THE LIFTING WING WITH MINIMUM DRAG
IN SUPersonic FLOW

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DELFt - THE NETHERLANDS

October 1977
DELTFT UNIVERSITY OF TECHNOLOGY

DEPARTMENT OF AEROSPACE ENGINEERING

Report LR-234

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SUMMARY

In this report an attempt is made to obtain some insight in a proper formulation of a drag minimisation problem for thin wings in supersonic flow. If the planform is not fixed the introduction of a few geometrical constraints is shown to be necessary. In analogy with slender body theory an area of "entrainement" can be introduced. A planform with minimum drag is proved to be a planform with maximum area of entrainement. At the same time some interesting features of the velocity field around an optimum wing and of the corresponding load distribution on the optimum wing are shown.
1. INTRODUCTION.

The minimisation of the drag for given lift is a longstanding problem in linearized supersonic potential flow theory. In subsonic flow the drag depends only on the spanwise distribution of lift. In supersonic flow however both the spanwise and lengthwise distribution of the lift are of importance to the drag. In supersonic flow it is usual to separate from the total drag a part which in analogy with subsonic flow corresponds to the spanwise distribution of trailing vortices in the wake. This part, which is only dependent of the lift distribution in spanwise direction is called vortex drag. The remainder of the drag can be identified with the production of waves by the wing and is called wave drag. This part of the drag depends on the distribution of lift over the whole planform. Most of the literature on drag minimisation in supersonic flow is concerned with finding the optimum distribution of lift over a fixed planform e.g. [6],[7],[14],[15] and [18].

If, however, the planform is free it is obvious that as in subsonic flow also in linearized supersonic non-viscous flow some geometrical constraints have to be made in order to avoid a meaningless optimum with vanishing lift on a planform of infinite extent. At this point some arbitrariness enters into the problem because it is obvious that there are indefinite many possible constraints which limit the dimensions of the planform and since in general the result depends on the constraints there are just as many optima to be found.

In this report in section 2 it will be shown how for wings with subsonic leading edges and fixed span the optimum to be found depends on the way the chordwise dimensions are restricted. In section 3 it will be shown that in general it is not correct to minimize vortex drag and wave drag separately. Only for special planforms minimum total drag corresponds to minimum vortex drag. It will also be pointed out that the optimum velocity field shows a velocity jump (shock like phenomenon) across the envelope of the characteristic cones originating from the leading edge. This velocity jump corresponds
for a wing with subsonic leading edges to a square root singularity in
the downwash in the vertex of the wing planform. On the other hand, if
supersonic edges were considered a continuously curved leading edge
would provide the necessary jump.
If the lengthwise dimensions are restricted by fixing the maximum span
points, it is shown in section 2 that a wing planform with sonic trailing
edges is a left-right symmetric optimum planform within the given
constraints. For this type of planform the optimum spanwise lift-
distribution, the minimum value of the drag and the velocity jump
across the machcone will be calculated in section 4.
From these results it will be shown that up to and within the not so
slender body approximation the wave drag and vortex drag can indeed
be minimized separately.
2. GENERAL CONSIDERATIONS.

2.1. Problem formulation.
Consider the supersonic flow with undisturbed velocity $U_0$ about a thin lifting wing of finite span. The disturbance velocity potential is called $\varphi$ and the velocities in respectively $x$, $y$ and $z$ direction will be denoted by $u = \varphi_x$, $v = \varphi_y$ and $w = \varphi_z$ (the subscripts denote differentiation). In linear theory the potential satisfies the wave equation:

$$\beta^2 \varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 \quad (2.1)$$

where $\beta = M - 1$, $M = \frac{U_0}{a_o}$ is the mach number and $a_o$ is the speed of sound of the unperturbed flow.

Introducing a control surface $S_c : x = f(y,z)$ ($f(y,z)$ is a double valued function) around the wing and making use of momentum methods we can write the forces on the wing in terms of the derivatives of $\varphi$ on the control surface. Expanding up to and including terms of the second order in the perturbation velocities the following expression for the drag can be derived (only terms quadratic in the first order perturbation velocities appearing in the integrand).

![Diagram](image-url)
\[ D = c_D \frac{1}{2} \rho_0 U_0^2 S_w = \rho_0 \int \frac{1}{2} \left( \frac{\beta^2 y_x^2 + y_y^2 + y_z^2}{1 + \beta^2 y_y + \beta^2 z} \right) + y_x y_y f_y + y_z f_z \, dS \, (2.2) \]

The lift is of first order in the perturbation velocities:

\[ L = c_L \frac{1}{2} \rho_0 U_0^2 S_w = \rho_0 \int \frac{-y_z U_0 - y_x U_0 f_z}{1 + \beta^2 y + \beta^2 z} \, dS \, (2.3) \]

\[ c_D \text{ and } c_L \text{ are respectively drag- and lift coefficients and } S_w \text{ is the surface area of the wing planform.} \]

Introducing:

\[ \Phi(y, z) = \Psi \{ \Phi(y, z), y, z \} \]

\[ \Phi_y = y_x f_y \]

\[ \Phi_z = y_z f_z \]

the expressions for drag and lift reduce to:

\[ D = \rho_0 \int \frac{1}{2} \left[ \left( \Phi_y^2 + \Phi_z^2 \right) + \left( \beta^2 \frac{f_y}{y_y} + \beta^2 \frac{f_z}{y_z} \right) y_x^2 \right] \, dS \, (2.5) \]

\[ L = \rho_0 \int \frac{-U_0 \Phi_z}{1 + \beta^2 y + \beta^2 z} \, dS \, (2.6) \]

It should noticed that:

\[ \frac{dS}{1 + \beta^2 y + \beta^2 z} = \alpha \, S_p \, (2.7) \]

represents the projection of a control surface element on a plane \( x \) = constant.

The integrand in the expression for the drag can be simplified by a proper choice of the control surface.

If we choose now \( \beta^2 - \frac{f_y^2}{y_y} - \frac{f_z^2}{y_z} = 0 \) (cf. ref. [16] and [19]).
indicating that the control surface has everywhere the characteristic
direction, equations (2.5) and (2.6) reduce to:

\[ D = \frac{\rho_0}{2} \int \int \left( \Phi_y^2 + \Phi_z^2 \right) \frac{dS_c}{1 + \rho_2^2 + \rho_2^2} \quad (2.8) \]

\[ I = -\rho_0 U_0 \int \int \Phi_z \frac{dS_c}{1 + \rho_2^2 + \rho_2^2} \quad (2.9) \]

Since the control surface is a closed surface it is composed of two
characteristic surfaces, belonging to two different families. The two
parts consist of the envelope of the bicharacteristics which form the
separation between the disturbed and the undisturbed region in
respectively direct and reversed flow. These characteristic surfaces
intersect in a space curve \( C_1^* \). In the case of a wing with subsonic
leading edges the part related to the direct flow forms a circular
cone with its vertex in the apex of the wingplanform.

This cone is the so called downstream machcone.

In order to avoid problems at the trailing edge related to a type of
Kutta-condition we restrict ourselves to wings with supersonic trailing
edges. These edges determine the shape of that part of the integration
surface which is related to the reversed flow. We call this part the
upstream characteristic surface.

On the downstream machcone \( \varphi = 0 \) so \( \Phi = 0 \) and so are all the derivatives
along the cone. It follows that the integrands in (2.8) and (2.9) vanish
on this part of the integration surface. Taking into account eq. (2.7),
(2.8) and (2.9) can be written in the form:

\[ D = \frac{\rho_0}{2} \int \int \left( \Phi_y^2 + \Phi_z^2 \right) dS_p \quad (2.10) \]

\[ I = -\rho_0 U_0 \int \int \Phi_z dS_p \]
The region of integration has been simplified to that part of the plane $z = \text{constant}$ enclosed by the projection $C_1$ of the curve $C_1^*$ and by the projection of the trailing-edge which forms the slit $C_2$ (see fig. 1). With these expressions for lift and drag we formulate a variational problem for the transformed velocity potential $\Phi$. 

2.2. Variational formulation.
Introducing a Lagrangian multiplier $\lambda$ one obtains:

$$\mathcal{D} - \lambda \mathcal{L} = \frac{\rho_0}{2} \iint_{S_{\mathcal{P}}} \left( \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} + 2 \lambda U_0 \frac{\partial \Phi}{\partial z} \right) \, dS_{\mathcal{P}} \tag{2.11}$$

Applying calculus of variations on this integral yields the Euler-Lagrange equation (cf. ref. [4]):

$$\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{2.12}$$

with the boundary conditions:

$$\Phi = \varphi + \varphi \frac{f}{z} = \text{const.} = -\lambda U_0 \text{ on } C_2$$

and

$$\Phi = 0 \text{ on } C_1' \tag{2.13}$$

The value of $\lambda$ is to be determined from the condition:

$$\mathcal{L} = -\rho_0 U \int \frac{\Phi}{z} \, dS_{\mathcal{P}} \tag{2.14}$$

and the corresponding value of the drag is:

$$\mathcal{D} = \frac{1}{2} \lambda \mathcal{L} \tag{2.15}$$

It will be proven that the smallest value of the total drag corresponds to the largest area of "entrainement" enclosed by the curve $C_1$. The formulation of a proper variational problem for the form of the curve $C_1$ seems not to be appropriate here since it is clear that no stationary value will appear, because the smallest value is obtained when the
curve $C_1$ is at infinity (the slender body case). A more direct approach will lead to a clear conclusion (c.f. ref. [19]).

![Diagram](image)

**fig.2**

We introduce a curve $C_3$ which encloses the area $S_p^* = S_p + \delta S_p$ with the restriction that $C_3$ does not intersect $S_p$. So the area enclosed by $C_3$ is larger than the area enclosed by $C_1$. We compare now the drag $D$ resulting from $\Phi$ which corresponds to $C_3$ with the drag $D$ as a result from $\Phi$ and $C_1$.

\[
\begin{align*}
\Phi &= 0 \quad \text{on } C_3 & \Phi &= 0 \quad \text{on } C_1 \\
\Phi_2 &= -\kappa U_0 \quad \text{on } C_2 & \Phi_2 &= -\kappa U_0 \quad \text{on } C_2
\end{align*}
\]

\[
L_t = -\rho U_0 \int_{\Gamma_2} \Phi_2 \, dS_p^* = -\rho U_0 \int_{\Gamma_2} \Phi_2 \, dS_p = L
\]

(2.16) (2.17)
From $\int_0^L \Delta \phi ds = \int_0^L \Delta \phi ds$ so: 

$$
\int_0^L \Delta \phi_1 ds = \int_0^L \Delta \phi_2 ds = \int_0^L \phi \frac{\partial \phi}{\partial n} dc = \int_0^L \phi \frac{\partial \phi}{\partial n} dc \quad (2.18)
$$

$$
\overline{D} = \frac{P_0}{2} \int_0^\infty (\phi_1 + \phi_2)^2 dS_p = \frac{P_0}{2} \int_0^\infty \phi \frac{\partial \phi}{\partial n} dc \quad \hat{n} = \text{unit normal vector directed outwards}
$$

$$
D - \overline{D} = \frac{P_0}{2} \int_0^\infty (\phi_1 \frac{\partial \phi}{\partial n} - \phi_2 \frac{\partial \phi}{\partial n}) dc = \frac{P_0}{2} \int_0^\infty \phi \frac{\partial \phi}{\partial n} (\phi - \phi) dc = 
$$

$$
= \frac{P_0}{2} \left[ \int_0^L \left( (\phi_1 - \phi_2) \frac{\partial \phi}{\partial n} + (\phi_2 - \phi_1) \frac{\partial \phi}{\partial n} \right) dc + \int_0^L \left( (\phi_1 - \phi_2) \frac{\partial \phi}{\partial n} + (\phi_2 - \phi_1) \frac{\partial \phi}{\partial n} \right) dc + \int_0^\infty \phi \frac{\partial \phi}{\partial n} dc \right] = 
$$

$$
= \frac{P_0}{2} \left[ \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc + \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc + \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc \right] 
$$

$$
= \frac{P_0}{2} \left[ \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc + \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc + \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc \right] 
$$

Since:

$$
\int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc + \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc + \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc = 0
$$

equation (2.19) becomes:

$$
D - \overline{D} = \frac{P_0}{2} \left[ \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc + \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc + \int_0^L \phi \frac{\partial \phi}{\partial n} dc - \int_0^L \phi \frac{\partial \phi}{\partial n} dc \right]
$$
This result can be simplified to:

\[ \mathcal{D} = \frac{\rho_0}{2} \left[ \iint_{S_p} \left( \nabla \Phi \cdot \nabla \Phi \right)^2 dS_p - \int_{C_l} \Phi \frac{\partial \Phi}{\partial n} dc + \int_{\hat{C}_3} \Phi \frac{\partial \Phi}{\partial n} dc \right] \]

For the region between \( C_l \) and \( \hat{C}_3 \) in the integral (II) along the curve \( C_l \) is an inward directed normal vector, so:

\[ \mathcal{D} = \frac{\rho_0}{2} \left[ \iint_{S_p} \left( \nabla \Phi \cdot \nabla \Phi \right)^2 dS_p + \int_{C_l + C_3} \Phi \frac{\partial \Phi}{\partial n} dc \right] \]

\[ = \frac{\rho_0}{2} \left[ \iint_{S_p} \left( \nabla \Phi \cdot \nabla \Phi \right)^2 dS_p + \int_{S_p^* - S_p} \left( \nabla \Phi \right)^2 dS_p \right] \approx 0 \tag{2.20} \]

For fixed lift and span the smaller minimum drag corresponds to the larger area \( S_p \). It is clear then that the drag decreases if the length of the wing tends to infinity, confirming a well known result of slender wing theory where the wave drag vanishes.

