Geoid Anomalies due to Low-Viscosity Zones in Glacial Isostatic Adjustment Modeling

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Preface

This master thesis work was carried out at the Section of Astrodynamics and Satellite Systems at the faculty of Aerospace Engineering of Delft University of Technology. Earth oriented research in the group belongs to Delft Earth Oriented Space Research (DEOS). As much as a master thesis project is a solo project, it can never be completed without the help of others. I would like to acknowledge the following persons.

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A vital contribution came from prof. P.P. Wu of the University of Calgary, who gave me the opportunity to compare results with his model. For me, this was an important step in the computation process, allowing me to develop important insights and to gain confidence in the results. Also I am grateful for his quick and helpful correspondence and the possibility to contribute to a journal paper.

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Finally I was happy to have the support of my mother, father, Pa, friends and family. Most of their questions about how the thesis work was progressing were out of sincere interest I believe.

Wouter van der Wal
Abstract

Glacial isostatic adjustment (GIA) is the response of the Earth due to growth and melting of ice sheets. The effects of melting of the last Pleistocene ice sheets, which started roughly 18,000 years ago and took approximately 10,000 years, are an average global sea level rise of 120 m and a rebound of hundreds of meters in parts of Canada and Scandinavia. Isostatic equilibrium of former glaciated areas is still not reached, as is indicated by a large negative displacement of the geoid, a gravity equipotential surface that coincides with mean sea level.

In this study, present day geoid anomalies due to ice loading are computed using the pseudo-spectral method for solving the sea-level equation, on a radially symmetric, Maxwell visco-elastic Earth. Usually the top layer of the Earth model is taken to be elastic, however, seismic observations indicate the presence of one or more low-viscosity zones (LVZ) in the crust. The effect of depth, thickness and viscosity of such layers on the GIA induced geoid anomalies are investigated in this study, using a variety of Earth models. It is found that they introduce a perturbation signal on the geoid anomaly of up to 1 meter with a characteristic shape in the former glaciated area. Lowering the viscosity or increasing the thickness increases the perturbations; the effect of depth is more ambiguous. Inclusion of more ice age cycles in the simulation is shown to be significant.

The GOCE satellite, due for launch in 2006, will measure the Earth’s gravity field in high detail allowing the geoid to be determined with centimeter accuracy on a spatial scale of 100 kilometer. Also existing geoid models are moving towards centimeter accuracy. The effect of the presence of a low viscosity zone is therefore in principal detectable in future geoid models. However, topographic and hydrological effects are also represented in the geoid. Further work needs to be done before a comparison with the simulations of this study can be used to constrain the presence of a LVZ. However, with the relations between LVZ parameters and their geoid perturbation signal put forward in this study it is expected that a future GOCE derived accurate geoid can be used to determine the presence of a LVZ.
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Chapter 1

Introduction

The timescale of geodynamic processes ranges from seconds, for earth quakes to millions of years for mantle convection. On intermediate timescales we find the loading of the Earth by Pleistocene ice sheets. In the last 2.4 million years, the Earth underwent several ice ages during which large ice masses covered vast areas in the northern hemisphere. Below the ice sheets, adjustment took place to balance the weight of the ice sheets, mainly by flow of mantle material. Melting of the ice has been recorded to occur in short time, compared to the build up. Therefore, after the ice sheets had disappeared, mantle material had not yet had the time to completely restore the equilibrium. At present, 18,000 years after the last ice age reached its maximum, we can observe land uplift of a centimeter per year in the center of Scandinavia. Gravity anomalies are observed over the former ice-covered land, indicating that equilibrium has not been reached yet.

The effects of Glacial Isostatic Adjustment (GIA) as the total spectrum of responses of the Earth to glacial loading or unload is called, manifest in the height of the geoid, or the geoid anomaly. The geoid is the equipotential surface which coincides with mean sea level. It is an important surface in geodesy for height system. In this study we are interested in the geoid because it has recorded the total of the Earth’s response due to GIA. The goal for the near-future is to reach an accuracy on the centimeter level for the determination of the geoid. With two gravity measuring satellite mission underway and a high-resolution satellite mission, the Gravity Field and Ocean Steady-state Explorer (GOCE) due for launch in 2006, that goal is within reach. But also existing (regional) geoid models, which are a combination of surface gravity measurements and all kinds of corrections, are approaching the centimeter level accuracy.

