A DEPTH INTEGRATED MODEL FOR SUSPENDED TRANSPORT

by

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Synopsis

A new depth averaged model for suspended sediment transport in open channels has been developed based on an asymptotic solution to the two dimensional convection-diffusion equation in the vertical plane. The solution for the depth averaged concentration is derived from the bed boundary condition and the computation of transport rate and entrainment rate are performed therefore. Expressions are derived for adaptation length and time. The model is economical and easy to apply even in unsteady flow situations and compares favourably with the full two dimensional solution for steady flow. The stability of bed level change calculations including numerical effects can be analysed prior to application of the model. The extension to three-dimensions is outlined.

Contents

Synopsis

Contents

Notation

Chapter 1. Introduction

1.1. General

1.2. Acknowledgements

Chapter 2. Theoretical Background

2.1. The Mass-Balance Equation

2.2. Boundary Conditions

2.3. The Depth Averaged Equation

Chapter 3. Preliminary Considerations I

3.1. Scales and Magnitudes

3.2. Asymptotic Solution - Case A

3.3. Asymptotic Solution - Case B

3.4. Implications

Chapter 4. Preliminary Considerations II

4.1. Definitions

4.2. Transformed Equations

4.3. Assumptions

4.4. The Diffusion Coefficient

4.5. The Operator $D^{-1}[\quad]$

Chapter 5. The General Solution for slowly Varying Flow

5.1. The General Solution (case A)

5.2. The Bed Boundary Condition

5.3. The Sediment Transport Rate

5.4. The Entrainment Rate

Chapter 6. Some Features of Interest
6.1. Adaptation Length and Time
6.2. Concentration Profiles
6.3. Validity of the Approximation
6.4. Adaptation from Zero Concentration
6.5. Expressions for Transport Rate
6.6. Expressions for Entrainment Rate
6.7. Depth-Averaged Solutions
6.8. Zero Entrainment
6.9. An Alternative Bed Boundary Condition

Chapter 7. The Complete First Order Quasisteady Solution
7.1. The Vertical Velocity Component
7.2. The Vertical Velocity in Quasisteady Flow
7.3. The Variation of ϕ_o in Quasisteady Flow
7.4. The Quasisteady Solution
7.5. Discussion

Chapter 8. Calculation of Bed Level Change
8.1. Basic Considerations
8.2. The Basic Equations
8.3. The Stability of Linearised Equations
8.4. Numerical Stability

Chapter 9. Computations
9.1. General
9.2. Computation of Profile Functions and Coefficients
9.3. Unsteady Flow Calculation
9.4. Suspended Sediment Transport due to a Flood Wave
9.5. Bed Level Change Calculations (Quasisteady Flow)
9.6. The Siltation of a Dredged Trench

Chapter 10. Conclusion

10.1. Summary

10.2. Comments

10.3. Extension to Three Dimensions

References

Appendix A

Appendix B

Appendix C

Notation

a	water depth	(m)
a_b	constant of proportionality for bed load formula	
a_s	constant of proportionality for equilibrium suspended load formula	
a_{ij}	concentration profile function	(-)
b_b	exponent in bed load formula	
b_s	exponent in equilibrium suspended load formula	
c	concentration of suspended sediment by volume	(-)
$\bar{c}$	depth averaged concentration as defined by (4.2)	(-)
c_a	concentration at the reference level $z = z_a + z_b$	(-)
c_e	equilibrium concentration profile	(-)
$\bar{c}_e$	mean equilibrium concentration as defined by (4.2)	(-)
c_i	i^{th} term of asymptotic solution for c	(-)
e_i	concentration profile function	(-)
f	as defined by (7.16)	(-)
g	acceleration due to gravity	(m/s^2)
g_2	as defined by (7.27)	(-)
h	water depth above reference level	(m)
p	normalised velocity profile	(-)
p_b	porosity of bed	(-)
q	water discharge per unit width	(m^2/s)
r	as defined by (7.13)	(-)
s_b	bed load transport per unit width	(m^2/s)
s_s	suspended load transport per unit width	(m^2/s)
s_t	total sediment trabsport per unit width	(m^2/s)
s_e	equilibrium suspended load transport per unit width	(m^2/s)
t	time	(s)
t'	dimensionless time	(-)
u	horizontal velocity in x-direction	(m/s)
u'	dimensionless velocity	(-)
$\bar{u}$	depth averaged velocity defined by (4.2)	(m/s)
u_*	shear velocity	(m/s)

$\bar{v}$	true depth averaged velocity	(m/s)
w	vertical velocity component	(m/s)
w'	dimensionless vertical velocity	(-)
w_s	particle fall velocity	(m/s)
x	horizontal coordinate (longitudinal)	(m)
x'	dimensionless horizontal coordinate	(-)
y	horizontal coordinate (lateral)	(m)
z	vertical coordinate	(m)
z'	dimensionless vertical coordinate	(-)
z_a	height of reference level above bed	(m)
z_b	elevation of bed	(m)
z_s	elevation of water surface	(m)
A	constant given by (7.7)	(-)
B	length scale in lateral direction	(m)
C	Chezy coefficient	($m^{\frac{1}{2}}/s$)
D	differential operator by (4.14)	
D^{-1}	integral operator defined in section 4.5.	
D_v	virtual diffusion coefficient	(m^2/s)
E	scale for turbulent diffusion coefficient for sediment	(m^2/s)
E'	dimensionless diffusion cofficient	(-)
H	depth scale	(m)
L	longitudinal length scale	(m)
L_A	adaptation length	(m)
P	as defined in (7.5)	(-)
T	time scale	(s)
T_A	adaptation time	(s)
U	velocity scale (longitudinal)	(m/s)
Z	suspension parameter as defined by (7.17)	(-)

α	pseudoviscosity	(m^2/s)
α_{ij}	dimensionless transport due to profile a_{ij}	(-)
β	given by (7.8)	(-)
γ_{ij}	value of a_{ij} at $\zeta = o$	(-)
δ	small parameter	(-)
ε	turbulent diffusion coefficient for sediment particle	(m^2/s)
ε'	dimensionless diffusion coefficient defined by (4.15)	(-)
ζ	transformed vertical coordinate defined by (4.1)	(-)
θ	weighting factor	(-)
κ	von Karmann's constant	(-)
λ_i	dimensionless transport due to profile e_i	(-)
μ_i	value of e_i at $\zeta = o$	(-)
ζ	transformed longitudinal coordinate defined by (4.6)	(-)
ρ	as defined by (8.32)	(-)
σ	Courant number	(-)
τ	transformed time defined by (4.6)	(-)
ϕ	$(= a_{11}(\zeta))$ normalised equilibrium concentration profile	(-)
ψ	as defined by (3.5)	

1. INTRODCUTION

1.1. General

The transport of suspended sediment in a stream by convection and turbulent diffusion under gravity can be expressed mathematically in the form of a linear partial differential equation for the local sediment concentration in space and time. If the flow field in known in advance, this equation can be solved if an empirical boundary condition is applied at the bed where sediment exchange takes place. The numerical procedure necessary to obtain such a solution is sufficiently expensive and time consuming to preclude, for the time being, application in many mathematical models, especially in three dimensional flow fields.

The objective of a sediment transport calculation is usually to make predictions of morphological changes. The level of sophistication of the mathematical model used must also be decided upon in this context. The principal feature that distingsuishes suspended sediment transport from bed load transport is the time taken for the suspension to adapt to changes in flow conditions.

The suspended sediment concentration profile is not entirely determined by local conditions. For a given flow there is a local adaptation length and an adaptation time that characterise the response of the concentration profile and therefore the sediment entrainment or deposition rate to a change in the flow conditions. However, there are many instances where the time and length scales of the problem under consideration far exceed the adaptation lengths and times of the suspension. In such cases there is little to be gained by using sophisticated models that take the adaptation phenomena into account. Rather, it would then be more appropriate to consider the suspended load to be a part of the total sediment load which is predicted by a formula based on local conditions.

There are also instances where the time and length scales of a problem are small enough to make it necessary that the mathematical model reflects the transient nature of suspended sediment, where solving the full convection-diffusion equation is still too expensive.

It is therefore necessary to develop simplified models which, while being easier to apply, still retain the essential characteristics of the convection-diffusion process. It is also necessary to study the assumptions on which such a simplified model is based so that an understanding is reached about the limits of its applicability. As many flow models used in morphological computations are based on depth averaged quantities it seems reasonable that the corresponding model for suspended sediment should also be based on depth averaged quantities.

This report describes one such model based on an asymptotic solution of the two-dimensional (in the vertical plane), unsteady convection diffusion equation. The model is developed for uniform or nearly uniform sediment which can be represented by a single fall velocity and the transport process is described by a partial differential equation for the depth averaged concentration in terms of the other depth averaged quantities, the horizontal coordinate and time. The equation incorporates the bed boundary condition explicitly. Once the depth averaged concentration is found it is possible to compute the transport rate and the sediment entrainment rate at the bed. The coefficients of the equations can be determined in advance if standardised profiles are used for velocity and for the diffusion coefficient for sediment. The vertical component of velocity can be taken into account. Expressions have also been derived for adaptation length and time.

Numerical and analytical solutions have been obtained and compared with existing numerical solutions of the full convection-diffusion equation for steady conditions. The comparison has been favourable not only for prediction of concentration levels but also for calculation of bed levels. The stability of the bed level computations is also investigated. It has been demonstrated that the model could be applied to unsteady flow situations.

It should be noted that the work reported here is only the first step towards developing a depth averaged model for a three-dimensional flow field. Possibilities of further work have been discussed in the last chapter.

1.2. Acknowledgements

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2. THEORETICAL BACKGROUND

2.1. The Mass-Balance Equation

The partial differential equation that governs the transport of suspended sediment by convection and turbulent diffusion under gravity is

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = w_s\frac{\partial c}{\partial z} + \frac{\partial}{\partial x}(\varepsilon\frac{\partial c}{\partial x}) +$$

$$\frac{\partial}{\partial y}(\varepsilon\frac{\partial c}{\partial y}) + \frac{\partial}{\partial z}(\varepsilon\frac{\partial c}{\partial z}) \qquad (2.1)$$

where z is the vertical coordinate.

If the diffusion terms other than the vertical diffusion term are neglected and the equation is written for a two-dimensional flow in the vertical plane

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + w\frac{\partial c}{\partial z} = w_s\frac{\partial c}{\partial z} + \frac{\partial}{\partial z}(\varepsilon\frac{\partial c}{\partial z}) \qquad (2.2)$$

The turbulent diffusion coefficient ε is not exactly equal to the eddy viscosity for the motion of water (Coleman, 1970). Modified expressions based on flume and field measurements have been suggested (DHL, 1978). Equation (2.2) has been solved numerically for steady conditions (Kerssens 1974, DHL 1980) using a transformed grid in the vertical direction for greater resolution of the region near the bed.

2.2. Boundary conditions

The concentration profile at the upstream boundary must be known at all time steps. The surface boundary condition is that there is no sediment flux across the surface.

$$\text{i.e., } \left(w_s\, c + \varepsilon \frac{\partial c}{\partial z}\right)_{\text{surface}} = 0 \qquad (2.3)$$

The bed boundary condition is either

$$[c]_{\text{bed}} = \text{f(flow and sediment parameters)} \qquad (2.4)$$

$$\text{or } \left(\frac{\partial c}{\partial z}\right)_{\text{bed}} = \text{f(flow and sediment parameters)} \qquad (2.5)$$

or a combination of (2.4) and (2.5) (DHL, 1980)

The bed boundary condition is not applied at the bed ($z = z_b$) itself but at some small distance $z = z_b + z_a$ above it (see fig. 4.1).

2.3. The Depth Averaged Equation

If the mass balance equation (2.2) is integrated vertically, using the surface boundary condition (2.3) while neglecting the vertical velocity

$$\frac{\partial}{\partial t}(h\,\bar{c}) + \frac{\partial}{\partial x}(h\,\overline{uc}) = E \qquad (2.6)$$

where h is the depth of flow, $\bar{c}$ the mean concentration,

$$h\,\overline{uc} = \int_{z_a + z_b}^{z_a + z_b + h} uc\, dz \qquad (2.7)$$

and the entrainment rate E is given by

$$E = -\left(w_s c + \varepsilon \frac{\partial c}{\partial z}\right)_{z = z_a + z_b} \qquad (2.8)$$

If (2.6) is to used to compute concentration levels some assumptions have to be made about the concentration profiles in order to calculate $\overline{uc}$ and E.

It has been shown (Taylor 1953, Elder 1959) that the transport and dispersion of a dissolved substance in a fluid flowing in a conduit can be represented by a single virtual diffusion coefficient that takes into account the combined action of sheared convection and turbulent diffusion. A similar approach, if applied to suspended sediment, will lead to

$$\overline{uc} = h\,\alpha\,\bar{u}\bar{c} - h\,D_v \frac{\partial \bar{c}}{\partial x} \qquad (2.9)$$

where D_v is the virtual diffusion coefficient and α is given by

$$\alpha = (\overline{u\,c_e})/(\bar{u}\,\bar{c}_e) \qquad (2.10)$$

where the subscript 'e' refers to equilibrium conditions.

Under certain conditions it is possible to justify (2.9) theoretically and even obtain an expression for D_v (Vreugdenhil, 1982). It is, however, necessary to use an empirical expression for the entrainment rate E to make it possible to solve for the mean concentration. The bed boundary condition (2.4) or (2.5) will be then implicitly be included in that empirical expression.

Vermaas (1982) obtained empirical expressions for D_v and E by comparing (2.6) with the numerical solution of (2.2) for a steady uniform flow with zero upstream concentration. It was found that D_v had only a small effect on the results and that E was proportional to $(\bar{c}_e - \bar{c})$.

3. PRELIMINARY CONSIDERATIONS I

3.1. Scales and Magnitudes

Let the flow under consideration be characterised by the following scales.

Horizontal distance	L
Vertical distance	H
Time	T
Horizontal velocity	U
Vertical velocity	UH/L
Turbulent Diffusion coefficient	E

Equation (2.2) may now be made dimensionless so that

$$\frac{H}{W_s T}\frac{\partial c}{\partial t'} + \frac{HU}{LW_s}\left(u'\frac{\partial c}{\partial x'} + w'\frac{\partial c}{\partial z'}\right) = \frac{\partial c}{\partial z'} + \frac{E}{w_s H}\frac{\partial}{\partial z'}\left(E'\frac{c}{\partial z'}\right)$$

where all quantities marked with (') have been made dimensionless using the corresponding scale.

The order of magnitude of E is roughly

$$E \sim \tfrac{1}{4}\,\kappa\, u_* H = \tfrac{1}{4}\kappa\frac{\sqrt{g}}{C}\,UH \sim 0.005\ UH$$

and

$$\frac{E}{w_s H} \sim 0.005\,\frac{U}{w_s} \sim 1$$

where C in the Chezy coefficient.

The terms on the right hand side of (3.1) are both of the same order of magnitude and are responsible for the vertical readjustment of the concentration distribution.

The magnitude of the terms on the left hand side depend on the values of the parameters $\frac{H}{w_s T}$ and $\frac{UH}{Lw_s}$. If both these parameters are of $O(\delta)$ or smaller it is possible to construct an asymptotic solution to (3.1).

Two possibilities are considered in the subsequent analysis.

<u>Case A</u>

$$\left.\begin{aligned} \frac{UH}{Lw_s} &= \delta << 1 \\ \frac{H}{w_s T} &= \delta << 1 \end{aligned}\right\} \tag{3.2}$$

This implies that $T = L/U$ so that the time scale is internal.

<u>Case B</u>

$$\left.\begin{aligned} \frac{UH}{Lw_s} &= \delta << 1 \\ \frac{H}{w_s T} &= \delta^2 \end{aligned}\right\} \tag{3.3}$$

This implies that the unsteady term is smaller than the convection terms and that

$$T \sim w_s L^2/HU^2 \sim EL^2/H^2U^2$$

which corresponds to the assumption made by Daubert, A., (1975)

3.2. <u>Asymptotic Solution - Case A</u>

Using (3.2), (3.1) may be written as

$$\left(\frac{\partial c}{\partial t'} + u' \frac{\partial c}{\partial x'} + w' \frac{\partial c}{\partial z'}\right) = \frac{\partial c}{\partial z'} + \frac{E}{w_s H} \frac{\partial}{\partial z'} \left(E' \frac{\partial c}{\partial z'}\right) \tag{3.4}$$

which will admit a solution of the form

$$c = \sum_{i=o}^{n} \delta^i \psi_i + O(\delta^{n+1}) \tag{3.5}$$

where

$$\frac{\partial \psi_o}{\partial z'} + \frac{E}{w_s H} \frac{\partial}{\partial z'} (E' \frac{\partial \psi_o}{\partial z'}) = o \tag{3.6}$$

and

$$\frac{\partial \psi_i}{\partial z'} + \frac{E}{w_s H} \frac{\partial}{\partial z'} (E' \frac{\partial \psi_i}{\partial z'}) = (\frac{\partial}{\partial t'} + u' \frac{\partial}{\partial x'} + w' \frac{\partial}{\partial z'}) \psi_{i-1}$$

$$\text{for } i \geqslant 1 \tag{3.7}$$

It is possible to revert (3.6) and (3.7) to the original coordinates by writing

$$c = \sum_{i=o}^{n} \delta^i \psi_i = \sum_{i=o}^{n} c_i \tag{3.8}$$

Consequently,

$$w_s \frac{\partial c_o}{\partial z} + \frac{\partial}{\partial z} (\varepsilon \frac{\partial c_o}{\partial z}) = o \tag{3.9}$$

and

$$w_s \frac{\partial c_i}{\partial z} + \frac{\partial}{\partial z} (\varepsilon \frac{\partial c_i}{\partial z}) = (\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z}) c_{i-1}$$

$$\text{for } i \geqslant 1 \tag{3.10}$$

It should be noted that c_{i+1} will be an order of magnitude smaller than c_i.

3.3. Asymptotic Solution - Case B

For Case B, (3.1) may be written as

$$\delta^2 \frac{\partial c}{\partial t'} + (u' \frac{\partial c}{\partial x'} + w' \frac{\partial c}{\partial z'}) = \frac{\partial c}{\partial z'} + \frac{E}{w_s H} \frac{\partial}{\partial z'} (E' \frac{\partial c}{\partial z'}) \qquad (3.11)$$

Assuming $c = \sum_{i=o}^{n} \delta^i \psi_2 = \sum_{i=o}^{n} c_i$ (3.8)

$$\frac{\partial \psi_o}{\partial z'} \quad \frac{E}{w_s H} \frac{\partial}{\partial z'} (E' \frac{\partial \psi_o}{\partial z} = o \qquad (3.6)$$

$$\frac{\partial \psi_1}{\partial z'} + \frac{E}{w_s H} \frac{\partial}{\partial z} (E' \frac{\partial \psi_1}{\partial z'}) = (u' \frac{\partial}{\partial x'} + w' \frac{\partial}{\partial z'}) \psi_o \qquad (3.12)$$

and

$$\frac{\partial \psi_i}{\partial z'} + \frac{E}{w_s H} \frac{\partial}{\partial z'} (E' \frac{\partial \psi_i}{\partial z'}) = \frac{\partial \psi_{i-2}}{\partial t'} + (u' \frac{\partial}{\partial x'} + w' \frac{\partial}{\partial z'}) \psi_{i-1} \qquad (3.13)$$

for $i \geqslant 2$

Reverting to the original coordinates

$$w_s \frac{\partial c_o}{\partial z} + \frac{\partial}{\partial z} (\varepsilon \frac{\partial c_o}{\partial z}) = o \qquad (3.9)$$

$$w_s \frac{\partial c_1}{\partial z} + \frac{\partial}{\partial z} (\varepsilon \frac{\partial c_1}{\partial z}) = (u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z}) c_o \qquad (3.14)$$

$$w_s \frac{\partial c_i}{\partial z} + \frac{\partial}{\partial z} (\frac{\partial c_i}{\partial z}) = \frac{\partial c_{i-2}}{\partial t} + (u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z}) c_{i-1} \qquad (3.15)$$

for $i \geqslant 2$

3.4. Implications

$\frac{UH}{Lw_s} \ll 1$ implies that the time taken for a particle to settle is much smaller than the time it takes to be convected along a distance L. Similarly $\frac{H}{w_s T} \ll 1$ implies that the settling time is less than the time scale. It is still difficult to say just how small the parameters should be in order to make the solution converge.

However, if (3.8) is substituted in (2.2) and the result is compared with the final equations for case A and case B (3.9, 3.10, 3.14 and 3.15) it can be demonstrated that the following quantities have been neglected (or assumed to be $O(\delta^{n+1})$ or smaller).

$$\text{Case A:} \quad \frac{\partial c_n}{\partial t} + u\frac{\partial c_n}{\partial x} + w\frac{\partial c_n}{\partial z} = O(\delta^{n+1}) \tag{3.16}$$

$$\text{Case B:} \quad \left.\begin{aligned} \frac{\partial c_{n-1}}{\partial t} + u\frac{\partial c_n}{\partial x} + w\frac{\partial c_n}{\partial z} &= O(\delta^{n+1}) \\ \frac{\partial c_n}{\partial t} &= O(\delta^{n+2}) \end{aligned}\right\} \tag{3.17}$$

It is not intended that this solution is applied for values of n greater than 1 or 2. Furthermore, it can be shown that Case A is more general than Case B. Therefore, in subsequent chapters, the analysis will be given in detail only for Case A. The corresponding expressions for Case B are given where they are of interest.

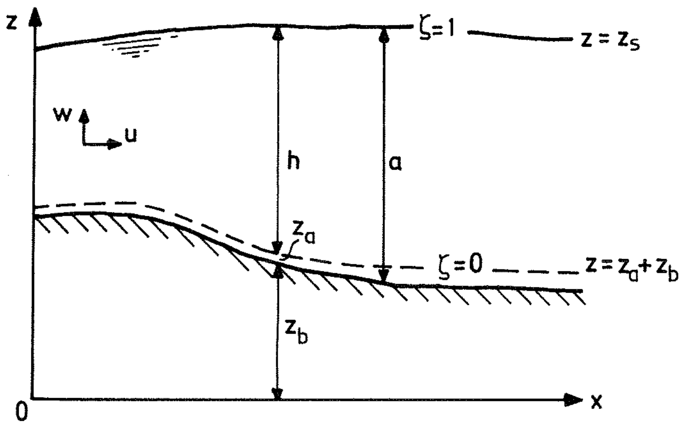


Fig. 4.1. The Flow Field

u = horizontal velocity component

w = vertical velocity component

z_s = surface elevation

z_b = bed elevation

a = depth of flow

h = depth above reference level

z_a = a - h

4. PRELIMINARY CONSIDERATIONS II

4.1. Definitions

Figure (4.1) shows the two-dimensional flow field.

a = full depth of flow

h = a - z_a = depth of suspended sediment transport

z_s = the elevation of the water surface

The bottom boundary condition is applied at $z = z_a + z_b$.
All transport below $z = z_a + z_b$ is included in the bed level transport.