For a fixed planform the area of entrainment \( S_p \) decreases with increasing mach number.

### 2.3. Constraints.

So far every argument would apply both to wings with supersonic and subsonic leading edges. Although the value of the lower bound of the drag does not show any discontinuous behaviour in the transit from subsonic to supersonic leading edges, some qualitative differences are of importance.

Firstly as we shall see in section 3.2., the solutions for wings with subsonic edges have to admit a square root singularity in the downwash distribution at the apex of the wing which can be avoided if supersonic edges are allowed.

Secondly, the solution \( \Phi \) of (2.12) depends on the boundary conditions (2.13) and thus on the form of \( C_l \). In the case of supersonic leading
edges the form of the curve is determined both by the leading edge and the trailing edge. However, for wings with subsonic leading edges the value of the lowerbound of the drag depends only on the form of the trailing edge. The optimal form of the edges depends on the imposed type of geometrical constraints.

It has been shown in section 2.2. that the minimum value of the drag decreases when the area of "entrainment" $S_P$ increases. For a good basis of comparison for the different wing planforms one should in addition to the already fixed span also limit the dimensions of the wing in $x$-direction. In order to obtain a minimum total drag for given lift within the constraints of limited length and span the planform of the wing should be formed in such a way that the area enclosed by the curve is as large as possible.

In this context it is interesting to note that just fixing the surface area of the wing planform is not a sufficient geometrical constraint to obtain a useful optimum because in that case the optimum solution is a lifting line of infinite extent.

So the geometrical constraints should be formulated in such a way that a planform with finite dimensions is the result of the optimisation procedure.

One way to attain this is fixing the span and the dimensions in streamwise direction and also demanding supersonic trailing edges in order to avoid problems with some kind of Kutta condition and in order to avoid a lifting line solution. Within the latter framework some freedom remains in the way the streamwise dimensions of the planform are limited.

A few results of examples for left-right symmetric planforms with subsonic leading edges are shown in figure 3. Figure a is the result of fixed length and fixed span, figure b is the planform corresponding to fixed maximum chord and fixed span. Figure c results from fixed maximum spanpoints and fixed span. For all three figures the leading edge is still to be determined.
It is expected that the solution of the so-called Goursat-type boundary value problem for the wave equation, which results from the solution of the Laplace problem (2.12) and (2.13) determines the camber and twist on the planform and the form of the leading edge of the planform. But little can be found in the literature about the existence and uniqueness of this type of boundary value problem.
3. BOUNDARY CONDITIONS.

3.1. Conditions at \( C_e \).

If \( \psi_z^2 = \text{constant at the trailing edge} \) the boundary condition (2.13)
\[ \frac{\partial \varphi}{\partial z} = -\lambda u \text{changes into } \psi_z = \text{const. at the trailing edge which implies} \]
constant downwash along the trailing edge. If \( \psi_z, \rho_z = 0 \) then \( (\varphi_z)_{e} = -\lambda u_{o} = -\kappa_{e} \frac{u_{o}}{L} \)
and thus \( D = \frac{1}{2} (\kappa_{e})^{2} L \). The possibilities which lead to this situation are:

\( \psi_z = 0 \) at the trailing edge which implies unloaded trailing edge,
\( \rho_z = 0 \) which corresponds to a sonic trailing edge and in the case of
a subsonic trailing edge both \( \psi_z \) and \( \rho_z \) are zero.

The trailing edge is called sonic if its direction everywhere coincides
with the characteristic direction.

Noting that only the velocity potential distribution in spanwise
direction across the wake does not change downstream of the trailing
edge, we can ascertain that the boundary condition \( \varphi_z = \text{constant at} \)
the trailing edge in general does not correspond to the case of minimum
vortex drag. Minimum vortex drag is obtained as in subsonic theory if
the downwash across the wake far downstream is constant. The optimum
wing is also a wing with minimum vortex drag only if an elliptic span-
wise lift distribution is the result of the solution of the Laplace
problem. Elliptic spanwise lift distribution is obtained only if \( C_{1} \)
is an ellips with properly chosen foci and thus if the wing edges have
a special form. Since the total drag is the sum of vortex drag and
wave drag it is clear that for a wing which has minimum total drag under
given geometrical constraints and also has minimum vortex drag, the
wave drag has to show an extreme value too. Only for a planform, which
possesses the property of an elliptic \( C_{1} \) the separate optimisation of
wave drag and vortex drag is compatible and leads to a minimum vortex
drag.

3.2. Condition at \( C_{1} \).

Apart from the characteristic constant downwash (equation 2.13) there
is another feature which wings should exhibit if they are optimal with
regard to drag.

From the theory of harmonic functions we know that the type of boundary
value problem for Laplace's equation, given by equations (2.12) and (2.13), yields a solution which shows a non-zero normal derivative on $C$. This implies a finite jump of the normal disturbance velocity across the mach cone, a shocklike phenomenon. Thus, the solutions of the wave equation, we are looking for, belong to the class of weak solutions [4]. We shall analyse how this kind of behaviour in the neighbourhood of the machcone could be created.

The disturbance velocity potential can be written in the upper halfspace as an integral over a source distribution where the local source strength is proportional to the local downwash ($\frac{3}{2}$ for convenience).

$$\varphi(x, y, z) = -\frac{1}{(x-y)^{2} - (y-n)^{2} - z^{2}} \int \frac{w(\xi, \eta) d\xi d\eta}{(x-\xi)^{2} - (y-\eta)^{2} - z^{2}}$$

(3.1)

![Diagram](image)

*Fig. 4*
The region of integration $\Delta$ is that part of the plane $z=0$ which belongs to the region of dependence of the fieldpoint $(x,y,z)$. The behaviour of the velocity component $u$ is considered when the fieldpoint is shifted along a line parallel to the $x$-axis through the machcone. By differentiation of equation (3.1) with respect to $x$ we obtain:

$$u = \varphi_x = \frac{1}{\pi} \int_{\Delta} \frac{(x-\xi) w(\xi,\eta) d\xi d\eta}{\{(x-\xi)^2 - (y-\eta)^2 - z^2\}^{3/2}} \quad (3.2)$$

The symbol $\int^*$ indicates that the finite part of the integral as defined by Hadamard $[13]$ has to be taken. In order to transform the integral in (5.2) into a more convenient form a new integration variable $S$ will be defined by:

$$S = \xi + \sqrt{(y-\eta)^2 + z^2}, \quad \xi = S - \sqrt{(y-\eta)^2 + z^2} = g(S, \eta) \quad (3.3)$$

where curves $S$ = constant are hyperbolae in the $\xi, \eta$ plane.

Upon substitution of (5.3), the expression (5.2) becomes:

$$\varphi_x = \frac{1}{\pi} \int_{r}^{\infty} \frac{dS}{(x-S)^{3/2}} \int_{\eta = \rho_1(S)}^{\rho_2(S)} \left[ \frac{\rho_1(S)}{\xi - S + \sqrt{(y-\eta)^2 + z^2}} \right] w(g(S, \eta), \eta) d\eta \quad (3.4)$$

where:

$$r = \sqrt{y^2 + z^2}, \quad \rho_1(S) = \frac{r^2 - S^2}{2(y-S)}, \quad \rho_2(S) = \frac{r^2 - S^2}{2(y+S)}$$

Performing the integration with respect to $S$ yields:

$$\varphi_x = \frac{1}{\pi} \int_{r}^{\infty} \frac{F(S) dS}{(x-S)^{3/2}} \quad (3.5)$$
Integrating by parts we obtain:

$$\psi = -\frac{2}{\pi} \int_r^\infty \frac{F'(\delta) d\delta}{\sqrt{\delta - r}}$$

(3.6)

This expression can be interpreted as an integral equation for the behaviour of $F'(\delta)$ in the limit for: $\delta \to r$, $\delta \geq r$

The equation (3.6) can be solved by means of a so called null transform (see [17]). A null transform of order zero to the kernel of this integral equation is:

$$\int_r^\infty \frac{d\delta}{(\delta - r)^{3/2}}$$

(3.7)

From the inversion of equation (3.6) for finite $\psi$, we obtain:

$$F'(\delta) \propto \frac{d}{d\delta} \sqrt{\delta - r}$$

(3.8)

so

$$F'(\delta) \propto \sqrt{\delta - r}$$

(3.9)

It has to be noted that with respect to the length of the integration interval in $F'(\delta)$ a qualitative difference arises between wings with a continuously curved leading edge and wings where the leading edges intersect in the apex.

For wings with a continuously curved leading edge which is therefore at least partially supersonic the length of the integration interval behaves like $\sqrt{\delta - r}$ and the required behaviour of $F'(\delta)$ can be obtained with a finite downwash on the wing in the neighbourhood of the apex. On the other hand for wings with a pointed planform and therefore an apex angle which is not equal to zero or $\pi$, the length of the integration interval is linear in $(\delta - r)$. If $\omega$ behaves like $(\delta - r)^{1/2}$ then $F'(\delta) \propto (\delta - r)^{1/2}$ since the remaining part of the integrand (including the possible square root type singular behaviour of $\omega$ near the leading edges of the wing) makes a finite contribution to the
integral. This contribution is directly proportional to the length of the integration interval. For this type of planform the downwash on the wing has to admit a square root singularity at the apex of the wing to obtain the required velocity behaviour near the machcone. The loading near the vertex of the wing is proportional to the square root out of the distance to the apex.

It was shown that in both cases a velocity jump across the machsurface occurs if the loading in chordwise direction is of elliptical nature in the neighbourhood of the "apex". This result agrees with the behaviour prescribed by an elliptic load distribution in chordwise direction as is found to be a sufficient condition to yield a minimum wavetrap in lit. [1], [14] and [15].

As peculiarity it may be mentioned that the character of the lift distribution near the apex of the wing corresponds to what in ref [3] and [8] is prescribed to generate a minimum sonic boom.

If we want to apply the theory of homogeneous functions to this problem of determining an optimum wing the necessary kind of behaviour near the machcone prescribes as will be shown a non-integer degree of homogeneity.

Consider the behaviour of a solution of the wave equation in the neighbourhood of the mach cone.

Introducing new variables:
\[
x_i = \sqrt{x^2 - \beta^2 (y^2 + z^2)}
\]
\[
y_i = y
\]
\[
z_i = z
\]

and putting:
\[
\varphi(x, y, z) = \varphi(x_i, y_i, z_i)
\]

the wave equation (2.1) can be rewritten in new coordinates as:
\[
\beta^2 \varphi_x x_i - \varphi_{yy} - \varphi_{zz} + 2 \beta^2 \frac{y}{x_i} \varphi_{y} y_i + 2 \beta^2 \frac{z}{x_i} \varphi_{z} z_i + 2 \beta^2 \frac{1}{x_i} \varphi_{x} x_i = 0
\]
In the neighbourhood of the Mach cone at some distance from the x-axis it is supposed that:
\[
\frac{x_1}{y_1} = O(e^{1/2}) \quad \frac{x_1}{\bar{z}_1} = O(e^{1/2})
\]
where \( e \) is a small parameter.

