steady state analysis a first comparison can be made to see how the mechanistic models perform compared to these experimental results. First multiple analyses are performed to see whether they show a critical condition at all. The results are described and presented in various ways and are physically interpreted to see whether the models make sense. Thereafter the models are compared to experimental results. Simulations are performed as close to the experimental set up as possible when using a steady state analysis. The comparison is made and discussed and finally the conclusions are summarized as a closure to this chapter.

3.1 Governing equations

For the steady state analysis there are several simplifications applied to the mass and momentum equations. First recall the mass and momentum equations for each phase.

\[
\begin{align*}
\frac{\partial \varepsilon_l}{\partial t} + \frac{\partial (\varepsilon_l u_l)}{\partial x} &= 0, \\
\frac{\partial \varepsilon_g}{\partial t} + \frac{\partial (\varepsilon_g u_g)}{\partial x} &= 0, \\
\rho_l \frac{\partial (\varepsilon_l u_l)}{\partial t} + \rho_l \frac{\partial (\varepsilon_l u_l^2)}{\partial x} + \varepsilon_l \frac{\partial P}{\partial x} &= - \frac{\tau_l S_l}{A} - \frac{\tau_l S_l}{A} - \rho_l \varepsilon_l g \sin \alpha, \\
\rho_g \frac{\partial (\varepsilon_g u_g)}{\partial t} + \rho_g \frac{\partial (\varepsilon_g u_g^2)}{\partial x} + \varepsilon_g \frac{\partial P}{\partial x} &= - \frac{\tau_g S_g}{A} - \frac{\tau_g S_g}{A} - \rho_g \varepsilon_g g \sin \alpha.
\end{align*}
\]

(3.1)

All the time derivatives are equal to zero since this is a steady state analysis, as well as the derivatives w.r.t. \(x\).
\[
\frac{\partial}{\partial x} (\varepsilon_l u_l) = 0, \\
\frac{\partial}{\partial x} (\varepsilon_g u_g) = 0, \\
\rho_l \frac{\partial}{\partial x} (\varepsilon_l u_l^2) + \varepsilon_l \frac{\partial P}{\partial x} = -\frac{\tau_l}{A} + \frac{\tau_l S_l}{A} - \rho_l \varepsilon_l g \sin \alpha, \\
\rho_g \frac{\partial}{\partial x} (\varepsilon_g u_g^2) + \varepsilon_g \frac{\partial P}{\partial x} = -\frac{\tau_g}{A} + \frac{\tau_g S_g}{A} - \rho_g \varepsilon_g g \sin \alpha.
\]

(3.2)

The mass conservation equation and momentum equation per phase are combined. This gives us a system of two equations, one for each phase respectively. Note that the equations are divided by the holdup term.

\[
\frac{\partial P}{\partial x} = -\frac{\tau_l S_l}{\varepsilon_l A} + \frac{\tau_l S_l}{\varepsilon_l A} - \rho_l g \sin \alpha, \\
\frac{\partial P}{\partial x} = -\frac{\tau_g S_g}{\varepsilon_g A} + \frac{\tau_g S_g}{\varepsilon_g A} - \rho_g \varepsilon_g g \sin \alpha.
\]

(3.3)

This derivation is actually the same as the derivation for the holdup equation presented earlier in this report. Substituting both equations for the pressure gradient gives us the holdup equation, as shown in equation 3.4,

\[
-\frac{\tau_l S_l}{\varepsilon_l A} + \frac{\tau_l S_l}{\varepsilon_l A} - \rho_l g \sin \alpha = -\frac{\tau_g S_g}{\varepsilon_g A} + \frac{\tau_g S_g}{\varepsilon_g A} - \rho_g \varepsilon_g g \sin \alpha.
\]

Using the fact that \( A_g = A \varepsilon_g \) and \( A_l = A \varepsilon_l \) gives,

\[
0 = \tau_g S_g \frac{1}{A_g} - \tau_l S_l \frac{1}{A_l} + \tau_g S_g \left[ \frac{1}{A_l} + \frac{1}{A_g} \right] + (\rho_l - \rho_g) g \sin \alpha = F (\varepsilon_l, U_g, S, U_l, S, D, \text{fluid properties})
\]

(3.4)

Now for convenience the whole righthand side of the holdup equation presented in equation 3.4 is called \( F \). So in the end,

\[
F (\varepsilon_l, U_g, S, U_l, S, D, \text{fluid properties}) = 0 \tag{3.5}
\]

This is what is required for the steady state analysis. Note that all the closure relations are captured by \( F \). We are looking for solutions of the holdup for which \( F \) is equal to zero, since the other flow and fluid properties will be fixed. So,

\[
F (\varepsilon_l, U_g, S, U_l, S, D, \text{fluid properties}) = f (\varepsilon_l) = 0 \tag{3.6}
\]

### 3.2 Computational model

To perform the steady state analysis the model is implemented using MatLab. The program is set up using two scripts. One script calculates the value of \( F \) in which the two closure relation sets by Biberg are programmed.

The superficial liquid velocity is set to zero and the superficial gas velocity is varied from 1 m/s to 12 m/s. The gas and liquid are taken as air and water respectively. The inclination angle is fixed at 1.3, 1.7 or 2.1, which are the same angles used by Milos Birvalski. This makes it possible to compare the steady state analysis results to his experimental results.
For a selected inclination angle a loop is performed that increments the superficial gas velocity by 0.2 m/s. For each superficial gas velocity the liquid holdup for which $F = 0$ is calculated. Close to the critical condition the superficial gas velocity is increased by 0.005 m/s per loop cycle.

### 3.3 Experimental results by Birvalski

Birvalski[27] performed several experiments using a V-section or low elbow pipe. The pipe was made out of transparent Perspex. In figure 3.1 a schematic of the V-section experimental setup is shown.

An amount between 50 and 600 milliliters of liquid was poured into the pipe, depending on the experiment. The liquid is at rest at the bottom in the lower elbow section and then air was blown into the pipe. The liquid is then blown upwards in the pipe section and just before the liquid is blown out of the pipe the critical condition is reached (by definition). The experimental set up used a pipe with a diameter of 5.08 centimeters (or 2 inches). The inclination angles used during the experiments were 1.3°, 1.7° and 2.1°.

In figures 3.2 and 3.3 the results for multiple experiments are shown for the critical superficial gas velocity and for the critical liquid holdup, respectively.

In the results a very slight dependence on the liquid volume is noticed: an increase in the water volume seems to slightly increase the critical superficial gas velocity. A similar behaviour is found for the critical liquid holdup results.

The inclination angles of 1.3°, 1.7° and 2.1° have a larger influence on the critical condition. In figure 3.2 it is clearly shown that for a steeper inclination angle a higher critical superficial gas velocity appears. From a physical point of view this is expected since for a steeper inclination angle it becomes harder to overcome gravity and therefore a higher superficial gas velocity is needed.

![Figure 3.1: A schematic of the V-section used in the experiments by Birvalski [1]](image)
Figure 3.2: The measured critical superficial gas velocity for water-air, for different liquid volumes and 1.3°, 1.7° and 2.1° pipe inclination angles [1]

Figure 3.3: The measured critical liquid holdup for water-air, for different liquid volumes and 1.3°, 1.7° and 2.1° pipe inclination angles [1]
3.4 Superficial gas velocity

A study is performed to see the effect of the superficial gas velocity on the holdup equation. As the parameter the superficial gas velocity is varied between 1 m/s and 12 m/s. The superficial liquid velocity is set to zero. The closure models by Biberg are all functions of the liquid holdup, as described in the previous chapter on closure relation models. The steady-state holdup solutions for the three inclination angles that Milos also used during his experiments are shown in figures 3.4 for \( \alpha = 1.3 \), figure 3.5 for \( \alpha = 1.7 \) and figure 3.6 for \( \alpha = 2.1 \).

Note that these solutions do not mean that they are physically stable and can thus be realized in an actual pipe; more on this in chapters 4 and 5. Overall the figures 3.4, 3.5 and 3.6 show a similar behaviour. For lower superficial gas velocities the two models (Biberg laminar liquid and Biberg turbulent liquid) both show two holdup values as a solution. There is a small region where three solutions are found for the holdup and each model has a specific superficial gas velocity as a local maximum. This superficial gas velocity is the critical superficial gas velocity. Once the critical superficial velocity is passed the liquid will be blown out of the pipe and therefore only a liquid holdup of zero is a steady state solution. Moreover, for zero-net liquid flow the zero liquid holdup is always a (trivial) solution to the holdup equation.

For the inclination angles of 1.3\(^{\circ}\), 1.7\(^{\circ}\) and 2.1\(^{\circ}\) the critical superficial velocity is always lower than the critical superficial velocity of the turbulent liquid model shows. In table 3.1 the values of the critical conditions are given.

The closure relations for laminar liquid and turbulent liquid only differ in the model used for the liquid wall shear stress. These shear stress models are used in the holdup equation. The holdup equation can actually also be seen as a force balance. It is interesting to see how the shear stresses differ, see figure 3.7. In this figure the interfacial, liquid wall and gas wall shear stresses are plotted against the superficial gas velocity. It is now clear that the liquid wall shear stress for zero net liquid flow is equal to zero for the turbulent liquid closure model. The laminar liquid closure model for the liquid wall shear stress shows a negative shear stress. This means that there is backflow along the wall, so the liquid flows downward. This results in a negative shear stress, which influences the force balance of the whole system. The laminar liquid model and the turbulent liquid model give a similar shape for both the interfacial shear stress and the gas wall shear stress. The laminar liquid model shows lower shear stresses for all the shear stresses compared to the shear stresses for the turbulent liquid model.