The new vertical coordinate ζ is defined as

$$\zeta = \frac{z - (z_b + z_a)}{h} \tag{4.1}$$

The vertical mean of any quantity is defined as

$$\bar{f} = \int_0^1 f \, d\zeta = \frac{1}{h} \int_{z_b + z_a}^{z_b + a} f \, dz \tag{4.2}$$

$$\text{where } \frac{\partial}{\partial \zeta} = h \frac{\partial}{\partial z} \tag{4.3}$$

It should be noted that ζ is not necessarily independent of x and t. Therefore if $f(\zeta)$ is a function of ζ only,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} \tag{4.4}$$

where $\frac{\partial \zeta}{\partial x}$ can be shown to be

$$\frac{\partial \zeta}{\partial x} = - \frac{1}{h} \left(\zeta \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} (z_a + z_b)\right) \tag{4.5}$$

Similar expressions can be obtained for $\frac{\partial\zeta}{\partial t}$ and other deviations of ζ.

New coordinates are introduced in horizontal direction as well as in time.

$$\left.\begin{aligned} \frac{\partial}{\partial\xi} &= \frac{\bar{u}h}{w_s}\frac{\partial}{\partial x} \\ \frac{\partial}{\partial\tau} &= \frac{h}{w_s}\frac{\partial}{\partial t} \end{aligned}\right\} \tag{4.6}$$

where

$$\bar{u} = \int_0^1 u \, d\zeta \tag{4.7}$$

It should be mentioned that the transformation (4.6) can give rise to difficulties because of the fact that $\bar{u}h$ and h are not constant. These difficulties are avoided because no differential or integral operations are carried out in the ξ - τ space. Therefore the transformation (4.6) is only used as a convenient shorthand and to obtain simpler expressions. The final solution is obtained only after reverting to x - t coordinates. Care has to be exercised in returning to x - t coordinates. For example

$$\frac{\partial^2}{\partial\xi^2} = \frac{\bar{u}h}{w_s}\frac{\partial}{\partial x}\left(\frac{\bar{u}h}{w_s}\frac{\partial}{\partial x}\right)$$

It is possible to express the horizontal velocity component u(x, t, z) as

$$u = \bar{u}\,(\xi,\ \tau)\,p\,(\zeta) \qquad (4.8)$$

where $\bar{u}$ is independent of ζ. $p(\zeta)$ however is not necessarily independent of x and t. From the definition of $\bar{u}$ it can be shown that

$$\int_o^1 p\,d\,\zeta = 1 \qquad (4.9)$$

4.2. Transformed Equations

Equation (3.9), (3.10). (3.14) as (3.15) may now be transformed using (4.3), (4.6) and (4.8).

Case A $\quad D[c_o] = o \qquad (4.10)$

$$D[c_i] = \left(\frac{\partial}{\partial\tau} + p\,\frac{\partial}{\partial\xi} + \frac{w}{w_s}\,\frac{\partial}{\partial\zeta}\right)\,c_{i-1} \text{ for } i \geqslant 1 \qquad (4.11)$$

Case B $\quad D[c_o] = o \qquad (4.10)$

$$D[c_1] = \left(p\,\frac{\partial}{\partial\xi} + \frac{w}{w_s}\,\frac{\partial}{\partial\zeta}\right)\,c_o \qquad (4.12)$$

$$D[c_i] = \frac{\partial c_{i\,-\,2}}{\partial\tau} + \left(p\,\frac{\partial}{\partial\xi} + \frac{w}{w_s}\,\frac{\partial}{\partial\zeta}\right)\,c_{i\,-\,1} \text{ for } \geqslant 2 \qquad (4.13)$$

where

$$D = \frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\zeta}\,\left(\varepsilon'\,\frac{\partial}{\partial\zeta}\right) \qquad (4.14)$$

and

$$\varepsilon' = \frac{\varepsilon}{w_s h} \qquad (4.15)$$

The boundary condition at the surface is

$$\sum_{i=o}^{n} c_i + \varepsilon' \frac{\partial}{\partial \zeta} \left(\sum_{i=o}^{n} c_i \right) = o \quad \text{at} \quad \zeta = 1 \tag{4.16}$$

4.3. Assumptions

a) Let $$\int_o^1 c_o \, d\zeta = \bar{c}(\xi, \tau) \tag{4.17}$$

i.e. it is assumed that the higher order terms do not contribute to the mean concentration.

As c_o has to satisfy (4.19) which is an ordinary differential equation in ζ, it is possible to write

$$c_o = \bar{c} \, (\xi, \tau) \, \phi_o(\zeta) \tag{4.18}$$

where

$$\int_o^1 \phi_o(\zeta) \, d\zeta = 1 \tag{4.19}$$

and

$$\frac{\partial \phi_o}{\partial \zeta} + \frac{\partial}{\partial \zeta} \left(\varepsilon' \frac{\partial \phi_o}{\partial \zeta} \right) = o \tag{4.20}$$

ϕ_o is then the normalised equilibrium profile which ensures that c_o does not contribute to any net vertical movement of sediment. It also follows that if the suspended sediment is in equilibrium

$$c_e = \bar{c}_e \, \phi_o(\zeta) \tag{4.21}$$

b) It is assumed that the consequence of assumption (4.17) as well as the surface boundary condition (4.16) will hold for all values of $n \geqslant o$.

The consequence of (4.17) is that

$$\sum_{i=1}^{n} \int_{o}^{1} c_i \, d\zeta = o \tag{4.22}$$

The surface boundary condition is that

$$\sum_{i=o}^{n} \left\{ c_i + \varepsilon' \frac{\partial c_i}{\partial \zeta} \right\}_{\zeta = 1} = o \tag{4.16}$$

If these are to hold for all values of n

$$\int_{o}^{1} c_i \, d\zeta = o \qquad \text{for all } i \geqslant 1 \tag{4.23}$$

and

$$\left[c_i + \varepsilon' \frac{\partial c_i}{\partial \zeta} \right]_{\zeta = 1} = o \tag{4.24}$$

For i = o, substituting (4.18) in (4.24)

$$\left[\phi_o + \varepsilon' \frac{\partial \phi_o}{\partial \zeta} \right]_{\zeta = o} = o \tag{4.25}$$

If ε' is known, (4.20), (4.19) and (4.25) allows $\phi_o(\zeta)$ to be determined completely.

4.4. The Diffusion Coefficient

The diffusion coefficient for momentum in a uniform channel flow with a logarithmic velocity profile is

$$\frac{\varepsilon_m}{u_* a} = \frac{(z - z_b)}{a} \left\{1 - \frac{z - z_b}{a}\right\} \tag{4.26}$$

This distribution of diffusion coefficient was used by Rouse (1937) to obtain an analytical expression for the equilibrium concentration profile, as the solution of (3.9) using a known concentration at some reference level to obtain the constant of integration. However, in the interest of obtaining a better fit with laboratory and field measurements the following modification have been suggested (DHL, 1980) in the expression for ε.

$$\frac{\varepsilon}{u_* a} = 4 \left\{\alpha_1 + \alpha_2 \left(\frac{w_s}{u_*}\right)^{\alpha_3}\right\} \left\{\frac{z - z_b}{a}\right\} \left\{1 - \frac{(z - z_b)}{a}\right\}$$

$$\text{for } \frac{z - z_b}{a} \leq 0.5 \tag{4.27}$$

$$\frac{\varepsilon}{u_* a} = \alpha_1 + \alpha_2 \left(\frac{w_s}{u_*}\right)^{\alpha_3} \qquad \text{for } \frac{z - z_b}{a} \geq 0.5 \tag{4.28}$$

where

$\alpha_1 = 0.1$, $\alpha_2 = 0.38$ and $\alpha_3 = 4.31$ for flumes
$\alpha_1 = 0.13$, $\alpha_2 = 0.20$ and $\alpha_3 = 2.12$ for natural channels: (4.29)

with suitable rearrangement it can be shown that

$$\varepsilon' = \frac{\varepsilon}{w_s h} = f\left(\zeta, \frac{w_s}{u_*}, \frac{z_a}{a}\right) \tag{4.30}$$

4.5. The Operator $D^{-1}[\ ]$ (Definition)

If the function $F(\zeta)$ satisfies the differential equation

$$D[F(\zeta)] = G(\zeta), \tag{4.31}$$

with the boundary conditions

$$\left[F + \varepsilon' \frac{dF}{d\zeta}\right]_{\zeta - 1} = o \tag{4.32}$$

and

$$\int_o^1 F \, d\zeta = o \tag{4.33}$$

then let $F(\zeta)$ be described by the convention

$$F(\zeta) = D^{-1}[G(\zeta)] \tag{4.34}$$

It can be demonstrated that (Appendix A)

$$F(\zeta) = - \int_\zeta^1 G \, d\zeta + \phi_o \int_\zeta^1 \frac{G}{\phi_o} \, d\zeta + B \, \phi_o \tag{4.35}$$

where the constant B is obtained from (4.33).

5. THE GENERAL SOLUTION FOR SLOWLY VARYING FLOW

In this section it is assumed that the vertical velocity w is zero and that the shear stress changes slowly enough to make it possible to neglect $\frac{\partial\phi_o}{\partial x}$ and $\frac{\partial\phi_o}{\partial t}$. The detailed analysis only treats Case A.

5.1. The General Solution (Case A)

The solution is

$$c = \sum_{i=o}^{n} c_i \tag{3.8}$$

where

$$c_o = \bar{c}\ (\xi,\ \tau)\ \phi_o(\zeta) \tag{4.18}$$

and

$$D[c_i] = (\frac{\partial}{\partial\tau} + p\ \frac{\partial}{\partial\xi})\ c_{i-1} \qquad i \geqslant 1 \tag{5.1}$$

(5.1) is from (4.11) by neglecting $\frac{w}{w_s}\ \frac{\partial c_{i-1}}{\partial\zeta}$

Substituting (4.18) in (5.1) for $i = 1$

$$D[c_1] = \phi_o\ \frac{\partial\bar{c}}{\partial\tau} + p\ \phi_o\ \frac{\partial\bar{c}}{\partial\xi} \tag{5.2}$$

where $\frac{\partial\bar{c}}{\partial\tau}$ and $\frac{\partial\bar{c}}{\partial\xi}$ are completely independent of ζ.

$$c_1 = D^{-1}\ [\phi_o]\ \frac{\partial\bar{c}}{\partial\tau} + D^{-1}\ [p\ \phi_o]\ \frac{\partial\bar{c}}{\partial\xi} \tag{5.3}$$

automatically ensures that c_1 satisfies the boundary condition (4.23) and (4.24) for $i = 1$.

$$\text{i.e.,}\ c_1 = a_{21}(\zeta)\ \frac{\partial\bar{c}}{\partial\tau} + a_{22}(\zeta)\ \frac{\partial\bar{c}}{\partial\xi} \tag{5.4}$$

where

$$\left.\begin{aligned} a_{21}(\zeta) &= D^{-1}\,[\phi_o] \\ a_{22}(\zeta) &= D^{-1}\,[p\,\phi_o] \end{aligned}\right\} \qquad (5.5)$$

Similarly, substituting (5.4) in (5.1) for i = 2 and so on it is possible to obtain the general solution

$$c_{i-1} = a_{i1}\,\frac{\partial^{i-1}\bar{c}}{\partial\tau^{i-1}} + a_{i2}\,\frac{\partial^{i-1}\bar{c}}{\partial\tau^{i-2}\partial\xi} + \ldots\ldots + a_{ii}\,\frac{\partial^{i-1}\bar{c}}{\partial\xi^{i-1}} \qquad (5.6)$$

where

$$\left.\begin{aligned} a_{i1}(\zeta) &= D^{-1}\,[a_{i-1,1}] \\ a_{ij}(\zeta) &= D^{-1}\,[p\,a_{i-1,j-1} + a_{i-1,j}] \quad 1 < j < i \\ a_{ii}(\zeta) &= D^{-1}\,[p\,a_{i-1,i-1}] \end{aligned}\right\} \qquad (5.7)$$

The n^{th} order solution for c may now be assembled from (3.8).

$$c = \sum_{i=1}^{n+1} \sum_{j=1}^{i} a_{ij}(\zeta)\,\frac{\partial^{i-1}\bar{c}}{\partial\tau^{i-j}\,\partial\tau^{j-1}}$$

where

$$a_{11}(\zeta) = \phi_o(\zeta) \qquad (5.9)$$

(5.8) satisfies the complete two dimensional equation (2.2) subject to the error shown in (3.16). The boundary condition at the surface is satisfied. The bed boundary condition is yet to be applied.

It should also be noted that if $p(\zeta)$ and $\phi_o(\zeta)$ are known all the functions $a_{ij}(\zeta)$ can be evaluated.

The zero order solution is

$$c = a_{11}(\zeta)\ \bar{c}\ (\xi,\ \tau) = \phi_o(\zeta)\ \bar{c}\ (\xi,\ \tau) \tag{5.10}$$

The represents a concentration distribution which is always in vertical equilibrium.

The first and second order solutions for Case A are respectively,

$$c = a_{11}\bar{c} + a_{21}\frac{\partial\bar{c}}{\partial\tau} + a_{22}\frac{\partial\bar{c}}{\partial\xi} \tag{5.11}$$

$$c = a_{11}\bar{c} + a_{21}\frac{\partial\bar{c}}{\partial\tau} + a_{22}\frac{\partial\bar{c}}{\partial\xi} + a_{31}\frac{\partial^2\bar{c}}{\partial\tau^2} + a_{32}\frac{\partial^2\bar{c}}{\partial\tau\partial\xi} + a_{33}\frac{\partial^2\bar{c}}{\partial\xi^2} \tag{5.12}$$

The corresponding expressions for Case B are

$$c = a_{11}\bar{c} + a_{22}\frac{\partial\bar{c}}{\partial\xi} \tag{5.13}$$

$$c = a_{11}\bar{c} + a_{21}\frac{\partial\bar{c}}{\partial\tau} + a_{22}\frac{\partial\bar{c}}{\partial\xi} + a_{33}\frac{\partial^2\bar{c}}{\partial\xi^2} \tag{5.14}$$

5.2. The Bed Boundary Condition

There are several types of bed boundary condition that could be applied. In this analysis only one type (for the value of c at $\zeta = o$) is applied. It must be noted that any other type of boundary condition could also be applied equally well (see section 6.9).

It is assumed that c_a the value of the concentration at $z = z_b + z_a$ (or $\zeta = o$) is known in terms of the local flow and sediment parameters. In other words c_a is known in advance.

$$c_a = f(u_*, w_s, \bar{u}, h, \text{etc} \ldots) \tag{5.15}$$

from (5.8)

$$c_a(\xi, \tau) = \sum_{i=1}^{n+1} \sum_{j=1}^{i} \gamma_{ij} \frac{\partial^{i-1} \bar{c}}{\partial\tau^{i-j} \partial\xi^{j-1}} \tag{5.16}$$

where

$$\gamma_{ij} = a_{ij}(o) \tag{5.17}$$

Often it would be reasonable to assume that the reference concentration c_a is the same as the equilibrium concentration at the same level in a uniform flow with the same flow parameters.

The equilibrium concentration profile must satisfy (4.18), (4.19) and (4.20). Therefore

$$c_e = \bar{c}_e(\xi, \tau)\,\phi_o(\zeta) = \bar{c}_e(\xi, \tau)\; a_{11}(\zeta) \tag{5.18}$$

where the subscript 'e' refers to equilibrium conditions.

$$\text{at } \zeta = o, \quad c_e = c_a = \gamma_{11}\,\bar{c}_e \tag{5.19}$$

(5.16) is a partial differential equation for $\bar{c}$ in ξ and τ with known coefficients. If c_a is known, this equation could be solved numerically by applying a sufficient number of boundary conditions for $\bar{c}$ and derivatives of $\bar{c}$.

In practice, however, it is not possible to work in the transformed coordinates τ and ξ. Therefore it is necessary to revert to the original coordinates.

Assuming that $c_a = \gamma_{11}\, \bar{c}_e$, (5.16) will become

a) Zero Order $\bar{c}_e = \bar{c}$ (5.20)

b) First Order (Case A)

$$\gamma_{11}\,\bar{c}_e = \gamma_{11}\,\bar{c}_e + \gamma_{21}\,\frac{h}{w_s}\,\frac{\partial \bar{c}}{\partial t} + \gamma_{22}\,\frac{\bar{u}h}{w_s}\,\frac{\partial \bar{c}}{\partial x} \tag{5.21}$$

c) Second Order (Case A)

$$\gamma_{11}\,\bar{c}_e = \gamma_{11}\,\bar{c} + \gamma_{21}\,\frac{h}{w_s}\,\frac{\partial \bar{c}}{\partial t} + \gamma_{22}\,\frac{\bar{u}h}{w_s}\,\frac{\partial \bar{c}}{\partial x} + \gamma_{31}\,\frac{h}{w_s}\,\frac{\partial}{\partial t}\left(\frac{h}{w_s}\,\frac{\partial \bar{c}}{\partial t}\right)$$
$$+ \gamma_{32}\,\frac{h}{w_s}\,\frac{\partial}{\partial t}\left(\frac{uh}{w_s}\,\frac{\partial \bar{c}}{\partial x}\right) + \gamma_{33}\,\frac{\bar{u}h}{w_s}\,\frac{\partial}{\partial x}\left(\frac{\bar{u}h}{w_s}\,\frac{\partial \bar{c}}{\partial x}\right) \tag{5.22}$$

The corresponding expressions for Case B are

a) Zero Order $\bar{c}_e = \bar{c}$ (5.20)

b) First Order

$$\gamma_{11}\,\bar{c}_e = \gamma_{11}\,\bar{c} + \frac{\bar{u}h}{w_s}\,\gamma_{22}\,\frac{\bar{c}}{x} \tag{5.23}$$

c) Second Order

$$\gamma_{11}\,\bar{c}_e = \gamma_{11}\,\bar{c} + \gamma_{21}\,\frac{h}{w_s}\,\frac{\partial \bar{c}}{\partial t} + \gamma_{22}\,\frac{\bar{u}h}{w_s}\,\frac{\partial \bar{c}}{\partial x}$$
$$+ \gamma_{33}\,\frac{\bar{u}h}{w_s}\,\frac{\partial}{\partial x}\left(\frac{\bar{u}h}{w_s}\,\frac{\partial \bar{c}}{\partial x}\right) \tag{5.24}$$

As $a_{ij}(\zeta)$ are constructed from $p(\zeta)$ and $\phi_o(\zeta)$ the coefficients γ_{ij} are functions of $\frac{\bar{u}}{u_*}$, $\frac{w_s}{u_*}$ and $\frac{z_a}{a}$.

It is assumed that $p(\zeta)$ is a family of curves entirely determined by the shear stress and the mean velocity.

$$\gamma_{ij} = \gamma_{ij} \left(\frac{\bar{u}}{u_*}, \frac{w_s}{u_*}, \frac{z_a}{a}\right) \tag{5.25}$$

Thus $\bar{c}$ is finally fixed by the application of the bottom boundary condition and $\bar{c}$ is obtained by solving the resulting partial differential equation. The precise equation to be solved (e.g. any one of the equation (5.20) to (5.24)) is determined by the type and order of the approximation.

5.3. The Sediment Transport Rate

The rate of transport of suspended sediment s_s is given by

$$s_s = \int_{z_b + z_a}^{z_b + a} uc\, dz \tag{5.26}$$

$$= h \int_o^1 uc\ d\zeta \tag{5.27}$$

Substituting (5.8) in (5.27)

$$s_s = \bar{u}h \sum_{i=1}^{n+1} \sum_{j=1}^{i} \alpha_{ij} \frac{\partial^{i-1} \bar{c}}{\partial\tau^{i-j}\, \partial\xi^{j-1}} \tag{5.28}$$

where

$$\alpha_{ij} = \int_o^1 p\ a_{ij}\ d\zeta \tag{5.29}$$

In the original coordinates (Case A)

Zero Order $s_s = \alpha_{11}\ \bar{u}h\,\bar{c}$ (5.30)

First Order

$$s_s = \alpha_{11}\ \bar{u}h\,\bar{c} + \alpha_{21}\,\frac{h^2\bar{u}}{w_s}\,\frac{\partial\bar{c}}{\partial t} + \alpha_{22}\,\frac{h^2\bar{u}^2}{w_s}\,\frac{\partial\bar{c}}{\partial x} \qquad (5.31)$$

The first order expression for Case B is

$$s_s = \gamma_{11}\ \bar{u}h\,\bar{c} + \gamma_{22}\,\frac{h^2\bar{u}^2}{w_s}\,\frac{\partial\bar{c}}{\partial u} \qquad (5.32)$$

Equation (5.31) could also be expressed as

$$s_s - s_e = \alpha_{11}\ \bar{u}h\,(\bar{c} - \bar{c}_e) + \alpha_{21}\,\frac{h^2\bar{u}}{w_s}\,\frac{\partial\bar{c}}{\partial t} + \alpha_{22}\,\frac{h^2\bar{u}^2}{w_s}\,\frac{\partial\bar{c}}{\partial x} \qquad (5.31a)$$

where

$s_e = \gamma_{11}\ \bar{u}h\,\bar{c}_e$ is the equilibrium transport rate.

As $\bar{c}$ in governed by (5.21) and $\bar{c}_e$ is a function of the local parameters only, (5.31a) explains why the $s_s - s_e$ diagram exhibits a hysteresis-like behaviour. If $\bar{c}$ is obtained from the quasisteady equation (5.23) this behaviour would again be different.