This accuracy enables the investigation of small scale signals recorded in the geoid. One such a signal is expected to come from GIA, when the uppermost uppermost, rigid layer of the Earth is replaced by a more ductile layer sandwiched between two brittle layers. For several reasons, explained in this report, it is expected that such a shallow low-viscosity zone (LVZ) is present at some areas. The goal of this study is to use mathematical models of the GIA process, using an Earth model with a LVZ in different shapes, to simulate the effect that this layer has on the geoid heights. This is so-called forward modelling with the ultimate goal to derive from accurate geoid models the possible presence of a LVZ. Thus, the long-term goal is to contribute to the knowledge of the Earth’s interior. A more accurate viscosity profile of the Earth contributes to issues as diverse as mantle convection and plate tectonics and present day ice mass balance.

In the following the structure of this report will be clarified. The first chapter introduces the main terms used in this thesis and presents a physical background of GIA. The mathematical model is built up from the theory of the response of the Earth, in chapter 2. Using this analysis, the response of the Earth to a realistic model of the ice sheet chronology can be calculated, by solving the so-called sea level equation. This is described in detail in chapter 3. Chapter 4 describes the main input parameters: the radial viscosity profile of the Earth and the model for the glaciation history. In chapter 5 the main results are presented and discussed. The report ends with a chapter of conclusions and recommendations for future research.
Chapter 2

Introduction to Global Isostatic Adjustment and the Geoid

This thesis describes the effect of shallow low-viscosity zones on the geoid height due to Glacial Isostatic Adjustment (GIA) or Post-Glacial Rebound (PGR) as it is also called. GIA is the term for the response of the Earth to glacial loads, the most significant of which occurred in the Pleistocene. Noticeable effects of melting of the last great ice sheets are the increase in sea level by roughly 120 meter since the Last Glacial Maximum (LGM) and the rebound of the former glaciated areas in Scandinavia and Canada. GIA is of special interest to geophysicists because it is a unique way of retrieving information about the interior of the Earth, in particular the viscosity of the mantle. Besides its usefulness in determining the Earth’s interior, knowledge of the current rebound process is also important to accurately estimate the GIA part of global sea level rise.

GIA simulations strongly lean on the available knowledge of the Earth’s interior. The first section of this chapter presents the layers of the Earth which will be used throughout this report. The viscosity profile belongs to the category of Earth’s interior parameters, but it is postponed until chapter 4, where it is presented together with all the other input models. Once the layering profile of the Earth is outlined, the physical context of the growth and retrieval of the Pleistocene ice sheets is sketched in section 2.3. The radial displacement and changing gravity potential still influences geophysical observables at the surface. This thesis focuses on of those, namely the GIA induced effect on the geoid which is the equipotential surface that is equal to the mean sea surface. Since the reader may not be completely familiar with the concept of the geoid and how it is embedded in geodesy, section 2.4 deals with these matters. It is also explained there how the geoid height is determined from surface gravity measurements and satellite gravimetry.

2.1 Radial Structure of the Earth’s Interior

At the Earth’s surface, oceans and mountains contribute to a diverse scenery; inside the Earth a largely radially symmetric Earth is brought about by gravity. Most knowledge of the radial structure comes from indirect measurements, the most useful of which are arrival times of seismic body waves: compressive (p-) waves and transversal (s-) waves that radiate from the epicenter of an Earth-quake. In a p-wave, the particles move in the longitudinal direction, whereas an s-wave is formed by transverse movement of the particles. Their propagating velocity is dependent on density and elasticity parameters of the material which carries the wave. For the p-wave velocity holds [Fowler, 1990], p. 433,434:

\[
\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}},
\]

(2.1)

and for the shear wave velocity:

\[
\beta = \sqrt{\frac{\mu}{\rho}},
\]

(2.2)
in which $\lambda$ and $\mu$ are the Lamé parameters which describe the elastic behavior of a homogeneous, isotropic material.