In order to get a better understanding of the shear stresses figure 3.8 is used. Here the shear stresses are normalized using the dynamic pressure of the gas phase. This is done since the superficial gas velocity is the parameter studied in this analysis. On the x-axis the liquid holdup, which is dimensionless by itself, is plotted. These figures

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>Laminar liquid model</th>
<th>Turbulent liquid model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 1.3^{\circ} )</td>
<td>critical ( \varepsilon_l ) 0.0795 [-]</td>
<td>0.0780 [-]</td>
</tr>
<tr>
<td></td>
<td>critical ( U_{g,S} ) 7.7944 [m/s]</td>
<td>8.1695 [m/s]</td>
</tr>
</tbody>
</table>

| \( \alpha = 1.7^{\circ} \) | critical \( \varepsilon_l \) 0.0750 [-] | 0.07225 [-] |
|                  | critical \( U_{g,S} \) 8.4096 [m/s] | 8.8072 [m/s] |

| \( \alpha = 2.1^{\circ} \) | critical \( \varepsilon_l \) 0.0710 [-] | 0.0683 [-] |
|                  | critical \( U_{g,S} \) 8.9165 [m/s] | 9.3320 [m/s] |
show the normalized shear stresses of the system with respect to the holdup.
The gas wall shear stress of both the laminar liquid and the turbulent liquid model are right on top of each other. The liquid wall shear stress again shows a very different behaviour for the laminar liquid model and the turbulent liquid model. This is expected since these models differ the most for the liquid wall shear stress. For higher liquid holdup values the liquid wall shear stress decreases according to the laminar liquid closure model. Also this behaviour is expected from the model. For higher liquid holdup values and zero net liquid inclined upward flow the velocity of the backflow along the wall will increase. The increase of the backflow velocity along the wall results in a higher negative wall shear stress according to the laminar liquid closure relation.

The interfacial shear stress normalized by the superficial gas dynamic pressure now shows that it behaves differently for the laminar liquid model compared to the turbulent liquid model. For higher liquid holdup values the laminar liquid model shows increasingly lower values for the interfacial shear stress as compared to the turbulent liquid model. Note that the values of the normalized interfacial shear stress and the normalized liquid wall shear stress differ by an order of ten in magnitude.

Figure 3.4: Steady state liquid holdup fraction versus the superficial gas velocity for an inclination angle $\alpha = 1.3^\circ$. 

Steady state: $\alpha = 1.3$
Steady state: $\alpha = 1.7$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{steady_state_17}
\caption{Steady state liquid holdup versus the superficial gas velocity for an inclination angle $\alpha = 1.7^\circ$.}
\end{figure}

Steady state: $\alpha = 2.1$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{steady_state_21}
\caption{Steady state liquid holdup versus the superficial gas velocity for an inclination angle $\alpha = 2.1^\circ$.}
\end{figure}
Figure 3.7: The interfacial (top), liquid wall (middle) and gas wall (bottom) shear stresses for a fixed inclination angle $\alpha = 1.7^\circ$. 

- **Interfacial shear stress versus superficial gas velocity**
- **Liquid wall shear stress versus superficial gas velocity**
- **Gas wall shear stress versus superficial gas velocity**
Figure 3.8: The interfacial (top), liquid wall (middle) and gas wall (bottom) shear stresses, normalized using the dynamic pressure of the gas phase, versus the liquid holdup \( \varepsilon_l \), for a fixed inclination angle \( \alpha = 1.7^\circ \).
3.5 Effect of the inclination angle

In addition to varying the superficial gas velocity the effect of changing the inclination angle has also been studied. The inclination angle will be varied while the superficial gas velocity is set to the critical value at $1.7^\circ$ for each model. The results of this analysis are shown in figure 3.9.

![Graph showing liquid holdup fraction versus the inclination angle. The superficial gas velocity is fixed at the critical condition that occurs at $\alpha = 1.7^\circ$ for both models.](image)

Both models show a turning point for an inclination angle $\alpha = 1.7^\circ$, as expected. When the inclination angle is lower than $1.7^\circ$ only the trivial solution is present. At the critical condition (which has been set at $1.7^\circ$ deg for the chosen superficial gas velocity) there are two solutions to the holdup equation namely, the trivial solution and the critical solution. Increasing the inclination angle above $1.7^\circ$ gives three solutions: the trivial solution, one giving a higher liquid holdup than the critical liquid holdup, and the other solution being lower than the liquid holdup solution.

3.6 Biberg models versus the Birvalski experiments

So far the critical conditions have been analysed only by using the mechanistic models of Biberg, one using a laminar liquid model and another one using a turbulent liquid model. Fortunately Milos Birvalski performed experiments\[1\] to find the stable steady state solution. These experiments were performed using a V-shaped pipe. The diameter of the pipe was 5.08 centimeters. In the lower ‘elbow’ section a volume of water is placed and then air is blown through the pipe. The gas flow rate just before the water would be blown out of the pipe is the critical value.

Birvalski already compared the turbulent liquid model of Biberg with his experiments. This comparison is redone here, but now also the laminar liquid model is added to the comparison. The experiments were performed for three inclination angles, namely $\alpha = 1.3^\circ, 1.7^\circ$ and $2.1^\circ$, and five different liquid volumes, namely 100 ml, 150 ml, 200 ml, 300 ml, 400 ml. The average of the measured values by Birvalski for each combination of liquid volume and inclination angle have been calculated. These average values are plotted in figures 3.10 and 3.11. Horizontal lines that represent the average of all the different liquid volumes are shown as well. The figures also show the results of the mechanistic models of Biberg, for both the turbulent liquid model and the laminar liquid model.

The simulations have been performed at conditions that are as close as possible to those used in the experiments. However, the steady state analysis with the mechanistic models does not include the V-section. The pipe diameter, fluid properties and
Figure 3.10: The critical superficial gas velocity $U_{gs}$. A comparison between the experimental results by Birvalski [1] and the mechanistic models by Dag Biberg for three inclination angles, $\alpha = 1.3^\circ$(red), $1.7^\circ$(blue) and $2.1^\circ$(green). On the x-axis the different liquid volumes are shown and the results of the mechanistic models for the laminar liquid and turbulent liquid models.

inclination angles have all been chosen in agreement with the experimental setup. Figure 3.10 shows that the laminar liquid model predicts the critical superficial gas velocity better than the turbulent liquid model. Table 3.2 gives an error of about 5% for the turbulent liquid model while an error below 1% is found for inclinations with the laminar liquid model. Moreover, the laminar liquid model predicts a critical superficial gas velocity within the region of the bounds of all the results found for each inclination angle. This shows that the laminar liquid model is much more accurate compared to the turbulent liquid model for finding the critical superficial velocity w.r.t. these experimental conditions.

Figure 3.11: The critical liquid holdup $\varepsilon_l$. A comparison between the experimental results by Birvalski [1] and the mechanistic models by Dag Biberg for three inclination angles, $\alpha = 1.3^\circ$(red), $1.7^\circ$(blue) and $2.1^\circ$(green). On the x-axis the different liquid volumes are shown and the results of the mechanistic models for the laminar liquid and turbulent liquid models.

Figure 3.11 shows the average liquid holdup values found in the experiments by Birvalski. Also the liquid holdup values found in the simulations using Biberg’s mechanistic models are shown. Unfortunately for the liquid holdup these models do not
Table 3.2: A comparison for the critical superficial velocities $U_{sg}$, between the experiments by Birvalski [1] and the turbulent liquid and laminar liquid models by Biberg.

<table>
<thead>
<tr>
<th>Milos Birvalski Average results experiments $U_{sg}$</th>
<th>Turbulent model Biberg</th>
<th>Laminar model Biberg</th>
<th>Error Turbulent model [%]</th>
<th>Error Laminar model [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.3^\circ$</td>
<td>7.847</td>
<td>8.16945</td>
<td>4.11</td>
<td>-0.67</td>
</tr>
<tr>
<td>$1.7^\circ$</td>
<td>8.358</td>
<td>8.80720</td>
<td>5.37</td>
<td>0.62</td>
</tr>
<tr>
<td>$2.1^\circ$</td>
<td>8.834</td>
<td>9.33195</td>
<td>5.63</td>
<td>0.93</td>
</tr>
</tbody>
</table>

Table 3.3: A comparison for the critical liquid holdup $\varepsilon_l$, between the experiments by Birvalski [1] and the turbulent liquid and laminar liquid models by Biberg.

<table>
<thead>
<tr>
<th>Milos Birvalski Average results experiments $\varepsilon_l$</th>
<th>Turbulent model Biberg</th>
<th>Laminar model Biberg</th>
<th>Error Turbulent model [%]</th>
<th>Error Laminar model [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.3^\circ$</td>
<td>0.022</td>
<td>0.078</td>
<td>250</td>
<td>257</td>
</tr>
<tr>
<td>$1.7^\circ$</td>
<td>0.023</td>
<td>0.0723</td>
<td>218</td>
<td>227</td>
</tr>
<tr>
<td>$2.1^\circ$</td>
<td>0.025</td>
<td>0.0683</td>
<td>176</td>
<td>187</td>
</tr>
</tbody>
</table>

predict the critical condition very accurately. While the values for the liquid holdup fraction in the experiments are quite consistent (except maybe for the $\alpha = 1.3^\circ$ which varies the most for different liquid volumes) they are about a third of the predictions by the simulations. The laminar liquid model performs even slightly worse than the turbulent liquid model. Both models give almost similar liquid holdup fractions, which are more than twice the experimental values (table 3.3). The largest deviations occur for $\alpha = 1.3^\circ$. For an increasing inclination angle the deviations seem to decrease. However, the differences between the experiments and simulations remain large.

3.7 Discussion and conclusions

Figures 3.4, 3.5 and 3.6 have shown that both mechanistic models behave very similar. This is expected since these models largely overlap. The main difference is that the liquid is modeled differently, as explained in the previous chapter.

Both models show the following three regions:

1. for lower superficial gas velocities two solutions for the liquid holdup are present
2. small region where three liquid holdup solutions are present
3. above the critical solution only the trivial solution is present, which is the one with zero liquid holdup.