5.4. The Entrainment Rate

The sediment entrainment rate E is given by

$$E = \frac{\partial(\bar{c}h)}{\partial t} + \frac{\partial(h\,\overline{uc})}{\partial x}$$

$$E = \frac{\partial(\bar{c}h)}{\partial t} + \frac{\partial s_s}{\partial x} \qquad (5.33)$$

The first order expressions are

Case A:

$$E = \frac{\partial(\bar{c}h)}{\partial t} + \frac{\partial}{\partial x}\left[\alpha_{11}\, h\bar{u}\bar{c} + \alpha_{21}\,\frac{h^2\bar{u}}{w_s}\,\frac{\partial\bar{c}}{\partial t} + \alpha_{22}\,\frac{h^2\bar{u}^2}{w_s}\,\frac{\partial\bar{c}}{\partial x}\right]$$

Case B:

$$E = \frac{\partial(\bar{c}h)}{\partial t} + \frac{\partial}{\partial x}\left[\alpha_{11}\, h\bar{u}\bar{c} + \alpha_{22}\,\frac{h^2\bar{u}^2}{w_s}\,\frac{\partial c}{\partial x}\right] \qquad (5.35)$$

Case B corresponds exactly to (2.6) where the virtual diffusion coefficient D_v is

$$D_v = -\,\alpha_{22}\,\frac{h\bar{u}^2}{w_s} \qquad (5.36)$$

or

$$D_v = (-\,\alpha_{22}\,\frac{\bar{u}}{u_*}/\frac{w_s}{u_*}).\bar{u}h \qquad (5.37)$$

$$D_v = (-\,\alpha_{22}\,\frac{C}{\sqrt{g}}/\frac{w_s}{u_*}).\bar{u}h \qquad (5.38)$$

where C is the Chezy coefficient.

6. SOME FEATURES OF INTEREST

6.1. Adaptation length and time

Consider a steady uniform flow where the suspended sediment is not in equilibrium. Then equation (5.21) may be written as

$$\bar{c}_e = \bar{c} + T_A \frac{\partial \bar{c}}{\partial t} + L_A \frac{\partial \bar{c}}{\partial x} \tag{6.1}$$

where

$$T_A = \frac{\gamma_{21}}{\gamma_{11}} \frac{h}{w_s} \tag{6.2}$$

and

$$L_A = \frac{\gamma_{22}}{\gamma_{11}} \frac{\bar{u}h}{w_s} \tag{6.3}$$

In steady uniform flow T_A and L_A will be constants and (6.1) will have straight characteristics in the x-t plane. It can also be shown that $(\bar{c} - \bar{c}_e)$ will decay exponentially with an adaptation length L_A and adaptation time T_A. The adaptation length/time is defined as the interval required to make $(\bar{c} - \bar{c}_e)$ decrease by a factor 'e'.

The characteristic speed of propagation is $\frac{\bar{u}\,\gamma_{22}}{\gamma_{21}}$. Some idea of the orders of magnitude of L_A and T_A could be obtained from figure 9.4. Let $z_a/a = 0.01$ and the Chezy coefficient = 60 $m^{\frac{1}{2}}/s$ ($\bar{u}/u_* = 19$). If the fall velocity is $w_s = 0.015$ m/s consider the combinations of $\bar{u} = 0.5$ and 1.0 m/s and h = 10 and 20 m.

$\bar{u} = 0.5$ m/s, $h = 10$ m, $L_A = 114$ m, $T_A = 110$ s

$\bar{u} = 1.0$ m/s, $h = 10$ m, $L_A = 304$ m, $T_A = 285$ s

$\bar{u} = 0.5$ m/s, $h = 20$ m, $L_A = 228$ m, $T_A = 220$ s

$\bar{u} = 1.0$ m/s, $h = 20$ m, $L_A = 608$ m, $T_A = 570$ s

It should also be noted that from the definition of L_A and T_A the length and time required for 95 percent adaptation are $L_A \ln(20)$ and $T_A \ln(20)$ respectively.

The values of L_A and T_A should be compared with the dimensions of the major features of the problem under consideration and with the mesh size of the computational grid and the time step of the calculation when a decision has to be made about the relative importance of adaptation phenomena.

6.2. Concentration Profiles

Consider a steady non-uniform flow. Then the concentration profile (first order) is (from 5.11).

$$c = a_{11}(\zeta)\,\bar{c} + a_{22}(\zeta)\,\frac{\partial \bar{c}}{\partial \xi} \tag{6.5}$$

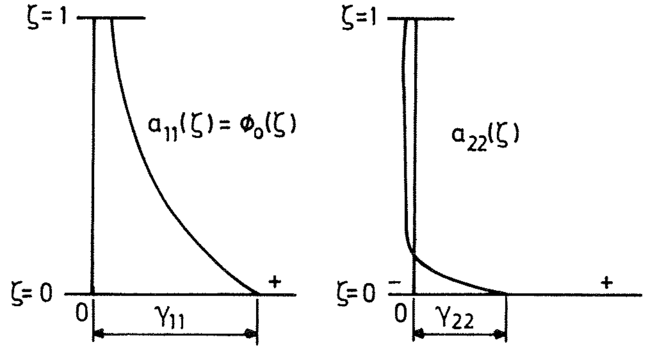
i.e.

$$c = \phi_o(\zeta)\,\bar{c} + a_{22}(\zeta)\,\frac{\bar{u}h}{w_s}\,\frac{\partial \bar{c}}{\partial x} \tag{6.6}$$

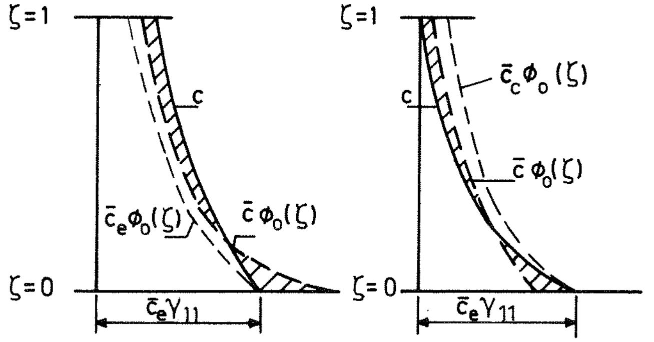
Figure (6.1) shows the typical shapes of $\phi_o(\zeta)$ and $a_{22}(\zeta)$.

The bed boundary condition (5.21) for steady flow is

$$\gamma_{11}\bar{c}_e = \gamma_{11}\bar{c} + \gamma_{22}\,\frac{\bar{u}h}{w_s}\,\frac{\partial \bar{c}}{\partial x} \tag{6.7}$$



Typical Profiles



Sedimentation

$\bar{c}_e < \bar{c}$

$c = \phi_o \bar{c} + a_{22} \frac{\bar{u}h}{w_s} \frac{\partial \bar{c}}{\partial x}$

$\frac{\partial \bar{c}}{\partial x} < o$

Erosion

$\bar{c}_e > \bar{c}$

$c = \phi_o \bar{c} + a_{22} \frac{\bar{u}h}{w_s} \frac{\partial \bar{c}}{\partial x}$

$\frac{\partial \bar{c}}{\partial x} > o$

Fig. 6.1. Concentration Profiles

or

$$\frac{\bar{u}h}{w_s}\frac{\partial \bar{c}}{\partial x} = \frac{\gamma_{11}}{\gamma_{22}}(\bar{c}_e - \bar{c}) \tag{6.8}$$

Figure (6.1) also shows how the concentration profile is modified during

$$\left.\begin{array}{llll} \text{(a) Sedimentation} & \bar{c} - \bar{c}_e > o & \text{or} & \dfrac{\partial \bar{c}}{\partial x} < o \\ \text{and (b) erosion} & \bar{c} - \bar{c}_e < o & \text{or} & \dfrac{\partial \bar{c}}{\partial x} > o \end{array}\right\}$$

When the solution is convergent, higher order solutions will give better and better approximation of the true concentration profile.

6.3. Validity of the Approximation

The solution is based on the assumption that c_i is an order of magnitude smaller than c_{i-1} (see 3.8). Therefore in the 1^{st} order steady solution (6.5),

$$\left| a_{22}(\zeta)\,\frac{\bar{u}h}{w_s}\frac{\partial \bar{c}}{\partial x}\right| << \left| a_{11}(\zeta)\,\bar{c}\right| \tag{6.10}$$

or

$$\left|\frac{a_{22}}{a_{11}}\cdot\frac{\bar{u}h}{\bar{c}\,w_s}\,\frac{\partial \bar{c}}{\partial x}\right| << 1 \tag{6.11}$$

The largest value of $\frac{a_{22}}{a_{11}}$ (figure 6.1) appears to occur at $\zeta = o$.

$$\therefore \left|\frac{\gamma_{22}}{\gamma_{11}}\,\frac{\bar{u}h}{\bar{c}\,w_s}\,\frac{\partial \bar{c}}{\partial x}\right| << 1 \tag{6.12}$$

Substituting from (6.8)

$$\left|\frac{\bar{c}_e - \bar{c}}{\bar{c}}\right| << 1 \tag{6.13}$$

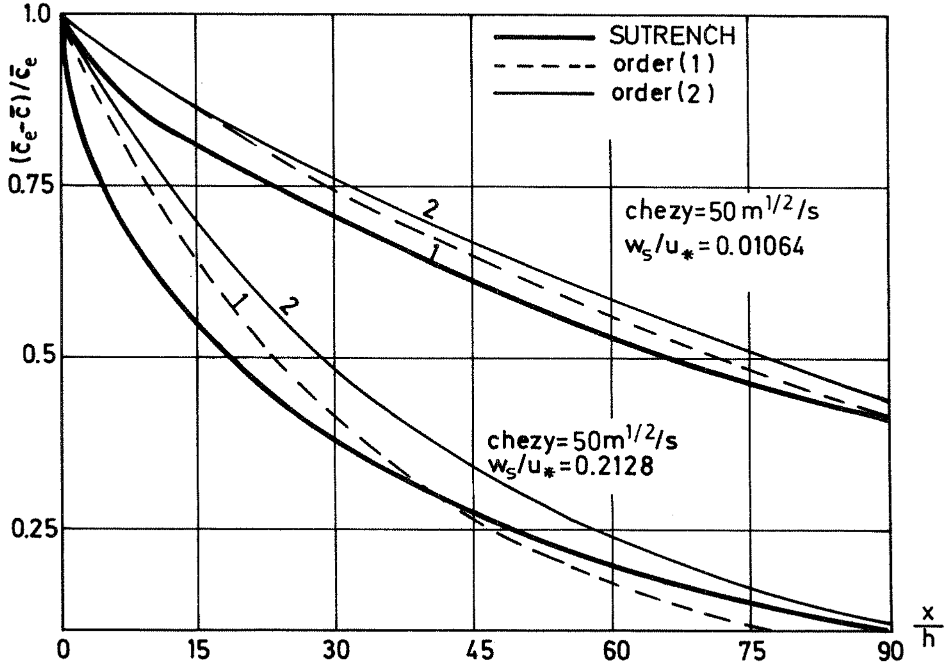


Fig. 6.2. Adaptation from zero concentration (uniform flow, z_a/a = 0.0125)

Therefore the error in the solution should increase as the local mean concentration moves away from the mean equilibrium concentration. The worst case is when $\bar{c} = o$.

6.4. Adaptation from zero concentration

Notwithstanding the conclusions of the previous section, the analysis was applied to a steady uniform flow where the initial mean concentration was zero. The solution could be obtained analytically because of the constant coefficients. Figure (6.2) shows the decay of $\frac{\bar{c}_e - \bar{c}}{\bar{c}_e}$ for two values of $\frac{w_s}{u_*}$. The first and second order analytical solutions as well as the numerical solution of the two dimensional mass balance equation (2.2) (Vermaas, 1982) are shown. It can be seen that while errors of adaptation rate are present when $\bar{c}$ is small, the comparison improves considerably as $\bar{c}$ increases.

It should be mentioned here that when the mean concentration is zero, the first order solution (6.5) will give negative values of concentration in the upper part of the flow! Agreement between the full numerical solution and the asymptotic solution (from the point of view of the adaptation rate) is quite reasonable for $(\bar{c}_e - c)/\bar{c}_e < 0.5$.

which is

$$\frac{\bar{c}_e - \bar{c}}{\bar{c}} < 1$$

6.5. Expressions for Transport Rate

In a steady non-uniform flow, the first order expression for the suspended sediment transport rate can be obtained from (5.31).

$$s_s = \alpha_{11}\ \bar{u}\,h\,\bar{c} + \gamma_{22}\ \frac{\bar{u}^2\bar{h}^2}{w_s}\ \frac{\partial\bar{c}}{\partial x} \tag{6.14}$$

Substituting for $\frac{\partial\bar{c}}{\partial x}$ from (6.8)

$$s_s = \alpha_{11}\ \bar{u}\,h\,\bar{c} + \alpha_{22}\ \bar{u}h\ \frac{\gamma_{11}}{\gamma_{22}}\ (\bar{c}_e - \bar{c}) \tag{6.15}$$

If $c = \bar{c} - \bar{c}_e$ and the equilibrium transport rate S_e is given by

$$s_e = \alpha_{11}\ \bar{u}\,h\,\bar{c}_e \tag{6.16}$$

$$s_s = s_e + \bar{u}\,h\,\Delta c\ (\alpha_{11} - \frac{\alpha_{22}\ \gamma_{11}}{\gamma_{22}}) \tag{6.17}$$

The deviation from equilibrium could be expressed in the dimensionless form

$$\frac{s_s - s_e}{s_e} = \frac{\Delta c}{\bar{c}_e}\ (1 - \frac{\alpha_{22}\ \gamma_{11}}{\alpha_{11}\ \gamma_{22}}) \tag{6.18}$$

The typical order of magnitude of $(1 - \frac{\alpha_{22}\ \gamma_{11}}{\alpha_{11}\ \gamma_{22}}) \sim 1.2$.
(6.18) gives an indication of the error present when fitting a transport formula to field measurements of suspended sediment which has not reached equilibrium. This point is of significance because the bed boundary condition in a suspended sediment calculation is often obtained from such a formula.

6.6. Expressions for the Entrainment Rate

The entrainment rate for steady flow may be derived from (6.15) as

$$E = \frac{\partial s_s}{\partial x} = \frac{\partial}{\partial x}\left[\gamma_{11}\,\bar{u}\,h\,\bar{c} + \gamma_{22}\ \frac{\bar{u}^2\bar{h}^2}{w_s}\ \frac{\partial\bar{c}}{\partial x}\right] \tag{6.19}$$

If the coefficients α and γ are assumed to be nearly constant and $\bar{u}h$ = constant it is possible to substitute from (6.8) and (6.17) to obtain

$$\frac{E}{w_s} = \frac{\gamma_{11}}{\gamma_{22}} \alpha_{11} \left(1 - \frac{\alpha_{22}}{\alpha_{11}} \frac{\gamma_{11}}{\gamma_{22}}\right)(\bar{c}_e - \bar{c}) + \alpha_{22} \frac{h\bar{u}}{w_s} \frac{\gamma_{11}}{\gamma_{22}} \frac{\partial \bar{c}_e}{\partial x} \tag{6.20}$$

If the equilibrium transport formula is

$$s_e = \alpha_{11}\ \bar{u}h\ \bar{c}_e = a\,\bar{u}^b \tag{6.21}$$

then

$$\frac{E}{w_s} = \frac{\gamma_{11}}{\gamma_{22}} \alpha_{11} \left(1 - \frac{\alpha_{22}}{\alpha_{11}} \cdot \frac{\gamma_{11}}{\gamma_{22}}\right)(\bar{c}_e - \bar{c}) + \alpha_{22} \frac{h}{w_s} \frac{\gamma_{11}}{\gamma_{22}}\, b\ \bar{c}_e \frac{\partial \bar{u}}{\partial x} \tag{6.22}$$

(6.22) shows that $E \propto (\bar{c}_e - c)$ is strictly valid only for uniform flow. However, the modified expression containing $\frac{\partial \bar{u}}{\partial x}$ could easily be included in the solution if a depth averaged entrainment equation of the type (2.6).

It is also possible to obtain the entrainment rate directly from the concentration profile as

$$E = -\left(w_s c + \varepsilon \frac{\partial c}{\partial z}\right)_{z\,=\,z_b\,+\,z_a} \tag{6.23}$$

which may also be written as

$$\frac{E}{w_s} = -\left(c + \varepsilon' \frac{\partial c}{\partial \zeta}\right)_{\zeta\,=\,o} \tag{6.24}$$

If c is represented by the steady first order solution

$$c = a_{11}(\zeta)\,\bar{c} + a_{22}(\zeta)\,\frac{\partial \bar{c}}{\partial \xi} \qquad (6.25)$$

it is possible to evaluate E from (6.24).

As $a_{11}(\zeta) = \phi_o(\zeta)$ and

$$\phi_o + \varepsilon' \frac{\partial \phi_o}{\partial \zeta} = o \quad \text{for all} \quad \zeta, \qquad (6.26)$$

only $a_{22}(\zeta)$ will contribute to entrainment.

$a_{22}(\zeta)$ satisfies (5.5) which means that (see section 4.5)

$$\frac{\partial a_{22}}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(\varepsilon' \frac{\partial a_{22}}{\partial \zeta}\right) = p\,\phi_o \qquad (6.27)$$

where

$$\left[a_{22} + \varepsilon' \frac{\partial a_{22}}{\partial \zeta}\right]_{\zeta = 1} = o \qquad (6.28)$$

By integrating (6.27) between the limits $\zeta = o$ and 1,

$$\left[a_{22} + \varepsilon' \frac{\partial a_{22}}{\partial \zeta}\right]_{\zeta = o} = -\,\alpha_{11} \qquad (6.29)$$

(see definition of α_{11} - (5.28))

From (6.24), (6.25) and (6.26)

$$\frac{E}{w_s} = -\frac{\partial \bar{c}}{\partial \xi}\left(a_{22} + \varepsilon' \frac{\partial a_{22}}{\partial \zeta}\right)_{\zeta = o} \qquad (6.30)$$

i.e.

$$\frac{E}{w_s} = \alpha_{11} \frac{\bar{u}h}{w_s} \frac{\partial \bar{c}}{\partial x} \tag{6.31}$$

Substituting from (6.8)

$$\frac{E}{w_s} = \alpha_{11} \frac{\gamma_{11}}{\gamma_{22}} (\bar{c}_e - \bar{c}) \tag{6.32}$$

Now it is apparent that (6.32) does not agree with (6.19). This discrepancy may be explained in the following way.

In the first order solution $c = c_o + c_1$, $c_o \sim O(1)$ and $c_1 \sim O(\delta)$. c_o does not produce any entrainment and therefore the expression (6.32) $\sim O(\delta)$.

In the derivation of (6.19) by differentiating the transport rate gives rise to two terms

$$\frac{E}{w_s} = \frac{E_o}{w_s} + \frac{E_1}{w_s} \tag{6.33}$$

where

$$\frac{E_o}{w_s} = \gamma_{11} \frac{11}{22} (\bar{c}_e - \bar{c}) = \gamma_{11} \frac{\bar{u}h}{w_s} \frac{\partial \bar{c}}{\partial x} \sim O(\delta)$$

and

$$\frac{E_1}{w_s} = \gamma_{22} \frac{\bar{u}h^2}{w_s^2} \frac{\partial \bar{c}}{\partial x} \sim O(\delta^2) \tag{6.35}$$

The expression for E_1 can also be obtained by starting with the <u>second order</u> concentration profile.

Thus (6.19) and (6.32) are not contradictory.

The absence of an O(1) term in the entrainment makes it desirable to include the $O(\delta^2)$ term to maintain the same precision (relatively) as the first order solution which has terms of O(1) and $O(\delta)$.

6.7. Depth-Averaged Solutions

There are two possible approaches to solving for the mean concentration. One approach is to solve the entrainment equation.

$$E = \frac{\partial(\bar{c}h)}{\partial t} + \frac{\partial}{\partial x}\left[\alpha_{11} h \bar{u} \bar{c} + \alpha_{21} \frac{h^2\bar{u}}{w_s} \frac{\partial \bar{c}}{\partial t} + \alpha_{22} \frac{h^2\bar{u}^2}{w_s} \frac{\partial \bar{c}}{\partial x}\right] \qquad (5.34)$$

or the equivalent steady flow equation

$$E = \frac{\partial}{\partial x}\left[\alpha_{11} h \bar{u} \bar{c} + \alpha_{22} \frac{h^2\bar{u}^2}{w_s} \frac{\partial \bar{c}}{\partial x}\right] \qquad (6.36)$$

by substituting an appropriate expression for E.
Expression of the type $E \propto (\bar{c}_e - \bar{c})$ are widely used. The use of such expressions seem to be justified in the light of (6.32), although the modification suggested in (6.22) would give a better approximation in non uniform flow.
In fact, if (6.32) is substituted into (6.36) while at the same time the diffusion term $\alpha_{22} \frac{h^2\bar{u}^2}{w_s} \frac{\partial \bar{c}}{\partial x}$ is neglected

$$w_s \, \alpha_{11} \frac{\gamma_{11}}{\gamma_{22}} (\bar{c}_e - \bar{c}) = \alpha_{11} \, \bar{u}h \frac{\partial \bar{c}}{\partial x} \qquad (6.37)$$

which reduces to the first order bed boundary condition

$$\gamma_{11} \, \bar{c}_e = \gamma_{11} \, \bar{c} + \gamma_{22} \frac{\partial \bar{c}}{\partial x} \qquad (6.7)$$

The solution of (6.7) to obtain the first order steady mean concentration is the alternative approach suggested in this report.

This reasoning leads to the following conclusion.

a) Solving the entrainment equation without the diffusion term and $E \propto (\bar{c}_e - \bar{c})$ will make it possible to obtain a solution accurate to $O(\delta)$ for both $\bar{c}$ and E. The equation to be solved is a first order ordinary differential equation in $\bar{c}$. Upstream boundary condition are sufficient.

c) If a more accurate expression (6.19) is used for the entrainment rate and the diffusion term is included, both $\bar{c}$ and E could be obtained to a higher degree of accuracy $(O(\delta^2))$. The equation to be solved is, however, a second order ordinary differential equation which requires a downstream boundary condition (for no obvious physical reason) because α_{22} is negative.

c) If the bed boundary condition (6.7) is solved $\bar{c}$ is obtained to an accuracy of $O(\delta)$. If this concentration is used to obtain the entrainment rate from (6.36) the answer will be of a higher order of accuracy then (a) and probably of the same accuracy as (b). The equation to be solved does not require a downstream boundary condition.

As the calculation of the entrainment rate is likely to be the most important objective of a suspended sediment transport calculation, the alternative (c) seems to have the most promise. It should however be noted that both approaches, i.e., the entrainment equation and the bed boundary condition, lead to the same answer asymptotically. It should also be noted that the bed boundary condition cannot be applied in the case of zero entrainment e.g. armoured beds, fixed beds etc.