Introduction of the stretched variables:
\[
\bar{x} = \frac{x_1}{e^{1/2}}, \quad \bar{y} = y_1, \quad \bar{z} = z_1 \quad \text{and} \quad \varphi^* (\bar{x}, \bar{y}, \bar{z}) = \varphi (x_1, y_1, z_1)
\]
transforms equation (3.12) into:
\[
(\partial^2 \varphi^* \frac{\partial}{\partial \bar{x}^2} + 2\beta^2 \left[ \frac{1}{\bar{x}} \varphi^* + \frac{\bar{y}}{\bar{x}} \varphi^* \frac{\partial}{\partial \bar{y}} + \frac{\bar{z}}{\bar{x}} \varphi^* \frac{\partial}{\partial \bar{z}} \right]) - e \left[ \varphi^* \frac{\partial^2 \varphi^*}{\partial \bar{y}^2} + \varphi^* \frac{\partial^2 \varphi^*}{\partial \bar{z}^2} \right] = 0
\]
(3.13)

A solution of (3.13) can be written as:
\[
\varphi^* = \varphi_o^* + e \varphi_i^* + e^2 \varphi_2^* + O(e^3)
\]
(3.14)

Substitution of (3.14) in (3.13) and ordering to increasing powers of gives:
\[
(\partial^2 \varphi_o^* \frac{\partial}{\partial \bar{x}^2} + 2\beta^2 \left[ \frac{1}{\bar{x}} \varphi_o^* + \frac{\bar{y}}{\bar{x}} \varphi_o^* \frac{\partial}{\partial \bar{y}} + \frac{\bar{z}}{\bar{x}} \varphi_o^* \frac{\partial}{\partial \bar{z}} \right]) = 0
\]
and for \( i = 1, 2, \ldots \)
\[
(\partial^2 \varphi_i^* \frac{\partial}{\partial \bar{x}^2} + 2\beta^2 \left[ \frac{1}{\bar{x}} \varphi_i^* + \frac{\bar{y}}{\bar{x}} \varphi_i^* \frac{\partial}{\partial \bar{y}} + \frac{\bar{z}}{\bar{x}} \varphi_i^* \frac{\partial}{\partial \bar{z}} \right]) = \varphi_i^* \frac{\partial^2 \varphi_i^*}{\partial \bar{y}^2} + \varphi_i^* \frac{\partial^2 \varphi_i^*}{\partial \bar{z}^2}
\]

The equation for the zero’th order part can be reduced to:
\[
\frac{1}{\bar{x}} \left[ 2 \frac{\partial}{\partial \bar{x}} \left( \bar{x} \varphi_o^* + \bar{y} \varphi_o^* + \bar{z} \varphi_o^* \frac{\partial}{\partial \bar{y}} \right) - \bar{x} \varphi_o^* \frac{\partial}{\partial \bar{x}} \right] = 0
\]
(3.15)
and since:
\[
\bar{x} \varphi_o^* \frac{\partial}{\partial \bar{x}} \left[ \bar{x} \varphi_o^* \right] - \varphi_o^* = \varphi_o^*
\]
one obtains: for (3.15)

\[
\frac{1}{\varepsilon} \left[ \frac{\partial}{\partial \xi} \left( \varepsilon \phi_0 \phi_0^* + \varepsilon \phi_0 \phi_0^* + \frac{\varepsilon}{2} \phi_0^* \right) \right] = 0 \tag{3.16}
\]

If \( \phi \) is a function homogeneous of degree \( n \) in \((x, y, z)\) then:

\[
x \phi_x + y \phi_y + z \phi_z = n \phi = \varepsilon \phi_0 \phi_0^* + \varepsilon \phi_0 \phi_0^* + \varepsilon \phi_0^* + O(\varepsilon) = n \phi_0^* + O(\varepsilon) \tag{3.17}
\]

Substitution of (3.17) in (3.16) yields:

\[
\frac{1}{\varepsilon} \frac{\partial}{\partial \xi} \left[ (n + \frac{1}{2}) \phi_0^* - \frac{\varepsilon}{2} \phi_0^* \right] = 0 \tag{3.18}
\]

Integrating equation (3.18) results in:

\[
(n + \frac{1}{2}) \phi_0^* - \frac{\varepsilon}{2} \phi_0^* = C_1(\overline{y}, \overline{z}) \tag{3.19}
\]

The solution of this differential equation is:

\[
\phi_0^* = C_2(\overline{y}, \overline{z}) \overline{z}^{(2n+1)} + \frac{C_1(\overline{y}, \overline{z})}{n + \frac{1}{2}} \tag{3.20}
\]

From the boundary condition \( \phi_0^* = 0 \) for \( \overline{z} = 0 \) one finds:

\[
C_1(\overline{y}, \overline{z}) = 0
\]

so:

\[
\phi_0^* = C_2(\overline{y}, \overline{z}) \overline{z}^{(2n+1)} \tag{3.21}
\]

Where \( C_2(\overline{y}, \overline{z}) \) is to be determined by some kind of matching procedure.

From (3.21) follows by differentiation:

\[
\frac{\partial \phi_0^*}{\partial \overline{z}} = \frac{\partial \phi_0^*}{\partial \overline{z}} \overline{z} + \frac{1}{\varepsilon} = C_2(\overline{y}, \overline{z}) (2n+1) \overline{z}^{(2n-1)} + O(1) \tag{3.2}
\]

Inspection shows that the first term in \( \phi_0^* \) can be finite and non zero on the machcone \( (\overline{z} = 0) \) only if \( \overline{z}^{(2n-1)} \overline{z} = 1 \) so \( n = \frac{1}{2} \).

Since it is the leading term in this approximation which determines the behaviour in the limit for \( \overline{z} \rightarrow 0 \), apparently the wave equation prescribes the degree of homogeneity needed for a prescribed behaviour.
of $\frac{\partial V}{\partial x}$ on the mach cone. It may be noted that solutions of positive integer degree of homogeneity (as introduced by Germain and Fenain [5] and [9]) cannot account for the required discontinuity in the normal velocity on the mach cone, because as can be deduced from equation (3.22) the wave equation prescribed that all derivatives up to and including the $n^{th}$ are zero for $x_1 = 0$. Therefore and because of the resulting qualitative different loading on the wing planform it is clear that if we limit our analysis to the class of homogeneous solutions of integer degree of homogeneity, we can only hope that the value of the drag converges to the lower bound as defined above without ever reaching this value.
4. PLANFORM WITH SONIC TRAILING EDGES.

4.1. Introduction.
We have found that for wings with subsonic leading edges the lower bound of the drag for a family of wings with the same lift, span and limited chord wise dimensions depends only on the form of the trailing edge. The optimum form of the trailing edge depends on the geometrical constraints. If for a left-right symmetric planform the points were the span is maximum are fixed, the optimum trailing edge is a sonic trailing edge and the resulting planform is a diamond-like planform shape. For this planform the curve $C_1$ is a circle and the Laplace problem (2.12), (2.13) can be solved.

4.2. Formulation of the Laplace problem.
The problem to be solved is:

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (4.1)$$

with the boundary conditions:

$$\phi = 0 \text{ for } y^2 + z^2 = a^2$$

$$\phi = 0 \text{ for } z = 0, \ b \leq |y| \leq a \quad (4.2)$$

$$\phi \bigg|_{z} = -\gamma \nu_0 \text{ for } z = 0, \ |y| \leq b$$

Once the solution of this boundary value problem is determined, the multiplier $\gamma$ can be expressed as function of the lift $L_1$ by means of:

$$L_1 = -\rho_0 \nu_0 \int \int_{S_p} \phi_2 \ dS_p \quad (4.3)$$

and for the drag results:

$$D = \frac{\gamma}{2} L_1 \quad (4.4)$$
4.3. Solution of the Laplace problem.

After defining the complex variable \( p = y + iz \) the conformal mapping:

\[
 u + iv = \frac{p}{p} = \frac{2p}{1 + \left(\frac{p}{a}\right)^2}
\]

(4.5)

can be introduced. This transformation maps the inner region of the circle \( |p| < a \) on the whole \( \mathbb{C} \) plane. (see fig. 6)

![Diagram](image)

Substitution of the transformation (4.5) in (4.1) and (4.2) and putting

\[
 \Phi^*(u,v) = \Phi(y,z)
\]

leads to the transformed problem:

\[
 \Phi^*_{uu} + \Phi^*_{vv} = 0
\]

(4.6)

with the boundary conditions:

\[
 \Phi^* = 0 \quad \text{for} \quad |u| \gg \frac{2b}{1 + \left(\frac{b}{a}\right)^2}, \quad \nu = 0
\]

(4.7a)
\[ \Phi^*_{uv} = -\gamma U_o \frac{a^2}{\omega^2 \sqrt{1 - \left(\frac{U}{U_o}\right)^2}} \left(\frac{1}{\sqrt{1 - \left(\frac{U}{U_o}\right)^2}} - 1\right) \text{ for } v = 0, \ |u| \leq \frac{2b}{1 + \left(\frac{b}{a}\right)^2} \] (4.7b)

Writing the solution \( \Phi^* \) in the form of a dipole distribution on the interval \( v = 0, \ |u| \leq k = \frac{2b}{1 + \left(\frac{b}{a}\right)^2} \) an integral equation for the solution is obtained:

\[ \left(\Phi^*_{uv}\right)_{v=0} = -\frac{1}{2\pi} \int_{-k}^{+k} \frac{\partial}{\partial u_o} \frac{\Delta \Phi^*}{u - u_o} \, du_o \] (4.8)

\( \Delta \Phi^*_{u_o} = \Phi^*_{+u_o} - \Phi^*_{-u_o} \) is equal to the jump of the disturbance velocity potential across the trailing edge of the wing and thus across the wake.

The inversion of this integral equation is obtained by means of a null transform of order one to the kernel of the equation (see ref [17]).

With the boundary conditions \( \Delta \Phi^*(k) = \Delta \Phi^*(-k) = 0 \) one gets:

\[ \frac{\partial}{\partial u_o} \left( \Delta \Phi^* \right) = \frac{2}{\pi \sqrt{k^2 - u_o^2}} \int_{-k}^{+k} \frac{\sqrt{k^2 - u_o^2}}{u_o - u} \left(\Phi^*_{uv}\right)_{v=0} \, du \] (4.9)

Upon substitution of (4.7) there is obtained:

\[ \frac{\partial}{\partial u_o} \left( \Delta \Phi^* \right) = \frac{2\gamma U_o a^2}{\pi \sqrt{k^2 - u_o^2}} \int_{-k}^{+k} \frac{\sqrt{k^2 - u_o^2}}{u_o(u_o - u)} \left[ \frac{1}{\sqrt{1 - \left(\frac{U}{U_o}\right)^2}} - 1 \right] \, du \] (4.10)

Evaluation of this integral (see appendix A) yields:

\[ \frac{\partial}{\partial u_o} \left( \Delta \Phi^* \right) = \frac{4\gamma U_o a^2}{\pi} \left[ E\left(\frac{k^2}{a^2}, u_o\right) - K\left(\frac{k^2}{a^2}ight) \right] \frac{\sqrt{k^2 - u_o^2}}{u_o^3} - \frac{\sqrt{k^2 - u_o^2}}{u_o^3} \left[ \frac{(k^2, k^2)}{u_o^3} \right] \] (4.11)

In (4.11) \( K, E \) and \( \Pi \) are complete elliptic integrals of respectively the first, the second and the third kind. In order to obtain the potential distribution, (4.11) has to be integrated with respect to \( t \).