One could wonder whether the high liquid holdup solutions for a low superficial gas velocity are physically obtainable. That is because slug and bubble effects are not included in these models, those situations simply fall beyond the scope of these stratified two-phase flow models. The mechanistic models do show a critical condition. Also the fact that the models show that for superficial gas velocities higher than the critical superficial gas velocity the only solution is that there is no liquid in the pipes anymore. This underlines the presence of this critical condition and shows that the models are capable of predicting such a critical condition. Also the when the inclination angle is increased the critical superficial gas velocities increase as well. This complies with the expectation since the heavier phase, the liquid, will be pulled down harder by gravity for higher inclination angles.
3.7.1 Comparison of the laminar liquid model and the turbulent liquid model

The liquid wall shear stress versus the superficial gas velocity was shown in figure 3.7; here there is a large difference between the laminar liquid model and the turbulent liquid model. The turbulent liquid model shows that for a steady state where the net liquid velocity equals zero, this model gives a liquid wall shear stress of zero. In reality however it has been observed that there can indeed be backflow along the liquid wall. This would result in a negative liquid wall shear stress. This is clearly modeled by the laminar liquid model, but not by the turbulent liquid model. The presence of a negative liquid wall shear stress has its effect on the overall momentum balance and on the critical condition predicted by the model. This is shown by the other shear stress graphs in figure 3.7. The laminar liquid model gives values for the interfacial shear stress and for the gas wall shear stress that are lower than those found with the turbulent liquid model.

When the shear stresses, normalized with the dynamic pressure based on the superficial gas velocity, are shown against the liquid holdup in figure 3.8, it can be seen that the interfacial shear stress is the most affected by the different liquid models. Also for higher liquid holdup values the liquid wall shear stress becomes more negative. The laminar liquid model predicts a higher backflow velocity along the wall, increasing the negative liquid wall shear stress, than the turbulent liquid model.

When first the superficial gas velocity is chosen for each mechanistic model such that it has the critical value for the considered inclination angle, and then the inclination angle is changed, the model predicts that for lower angles than the critical inclination angle, only the trivial liquid holdup will be a solution to the holdup equation. Physically this means that for that superficial gas velocity in combination with a lower inclination angle the liquid would be blown out of the pipe. Higher inclination angles than the critical inclination angle give two solutions in addition to the trivial solution. The mechanistic models were compared to the experiments performed by Birvalski [1] and the results are shown in figures 3.10 and 3.11 and tables 3.2 and 3.3. The laminar liquid model proved to be more accurate in predicting the critical superficial gas velocity than the turbulent liquid model when compared to the experimental results. The difference between the two models is only the closure relation for the liquid wall shear stress as explained in the previous chapter. A large difference between those liquid wall shear stress models is that only the laminar liquid model does model backflow conditions for zero-net-liquid flow. The turbulent liquid model calculates a friction factor based on a velocity distribution just like the laminar model does, so this friction factor does include the physical phenomenon of backflow along the wall. However, this friction factor is then multiplied by the dynamic pressure of the liquid, which included the average liquid velocity. The average liquid velocity is zero for the critical condition, therefore the liquid wall shear stress will be zero for the turbulent liquid model at critical conditions.

The predictions by both models for the critical liquid holdup are far off from the experimental values. The predicted values by the mechanistic models are about two and a half times higher than those found by Birvalski in his experiments. The reason for this could be that the simulations did not model the experimental conditions closely enough. The simulations only look at the possible steady states and they do not include a liquid volume distribution since there is no spatial discretization. The simulations also do not include the effect of a V-section, which is present in the experimental set up.

Another possible explanation is that the geometrical closure relations are too simplistic. The geometrical closure relations assume a flat interface while in practice it is well possible that the interface is somewhat curved. This would increase the interfacial perimeter resulting in a larger shear force exerted by the gas on the liquid. This would potentially reduce the liquid holdup value needed to acquire critical conditions. In the previous chapter multiple models were introduced. The steady state behaviour
of the models was analyzed in this chapter. Two mechanistic models are compared to each other and to experiments performed by Birvalski. Their performance is analyzed through multiple parameter variations and through comparison with experiments.

From the superficial gas velocity variation in the liquid holdup equation we saw that the laminar liquid model always showed a slightly lower critical superficial gas velocity than the turbulent liquid model. The critical liquid holdups predicted by the mechanistic models are similar, although the turbulent liquid model showed a slightly lower prediction. For all the simulations both models showed a similar overall behaviour. Moreover the results of the simulations are all physically explainable taking the assumptions into account. For superficial gas velocities higher than the critical superficial gas velocity the models would both predict a zero liquid holdup, which physically means that the liquid would be blown out of the pipe. This result is consistent with the definition of the critical condition. For lower superficial gas velocities a (very) high liquid holdup would appear as a solution. Since the gas area would be rather small, the actual gas velocity would be very high compared to the superficial gas velocity, resulting in much higher interfacial and gas wall shear stresses. Since the models strictly model stratified two-phase flow systems these solutions can be expected. Whether they are physically possible cannot be verified with the applied steady state analysis for stratified flow, since it would require different models that predict the transition to different flow regimes such as slug and bubble flow.

The main difference between the laminar liquid model and the turbulent liquid model is, as the names suggest, in the liquid wall shear stress modeling. This is clearly shown by the simulation results for the (normalized) shear stresses. The turbulent liquid model has a liquid wall shear stress of zero when the average liquid velocity is zero, which is the case at the critical condition by definition. The laminar liquid model does include backflow effects during the zero-net-liquid flow, resulting in a negative liquid wall shear stress. As a result the critical superficial gas velocity prediction by the laminar liquid model is closer to the empirical results by Birvalski than the turbulent liquid model predictions. The predictions for the critical liquid holdup by the mechanistic models are both about two and a half times higher than the critical liquid holdup values found in the experiments. This could be caused by several simplifications made during a steady state analysis compared to the experimental set up. For instance the V-section of the pipe in the experiments is not included in the simulations. Another explanation could be that the geometrical closure relations are too simplistic to accurately predict the critical liquid holdup condition. The geometrical closure relations assume a flat interface between the gas and the liquid, while in practice this interface might be curved. This would increase the interfacial perimeter, resulting in a larger interfacial shear force exerted by the gas upon the liquid allowing for a smaller liquid holdup to achieve critical conditions.

### 3.7.2 Final conclusion

The steady state analysis showed that both mechanistic models presented intuitively a correct physical behaviour. The laminar liquid model was shown to be superior. This model includes a sophisticated liquid wall shear stress closure relation. This characteristic of the laminar liquid model resulted in highly accurate critical superficial gas velocity predictions.
4. Holdup stability analysis

The steady state analysis showed that there exist multiple solutions for the liquid holdup $\varepsilon_l$ for stratified gas-liquid upward inclined pipe flow with zero net-liquid transport. This was shown using the holdup equation, which is a force balance of the shear stresses and the weight of the fluid. The mechanistic models, both the laminar liquid model and the turbulent liquid model, showed this behaviour.

In order to see whether these steady state solutions could be obtained in reality a stability analysis is required. This will give better understanding of the dynamical behaviour of the two-phase mechanistic models. Literature has shown that the lower nonzero liquid holdup solution, also called the second solution, is linearly unstable. The third solution has been shown to be linearly stable. Barnea and Taitel [28] used different closure relations in their analysis, not including backflow along the wall. Also Birvalski [1] neglected backflow in his stability analysis. A stability analysis using closure relations that include liquid backflow along the wall has not been reported in the literature so far.

The turbulent liquid closure relations (only for nonzero average liquid velocity) and the laminar liquid closure relations presented in chapter ?? include liquid backflow along the wall.

In this chapter first the transient equations will be derived, which are governed by the mass and momentum conservation equations set up for each phase. The behaviour of the models with respect to time is observed, to see which of the steady states are viable according to the transient system for one-dimensional two-phase flow. The goal of this analysis is to see whether the steady states solutions as found in the previous chapter could be stable and under which conditions this could occur.

4.1 The transient system

The analysis starts with the well-known conservation equations, shown in equation 4.1,

$$
\frac{\partial \varepsilon_l}{\partial t} + \frac{\partial (\varepsilon_l u_l)}{\partial x} = 0,
\frac{\partial \varepsilon_g}{\partial t} + \frac{\partial (\varepsilon_g u_g)}{\partial x} = 0,
\rho_l \frac{\partial (\varepsilon_l u_l)}{\partial t} + \rho_l \frac{\partial (\varepsilon_l u_l^2)}{\partial x} + \varepsilon_l \frac{\partial P}{\partial x} = -\frac{\tau_l S_l}{A} + \frac{\tau_l S_i}{A} - \rho_l \varepsilon_l g \sin \alpha,
\rho_g \frac{\partial (\varepsilon_g u_g)}{\partial t} + \rho_g \frac{\partial (\varepsilon_g u_g^2)}{\partial x} + e_g \frac{\partial P}{\partial x} = -\frac{\tau_g S_g}{A} + \frac{\tau_g S_i}{A} - \rho_g \varepsilon_g g \sin \alpha.
$$

In these four partial differential equations there are five unknowns, namely: $\varepsilon_l, \varepsilon_g, u_l, u_g$ and $P$. However, there is also a fifth relation that connects the liquid holdup to the gas holdup,

$$
\varepsilon_l + \varepsilon_g = 1.
$$

(4.1)
This complex system will be simplified since a few assumptions can be made.