6.8. Zero Entrainment

It is obvious that the bed boundary condition (5.16) cannot be applied where there is zero entrainment.
Consider the first order solution for the concentration profile

$$c = a_{11}(\zeta)\ \bar{c} + a_{22}(\zeta)\ \frac{\bar{u}h}{w_s}\frac{\partial\bar{c}}{\partial x} \qquad (6.7)$$

as $a_{11} = \phi_o$ cannot give rise to entrainment, the other term on the right hand side also should not give rise to any entrainment. The only possible solution is therefore trivial, i.e. $\bar{c}$ = constant.

If a non-trivial solution is required, the second order solution has to be used. Then it can be shown that

$$E = \frac{\partial}{\partial x}\left(\alpha_{11}\ \bar{u}\,h\,\bar{c} + \alpha_{22}\ \frac{\bar{u}^2h^2}{w_s}\frac{\partial\bar{c}}{\partial x}\right) = o \qquad (6.38)$$

Because α_{22} is negative the solution of (6.38) requires the application of a downstream boundary condition which so restricts the influence of the diffusion term that in uniform flow it gives rise to the same trivial solution $\bar{c}$ = constant. It should be noted that (6.38) can also be obtained by differentiating the first order expression for the transport rate.

6.9. An Alternative Bed Boundary Condition

The boundary condition described in section (5.2) assumes that the value of the concentration c_a at $z = z_a + z_b$ is known in advance. If this value is the same as the equilibrium value then, for example, the first order steady equation

$$\gamma_{11}\ \bar{c}_e = \gamma_{11}\ \bar{c} + \gamma_{22}\ \frac{\bar{u}h}{w_s}\frac{\partial\bar{c}}{\partial x} \qquad (6.7)$$

may be derived.

If an alternative boundary condition, i.e., that the concentration gradient at $z = z_a + z_b$ is equal to the equilibrium value, is used the first order steady solution would yield

$$\left(\frac{\partial a_{11}}{\partial \zeta}\right)_{\zeta = o} \bar{c}_e = \left(\frac{\partial a_{11}}{\partial \zeta}\right)_{\zeta = o} \bar{c} + \frac{\bar{u}h}{w_s} \left(\frac{\partial a_{22}}{\partial \zeta}\right)_{\zeta = o} \frac{\partial \bar{c}}{\partial x} \qquad (6.39)$$

From (4.25) and (5.5) it can be shown that

$$\left(\frac{\partial a_{11}}{\partial \zeta}\right)_{\zeta = o} = - \frac{\phi_o(o)}{\varepsilon'(o)} = - \frac{\gamma_{11}}{\varepsilon'(o)} \qquad (6.40)$$

and

$$\left(\frac{\partial a_{22}}{\partial \zeta}\right)_{\zeta = o} = - \frac{1}{\varepsilon'(o)} \int_o^1 p \, \phi_o \, d\zeta - \frac{a_{22}(o)}{\varepsilon'(o)} = - \frac{(\alpha_{11} + \gamma_{22})}{\varepsilon'(o)} \qquad (6.41)$$

Thus (6.39) will reduce to

$$\gamma_{11} \bar{c}_e = \gamma_{11} \bar{c} + (\gamma_{22} + \alpha_{11}) \frac{\bar{u}h}{w_s} \frac{\partial \bar{c}}{\partial x} \qquad (6.42)$$

Therefore, the use of this boundary condition will lead to larger adaptation lengths than before. For very fine sediment ($w_s \to o$), it can be shown that $\alpha_{11} \to 1$ while γ_{22}/w_s remains finite. So the adaptation length will become infinite.

7. THE COMPLETE FIRST ORDER QUASISTEADY SOLUTION

The analysis given in this chapter takes into account the vertical velocity and the changes of shape of both the velocity profile 'p' and the equilibrium concentration profile ϕ_o. For the sake of simplicity equations are derived only for quasisteady motion.

7.1. The Vertical Velocity Component

The vertical velocity must satisfy the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \qquad (7.1)$$

with the surface boundary condition

$$[w]_{\zeta = 1} = \frac{\partial z_s}{\partial t} + \bar{u}\, p\,(1)\, \frac{\partial z_s}{\partial x} \qquad (7.2)$$

where $z_s = z_a + z_b + h$ is the elevation of the water surface.

As the bed level changes slowly it may be assumed that

$$\frac{\partial z_s}{\partial t} = \frac{\partial h}{\partial t}\,(1 + \beta) \qquad (7.3)$$

where

$$\beta = \frac{z_a}{h}$$

The normalised velocity profile $p(\zeta)$ is not necessarily constant in shape. In some circumstances, such as fully rough flow, the shape of $p(\zeta)$ could be assumed to depend only on one parameter $f_* = \frac{\kappa \bar{u}}{u_*}$.

Then

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} + \frac{\partial p}{\partial f_*} \cdot \frac{\partial f_*}{\partial x} \tag{7.4}$$

The integration of (7.1) is given in Appendix B and leads to

$$w = \frac{\partial h}{\partial t} + P(\zeta)\,\frac{\partial(\bar{u}h)}{\partial x} - \bar{u}h\,p(\zeta)\,\frac{\partial \zeta}{\partial x} + \bar{u}h\,\frac{\partial P}{\partial f_*} \cdot \frac{\partial f_*}{\partial x} \tag{7.5}$$

where

$$P(\zeta) = \int_{\zeta}^{1} p(\zeta)\, d\zeta$$

If the velocity profile is logarithmic Appendix B also shows that

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(\bar{u}h) = A\,\frac{\partial}{\partial x}(\bar{u}h/f_*) \tag{7.6}$$

where

$$A = \beta \ln \left(\frac{\beta + 1}{\beta}\right) \tag{7.7}$$

and

$$\beta = \frac{z_a}{h} \quad \text{(a constant)} \tag{7.8}$$

$\bar{u}$ as defined by (4.2) is not the true depth averaged velocity. It can be shown (Appendix B) that the true depth averaged velocity $\bar{v}$ is given by

$$\bar{v} = (1 - A/f_*)\,\bar{u} \tag{7.9}$$

where

$$\bar{v} = \frac{1}{a} \int_{z_b}^{z_b + a} u\,dz$$

7.2. The Vertical Velocity in Quasisteady Flow

When $\frac{\partial h}{\partial t} = o$, (7.6) becomes

$$\frac{\partial}{\partial x} (\bar{u}h) = A \frac{\partial}{\partial x} (\frac{\bar{u}h}{f_*}) \tag{7.10}$$

which may also be written as

$$\bar{u}h\ (1 - A/f_*) = \text{constant} \tag{7.11}$$

It is shown in Appendix B that the vertical velocity is now gives by

$$w = r(\zeta)\ \frac{\bar{u}h}{f_*} \frac{\partial f_*}{\partial x} - p(\zeta)\ \bar{u}h\ \frac{\partial \zeta}{\partial x} \tag{7.12}$$

where (for logarithmic profiles)

$$r(\zeta) = -\ P(\zeta)/(1 - A/f_*) + (1 - \zeta) \tag{7.13}$$

The vertical velocity can also be expressed as

$$\frac{w}{w_s} = \frac{r(\zeta)}{f_*} \frac{\partial f_*}{\partial \xi} - p(\zeta) \frac{\partial \zeta}{\partial \xi} \tag{7.14}$$

7.3. The Variation of ϕ_o in Quasisteady flow

The equilibrium concentration profile ϕ_o can be shown (see Appendix B) to be of the general form

$$\phi_o = B \exp\ (Z\ f(\zeta)) \tag{7.16}$$

where B is a constant obtained from the normalising condition and f is a function of ζ only. The suspension parameter Z is given by

$$Z = \frac{a\,w_s}{4\,\varepsilon_{max}} \tag{7.17}$$

It can be shown that the suspension parameter is a function of $\frac{w_s}{u_*}$ only and that B is a function of Z. The simplest expression for Z is (Rouse, 1937)

$$Z = \frac{w_s}{\kappa\,u_*}$$

However, if (4.27) and (4.28) are used, more complicated expressions will result. In general the variation of ϕ_o along the x-direction could be expressed as

$$\frac{\partial\phi_o}{\partial x} = \frac{\partial\phi_o}{\partial\zeta}\cdot\frac{\partial\zeta}{\partial x} + \frac{\partial\phi_o}{\partial Z}\cdot\frac{\partial Z}{\partial x} \tag{7.19}$$

as w_s is constant

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial(\frac{w_s}{u_*})}\cdot\frac{\partial(\frac{w_s}{u_*})}{\partial x} = -\frac{w_s}{u_*}\cdot\frac{\partial Z}{\partial(\frac{w_s}{u_*})}\cdot\frac{1}{u_*}\frac{\partial u_*}{\partial x} \tag{7.20}$$

Complete expressions for $f(\zeta)$, $\frac{\partial Z}{\partial(\frac{w_s}{u_*})}$ and $\frac{\partial\phi_o}{\partial Z}$ are given in Appendix B.

As f_* is defined by

$$f_* = \frac{\bar{u}\kappa}{u_*} \tag{7.21}$$

$$\frac{1}{f_*}\frac{\partial f_*}{\partial x} = \frac{1}{u}\frac{\partial\bar{u}}{\partial x} - \frac{1}{u_*}\frac{\partial u_*}{\partial x} \tag{7.22}$$

However, from (7.11)

$$\frac{1}{h}\frac{\partial h}{\partial x} + \frac{1}{\bar{u}}\frac{\partial \bar{u}}{\partial x} = -\frac{1}{(1 - A/f_*)} \cdot \frac{A}{f_*}\frac{1}{f_*}\frac{\partial f_*}{\partial x} \tag{7.23}$$

$$\therefore \frac{1}{u_*}\frac{\partial u_*}{\partial x} = -\frac{1}{h}\frac{\partial h}{\partial x} - \frac{1}{(1 - A/f_*)} \cdot \frac{1}{f_*}\frac{\partial f_*}{\partial x} \tag{7.24}$$

Substituting in (7.20) and (7.19)

$$\frac{\partial \phi_o}{\partial x} = \frac{\partial \phi_o}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} + \frac{\partial \phi_o}{\partial Z}\frac{\partial Z}{(\frac{w_s}{u_*})}\left[\frac{1}{h}\frac{\partial h}{\partial x} + \frac{1}{(1 - A/f_*)} \cdot \frac{1}{f_*}\frac{\partial f_*}{\partial x}\right]\frac{w_s}{u_*} \tag{7.25}$$

or

$$\frac{\partial \phi_o}{\partial x} = \frac{\partial \phi_o}{\partial \zeta} \cdot \frac{\partial \zeta}{\partial x} + g_2(\zeta)\left[\frac{1}{h}\frac{h}{x} + \frac{1}{(1 - A/f_*)}\frac{1}{f_*}\frac{\partial f_*}{\partial x}\right] \tag{7.26}$$

where

$$g_2(\zeta) = \frac{w_s}{u_*}\frac{\partial \phi_o}{\partial Z} \cdot \frac{\partial Z}{\partial(\frac{w_s}{u_*})} \tag{7.27}$$

7.4. The Quasisteady Solution

The first order solution is (from 4.11)

$$c = c_o + c_1 \tag{7.28}$$

where

$$D[c_1] = p\frac{\partial c_o}{\partial \xi} + \frac{w}{w_s}\frac{\partial c_o}{\partial \zeta} \tag{7.29}$$

and

$$c_o = \bar{c}\ \phi_o(\zeta) \tag{7.30}$$

Substituting in (7.30) and (7.26)

$$D[c_1] = p\ \phi_o \frac{\partial\bar{c}}{\partial\xi} + p\,\bar{c}\frac{\partial\phi_o}{\partial\xi} + \frac{w}{w_s}\frac{\partial\phi_o}{\partial\zeta}\bar{c} \tag{7.31}$$

Using (7.14) and (7.26)

$$D[c_1] = p\ \phi_o \frac{\partial\bar{c}}{\partial\xi} + \left\{p\ \frac{\partial\phi_o}{\partial\zeta}\cdot\frac{\partial\zeta}{\partial\xi} + p\,g_2\left[\frac{1}{h}\frac{\partial h}{\partial\xi} + \frac{1}{(1-A/f_*)}\frac{1}{f_*}\frac{\partial f_*}{\partial\zeta}\right]\right.$$

$$\left. + \frac{r}{f_*}\frac{\partial f_*}{d\xi}\frac{\partial\phi_o}{\partial\zeta} - p\ \frac{\partial\phi_o}{\phi\zeta}\frac{\partial\zeta}{\partial\xi}\right\}\bar{c} \tag{7.32}$$

$$= p\ \phi_o \frac{\partial\bar{c}}{\partial\xi} + \left\{p\,g_2\cdot\frac{1}{h}\frac{\partial h}{\partial\xi} + \left(\frac{p\,g_2}{(1-A/f_*)} + \frac{r\,\partial\phi_o}{\partial\zeta}\right)\frac{1}{f_*}\frac{\partial f_*}{\partial\xi}\right\}\bar{c} \tag{7.33}$$

Equation (7.33) may now be integrated with the boundary condition (4.23) and (4.24) to yield

$$c_1 = a_{22}(\zeta)\frac{\partial\bar{c}}{\partial\xi} + \left\{\frac{e_1(\zeta)}{h}\frac{\partial h}{\partial\xi} + \frac{e_2(\zeta)}{f_*}\frac{\partial f_*}{\partial\xi}\right\}\bar{c} \tag{7.34}$$

where

$$e_1(\zeta) = D^{-1}[p\,g_2] \tag{7.35}$$

$$e_2(\zeta) = D^{-1}\left[p\,g_2/(1 - A/f_*) + r\frac{\partial\phi_o}{\partial\zeta}\right] \tag{7.36}$$

The first order solution may now be written as

$$c = (a_{11} + \frac{e_1}{h}\frac{\partial h}{\partial \xi} + \frac{e_2}{f_*}\frac{\partial f_*}{\partial \xi})\,\bar{c} + a_{22}\,\frac{\partial \bar{c}}{\partial \xi} \tag{7.37}$$

The bottom boundary condition is now

$$c_a = (\gamma_{11} + \frac{\mu_1}{h}\frac{\partial h}{\partial \xi} + \frac{\mu_2}{f_*}\frac{\partial f_*}{\partial \xi})\,\bar{c} + \gamma_{22}\,\frac{\partial \bar{c}}{\partial \xi} \tag{7.38}$$

$$\text{when } \mu_1 = e_1(o) \text{ and } \mu_2 = e_2(o) \tag{7.39}$$

If $c_a = \gamma_{11}\,\bar{c}_e$ and reverting to the original coordinates

$$\gamma_{11}\,\bar{c}_e = (\gamma_{11} + \frac{\mu_1\bar{u}}{w_s}\frac{\partial h}{\partial x} + \mu_2\,\frac{\bar{u}h}{w_s f_*}\frac{\partial f_*}{\partial x})\,\bar{c} + \gamma_{22}\,\frac{\bar{u}h}{w_s}\frac{\partial \bar{c}}{\partial x} \tag{7.40}$$

It can be seen in figure 9.3 that the values μ_1 and μ_2 are rather small when compared with γ_{11}. The relative importance of these two additional terms ($\mu_1\,\frac{\bar{u}}{w_s}\frac{\partial h}{\partial x}$ and $\mu_2\,\frac{\bar{u}h}{w_s f_*}\frac{\partial f_*}{\partial x}$), which have appeared because of the variations in the concentration profile and the velocity profile in the x-direction, cannot be estimated because it depends entirely on $\frac{\bar{u}\partial h}{\partial x}$ and $\frac{\bar{u}h}{f_*}\frac{\partial f_*}{\partial x}$ which can vary from problem to problem. Both additional terms will become zero when the flow is truly uniform.

The suspended sediment transport rate could also be obtained as

$$s_s = \bar{u}h\left\{(\alpha_{11} + \lambda_1\,\frac{\bar{u}}{w_s}\frac{\partial h}{\partial x} + \lambda_2\,\frac{\bar{u}h}{w_s f_*}\frac{\partial f_*}{\partial x})\,\bar{c} + \alpha_{22}\,\frac{\bar{u}h}{w_s}\frac{\partial \bar{c}}{\partial x}\right\} \tag{7.41}$$

where

$$\lambda_1 = \int_o^1 e_1\,p\,d\zeta \tag{7.42}$$

and

$$\lambda_2 = \int_0^1 e_2 \, p \, d\zeta \tag{7.43}$$

The term $\frac{1}{f_*} \frac{\partial f_*}{\partial x}$ is related to the Chezy coefficient C by

$$\frac{1}{f_*} \frac{\partial f_*}{\partial x} = \frac{1}{C} \frac{\partial C}{\partial x} \tag{7.44}$$

Furthermore, if the actual roughness height is kept constant then

$$\frac{\partial f_*}{\partial x} = \frac{1}{h} \frac{\partial h}{\partial x} \tag{7.45}$$

Therefore, for constant roughness height, the boundary condition (7.40) becomes

$$\gamma_{11} \, \bar{c}_e = \left\{ \gamma_{11} + (\mu_1 + \mu_2/f_*) \frac{\bar{u}}{w_s} \frac{\partial h}{\partial x} \right\} \bar{c} + \gamma_{22} \frac{\bar{u}h}{w_s} \frac{\partial \bar{c}}{\partial x} \tag{7.46}$$

7.5. Discussion

The analysis given in this chapter takes into account the variation in the x-direction of the equilibrium profile ϕ_o as well as the component of vertical velocity that is generated by the changing shape of the velocity profile. The major contribution (usually) to the vertical velocity is due to the shape of the bed and the water surface. It can be seen that this particular term is cancelled out by the rate of change of ϕ_o induced by the transformed vertical coordinate. Therefore it could be said that this component of the vertical velocity is implicit in the coordinate transformation and also that this component is therefore included in the analysis of Chapter 5 also.

There are, however, some basic assumptions underlying this analysis that should be stated clearly.

a) The equilibrium profile ϕ_o and therefore all other concentration profiles $a_{21}(\zeta)$, $e_1(\zeta)$ etc., are based on an assumption that the turbulent diffusion coefficient is the same as that found in uniform flow. The suspension parameter $w_s/\kappa u_*$ relates the shape of the profiles directly to the bottom shear stress. This would obviously be wrong in rapidly changing flow situations.

b) In the analysis it is assumed that the shape of the velocity profile is governed by a single parameter $\kappa\bar{u}/u_*$. This would also not be true in rapidly changing flow.
However, a two or three parameter velocity profile could be accommodated quite easily. It has been found (DHL, 1980) that the results of computations based on a two dimensional model were not very sensitive to refinements in the description of the flow field.

8. CALCULATION OF BED LEVEL CHANGE

8.1. Basic considerations

Changes in bed level are the result of non-uniformity of sediment transport along the channel. In the case of suspended sediment transport the time rate of change of the amount of sediment held in suspension will also influence bed level change, but this effect is usually very small. Ihe interaction between bed level change and the flow must be taken into account in formulating a mathematical model. It has however been demonstrated (de Vries, 1965) that for bed load transport the celerity of small disturbances in the bed geometry is usually very much smaller than the two characteristic celerities of water motion and that it is possible therefore to uncouple the hydraulic computation from the bed level computation: Thus it is possible to carry out the hydraulic computation and the bed level adjustment alternately and to take large time steps where the flow conditions permit. These considerations are likely to apply to suspended sediment also.

It is necessary to analyse the equation governing bed level change due to suspended sediment to determine.

a) The order of magnitude of the celerity of bed disturbances and the stability of the equation for bed level calculations.

b) The possibility of instabilities being introduced by the numerical scheme and whether it is necessary to introduce a pseudoviscosity term (Vreugdenhil and de Vries, 1973) to ensure stability.

8.2. The Basic Equation

It is assumed that the calculation of concentration could be carried out on a quasi-steady basis. Then the full first order solution when applied to the bottom boundary condition is

$$\gamma_{11}\,\bar{c}_e = (\gamma_{11} + \mu_1 \frac{\bar{u}}{w_s}\frac{\partial h}{\partial x} + \mu_2 \frac{\bar{u}h}{w_s f_*}\frac{\partial f_*}{\partial x})\,\bar{c} + \gamma_{22}\frac{\bar{u}h}{w_s}\frac{\partial \bar{c}}{\partial x} \qquad (7.40)$$

$\bar{c}_e$ could be determined (in terms of the flow parameters) if a transport formula is available for the equilibrium sediment transport rate s_e. All the other coefficients of the first order differential equation (7.40) are also fully determined by the flow field. The equilibrium transport rate is given by

$$s_e = \alpha_{11}\,\bar{u}h\,\bar{c}_e \qquad (8.1)$$

If the roughness height is constant, then there is a relationship between $\frac{\partial h}{\partial x}$ and $\frac{\partial f_*}{\partial x}$ (see 7.45).

Then

$$\gamma_{11}\,\bar{c}_e = \left\{\gamma_{11} + (\mu_1 + \mu_2/f_*)\,\frac{\bar{u}}{w_s}\frac{\partial h}{\partial x}\right\}\bar{c} + \gamma_{22}\frac{\bar{u}h}{w_s}\frac{\partial \bar{c}}{\partial x} \qquad (7.46)$$

If the mean concentration $\bar{c}$ is known, then the suspended sediment transport rate may be calculated from

$$s_s = \bar{u}h\left\{(\alpha_{11} + \lambda_1 \frac{\bar{u}}{w_s}\frac{\partial h}{\partial x} + \lambda_2 \frac{\bar{u}h}{w_s f_*}\frac{\partial f_*}{\partial x})\,\bar{c} + \alpha_{22}\frac{\bar{u}h}{w_s}\frac{\partial \bar{c}}{\partial x}\right\} \qquad (7.41)$$

For constant roughness height

$$s_s = \bar{u}h\left\{(\alpha_{11} + [\lambda_1 + \lambda_2/f_*]\,\frac{\bar{u}}{w_s}\frac{\partial h}{\partial x})\,\bar{c} + \alpha_{22}\frac{\bar{u}h}{w_s}\frac{\partial \bar{c}}{\partial x}\right\} \qquad (8.2)$$

For a specific flow situation it is usually possible to construct transport formulae for the suspended sediment and for the bed load separately. Let these formulae be

$$s_e = a_s \bar{u}^{b_s} \tag{8.3}$$

and

$$s_b = a_b \bar{u}^{b_b} \tag{8.4}$$

where the constants a_s, b_s, a_b and b_b are determined from field measurements.

