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MEAN AND LOW FREQUENCY WAVE DRIFTING FORCES ON FLOATING STRUCTURES

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Abstract—A recently developed method, based on three-dimensional potential theory, to compute the mean wave drifting forces on a free floating structure in regular waves, is extended to include low frequency oscillatory components which arise when the structure is floating in regular wave groups consisting of two regular waves with small difference frequency. This completes the information necessary for the determination of the wave drifting forces under arbitrary irregular wave conditions.

In regular wave groups the drifting forces not only depend on the first order velocity potential and the first order body motions, but also on the wave exciting forces due to the low frequency part of the second order potential. For the general three-dimensional case the latter contribution can only be determined numerically and at the expense of long computation times. Since this contribution is generally not large compared to components which may be determined using linear potential theory it is included using a simple approximation. Results of the method of approximation are compared with some two-dimensional cases for which exact solutions are known.

Results of computations of the total mean and low frequency surge forces on a rectangular barge and a column stabilized semi-submersible platform are presented. For both structures, the computed mean surge drifting forces in regular head waves are compared with results of model tests.

The computed components of the total mean drifting forces are presented. It appears that for both the barge and the semi-submersible the same components are of importance.

For the semi-submersible, the computed low frequency second order surge forces in head waves are compared with results obtained from a test in irregular head waves using cross-bi-spectral analysis methods.

NOMENCLATURE

ω_i, ω	frequency in rad/sec
ε_i	random phase angle
ζ_i	amplitude of component with frequency ω_i
N	a large number
P_{ij}, Q_{ij}	in- and out-of-phase components of the second order low frequency force
F_{ij}	amplitude of second order force
φ	total velocity potential including all orders
ε	a small parameter
$\varphi^{(1)}$, $\varphi^{(2)}$	first and second order approximations to the total potential
ρ	specific mass of water
g	acceleration of gravity
$R^{(1)}$	matrix containing first order oscillatory angular motions, see Pinkster and Van Oortmerssen (1977)
$\mathbf{n}'$	normal vector, positive outwards from body referenced to body axes
dl	length element along the waterline
m	mass of body in air
S	instantaneous wetted surface
S_0	mean wetted surface
$\mathbf{X}^{(1)}$	first order linear motion vector of a point on S_0
dS	surface element of S or S_0
$\mathbf{x}$	position vector of surface element in body axes
k, k_i	wave number
h	waterdepth

X_3	vertical co-ordinate of a point positive if above the mean surface
ζ_a	amplitude of a regular wave
$F^{(1)}$	first order wave exciting force or moment
g_{ij}	transformed gravity constant
$\ddot{\mathbf{X}}^{(1)}_g$	first order linear acceleration vector of the centre of gravity relative to the fixed system of co-ordinate axes
$\ddot{\mathbf{x}}_r^{(1)}$	first order angular acceleration vector
t	time
∇	displaced volume
$\mathbf{v}^{(1)}$	first order velocity of a point on S_0 relative to the fixed system of co-ordinate axes
$\mathbf{N}^*$	normal rector, including mean and oscillatory parts, of a surface element dS relative to the fixed system of co-ordinate axes
$\mathbf{N}^{(1)}$	first order oscillatory component of the normal vector $\mathbf{N}$ of a surface element dS relative to the fixed system of co-ordinate axes
$S\zeta(\omega)$	spectral density of irregular waves
μ	frequency of second order force
$\zeta^{(1)}_r$	first order relative wave height at a point along the waterline including effect of undisturbed waves, diffraction and body motions
L	length of the barge
d	draft of horizontal two-dimensional cylinder with breadth equal to $2d$
M	matrix containing mass of body
I	matrix containing moment of inertia
WL	waterline
$0{-}X_1{-}X_2{-}X_3$	fixed system of rectangular co-ordinate axes. Positive X_3-axis vertically upwards. Origin in mean free surface
$G{-}x_1{-}x_2{-}x_3$	system of body axes. Origin in center of gravity G. x_3-axis positive upwards. x_2-axis positive to port. x_1-axis positive forward.
$G{-}X_1{-}X_2{-}X_3$	system of axes parallel to $0{-}X_1{-}X_2{-}X_3$ system but with origin in center of gravity of the body. Coincides with $G{-}x_1{-}x_2{-}x_3$ system in the mean position.

INTRODUCTION

THE MEAN and low frequency second order wave drifting forces acting on structures moored in waves are usually of interest from the point of view of mooring loads and horizontal motions. Investigations into the nature of the wave drifting forces have generally stemmed from the need to include these effects in the total load on structures moored by anchor lines or cables (Hsu and Blenkarn, 1970; Remery and Hermans, 1972; Rye *et al.*, 1975; Arai *et al.*, 1976).

Such investigations were initially prompted by the fact that the observed horizontal motions of vessels moored in irregular waves and the associated mooring line forces showed low frequency components which could not be explained by linear wave theory alone. Van Oortmerssen (1976) demonstrated that the low frequency horizontal motions of a vessel moored to a jetty in irregular waves were in many cases directly attributable to the second order wave drifting force, thus underlining the necessity for reliable means of determining this part of the total wave forces.

The influence of the second order wave drifting forces is not only restricted to the horizontal plane. This is, for instance, indicated by Numata *et al.* (1976) in connection with the steady tilt of semi-submersible vessels in regular waves.

The presence of low frequency components in the vertical motions of large volume structures with small water plane area during model tests in irregular waves infers that non-linear hydrodynamic effects play a large part here also.

Up to now, however, the majority of the investigations concerning the mean and low frequency wave drifting forces are concerned with forces and motions in the horizontal plane. Faltinsen and Loken (1978) reviewed the available methods for calculating the horizontal drifting forces on fixed and floating objects. A comparable study on the methods to calculate the vertical wave drifting forces has not been presented yet.

This paper deals with a method to calculate the mean and low frequency order wave drifting forces which is applicable to all six degrees of freedom.

The method is based on direct integration of all contributions to the second order forces over the instantaneous wetted surface of the hull of the structure.

The expressions obtained are evaluated using numerical methods based on three-dimensional potential theory which includes the effect of water depth.

In the following the procedure to obtain the theoretical expression for the second order wave forces is reviewed.

The theoretical expressions are used to obtain quadratic transfer functions for the second order forces. The relationship between the slowly varying second order forces in irregular waves and the irregular waves is given.

In the Appendix the method of approximating the contribution due to the low frequency part of the second order potential is presented.

SECOND ORDER WAVE EXCITING FORCES ON FLOATING STRUCTURES

We assume that the fluid is homogeneous, inviscid, incompressible and irrotational. The fluid motions may be described by a velocity potential φ:

$$\varphi = \varepsilon\varphi^{(1)} + \varepsilon^2\varphi^{(2)} + \dots \quad (1)$$

In this expression the first order potential $\varphi^{(1)}$ and the second order potential $\varphi^{(2)}$ include the effects from the incoming waves, the diffraction due to the body and the body motions (excluding effects of low frequency motions which occur in response to the low frequency second order forces). If the potentials are defined relative to a fixed system of co-ordinates $0\text{-}X_1\text{-}X_2\text{-}X_3$, the pressure in a point on the hull can be determined using the Bernoulli equation:

$$p = -\rho g\, X_3 - \rho\varphi_t - \tfrac{1}{2}\rho \left| \nabla \varphi \right|^2 + p^{(0)}, \quad (2)$$

in which $p^{(0)}$ is the static pressure. This will be disregarded hereafter since it has no influence upon the final results.

Assuming that the body is carrying out small amplitude motions about the mean position, the first and second order components of the pressure in a point on the hull may be expressed in terms of the hydrostatic component due to the vertical motion of the point and in terms of the derivatives of the first and second order potentials taken at the mean position of the point. The pressure p becomes:

$$p = \varepsilon p^{(1)} + \varepsilon^2 p^{(2)}, \quad (3)$$

where

$$p^{(1)} = -\rho g\, X^{(1)}{}_3 - \rho\varphi^{(1)}{}_t \quad (4)$$

is the total first order pressure including the oscillatory hydrostatic component and

$$p^{(2)} = -\rho g\, X^{(2)}{}_3 - \rho\varphi^{(2)}{}_t - \tfrac{1}{2}\rho \left|\nabla\varphi^{(1)}\right|^2 - \rho(\mathbf{X}^{(1)} \cdot \nabla\varphi^{(1)}{}_t) \quad (5)$$

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is the total second order contribution, also including a hydrostatic component due to second order vertical motions. In further calculations the latter component is disregarded since it is part of the second order reaction force. The total fluid force acting on the vessel referenced to a rectangular system of co-ordinates G–X_1–X_2–X_3 which coincides with a body bound G–x_1–x_2–x_3 system of co-ordinates in the mean position follows from:

$$\mathbf{F} = -\iint_S p\,\mathbf{N}\,\mathrm{d}S. \tag{6}$$

In this expression the wetted surface S consists of a mean wetted surface S_0 which extends up to the mean waterline on the *structure*, and an oscillating part between the mean waterline and the instantaneous wave profile along the hull.