From wave arrival times measured by seismometers positioned all over the world, a three-dimensional seismic velocity-profile can be constructed. A third equation is required to derive density and Lamé parameter throughout the Earth by means of inversion techniques. This equation can be the Adams-Williamson equation [Fowler, 1990], p.110:

\[
\frac{d\rho}{dr} = -\frac{G}{r_0^2}\frac{1}{(\alpha^2 - \frac{4}{3}r^2)} \int_{r_0 = 0}^{r = r_0} 4\pi \rho r_1^2 dr_1,
\]

in which $r_0$ denote the Earth’s radius, but also empirical relations exist for density and seismic velocities. Assuming an average density of rocks at the surface, equation 2.3 can be integrated to give density as function of radius inward. The elastic parameters can be obtained from equations 2.1 and 2.2. Two constrains exist on the integrated density distribution: total mass and moment of inertia of the Earth can be obtained in a totally independent way from astronomic measurements. The Adams-Williamson approach was found to give a too low mass; this deficit originated from phase changes occurring at sharp depths in the upper 1500 km of the Earth. At a phase change boundary, the pressure is just high enough to transform the crystal lattice of the material into a more efficient, denser packing.

Other observational data sources for the Earth’s interior are given in the following itemization [Brown and Mussett, 1993], [Fowler, 1990], [Lambeck, 1988]:

- Shear wave velocity is affected by crystal anisotropy. Convective flow aligns the crystals along a preferred direction and is therefore ‘visible’ as differences in horizontal and vertical polarized shear wave velocity arrival times.
- A strong enough Earth quake excites surface waves in the order of the Earth’s radius resulting in radial, toroidal and spheroidal oscillations of the whole Earth. Observations of a free-oscillation frequency provide an additional relation between density and seismic velocity.
- Besides body waves, an Earth quake excites surface waves. The longer the wavelength of a surface wave, the deeper it penetrates. The elastic modulus and density change with depth, therefore the velocity of the surface wave depends on wavelength. This phenomenon is called dispersion and yields useful information on the shear wave velocity.
- At some sites layers come to the surface which are beyond direct observation elsewhere.

From the surface downwards the following layers are found in the Earth:

**Crust**

The crust has a diverse chemical composition and is the most lateral heterogeneous layer, with a major distinction between oceanic and continental crust. Oceanic crust forms at mid-oceanic ridges and spreads outwards until it subducts when it meets a heavier continental plate, which happens roughly after 80 Myears, compared to an age of 2000 Myears for continental crust. Oceanic crust is 2 to 37 km thick and contains from top to bottom: sea water, sediments and a volcanic layer with a rather uniform constitution of basaltic materials. Gravity measurements tell us that the oceanic and continental crust at a certain depth exert the same pressure, which indicates that an oceanic crust balances a continental crust which is roughly 5 times as thick [Fowler, 1990], p.284.

Continental crust shows much more variation than oceanic crust, with a thickness ranging from 20 to 70-80 km beneath mountains. Despite its variability, seismic profiling studies have indicated a two-layer composition: a brittle upper crust of 20-25 km thickness and a weaker lower crust of 15-20 km thickness [Schubert et al., 2001], [Brown and Mussett, 1993]. In some places the seismic velocity increase in the crust is gradual and several low-velocity layers are found [Fowler, 1990], p.353. In general, continental crust contains more silicates and is therefore more deformable than oceanic crust.

**Lithosphere**
The lithosphere is comprised of the crust and that part of the upper mantle which supports non-hydrostatic stresses up to a few hundred MPa for long periods (> 1 kyear). This definition is a mechanical one; seismologists would include that part of the mantle with high seismic velocities \[\text{Lambeck, 1988}, \text{p.85}, \text{Ranalli and Murphy, 1987}\]. The lithosphere should be considered a useful definition when dealing with loading or convection problems, whereas crust and mantle are layers that are observable in terms of chemical composition or phase changes.

Separating the crust and the mantle is the Mohorovicic discontinuity which manifests in a discontinuous increase in p-wave and s-wave velocities and also marks a transition from continental crustal density of \(2,600 \frac{\text{kg}}{\text{m}^3}\) to \(3,350 \frac{\text{kg}}{\text{m}^3}\). \[\text{Schubert et al., 2001}, \text{p.73}\]. The rigidity is approximately equal to 64 GPA in all of the lithosphere \[\text{Klemann and Wolf, 1999}\].