Taking into account the boundary conditions \( \Delta \Phi^*(k) = \Delta \Phi^*(-k) = 0 \)
one obtains (see appendix B):

\[
\Delta \Phi^*(u_o) = \sqrt{k^2 - u_o^2} \frac{\gamma u_o}{\pi} \left[ K(k) + \frac{a^2 - u_o^2}{u_o^2} \prod \left\{ \left( \frac{k}{u_o} \right)^2, \frac{1}{a^2} \right\} \right]
\]

(4.12)

In order to determine \( \lambda \) the lift has to be calculated:

\[
L = \rho_o U_o \int_{-b}^{b} \Delta \Phi \, dy = \rho_o U_o \int_{-k}^{+k} \Delta \Phi^* \frac{\partial y}{\partial u_o} \, du_o
\]

(4.13)

From (4.5) one obtains for \( \frac{\partial y}{\partial u_o} \):

\[
\frac{\partial y}{\partial u_o} = \frac{a^2}{u_o^2} \frac{1 - \sqrt{1 - \left( \frac{u_o}{a} \right)^2}}{\sqrt{1 - \left( \frac{u_o}{a} \right)^2}}
\]

(4.14)

Upon substitution of (4.14) and (4.12) equation (4.13) becomes:

\[
L = \lambda \frac{\gamma \rho_o U_o^2}{\pi} a^2 \int_{-k}^{+k} \sqrt{k^2 - u_o^2} \frac{1 - \sqrt{1 - \left( \frac{u_o}{a} \right)^2}}{\sqrt{1 - \left( \frac{u_o}{a} \right)^2}} \left[ K(k) + \frac{a^2 - u_o^2}{u_o^2} \prod \left\{ \left( \frac{k}{u_o} \right)^2, \frac{1}{a^2} \right\} \right] du_o
\]

(4.15)

Evaluation of this integral requires some tricks as is shown in appendix C. The result is:

\[
L = \lambda \frac{\gamma \rho_o U_o^2}{\pi} a^2 \left[ \left( \frac{\pi}{2} \right)^2 - \left\{ E(k) \right\}^2 - \frac{1 - \left( \frac{k}{a} \right)^2}{\left( \frac{k}{a} \right)^2} \left\{ K(k) - E(k) \right\}^2 \right]
\]

(4.16)

from which \( \lambda \) is found to be:

\[
\lambda = \frac{\pi L}{\gamma \rho_o U_o^2 a^2} \frac{1}{\left( \frac{\pi}{2} \right)^2 - \left\{ E(k) \right\}^2 - \frac{1 - \left( \frac{k}{a} \right)^2}{\left( \frac{k}{a} \right)^2} \left\{ K(k) - E(k) \right\}^2}
\]

(4.17)
Having found \( \lambda \) we are able to calculate the drag. By means of (4.4) we get:

\[
D = \frac{\pi L^2}{8 \rho_0 U^2 \alpha^3} \left\{ \left( \frac{1}{2} \right)^2 - \left( \frac{k}{\alpha} \right)^2 - \frac{1 - \frac{k}{\alpha}}{\left( \frac{k}{\alpha} \right)^2} \right\} \frac{1}{\left( \frac{k}{\alpha} \right)^2} \left( E \left( \frac{k}{\alpha} \right) - 2 \sqrt{1 - \left( \frac{k}{\alpha} \right)^2} \right)^2 \] (4.18)

or with

\[
\left( \frac{b}{a} \right)^2 = \frac{2 - \frac{k}{\alpha} - 2 \sqrt{1 - \left( \frac{k}{\alpha} \right)^2}}{\left( \frac{k}{\alpha} \right)^2}
\]

\[
D = \frac{L^2}{\pi b^2 \rho_0 U^2} \frac{\pi^2}{16} \left( \frac{1}{\left( \frac{1}{2} \right)^2} - \left( \frac{k}{\alpha} \right)^2 - \frac{1 - \frac{k}{\alpha}}{\left( \frac{k}{\alpha} \right)^2} \right) \left( E \left( \frac{k}{\alpha} \right) - 2 \sqrt{1 - \left( \frac{k}{\alpha} \right)^2} \right)^2 \] (4.19)

The lift distribution in spanwise direction can also be found.

\[
L(u) = \rho_0 U_0 \Delta \Phi^\pm \frac{L^2}{\alpha^4} \frac{k}{\alpha} u_0 - \frac{u_0}{u_0} \frac{2 - \frac{k}{\alpha} - \frac{2 \sqrt{1 - \left( \frac{k}{\alpha} \right)^2}}{\left( \frac{k}{\alpha} \right)^2} \left( E \left( \frac{k}{\alpha} \right) - 2 \sqrt{1 - \left( \frac{k}{\alpha} \right)^2} \right)^2}{\left( \frac{1}{2} \right)^2 - \left( \frac{k}{\alpha} \right)^2 - \frac{1 - \frac{k}{\alpha}}{\left( \frac{k}{\alpha} \right)^2} \right} \] (4.20)

where

\[
\frac{k}{\alpha} = \frac{2 \frac{b}{a}}{1 + \left( \frac{b}{a} \right)^2} \quad \text{and} \quad \frac{u_0}{\alpha} = \frac{2 \frac{b}{a} \frac{y}{b}}{1 + \left( \frac{b}{a} \right)^2 \left( \frac{y}{b} \right)^2}
\] (4.21)

Because of the simple nature of the curve \( C_1 \) it is a rather straightforward procedure to calculate the velocity potential distribution in the whole \( \bar{P} \) plane.

Formula (4.12) shows the behaviour of \( \Phi^\pm \Phi^\pm \) on the slit \( \nu = \nu_0, |\nu| \leq k \).

Since \( \Phi^\pm \) is antisymmetric in \( \nu \) (lift-problem), \( \Phi^\pm \) has to be zero everywhere on the real axis in the \( \bar{P} \) plane except on the slit \( C_2 \).

Here \( \Phi^\pm = \frac{\Delta \Phi^\pm}{2} \) and \( \Phi^\pm = - \frac{\Delta \Phi^\pm}{2} \).

The complete solution is obtained as the real part of the complex function \( \chi(\bar{P}) \):

\[
\Phi^\pm(u, \nu) = \Re \left\{ \chi(\bar{P}) \right\} = \Re \left[ \frac{\Delta \Phi^\pm(\bar{P})}{2} + \chi^\pm(\bar{P}) \right]
\] (4.22)
In (4.22) \( \frac{\Delta \Phi^*(\vec{p})}{2} \) is the complex value of the function \( \frac{\Delta \Phi}{2} \) given in formula (4.12) in a point corresponding to the complex coordinate \( \vec{p} \). \( \chi^*(\vec{p}) \) is still to be determined. The real part of \( \chi^*(\vec{p}) \) has to be zero on the real axis and the derivative of \( \chi^*(\vec{p}) \) provides the proper downwash on the slit \( C_2 \).

Thus for \( \chi(\vec{p}) \) can be written:

\[
\chi(\vec{p}) = \sqrt{k^2 - p^2} \left[ K\left(\frac{k}{a}\right) + \frac{\alpha^2 - p^2}{p^2} \frac{1}{\pi} \left\{ \frac{k}{p}, \frac{k}{a} \right\} \right] \frac{2 \eta U_o}{\pi} + \chi^*(\vec{p}) \quad (4.23)
\]

That branch of the square root in (4.23) is taken which possesses a positive value on the upperside of the slit \( C_2 \) (see appendix D). In order to determine \( \chi^*(\vec{p}) \) the velocity distribution will be calculated. By differentiation of (4.23) one obtains:

\[
\frac{d \chi(\vec{p})}{d \vec{p}} = U - i V = \frac{\partial \Phi^*}{\partial u} - i \frac{\partial \Phi^*}{\partial v} = \frac{2 \eta U_o \alpha^2}{\pi} \left[ \frac{E(k/a) - K(k/a)}{p \sqrt{k^2 - p^2}} \right] \frac{1}{\pi} \left\{ \frac{k}{p}, \frac{k}{a} \right\} + \frac{d \chi^*(\vec{p})}{d \vec{p}} \quad (4.24)
\]

Analysis of the behaviour of the elliptic integral of the third kind \( \frac{1}{\pi} \left\{ \frac{k}{p}, \frac{k}{a} \right\} \) shows a jump in the imaginary part of this integral across the slit \( C_2 \) (see appendix E). Thus the velocity distribution on the slit is given by:

\[
U^+ = \frac{2 \eta U_o \alpha^2}{\pi} \left[ \frac{E(k/a) - K(k/a)}{u \sqrt{k^2 - u^2}} \right] \frac{1}{\pi} \left\{ \frac{k}{u}, \frac{k}{a} \right\} + \Re\left\{ \frac{d \chi^*(\vec{p})}{d \vec{p}} \right\}
\]

\[
U^- = -U^+
\]

\[
V = \frac{-\eta U_o \alpha^2}{u^2 \sqrt{1 - \frac{1}{U_o^2}}} + \Im\left\{ \frac{d \chi^*(\vec{p})}{d \vec{p}} \right\} = \frac{-\eta U_o \alpha^2}{u^2 \sqrt{1 - \frac{1}{U_o^2}}} + \frac{\eta U_o \alpha^2}{u^2} \quad (4.25)
\]
The right hand side of equation (4.25) is obtained from the boundary conditions (4.7). From this equation (4.25) one obtains

\[ \chi^*(\phi) = i \frac{\lambda U_0 a^2}{\rho} \]  

Substitution of (4.26) in (4.23) yields the complete solution:

\[ \Phi^*(u, \nu) = \text{Re} \left[ \sqrt{k^2 - \frac{a^2}{\rho^2}} \left( K\left(\frac{k}{a}\right) + \frac{a^2 - \frac{a^2}{\rho^2}}{\rho^2} \pi \left( \frac{K\left(\frac{k}{a}\right)}{\frac{k}{a}} \right) \right) \frac{2\lambda U_0}{\pi} + i \frac{\lambda U_0 a^2}{\rho} \right] \]

4.4. Discussion.

In this section some properties of the drag, the lift distribution and the velocity near the mach cone are discussed in order to obtain some qualitative insight in the formulas derived in the previous section.

In examining formula (4.19) one of the first things to be noticed is that if an area of reference has to be found to make the lift- and drag force dimensionless, this formula shows that as far as optimisation is concerned the area of the circumscribing circle of the span \((\pi b^2)\) should be preferred above the usual projected wing planform area, because \(\pi b^2\) is a fixed quantity for a larger class of geometrical constraints.

If one compares the value of the lower bound of the drag of a wing with subsonic leading edges, a sonic trailing edge and a finite span as given by formula (4.19) with the drag of a flat plate wing with sonic trailing edges and with straight leading edges through the same maximum span points producing the same lift, a gain of the order of a few percent can be obtained (see fig. 7).

Figure 7 also shows curves of minimum drag under the constraint of fixed lift obtained by adding to the latter flat plate planform the proper camber and twist which corresponds to solutions of positive integer degree of homogeneity up to and within the degree two and three respectively. Compared with a flat plate wing with straight leading edges, the same maximum span points and a straight trailing edge perpendicular to the oncoming undisturbed flow direction the gain is much larger (see fig. 8).
In the same way it is possible to compare the results of ref. [10] and [11] with the results obtained here. Refering to chapter 2 the value of the drag here obtained should be lower than that of reference [11]. But a different behaviour is found, see figure 8. So the conclusion could be that the result of ref. [11] obtained by "analogie electrique" should be modified. The results obtained here agree indeed with the results obtainable by combining solution of the wave equation of positive integer degree of homogeneity in a proper way to yield a minimum drag, (see fig. 8).

In fig. 9 the spanwise lift distribution is shown for different values of the common slenderness parameter $\beta \tau$ (where $\tau = \frac{b}{L}$ and $L^*$ is the value of the x-coordinate at $y = b$) which is related to the parameter $\frac{b}{a}$ by $\beta \tau = \frac{b}{a} \frac{b}{L^*}$ (see fig. 10)
It is noted that for small values of $\beta r$ even up to 0.5 the deviation from an elliptical distribution is hardly noticeable. Only in the neighbourhood of $\beta r = 1$ the distribution is qualitatively different from elliptical.

This feature can be examined by expanding the lift distribution (4.20) in powers of a small parameter $\frac{b}{a}$.

The result is:

$$L\left(\frac{y}{b}\right) = \frac{2}{\pi} \frac{L}{b} \sqrt{1 - \left(\frac{y}{b}\right)^2} \left[1 + \left(\frac{b}{a}\right)^6 \left(-\frac{3s}{32} - \frac{1}{2} \left(\frac{y}{b}\right)^2\right) + O\left(\left(\frac{b}{a}\right)^8\right)\right]$$

(4.27)

The deviation from elliptical appears only in the sixth power of the parameter $\frac{b}{a}$. Formula (4.27) shows that within the class of wing planforms with sonic trailing edges a wing with minimum total drag shows an elliptical spanwise lift distribution up to and within the order of $\left(\frac{b}{a}\right)^4$. This type of lift distribution corresponds to minimum vortex drag. So it is correct up to and within the order of $\left(\frac{b}{a}\right)^4$ to minimize the vortex and the wave drag separately.