### 4.1.1 Quasi-steady state gas phase

Since the gas responds much faster to changes than the liquid it can be assumed that the gas is in a quasi-steady state without changing the physical behaviour too much. This means that the time derivative drops out for the continuity and momentum equation. Moreover, those equations take the following form, see 4.3 and 4.4,

\[ u_g \rho_g \varepsilon_g A = \text{constant}, \]  
\[ \rho_g \frac{\partial \left( \varepsilon_g u_g^2 \right)}{\partial x} + \varepsilon_g \frac{\partial P}{\partial x} = -\frac{\tau_{g g} S_g}{A} - \rho_g \varepsilon_g g \sin \alpha. \] (4.4)

These two equations can be substituted into the liquid momentum equation, which will then become,

\[ \rho_l \frac{\partial u_l}{\partial t} + \rho_l u_l \frac{\partial u_l}{\partial x} - u_g \rho_g \frac{\partial \varepsilon_g}{\partial x} = -\frac{\tau_{l l} S_l}{A \varepsilon_l} + \frac{\tau_{l g} S_g}{A \varepsilon_g} + \frac{\tau_{l i} S_i}{A \varepsilon_i} \left( \frac{1}{\varepsilon_l} + \frac{1}{\varepsilon_g} \right) - \left( \rho_l - \rho_g \right) g \sin \alpha. \] (4.5)

In this manner the resulting system consists of only two partial differential equations, where the liquid holdup and liquid velocity are the unknowns and the gas velocity is an input variable.

### 4.1.2 Uniform film thickness: pipe length discretization

A second simplification of the system is that the liquid level along the pipe is constant for a certain pipe length. The space derivative term disappears and only the time derivative remains. The system with a finite pipe length with uniform liquid holdup becomes:

\[ \frac{d \varepsilon_l}{dt} = \frac{u_l S - u_l \varepsilon_l}{L}, \] (4.6)
\[ \rho_l \frac{du_l}{dt} + \rho_l \frac{u_l S - u_l \varepsilon_l}{L} \left( \frac{u_l - u_l S}{\varepsilon_l} \right) = -\frac{\tau_{l l} S_l}{A \varepsilon_l} + \frac{\tau_{l g} S_g}{A \varepsilon_g} + \frac{\tau_{l i} S_i}{A \varepsilon_i} \left( \frac{1}{\varepsilon_l} + \frac{1}{\varepsilon_g} \right) - \left( \rho_l - \rho_g \right) g \sin \alpha. \] (4.7)

In these equations the term \( u_l S \) is the initial superficial liquid velocity. Rewriting gives,

\[ \frac{d \varepsilon_l}{dt} = \frac{u_l S - u_l \varepsilon_l}{L}, \] (4.8)
\[ \frac{du_l}{dt} = -\frac{u_l S}{\varepsilon_l L} \left( u_l - \frac{u_l S}{\varepsilon_l} \right) - \frac{\tau_{l l} S_l}{\rho_l A \varepsilon_l} + \frac{\tau_{l g} S_g}{\rho_l A \varepsilon_g} + \frac{\tau_{l i} S_i}{\rho_l A} \left( \frac{1}{\varepsilon_l} + \frac{1}{\varepsilon_g} \right) - \left( \frac{1}{\rho_l} - \frac{1}{\rho_g} \right) g \sin \alpha. \] (4.9)

Note how the closure models are needed only in the equation for the time derivative of the liquid velocity. Since these closure relations only depend on the liquid holdup, both differential equations depend on both, the liquid velocity and the liquid holdup.

Assuming a zero-net-liquid flow this nonlinear system becomes as shown in equations 4.10 and 4.11,
\[
\frac{d\varepsilon_l}{dt} = -\frac{u_l\varepsilon_l}{L}, \quad (4.10)
\]
\[
\frac{du_l}{dt} = \frac{F}{\rho_l}. \quad (4.11)
\]

Where,
\[
F = -\frac{\tau_l S_l}{A\varepsilon_l} + \frac{\tau_c S_c}{A\varepsilon_c} + \frac{\tau_g S_g}{A} \left( \frac{1}{\varepsilon_l} + \frac{1}{\varepsilon_g} \right) - (\rho_l - \rho_g) g \sin \alpha = f(\varepsilon_l), \quad (4.12)
\]

which contains all the closure models. Note that the liquid velocity disappeared in equation 4.11 after the zero-net-liquid flow was assumed.

### 4.2 Typical behaviour of the transient system

As a first indication we consider the expected behaviour of this transient system. From the steady state analysis we know that there is always a trivial solution, where the liquid holdup in the pipe flow is zero. Physically this means that the pipe does not contain liquid. This could occur if either the liquid would flow back downward along the pipe when the interfacial shear stress is not high enough. Or it would mean that the interfacial shear stress is too high, blowing all the liquid out of the pipe.

The steady state analysis does however also predict two other possible solutions. These solutions are only available for certain superficial gas velocities combined with specific holdup values according to the steady state analysis. The steady state analysis also revealed that for certain superficial gas velocities multiple liquid holdup values were possible. Therefore it will be interesting to see how the liquid holdup will develop using this transient simulation.

![Figure 4.1](image)

See figure 4.1 which was adopted from Birvalski [1]. Here the superficial gas velocity is fixed at \( u_{g, s} = 8.8 \text{ m/s} \), which is close to the critical superficial gas velocity but not the exact critical superficial gas velocity. Different initial hold up values are used, which show a different behaviour of the transient system. There is a certain domain within which we see that, according to the findings of Birvalski, the solution oscillates towards the steady state with the highest holdup. When the starting or initial liquid holdup value is outside this domain the final solution of the system is the one with the zero liquid hold up. This physically means that either the liquid is blown out upward or has flown back downward.

It is interesting to further consider the domain with the oscillations. When the liquid velocity is fluctuating heavily in this simulation, the velocity can become negative for part of the time. Note that this system does not take a velocity distribution into account, and therefore the liquid wall shear stress is directly coupled to this instantaneous liquid velocity. Moreover it seems that the third solution is not reached.

This figure does not show the variation of the instantaneous liquid velocity nor the liquid holdup \( \varepsilon_l \) over time. This makes it unclear how fast the system is reacting and how the flow properties are fluctuating. It does show how the average liquid velocity and liquid holdup behave with respect to each other over time. The two dots and the star shown in figure 4.1 are the initial conditions for the stability analysis. The triangle represents the lower nonzero liquid holdup solution obtained using the steady state analysis for these flow conditions. The square represents the higher liquid holdup solution obtained using the steady state analysis. Now that it is known what could be expected, the laminar liquid model and the turbulent liquid model will be used in the simulation in order to analyze their time-dependent behaviour.

The triangle in figure 4.1 shows the lower liquid holdup solution obtained with the steady state analysis. This means that the simulation was not performed at exactly the critical condition. The exact critical condition is located at one exact point. This
would mean that these simulations are not performed at the exact critical condition but at conditions close to the critical conditions. During the steady state analysis the critical conditions found were also approximations towards the critical condition. This would mean that there will be a second solution can be found when analyzing the dynamical behaviour using the critical condition approximations found that resulted from the steady state analysis. The flow conditions used during the stability analysis will therefore be close to the critical flow conditions, but these will not be the exact critical flow conditions.

4.3 Effect of initial liquid holdup $\varepsilon_{l,\text{init}}$

The previous section, on the expected outcome of this analysis, shows that the initial (starting) liquid holdup can change the liquid holdup $\varepsilon_l$ which is obtained when the solution is converged. This analysis is performed to obtain a better understanding of the transient behaviour of critical condition. For our analysis the two previously discussed closure models are used namely, Biberg laminar liquid and Biberg turbulent liquid.

4.3.1 Simulation set-up

Five different cases are discussed, using five different initial liquid holdup values $\varepsilon_l$. See figures 4.2 to 4.6. During all the simulations the almost critical superficial gas velocity $U_{g,S,\text{lam}} = 8.4095$ m/s and $U_{g,S,\text{tur}} = 8.8070$ m/s are used. These values are close to critical flow conditions for an inclination angle equal to $\alpha = 1.7^\circ$. The pipe length is also held constant at $l = 0.05$ m long. The pipe diameter is set to 5.08 cm, which is the same as used by Birvalski. The fluid properties used are those of water for the liquid and of air for the gas.
4.3.2 Results

Figure 4.2 shows the system for the lowest values of the initial liquid holdup $\varepsilon_{l,\text{init}}$. The liquid holdup is too low which leads to the trivial solution. Looking at the liquid holdup versus time it is clear that the liquid level drops immediately to zero. The velocity shows a peak, indicating that the liquid is blown out of the pipe according to both models. Both models differ slightly but overall they look quite similar.

The second case, presented in figure 4.3, shows that for an initial liquid holdup $\varepsilon_{l,\text{init}} \approx 0.07$ according to these models the critical condition is obtained. The average liquid velocity drops and becomes even negative, which increases the liquid level (and therefore the liquid holdup). The liquid holdup $\varepsilon_l$ increases, and with a small overshoot the critical liquid holdup value is obtained for each model respectively. The response of the turbulent model is about 40% slower than the laminar liquid model. Note that even though the average liquid velocity becomes negative for a short period, the amplitude is three orders of magnitude smaller than the superficial gas velocity.

Figure 4.4, shows the third case. This is where the initial liquid holdup $\varepsilon_{l,\text{init}}$ is larger than the final liquid holdup. The average liquid velocity shows a peak and then settles down at zero, giving a zero net liquid flow. This increase in velocity is paired with a drop in liquid holdup.

The fourth case studied is shown in figure 4.5. This case presents the behaviour of the largest initial liquid holdup that would still result in a non-zero liquid holdup. A relatively high(er) liquid holdup means a relatively low(er) gas area. And with the superficial gas velocity being a constant (constant gas mass flow), this results in a relatively high(er) actual gas velocity. This in turn increases the interfacial friction, speeding up the average liquid velocity. Therefore the gas ‘drags’ the liquid upward, reducing the liquid holdup, which eventually reduces the liquid velocity again. This is clearly seen in the third case as well (see figure 4.4). We now consider figure 4.5. When the average liquid velocity has dropped to zero after increasing and decreasing, the liquid holdup is just within the proper domain and follows the path presented in case 2 (figure 4.3).