Also taking into account that the body is carrying out small oscillatory angular motions (pitch, roll and yaw) due to which the components of the normal vector N of a surface element dS along the axes of the fixed system of co-ordinates 0–X_1–X_2–X_3 contain a constant and an oscillatory part, it can be shown (see Pinkster, 1977) that the second order wave exciting forces follow from:

$$\mathbf{F}^{(2)} = -\tfrac{1}{2}\rho g \int_{WL} \zeta^{(1)2}{}_r\,\mathbf{n}\,\mathrm{d}l + R^{(1)}\,.\,M\,.\,(\ddot{\mathrm{x}}^{(1)}{}_g) - \iint_{S_0} \{-\tfrac{1}{2}\rho\,|\nabla\varphi^{(1)}|^2 - \rho\varphi^{(2)}{}_t$$
$$- \rho(\mathbf{X}^{(1)}\,.\,\nabla\dot{\varphi}^{(1)}{}_t)\}\,\mathbf{n}\,\mathrm{d}S. \tag{7}$$

A similar expression can be deduced for the second order moment:

$$\mathbf{M}^{(2)} = -\tfrac{1}{2}\rho g \int_{WL} \zeta^{(1)2}{}_r\,(\mathbf{x}\times\mathbf{n})\,\mathrm{d}l + R^{(1)}\,.\,(I\,.\,\dot{\mathrm{x}}^{(1)}{}_r) - \iint_{S_0} \{-\tfrac{1}{2}\rho\,|\nabla\varphi^{(1)}|^2 - \rho\varphi^{(2)}{}_t$$
$$- \rho(\mathbf{X}^{(1)}\,.\,\nabla\varphi^{(1)}{}_t)\}\,(\mathbf{x}\times\mathbf{n})\,\mathrm{d}S. \tag{8}$$

The expressions shown here differ from earlier eressions for the second order force and moment in so far that they have to be evaluated on the surface of the body and give insight in the mechanism by which the waves and vessel interact to produce the second order force.

We will see this upon closer inspection of the expression for the second order wave force and restrict ourselves to the surge force in irregular head waves.

The expression for the surge force becomes:

$$F^{(2)}{}_1 = -\tfrac{1}{2}\rho g \int_{WL} \zeta^{(1)2}{}_r\,n_1\mathrm{d}l + x_5{}^{(1)}\,.\,m\,.\,\ddot{\mathrm{X}}^{(1)}{}_{3g} - \iint_{S_0} \{-\tfrac{1}{2}\rho|\nabla\varphi^{(1)}|^2 - \rho\varphi^{(2)}{}_t$$
$$- \rho(\mathbf{X}^{(1)}\,.\,\nabla\varphi^{(1)}{}_t)\}\,n_1\mathrm{d}S. \tag{9}$$

The preceding expression contains three components: The first part is a line integral around the static waterline of the vessel of the square of the relative wave height. The relative wave height is the wave height as measured from the vessel. This contribution arises from the fact that near the surface the pressure in the waves can be approximated by the hydrostatic pressure. The hydrostatic pressure increase at the mean waterline on the hull is proportional to the relative wave height and the additional area on which the pressure acts is also proportional to the relative wave height. This results in an inwardly directed force proportional to the square of the relative wave height which has the form of the first part of the preceding equation.

The second contribution to the second order wave drifting force is a consequence of the choice of the axes to which the second order force is referenced. The expressions (7) and (8) apply to forces and moments relative to a G–X_1–X_2–X_3 system of axes with X_3-axis vertically upwards. The surge force given in expression (9) is along a horizontal X_1-axis and not along the moving longitudinal x_1 body axis. As a result of this, first order vertical hydrodynamic and hydrostatic forces acting along the vertical body axis G–x_3 will give a longitudinal second order force component along the X_1-axis according to:

$$F^{(1)}_3 \,.\, x^{(1)}_5 \text{ (see Fig. 1)},$$

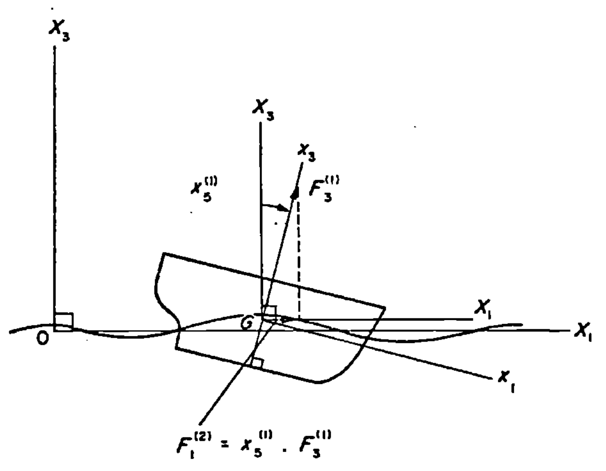

FIG. 1. Contribution of pitch angle and heave force to second order surge drifting force.

where: $F^{(1)}_3$ = first order vertical force,
$x^{(1)}_5$ = first order pitch angle.

Taking into account that the vertical force $F^{(1)}_3$ is the total fluid force which equals:

$$F^{(1)}_5 = m \,.\, \ddot{X}^{(1)}_{3g}, \tag{10}$$

this contribution becomes:

$$x^{(1)}_5 \,.\, m \,.\, \ddot{X}^{(1)}_{3g}. \tag{11}$$

The third contribution to the second order wave drifting force stems from the integration of all second order hydrodynamic pressures over the mean submerged part of the hull. These pressure contributions follow from the second order part of the Bernoulli equation (2) which, excluding the second order hydrostatic term, follow from:

$$p^{(2)} = -\rho\varphi^{(2)}{}_t - \tfrac{1}{2}\rho\,|\nabla\varphi^{(1)}|^2 - \rho(\mathbf{X}^{(1)}\cdot\nabla\varphi^{(1)}{}_t) \qquad (12)$$

in which: $\varphi^{(1)}$ = first order approximation for the total velocity potential $= \varphi^{(1)}{}_w + \varphi^{(1)}{}_d + \varphi^{(1)}{}_m$,

$\varphi^{(1)}{}_w, \varphi^{(1)}{}_d, \varphi^{(1)}{}_m$ = first order velocity potentials associated with respectively the undisturbed incoming wave, the diffraction potential and the body motion potential,

$\varphi^{(2)}$ = second order approximation for the total velocity potential consisting of contributions from the incoming waves, diffraction and body motions. Since we are restricting ourselves to the wave exciting forces this potential does not include the component due to second order motions of the body which are the reactions to the second order forces.

For vessels or floating structures at zero forward speed, the components of the second order wave drifting forces and moments which depend on product of first order quantities may be calculated using computer programs based on three-dimensional potential theory (Pinkster, 1976, 1977).

The components due to the second order potential $\varphi^{(2)}$ are determined using a simple approximation which is discussed in the Appendix.

The method is based on transformation of a first order wave exciting force in a long wave corresponding to the length of a wave group into a second order wave force. The accuracy of this method is indicated by comparison with some exact two-dimensional results. The method of approximation is applicable to three-dimensional cases and to shallow water.

Equations (7) and (8) for the second order wave forces and moments are not in a form which is easily manageable for practical cases. In Pinkster (1976) it is shown that, as is also the case with linear wave exciting forces, the second order wave exciting forces may be expressed in terms of (quadratic) transfer functions which, in turn, may be used to construct time records of the second order forces as indicated by Dalzell (1976) or to calculate the spectral density of the forces. A brief summary of the main characteristics of the low frequency part of the second order forces in relation to the waves is given in the following section.