If a different kind of trailing edge is involved the boundary curve $C_1$ will have a different form. For small values of $\frac{b}{a}$ the new contour $C_1'$ can be considered as a disturbed circle where the disturbance of this circle is of order $\left(\frac{b}{a}\right)$. The qualitative character of the solution of the new Laplace problem can be obtained up to and including terms of $O\left(\left(\frac{b}{a}\right)^4\right)$ from the solution of the problem for a unit circle by means of a conformal mapping. The mapping may be written as:

$$p = q \left(1 + \frac{b}{a} f(q) + \left(\frac{b}{a}\right)^2 g(q) + O\left(\left(\frac{b}{a}\right)^3\right)\right)$$

(4.28)
This transformation maps the normalized new contour $C_1^*$ in the $q = u + iv$ plane up to and within $O\left(\frac{b}{a}\right)^3$ onto the unit circle in the $p = y + iz$ plane. The conformal mapping (4.28) has to show some symmetry-properties. If the disturbed trailing edge is left right symmetric, the disturbed circle will be symmetric with respect to the real- and the imaginary axis. Because of this symmetry, only even powers of $q$ appear in $f(q)$ and $g(q)$. The solution for the disturbed circle $\Phi^*$ has to show a constant normal derivative on the slit $C_2$. From the transformation follows:

$$
\left(\frac{\partial \Phi^*}{\partial n}\right)_{C_2} = \left(\frac{\partial \Phi}{\partial n}\right)_{C_2} \left\{ 1 + \frac{b}{a} f'(u) + \left(\frac{b}{a}\right)^2 g(u) + \left(\frac{b}{a}\right)^3 u f''(u) + \left(\frac{b}{a}\right)^2 u g'(u) + O\left(\left(\frac{b}{a}\right)^3\right) \right\} \quad (4.29)
$$

On the slit in the normalized plane $u = O\left(\frac{b}{a}\right)$ so taking into account the fact that $f(u)$ and $g(u)$ are even functions of $u$, the boundary condition (4.29) simplifies to:

$$
\left(\frac{\partial \Phi^*}{\partial n}\right)_{C_2} = \left(\frac{\partial \Phi}{\partial n}\right)_{C_2} \left\{ 1 + \frac{b}{a} K_1 + \left(\frac{b}{a}\right)^2 K_2 + O\left(\left(\frac{b}{a}\right)^3\right) \right\} \quad (4.30)
$$

where $K_1$ and $K_2$ are the constant terms of the functions $f$ and $g$. Thus a solution for the circle with $\Phi^*$ is constant on $C_2$ and $\Phi = 0$ on the circle shows after transformation the proper behaviour up to and within $O\left(\left(\frac{b}{a}\right)^3\right)$. Substituting the transformation (4.28) into the lift distribution (4.27) and taking into account that $u = O\left(\frac{b}{a}\right)$ on the wing one obtains:

$$
L\left(\frac{u}{b}\right) = \frac{b}{\pi b} \sqrt{1 - \left|\frac{u}{b}\right|^2} + O\left(\left|\frac{b}{a}\right|^3\right) \quad (4.31)
$$

So for any trailing edge with the proper left-right symmetry the optimal lift distribution is elliptical for small $\frac{b}{a}$ at least up to and within $O\left(\left(\frac{b}{a}\right)^2\right)$. Thus the uncoupling of wave drag and vortex drag also is correct for other types of trailing edges at least up to this order. This confirms the result of Adams and Sears, see ref. [1]. From the theory of homogeneous flows it is known that a lifting flat
plate delta wing with straight leading edges and a straight trailing edge perpendicular to the undisturbed oncoming flow direction possesses an elliptic spanwise lift distribution:

\[ \mathcal{L}(\frac{y}{b}) = \frac{\alpha}{\pi b} \sqrt{1 - \left(\frac{y}{b}\right)^2} \]  \hspace{1cm} (4.32)

The drag of this flat plate is:

\[ D = \frac{L^2}{2\pi \rho U_0^2 b^4} \left[ 2 E' - \sqrt{1 - (\sigma \epsilon)^2} \right] \]  \hspace{1cm} (4.33)

where \( E' \) is the complete elliptic integral of the second kind with modulus \( k = \sqrt{1 - (\sigma \epsilon)^2} \).

Expanding (4.33) in terms of the small parameter \( \frac{b}{\alpha} \) related to \( \sigma \epsilon \)

by \( \sigma \epsilon = \frac{b}{\frac{2\beta + \frac{b}{\alpha}}{\alpha}} \) one obtains:

\[ D_{fl. pl} = \frac{L^2}{2\pi \rho U_0^2 b^4} \left[ 1 - \frac{1}{4} \left(\frac{b}{\alpha}\right)^2 \ln \frac{b}{\alpha} + O \left(\frac{b}{\alpha}\right)^3 \right] \]  \hspace{1cm} (4.34)

If formula (4.19) is expanded for small \( \frac{b}{\alpha} \) one finds:

\[ D_{opt} = \frac{L^2}{2\pi \rho U_0^2 b^4} \left[ 1 + \frac{1}{4} \left(\frac{b}{\alpha}\right)^2 - \frac{33}{32} \left(\frac{b}{\alpha}\right)^6 + O \left(\frac{b}{\alpha}\right)^9 \right] \]  \hspace{1cm} (4.35)

As was shown above this result is also valid up to and including terms of \( O \left(\frac{b}{\alpha}\right)^3 \) for the optimum wing with straight trailing edge. Comparing the result of (4.35) with that of (4.34) it is clear that the improvement of the drag is of \( O \left(\frac{b}{\alpha}\right)^3 \) although the spanwise lift distributions are the same up to terms of \( O \left(\frac{b}{\alpha}\right)^3 \). So the major contribution to the gain in drag with respect to the flat plate should result from a different chordwise behaviour of the pressure distribution.

One of the interesting features of the optimum wing is the velocity behaviour in the neighbourhood of the mach cone. In order to examine the optimum velocity behaviour near the mach cone for a wing with sonic trailing edges, the velocity distribution on the upstream characteristic
surface has to be calculated. Substitution of (4.26) in (4.24) yields:

\[
\frac{d\chi}{dp} = U - iV = \frac{2\lambda U_o a^2}{\pi} \left[ \frac{E(k_c)}{k_c} - K(k_c) \right] - \frac{\sqrt{k_c^2 - p^2}}{p \bar{p}} \cdot \frac{\sqrt{1/\bar{p}}}{\bar{p} a} \cdot \left\{ \frac{k_c^2}{k_c^2 - a^2} \right\} - \frac{i\lambda U_o a^2}{\bar{p}^2}
\]

(4.36)

The circle \( C_1 \) corresponds to that part of the real axis where \(|u| > a\). On this interval one finds for the velocity distribution \( u = \sigma \) and if the proper branch of the square root is taken the vertical velocity can be expressed as:

\[
V = \frac{2}{\pi} \frac{\alpha}{|u| \sqrt{u^2 - k^2}} \left[ \frac{2}{\pi} \frac{E(k_c)}{k_c} - K(k_c) \right] - \frac{2}{\pi} \frac{\sqrt{u^2 - k^2}}{|u| \sigma} \cdot \left\{ \frac{K(k_c)}{k_c} \right\} + \frac{1}{u^2}
\]

(4.37)

where

\[
\left\{ \frac{K(k_c)}{k_c} \right\} = \frac{\sigma \alpha \mu (u) - E(k_c)}{\sqrt{u^2 - k^2} \sqrt{v(u)}^2 - 1}
\]

(4.38)

and

\[
\beta = \arcsin \frac{a}{|u|}
\]

The corresponding value of \( \left( \frac{\partial \Phi}{\partial r} \right) \) in the \( p \)-plane is found by means of the relation:

\[
\left( \frac{\partial \Phi}{\partial r} \right)_{r=a} = - (V) \left| \frac{dp}{d\rho} \right|
\]

(4.39)

For the value of \( p \) on the curve \( C_1 \), will be introduced:

\[
p = ae^{i(\pi - \psi)}
\]

(4.40)

Substitution of (4.40) in (4.39), (4.38), (4.37) and (4.5) yields:

\[
\left( \frac{\partial \Phi}{\partial r} \right)_{r=a} = - \lambda U_o \cos \psi \left[ \frac{2}{\pi} \frac{K(k_c)}{k_c} - \frac{2}{\pi} \frac{\sqrt{1 - k_c^2 \sin^2 \psi}}{k_c^2 \sin \psi} K(k_c^2) \right]
\]

\[
\left. \frac{2}{\pi} \frac{K(k_c) E(k_c) - E(k_c) F(k_c)}{\cos \psi} \sin \psi + 1 \right]\]

(4.41)
After substitution of \( \Phi \) (4.17) in (4.41) the distribution of \( \frac{\partial \Phi}{\partial r} \bigg|_{r=a} \) for fixed lift is obtained. In Figure 11 a few of these distributions are shown for some values of \( \frac{b}{a} \).  
\[
\left( \frac{b}{a} = \frac{l - \sqrt{l^2 - (\frac{b}{a})^2}}{k} \right) (c' = \frac{L}{\frac{1}{2} \rho \omega^2 \pi b^2})
\]

![Figure 11](image)

**fig.11**

Expansion of (4.41) in a Fourier series yields:

\[
\left( \frac{\partial \Phi}{\partial r} \right)_{r=a} = -\gamma U_0 \sum a_n \cos \Psi + a_3 \cos 3\Psi + a_5 \cos 5\Psi + \ldots
\]

\[
= -\gamma U_0 \sum_{n=0}^{\infty} a_{2n+1} \cos (2n+1)\Psi
\]

\[ (4.42) \]
Where the Fourier coefficients $a_{2n+1}$ can be obtained by integration of (4.41) in the usual way. The first three coefficients are calculated explicitly; the result is:

$$a_1 = \frac{(k^2_a - 1)}{(k^2_a)} \left( \frac{K(k_a) - E(k_a)}{\frac{\pi}{2}} \right)^2 - \left( \frac{E(k_a)}{\frac{\pi}{2}} \right)^2 + 1$$

$$a_3 = \frac{(k^2_a - 1)}{(k^2_a)} \left( \frac{K(k_a) - E(k_a) - 2}{\frac{\pi}{2}} \right)^2 - \left( \frac{E(k_a)}{\frac{\pi}{2}} \right)^2 + \frac{16}{g(k_a)^2} \left( \frac{E(k_a)}{\frac{\pi}{2}} \right)^2$$

$$+ \frac{(k^2_a - 1)}{(k^2_a)} \left( \frac{(k^2_a)^2 - 3}{g(k_a)^4} \right) \left( \frac{K(k_a) - E(k_a)}{\frac{\pi}{2}} \right)^2 - \left( \frac{E(k_a)}{\frac{\pi}{2}} \right)^2 + \frac{32}{g(k_a)^4} \frac{E(k_a)}{\frac{\pi}{2}} \frac{K(k_a) - E(k_a)}{\frac{\pi}{2}}$$

(4.43)

The distribution of $\frac{\partial^2\Phi}{\partial r^2} = a$ corresponding to the sum of the first three terms in the Fourier series is shown in fig. 12 for those values of the parameter $\frac{b}{a}$, which corresponds to the values of fig. 11.
Expansion of the Fourier coefficients for small values of \( \frac{k}{a} \) with respect to this small parameter \( \frac{k}{a} \) shows that the first term corresponding to a dipole velocity distribution is \( O\left( \frac{k}{a} \right) \) the second term is \( O\left( \frac{k}{a} \right)^2 \) and so forth. This result can also be obtained from direct expansion of (4.41). Keeping only the lowest order terms one obtains:

\[
\left( \frac{\partial \Phi}{\partial r} \right)_{r=a} = -\gamma U_0 \cos \psi \left[ \frac{(k/a)^2}{2} + O\left( \frac{k}{a} \right)^3 \right]
\]  

(4.44)

From the definition of \( \Phi(y,z) = \Phi(x=x(y,z), y, z) \) it follows that:

\[
\Phi_y = \varphi_y + \varphi_x \Phi_y
\]

\[
\Phi_z = \varphi_z + \varphi_x \Phi_z
\]

(4.45)

The condition that \( \varphi = 0 \) along the mach cone implies at \( C_1 \):

(see formula (4.40) for the definition of \( \psi \))

\[
\Phi_y = \sin \psi \Phi_r
\]

\[
\Phi_z = \cos \psi \Phi_r
\]

(4.46)

\[
\frac{1}{2 \cos \psi} \Phi_r = t g \psi \varphi_y + \varphi_z
\]

From these three equations (4.46) \( \varphi_x, \varphi_y \) and \( \varphi_z \) can be solved at \( C_1 \)

The result is:

\[
\varphi_x = -\frac{\Phi_r}{2 \beta}
\]

\[
\varphi_y = \frac{\Phi_r}{2} \sin \psi
\]

(4.47)

\[
\varphi_z = \frac{\Phi_r}{2} \cos \psi
\]

From the velocity components (4.47) one finds for the normal velocity jump across the mach cone:
\[ \varphi_n = -\frac{\sqrt{1+\beta^2}}{2\beta} \Phi_r \]  

(4.48)

where \( \varphi \) is the inward directed normal.