The fifth and last case is shown in figure 4.6. This case shows how the system behaves if the initial liquid holdup is too high. The liquid will accelerate and the liquid holdup will drop, which in turn causes the average liquid velocity to decrease. However before the average liquid velocity reaches zero, the holdup has become too low and the liquid is blown out of the pipe, resulting in a zero liquid holdup.

4.3.3 Discussion

These cases show that there clearly is a domain of initial liquid holdup values that result in a non-zero liquid holdup. Cases 2 and 4 show the shape of these boundaries. In this case the boundaries for initial liquid holdup $\varepsilon_{l,\text{init}}$ are [0.696, 0.107] for the laminar liquid model and [0.686, 0.938] for the turbulent liquid model. Note that both models are simulated with a superficial gas velocity that is close to the critical value. Note that this analysis shows a different value for the liquid holdup after settling. The liquid holdups obtained by this system are $\varepsilon_l = 0.0804$ and 0.0757 for the laminar liquid model and the turbulent liquid model, respectively. Especially the results with the turbulent liquid model differs a lot from the steady state solution, since the exact critical conditions are not used during this analysis. The superficial gas velocity is chosen close to the critical gas velocity. From the steady state analysis it is known that the resulting liquid holdup $\varepsilon_l$ changes rapidly with relatively small changes in superficial gas velocity close to the critical condition. This explains the change in liquid holdup when the system is converged.
Figure 4.2: Liquid holdup response, $\varepsilon_{l, \text{init}, \text{Lam}} = 0.060$ and $\varepsilon_{l, \text{init}, \text{Tur}} = 0.062$

Figure 4.3: Liquid holdup response, $\varepsilon_{l, \text{init}, \text{Lam}} = 0.069$ and $\varepsilon_{l, \text{init}, \text{Tur}} = 0.068$
Figure 4.4: Liquid holdup response, \( \epsilon_{l, \text{init, lam}} = 0.09 \) Tur: 0.085

Figure 4.5: Liquid holdup response, \( \epsilon_{l, \text{init, lam}} = 0.107 \) and Tur: 0.094
Figure 4.6: Liquid holdup response, $\varepsilon_{l,\text{init, lam}} = 0.111$ and $\varepsilon_{l,\text{init, tur}} = 0.096$
4.4 Effect of pipe length

So far the length of the pipe used during the transient analysis of the liquid holdup has been fixed at \( l = 0.05 \) m. In the steady state analysis there was no pipe length involved and the pipe could be seen as infinitely long. Birvalski used a pipe length of three meters in his simulations. He stated that there was no effect of choosing the pipe length and that a length of three to ten meters all gave the same result. During a few quick simulations with different pipe lengths clear differences were noticeable. Therefore we have performed a study on the effect of the pipe length.

This section will study the effect of the pipe length. Different pipe lengths will be used in the simulations and the results will be compared. Finally a discussion will conclude this section.

4.4.1 Simulation set-up

The simulation is very similar to the one used in the previous section 4.3, which is done for consistency. The laminar liquid and the turbulent liquid closure relations are used for the analysis. The inclination angle \( \alpha \) is set to 1.7°. The superficial gas velocity is the approximated critical superficial gas velocity for this inclination angle. This superficial gas velocity was found in the steady state analysis, which differs slightly for each model (\( u_{g,5,lam} = 8.4095 \) m/s and \( u_{g,5,tur} = 8.8070 \) m/s). For the initial liquid holdup the value of \( \varepsilon_{l,init} = 0.1 \) is used. The fluid properties for the liquid and gas used are those of water and air respectively. Finally the pipe lengths used are 0.005, 0.05, 0.5 and 5 meter.

4.4.2 Results and discussion

The results of the simulations are shown in figures 4.7 and 4.8. First of all, it is immediately clear that the results are not the same for different pipe lengths. There are three major differences for longer or shorter pipe lengths. Firstly, longer pipe lengths show a higher maximum average liquid velocity and secondly, longer pipe lengths need more time to converge. Thirdly, shorter pipe lengths show a negative average liquid velocity or do suddenly not converge to the nonzero liquid holdup value. This third effect means that the pipe length apparently influences the region for which a solution converges to the critical liquid holdup.

4.4.2.1 Pipe length effects

The simulation with the five meter long pipe shows the highest average liquid velocity. Both the laminar liquid and the turbulent liquid model show this behaviour. Figure 4.8 shows that the average liquid velocity rises quickly after only a few seconds but it takes a long time for its value to settle down. Moreover, this simulation is not yet fully converged. Using a longer simulation time (so more than 80 seconds) will show that it converges to the same liquid holdup as the red line which represents the simulation using a pipe length of half a meter. The responses of the simulations with longer pipe lengths are overall slower. This could physically be explained by the fact that the amount of liquid in longer pipes is larger. This larger amount of liquid has a larger mass and will therefore respond slower to the system. Similarly, for shorter pipe lengths the liquid have less mass and therefore the response over time is much quicker.

As can be seen in figure 4.7 for the shortest pipe length, suddenly the liquid holdup value converges to zero. Therefore apparently the pipe length does play a role in the stability and even in the final outcome of the liquid holdup. The pipe length seems to influence the value of the initial liquid holdup \( \varepsilon_{l,init} \) for which the liquid holdup converges to a nonzero liquid holdup. This domain of initial liquid holdup values (discussed in the previous section) seems to vary with pipe length. Figures 4.9 and
Figure 4.7: The response of liquid velocity versus the liquid holdup $\varepsilon_l$ for different pipe lengths. $\varepsilon_{l,\text{init}} = 0.1$

4.10, show the maximum initial liquid holdup values that still result in a nonzero liquid holdup for each pipe length. Now it is clear that for shorter pipe lengths this domain reduces and the maximum initial liquid holdup is decreased when the pipe length decreases. For the longer pipe lengths the average liquid velocity and liquid holdup will oscillate before converging to the zero net liquid velocity and the critical liquid holdup. Moreover, the liquid holdup seems to be attracted to a specific point where the average liquid velocity is zero and the liquid holdup is approximately 0.07. This point of attraction is independent of the chosen pipe length. For very long pipe lengths the system seems to be always stable as the maximum initial liquid holdup goes above $\varepsilon_{l,\text{init}} > 0.25$. This is not interesting since for these kind of initial liquid holdups other physical phenomena will be encountered that are not covered by these models, such as slug flow. This point of attraction shown by these models will be analyzed and discussed more thoroughly in the next section. Overall it can be concluded that the pipe length does influence the analysis, contrary to earlier claims made in the literature [1]. Longer pipe lengths will be more stable and have a larger maximum initial liquid holdup that will result in a nonzero liquid holdup. It does take more time before simulations with longer pipe length to converge, due to the greater amount of liquid mass simulated.

As a final remark it should be noted that the pipe length seems to have some sort of a dampening effect on the way in which the liquid holdup and average liquid velocity behave with respect to time. For short pipe lengths fluctuations occur, which even could result in zero liquid holdup, as the flow is attracted towards the lower nonzero liquid holdup solution before it converges towards the higher liquid holdup solution or the zero liquid holdup solution. Longer pipe lengths do not show this behaviour and immediately converge towards the higher liquid holdup solution without any fluctuations as is shown in figure 4.10.
Figure 4.8: The response of the liquid velocity and the liquid holdup $\varepsilon_l$ versus the time. $\varepsilon_{l,\text{init}} = 0.1$.
Figure 4.9: The largest initial liquid holdup $\varepsilon_{l,\text{init}}$ for which the system still converges to a nonzero liquid holdup.

Figure 4.10: A close-up of figure 4.9.
4.5 Domain of nonzero liquid holdup

In the previous section it was shown that the length of the pipe chosen in the stability analysis does have an influence on the dynamical behaviour. Also it was seen that there is clearly some domain for which the system will converge towards the nonzero liquid holdup. This section will investigate how this domain of nonzero liquid holdup changes for the closure relations for the laminar liquid model and the turbulent liquid model. A study is performed into the flow parameters that influence this domain and how they affect the domain.

4.5.1 Area of attraction

Birvalski observed that there was an area of attraction which, was not mentioned in the literature before. This area of attraction was towards the second solution seen in the steady state analysis. This second solution, which is the lower nonzero liquid holdup solution, is linearly unstable. This area of attraction can be seen clearly in figure 4.6. Around $\varepsilon_l = 0.07$ the system seems to get attracted to a zero liquid velocity, but then suddenly the liquid accelerates and results in a zero liquid holdup.

When a lower initial liquid holdup $\varepsilon_l$ is used the average liquid velocity will become zero before the second solution. This is shown in figures 4.4 and 4.5. Therefore when the average liquid velocity becomes zero and the liquid holdup at that point is higher than the liquid holdup of the second solution, the system will converge towards the third solution. When the liquid solution becomes zero at or below the second solution liquid holdup value the average liquid velocity will increase and the system results in a zero liquid holdup.

Using this knowledge the lower boundary of the nonzero liquid holdup domain is known, namely the second liquid holdup solution found by the steady state analysis. This boundary is independent of the pipe length used for the stability analysis. This is an important finding, since it means that at the exact critical conditions there will be no domain of nonzero liquid holdup. So far approximations towards the critical flow conditions have been used in the analysis, approaching the critical solution using slightly lower superficial gas velocities (since slightly higher superficial gas velocities would result in a zero liquid holdup instantly). As a direct result of these approximations there was always a second solution and a third solution, since at the exact critical conditions these two solutions would be equal resulting in one possible solution. Without the second solution there will be no lower boundary for the domain of nonzero liquid holdup.

4.5.2 Parameters that influence the domain of nonzero liquid holdup

So far it was shown that the domain is influenced by two variables, the superficial gas velocity and the length of the pipe. Figure 4.11 shows the domains of nonzero liquid holdup. The area within the lines will result in a nonzero liquid holdup. The effect of the pipe length is now clear and for longer pipe length the domains will grow. For the longest pipe length the domain is even so large that it is questionable if the closure models are still accurate. Remember that these models were designed for low liquid holdup values. Moreover, notice that the lower boundary for the liquid holdup of the domain remains constant for all the lengths of pipe. Note that the final liquid holdup does not vary with the pipe length and they are represented by the asterisks in the figures.