Then from (8.1)

$$\bar{c}_e = a_s \frac{\bar{u}^{b_s}}{\alpha_{11} \bar{u} h} \tag{8.5}$$

If the porosity of the bed is p_b and the storage term $\frac{\partial(\bar{c}h)}{\partial t}$ is negligible, the rate of change of the bed level z_b could be expressed as

$$\frac{\partial z_b}{\partial t} + \frac{1}{(1 - p_b)} \left[\frac{\partial s_s}{\partial x} + \frac{\partial s_b}{\partial x} \right] = o \tag{8.6}$$

8.3. The Stability of the Linearised Equations

Consider a uniform steady flow where the suspended sediment is in equilibrium. Let u_o, h_o and c_o be the mean velocity, depth and mean concentration respectively. Consider a small ripple on the bed so that

$$h = h_o + h' \tag{8.7}$$

where

$$h' = H \exp(\lambda t + i k x) \tag{8.8}$$

Let the perturbations in the mean velocity and concentration be u' and c' respectively so that

$$\left.\begin{aligned} \bar{u} &= u_o + u' \\ \bar{c} &= c_o + c' \end{aligned}\right\} \tag{8.9}$$

If the discharge remains constant

$$\bar{u}h = u_o h_o = q \text{ (constant)} \tag{8.10}$$

Let the total sediment transport rate be

$$s_t = s_o + s' \tag{8.11}$$

where s_o is the unperturbed total transport.

As h' is small (8.5) could be linearised as

$$\bar{c}_e = c_o\left(1 - \frac{b_s h'}{h_o}\right) \tag{8.12}$$

The governing equation (for constant roughness height) (7.46) could be linearised to become

$$\frac{\partial c'}{\partial x} + \beta c' = a_1 h' + a_2 \frac{\partial h'}{\partial x} \tag{8.13}$$

where

$$\beta = \frac{\gamma_{11}}{\gamma_{22}} \frac{w_s}{q} \tag{8.14}$$

$$a_1 = - \frac{b_s c_o w_s \gamma_{11}}{h_o q \gamma_{22}} \tag{8.15}$$

$$a_2 = - \frac{(\mu_1 + \mu_2/f_*)}{\gamma_{22}} \frac{u_o c_o}{q} \tag{8.16}$$

From (8.8) $\frac{\partial h'}{\partial x} = i\,k\,h'$ (8.17)

and

$$\frac{\partial h'}{\partial t} = \lambda\, h' \quad (8.18)$$

Substituting in (8.13)

$$c' = \frac{(a_1 + a_2\ ik)}{(\beta + ik)}\ h'$$

where the homogeneous solution of (8.13) is neglected because it would decay along the x-axis

If (8.2) and (8.4) are combined and linearised

$$\frac{s'}{q} = b_1 c' + b_2 \frac{\partial h'}{\partial x} + b_3 \frac{\partial c'}{\partial x} + b_4 h' \quad (8.20)$$

where

$$b_1 = \alpha_{11} \quad (8.21)$$

$$b_2 = (\lambda_1 + \lambda_2/f_*)\ \frac{u_o c_o}{w_s} \quad (8.22)$$

$$b_3 = \frac{q}{w_s}\ \alpha_{22} \quad (8.23)$$

and

$$b_4 = -\ \frac{b_b s_{bo}}{h_o q} \quad (8.24)$$

Then

$$\frac{\partial s'}{\partial x} = q\,i\,k \left[\frac{(b_1 + b_3\ ik)(a_1 + a_2\ ik)}{(\beta + ik)} + b_2\ ik + b_4\right] h' \quad (8.25)$$

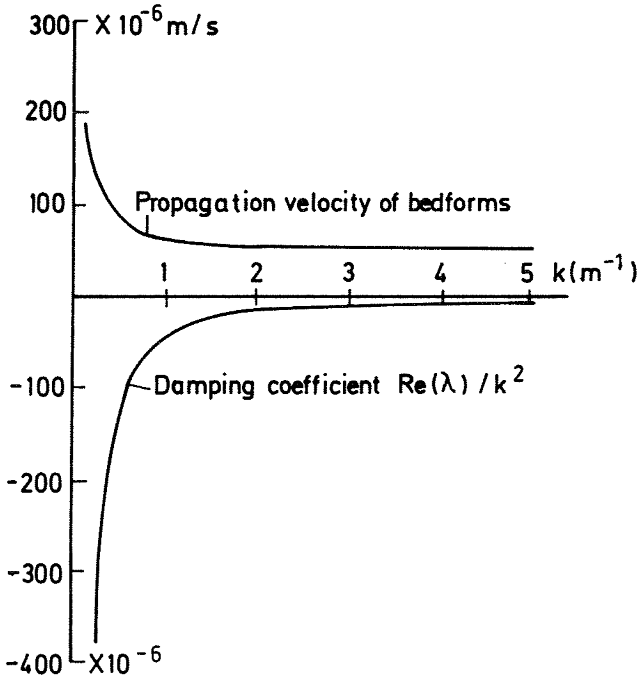


Fig. 8.1. Stability of linearised equations for bed level change (physical parameters as in section 9.6.).

Assuming that $\frac{\partial z_b}{\partial t} = - \frac{\partial h}{\partial t}$ (rigid lid assumption)

(8.6) becomes

$$(1 - p_b)\frac{\partial h}{\partial t} = \frac{\partial s'}{\partial x} \tag{8.26}$$

Substituting (8.18) and (8.25)

$$\lambda(1 - p_b) = q\,ik\left[\frac{(b_1 + b_3\,ik)(a_1 + a_2\,ik)}{(\beta + ik)} + b_2\,ik + b_4\right] \tag{8.27}$$

The propagation velocity is - Im(λ)/κ and the criterion for stability is

$$R_e(\lambda) < o \tag{8.29}$$

It is apparent that bed load introduces a positive propagation velocity which is independent of the wave-length of the bed disturbance and does not influence stability. The propagation velocity induced by suspended sediment transport is not independent of the wave-length of the disturbance.
The number of parameters involved in this analysis is very large. Therefore it is not possible to investigate the stability of the equations with respect to every parameter. It is however possible to investigate stability with respect to a specific problem and its associated parameters. This analysis was applied to the parameters of the computation of the siltation of a dredged trench described in sections 9.5 and 9.6. The results are shown in figure 8.1.

8.4. Numerical Stability

If $h' = h_j^n$, $c' = c_j^n$ etc.
at $x = j\,\Delta x$ and $t = n\,\Delta t$ the equation (8.13) and (8.20) may be written as difference relations.

$$\frac{c^n_{j+1} - c^n_{j-1}}{2\,\Delta x} + \beta c^n_j = a_1 h^n_j + a_2 \frac{h^n_{j+1} - h^n_{j-1}}{2\,\Delta x} \tag{8.30}$$

$$\frac{s^n_j}{q} = b_1 c^n_j + b_2 \frac{h^n_{j+1} - h^n_{j-1}}{2\Delta x} + b_3 \frac{c^n_{j+1} - c^n_{j-1}}{2\Delta x} + b_4 h^n_j \tag{8.31}$$

Assume that the bed disturbance is of the form

$$h^n_j = H \rho^n e^{ikx} \tag{8.32}$$

and that the other quantities are also of a similar form, e.g.,

$$c^n_j = C \rho^n e^{ikx} \quad \text{etc.} \tag{8.33}$$

From (8.30), (8.33) and (8.32)

$$C = \frac{a_1 + a_2 \, im}{\beta + im} \; H \tag{8.43}$$

where

$$m = \frac{\sin (k\Delta x)}{\Delta x} \tag{8.35}$$

Similarly (8.31) becomes

$$\frac{S}{q} = \left[\frac{(b_1 + b_3 \, im)(a_1 + a_2 \, im)}{(\beta + im)} + b_2 \, im + b_4\right] H \tag{8.36}$$

where

$$s^n_j = S \rho^n e^{ikx} \tag{8.37}$$

The calculation of the bed levels for the next step could be written as

$$(1 - p_b)\,\frac{h_j^{n+1} - h_j^n}{\Delta t} - \frac{s_{j+1}^n - s_{j-1}^n}{2\,\Delta x} - \frac{\alpha(1 - p_b)}{2\,\Delta t}(h_{j+1}^n - 2\,h_j^n + h_{j-1}^n) = o$$

(8.35)

where α is the pseudoviscosity which might have to be introduced to obtain stability. (8.38) could also be expressed as

$$(1 - p_b)(\rho - 1 - \alpha(\cos(k\,\Delta\,x))\,\frac{H}{\Delta t} - im\;S = o \tag{8.39}$$

or

$$\rho = 1 + \alpha(\cos(k\,\Delta\,x) - 1) + \frac{\Delta t}{(1 - p_b)}\;q\;im\left[\frac{(b_1 + b_3\,im)(a_1 + a_2\,im)}{(\beta + im)} + \right.$$

$$\left. + b_2\,im + b_4\right] \tag{8.40}$$

The condition for stability is

$$|\rho| \leqslant 1 \quad \text{for all} \quad k\,\Delta\,x \tag{8.41}$$

The relevant range is $o \leqslant k\,\Delta\,x \leqslant \pi$

If $k\,\Delta\,x = \pi$, (8.41) becomes

$$\alpha \leqslant 1 \tag{8.42}$$

The extent of additional damping could be obtained be comparing with the analytical solution in section 8.3. The relative damping factor for a single step is

$$d = \frac{|\rho|}{\exp(Re(\lambda\Delta t))} \tag{8.43}$$

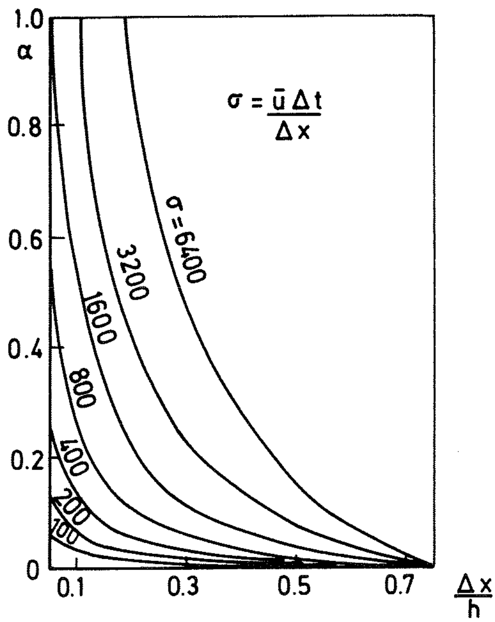


Fig. 8.2. Numerical stability: minimum values of α (parameters given in section 9.6).

The ratio of celerities is

$$c_r = \frac{\arg(\rho)}{\mathrm{Im}(\lambda\Delta t)} \qquad (8.44)$$

Fig. 8.2. shows the minimum values of α required to obtain stability for all k in the problem that is analysed in section 8.3 for various values of $\frac{\Delta x}{h}$ and the Courant number $\sigma = \frac{u\,\Delta t}{\Delta x}$.

This analysis offers a reasonably straightforward means of determining the main stability parameters for a specific bed level change calculation with both bed level and suspended load. Figure 8.2. shows the marked influence of the ratio of $\Delta x/h$ in determining how large a time step could be taken in the bed level computation large values of $\Delta x/h$ are usually present in river models. In calculating trench siltation problems however it becomes necessary to use a small value of Δx so that the trench geometry could be represented accurately. This will thus place a limitation on the size of time step that could be employed. In river models however the time step is more likely to be restricted by the necessity to represent the incoming hydrograph accurately.

9. COMPUTATIONS

9.1. General

The solution of the depth averaged equations that are developed in this report requires the prior knowledge of the coefficients γ_{ij}, α_{ij}, μ_i etc. as well as the mean equilibrium concentration $\bar{c}_e$. If a transport formula is available for the equilibrium transport rate it is possible to obtain $\bar{c}_e$ from the local hydraulic conditions (see 8.5). If the coefficients and $\bar{c}_e$ are known it is easy to set up a numerical scheme for the solution of the differential equation that governs the mean concentration $\bar{c}$. Analytical expressions for obtaining the coefficients have been given in the text. It can be seen that all the coefficients could be obtained if the following quantities are known.

a) z_a the level where the bottom boundary condition is applied. This is determined by the dimensionless quantity z_a/a or $z_a/h = \beta$.

b) The normalised velocity profile $p(\zeta)$. If the flow is assumed to be fully rough the shape of $p(\zeta)$ can be shown to depend on only the parameter $f_* = \frac{\kappa \bar{u}}{u_*}$ (see Appendix B) and β.

c) The normalized equilibrium concentration profile $\phi_o(\zeta)$. This profile can be shown (Appendix B) to depend on the parameters $\frac{w_s}{u_*}$ and β.

Thus all the coefficients could be shown to depend on 3 dimensionless parameters $\beta, \frac{\bar{u}}{u_*}$ and $\frac{w_s}{u_*}$.

It is possible, however, to reduce this to two parameters by assuming that in a given problem a constant value of β is used. The derivations of Appendix B assume that β is constant.

9.2. Computation of Profile Functions and Coefficients

Once $p(\zeta)$ and $\phi_o(\zeta)$ are known, the concentration profile function $a_{ij}(\zeta)$ and $e_i(\zeta)$ can be obtained by the application of the operator $D^{-1}[\;]$. It is shown that (see Appendix A) $D^{-1}[\;]$ involves two indefinite integrals and one definite integral in the dimensionless vertical coordinate ζ. Some care has to be taken in evaluating these integrals because of the rapid variation of the integrand near the bed.

The results presented in this report have been obtained from the following numerical procedure. The functions were evaluated on a 201 point grid from $\zeta = o$ to $\zeta = 1$. All integrations were performed twice; once over a fine grid $\Delta\zeta = 0.005$ and once over a coarse grid $\Delta\zeta = 0.01$, using Simpson's rule. The error in the integral over the coarse grid should be 16 times as large as the error on the fine grid. Thus an estimate of the error could be made at alternate grid points. The error estimate at the other grid points were obtained by linear interpolation and then the integral was corrected using these error estimates.

For a given value of z_a/a, the values of the coefficients were evaluated for a range of values of $\bar{u}/u_*$ and w_s/u_*. It was found that the values so obtained could be fitted by the following functions of $\frac{\bar{u}}{u_*}$ and w_s/u_*, for a fiven value of β.

$$\frac{\gamma_{21}}{\gamma_{11}} = \frac{w_s}{u_*} \exp\left[\sum_{i=1}^{4} a_i \left(\frac{w_s}{u_*}\right)^{i-1}\right] \qquad (9.1)$$

$$\frac{\gamma_{22}}{\gamma_{11}} = \frac{w_s}{u_*} \exp\left[\sum_{i=1}^{4} \left(b_i + \frac{u_*}{\bar{u}} c_i\right)\left(\frac{w_s}{u_*}\right)^{i-1}\right] \qquad (9.2)$$

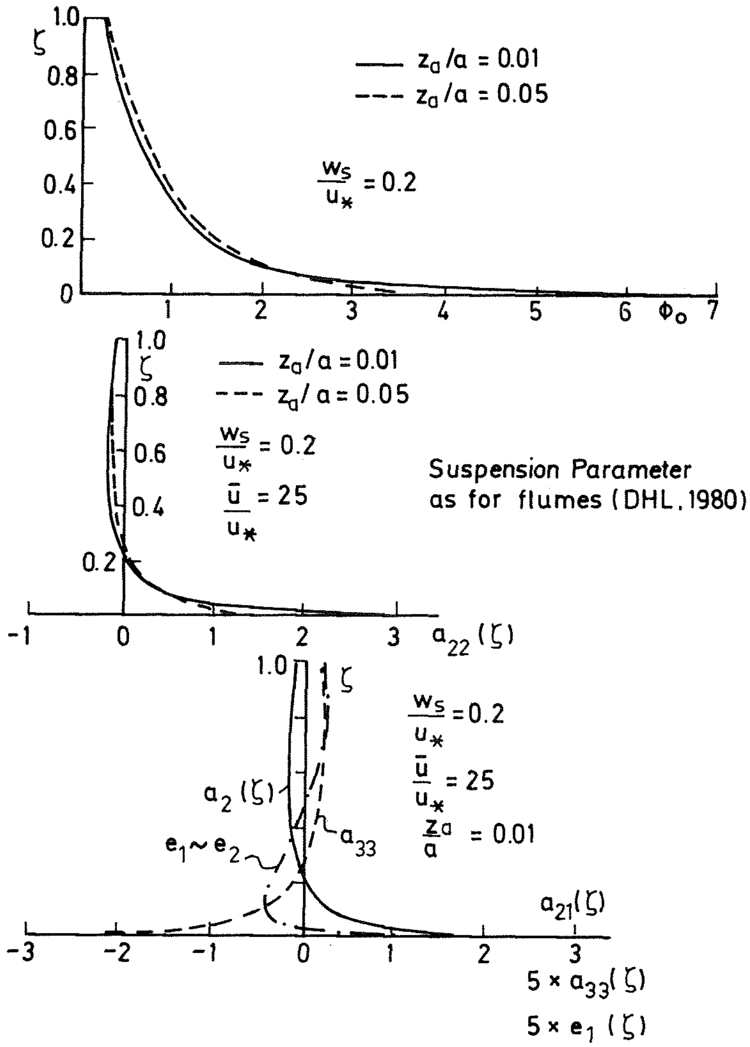


Fig. 9.1. Typical Profile functions

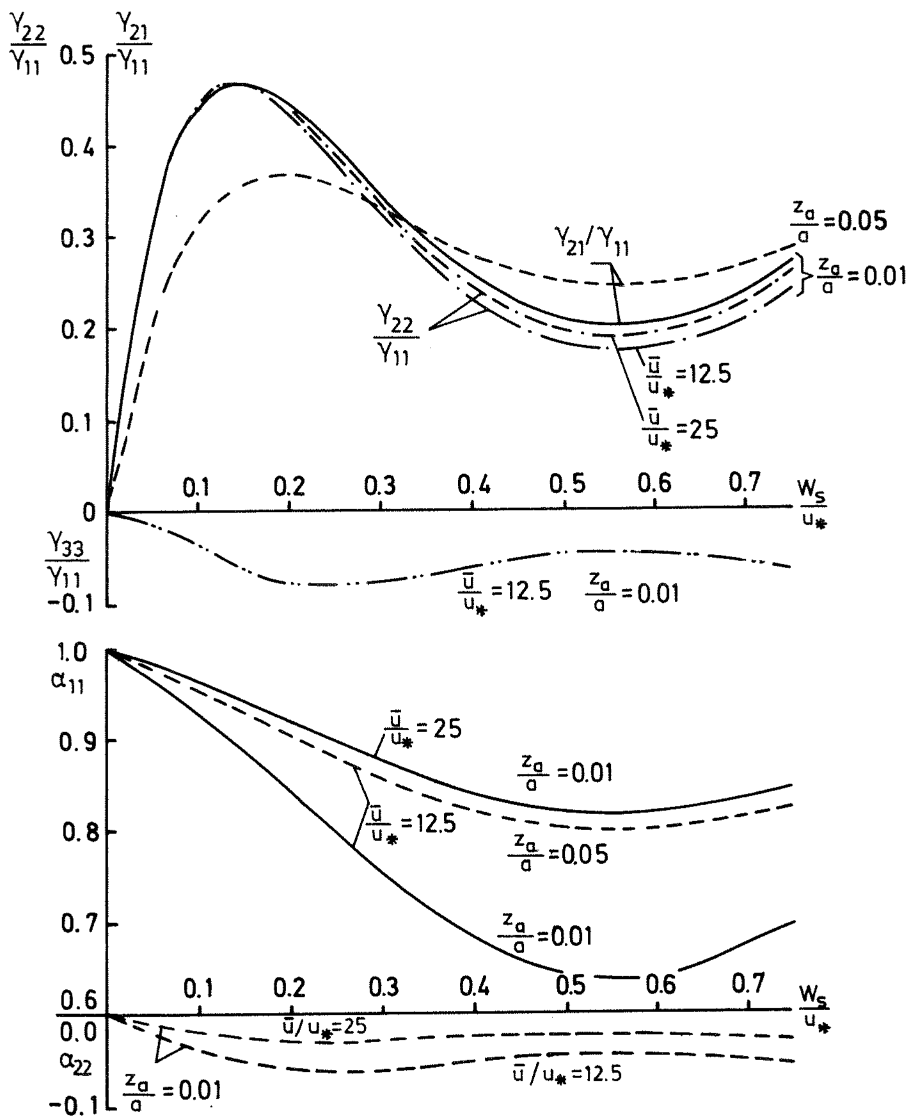


Fig. 9.2. γ and α coefficients
(for natural channels)

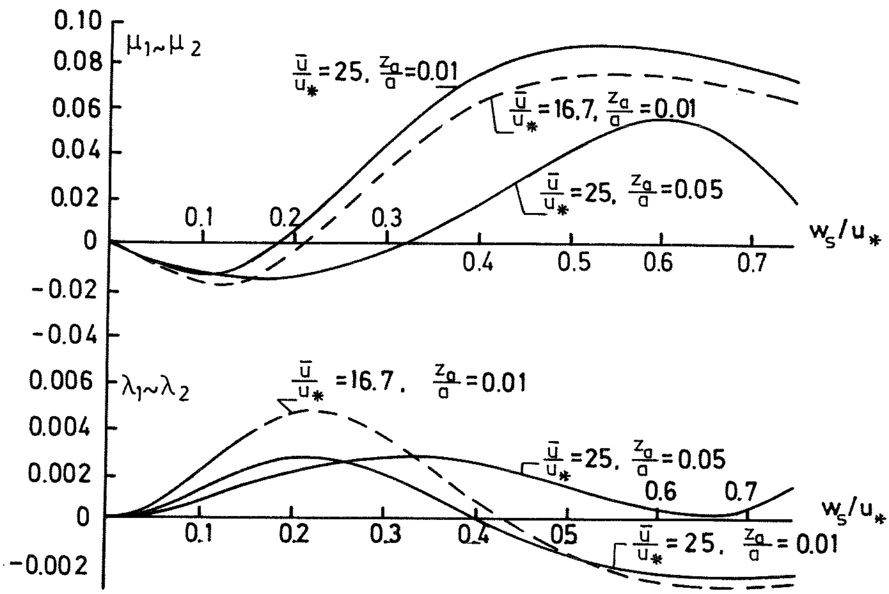


Fig. 9.3. <u>μ and λ coefficients</u>
(for natural channels)

$$\frac{\gamma_{33}}{\gamma_{11}} = \left(\frac{w_s}{u_*}\right)^2 \exp\left[\sum_{i=1}^{4} \left(d_i + \frac{u_*}{\bar{u}}\right)\left(\frac{w_s}{u_*}\right)^{i-1}\right] \tag{9.3}$$

$$\frac{\mu_1}{\gamma_{11}} = \left(\frac{w_s}{u_*}\right) \sum_{i=1}^{4} \left(f_i + \frac{u_*}{\bar{u}} g_i\right)\left(\frac{w_s}{u_*}\right)^{i-1} \tag{9.4}$$

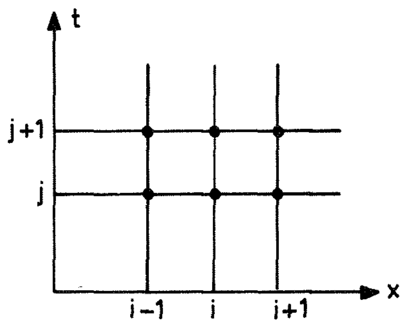
$$\alpha_{11} = \sum_{i=1}^{4} \left(h_i + \frac{u_*}{\bar{u}}\right)\left(\frac{w_s}{u_*}\right) \tag{9.5}$$

The values of $\frac{\mu_2}{\gamma_{11}}$, λ_1, λ_2, α_{21}, α_{22}, α_{33} can also be fitted with expression of the type (9.4). The constants a_i, b_i etc. can be fitted to obtain the coefficients to an accuracy of 2 percent for a range of $\frac{w_s}{u_*}$ = o to 0.75 and $\frac{u}{u_*}$ = 12 to 25.