LOW FREQUENCY SECOND FORCES IN IRREGULAR WAVES

In irregular waves the wave elevation in a point may be written as follows:

$$\zeta(t) = \sum_{i=1}^{N} \zeta_i \cos(\omega_i t + \varepsilon_i). \qquad (13)$$

The square of the wave elevation is:

$$\zeta^2(t) = \sum_{i=1}^{N}\sum_{j=1}^{N} \zeta_i\zeta_j \cos(\omega_i t + \varepsilon_i)\cos(\omega_j + \varepsilon)_j. \qquad (14)$$

The low frequency part of the square of the wave elevation is:

$$\zeta^{(2)}{}_{l}(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} \tfrac{1}{2}\zeta_i\zeta_j \cos\{(\omega_i - \omega_j)t + (\varepsilon_i - \varepsilon_j)\}. \tag{15}$$

The low frequency second order wave drifting forces are related to the square of the wave elevation and may be written as follows:

$$F^{(2)}(t) = \sum_{i=1}^{N} \sum_{j=1}^{N} \zeta_i\zeta_j P_{ij} \cos\{(\omega_i - \omega_j)t + (\varepsilon_i - \varepsilon_j)\}$$

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \zeta_i\zeta_j Q_{ij} \sin\{(\omega_i - \omega_j)t + (\varepsilon_i - \varepsilon_j)\}, \tag{16}$$

where:

P_{ij} = second order transfer function for the part of the force which is in-phase with the low frequency part of the wave height squared,

Q_{ij} = out-of-phase part of the second order transfer function.

These transfer functions are evaluated based on (7) and (8) and each contain contributions from the components discussed in the previous section.

As indicated by the subscripts i and j, P_{ij} and Q_{ij} are quantities which are a function of two frequencies ω_i and ω_j. In a physical sense, P_{ij} and Q_{ij} give the in- and out-of-phase components of the low frequency oscillating part of the second order wave exciting forces when the structure is placed in a wave train consisting of two regular waves with frequencies ω_i and ω_j. This is known as a regular wave group. The frequency of the oscillating part of the force equals the difference frequency $\omega_i - \omega_j$. This is also the frequency associated with the envelope of the group.

The total second order force in a regular wave group also contains a constant part which corresponds to the sum of the mean or constant forces exerted by each of the two regular waves which make up the regular wave group. In irregular waves the situation is the same. The mean force included in (16) is found by letting $\omega_i = \omega_j$:

$$F^{(2)}_{\text{mean}} = \sum_{i=1}^{N} \zeta^2_i P_{ii}. \tag{17}$$

The frequency transfer functions P_{ij} and Q_{ij} can be determined in such a way that the following relationships exist:

$$P_{ij} = P_{ji}, \tag{18}$$

$$Q_{ij} = -Q_{ji}. \tag{19}$$

The low frequency force in a regular wave group consisting of regular waves with frequency, amplitude and phase ω_1, ζ_1, ε_1 and ω_2, ζ_2, ε_2, respectively, is:

$$F^{(2)}(t) = P_{11}\zeta^2_1 + P_{22}\zeta^2_2 + (P_{12} + P_{21})\,\zeta_1\,\zeta_2 \cos\{(\omega_1 - \omega_2)t + (\varepsilon_2 - \varepsilon_2)\}$$

$$+ (Q_{12} - Q_{21})\,\zeta_1\zeta_2 \sin\{(\omega_1 - \omega_2)t + (\varepsilon_1 - \varepsilon_2)\}. \tag{20}$$

For the sake of brevity, in this paper use will be made of the amplitudes of the second order transfer functions. These are defined as follows:

$$F_{ij} = \sqrt{P^2_{ij} + Q^2_{ij}}. \tag{21}$$

For F_{ij} the following relationship exists:

$$F_{ij} = F_{ji}. \tag{22}$$

SPECTRAL REPRESENTATION OF MEAN AND LOW FREQUENCY SECOND ORDER WAVE EXCITING FORCES

Equations (16) and (17) give the total and mean second order wave exciting forces in irregular waves as a function of time for discrete values of the frequencies ω_i and ω_j.

If the spectral density $S\zeta(\omega)$ of the irregular waves is given, the mean second order wave drifting force may be determined from:

$$F^{(2)}_{\text{mean}} = 2 \int_0^\infty S_\zeta(\omega) \left[\frac{F^{(2)}}{\zeta_1\zeta_2}(\omega, \omega)\right] d\omega, \tag{23}$$

while the spectral density of the low frequency oscillatory part follows from:

$$S_F(\mu) = 8 \int_0^\infty S_\zeta(\omega)\, S_\zeta(\omega + \mu) \left[\frac{F^{(2)}}{\zeta_1\zeta_2}(\omega, \omega + \mu)\right]^2 d\omega, \tag{24}$$

where:

$\dfrac{F^{(2)}}{\zeta_1\zeta_2}(\omega, \omega)$ = mean second order wave drifting force in regular waves of frequency ω. Equivalent of F_{ii} in (21) with $\omega = \omega_i$,

$\dfrac{F^{(2)}}{\zeta_1\zeta_2}(\omega, \omega + \mu)$ = amplitude of the low frequency component of the wave drifting force. Equivalent of F_{ij} in (21) with $\omega_i = \omega + \mu$ and $\omega_j = \omega$ while $\mu \geq 0$.

MEAN SECOND ORDER WAVE DRIFTING FORCES IN REGULAR WAVES: COMPARISON BETWEEN EXPERIMENTS AND COMPUTATIONS

In order to check the method of calculation described in this paper, model tests were carried out in regular waves with the model of a rectangular barge and a column stabilized semi-submersible platform.

The lay-out and main particulars of the models are given in Fig. 2.

The model scale amounted to 1 : 50 for the barge and 1 : 40 for the semi-submersible. The waterdepth corresponded to 50 m and 40 m, respectively, in reality.

The tests were carried out in regular head waves and the models were moored by soft

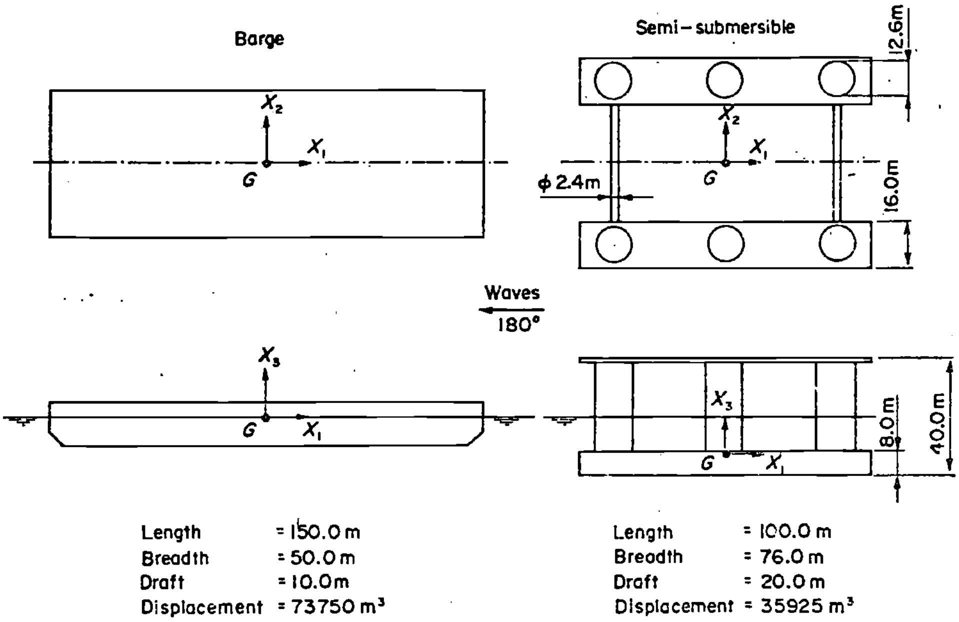

FIG. 2. The vessels.

spring mooring systems. This allowed the models to execute freely motions with wave frequency while applying a constant mooring force equal and opposite to the mean surge drifting force in the regular waves.

The results of the calculations are compared with measurements in Fig. 3.