Another parameter which influences the domain is the superficial gas velocity. The superficial gas velocity will change the left boundary of the domain, since it will change the second solution of the holdup equation as shown in the steady state analysis. Also the inclination angle therefore influences the domain of nonzero liquid holdup. For all initial conditions of the liquid holdup that lay within the domain of attraction of the nonzero liquid holdup, there will be convergence to the third (and highest) solution.
of the holdup equation. This domain is determined by the pipe length, superficial gas velocity, inclination angle, pipe diameter and fluid properties.
Figure 4.11: Domains of nonzero liquid holdup using the following conditions: $U_{g,S, lamin} = 8.4$ m/s, $U_{g,S,turb} = 8.8$ m/s, $U_{l,s} = 0$ m/s, inclination angle $\alpha = 1.7^\circ$ and fluid properties of water and air for liquid and gas respectively.
4.6 Discussion and conclusions

Throughout this chapter the dynamical behaviour of the one-dimensional two-phase inclined upward stratified zero-net-liquid flow has been analysed. The laminar liquid and turbulent liquid closure relations both allow for liquid backflow along the wall, which occurs close to and at the critical flow conditions. At first simulations have been performed using flow conditions close to the critical flow conditions. The system is analysed for different initial liquid holdup conditions, while the other properties were kept constant. This showed different behaviour for various values of the initial liquid holdup $\varepsilon_{l,\text{init}}$ and several typical behaviour cases have been distinguished and discussed.

A domain of initial values of the liquid holdup was found to converge to the higher nonzero liquid holdup solution found the in steady state analysis. The value of the converged nonzero liquid holdup depends on the the same properties as the steady state system, the superficial gas velocity, pipe inclination angle, pipe diameter and fluid properties. The converged nonzero liquid holdup value is therefore independent of the chosen pipe length, which is a new variable which was introduced in the dynamic stability analysis.

A new finding in this work is the effect of the pipe length parameter used during the stability analysis. Longer pipe lengths have a dampening effect on the average liquid velocity and on the liquid holdup over time. For longer pipe lengths the system directly converges, without oscillations, towards the higher nonzero liquid holdup found in the steady state analysis. Shorter pipe lengths show a different behaviour in which the flow is first attracted towards the lower nonzero liquid holdup solution found in the steady state analysis. This lower solution is not linearly stable, and the flow than either converges to the higher nonzero liquid holdup solution or to a zero liquid holdup. This depends on the initial conditions of the transient simulation.

A domain of nonzero liquid holdup is defined. For initial conditions within this domain the system will converge to the higher nonzero liquid holdup predicted in the steady state analysis. The lower boundary for initial liquid holdup $\varepsilon_{l,\text{init}}$ is shown to be equal to the lower nonzero liquid holdup solution found in the steady state. This lower boundary is therefore independent of the pipe length. The rest of the domain is dependent on the pipe length. The domain becomes larger when the pipe length is increased.

However, when the flow conditions used during the simulations (i.e. superficial gas velocity etcetera), are taken closer towards the exact critical flow conditions the lower and higher nonzero liquid holdup solutions in the steady state analysis become closer to each other. This decreases the size of the domain for nonzero liquid holdup since the lower boundary approximates the higher liquid holdup.

Concluding, the stability analysis shows that the system can be stable and that there exists a domain for which initial flow conditions will converge towards a nonzero liquid holdup with a zero average liquid velocity. This domain decreases when the flow conditions approximate exact critical conditions. Note that the exact critical condition is a pure theoretical case. Still at the exact critical conditions the flow will only converge to a nonzero liquid holdup if the initial liquid holdup is higher than the critical liquid holdup. Also the pipe length needs to be chosen large enough such that the system will converge towards the critical holdup directly without oscillations. For flow conditions close to, but lower than, the exact critical flow conditions the linear stability analysis shows that a nonzero liquid holdup can be stable.
5. Wave stability analysis

In this chapter a wave stability analysis is performed using the method of characteristics. There are some significant differences compared to the transient analysis presented in the previous chapter, equation 3.1. Still the gas is assumed to be in a quasi-steady state, since the gas responds much faster to changes than the liquid. However the (liquid) holdup is not assumed to be constant along a certain pipe length. Therefore a discretization in space is required for this analysis. Another simplification which is needed to create a hyperbolic system assumes that the surface tension is zero, which is a valid assumption according to Barnea and Taitel [29].

\[ \sigma = 0 \] (5.1)

This analysis is inherently different from the previous stability analysis. In this wave stability analysis the focus lies on the growth or decay of waves on the surface between the liquid and the gas. The main question is, how the waves behave in time and space when the flow conditions are near critical.

In the previous stability analysis it was possible to see whether a pipe would become empty (zero liquid holdup) over time. This is not possible in this model, because this is not included in this type of model. This wave stability analysis simulates how waves would propagate through time and space and whether for instance their amplitude would grow or decay over time. In the literature this analysis has not been performed for these sets of closure relations.

5.1 The non-linear wave equation

First the equations for the wave stability analysis will be derived. The conservation equations have to be rewritten slightly compared to the equations presented in equation 3.1. In these set of conservation equations the liquid holdup is rewritten to the liquid height level \( h_l \). This system is shown in equation 5.2,

\[
\frac{\partial h}{\partial t} + u_l \frac{\partial h}{\partial x} + \frac{A_l}{dA_l/dh} \frac{\partial u_l}{\partial x} = 0
\]

\[
\rho_l \left( \frac{\partial u_l}{\partial t} + u_l \frac{\partial u_l}{\partial x} + g \cos \alpha \frac{\partial h}{\partial x} \right) = -\frac{\partial P_l}{\partial x} - \frac{\tau_l S_l}{A_l} + \frac{\tau_i S_i}{A_l} - g \rho_l \sin \alpha
\]

\[
\rho_g \left( \frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + g \cos \alpha \frac{\partial h}{\partial x} \right) = -\frac{\partial P_g}{\partial x} - \frac{\tau_g S_g}{A_g} - \frac{\tau_l S_l}{A_g} - g \rho_g \sin \alpha
\] (5.2)

and,

\[
A_g u_g = A u_g s
\] (5.3)

Now the pressure drops are assumed to be equal in the momentum equations for both phases. When with the superficial gas velocity definition in equation 5.3, the system becomes,
\[
\frac{\partial h}{\partial t} + u_l \frac{\partial h}{\partial x} + \frac{A_l}{\partial A_l/\partial h} \frac{\partial u_l}{\partial x} = 0
\]
\[
\frac{\partial u_l}{\partial t} + G \frac{\partial h}{\partial x} + u_l \frac{\partial u_l}{\partial x} = -\sigma \frac{\partial^3 h}{\rho_l \partial x^3} - E = -E
\] (5.4)

Where,
\[
G = \left(\rho_l - \rho_g\right) g \cos \alpha - \frac{\rho_l A_l^2 U^2 g S}{\rho_l} \frac{dA_l/\partial h}{\partial A_l/\partial h}
\]
\[
E = \frac{-F}{\rho_l}
\] (5.5)

Note that the term in the second equation of the system 5.4 disappears due to the assumption that the surface tension \(\sigma\) equals zero. This system has an hyperbolic character and therefore the characteristic velocities can be derived to be,
\[
C_H = u_l + \sqrt{G \frac{A_l}{dA_l/\partial h}} C_L = u_l - \sqrt{G \frac{A_l}{dA_l/\partial h}}
\] (5.6)

where \(C_H\) and \(C_L\) are the higher and lower characteristic velocity respectively. Note that these characteristic velocities do not have to be constant with time and space, which is typical for a non-linear method of characteristics. Along these characteristics the conservation equations become ordinary differential equations, resulting in an overall solvable system. The system of ordinary differential equations is shown in equation 5.7,
\[
\frac{dh}{dt} - \sqrt{\frac{1}{G} \frac{dA_l}{\partial h}} \frac{du_l}{dt} = \sqrt{\frac{1}{G} \frac{dA_l}{\partial h}} \times E = 0 \quad \text{along,} \quad \frac{dx}{dt} = C_L
\]
\[
\frac{dh}{dt} + \sqrt{\frac{1}{G} \frac{dA_l}{\partial h}} \frac{du_l}{dt} + \sqrt{\frac{1}{G} \frac{dA_l}{\partial h}} \times E = 0 \quad \text{along,} \quad \frac{dx}{dt} = C_H
\] (5.7)

### 5.2 Computational model

In order to work with the equations derived in the previous section a computational model is created. The Matlab application is used to perform the computations. Time and space are discretized, which is described in section 5.2.1. The equations shown in 5.7 are rewritten according to this discretization. Finally the method is described that is used to accurately determine the grid nodes.