The typical shapes of the profile functions $a_{ij}(\zeta)$ and $e_i(\zeta)$ used to build up the concentration profile are shown in figure 9.1. The values of the main coefficient are shown in figures 9.2 and 9.3. The coefficients required to construct the fitted function for γ_{ij}, μ_i etc. are given in Appendix C. Values of the adaptation length and time as defined in section 6.1 are shown in figure 9.4.

9.3. Unsteady Flow Calculation

If the effects of the changing shape of the velocity profile and the equilibrium profile are neglected, the basic equation that governs the first order variation of the mean concentration is (from chapter 5).



The six-point scheme

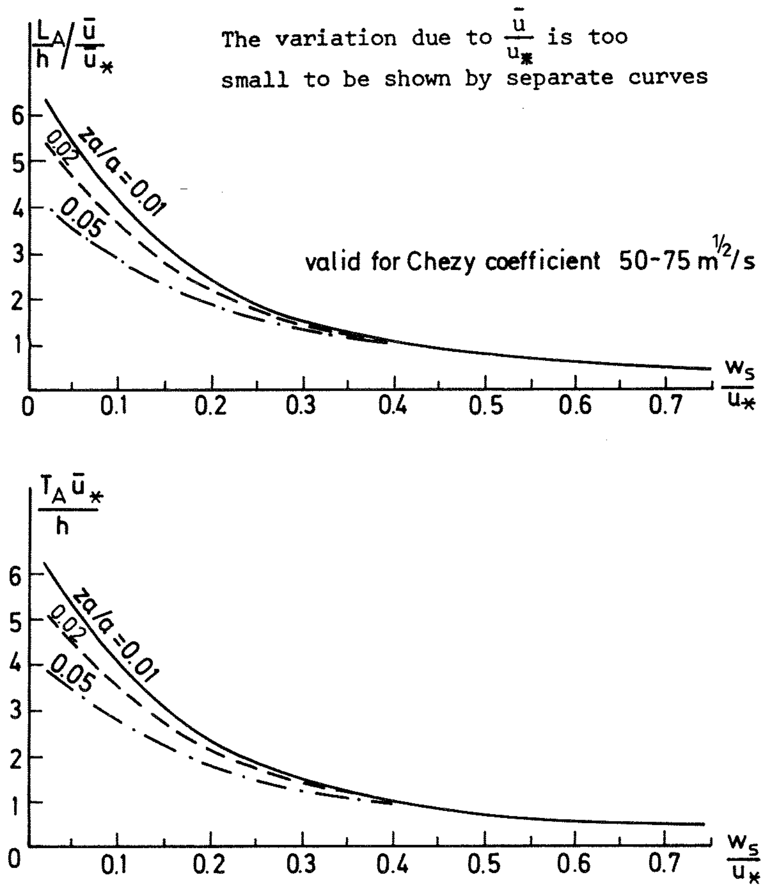


Fig. 9.4. Adaptation length and time
(for natural channels)

$$\gamma_{11}\,\bar{c}_e = \gamma_{11}\,\bar{c} + \gamma_{21}\,\frac{h}{w_s}\,\frac{\partial \bar{c}}{\partial t} + \gamma_{22}\,\frac{\bar{u}h}{w_s}\,\frac{\partial \bar{c}}{\partial x} \tag{5.21}$$

which may also be written as

$$\bar{c}_e = \bar{c} + L_A\,\frac{\partial \bar{c}}{\partial x} + T_A\,\frac{\partial \bar{c}}{\partial t} \tag{9.6}$$

where

$$L_A = \frac{\gamma_{22}}{\gamma_{11}}\,\frac{\bar{u}h}{w_s} \quad \text{and} \quad T_A = \frac{\gamma_{21}}{\gamma_{11}}\,\frac{h}{w_s} \tag{9.7}$$

The six-point scheme shown above is used for expressing this equations in finite-difference form.

$$\frac{\partial \bar{c}}{\partial t} = \frac{c_i^{j+1} - c_i^j}{\Delta t} \tag{9.8}$$

$$\frac{\partial \bar{c}}{\partial x} = \left\{(1 - \theta)(c_{i+1}^j - c_{i-1}^j) + \theta\,(c_{i+1}^{j+1} - c_{i+1}^{j+1})\right\}/2\Delta x \tag{9.9}$$

$$\left.\begin{aligned} \bar{c} &= (1 - \theta)\,c_i^j + \theta\,c_i^{j+1} \\ \bar{c}_e &= (1 - \theta)\,c_{e_i}^j + \theta\,c_{e_i}^{j+1} \\ T_A &= (1 - \theta)\,T_i^j + \theta\,T_i^{j+1} \\ L_A &= (1 - \theta)\,L_i^j + \theta\,L_i^{j+1} \end{aligned}\right\} \tag{9.10}$$

As $\bar{u}$, h, u_* etc are known beforehand, - $\bar{c}_e$, T_A, L_A could be calculated from (9.10) using the fitted relations to obtain $\frac{\gamma_{21}}{\gamma_{11}}$ and $\frac{\gamma_{22}}{\gamma_{11}}$ for each point (i, j).

Using (9.8), (9.9) and (9.10), (9.6) could be written as

$$a_i^j = b_{i-1}^j \ c_{i-1}^{j+1} + b_i^j \ . \ c_i^{j+1} \ c_{i+1}^{j+1} \qquad 2 \leqslant i \leqslant n-1 \tag{9.11}$$

where

$$b_{i-1}^j = - L_A\theta/2\Delta x, \ b_i^j = T_A/\Delta t + \theta, \ b_{i+1}^j = \theta L_A/2\Delta x \tag{9.12}$$

and

$$a_i^j = \bar{c}_e + \frac{T_A}{\Delta t} c_i^j - (1 - \theta) \ c_i^j - (1 - \theta) \ \frac{L_A}{2\Delta x} \ (c_{i+1}^j - c_{i-1}^j) \tag{9.13}$$

If the upstream and downstream boundaries are $i = o$ and $i = n$, the equations (9.11) will hold for $i = 2$ to $n - 1$. θ is the weighting factor which will normally be taken to be 0.55.

At the upstream boundary c_o^j is known as well as c_o^{j+1}. Therefore

$$a_1^j = b_1^j c_1^{j+1} + b_2^j c_2^{j+1} \tag{9.13}$$

where

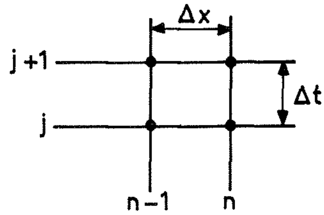
$$b_1^j = T_A/\Delta t + \theta, \ b_2^j = \theta \ L_A/2\Delta x \tag{9.14}$$

and

$$a_1^j = \bar{c}_e + \frac{T_A}{\Delta t} c_1^j - (1 - \theta) \ c_1^j - (1 - \theta) \ \frac{L_A}{2\Delta x} \ (c_2^j - c_o^j) + \frac{L_A}{2\Delta x} \ c_o^{j+1} \tag{9.15}$$

At the downstream boundary a four point scheme is required. Then

$$\frac{\partial c}{\partial t} = \frac{(c_n^{j+1} + c_{n-1}^{j+1}) - (c_n^i - c_{n-1}^j)}{2\Delta t} \tag{9.16}$$



$$\frac{\partial c}{\partial x} = \left\{(1 - \theta)(c_n^j - c_{n-1}^j) + \theta\,(c_n^{j+1} - c_{n-1}^{j+1})\right\}/\Delta x \tag{9.17}$$

$$\bar{c} = (1 - \theta)(c_n^j + c_{n-1}^j)/2 + \theta\,(c_n^{j+1} + c_{n-1}^{j+1})2 \tag{9.18}$$

$$\bar{c}_e = (1 - \theta)(c_{e_n}^j + c_{e_{n-1}}^j)/2 + \theta\,(c_{e_n}^{j+1} + c_{e_{n-1}}^{j+1})/2 \tag{9.19}$$

And expressions similar to (9.18) + (9.19) for T_A, L_A etc. Substituting in (9.6)

$$a_n^j = b_{n-1}^j\ c_{n-1}^{j+1} + b_n^j\ c_n^{j+1} \tag{9.20}$$

where b_{n-1}^j and b_n^j can be evaluated from the preceding equations.

Equations (9.11), (9.13) and (9.20) represent a set of n simultaneous equations for c_i^{j+1} from i = 1 to n. a_i^j is known if the concentrations c_i^j at the previous time step are known. The solution requires the inversion of a tri-diagonal matrix. The boundary conditions required are

a) c_o^j for all values of j (upstream condition)

and

b) c_i^o for all values of i (initial condition)

Once the mean concentration is obtained, the sediment transport rate can be calculated from equation (5.31), by putting

$$\frac{\partial \bar{c}}{\partial x} = (c_{i+1}^j - c_{i-1}^j)/2\Delta x \tag{9.21}$$

$$\frac{\partial \bar{c}}{\partial t} = (c_i^{j+1} - c_i^{j-1})/2\Delta t \tag{9.22}$$

and $\bar{c} = c_i^j$ to obtain s_i^j.

The entrainment rate is obtained from (5.33) by using

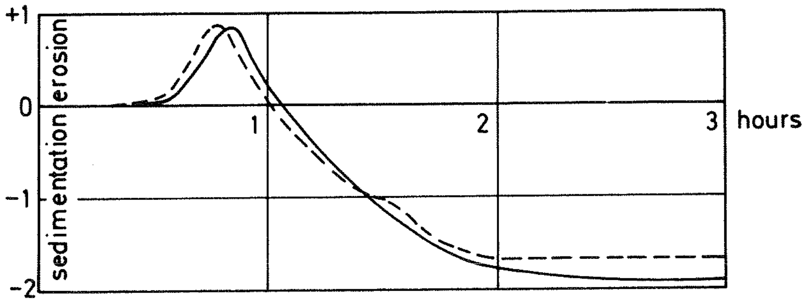
$$\frac{\partial(\bar{c}h)}{\partial t} = (c_i^{j+1}\ h_i^{j+1} - c_i^{j-1}\ h_i^{j-1})/2\Delta t \tag{9.23}$$

9.4. Suspended Sediment Transport due to a Flood Wave

The numerical scheme described above was applied to a flood wave down a 20 km stretch of river of uniform bed slope 5 x 10^{-5} and a Chezy coefficient of 50 $m^{\frac{1}{2}}/s$. A rather rapid flood was generated and computed using the ICES sybsystem FLOWS (Booij, 1980). The mean velocities and depths were calculated for all grid points at all time levels and stored on disc for use in the sediment transport computation as it progressed. A fifth power law was used yo compute the equilibrium rate for suspended sediment (i.e. $s_e \alpha \bar{u}^5$). The constant of proportionality was chosen arbitrarily because the purpose of the computation was to compare different types of solutions. Therefore no units are presented for the values of transport rated and mean concentration computed. A comparison is made of the results of the following computations:

a) An equilibrium calculation $\bar{c} = \bar{c}_e$
b) A quasisteady calculation (first order) $\frac{\partial \bar{c}}{\partial t} = o$
c) A full first order unsteady flow calculation

The initial condition was a uniform river with sediment in equilibrium all along it. The upstream boundary condition was found to extend someway down the river, especially when a zero upstream concentration was assumed. As the characteristic speed of propagation of $\bar{c}$ was $\frac{\bar{u}\ \gamma_{22}}{\gamma_{11}} \sim \bar{u}$, it was possible to take a long stretch of river and escape the effect of the upstream boundary condition.



Erosion

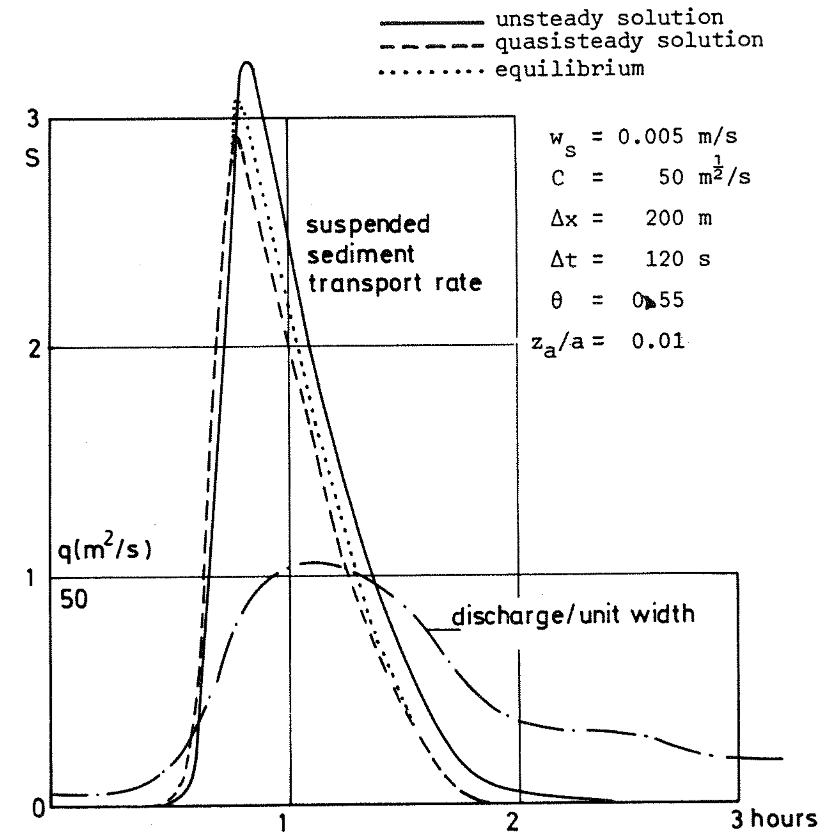


Fig. 9.5. Sediment transport by flood wave (8 km)

The upstream boundary condition used in the computation was that the suspended sediment was in equilibrium at $x = o$. The effect of adding a second order term (corresponding to second order Case B) was also examined and found to be not significant.

As $\frac{\partial \bar{c}}{\partial x}$ and $\frac{\partial \bar{c}}{\partial t}$ happened to be of the opposite sign it was found that the equilibrium solution (a) was closer to the 'complete' unsteady solution than the quasisteady solution (b). The difference in the total bed level change calculated by (b) and (c) was small and insignificant in the context of the errors inherent in morphological computations. Figure 9.5 shows the variation of mean velocity, depth, sediment transport rate and bed level with time at a station x = 8 km. Figure 9.6 does the same at x = 16 km. Figure 9.7 shows the relationship between the actual transport rate and the equilibrium transport rate.

In the case of tidal flow $\frac{\partial \bar{c}}{\partial x}$ and $\frac{\partial \bar{u}}{\partial t}$ are usually of the same sign. Thus it could be expected that $\frac{\partial \bar{c}}{\partial x}$ and $\frac{\partial \bar{c}}{\partial t}$ will also be of the same sign. Then these two terms will reinforce each other and the differences between the equilibrium solution and the unsteady solution should become much larger.

9.5. Bed Level Change Calculations (Quasisteady flow)

The first order equation governing the mean concentration, in a flow when the roughness height is assumed to remain constant, is given by (7.46) which may also be expressed as

$$\bar{c}_e = (1 + G_A \frac{\partial h}{\partial x})\, \bar{c} + L_A \frac{\partial \bar{c}}{\partial x} \tag{9.25}$$

where

$$G_A = (\mu_1 + \mu_2/f_*) \frac{\bar{u}}{\gamma_{11} w_s} \tag{9.26}$$

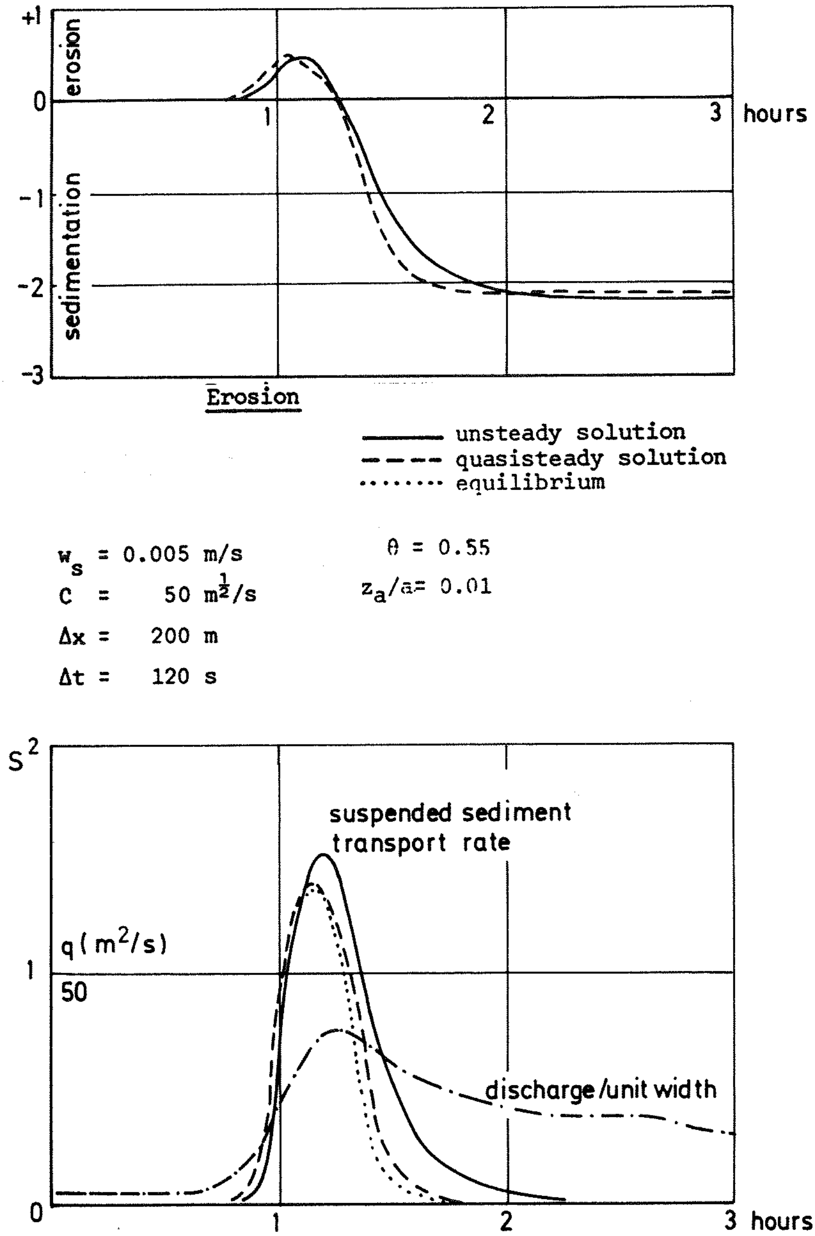


Fig. 9.6. Sediment transport by flood wave (16 km)

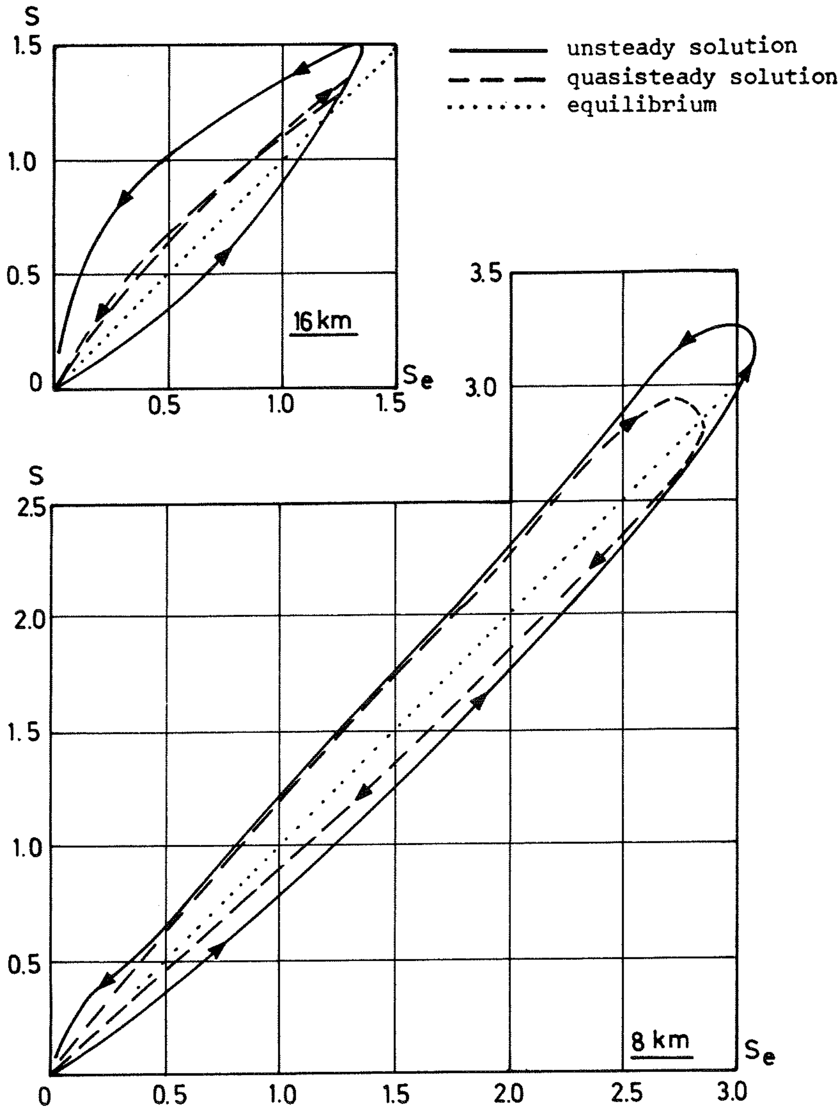


Fig. 9.7. S vs S_e during passage of flood

Equation (9.25) is solved at each time level independently. The following difference scheme was used to set up the simultaneous equations to solve for c_i^j from i = 1 to n.

$$\left.\begin{aligned} \frac{\partial \bar{c}}{\partial x} &= (c_{i+1}^j - c_{i-1}^j)/2\Delta x \\ \frac{\partial h}{\partial x} &= (h_{i+1}^j - h_{i-1}^j)/2\Delta x \end{aligned}\right\} \text{for } i = 1 \text{ to } n - 1 \tag{9.27}$$

$$\left.\begin{aligned} G_A &= G_{A_i}^j \\ L_A &= L_{A_i}^j \\ \bar{c}_e &= c_{e_i}^j \end{aligned}\right\} i = 1 \text{ to } n - 1 \tag{9.28}$$

Substituting (9.27) and (9.28) in (9.25) will yield n-1 equations in c_i^j for i = 1, n. The last equation is obtained by writing the differential equation for $x = (n-\frac{1}{2})\Delta x$.