The results are given as mean wave drifting force transfer functions which have been made non-dimensional using the displaced volume. The wave frequency has also been

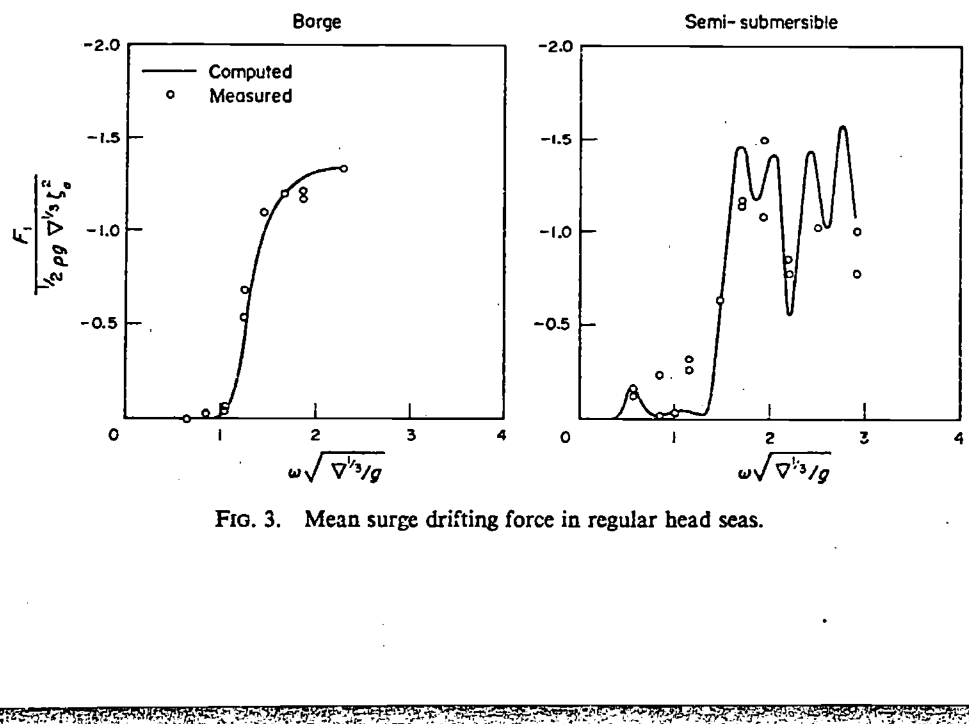

FIG. 3. Mean surge drifting force in regular head seas.

made non-dimensional using the displaced volume. This makes it possible to compare the drift forces on the barge and the semi-submersible on the basis of equal displacement.

The results given in the figures show good agreement between theory and experiment for both the barge and the semi-submersible. The agreement for the barge is not so surprising since one expects potential effects to dominate the wave drift forces for this type of hull. More surprising is the good agreement obtained for the semi-submersible. It appears that the wave drift force on such a structure is also dominated by potential effects.

This has been a matter of conjecture as is apparent from results given by Huse (1976) and Pijfers and Brink (1977).

Noteworthy in the results for the semi-submersible is the wavy character of the transfer function. It appears that this is due to the interference effect between the six columns. The low value of the transfer function at a non-dimensional frequency of 2.2 coincides with a wave length equal to the column spacing.

RELATIVE MAGNITUDE OF THE COMPONENTS OF THE COMPUTED MEAN SURGE DRIFTING FORCE IN REGULAR WAVES

In regular waves, the influence of the second order potential $\varphi^{(2)}$ on the second order wave drifting forces vanishes leaving only the following components:

I: Relative wave height contribution:

$$-\tfrac{1}{2}\rho g \int_{WL} \zeta^{(1)2}{}_{r}\, n_1 \,\mathrm{d}l. \tag{25}$$

II: Pressure drop due to velocity squared:

$$-\iint_{S_0} -\tfrac{1}{2}\rho\, \left|\nabla\varphi^{(1)}\right|^2 n_1 \,\mathrm{d}S. \tag{26}$$

III: Pressure due to product of gradient of first order pressure and first order motion:

$$-\iint_{S_0} -\rho\,(\mathbf{X}^{(1)} \cdot \nabla\varphi^{(1)}{}_{t})n_1 \,\mathrm{d}S. \tag{27}$$

IV: Contribution due to products of angular motions and inertia forces:

$$x^{(1)}{}_{5}\,.m.\ddot{X}^{(1)}{}_{3g}. \tag{28}$$

For the barge and the semi-submersible the components are indicated in Fig. 4 by the appropriate numerals.

The results shown in these figures show that in general the contribution I due to the relative wave height is dominant while the three remaining terms tend only to reduce somewhat the effect of contribution I. For high frequencies terms III and IV vanish since the body motions, on which these terms are dependent, reduce to zero in short waves. This leaves only the terms I and II.

For the semi-submersible it is apparent that it is the wave drift forces acting on the columns which form the greater part of the total horizontal wave drift forces.

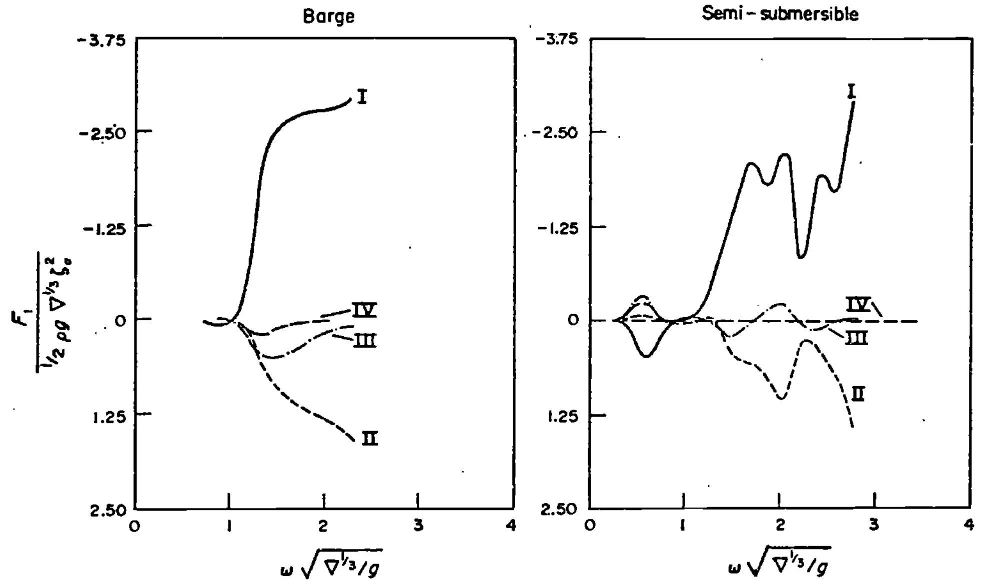

FIG. 4. Components of the computed mean surge drifting force.

Finally, the results of the computed mean vertical wave drifting force are given for both structures in Fig. 5.

The results indicate that the mean force on the barge is downwards while on the semi-submersible it is upwards. Since both vessels have vertical walls at the waterline, it is clear that contribution I is zero. It further appears that contribution II and also partly contribution III and IV are responsible for the vertical force.

In the case of the semi-submersible it will be clear that the vertical force is due to the second order pressures acting on the floaters.

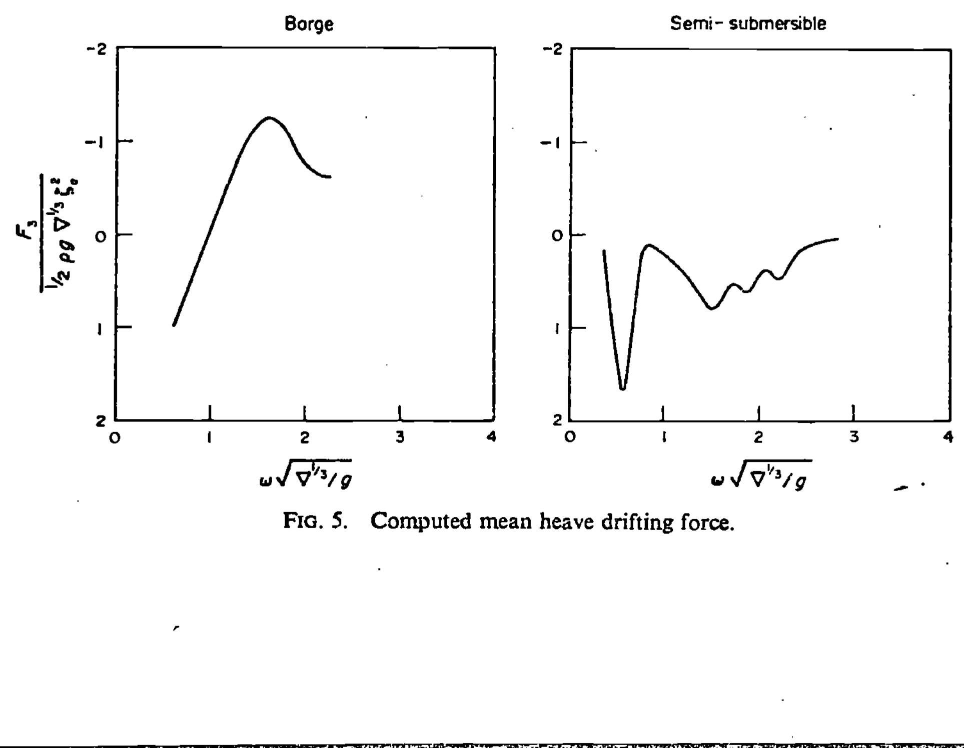

FIG. 5. Computed mean heave drifting force.