#### 5.2.1 Numerical discretization scheme

In order to be able to track waves and see wave propagation through the domain a space and time discretization is required. This is different from the steady-state (where \(d/dx = 0\) and \(d/dt = 0\)) and the linear stability analysis (where \(d/dx = 0\)). The characteristic velocities are used to determine the next grid node in time and space. The scheme that determines \(x_{i,k+1}\) and \(t_{i,k+1}\) is shown in equation 5.8,
\[
\frac{x_{i,k+1} - x_{i,k}}{t_{i,k+1} - t_{i,k}} = C_{H,k}
\]
\[
\frac{x_{i,k+1} - x_{i,k}}{t_{i,k+1} - t_{i,k}} = C_{L,k}
\] (5.8)

Figure 5.1 shows a physical interpretation of the variables. The \(k\)-subscript describes the \(k\)th timestep. Note that the physical time can differ between grid nodes with the same timestep due to the nonlinear nature of the system. The \(i\)-subscript describes the \(i\)th node in space discretization.
This system presented in equations 5.8 is rewritten into a matrix-form like this,

\[
\begin{bmatrix}
1 & -C_{L+1,k} \\
1 & -C_{H+1,k}
\end{bmatrix}
\begin{bmatrix}
x_{i,k+1} \\
x_{i+1,k+1}
\end{bmatrix} = \begin{bmatrix}
x_{i+1,k} - C_{L+1,k} t_{i+1,k} \\
x_{i,k} - C_{H+1,k} t_{i,k}
\end{bmatrix}
\]

(5.9)

Solving the system in 5.9 determines the location in space and time for the next time step \((k+1)\). The liquid height \(h\) and average liquid velocity \(u_i\) are now evaluated for the next time step \((k+1)\). See equation 5.10 below.

\[
\begin{align*}
\frac{h_{i,k+1} - h_{i+1,k}}{G_{i+1,k}} &= \sqrt{\frac{1}{G_{i+1,k}} \left( \frac{dA_i}{dh} \right)_{i+1,k} (u_{i,k+1} - u_{i+1,k})} - \sqrt{\frac{1}{G_{i+1,k}} \left( \frac{dA_i}{dh} \right)_{i+1,k} E_{i+1,k} (t_{i+1,k} - t_{i+1,k}) = 0} \\
\frac{h_{i,k+1} - h_{i,k}}{G_{i,k}} &= \sqrt{\frac{1}{G_{i,k}} \left( \frac{dA_i}{dh} \right)_{i,k} (u_{i,k+1} - u_{i,k})} - \sqrt{\frac{1}{G_{i,k}} \left( \frac{dA_i}{dh} \right)_{i,k} E_{i,k} (t_{i+1,k} - t_{i,k}) = 0}
\end{align*}
\]

Also this system is rewritten into a matrix-form like this,

\[
\begin{bmatrix}
1 - \sqrt{\frac{1}{G_{i+1,k}} \left( \frac{dA_i}{dh} \right)_{i+1,k}} \\
1 - \sqrt{\frac{1}{G_{i,k}} \left( \frac{dA_i}{dh} \right)_{i,k}}
\end{bmatrix}
\begin{bmatrix}
h_{i+1,k} \\
h_{i,k+1}
\end{bmatrix} = \begin{bmatrix}
h_{i+1,k} - \sqrt{\frac{1}{G_{i+1,k}} \left( \frac{dA_i}{dh} \right)_{i+1,k} (u_{i,k+1} + \frac{E_{i+1,k}}{\rho} (t_{i,k+1} - t_{i+1,k}))} \\
h_{i,k} + \sqrt{\frac{1}{G_{i,k}} \left( \frac{dA_i}{dh} \right)_{i,k} (u_{i,k} + \frac{E_{i,k}}{\rho} (t_{i,k+1} - t_{i,k}))}
\end{bmatrix}
\]

(5.11)

5.2.2 Grid iteration

For every step in time, when going from level \((k)\) to \((k+1)\), first the new node locations in space and physical time will have to be determined. See figure 5.2. The next node location is determined using the higher and lower characteristic velocities of timestep \((k)\). This is an approximation since in the velocities are assumed to be constant.

In an effort to improve the location of the next node, the characteristic velocities are calculated at the location of the node of the first guess. The average of the characteristic velocities at timestep \((k)\) is determined and the first guess of the next time step \((k+1)\) is calculated. Note that the timestep and physical time are two different items. The time may vary along a timestep; a timestep is virtual, while the time is physical. This average characteristic velocity is then used to calculate the new location of the new node at the next time step \((k+1)\). This iterative process is depicted in figure 5.2. This iterative process improves the accuracy with which the next node in time and space is calculated. This improves the numerical stability allowing a longer physical simulation time before the simulation will diverge due to numerical errors.
Figure 5.2: A schematic of the grid iteration process.

Figure 5.3 shows an example of two simulations. For the same flow conditions, the red simulation did not perform the grid iteration whereas the blue simulation did do that. In the time interval between 0 and 20 sec, the red lines and blue lines coincide. Thereafter the red simulation breaks due to numerical error. The red model is unable to generate an accurate grid; note that every time step introduces an error since the location of the next grid point is calculated using a linear approximation. Due to the iteration, the grid nodes of the next timestep are calculated with an increased accuracy, which allows for more timesteps before the model collapses.

5.2.3 Grid refinement study

The effect of the size of the initial spatial discretization is studied. This should give insight on how the model behaves for different initial $d_x$, where $d_x$ is the distance between grid nodes on the first timestep. Over time this distance may vary from node to node due to the change in characteristic velocities, triggered by the initial wave. In figure 5.4 two simulations are shown. Both simulations have the same flow conditions; the blue simulation has $d_{x_{blue}} = 0.0001$ and the red simulation has $d_{x_{red}} = 0.001$. Decreasing the spatial discretization has a direct effect on the physical time between virtual timesteps. This is shown in figure 5.4. The blue simulation covers 8 seconds, while 30,000 timesteps are made. The red simulation covers almost 30 seconds with 10,000 timesteps. After 20,000 timesteps the blue simulation collapses since the accumulated error in the grid through all the virtual timesteps becomes too large. Looking back at figure 5.1 it is understandable that for a smaller space between $x_{i,k}$ and $x_{i+1,k}$ the next point in time will also be smaller since the characteristic velocities are roughly the same. However, when $d_x$ becomes too large, information is lost and the grid will not be able to capture the shape of the wave accurately enough. Moreover, the error introduced by the linear approximation (even though there is an extra grid iteration) will increase as well. For all simulations an initial $d_x = 0.001$ is applied which has been shown to give accurate results before the errors break the model down.
Figure 5.3: An example of grid iteration. Red: Grid iteration off, Blue: Grid iteration on. $dx = 0.001$, timesteps $= 15,000$.

Figure 5.4: A grid refinement study, blue: $dx = 0.0001\text{m}$ timesteps: 30,000, red: $dx = 0.001\text{m}$ timesteps: 10,000.
5.3 Wave propagation

The initial flow conditions in the pipe are set close to the critical condition for each set of closure relations. As explained in section 5.2.3, the initial spatial distance between nodes is set to $dx = 0.001$. The superficial gas velocities per set of closure relations are set to $U_{g,S,\text{laminar}} = 8.409$ m/s and $U_{g,S,\text{turbulent}} = 8.807$ m/s. The pipe inclination angle is set to $\alpha = 1.7$, which is consistent with the previous analysis. For both models the liquid holdup is set to $\varepsilon_l = 0.077$, which allows to compare these models. All conditions are close to, but a little lower than, the critical condition. Thus it is possible to see how waves behave close to the critical condition.

Figure 5.5 shows the results are plotted in three dimensions. Each line presents the values at a virtual time step.

In figure 5.5 an initial wave with an amplitude of $\varepsilon_l = 0.125$ is simulated at the previously stated flow conditions. The red lines represent the simulation using the laminar closure relations and the blue lines represent the simulation using turbulent closure relations. Both models show an immediate decay of the maximum in the wave amplitude with respect to the physical time. This is also seen in figure 5.6, where the maximum of the wave amplitude is shown against the physical time. The area of the wave remains constant over time, which is due to the mass conservation law. This simulation shows that a wave with an amplitude that is $\pm 150\%$ higher than the critical liquid holdup will decay over time at flow conditions which approach the critical condition.
Figure 5.5: Wave propagation with flow conditions close to the critical condition. Amplitude initial wave: $\varepsilon_l = 0.125$ Red: laminar closure relations, Blue: turbulent closure relations.

Figure 5.6: Maximum in the wave amplitude of the wave shown in figure 5.5 versus the physical time. Amplitude initial wave: $\varepsilon_l = 0.125$ Red: laminar closure relations, Blue: turbulent closure relations.

5.4 Roll-over waves

In section 5.3 a simulation was presented showing the decay of the wave amplitude with respect to time. An initial wave with an amplitude of $\varepsilon_l = 0.125$ was used for this simulation. This section shows how the system behaves for waves with a higher initial amplitude. Also the behaviour of waves with a higher superficial gas velocity than the critical superficial velocity presented.

5.4.1 High initial wave amplitude

A simulation has been performed with an initial wave that has an amplitude of $\varepsilon_l = 0.18$. The result is presented in figure 5.7. The model with the turbulent closure relations is unable to simulate the high amplitude wave and it does not present the
liquid holdup values for the wave. Therefore only the model with the laminar closure relations is presented. At first the wave amplitude drops, but after 6 physical seconds the wave amplitude suddenly increases. This is also seen in figure 5.8. Shortly thereafter the model collapses due to discretization errors as an effect of the high wave. The grid created in the virtual timesteps after 7 seconds is unable to cope with these high liquid holdup values. These high liquid holdup values cannot be captured with this model, because physically different phenomena will occur. The flow can no longer be assumed to be stratified or stratified-wavy and slugs could start to occur. These phenomena are not included in the model that is derived for this simulations. We can conclude that the flow will become unstable for waves with a high amplitude in the liquid holdup.

**Figure 5.7:** Wave propagation with flow conditions close to the critical condition. Amplitude initial wave: \( \varepsilon_l = 0.17 \). Red: laminar closure relations.

**Figure 5.8:** Maximum in the wave amplitude of the wave shown in figure 5.7 versus the physical time. Red: laminar closure relations.
5.4.2 High superficial gas velocity

A simulation is performed with an increased superficial gas velocity. The same initial wave that was used in figure 5.5 is used in this simulation. The superficial gas velocities for each model have been increased by 1 m/s and are now set to $U_{g,S,\text{laminar}} = 9.409$ m/s and $U_{g,S,\text{turbulent}} = 9.807$ m/s. The results are presented in figures 5.9 and 5.10. Similar to the simulations with a high initial wave amplitude (presented in section 5.4.1) the model with turbulent closure relations is unable to simulate these conditions. Figure 5.9 shows that the wave starts to increase in amplitude immediately. See also figure 5.10, which shows that after 4 physical seconds the model breaks down. Similar to the simulation with a high initial wave amplitude the model is not capable to handle these high liquid holdup values.