Mantle

**Upper Mantle or Astenosphere.** Mantle composition is derived from meteorite composition and mantle material arriving at the surface in some areas. The composition is further restricted by seismic anisotropy and wave velocities \[\text{Schubert et al., 2001}, \text{p.80}\]. The former suggests that the upper 220 km of the mantle consists mainly of olivine, a silicate with chemical composition \(Mg_2Fe_2SiO_4\). Several phase changes of olivine occur in the mantle which are associated with seismic discontinuities at 400 and 670 km depth \[\text{Ranalli, 1995}, \text{p.150}, \text{Schubert et al., 2001}, \text{p.84}\]. In between the upper and lower mantle stretches the transition zone, characterized by a sharp increase in density. There is no agreement about whether the transition zone is a pure phase boundary or that a chemical transition occurs.

**Lower Mantle.** The most common components of the lower mantle are silicate perovskite and magnesiowüstitide. At pressures higher than those in the transition zone no phase changes occur, so that the lower mantle is quite homogeneous. The instability of the materials at low pressure make it difficult to carry out stress experiments in a laboratory setting \[\text{Schubert et al., 2001}, \text{p.93}\]. In the bottom 300 km of the lower mantle, denoted the \(D''\)-layer, the interaction between the core and the mantle takes place. Seismically, it is seen as a layer with sharp increase in seismic velocity followed by a moderate increase and a zone of low velocities. In the core, temperature is higher than the melting temperature, so that the \(D''\)-layer is interpreted as being a hot boundary layer where heat is conducted out of the core and where mantle convection is initiated. The instability of the \(D''\) layer induces stresses in the lower mantle and lateral density variations which in turn influence the gravity potential \[\text{Schubert et al., 2001}, \text{p.94}, \text{Lambeck, 1988}, \text{p.461}\]. The Core Mantle Boundary (CMB) is believed to be no wider than 10 km. Little is known about the conditions on the CMB \[\text{Schubert et al., 2001}, \text{p.98}\].

Core

The outer core is found to be fluid because since it does not allow s-waves to propagate through it. The main constituent is molten iron, which, by movement around the solid inner core, drives the Earth's magnetic field. The boundary between outer and inner core occurs where the molten iron freezes. This point was located at a radius of 1,220 km by observations of seismic waves traveling through the inner core being reflected. The average core density is \(11,000 \frac{\text{kg}}{\text{m}^3}\) \[\text{Brown and Mussett, 1993}\].

### 2.2 Ice Ages

In the Pleistocene, spanning the past 2.4 million years, the climate has undergone some dramatic changes which are usually attributed to changes in the Earth's orbit. Measurements of the \(^{18}O\) isotope content in sea sediments show a period of the formation of continental ice of roughly 90,000 years and a decay period of roughly 10,000 years. The relation of the presence of the \(^{18}O\) isotope to glaciation is as follows: The ocean contains the oxygen isotopes \(^{18}O\) and \(^{16}O\) in a fixed ratio. The lighter isotope \(^{16}O\) evaporates more easily from the oceans and consequently more of this isotope will precipitate in the form
of snow. Thus, during an ice age the ocean contains more of the heavier isotope $^{18}O$, which is likewise recorded in ocean sediments.

The total period of an ice age of approximately 100,000 years closely matches the period of change in ellipticity of the Earth’s orbit. When the orbit becomes more elliptic, the Earth will experience shorter but more intense summers and longer and colder winters. However, it is not sure if this alone is enough to bring about an ice age. The orbital changes might disturb the ocean current system which has a stronger effect than the change in insolation of an elliptical orbit alone. Some studies suggest that the explanation for an ice age could be a change in inclination of the Earth’s orbit which sent the Earth through denser or less dense interplanetary dust clouds, thereby changing insolation of the Earth [Vermeersen and Sabadini, 1999], [Tushingham and Peltier, 1991].