A simple geometrical observation learns that in this case of a circular

\( C_1 \), the following relation for \( \beta \) holds:

\[ \beta = \sqrt{M^2 - 1} = \frac{l}{2a} \]  

(4.49)

where \( l \) is the length of the wing.

Substitution of (4.49) and (4.44) in (4.48) yields:

\[ \varphi_n = \frac{1}{2} U_0 \cos \psi \sqrt{1+\beta^2} \left[ \left( \frac{b}{l} \right) \left( \frac{1}{\frac{b}{a}} \right) + \left( \frac{1}{l} \right) \left( \frac{b}{a} \right)^2 \right] \]  

(4.50)

It should be noted from (4.23) that both in the case of slender wings 

(\( \text{small} \ \frac{b}{l} \)) and \( M \rightarrow 1 \) (\( \text{small} \ \frac{b}{a} \)) the velocity jump vanishes. So in 

this region of values of the parameters \( \frac{b}{l} \) and \( \frac{b}{a} \) it would be 

expected that the optimisation procedure by means of solutions of the 

wave equation of integer degree of homogeneity better approaches the 

lower bound. This is indeed confirmed in the figures 7 and 8.
5. CONCLUSIONS.

One of the principal conclusions that can be drawn from the considerations in this report is the following. Although it is always possible to distinguish wave drag and vortex drag it has been shown that in general it is incorrect to split the total drag into wave drag and vortex drag, in order to optimise both parts separately. Only for special planforms these optimisation processes are compatible and a minimum total drag is obtained. It has been shown too that the separately optimisation procedure is indeed allowable within the framework of slender planforms up to and including the order of the so-called "not so slender" terms.

The velocity field around an optimum wing was studied as well. Examination showed a jump in the normal component of the disturbance velocity across the mach cone. Solutions of the wave equation of positive integer degree of homogeneity cannot possibly account for this jump. It has been found that a homogeneous solution of degree one-half provides the necessary velocity jump.

It has been shown that if the planform is not fixed a few arbitrary geometrical constraints have to be introduced in order to obtain a proper formulation of the optimisation problem. The optimum planform was found to be the planform with the maximum "area of entrainment". If the geometrical constraints allow the optimum wing to possess a supersonic leading edge then the "area of entrainment" is determined both by the leading and by the trailing edge. If however only subsonic leading edges are allowed, then the "area of entrainment" is only determined by the trailing edge.

Apart from the described difference a few other qualitative differences exist between optimum wings with subsonic and supersonic leading edges. The velocity jump across the downstream characteristic surface can be provided by a continuously curving of the leading edge if supersonic leading edges are allowed. In this case the optimum is obtained by distributing the downwash on the wing planform in such a way that the disturbance velocity potential satisfies the Laplace equation in the upstream characteristic surface.
However, if only subsonic leading edges are allowed and the span and trailing edge are given, then apart from the necessary square root singularity in the downwash the uniqueness of the solution is questionable. Since, although the vague constraint of subsonic leading edges suggests a lot of freedom and it seems that for every leading edge and thus planform there will be a downwash distribution which corresponds to the optimum and maybe for every downwash distribution a planform can be found in such a way that the disturbance velocity potential is the solution of a Laplace equation in the upstream characteristic surface, the counter example of the flat plate with sonic trailing edges (appendix F) proves this impression to be false. And thus the described freedom only exists within a limited set of combinations of planforms and downwash distributions and may be this set contains only one element or is an empty set, for the given geometrical constraints.
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Appendix A

Evaluation of the integral:

\[ \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u^2(u_0-u)} \left[ \frac{1}{\sqrt{1-(u_0^2)}} - 1 \right] \, du = (A.1) \]

The integral (A.1) can be split up into six integrals:

\[ \frac{1}{u_0^2} \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u} \left[ \frac{1}{\sqrt{1-(u_0^2)}} - 1 \right] \, du + \]

\[ \frac{1}{u_0} \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u^2} \left[ \frac{1}{\sqrt{1-(u_0^2)}} - 1 \right] \, du + \]

\[ \frac{1}{u_0^2} \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u_0-u} \left[ \frac{1}{\sqrt{1-(u_0^2)}} - 1 \right] \, du = \]

\[ \frac{1}{u_0} \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u \sqrt{1-(u_0^2)}} \, du - \frac{1}{u_0^2} \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u} \, du + \frac{1}{u_0} \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u \sqrt{1-(u_0^2)}} \, du \]

\[ \frac{1}{u_0} \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u^2} \, du + \frac{1}{u_0^2} \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u_0-u} \, du - \frac{1}{u_0} \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u} \, du \]  

The result of the first two integrals in (A.2) is zero because the integrands are antisymmetric with respect to \( u \) on the symmetric integration interval. The third and fourth integral are easily evaluated by means of the substitution \( u = k \sin \varphi \), the result is:

\[ \frac{1}{u_0} \left[ \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u^2 \sqrt{1-(u_0^2)}} \, du - \int_{-k}^{+k} \frac{\sqrt{k^2-u^2}}{u} \, du \right] = \frac{\pi - 2 \, E \left( \frac{k}{a} \right)}{u_0} (A.3) \]
where \( E(\frac{k}{a}) \) is a complete elliptic integral of the second kind with the parameter \( \frac{k}{a} \).

The last two integrals of (A.2) require some further examination because of the singularity in the integrand. By substitution of \( u = k \sin \psi \) these two terms in (A.2) can be rewritten as:

\[
\frac{k^2}{u_o^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1 - \sin^2 \psi)}{(u_o - k \sin \psi) \sqrt{1 - \left(\frac{k}{a}\right)^2 \sin^2 \psi}} \, d\psi - \frac{k^2}{u_o^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(1 - \sin^2 \psi)}{u_o - k \sin \psi} \, d\psi
\]  
(A.4)

Performing some algebra and making use of the symmetry properties of the integrand (A.4) can be rewritten as:

\[
\frac{2}{u_o} \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1 - \left(\frac{k}{u_o}\right)^2 \sin^2 \psi}} + \frac{2}{u_o} \left(\frac{k}{u_o}\right)^2 \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{(1 - \left(\frac{k}{u_o}\right)^2 \sin^2 \psi) \sqrt{1 - \left(\frac{k}{a}\right)^2 \sin^2 \psi}}
\]

\[
- \frac{2}{u_o} \int_{0}^{\frac{\pi}{2}} d\psi - \frac{2}{u_o} \left(1 - \left(\frac{u_o}{k}\right)^2\right) \frac{k}{u_o} \left\{ \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{u_o - \sin \psi} + \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{u_o + \sin \psi} \right\} =
\]

\[
\frac{2}{u_o} K(\frac{k}{a}) + \frac{2}{u_o} \left(\frac{k^2 - u_o^2}{k^2}\right) \Pi \left\{ \left(\frac{k}{u_o}\right)^2, \frac{k^2}{a^2}\right\} - \frac{\pi}{u_o} -
\]

\[
- \frac{k}{u_o^2} \left(\frac{k^2 - u_o^2}{k^2}\right) \left\{ \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{u_o - \sin \psi} + \int_{0}^{\frac{\pi}{2}} \frac{d\psi}{u_o + \sin \psi} \right\}
\]  
(A.5)

In (A.5) \( K(\frac{k}{a}) \) is a complete elliptic integral of the first kind and \( \Pi \left\{ \left(\frac{k}{u_o}\right)^2, \frac{k^2}{a^2}\right\} \) is a complete elliptic integral of the third kind \([2]\).

Depending on the sign of \( u_o \) the integrand in one of the last two integrals of (A.5) is infinite on the integration interval.

Suppose \( u_o > 0 \) then

\[
\int_{0}^{\frac{\pi}{2}} \frac{d\psi}{u_o/k + \sin \psi} = \frac{1}{\sqrt{1 - \left(\frac{u_o}{k}\right)^2}} \ln \left| 1 + \sqrt{1 - \left(\frac{u_o}{k}\right)^2} \right| \frac{u_o}{k}
\]  
(A.6)
and the other integral can be split into three parts:

\[
\int_{\frac{u_0}{k}}^{\frac{u_0}{k} + \epsilon} \frac{d\varphi}{u_0 - \sin \varphi} = \int_{0}^{\arcsin \left( \frac{u_0}{k} \right)} \frac{d\varphi}{u_0 - \sin \varphi} + \int_{\arcsin \left( \frac{u_0}{k} - \epsilon \right)}^{\arcsin \left( \frac{u_0}{k} + \epsilon \right)} \frac{d\varphi}{u_0 - \sin \varphi} = \\
= \frac{1}{\sqrt{1 - \left( \frac{u_0}{k} \right)^2}} \left[ \ln \left\{ \frac{\epsilon}{-1 + \frac{u_0}{k} \left( \frac{u_0}{k} - \epsilon \right) + \sqrt{1 - \left( \frac{u_0}{k} \right)^2} \sqrt{1 - \left( \frac{u_0}{k} - \epsilon \right)^2}} \right\} - \ln \left\{ \frac{\frac{u_0}{k}}{-1 + \sqrt{1 - \left( \frac{u_0}{k} \right)^2}} \right\} \right] + \\
+ \frac{1}{\sqrt{1 - \left( \frac{u_0}{k} \right)^2}} \left[ -\ln \left\{ \frac{\epsilon}{1 - \frac{u_0}{k} \left( \frac{u_0}{k} + \epsilon \right) + \sqrt{1 - \left( \frac{u_0}{k} \right)^2} \sqrt{1 - \left( \frac{u_0}{k} + \epsilon \right)^2}} \right\} \right] + \\
\varphi = \arcsin \left( \frac{u_0}{k} \right) + \epsilon
\]

(A.7)

substitution of \( t = \frac{\sin \varphi - \frac{u_0}{k}}{\epsilon} \) in the last integral in (A.7) transforms this integral into:

\[
\int_{\arcsin \left( \frac{u_0}{k} - \epsilon \right)}^{\arcsin \left( \frac{u_0}{k} + \epsilon \right)} \frac{d\varphi}{u_0 - \sin \varphi} = \int_{1}^{\frac{1}{\cos \left( \frac{u_0}{k} \right) \left( \cos \left( \frac{u_0}{k} \right) \right)^2}} \frac{\left( \frac{u_0}{k} + \epsilon t \right)^2}{-t} \, dt = \\
= \sqrt{1 - \frac{u_0^2}{k}} \int_{-1}^{1} \frac{dt}{t} + \frac{\frac{u_0}{k}}{\sqrt{1 - \frac{u_0^2}{k}}} (-\epsilon) \int_{-1}^{1} \, dt + O(\epsilon^2)
Substitution of the last result in (A.7) and performing the limit procedure for $\varepsilon \to 0$ leads to the following:

\[
\int_{0}^{\frac{\pi}{2}} \frac{d\varphi}{\frac{u_0}{k} \sin \varphi} = \frac{1}{\sqrt{1 - \left(\frac{u_0}{k}\right)^2}} \ln \left(1 - \frac{\frac{u_0}{k}}{\sqrt{1 - \left(\frac{u_0}{k}\right)^2}}\right) \quad (A.8)
\]

Summing of (A.6) and (A.8) yields:

\[
\int_{0}^{\frac{\pi}{2}} \frac{d\varphi}{\frac{u_0}{k} + \sin \varphi} + \int_{0}^{\frac{\pi}{2}} \frac{d\varphi}{\frac{u_0}{k} - \sin \varphi} = \frac{1}{\sqrt{1 - \left(\frac{u_0}{k}\right)^2}} \ln \left(1 - \frac{1 - \left(\frac{u_0}{k}\right)^2}{\left(\frac{u_0}{k}\right)^2} \right) = O(A.9)
\]

Substitution of (A.9) in (A.5) and thereupon substitution of (A.5) and (A.3) in (A.2) yields the final result for (A.1):

\[
\int_{-k}^{k} \frac{1}{\sqrt{\frac{k^2 - u_0^2}{u_0(u_0 - u)}} \left(\frac{1}{\sqrt{1 - (\frac{u_0}{k})^2}} - 1\right)} \, du = \frac{2}{u_0} \left[K\left(\frac{k}{a}\right) - E\left(\frac{k}{a}\right) + \frac{k^2 - u_0^2}{u_0^2} \left(\frac{k}{u_0}\right)^2 \right]
\]
Appendix B.