Then

$$\frac{\partial \bar{c}}{\partial x} = (c_n^j - c_{n-1}^j)/\Delta x \tag{9.29}$$

$$\frac{\partial h}{\partial x} = (h_n^j - h_{n-1}^j)/\Delta x \tag{9.30}$$

$$G_A = (G_{A_n}^j + G_{A_{n-1}}^j)/2 \qquad \text{etc.} \tag{9.31}$$

Therefore it is possible to express (9.25) as n simultaneous equations for c_i^j for i = 1 to n. c_o^j is the known boundary condition. The coefficients of the equations depend on the known values h_i^j, u_i^j, $c_{e_i}^j$, $L_{A_i}^j$ and $G_{A_i}^j$.

Once c_i^j is known the sediment transport rate in calculated from (8.2) and (8.4).

$$S_T = a_b \bar{u}^{b_b} + \bar{u}h \left\{ \left[\alpha_{11} + (\lambda_1 + \lambda_2/f_*) \frac{\bar{u}}{w_s} \frac{\partial h}{\partial x} \right] \bar{c} + \alpha_{22} \frac{\partial h}{w_s} \frac{\partial \bar{c}}{\partial x} \right\} \tag{9.32}$$

by expressing it in finite difference form.

The new bed level is calculated from

$$z_{b_i}^{j+1} = z_{b_i}^j - \frac{1}{(1 - p_b)} \frac{(s_{i+1}^j - s_{i-1}^j)}{2\Delta x} t + 0.5\alpha (z_{b_{i+1}}^j - 2 z_{b_{i-1}}^j) \tag{9.33}$$

The term $0.5\,\alpha\,(Z_{b_{i+1}}^j - 2\,Z_{b_i}^j + Z_{b_{i-1}}^j)$ is an artificially introduced "pseudoviscosity' term. The smallest possible value of α compatible with stability is used. Once the new bed profile is known it is possible to compute the new flow yield etc, using an appropriate procedure. Hence u_i^{j+1}, h_2^{j+1} etc may be obtained.

9.6. The Siltation of the Dredged Trench

In order to test the effectiveness of the depth averaged approach, the numerical cheme described above was applied to a siltation problem for which there was already experimental data as well as a numerical model that solved the full convection-diffusion equation in two dimension. The problem selected was the siltation of a dredged trench with mild (1:10) side slopes in a laboratory flume reported in DHL (1980). The numerical solution by the SUTRENCH model is described in detail in DHL (1980). Only the details relevant for a proper comparison are given below.

Incoming bed load	s_{bo} = 0.010 kg/sm
Incoming suspended load	s_{eo} = 0.030 kg/sm
Upstream flow velocity	$\bar{v}_o$ = 0,51 m/s
Upstream flow depth	a_o = 0.39 m
Maximum diffusion coefficient	ε_{max} = 0.00165 m^2/s
Particle fall velocity	w_s = 0.013 m/s
Equivalent sand roughness	κ_s = 0.025 m
Porosity of Bed sediment	p_b = 0.4
Density of sediment	ρ_s = 2650 kg/m^3
Density of water	ρ = 1000 kg/m^3
Boundary level above bed	z_a = 0.0125 m
Longitudinal space interval	Δx = 0.25
Time step	Δt = 900 s
Vertical grid interval	Δz (variable) 10 steps

The bed boundary condition

$$c_a = \gamma_{11}\,\bar{c}_e \quad \text{for} \quad \frac{\partial s}{\partial x} \geq o$$

$$\left(\frac{\partial c}{\partial z}\right)_{z = z_a + z_b} = o \quad \text{for} \quad \frac{\partial s}{\partial x} < o$$

The water surface was assumed to be horizontal (rigid lid). The diffusion-coefficient was assumed to be the parabolic-constant type which is also used in this report. The following transport formulae were used

$$s_b = a_b\,\bar{v}^{\,b_b}$$

where

b_b = 3 and s_b = 0.010 kg/sm when $\bar{v}$ = 0.5

and

$$s_e = a_s \bar{v}^{b_s}$$

when $b_s = 5.5$ and $s_e = 0.030$ kg/sm when $\bar{v} = 0.5$.

In the computation using the depth averaged model idential values were used with the following differences.

1) It was not possible to use $z_a = 0.0125$ m because it was necessary to use z_a/a = constant. Therefore an intermediate value of $z_a/a = 0.02667$ was used.
2) As the coefficients were calculated for fully rough flow, $k_s = 0.025$ was used with $z_o = k_s/30$ when calculating the velocity profiles (see Appendix B). The maximum diffusion coefficient was not varied independently.
3) A single boundary condition
$c_a = \gamma_{11} \bar{c}_e$ was used
4) The proportion between bed load and suspended load was varied (while keeping the total load constant) in order to fit the propagation velocity of the trench to the measured values. This was the only calibration employed. The final values used were

$s_b = 0.018$ kg/sm
$s_e = 0.022$ kg/sm

Three variants of the depth averaged solution were applied.

1) The full quasisteady solution where u_* was determined by keeping k_s = constant along the flume.
2) The full solution while keeping $\kappa \frac{\bar{u}}{u_*} = f_*$ constant along the flume (equal to the upstream value).
3) A restricted solution where $\mu_1 = \mu_2 = \lambda_1 = \lambda_2 = o$, which corresponds to the solution discussed in chapter 5.

The value of α used in all computation was 0.05, with a time step of 450 seconds.

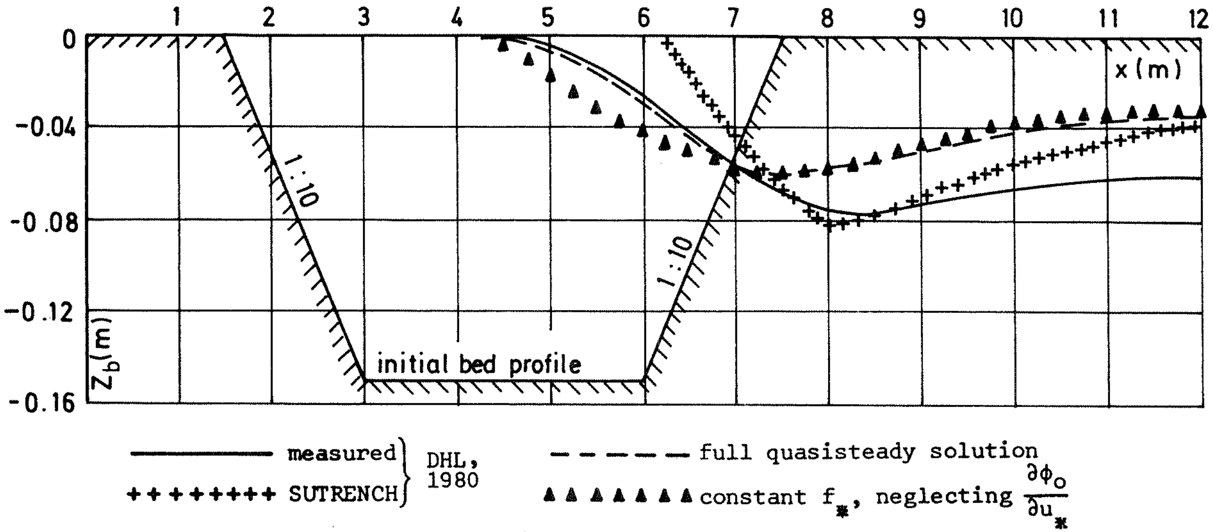


Fig. 9.9. Siltation of a trench: bed profile at 15 hours

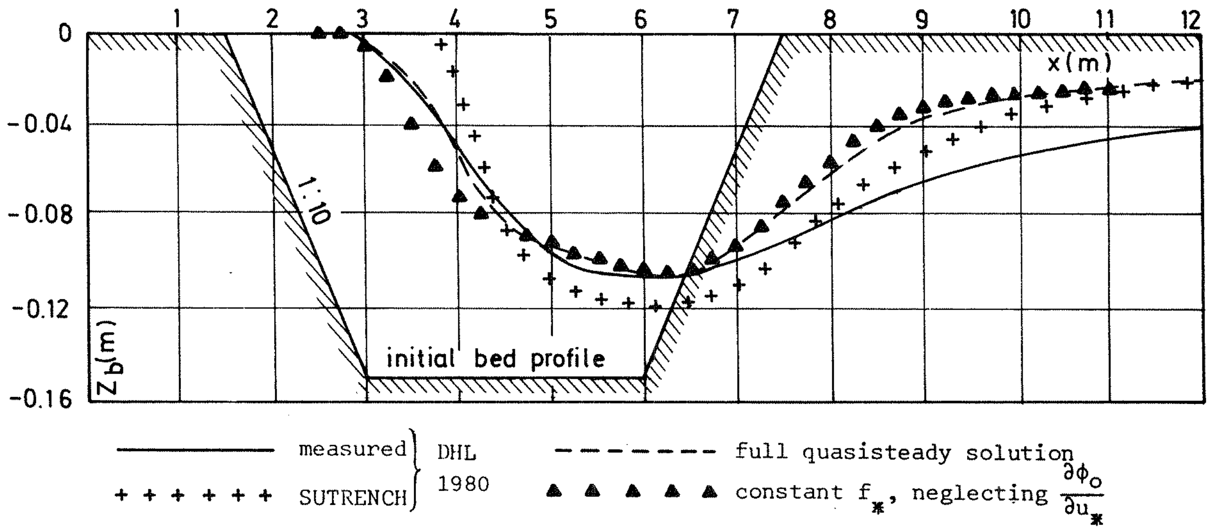


Fig. 9.8. Siltation of a trench: bed profile at 7.5 hours

The difference between (1) and (2) were found to be extremely small. Therefore the results of (1) and (3) with the results of the SUTRENCH calculation and the measurements from DHL (1980). Figures (9.8) and (9.9) show the bed profiles after 7.5 hours and 15 hours respectively. Figure 9.10 shows the total sediment transport at t = o over the undisturbed trench.

The results show very good agreement between experiment and the depth averaged model on the upstream side of the trench. The agreement is not so good further downstream where the model has underestimated the erosion as is the case to a lesser extent with SUTRENCH.

The differences between the depth averaged model and SUTRENCH could also be caused by the different boundary conditions and the differences in apportioning bed load and suspended load. This could be thought of as two different methods of calibration.

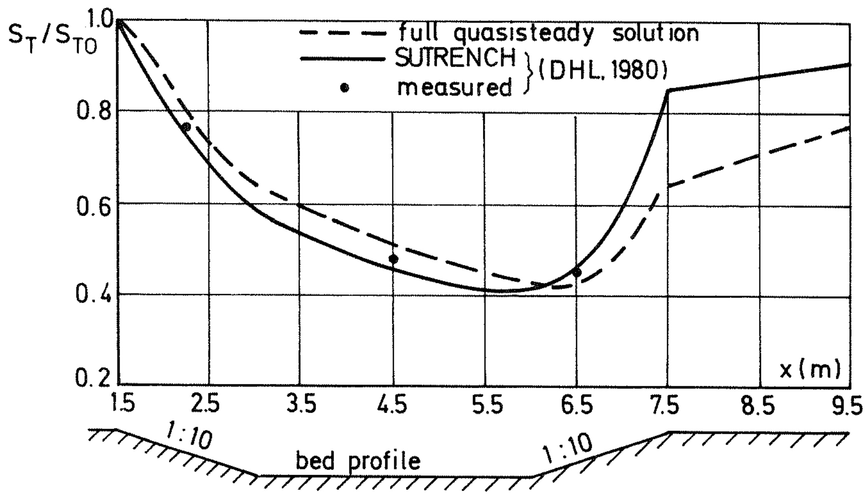


Fig. 9.10. Total sediment transport at t = o

10. CONCLUSION

10.1. Summary

A general asymptotic solution has been found for the two-dimensional convection-diffusion equation the transport of suspended sediment in an open channel. The concentration is expressed in terms of a series of previously determined profile functions multiplied by the mean concentration and its derivatives in time and space. The profile functions are based on known families of equilirbium concentration profiles and velocity profiles.

The asymptotic solution is based on a small parameter UH/Lw_s. Although it is not possible to conclude how large this parameter may be allowed to become, it has been shown that the solution is valid when the deviation of the mean concentration from the equilibrium mean value is not too large (say 50%).

The application of the bottom boundary condition gives rise to a partial differential eqguation which for the first order of approximation may be expressed as

$$\bar{c}_e = \bar{c} + T_A \frac{\partial \bar{c}}{\partial t} + L_A \frac{\partial \bar{c}}{\partial x} \qquad (10.1)$$

Expressions have been derived for the adaptation time T_A and the adaptation length L_A in terms of the local hydraulic and sediment parameters. The mean equilibrium concentration $\bar{c}_e$ is obtained from a suitable transport formula Equation (10.1) however does not take into account the effects of rapid changes of velocity profiles or equilibrium concentration profiles. Higher order approximations will yield equations with higher derivations of $\bar{c}$. In steady or quasisteady flow

$$\bar{c}_e = \bar{c} + L_A \frac{\partial \bar{c}}{\partial x} \tag{10.2}$$

Expressions have also been obtained for sediment transport rate and entrainment rate, again based on $\bar{c}$ and its derivatives and previously derived coefficients. The first order expression for entrainment in quasisteady flow is found to correspond directly to the depth averaged convection diffusion equation where the effect of convection and diffusion due to the sheared flow is represented by a virtual diffusion coefficient. It should be noted that the expression for entrainment is only used to evaluate entrainment after $\bar{c}$ has been obtained from (10.1) or (10.2). It has also been shown that the assumption of entrainment being proportional to $(\bar{c}_e - \bar{c})$ is only strictly true in uniform flow.
It has been concluded that when concentrations are obtained by solving the entrainment equation, the assumed expression for entrainment plays a much larger role in the solution than the virtual diffusion coefficient.

The application of the unsteady first order equation to a severe flood wave gave an indication that adaptation processes did not play an important part in the final morphological effect of the flood.

The solution was extended to obtain the full solution for quasisteady flow. This solution took into account all changes in velocity profiles and equilibrium concentration profiles. The resulting differential equation was of the type

$$\bar{c}_e = \bar{c}\,(1 + H_A \frac{\partial h}{\partial x} + F_A \frac{\partial f_*}{\partial x}) + L_A \frac{\partial \bar{c}}{\partial x} \tag{10.3}$$

which is essentially a modified version of (10.2) with similar properties. H_A and F_A are also determined from the flow and sediment parameters.

The stability of this equation was investigated when coupled with the equation for bed level change. A procedure has been devised to determine in advance the value of pseudoviscosity required to eliminate numerical instabilities in a specific bed level change calculation.

The full quasisteady solution was compared with an existing numerical solution of the full convection diffusion equation for the siltation of a dredged trench for which flume measurements were also available. Satisfactory agreement was obtained subject to calibration which was carried out by changing to proportion of suspended load to bed load in the incoming flow while keeping the total load constant. The additional terms in equation (10.3) were found to have an influence especially on the shape of the upstream slope of the trench.

10.2. Comments

It has been demonstrated that the asymptotic approach can, in some circumstances atleast, be used as a cheaper alternative to the full two dimensional model. This approach uses fewer empirical inputs than methods that use straightforward depth averaging in conjunction with an empirical entrainment function. The model describes how the mean concentration adapts in time and space towards the local mean equilibrium concentration.

The single computation carried out for a rather severe flood wave points to the possibility that no real advantage can be gained by applying an unsteady transport calculation for calculating bed level changes. As this effect was due to the fact that in this instance insteadiness tended to cancel out the effects of non-uniformity in the flow, no firm conclusion could be drawn until many more computations are made.

The advantages of using the unsteady solution should be greatest in the case of tidal flows where unsteadiness and non-uniformity could be expected to reinforce each other.

Although the model was able to give a satisfactory prediction of the siltation of a dredged trench in a laboratory flume it should be noted that for this application the basic assumption of small deviations from the mean equilibrium concentration would have been extended to the limit. The model could be expected to perform atleast as well when it is applied to larger scale problems. Two of the major advantages of the model are that it is able to predict adaptation lengths and times and that it makes it possible to estimate propagation velocities and numerical stability prior to its application.

The usefulness of the model must ultimately be proven in the field. It will be necessary to modify and extend the basic model described in this report before it could be applied to a real flow situation. These extensions and modifications will have to be carried out in the context of the specific problem to be solved. As the model is two-dimensional, its applicability in the field will be extremely limited. Thus it is essentual that the theory be extended to cover a three dimensional flow.

10.3. Extension to Three Dimensions

The three dimensional solution requires the inclusion of the following terms in the convection diffusion equation.

$$v\frac{\partial c}{\partial y}, \ \frac{\partial}{\partial y}\left(\varepsilon\frac{\partial c}{\partial y}\right) \ \text{and} \ \frac{\partial}{\partial x}\left(\varepsilon\frac{\partial c}{\partial x}\right)$$

If the length scale in the y-direction is B and the velocity scale for v is UB/L, it can be shown that the above terms could be included in the dimensionless equation (3.1) as

$$\frac{HU}{L\,w_s} v' \frac{\partial c}{\partial y} , \frac{E}{w_s H} \left(\frac{H}{B}\right)^2 \frac{\partial}{\partial y'} \left(E' \frac{\partial c}{\partial y}\right) \text{ and } \frac{E}{w_s H} \left(\frac{H}{L}\right)^2 \frac{\partial}{\partial x'} \left(E' \frac{\partial c}{\partial x'}\right)$$

respectively.

As it has been shown that $E/w_s H \sim O(1)$, it would be justified to assume that $(H/L)^2 \sim O(\delta^2)$ and ignore longitudinal diffusion.
It is unlikely that $(H/B)^2$ will be larger than $O(\delta)$. Therefore ir could be assumed that lateral diffusion is of the same order (or smaller) as the convection terms. These arguments lead to a first order asymptotic solution where

$$c = c_o + c_1 \tag{10.4}$$

where

$$w_s \frac{\partial c_o}{\partial z} + \frac{\partial}{\partial z} \left(\varepsilon \frac{\partial c_o}{\partial z}\right) = o \tag{10.5}$$

and

$$w_s \frac{\partial c_1}{\partial z} + \frac{\partial}{\partial z} \left(\varepsilon \frac{\partial c_1}{\partial z}\right) = \frac{\partial c_o}{\partial t} + u \frac{\partial c_o}{\partial x} + v \frac{\partial c_o}{\partial y} - \frac{\partial}{\partial y} \left(\varepsilon \frac{\partial c_o}{\partial y}\right) \tag{10.6}$$

It would be perfectly feasible to construct a solution for (10.5) and (10.6) along the lines used for the two-dimensional solution if it possible to describe v in terms of a family or families of profiles. On application of the bottom boundary condition the basic differential for $\bar{c}$ could be obtained.

References

Booij, N. (1980) Report on the ICES subsystem FLOWS. Report no. 78-3, Communications in Hydraulics, Dept. of Civil Eng., Delft University of Technology, Revised Edition.

Coleman, N.L. (1970) Flume studies of the Sediment Transfer Coefficient. Water Resources Research, 6, 3, 801-809.

Delft Hydraulics Laboratory (1978) Morphological Computations for Suspended Sediment Transport. Research Report S 78-VI.

Delft Hydraulics Laboratory (1980) Storm Surge Barrier Oosterschelde: Computation of Siltation in Dredged Trenches. Report R 1267-V/M 1570.

Daubert, A. (1974), La dispersion dans les écoulements filaires. (Dispersion in uni-directional flow), la Houille Blanche, 29, 47-54.

Elder, J.W. (1959 The Dispersion of Marked Fluid in Turbulent Shear Flow. J. Fluid Mech., 5, 544-560.

Kerssens, P.J.M. (1974) Adaptation Length for Suspended Sediment Verticals (in Dutch), Delft University of Technology, Dept. of Civil Eng., August 1974.

Rouse, H. (1937) Modern Conceptions of the Mechanics of Turbulence, Trans. ASCE, 102, 532-536.

Taylor, G.I. (1954) The Dispersion of Matter in Turbulent Flow through a Pipe. Proc. Roy. Soc. A, 223, 446-468.

Vermaas, H. (1982) Private Communication.

Vreugdenhil, C.B. (1982) Private Communication.

Vreugdenhil, C.B. and Vries, M. de (1967) Computation of Non-steady Bedload-transport by a Pseudoviscosity Method. IAHR, Fort Collins.

Vries, M. de (1965) Considerations about Non-steady Bed-load transport in Open channels. IAHR, Leningrad, paper 3.8, 3.8.1.-3.8.8.