The abrupt disappearance of the ice sheets is as much a mystery as their growth. A series of growth and decay cycles of ice sheets caused a change in moment of inertia which led the rotation axis to drift away from the rotation pole. The change in seasons which resulted from this could help in the removal of the ice sheets. [Sabadini and Vermeersen, 1997]. Using C-14 radiocarbon dating of RSL data it was found that the Last Glacial Maximum (LGM) occurred around 18,000 years ago [Tushingham and Peltier, 1991]. At that time the Laurentide ice sheet covered all of Canada with a maximum average thickness of 2 km, and the Fennoscandian ice sheet of about 1 km covered North-western Europe. The Antarctic ice sheet was larger than at present and a small ice sheet resided in northern Eurasia and over Chile.

The ice sheets have clear left marks in the landscape, such as rocks pushed ahead by moving glaciers left as a remnant of their furthest inland position. However, the most useful data source for the spatial extent of ice sheets are relative sea level-time curves which constrain the mass of ice in the surroundings. section 5.4 goes into more detail on the trial and error process of constructing ice models.

### 2.3 Glacial Isostatic Adjustment

This section discusses the relation between ice sheet melting and sea level. The sea level is key for solving for gravitational parameters such as the free-air gravity anomaly and the geoid anomaly. At first sight the relation seems obvious: the water that becomes available from the melting of an ice sheet on land is added to the oceans and equally distributed over the entire ocean area. This eustatic sea level rise can simply be calculated by:

$$s_e(t) = \frac{M_i(t)}{\rho_w A_o}$$

with $M_i(t)$ the land ice mass on land that has melted up to time $t$, $\rho_w$ the water density and $A_o$ the ocean area. Melting of sea ice has no influence on sea level and therefore sea ice is left out of the equation. Since the onset of deglaciation 18,000 years ago, the eustatic sea level rise amounted up to 120-130 m. [Tushingham and Peltier, 1991].

Melting of an ice sheet may actually lead to sea level drop in the vicinity of the ice sheet. This can be explained by self-gravitation of the ice: the mass of the ice sheet attracts ocean water. When an ice sheet melts, the ocean water which was accumulated around it flows away causing a local sea level drop (compared to the situation before melting). From [Vermeersen and Sabadini, 1999] the following example is taken: if all the land ice in Greenland is concentrated on the south tip and instantly melts, the sea level will drop at the coast of Iceland, rise less than the eustatic value in New York and in Australia the sea level will rise with more than the eustatic value.

Another complication stems from the fact that the Earth cannot be regarded as rigid when it comes to ice melting. Consider first the situation on land. When an ice sheet forms, the material below deflects instantaneously under the weight. Mantle material starts flowing away from under the ice sheet causing an even deeper depression. Material flow velocity is determined by its viscosity: the higher the viscosity, the higher the pressure required for a certain flow rate. The material which flows away from underneath the ice sheet pushes up a ring around the glaciated area known as the peripheral
2.3 Glacial Isostatic Adjustment

bulge. Sinking of the ice covered material stops when a state of isostatic equilibrium is reached, which means that at a certain depth, the hydrostatic pressure is equal below and around the ice sheet.

For the precise mechanism of mantle flow two models exist: the half-space or whole mantle model, in which the mantle is thought to extend to infinite depth, and the channel flow model, in which adjustment takes place by horizontal flow in a thin layer. The two models lead to an entirely different rebound pattern: rebound in the channel model is slower for larger loads, whereas in the whole mantle rebound model rebound occurs faster for larger loads. The main difference is that the whole mantle model induces mainly vertical deformations so that the center of rebound precedes relaxation of the edges, while in the half-space model flow from the sides heightens the sides from the basin first. Both of the models are supported by observations. The whole mantle model predicts the observed up-down movement of sites just outside the ice margin, but the channel model gives good values for the remaining uplift in Fennoscandia [Schubert et al., 2001], p276.

When the ice sheet melts in a relatively short time span compared to the accumulation of the ice sheet, an immediate elastic rebound follows. Because of the mass deficit and consequently lower hydrostatic pressure at depth, mantle material starts to flow back, causing the peripheral bulge to collapse and the former glaciated area to rebound. The process of elastic and viscous deformation to loading and unloading of the Earth by ice formation and removal is termed Glacial Isostatic Adjustment (GIA).