The peak in the response function of the mean vertical wave drift force is at the natural pitch and heave frequency.

COMPUTATION OF THE LOW FREQUENCY SECOND ORDER FORCES IN REGULAR WAVE GROUPS

For the barge and the semi-submersible, the quadratic transfer functions of the low frequency second order surge drifting forces in regular head wave groups have been computed.

The results are presented in the form of amplitude response functions of the total surge force in Table 1.

Since these are symmetrical about the diagonal (see (22)) only one side is given.

The results can be used to calculate the mean surge drifting force and the spectrum of the low frequency surge drifting force in irregular head waves as indicated in (23) and (24). The quadratic transfer functions for the forces have not been made non-dimensional. They are expressed in tf/m² for the vessel sizes indicated in Fig. 2. The values given on the diagonals represent the mean drifting forces which have been given in non-dimensional form in Fig. 3.

The negative sign for the components on the diagonal indicates that the mean force in head waves (180°) is in the direction of the negative X_1-axis.

In regular wave groups the contribution due to the second order potential $\varphi^{(2)}$ is not equal to zero. The absolute value of this contribution is shown in Table 2 for the barge and the semi-submersible.

Another way of judging the influence of the second order potential is to compare the transfer function which includes all components with the transfer function excluding the effect of $\varphi^{(2)}$. The latter case is presented in Table 3.

Comparison of Table 3 and Table 1 reveals that near the diagonal, the influence of the second order potential is small in the frequency range where the mean drifting force is large (higher frequencies) and increases in relative importance in the range of frequencies where the mean drifting force is small (low frequencies).

This means that in long waves where diffraction effects are small, the low frequency second order excitation due to the second order potential may be of some importance, whereas in the range of short waves where the low frequency excitation is dominated by second order effects arising from products of first order, the second order potential $\varphi^{(2)}$

TABLE 1. TOTAL MEAN AND LOW FREQUENCY SURGE DRIFTING FORCE IN HEAD WAVES

Barge

ω_2 \ ω_1	0.5	0.6	0.7	0.8	0.9	1.0	1.1
0.5	0	2	14	47	46	25	71
0.6		−9	17	48	53	19	74
0.7			−22	36	41	20	49
0.8				−26	32	31	24
0.9	$\frac{F^{(2)}}{\zeta_1\zeta_2}$ in tf/m²				−27	31	26
1.0						−29	28
1.1							−29

Semi-submersible

ω_2 \ ω_1	0.5	0.6	0.7	0.8	0.9	1.0	1.1
0.5	0	8	11	7	8	20	15
0.6		−1	6	7	5	14	19
0.7			0	7	15	3	18
0.8				−11	23	20	13
0.9	Frequencies in				−25	21	9
1.0	rad/sec					−20	21
1.1							−24

TABLE 2. CONTRIBUTION OF SECOND ORDER POTENTIAL TO SURGE DRIFTING FORCE

Barge

ω_2 \ ω_1	0.5	0.6	0.7	0.8	0.9	1.0	1.1
0.5	0	6	8	5	15	17	20
0.6		0	6	6	8	14	17
0.7			0	6	5	11	14
0.8				0	6	4	12
0.9					0	6	5
1.0						0	6
1.1							0

$\frac{F^{(2)}}{\zeta_1\zeta_2}$ in tf/m²

Semi-submersible

ω_2 \ ω_1	0.5	0.6	0.7	0.8	0.9	1.0	1.1
0.5	0	7	12	11	3	19	12
0.6		0	6	10	6	12	15
0.7			0	6	9	0	15
0.8				0	6	8	6
0.9					0	7	6
1.0						0	7
1.1							0

Frequencies in rad/sec

TABLE 3. SURGE DRIFTING FORCE EXCLUDING EFFECT OF SECOND ORDER POTENTIAL

Barge

ω_2 \ ω_1	0.5	0.6	0.7	0.8	0.9	1.0	1.1
0.5	0	8	22	48	33	30	54
0.6		−9	21	54	48	28	68
0.7			−22	42	44	19	54
0.8				−26	37	34	34
0.9					−27	36	29
1.0						−29	34
1.1							−29

$\frac{F^{(2)}}{\zeta_1\zeta_2}$ in tf/m²

Semi-submersible

ω_2 \ ω_1	0.5	0.6	0.7	0.8	0.9	1.0	1.1
0.5	0	1	1	5	6	3	7
0.6		−1	1	3	6	5	12
0.7			0	3	10	3	8
0.8				−11	21	18	8
0.9					−25	21	11
1.0						−20	20
1.1							−24

Frequencies in rad/sec

is of less importance. It is noted that the approximation for the force component due to $\varphi^{(2)}$ as given in this paper is expected to be most accurate when diffraction effects and effects of body motions are small. From this it is concluded that in the range of frequencies where the second order potential $\varphi^{(2)}$ is of importance, the approximation for this force component is closest to the correct value. The component due to $\varphi^{(2)}$ generally tends to increase the surge force on the semi-submersible while it tends to decrease the surge force on the barge.

DETERMINATION OF THE QUADRATIC TRANSFER FUNCTION FROM A TEST IN IRREGULAR WAVES

A test was carried out in irregular waves with the model of the semi-submersible to determine the low frequency second order surge force in head waves. From the measured record of the surge force and the simultaneously obtained wave elevation, it is possible to determine, using cross-bi-spectral methods, the quadratic transfer function for the low frequency surge force of which calculated results were presented in the previous section.

MOORING SYSTEM

In order to obtain the correct surge force, it is necessary to restrain the model in such a way that it is allowed to carry out freely the first order wave induced motions while restricting completely the low frequency surge motions induced by the low frequency second order surge force.

This can be easily seen from (7) and (8) for the second order forces. This requires a system of restrain which has the properties of an ideal low pass filter. Such a system does not exert forces on the vessel with wave frequencies but is infinitely stiff for surge motions at frequencies below the wave frequencies.

An analogous situation exists for dynamic positioning systems for drillships. In that case the thrust must be controlled in such a manner that the vessel is kept on station and the low frequency horizontal motions are reduced to a minimum while it is not permitted, from the point of view of wear of the propulsion units, to vary the thrust with frequencies corresponding to the wave frequencies.

A dynamic restraining system which can aid in achieving this and which can be used in model tests to restrain the model in the manner described here is in development at the N.S.M.B. It is discussed in Pinkster (1978) in relation to the dynamic positioning of drillships.

Instead of using a dynamic restraining system, it is also possible to make use of a less complex system using linear springs to restrain the model in surge. The stiffness of the mooring system must then be great enough so that little or no dynamic magnification occurs in the measured low frequency surge force while at the same time the mooring stiffness is not so great that the natural surge frequency is too close to the frequency range of the waves. In such a case the first order wave induced surge motions will not be affected significantly which means that the low frequency part of the measured force will be correct.

This system of restraint was used to moor the semi-submersible in irregular head seas. The model was moored by lines running horizontally fore and aft from the pontoons. Each line incorporated a force transducer. The stiffness of the mooring system resulted in a natural surge frequency of 0.4 rad/sec corresponding to about 15.8 sec full scale.

The spectrum of the irregular waves is given in Fig. 6. As can be seen, the natural surge frequency is just below the frequencies in the spectrum. In order to show the influence of the stiffness of the mooring system on the first order motions, the response functions of surge, heave and pitch are given in Fig. 7. The results for the very stiff mooring system are

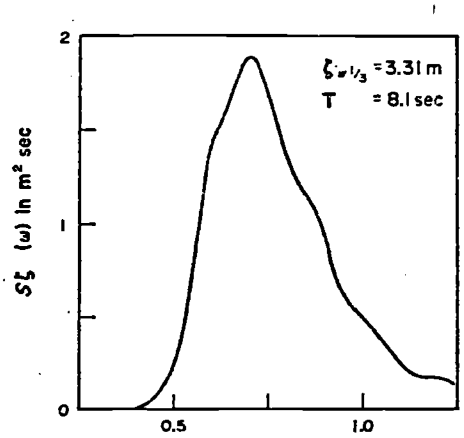

FIG. 6. Spectrum of irregular waves.