The potential distribution is given by:

$$
\Delta \Phi^*(u_0) = \frac{4 \pi u_0 a^2}{\sigma} \left[ \int \frac{E(k, \frac{u_0}{k}) - K(k)}{\frac{u_0}{k}} \, du_0 - \int \frac{\sqrt{k^2 - u_0^2}}{u_0^3} \prod \left\{ \left( \frac{k}{u_0} \right)^2, \frac{k}{a} \right\} \, du_0 \right] + C
$$

(B.1)

The integrals in this expression can be evaluated separately and the integration constant $C$ is determined by the boundary conditions:

$$
\Delta \Phi^*(k) = \Delta \Phi^*(-k) = 0
$$

(B.2)

The first integral in (B.1), is a simple one:

$$
\int \frac{du_0}{u_0 \sqrt{k^2 - u_0^2}} = \frac{1}{k} \ln \frac{u_0}{k + \sqrt{k^2 - u_0^2}}
$$

(B.3)

The second integral is a more complicated.

If \( \frac{k}{u_0} \gg 1 \), the elliptic integral of the third kind can be rewritten as:

$$
\prod \left\{ \left( \frac{k}{u_0} \right)^2, \frac{k}{a} \right\} = -\frac{k}{u_0} \frac{K(k)}{\sqrt{\left( \frac{k}{u_0} \right)^2 - 1}} \frac{Z(\frac{k}{u_0}, \frac{k}{a})}{\sqrt{\left( \frac{k}{u_0} \right)^2 - 1} \left( \frac{k}{a} \right)^2}
$$

(B.4)

where

$$
K(k) Z(\frac{k}{u_0}, \frac{k}{a}) = K(k) E(\frac{k}{a}, \frac{k}{a}) - E(\frac{k}{a}) F(\frac{k}{a}, \frac{k}{a})
$$

The angle $\theta$ is given by $\theta = \arcsin \frac{u_0}{k}$ and $F(\theta, k)$ and $E(\theta, k)$ are incomplete elliptic integrals of the first and second kind respectively.

Substitution of (B.4) in the second integral of (B.1) yields

$$
\int \frac{\sqrt{k^2 - u_0^2}}{u_0^3} \prod \left\{ \left( \frac{k}{u_0} \right)^2, \frac{k}{a} \right\} \, du_0 = \frac{1}{2k} K(k) \left( \frac{E(\frac{k}{a}, \frac{k}{a})}{\sqrt{\left( \frac{k}{u_0} \right)^2 - 1}} \right) - \frac{1}{2k} E(\frac{k}{a}) \left( \frac{F(\frac{k}{a}, \frac{k}{a})}{\sqrt{\left( \frac{k}{u_0} \right)^2 - 1}} \right)
$$

(B.5)
Integrating (B.5) by parts there is obtained:

\[
\frac{1}{u_0} \int \left[ \frac{k^2 - u_0^2}{u_0} \Pi\left\{ \left( \frac{k}{u_0} \right)^2, \frac{k}{a} \right\} \right] \, du_0 - \frac{\sqrt{1 - \left( \frac{u_0}{a} \right)^2}}{u_0} \left[ K\left( \frac{k}{a} \right) E\left( \frac{k}{a}, \frac{k}{a} \right) - E\left( \frac{k}{a} \right) F\left( \frac{k}{a}, \frac{k}{a} \right) \right] -
\]

\[
= -K\left( \frac{k}{a} \right) \int \frac{1 - \left( \frac{u_0}{a} \right)^2}{u_0 \sqrt{k^2 - u_0^2}} \, du_0 - E\left( \frac{k}{a} \right) \int \frac{du_0}{u_0 \sqrt{k^2 - u_0^2}}
\]

\[
= - \frac{1 - \left( \frac{u_0}{a} \right)^2}{u_0^2} \sqrt{k^2 - u_0^2} \, \Pi\left\{ \left( \frac{k}{u_0} \right)^2, \frac{k}{a} \right\} - K\left( \frac{k}{a} \right) \frac{1}{k} \ln \frac{u_0}{k + \sqrt{k^2 - u_0^2}} +
\]

\[
+ \frac{K\left( \frac{k}{a} \right)}{a^2} \left( - \sqrt{k^2 - u_0^2} \right) + E\left( \frac{k}{a} \right) \frac{1}{k} \ln \frac{u_0}{k + \sqrt{k^2 - u_0^2}} \quad (B.6)
\]

Substitution of (B.6) and (B.3) in (B.1) yields:

\[
\Delta \Phi^*(u_0) = \frac{\eta^2 u_0}{\pi} \sqrt{k^2 - u_0^2} \left[ K\left( \frac{k}{a} \right) + \frac{a^2 - u_0^2}{u_0^2} \, \Pi\left\{ \left( \frac{k}{u_0} \right)^2, \frac{k}{a} \right\} \right] + \zeta
\]

The boundary conditions (B.2) are satisfied if \( \zeta = 0 \).
Appendix C

Calculation of the lift force

The lift is given by:

\[
L = \frac{4 \pi \rho_0 U_0^2}{\pi} a^2 \int_{-k}^{+k} \left[ \frac{\sqrt{k^2 - u_0^2}}{u_0^2} \left( 1 - \frac{1 - \left(\frac{u_0}{\alpha}\right)^2}{\alpha^2} \right) \right] \left[ K(\frac{k}{\alpha}) - \frac{a^2 - u_0^2}{\alpha^2} \right] \frac{u_0}{\alpha^2} \left\{ \left( \frac{k}{\alpha} \right)^2, \frac{k}{\alpha} \right\} \, du_0
\]

\[
= \frac{4 \pi \rho_0 U_0^2}{\pi} a^2 \left[ K(\frac{k}{\alpha}) \right] \left\{ \pi - 2 E(\frac{k}{\alpha}) \right\} - a^2 \int_{-k}^{+k} \left[ \frac{1 - \left(\frac{u_0}{\alpha}\right)^2}{\alpha^2} \right] \left[ K(\frac{k}{\alpha}) \right] \left\{ \left( \frac{k}{\alpha} \right)^2, \frac{k}{\alpha} \right\} \, du_0
\]

\[
= \frac{4 \pi \rho_0 U_0^2}{\pi} a^2 \left[ K(\frac{k}{\alpha}) \right] \left\{ \pi - 2 E(\frac{k}{\alpha}) \right\} - a^2 \int_{-k}^{+k} \left[ \frac{1 - \left(\frac{u_0}{\alpha}\right)^2}{\alpha^2} \right] \left[ K(\frac{k}{\alpha}) \right] \left\{ \left( \frac{k}{\alpha} \right)^2, \frac{k}{\alpha} \right\} \, du_0
\]

where \( A = \arcsin \frac{u_0}{k} \).

Integrating by parts, the last integral in (C.1) can be rewritten as:

\[
\int_{-k}^{+k} \left[ \frac{1 - \left(\frac{u_0}{\alpha}\right)^2}{\alpha^2} \right] \left[ K(\frac{k}{\alpha}) \right] \left\{ \left( \frac{k}{\alpha} \right)^2, \frac{k}{\alpha} \right\} \, du_0 =
\]

\[
\int_{0}^{+k} \left[ K(\frac{k}{\alpha}) \right] \left\{ \left( \frac{k}{\alpha} \right)^2, \frac{k}{\alpha} \right\} \left\{ \frac{1 + \left(\frac{u_0}{\alpha}\right)^2}{\alpha^2} + \frac{1}{\alpha^3} \ln \frac{\frac{u_0}{\alpha}}{1 + \sqrt{1 - \left(\frac{u_0}{\alpha}\right)^2}} \right\} \, du_0
\]

\[
= \left[ \left[ K(\frac{k}{\alpha}) \right] \left\{ \left( \frac{k}{\alpha} \right)^2, \frac{k}{\alpha} \right\} \left\{ \frac{1 + \left(\frac{u_0}{\alpha}\right)^2}{\alpha^2} + \frac{1}{\alpha^3} \ln \frac{\frac{u_0}{\alpha}}{1 + \sqrt{1 - \left(\frac{u_0}{\alpha}\right)^2}} \right\} \right]_{u_0 = 0}^{+k} - K(\frac{k}{\alpha}) \int_{0}^{+k} \frac{1 + \left(\frac{u_0}{\alpha}\right)^2}{\alpha^2} \, du_0 + E(\frac{k}{\alpha}) \int_{0}^{+k} \frac{1 + \left(\frac{u_0}{\alpha}\right)^2}{\alpha^2} \, du_0
\]

\[
- K(\frac{k}{\alpha}) \int_{0}^{+k} \left[ \frac{1 + \left(\frac{u_0}{\alpha}\right)^2}{\alpha^2} \right] \left[ K(\frac{k}{\alpha}) \right] \left\{ \left( \frac{k}{\alpha} \right)^2, \frac{k}{\alpha} \right\} \, du_0
\]

\[
= \frac{4 \pi \rho_0 U_0^2}{\pi} a^2 \int_{-k}^{+k} \left[ \frac{1 - \left(\frac{u_0}{\alpha}\right)^2}{\alpha^2} \right] \left[ K(\frac{k}{\alpha}) \right] \left\{ \left( \frac{k}{\alpha} \right)^2, \frac{k}{\alpha} \right\} \, du_0
\]

(C.2)
For reasons of brevity (C.2) will be written as:

\[ J^* = J.P. + J(I) + J(II) + J(III) + J(IV) \]

Substitution of the boundary values in \( J.P. \) gives zero as result. Simple integration of \( J(I) \) and \( J(II) \) gives:

\[ J(I) = -\frac{K(k/a)}{\mu_0} \int_0^k \frac{1}{\sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}}} \left( \sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}} \right) d\nu = \frac{K(k)}{\kappa^2} \left\{ K(k) \right\} - \frac{\{K(k)\}^2}{\kappa^2} \frac{\pi}{2}, \quad (C.3) \]

\[ J(II) = \frac{E(k)}{\kappa} \int_0^k \frac{1}{\sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}}} \left( \sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}} \right) d\nu = \frac{E(k)}{\kappa} \left\{ E(k) - K(k) \right\} \quad (C.4) \]

\( J(IV) \) can be rewritten as:

\[ J(IV) = \frac{E(k/a^2)}{a^2} \int_0^k \ln \frac{\mu_0}{1 + \sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}}} d\nu(k/a^2) = \frac{E(k/a^2)}{a^2} \int_0^k \ln \frac{\mu_0}{1 + \sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}}} \sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}} d\nu = \]

\[ = \frac{E(k/a^2)}{a^2} \int_0^\pi \ln \frac{\mu_0}{\kappa a} \frac{1 + \sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}} \sin \nu}{\sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}} \sin \nu} d\nu = \]

\[ - \frac{E(k/a^2)}{a^2} \int_0^\pi \ln \frac{1 + \sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}} \sin \nu}{\sqrt{1 - \frac{(\nu \mu_0)^2}{\kappa^2}} \sin \nu} d\nu. \]
The last two integrals can be found in standard integral tables like ref. [12]. The result is: (see 4.386.3 and 4.317.7 of ref. [12]):

\[ J(\text{III}) = \frac{E(k)}{a^2} \left( -\frac{\pi}{4} K\left(\sqrt{1-\left(\frac{k}{a}\right)^2}\right) \right) \]  

(C.5)

Also, \( J(\text{III}) \) can be simplified as follows:

\[ J(\text{III}) = -\frac{K(k)}{a^2} \int_0^k \ln \frac{uk}{a} \frac{\sqrt{1-\left(\frac{uk}{a}\right)^2}}{\sqrt{k^2-uk^2}} du_0 = \]

\[ = -\frac{K(k)}{a^2} E\left(\frac{k}{a}\right) \ln \frac{k}{a} - \frac{K(k)}{a^2} \int_0^{\frac{\pi}{2}} \ln \sin \varphi \sqrt{1-\left(\frac{k}{a}\right)^2 \sin^2 \varphi} d\varphi + \]

\[ + \frac{\pi}{2} \frac{K(k)}{a^2} \int_0^{\frac{\pi}{2}} \ln \left(1 + \sqrt{1-\left(\frac{k}{a}\right)^2 \sin^2 \varphi}\right) \sqrt{1-\left(\frac{k}{a}\right)^2 \sin^2 \varphi} d\varphi \]  

(C.6)

The last two integrals in \( J(\text{III}) \) however cannot be found in standard tables of integrals. For the evaluation of these integrals both integrals are differentiated with respect to the parameter \( \left(\frac{k}{a}\right) \).