Appendix A   The Operator $D^{-1}[\ ]$

The operator $D^{-1}[\ ]$ was defined as

$$D^{-1}[g(\zeta)] = f(\zeta) \tag{a1}$$

if

$$D[f(\zeta)] = \frac{\partial f}{\partial \zeta} + \frac{\partial}{\partial \zeta}\left(\varepsilon' \frac{\partial f}{\partial \zeta}\right) = g(\zeta) \tag{a2}$$

where

$$\left[f + \varepsilon' \frac{\partial f}{\partial \zeta}\right]_{\zeta = 1} = 0 \tag{a3}$$

and

$$\int_0^1 f \, d\zeta = 0 \tag{a4}$$

The equation (a2) could be integrated once directly, using the boundary condition (a3). Then

$$f + \varepsilon' \frac{\partial f}{\partial \zeta} = - \int_\zeta^1 g(\zeta) \, d\zeta = G(\zeta) \tag{a5}$$

which also implies that $\dfrac{\partial G}{\partial \zeta} = g(\zeta)$ (a6)

As ϕ_o satisfies the equation

$$\phi_o + \varepsilon' \frac{\partial \phi_o}{\partial \zeta} = 0 \tag{a7}$$

the solution of (a5) may be written in the form

$$f = A(\zeta)\ \phi_o(\zeta) \tag{a8}$$

Substituting (a8) in (a2) and using the property (a7), we obtain

$$\frac{\partial A}{\partial \zeta} = \frac{G(\zeta)}{\varepsilon' \phi_o}$$

However $\frac{\partial}{\partial \zeta}\left(\frac{1}{\phi_o}\right) = -\frac{1}{\phi_o{}^2}\frac{\partial \phi}{\partial \zeta}$ (a9)

Again substituting in (a7)

$$\frac{\partial}{\partial \zeta}\left(\frac{1}{\phi_o}\right) = \frac{1}{\varepsilon' \phi_o} \tag{a11}$$

Thus

$$A = \int_\zeta^1 G \frac{\partial}{\partial \zeta}\left(\frac{1}{\phi_o}\right) d\zeta + \text{constant} \tag{a12}$$

$$= \left[\frac{G}{\phi_o}\right]_\zeta^1 + \int_\zeta^1 \frac{\partial G}{\partial \zeta} \cdot \frac{1}{\phi_o}\, d\zeta + \text{a constant} \tag{a13}$$

Substituting from (a5) and (a6)

$$A = \frac{G(\zeta)}{\phi_o(\zeta)} + \int \frac{g(\zeta)}{\phi_o}\, d\zeta + B \tag{a14}$$

where B is a constant.

The solution (a8) may now be written as

$$f(\zeta) = -\int_\zeta^1 g(\zeta) d\zeta + \phi_o(\zeta) \int \frac{g(\zeta)}{\phi_o(\zeta)}\, d\zeta + B\ \phi_o \tag{a15}$$

B may now be obtained from

$$\int_{o}^{1} f(\zeta)d\zeta \quad = o \tag{a4}$$

Therefore

$$D^{-1}[g] = - \int_{\zeta}^{1} g \; d\zeta \; + \; \phi_{o} \int_{\zeta} \frac{g}{\phi_{o}} \; d\zeta \; + \; B \; \phi_{o} \tag{a16}$$

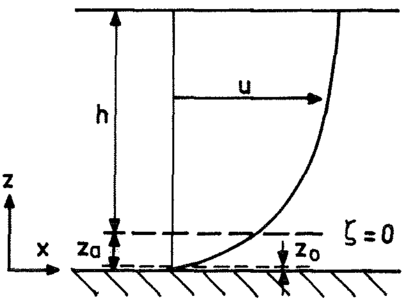
It should be noted that the solution does not require an explicit knowledge of ε'. $D^{-1}[g]$ can always be obtained if the equilibrium concentration profile (normalised) ϕ_o is known and the integration of $\int_{\zeta}^{1} \frac{g}{\phi_o} d\zeta$ does not cause difficulties at $\zeta = 1$.

In the Rouse (1937) profile $\phi_o(1) = o$. However, it turns out that the functions g that arise have the property that g/ϕ_o is finite at $\zeta = 1$. The modified profiles used in the computations in this report have the property $\phi_o(1) > o$.

Appendix B Velocity Profiles and the Equilibrium Concentration Profile

1) The Horizontal Velocity Profile

If the flow is fully rough the velocity profile may be expressed as



$$u = \frac{u_*}{\kappa} \ln \left(\frac{z}{z_o}\right) \qquad \text{(b1)}$$

By the definition (4.2)

$$\bar{u}h = \int_{z_a}^{z_a+h} u\,dz = \frac{u_*}{\kappa}\int_{z_a}^{z_a+h} (\ln z - \ln z_o)\,dz \qquad \text{(b2)}$$

$$\text{If } \beta = \frac{z_a}{h} \quad \text{(a constant)} \qquad \text{(b3)}$$

$$\bar{u}h = \frac{u_*}{\kappa} h \left\{\beta \ln \left(\frac{\beta+1}{\beta}\right) + \ln(\beta+1)h - \ln z_o - 1\right\} \qquad \text{(b4)}$$

$$\therefore\ u - \bar{u} = \frac{u_*}{\kappa}\left[\ln\left(\frac{\zeta+\beta}{\beta+1}\right) - \beta\ln\left(\frac{\beta+1}{\beta}\right) + 1\right] \qquad \text{(b5)}$$

As $p(\zeta) = \frac{u}{\bar{u}}$,

$$p(\zeta) = 1 + \frac{u_*}{\bar{u}\kappa}\left[\ln\left(\frac{\zeta+\beta}{1+\beta}\right) + \beta\ln\left(\frac{\beta}{\beta+1}\right) + 1\right] \qquad \text{(b6)}$$

$$\text{If } \frac{1}{f_*} = \frac{u_*}{\bar{u}\kappa}, \quad \text{then} \qquad \text{(b7)}$$

$$p = 1 + \frac{1}{f_*}\left[\ln\left(\frac{\zeta+\beta}{\beta+1}\right) - A + 1\right] \qquad \text{(b8)}$$

where

$$A = \beta \ln\left(\frac{\beta+1}{\beta}\right) \qquad \text{(b9)}$$

Therefore

$$\frac{\partial p}{\partial f_*} = - \frac{1}{f_*} (p - 1) \tag{b10}$$

and

$$\frac{\partial p}{\partial \zeta} = \frac{1}{f_* (\zeta + \beta)} \tag{b11}$$

2) <u>The Depth Averaged Continuity Equation</u>

The flow between the bed and the reference level z_a must be included in this equation. Let Δq be the flow below the reference level. Then

$$\Delta q = \frac{u_*}{\kappa} \int_{z_o}^{z_a} \ln \left(\frac{z}{z_o}\right) dz \tag{b12}$$

As z_o is usually very small,

$$\Delta q = \frac{u_* z_a}{\kappa} [\ln z_a - 1 - \ln z_o] \tag{b13}$$

From (b3) and (b4) it is possible to eliminate z_o from (b13)

$$\Delta q = \beta h \bar{u} \left[1 - \frac{1}{f_*} (\beta + 1) \ln \left(\frac{\beta + 1}{\beta}\right)\right]$$

Now the depth averaged continuity equation may be written

$$\frac{\partial}{\partial t} ((\beta + 1)h) + \frac{\partial}{\partial x} \left\{ \bar{u}h + \Delta q \right\} = o \tag{b15}$$

Substituting (b14)

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (\bar{u}h) = \beta \ln \left(\frac{\beta + 1}{\beta}\right) \frac{\partial}{\partial x} \bar{u}h/f_* \tag{b16}$$

$$= A \frac{\partial}{\partial x} (\bar{u}h/f_*) \tag{b17}$$

The total discharge is $\bar{u}h + \Delta q$.

The true mean velocity $\bar{v} = \dfrac{\bar{u}h + \Delta q}{(1+\beta)h}$ (b18)

which turns out to be $\bar{v} = (1 - \dfrac{A}{f_*})\bar{u}$ (b19)

In steady flow

$$\bar{u}h(1 + \frac{A}{f_*}) = \text{constant} \tag{b20}$$

3) <u>The Vertical Velocity Component</u>

The vertical velocity must satisfy

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

or

$$w - w_1 = \int_z^{z_s} \frac{\partial u}{\partial x}\, dz \tag{b22}$$

where w_1 is the vertical velocity at the surface and $z_s = z_b + z_a + h$ (see figure 4.1)

$$w_1 = \frac{\partial z_s}{\partial t} + \bar{u}p(1)\, \frac{\partial z_s}{\partial x} \tag{b23}$$

But, $$\int_z^{z_s} \frac{\partial u}{\partial x}\, dz = \frac{\partial}{\partial x} \int_z^{z_s} u\, dz - [u]_{z = z_s} \frac{\partial z_s}{\partial x} \tag{b24}$$

$$\therefore\ w = \frac{\partial z_s}{\partial t} + \frac{\partial}{\partial x} \int_z^{z_s} u\, dz \tag{b25}$$

$$= \frac{\partial z_s}{\partial t} + \frac{\partial}{\partial x} [\bar{u}h \; P(\zeta)] \qquad \text{(b26)}$$

where

$$P(\zeta) = \int_o^1 p(\zeta) d\zeta \qquad \text{(b27)}$$

or

$$p(\zeta) = - \frac{\partial P}{\partial \zeta} \qquad \text{(b28)}$$

Therefore (b26) becomes

$$w = \frac{\partial z_s}{\partial t} + P(\zeta)\frac{\partial (\bar{u}h)}{\partial x} - \bar{u}h \, p(\zeta) \frac{\partial \zeta}{\partial x} + \bar{u}h \frac{\partial P}{\partial f_*} \frac{\partial f_*}{\partial x} \qquad \text{(b28)}$$

As the bed level does not change quickly

$$w = (1+\beta) \frac{\partial h}{\partial t} + P(\zeta) \frac{\partial}{\partial x}(\bar{u}h) - \bar{u}h \, p(\zeta) \frac{\partial \zeta}{\partial x} + \bar{u}h \frac{\partial p}{\partial f_*} \cdot \frac{\partial f_*}{\partial x} \qquad \text{(b29)}$$

From (b17)

$$(1 + \beta)\frac{\partial h}{\partial t} + (1 - \frac{A}{f_*}) \frac{\partial}{\partial x}(\bar{u}h) + \bar{u}h \frac{A}{f_*^2} \frac{\partial f_*}{\partial x} = o \qquad \text{(b30)}$$

4) <u>The Quasisteady Vertical Velocity</u>

If $\frac{\partial h}{\partial t} = o$, then from (b30)

$$\frac{\partial}{\partial x} (\bar{u}h) = - \frac{\bar{u}h}{(1 - A/f_*)} \cdot \frac{A}{f_*} \cdot \frac{1}{f_*} \frac{\partial f_*}{\partial x} \qquad \text{(b31)}$$

Integrating (b6) and rearranging

$$P(\zeta) = \int_{\zeta}^{1} p \, d\zeta$$

$$= (1 - \zeta) + \frac{1}{f_*} [\beta(1 - \zeta) \ln (\frac{\beta}{\zeta + \beta}) + \zeta(1 + \beta) \ln (\frac{\beta + 1}{\zeta + \beta})] \tag{b32}$$

Then $\frac{\partial P}{\partial f_*} = \frac{(1 - \zeta) - P(\zeta)}{f_*}$ (b33)

Substituting (b31) and (b33) in (b29), the vertical velocity becomes

$$w = r(\zeta) \frac{\bar{u}h}{f_*} \frac{\partial f_*}{\partial x} - uh \, p(\zeta) \frac{\partial \zeta}{\partial x} \tag{b34}$$

where

$$r(\zeta) = - P(\zeta)/(1 - A/f_*) + (1 - \zeta) \tag{b35}$$

(b34) could also be written as

$$\frac{w}{w_s} = \frac{r(\zeta)}{f_*} \frac{\partial f_*}{\partial \xi} - p(\zeta) \frac{\partial \zeta}{\partial \xi} \tag{b36}$$

5) <u>The Diffusion Coefficient for Sediment Particles</u>

All expressions for the diffusion coefficient ε and the resulting equilibrium concentration profile ϕ_o have been obtained from DHL, (1980)

For instance

$$\frac{\varepsilon}{u_* a} = 4\left\{ \alpha_1 + \alpha_2 \left(\frac{w_s}{u_*}\right)^{\alpha_3} \right\} \left(\frac{z - z_b}{a}\right)\left(1 - \frac{z - z_b}{a}\right) \text{ for } \frac{z - z_b}{a} < 0.5 \tag{b37}$$

and

$$\frac{\varepsilon}{u_* a} = \alpha_1 + \alpha_2 \left(\frac{w_s}{u_*}\right)^{\alpha_3} \quad \text{for} \quad \frac{z - z_b}{a} \geq 0.5 \tag{b38}$$

where

$\alpha_1 = 0.1$, $\alpha_2 = 0.38$, $\alpha_3 = 4.31$ for Humus

$\alpha_1 = 0.13$, $\alpha_2 = 0.20$, $\alpha_3 = 2.12$ for natural channels

$$\left.\begin{array}{l} \text{Using } \zeta = \dfrac{z - (z_a + z_b)}{h}, \; z_a = \beta h \\ \\ \qquad \varepsilon' = \varepsilon / w_s h \quad \text{and} \quad a = (1 + \beta) h \end{array}\right\} \tag{b39}$$

it is possible to transform (b37) + (b38) to

$$\varepsilon' = \frac{1}{Z} \frac{(\zeta + \beta)}{(1 + \beta)} \left\{ 1 - \frac{(\zeta + \beta)}{(1 + \beta)} \right. \quad \text{for } \zeta < 0.5 \; (1 - \beta) \tag{b40}$$

and

$$\varepsilon' = \frac{1}{4Z} \quad \text{for } \zeta \geq 0.5 \; (1 - \beta) \tag{b41}$$

where

$$Z = \frac{1}{4(\beta + 1)} \frac{w_s}{u_*} \frac{1}{\left[\alpha_1 + \alpha_2 \left(\frac{w_s}{u_*}\right)^{\alpha_3}\right]} \tag{b42}$$

The maximum diffusion coefficient ε_{max} is given by

$$\varepsilon_{max} = \frac{w_s h}{4 Z} \tag{6.43}$$

$\dfrac{\partial Z}{\partial \left(\frac{w_s}{u_*}\right)}$ may be easily obtained from (b42).

6) The Normalised Equilibrium Profile

The equilibrium profile ϕ_o must satisfy

$$\phi_o + \varepsilon' \frac{\partial \phi_o}{\partial \zeta} = o \qquad (6.26)$$

where

$$\int_o^1 \phi_o \, d\zeta = 1 \qquad (4.19)$$

The following expressions for ϕ_o can be obtained from DHL, 1980.

$$\phi_o = B\left(\frac{1 - \zeta}{\zeta + \beta}\right)^{(1+\beta)Z} \qquad \zeta < 0.5\ (1 - \beta) \qquad (b44)$$

$$\phi_o = B \exp\ [- 4Z\{\zeta - 0.5(1 - \beta)\}]\ \zeta \geqslant 0.5\ (1 - \beta) \qquad (b45)$$

The constant B may be obtained from the normalising condition (4.19). It should however be noted that B is a function of Z.

$$\text{Thus if } \phi_o = B \exp\ [Z\ f(\zeta)] \qquad (7.16)$$

$$f(\zeta) = (1 + \beta) \ln \{(1 - \zeta)/(\zeta + \beta)\} \text{ for } \zeta < 0.5\ (1 - \beta) \qquad (b.46)$$

$$f(\zeta) = - 4\ \{\zeta - 0.5(1 - \beta)\} \qquad \text{for } \zeta \geqslant 0.5\ (1 - \beta) \qquad (b47)$$

Furthermore

$$\frac{\partial \phi_o}{\partial Z} = f(\zeta)\ \phi_o(\zeta) + \frac{\partial B}{\partial Z} \exp\ [Z\ f(\zeta)] \qquad (b48)$$

Integrating (b48) between $\zeta = o$ and $\zeta = 1$

$$\frac{\partial}{\partial Z}\left(\int_o^1 \phi_o \, d\zeta\right) \int_o^1 f(\zeta)\,\phi_o(\zeta) + \frac{1}{B}\frac{\partial B}{\partial Z}\int_o^1 \phi_o \, d\zeta \tag{b49}$$

As $\int_o^1 \phi_o \, d\zeta = 1$

$$\frac{\partial}{\partial Z}\left(\int_o^1 \phi_o \, d\zeta\right) = o \tag{b50}$$

$$\therefore \frac{\partial B}{\partial Z} = -B\int_o^1 f\,\phi_o \, d\zeta \tag{b51}$$

$$\therefore \frac{\partial \phi_o}{\partial Z} = f(\zeta)\,\phi_o(\zeta) - \left(\int_o^1 f\,\phi_o \, d\zeta\right) B \exp\,[Z\,f(\zeta)] \tag{b52}$$

or

$$\frac{\partial \phi_o}{\partial Z} = \phi_o(\zeta)\,\left[f(\zeta) - \int_o^1 f\,\phi_o \, d\zeta\right] \tag{b53}$$

Appendix C   Formulae for Coefficients

Each coefficient is built up from eight constants a_i, b_i from $i = 1$ to 4. The values of a_i and b_i required to compute eleven coefficients for each combination of values of w_s/u_* and $\bar{u}/u_*$ are given later in this appendix. The formulae to be used are as follows.

$\kappa = 1$	$\gamma_{21}/\gamma_{11} = (w_s/u_*)\ \exp(f)$
$\kappa = 2$	$\gamma_{22}/\gamma_{11} = (w_s/u_*)\ \exp(f)$
$\kappa = 3$	$\gamma_{33}/\gamma_{11} = (w_s/u_*)^2 \exp(f)$
$\kappa = 4$	$\mu_1/\gamma_{11} = w_s f/u_*$
$\kappa = 5$	$\mu_1/\gamma_{11} = w_s f/u_*$
$\kappa = 6$	$\alpha_{11} = f$
$\kappa = 7$	$\alpha_{21} = w_s f/u_*$
$\kappa = 8$	$\alpha_{22} = w_s f/u_*$
$\kappa = 9$	$\alpha_{33} = w_s f/u_*$
$\kappa = 10$	$\lambda_1 = w_s f/u_*$
$\kappa = 11$	$\lambda_2 = w_s f/u_*$

where

$$f = \sum_{i=1}^{4} (a_i + b_i \frac{u_*}{\bar{u}})(\frac{w_s}{u_*})^{i-1}$$

where a_i and b_i are obtained from the tables c1 to c3 for the corresponding value of κ for $z_a/a = 0.01$, 0.02 and 0.05. The computations were based on the suspension parameter for natural channels.

$z_a/a = 0.01$ for natural channels

k	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4
1	1.9779	0.000	- 6.3214	0.000	3.256	0.00	0.193	0.00
2	1.9782	0.543	- 6.3255	- 3.331	3.272	0.40	0.181	1.79
3	1.0944	5.632	- 4.3437	- 13.537	- 2.844	15.34	3.812	- 5.77
4	- 0.0109	- 0.808	- 4.8698	- 11.761	12.161	39.05	- 8.041	- 30.26
5	- 0.0107	- 0.819	- 4.8663	- 12.471	12.150	40.93	- 8.033	- 31.56
6	1.0000	0.114	0.0000	- 7.995	- 0.000	2.04	0.000	3.48
7	0.0000	- 3.852	0.0001	3.763	- 0.000	5.25	- 0.012	- 7.03
8	- 0.0098	- 4.254	- 0.0382	5.325	0.042	3.52	- 0.012	6.52
9	- 0.0004	- 0.006	- 0.0245	4.787	0.068	- 11.27	- 0.049	7.34
10	0.0007	- 0.307	- 0.0547	13.221	0.135	- 26.98	- 0.088	14.75
11	0.0008	- 0.311	- 0.0585	13.370	0.143	- 27.28	- 0.093	14.91

Table C1

$z_a/a = 0.02$ for natural channels

k	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4
1	1.7883	0.000	- 5.7793	0.000	2.860	0.00	0.226	0.00
2	1.7887	0.570	- 5.7832	- 3.000	2.872	0.56	0.217	1.43
3	0.9619	4.942	- 4.3581	- 10.455	- 2.423	11.06	3.440	- 3.70
4	- 0.0177	- 0.565	- 4.1797	- 9.906	10.487	31.50	- 6.955	- 23.95
5	- 0.0175	- 0.680	- 4.1743	- 10.959	- 10.470	34.28	- 6.9.42	- 25.85
6	1.0000	0.091	0.0000	- 7.040	- 0.000	3.16	- 0.000	1.76
7	0.0000	- 3.372	0.0000	4.239	- 0.000	2.16	0.000	- 4.48
8	0.0084	- 3.715	- 0.0313	5.522	0.036	0.68	- 0.012	- 3.98
9	- 0.0003	0.006	- 0.0189	3.780	0.052	- 8.98	- 0.037	5.89
10	0.0005	- 0.212	- 0.0420	- 10.435	0.105	- 22.13	- 0.070	- 12.64
11	0.0006	- 0.216	- 0.0472	- 10.637	0.116	- 22.55	- 0.076	- 12.88

Table C2

$z_a/a = 0.05$ for natural channels

k	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4
1	1.4856	0.000	- 4.9986	0.000	2.306	0.00	0.247	0.00
2	1.4859	0.576	- 5.0016	- 2.416	2.314	0.72	0.242	0.91
3	0.6944	4.006	- 4.2619	- 6.914	- 1.902	6.62	2.895	- 1.77
4	- 0.0198	- 0.300	- 3.1905	- 7.145	7.985	21.48	- 5.280	- 15.91
5	- 0.0195	- 0.320	- 3.1820	- 8.745	7.960	25.64	- 5.262	- 18.72
6	1.0000	0.059	0.0000	- 5.363	- 0.000	3.48	0.000	0.30
7	0.0000	- 2.535	- 0.0000	9.937	- 0.000	- 0.47	0.000	- 1.85
8	0.0059	- 2.776	- 0.0206	4.777	0.024	- 1.45	- 0.009	- 1.49
9	- 0.0002	0.013	- 0.0113	2.412	0.030	- 5.73	- 0.022	3.75
10	0.0002	- 0.110	- 0.0252	6.710	0.064	- 14.83	- 0.044	8.87
11	0.0004	- 0.115	- 0.0320	6.973	0.079	- 15.40	- 0.053	9.21

Table C3