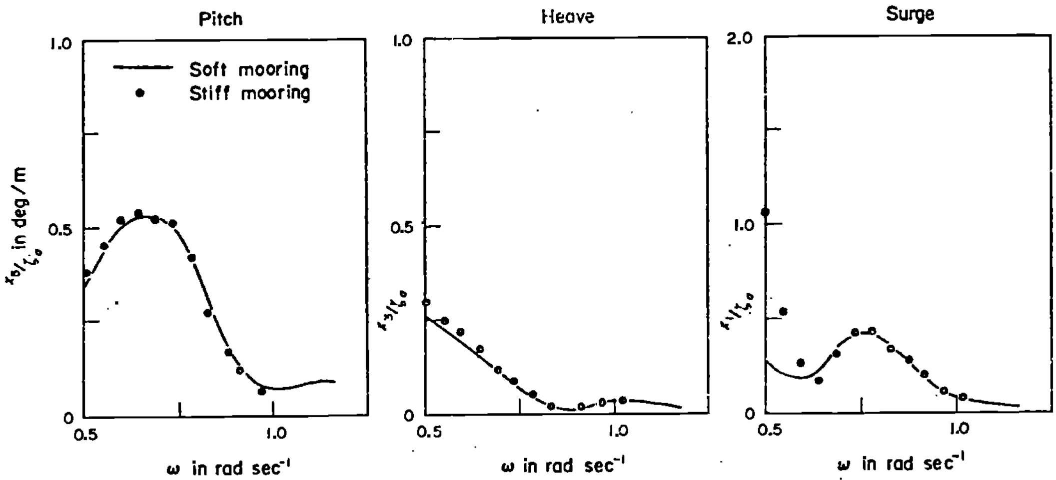

FIG. 7. First order motions of the semi-submersible in head waves.

compared to results for a very soft mooring system which gave a natural surge period of about 175 sec. The response functions were determined from tests in irregular head seas.

From the results it appears that the heave and pitch response functions are not affected by the stiff mooring system. The surge response function increases towards the lower frequencies as the frequency approaches the natural surge frequency of 0.4 rad/sec. In the frequency range of the major energy in the waves the surge response is unaffected.

From the comparison of the surge response function it appears that probably the mooring system was a little too stiff. The influence on the first order, wave frequency motions is, however, small.

From this it is concluded that the mean and the low frequency surge forces measured during the test are not significantly affected by the mooring system.

Assuming that the equation of motion for surge can be approximated by a second order differential equation it can be concluded that the dynamic magnification of the low frequency force is about 6% for a frequency of 0.1 rad/sec which corresponds to a period of 62.8 sec. Higher frequencies are not of practical interest since a real mooring system for such a vessel will almost certainly result in natural surge frequencies which are considerably lower than 0.1 rad/sec.

METHOD OF ANALYSIS

The total low frequency surge force measured in the mooring system was analysed by means of cross-bi-spectral methods based on work by Dalzell (1976).

By means of C.B.S. methods it is possible to determine the quadratic transfer function for the surge force in regular wave groups from the measured low frequency surge force and the irregular wave measured during the test. The wave elevation was measured some distance away from the model in a line parallel to the wave crests through the mean position of the center of gravity.

TEST DURATION

In order to have sufficient oscillations in the low frequency surge force on which to base the analysis, the test was carried out for a time duration corresponding to 6 hr in reality. During this time about 2700 waves passed the model. The low frequency surge force components with a frequency of 0.1 rad/sec (62.8 sec) contained about 340 oscillations.

COMPARISON OF RESULTS OF THEORETICAL COMPUTATIONS AND CROSS-BI-SPECTRAL ANALYSIS

The results are given in Fig. 8. In this figure the amplitude of the quadratic transfer functions of the mean surge drifting force and the low frequency surge drifting force with a frequency equal to 0.1 rad/sec are given in tf/m².

The computed values of the total force correspond to the numbers given in Table 1 on the diagonal (mean force) and in the first row beside the diagonal (low frequency force with frequency 0.1 rad/sec).

The computed values of the total force excluding the component of $\varphi^{(2)}$ correspond with the numbers given in Table 4 in the row next to the diagonal.

The results given in Fig. 8 shows a reasonable correlation between the measured and computed results.

The mean wave drifting forces compare well for frequencies higher than 0.75 rad/sec while the C.B.S. results are somewhat higher for lower frequencies.

The computed amplitude of the total low frequency force with a frequency of 0.1 rad/sec compares well with the results obtained by C.B.S. methods over the whole frequency range except for the spike at 0.92 rad/sec.

From the result it would appear that the second order potential contribution is indeed present at lower wave frequencies, since the total force excluding the effect of $\varphi^{(2)}$ gives a significantly lower value.

This conclusion is perhaps not fully supported, however, since in the same range of wave frequencies the C.B.S. results tend to over-estimate the mean surge force transfer function.

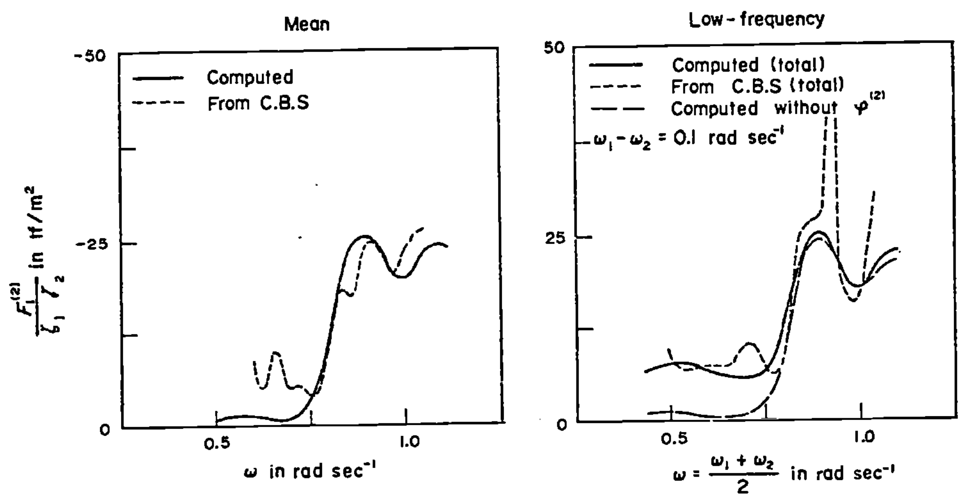

FIG. 8. Comparison of mean and low frequency surge drifting forces.

TABLE 4. LOW FREQUENCY DRIFTING FORCES ON A CYLINDER IN BEAM SEAS DUE TO THE SECOND ORDER POTENTIAL

Exact

	0.59	0.72	0.84	0.95	1.12	$\omega_1\sqrt{d/g}$
0.59	0	0.02	0.01	0.02	0.13	
0.72		0	0.03	0.01	0.10	
0.84			0	0.03	0.04	$\dfrac{F^{(2)}}{\rho g L\zeta_1\zeta_2}$
0.95				0	0.06	
1.12					0	
$\omega_2\sqrt{d/g}$						

Approximation

	0.59	0.72	0.84	0.95	1.12	$\omega_1\sqrt{d/g}$
0.59	0	0.03	0.11	0.20	0.33	
0.72		0	0.03	0.11	0.26	
0.84			0	0.04	0.18	$\dfrac{F^{(2)}}{\rho g L\zeta_1\zeta_2}$
0.95				0	0.10	
1.12					0	
$\omega_2\sqrt{d/g}$						

It should be considered, however, that the C.B.S. method of analysis obtains the quadratic transfer function for the force by an averaging process over the number of oscillations occurring in the measured force signal at the (low) frequency of interest. This means that the transfer function for the surge force at a frequency of 0.1 rad/sec has been obtained by averaging over about 340 oscillations. The mean force transfer function is obtained from averaging over an oscillatory component with frequency zero. The accuracy of results for the mean force transfer function becomes questionable when taking this into account. At present, there is no clear indication of the accuracy which may be expected from cross-bi-spectral methods other than that the accuracy increases with the length of the record.