\[ \int_0^{\frac{\pi}{2}} \ln(\sin \varphi) \sqrt{1-\left(\frac{k}{a}\right)^2 \sin^2 \varphi} d\varphi = \alpha \left(\frac{k}{a}\right) \]

Differentiation with respect to \( \frac{k}{a} \) yields:

\[ \frac{d(\alpha)}{d\left(\frac{k}{a}\right)} = \frac{\alpha}{k/a} - \frac{1}{k/a} \left( \frac{\pi}{2} \right) \frac{\ln \sin \varphi}{\sqrt{1-\left(\frac{k}{a}\right)^2 \sin^2 \varphi}} d\varphi = (\text{see 4.386.3 of ref. [12]}) \]

\[ \frac{\alpha}{k/a} - \frac{1}{k/a} \left[ -\frac{\pi}{2} K\left(\frac{k}{a}\right) \ln \frac{k}{a} - \frac{\pi}{4} K\left(\sqrt{1-\left(\frac{k}{a}\right)^2}\right) \right] \]
This is an ordinary differential equation for \( \alpha \left( \frac{k}{a} \right) \).
The solution is:

\[
\alpha \left( \frac{k}{a} \right) = C \left( \frac{k}{a} \right) + \frac{\pi}{4} \left\{ E(\sqrt{1 - \left( \frac{k}{a} \right)^2}) - K(\sqrt{1 - \left( \frac{k}{a} \right)^2}) \right\} - \frac{E \left( \frac{k}{a} \right) \left\{ \log \frac{k}{a} - E \left( \frac{k}{a} \right) + \frac{1 - \left( \frac{k}{a} \right)^2}{2} K \left( \frac{k}{a} \right) \right\}}{2}
\]

A boundary condition is found for \( \frac{k}{a} = 1 \): \( \alpha (1) = -1 \)
so \( C = 0 \) and the result is:

\[
\int_0^{\pi} m \sin \varphi \sqrt{1 - \left( \frac{k}{a} \right)^2} \sin \varphi \, d\varphi = \frac{\pi}{4} \left\{ E(\sqrt{1 - \left( \frac{k}{a} \right)^2}) - K(\sqrt{1 - \left( \frac{k}{a} \right)^2}) \right\} - \frac{E \left( \frac{k}{a} \right) \left\{ \log \frac{k}{a} - E \left( \frac{k}{a} \right) + \frac{1 - \left( \frac{k}{a} \right)^2}{2} K \left( \frac{k}{a} \right) \right\}}{2}
\]

Applying the same method on the other integral one obtains:

\[
\int_0^{\pi} m \left( 1 + \sqrt{1 - \left( \frac{k}{a} \right)^2} \right) \sin \varphi \sqrt{1 - \left( \frac{k}{a} \right)^2} \sin \varphi \, d\varphi = \frac{\pi}{4} \left\{ K(\sqrt{1 - \left( \frac{k}{a} \right)^2}) - E(\sqrt{1 - \left( \frac{k}{a} \right)^2}) \right\} + \frac{E \left( \frac{k}{a} \right) \left\{ \log \frac{k}{a} - E \left( \frac{k}{a} \right) + \frac{1 - \left( \frac{k}{a} \right)^2}{2} K \left( \frac{k}{a} \right) \right\}}{2}
\]

Substitution of these results in (C.6) yields:

\[
J(\text{III}) = \frac{\pi}{2} \left[ E(\sqrt{1 - \left( \frac{k}{a} \right)^2}) - K(\sqrt{1 - \left( \frac{k}{a} \right)^2}) - 1 \right] \left[ - \frac{K \left( \frac{k}{a} \right)}{\alpha^2} \right]
\]

Summing \( J(\text{I}) \), \( J(\text{II}) \), \( J(\text{III}) \) and \( J(\text{IV}) \) and making use of Legendre's relation:

\[
K \left( \frac{k}{a} \right) E(\sqrt{1 - \left( \frac{k}{a} \right)^2}) - K \left( \frac{k}{a} \right) K(\sqrt{1 - \left( \frac{k}{a} \right)^2}) + E \left( \frac{k}{a} \right) K(\sqrt{1 - \left( \frac{k}{a} \right)^2}) = \frac{\pi}{2}
\]

one obtains for \( J^* \):

\[
J^* = \left( \frac{K \left( \frac{k}{a} \right) - E \left( \frac{k}{a} \right)}{K^*} \right)^2 - \left( \frac{\pi}{2} - K \left( \frac{k}{a} \right) \right)^2 \quad \text{(C.7)}
\]

Substituting (C.7) in (C.1) one finds for the lift:
\[ T = n \frac{4\pi \sigma u^2}{\pi} a^2 \left[ K\left( \frac{R}{a} \right) \left( \frac{e - 2E}{E} \right)^2 + \left( \frac{E}{E - K} \right)^2 - \frac{K\left( \frac{R}{a} \right) - E\left( \frac{R}{a} \right)^2}{(E)^2} \right] \]
Appendix D.

Analysis of the mapping \( w = k^{-1}z \)

Figure D1 shows the \( w \) and the \( z \) plane, where that branch of the mapping is chosen where \( w \) is real and positive if \( z \) approaches the slit from the upperside. If a path in the \( z \) plane is followed numbered from 1 to 12 than a corresponding path in the \( w \) plane exists connecting the corresponding numbered points from 1 to 12.

A simple observation made from this figure is that if that branch of the mapping is chosen which is pictured in this figure, than \( \sqrt{k^{-1}z} \) behaves like \(-iz\) if \( z \) approaches infinity along the positive real axis.
Appendix E.

Analysis of the behaviour of the complete elliptic integral of the third kind with complex parameter.

The integral is defined as:

\[
\Pi \left\{ \frac{(k')^2}{k} \frac{1}{(e_0^2 - \sin^2 \varphi)^{1/2}} \right\} = \frac{\pi}{2} \int_0^{\pi/2} \frac{d\varphi}{\sin^2 \varphi \sqrt{1 - (\frac{e_0}{k})^2 \sin^2 \varphi}}
\]

\[
= \frac{\pi}{2} \int_0^{\pi/2} \frac{d\varphi}{(k' - k \sin \varphi) \sqrt{1 - (\frac{e_0}{k})^2 \sin^2 \varphi}} + \frac{\pi}{2} \int_0^{\pi/2} \frac{d\varphi}{(k' + k \sin \varphi) \sqrt{1 - (\frac{e_0}{k})^2 \sin^2 \varphi}}
\]

(E.1)

Substitution of \( t = k \sin \varphi \) yields:

\[
\Pi \left\{ \frac{(k')^2}{k} \frac{1}{(e_0^2 - \sin^2 \varphi)^{1/2}} \right\} = \frac{\pi}{2} \int_0^{k} \frac{dt}{(\frac{e_0}{k} - t) \sqrt{k^2 - t^2}} + \frac{\pi}{2} \int_0^{k} \frac{dt}{(\frac{e_0}{k} + t) \sqrt{k^2 - t^2}}
\]

(E.2)

If \( \tilde{p} \) approaches the real axis when \( |Re \tilde{p}| < k \) one of the integrands in (E.2) shows a singularity. It is supposed that \( 0 < Re \tilde{p} < k \).

In that case the second integrand is regular on the integration interval but the first integral will need some further examination.

Introduction of \( \tilde{p} = u + i \varepsilon \) in the first integral yields:

\[
\int_0^k \frac{dt}{(u-t+i\varepsilon) \sqrt{k^2-t^2}} = \int_0^k \frac{[u-t-i\varepsilon] dt}{[u-t]^2 + \varepsilon^2} \frac{1}{\sqrt{k^2-t^2}} \frac{1}{\sqrt{1-(\frac{e_0}{k})^2}}
\]

(E.3)

The integration interval will be divided into three parts:

\[0 \leq t \leq u-\delta \quad u + \delta \leq t \leq k \quad \text{and} \quad u - \delta \leq t \leq u + \delta\]

On the first two intervals the integrand behaves regular and in the limit when \( \varepsilon \to 0 \) the result of these two parts is:

\[
\int_0^{u-\delta} \frac{dt}{(u-t) \sqrt{k^2-t^2}} + \int_{u+\delta}^k \frac{dt}{(u-t) \sqrt{k^2-t^2}} \frac{1}{\sqrt{1-(\frac{e_0}{k})^2}}
\]

(E.4)
On the remaining third part in the limit when the length of the interval approaches zero the integral can be given by:

\[ \int_{u-\delta}^{u+\delta} \frac{(u-t-\epsilon) \, dt}{\sqrt{k^2 - \omega^2} \, \sqrt{1 - \frac{u^2}{\omega^2}}} = \int_{u-\delta}^{u+\delta} \frac{u-t-\epsilon}{\sqrt{(u-t)^2 + \epsilon^2}} \, dt \tag{E.5} \]

By substitution of \( u-t = \xi \) (E.6) can be simplified to:

\[ \int_{-\delta}^{\delta} \frac{1}{\sqrt{k^2 - \omega^2} \, \sqrt{1 - \frac{u^2}{\omega^2}}} \, \xi \, d\xi - \frac{i \epsilon}{\sqrt{k^2 - \omega^2} \, \sqrt{1 - \frac{u^2}{\omega^2}}} \int_{-\delta}^{\delta} \frac{d\xi}{\xi^2 + \epsilon^2} = \]

\[ = 0 + \frac{i}{\sqrt{k^2 - \omega^2} \, \sqrt{1 - \frac{u^2}{\omega^2}}} \arctan \frac{\delta}{\epsilon} \bigg|_{\xi = -\delta}^{\xi = \delta} = -\frac{i}{\sqrt{k^2 - \omega^2} \, \sqrt{1 - \frac{u^2}{\omega^2}}} \arctan \frac{\delta}{\epsilon} \tag{E.6} \]

The result (E.6) represents the imaginary part of E.3. One observes that (E.6) shows a different sign depending on whether \( \epsilon \) is positive or negative when \( \epsilon \) approaches zero.

\[ \lim_{\epsilon \to 0^+} \arctan \frac{\delta}{\epsilon} = \frac{\pi}{2} \quad \text{and} \quad \lim_{\epsilon \to 0^-} \arctan \frac{\delta}{\epsilon} = -\frac{\pi}{2} \]

Thus the final result is:

\[ \lim_{\epsilon \to 0^+} \Pi \left\{ \left( \frac{k}{(u+i\epsilon)} \right)^2 \frac{1}{\omega} \right\} = \Pi \left\{ \left( \frac{k}{\omega} \right)^2 \frac{1}{\omega} \right\} - \frac{i \pi u}{2 \sqrt{k^2 - \omega^2} \sqrt{1 - \frac{u^2}{\omega^2}}} \]

\[ \lim_{\epsilon \to 0^-} \Pi \left\{ \left( \frac{k}{(u+i\epsilon)} \right)^2 \frac{1}{\omega} \right\} = \Pi \left\{ \left( \frac{k}{\omega} \right)^2 \frac{1}{\omega} \right\} + \frac{i \pi u}{2 \sqrt{k^2 - \omega^2} \sqrt{1 - \frac{u^2}{\omega^2}}} \]
Appendix F.

For a wing with sonic trailing edges the boundary conditions:

\[ \tilde{\phi}_2 = \varphi_2 + \rho_a \frac{\partial \varphi_2}{\partial z} = \text{const.} = -\eta U_o \quad (F.1) \]

simplifies to:

\[ \varphi_2 = -\eta U_o \quad (F.2) \]

Here \( \varphi_2 \) is the downwash at the trailing edge of the wing. In linearized theory this can be written as:

\[ \varphi_{e,e} = -\alpha_{e,e} U_o \quad (F.3) \]

From (F.2) it follows that for the optimum wing \( \eta = \alpha_{e,e} \). It has also been found that for the optimum wing the drag can be expressed as:

\[ D = \frac{\eta}{2} L \quad (F.4) \]

For a wing with sonic trailing edges this changes into

\[ D = \frac{\alpha_{e,e}}{2} L \quad (F.5) \]

If this optimum wing is supposed to be a flat plate then \( \alpha_{e,e} = \alpha \) and (F.5) becomes:

\[ D = \frac{\alpha}{2} L \quad (F.6) \]

For every flat plate wing the following expression holds:

\[ D = \alpha \frac{L}{2} - \text{suction force} \quad (F.7) \]

which implies that for the optimum flat plate wing with sonic trailing edges the suction force has to be one half of the drag without suction force included. This can only be attained in the slender body limit.