Also in this case, where the superficial gas velocity is higher than the critical condition, the wave amplitude starts to grow. It is concluded that waves on the interfacial surface are unstable when the superficial gas velocity is higher than the critical superficial gas velocity.
5.5 Discussion and conclusions

This chapter covered the stability of a wave on the interface between a gas and a liquid in upward inclined pipe flow. The flow is in the stratified flow regime and has a relatively low liquid holdup. The conditions are the same as those used in the analysis presented in the previous chapter.

Due to the nonlinear nature of the wave stability analysis the characteristic velocities were not constant. The discretization in time and space proved to be challenging. However, applying an extra iteration step improved the robustness of the numerical method. This allowed for using a limited number of time steps to obtain an accuracy that is sufficient to perform the wave stability analysis.

The simulations showed that a wave with a liquid holdup with an amplitude that is about 150% larger than the critical liquid holdup decreased in amplitude with respect to time. This proves that a relatively small wave on the interfacial surface, with flow conditions near the critical conditions, damps out. Therefore it cannot be concluded that interfacial waves cause an instability per se.

For higher liquid holdup amplitudes of an initial wave the behaviour is different. The wave amplitude drops at first, but it then increases and it finally obtains too high values to satisfy the assumption that the flow is still stratified. It can therefore be concluded that higher waves can trigger the wave to grow. This results in a different flow regime and thus the flow becomes unstable.

A similar behaviour occurs when the superficial gas velocity is increased. From the start of the simulation the wave grows in amplitude. Thereafter the model is incapable to correctly represent the physical phenomena. Thus it is concluded that for a higher superficial gas velocity than the critical condition the flow becomes unstable.

It was also shown that not all waves result in an unstable system. It can therefore not be concluded that an interfacial wave, at flow conditions near, but lower than, the critical flow conditions, results in instability of the flow.
6. Conclusions and recommendations

This research was inspired by the earlier work of Birvalski on the critical flow conditions in two-phase pipe flow. The modeling of liquid accumulation in the low elbow sections of undulated pipelines that transport two-phase gas and liquid, as commonly used in for example the oil and gas industry, is a very challenging research topic. We have investigated how multiple steady states for the liquid holdup can occur in the pipe system, and how the stability of these states is with respect to time. Thereto different simplified analyses were carried out, and a comparison was made with existing experiments. In particular we have investigated at which critical value of the superficial gas velocity the liquid is swept out of the low spots in the pipeline.

1. What mechanistic two-phase stratified pipe flow models that allow for liquid backflow along the wall do currently exist?

In chapter two, various closure relations were presented. After careful analysis two models were found to meet the requirements. At and around the critical condition the average liquid flow is zero or nearly zero. The liquid volume itself is not zero and the liquid is dragged upwards by the interfacial shear stress of the gas which flows over it. Along the wall the liquid flows backwards. This up and back flow gives a zero net liquid velocity. This backflow along the wall results in a nonzero liquid wall shear stress. Many models for the wall shear stress consist of a friction factor which is multiplied with the dynamic pressure. Unfortunately this dynamic pressure is zero for a zero net liquid flow and therefore the liquid wall shear stress is neglected, which is a disadvantage of such models.

One fully analytical model, called the laminar liquid model in this thesis, does not model the liquid wall shear stress in this manner. It consists of the sum of two products: one term describes the wall friction due to the average liquid flow and the second term describes the effect of the interfacial shear stress on the wall shear stress. The second model that was considered contains a friction factor which is multiplied by the dynamic pressure of the liquid and is called the turbulent liquid model. Even though this model uses a friction factor in this manner, the model can still represent the physical phenomenon of backflow along the wall, though only when the liquid velocity is nonzero. This is promising for transient simulations, where the instantaneous liquid velocity may change. Therefore these two models were selected for further analysis in the critical condition and the dynamical behavior around the critical condition.
2. How well is the critical condition predicted with mechanistic closure relations that allow for liquid backflow along the wall?

In the third chapter the critical condition was analyzed using a steady-state analysis. The so-called holdup equation was derived, which could be seen as a momentum balance of the gravity and the shear stresses of the gas and liquid between each other and with the wall.

In the trivial case there is no liquid holdup in the upward inclined pipe system, which is a solution of the liquid holdup equation that is always present. There is exactly one point where two solutions are possible, and this is at the exact critical condition. Physically this point can be interpreted as the point where a slight increase of the superficial gas velocity would cause the liquid to be blown out of the pipe. Thus for higher superficial velocities than the critical condition the only steady-state solution that remains is the trivial solution of zero liquid holdup.

For superficial gas velocities lower than the critical condition there exist three liquid holdup values for which the system has a steady-state: the trivial zero liquid holdup solution, a lower nonzero liquid holdup and a higher nonzero liquid holdup. For lower values of the superficial gas velocity the lower liquid holdup solution decreases and the higher liquid holdup solution increases.

In the comparison with experimental results obtained by Birvalski, the laminar liquid closure relations turn out to predict the critical superficial gas velocity with the highest accuracy namely, with an error below $\pm 1\%$. The turbulent liquid model had an average error of $\pm 5\%$.

3. How does the liquid holdup behave in transient simulations close to the critical condition, using mechanistic closure relations that allow for liquid backflow along the wall?

It was thus shown that there is one exact critical condition and there are two solutions in the steady-state analysis close to the critical condition. The dynamical behaviour close to the critical condition where two steady-state solutions exist was analyzed in chapter four.

A transient analysis is set up using closure relations that allow for liquid backflow along the wall at flow conditions close to the critical condition. This is an analysis that was not performed in the literature before. One of the assumptions made is that the liquid holdup is constant along a certain pipe length. Therefore the model is unable to model the effect of waves on the interfacial surface. This analysis set all the spatial derivatives to zero, simplifying the system. Furthermore the gas is assumed to be in a quasi-steady state, since its dynamical response is assumed to be much faster than the response of the liquid. This results in a system of two nonlinear ordinary differential equations.

The analysis of this system shows that for both the laminar and turbulent closure relations at flow conditions close to the critical condition there are initial conditions for which the equations converge to the solution with the higher steady-state liquid holdup. The solution with the lower steady-state liquid holdup is unstable. As it turns out, the solution with the lower steady-state liquid holdup proves to be the lower boundary of the domain of initial liquid holdups for which the system converges to the solution with the higher steady-state liquid holdup solution.

This domain of higher liquid holdups is thus bounded by the lower steady-state liquid holdup solution. The upper boundary of this domain is determined by the pipe length and by the superficial gas velocity used in the simulation. For short pipe lengths this
domain becomes very small, while for long pipe lengths the domain grows making large initial liquid holdups stable as well. These large initial liquid holdups, however, are beyond the capabilities of the model. The effect of the pipe length has not been presented in the literature before.

It is concluded that close to the critical condition there is a domain (boundaries that are dependent on the lower steady-state liquid holdup, the pipe length and the superficial gas velocity) for which this simplified transient simulation shows that the solution with the highest steady-state liquid holdup is stable.

In chapter five a wave stability analysis was performed. The previous stability analysis was unable to model the behaviour of waves on the surface between the gas and the liquid. Assuming that the surface tension is equal to zero makes it possible to rewrite the mass and momentum conservation laws such that along two characteristics, the conservation equations become a system of ordinary differential equations. This is used to analyze the wave propagation behaviour on the interface when the conditions are close to the critical values.

A simulation with an initial wave with a liquid holdup amplitude that is 150% larger than the critical value is performed. This simulation showed that the wave amplitude would decay over time. This dynamical behaviour was similar for simulations using both laminar and turbulent liquid closure relations.

Simulations for an initial wave with a liquid holdup amplitude that is 200% larger than the critical value shows a different result. At first the wave amplitude decays over time, but thereafter it suddenly increases rapidly. For these high liquid holdup values the flow cannot be assumed to be in the stratified flow regime anymore and the models are no longer able to describe the flow.

When the superficial gas velocity is higher than the critical value, the wave amplitude grows over time. A similar situation occurs as found with the amplitude of the wave for the high initial liquid holdup wave, where the model is unable to correctly describe the flow at the high liquid holdup values that occur when the wave is growing. It is therefore concluded that a wave on the surface interface does not necessarily make the flow unstable.

**To what extent are mechanistic models able to describe the critical gas velocity to sweep out liquids in the low elbows of pipelines?**

Overall it can be concluded that including backflow along the wall does increase the performance of the models that are used for the prediction of the critical condition in a steady-state analysis, when compared to experimental results. Furthermore it has been shown that using these closure relations in dynamic simulations lead to the steady solutions with the higher liquid holdup that are stable for all flow conditions close to the critical condition.

**Recommendations**

More experiments of the ‘sweep out’ gas velocity will provide more information that can be used to validate the closure relations. Different pipe diameters, pipe lengths, liquid volumes and inclination angles will make it possible to validate and improve the closure relations. Also the transient data of two-phase pipe flow with near critical conditions is so far unavailable.

This research focused on the closure relations of the liquid wall shear stress mainly. The interfacial shear stress remains challenging to model for one-dimensional multiphase flow. Therefore new or improved mechanistic closure relations for the interfacial shear stress are needed to improve the modeling of multiphase flow. Especially
since the liquid wall shear stress is dependent on the interfacial shear stress. Improving the modeling of the interfacial shear stress will therefore contribute greatly in the overall modeling of multiphase flow. Finally the wave stability analysis presented in this research can be extended further. This research presents one type of initial wave. Analysis of multiple initial waves or different types of initial waves might give more insight in the stability of waves in multiphase flow.
Bibliography