CONCLUSIONS

It has been shown that the mean and low frequency second order wave drifting forces can be obtained through the method of direct integration of pressures on the instantaneous wetted surface of the hull of a floating structure. The theoretical expressions have been evaluated using numerical methods based on three-dimensional potential theory.

Comparisons of the computed mean surge drifting force in regular head waves with result of measurements show good agreement for the case of a rectangular barge and for a column stabilized semi-submersible.

This implies that, also for a semi-submersible, potential effects dominate in the wave drifting forces.

It appears that for the barge and the semi-submersible the contribution of the relative wave height to the mean wave drifting force is dominant.

Comparisons between the computed values of the mean and low frequency surge forces on the semi-submersible and the values obtained from a test in irregular head seas using cross-bi-spectral methods show reasonable correlation.

More information is needed concerning the accuracy of cross-bi-spectral methods when these are used to obtain transfer functions for very low frequency components in the measured forces.

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APPENDIX A

CONTRIBUTION OF THE LOW FREQUENCY SECOND ORDER POTENTIAL $\varphi^{(2)}$

When a body is floating in a regular wave group consisting of two regular waves with frequencies ω_i and ω_j, part of the second order wave exciting forces are due to the second order velocity potential $\varphi^{(2)}$ as was already indicated by (7) and (8).

The second order potential $\varphi^{(2)}$ has to be determined taking into account the following boundary conditions; see Pinkster (1979):

At the mean free surface,

$$g\varphi_{x^{(2)}_3} + \varphi^{(2)}_{tt} = -2.\nabla\varphi^{(1)}.\nabla\varphi^{(1)}_t + \varphi^{(1)}_t\left(\varphi_{x_3x^{(1)}_3} + \frac{1}{g}\varphi_{ttx^{(1)}_3}\right), \tag{29}$$

At the mean position of the body,

$$\nabla\varphi^{(2)}.\mathbf{n} = -\{X^{(1)}.\nabla)\nabla\varphi^{(1)})\}.\mathbf{n} + (\mathbf{v}^{(1)} - \nabla\varphi^{(1)}).\mathbf{N}^{(1)}, \tag{30}$$

At the sea floor,

$$\frac{\partial\varphi^{(2)}}{\partial n} = 0, \tag{31}$$

Everywhere within the fluid,

$$\Delta\varphi^{(2)} = 0, \tag{32}$$

Besides these conditions also a radiation condition has to be satisfied.

Faltinsen and Loken (1979) have given a solution for the low frequency wave exciting force due to second order potential $\varphi^{(2)}$ which is valid for the two-dimensional case with infinite waterdepth. For practical cases this may be used to estimate the wave drifting forces on a vessel in beam waves.

For the general three-dimensional case no analytical solution has been given and only numerical methods can be applied. The main difficulties lie in the inhomogeneous free surface boundary condition (29). The right-hand side of this equation contains the total solution to the first order problem. The total first order potential $\varphi^{(1)}$ contains contributions from the undisturbed incoming waves, diffraction waves and waves due to first order body motions.

If the total force due to the second order potential is to be determined numerically by means of distributions of sources then, in order to satisfy the free surface boundary condition of (29), the sources must not only be distributed over the hull but also over the free surface. This method will require a lot of computer space and computing time. It is not clear at this time whether the importance of this contribution justifies such a step.

As indicated by Newman (1974) and also inferred by results obtained by Faltinsen and Loken, it appears that for many practical cases the contribution due to the second order potential will be small. This conclusion is justified when regarding the low frequency wave drifting forces on structures which have very low natural frequencies due to their large effective mass in relation to the stiffness of the mooring systems. As a consequence of the low damping at the natural frequency, the wave drifting force components with the same frequency as the natural frequency are of importance.

It can be shown that, as the frequency approaches zero, the contribution to the drifting forces due to the second order potential vanishes leaving only contributions which are due to products of first order quantities.

Results obtained by Bowers (1975) on the low frequency surge motions of a barge in irregular head waves indicate that, as the natural surge frequency is increased by increasing the stiffness of the mooring system, the influence of contributions to the low frequency drifting force related to the second order potential increase also. The general conclusions to be drawn at this time is that for systems which have very low natural frequencies the second order potential will not be very important but for systems with higher natural frequencies it may be necessary to include a contribution due this potential. Systems with low natural frequencies in the horizontal motions are for instance vessels moored to single point mooring systems in deep water. In such cases natural periods in the order of several hundreds of seconds are not uncommon. Higher natural frequencies are found in, for instance, the vertical motions of semi-submersible type structures which have natural periods of 20–40 sec or in the horizontal motion of vessels moored to jetties. Due to the stiffness of the mooring systems natural periods of the horizontal motions of 30–100 sec occur in such cases.

APPROXIMATION FOR THE CONTRIBUTION OF THE SECOND ORDER POTENTIAL

From the afore-going it will be clear that at least an indication of the magnitude of the contribution of the second order potential to the low frequency wave drifting forces in relation to other components is desirable.

To this end an approximative method can be used based on linear potential theory which is applicable in both two and three dimensions. The approximation is based on the assumption that the major part of the low frequency second order force due to the second order potential is the wave exciting force component due to the contribution of the undisturbed incoming waves to the second order potential.

This means that in the right hand side of the free surface boundary condition of (29) only terms involving the first order velocity potential of the undisturbed incoming waves remain. The second order potential which satisfies this boundary condition and the boundary condition at the sea-floor of (31) as well as the equation of continuity (32) has, for instance been given by Bowers (1975).

If the first order velocity potential associated with the undisturbed incoming regular wave group consisting of two regular waves is as follows:

$$\varphi^{(1)} = - \sum_{i=1}^{2} \frac{\zeta_i g}{\omega_i} \frac{\cosh k_i (X_3 + h)}{\cosh k_i h} \sin (k_i X_1 + \omega_i t + \varepsilon_i). \tag{33}$$

Then the low frequency component of the second order velocity potential associated with these waves is as follows:

$$\varphi^{(2)} = - \sum_{i=1}^{2} \sum_{j=1}^{2} \zeta_i \zeta_j A_{ij} \frac{\cosh \{(k_i - k_j)(X_3 + h)\}}{\cosh (k_i - k_j)h} \sin \{(k_i - k_j)X_1 + (\omega_i - \omega_j)t + (\varepsilon_i - \varepsilon_j)\} \qquad \omega_i \geq \omega_j \tag{34}$$

in which A_{ij} is a coefficient depending on ω_i, ω_j and the waterdepth and is given by Bowers (1975).

The low frequency component of this second order potential represent long waves which are induced by the presence of the regular wave group. The phase of this long wave relative to the regular wave group is such that it has a trough where the wave group attains its maximum wave height. This is shown in Fig. 9.

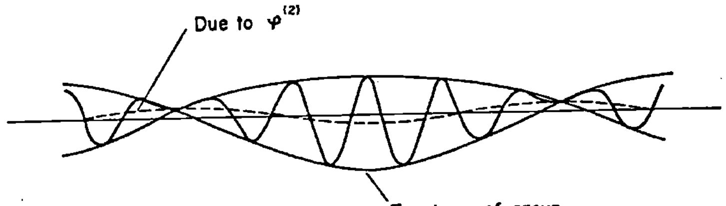

FIG. 9. Wave due to second order potential in a wave group.

The method for approximating the component of the second order wave exciting force due to this potential is based on the assumption that the long waves associated with the potential may be viewed as separate waves.

The potential associated with such a wave does not satisfy the boundary condition on the body which, for the simplified case, is assumed to be equivalent to the normal first order boundary condition. This means that the right hand side of (30) is zero. This means that an additional diffraction potential must be introduced. This potential need only satisfy the equation of continuity, the boundary condition at the sea-floor, the radiation condition and the homogeneous free surface condition:

$$g\varphi_{x_3} + \varphi_{tt} = 0. \tag{35}$$

This last condition gives rise to the well known dispersion relationship:

$$\omega^2 = kg \tanh kh. \tag{36}$$

The incoming waves have a wave number equal to $k_i - k_j$ and wave frequency equal to $\omega_i - \omega_j$.

These waves do not obey the dispersion law Equation (36). If the incoming waves have a frequency of $\omega_i - \omega_j$ then the diffracted waves have the same frequency but the wave number will be according to the relationship.

$$(\omega_i - \omega_j)^2 = kg \tanh kh. \tag{37}$$

In order to simplify the situation we allow the diffraction waves to have the same wave number $k_i - k_j$ as the incoming waves. This means that differences will occur in the diffracted waves further away from the vessel. Close to the body the situation will be unchanged, since the boundary condition at the body still has to be satisfied. The reason for this alteration in wave number of the diffraction waves will be apparent from the following.

We have now reduced the problem to the situation where we have to determine the wave exciting force on the body due to a wave which has a velocity potential as given by (34) while we allow the diffracted waves to have the same wave number and frequency as the incoming waves.

This is solved by considering the ordinary first order wave exciting force $F^{(1)}$ on the body in a regular wave with wave number k equal to $k_i - k_j$ in an ordinary gravity field with g as constant of gravity. For such a case the associated wave frequency ω will be in accordance with the dispersion relationship of (34). The frequency of this wave can be made equal to the frequency of the second order waves by selecting a different value for the constant of gravity:

$$g_{ij} = \frac{(\omega_i - \omega_j)^2}{(k_i - k_j)\tanh(k_i - k_j)h} \qquad \omega_i \gg \omega_j. \tag{38}$$

Since the wave exciting force is proportional to the constant of gravity, the initial force $F^{(1)}$ with wave with frequency ω which follows from (36) becomes a second order force with frequency $\omega_i - \omega_j$ by simply applying the factor

$$n_{ij} = \frac{g_{ij}}{g} \tag{39}$$

to the initial force.

This does not complete the transformation, however, since besides satisfying the requirement that wave number and wave frequency be equal, the amplitudes of the potentials must be equal. After the alteration of the constant of gravity, the transformed potential of the first order regular wave is:

$$\varphi = -\frac{\zeta_a g_{ij}}{(\omega_i - \omega_j)} \cdot \frac{\cosh \{(k_i - k_j)(X_3 + h)\}}{\cosh (k_i - k_j)h} \cdot \sin \{(k_i - k_j) X_1 + (\omega_i - \omega_j)t + (\varepsilon_i - \varepsilon_j)\} \qquad \omega_i \geqslant \omega_j. \tag{40}$$

The amplitude of the second order potential is given in (34). Equality of the amplitudes means that:

$$\frac{\zeta_a g_{ij}}{(\omega_i - \omega_j)} = \zeta_i \zeta_j A_{ij} \qquad \omega_i \geqslant \omega_j. \tag{41}$$

This means that the first order wave amplitude must be selected so that:

$$\zeta_a = \zeta_i \zeta_j \frac{A_{ij}(\omega_i - \omega_j)}{g_{ij}} \qquad \omega_i \geqslant \omega_j. \tag{42}$$

The first order force $F^{(1)}$ is determined for a value of unity for ζ_a.

Since forces are proportional to the wave amplitude, (42) gives a second correction factor which has to be applied to the force $F^{(1)}$ in order to give the required second order force $F^{(2)}$:

$$F^{(2)}_{ij} = n_{ij} \cdot \frac{\zeta_i \zeta_j A_{ij}(\omega_i - \omega_j) \cdot F^{(1}}{g_{ij}} \qquad \omega_j \geqslant \omega_i, \tag{43}$$

which taking into account (35) gives:

$$F^{(2)}_{ij} = f_{ij} \cdot F^{(1)}_{ij} \quad , \tag{44}$$

where

$$f_{ij} = \frac{\zeta_i \zeta_j A_{ij}(\omega_i - \omega_j)}{g} \qquad \omega_i \geqslant \omega_j. \tag{45}$$

COMPARISON BETWEEN THE EXACT RESULTS AND THE APPROXIMATION

It can be shown that this method of approximation gives exact results in two simple cases and gives a reasonable approximation for a third, more practical, case.

The first case concerns the second order pressure due to the second order potential in undisturbed irregular waves in a point $X_3 = -a$ below the still water level. The second order pressure is:

$$p^{(2)} = -\rho\, \varphi^{(2)}_t \quad . \tag{46}$$

For the low frequency component given in (34) the amplitude of the pressure is:

$$p^{(2)}_{ij} = \rho \zeta_i \zeta_j A_{ij}(\omega_i - \omega_j) \frac{\cosh \{(k_i - k_j)(-a + h)\}}{\cosh (k_i - k_j)\, h} \qquad \omega_i \geqslant \omega_j. \tag{47}$$

For the first order potential the pressure follows from:

$$p^{(1)} = -\rho\, \varphi^{(1)}_t. \tag{48}$$

The amplitude of the pressure using a first order potential component of the type given in (33), unit wave amplitude ζ_a and wave number $k_i - k_j$ is:

$$p^{(1)} = \rho g \frac{\cosh\{(k_i - k_j)(-a + h)\}}{\cosh(k_i - k_j)h} \qquad \omega_i \gg \omega_j. \tag{49}$$

Using the coefficient f_{ij} given in (45), gives the following approximation for the second order pressure amplitude:

$$p^{(2)}_{ij} = \rho\,\zeta_i\zeta_j\,A_{ij}\,(\omega_i - \omega_j)\,.\,\frac{\cosh\{(k_i - k_j)(-a + h)\}}{\cosh(k_i - k_j)h} \qquad \omega_i \gg \omega_j \tag{50}$$

which equals the exact value given in (47). The reason for this is that other contributions to the exact value which are neglected in the approximation namely, those due to diffraction and body motions are in this case zero.

The second case concerns the horizontal low frequency wave drifting force, due to the second order potential, acting on a vertical wall in deep water. It can be shown that the approximation is also equal to the exact result in this case. The reason for this is that the first order incoming waves and the first order outgoing waves are identical (total reflection) and the total second order potential consists of a contribution associated with the undisturbed incoming waves and a contribution due to the outgoing diffraction waves. Since the approximation gives the exact value for the second order potential associated with the incoming waves it also gives the exact value for the second order potential associated with the outgoing waves and hence the approximation is also the exact value.

The third example concerns the two-dimensional case of a free floating cylinder in beam waves as presented by Faltinsen and Loken (1979).

These authors solved the second order problem exactly and gave results on the contribution of the total second order potential to the low frequency second order sway force in regular wave groups in deep water. The method of approximation presented here was applied to the same case using results given by Vugts (1968) on the first order sway force in regular beam waves.

The coefficient f_{ij} of (45) becomes for deep water:

$$f_{ij} = -\frac{\omega_i(\omega_i - \omega_j)}{g}\,\zeta_i\zeta_j \qquad \omega_i \geqslant \omega_j. \tag{51}$$

The results are presented in the form of the amplitude of the low frequency second order force due to the second order potential for a range of combinations of ω_i and ω_j which are the frequencies of two waves making up a regular wave group. The exact results are compared with the approximation in Table 4.

Comparison of the results show that near the line $\omega_i = \omega_j$ the approximation is good but for larger differences between ω_i and ω_j the approximation is considerably higher than the exact value. Further study will be required to determine the reason for the large differences which occur at higher difference frequencies. At the present, however, it may be tentatively concluded that the method of approximation gives the right order of magnitude to the low frequency forces due to the second order potential for difference frequencies which are not too large. For the cylinder in beam seas large differences between the exact results and the approximation occurred for values of the non-dimensional difference frequency greater than about 0.1.

The approximation will give best results when the contribution to the second order potential of the first order diffraction waves and waves due to body motions are negligible. This requirement is probably satisfied more by vessels such as semi-submersibles than by ordinary ship or barge shapes.

APPENDIX B

SYMMETRY OF THE QUADRATIC TRANSFER FUNCTION OF THE COMPONENT DUE TO $\varphi^{(2)}$

The approximation for $F^{(2)}_{ij}$ as given in (44) is only defined for $\omega_i \gg \omega_j$. In order to conform with the definition given in (18)–(22) for the quadratic transfer function, the in- and out-of-phase parts of the force component due to $\varphi^{(2)}$ become:

$$P_{ij} = \tfrac{1}{2}P^{(2)}_{ij} \qquad \omega_i \gg \omega_j, \tag{52}$$

$$P_{ji} = P_{ij} \qquad , \tag{53}$$

$$Q_{ij} = \tfrac{1}{2}Q^{(2)}_{ij} \qquad \omega_i \geqslant \omega_j, \tag{54}$$

$$Q_{ji} = -\,Q_{ij} \tag{55}$$

In these equations $P^{(2)}_{ij}$ and $Q^{(2)}_{ij}$ represent the in- and out-of-phase components of $F^{(2)}_{ij}$ as determined by (44).

This transformation was applied to the example concerning the free floating cylinder in beam waves and also to the results concerning the barge and the semi-submersible.