At the end of the nineteenth century, observers began to relate the raised beaches in Scandinavia and tilting of the Great Lake in Canada to GIA and it was realized that flow or plastic yielding must occur in the mantle. In the 1930’s, viscosity estimates for the mantle were produced using simple creep models. Much later, ancient shorelines were used for estimating the relaxation time [Lambeck, 1988], p.534. In Fennoscandia extensive gravity studies showed negative gravity anomalies, which were interpreted as an indication of an ongoing rebound process. The significance of GIA at present is shown by GPS measurements in Scandinavia which clearly indicate an uplift of more than 1 centimeter in the center of the former glaciated area [Milne et al., 2001].

To complete the picture of sea level change due to land ice melting, consider the melting of an ice sheet located on land near the coast. Due to the self-gravitation effect, removal of the ice sheet lowers the sea surface. If the peripheral bulge is located in the sea, the ocean floor subsides which draws in extra water, increasing the gravitational potential. Melt water, added to the oceans, will distribute so that the sea surface is again a surface of constant potential, but extra mass pushes the ocean floor downwards. What used to be the coastline at the maximum of the ice age lies below sea level after deglaciation and depressions left by the vast ice sheets become filled with water. This occurred in what is now the Hudson Bay in Canada.

Obviously, there is no simple relation between melting of the ice sheet and sea level rise. To first order the sea level can be computed with the eustatic sea level change, but the aforementioned effects should be taken into account when computing detailed sea level

Figure 2.1 Raised beaches in Easter Gotland, Sweden. (from: Schubert et al. (2001), p.214)
and geoid changes. When speaking of sea level change, the Relative Sea Level (RSL) is meant, which is defined as the difference between sea surface and the bottom of the sea. Absolute sea level gives the sea level with respect to the center of mass of the Earth.

### 2.4 Geodetic Geoid

Sea level and the Earth’s gravity field are closely connected; in principal, gravity potential differences can not exist in the ocean surface because these would be removed by water flow. In reality non-conservative forces such as thermally induced currents and winds cause deviations of the ocean surface from the equipotential surface. Most of these effects average out, so a particular equipotential field called the geoid can be constructed that fits the time-averaged ocean surface within 1 to 2 cm [Lambeck, 1988], p. 5. For geodetic purposes the geoid is thought to extend inside the continents. In geodesy, equipotential surfaces play an important role because heights are measured relative to them. As such the geoid is the ‘fundamental surface of physical geodesy’ [Heiskanen and Moritz, 1967], p.49.

For comparing global measurements and computations the geoid must be related to a reference shape that is mathematically defined. The geodetic society adopted as a reference surface a rotating ellipsoid, which is defined as having a shape which best approximates the geoid and having the same gravity potential as the geoid at its surface. Deviations of the actual gravity field from the normal gravity field are sufficiently small so that they can be linearized. Still the departure of the geoid from the reference ellipsoid may amount to 100 m [Lambeck, 1988], p.17. There is a subtlety in the choice of words that needs to be clarified first. The gravitational potential \( V(x, y, z) \) refers to the potential field due to mass attraction only. Gravity potential \( U(x, y, z) \) includes the effect that the rotation \( \omega \) of the Earth has on a unit mass [Heiskanen and Moritz, 1967], p. 64:

\[
U(x, y, z) = V(x, y, z) + \frac{1}{2} \omega^2 (x^2 + y^2). 
\]

An ellipsoid is described by:

\[
S_0 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, 
\]

and its gravity field is completely determined by 4 parameters: the semi-major axis \( a \), the semi-minor axis \( b \), the total mass \( M \) and the angular velocity \( \omega \). This is a consequence of Stokes’ theorem which states that a function harmonic outside a surface \( S \) is uniquely determined by its values on that surface [Heiskanen and Moritz, 1967], p. 17. Usually the international reference ellipsoid is given in the parameters \( a \), geometric flattening \( f = \frac{a-b}{a} \), normal gravity acceleration at the equator \( \gamma_e \) and \( \omega \). Values for the reference system EOS1983 are given in table 2.4. The potential and gravity belonging to the reference ellipsoid are sometimes referred to as normal, but here they will be denoted by reference.

The geoid in continental areas is a theoretical concept, in order for it to be useful it must be related to observable parameters, see figure 2.4. We write the geoid potential \( W_0 \) as the sum of the reference gravity potential \( U \) and a perturbation potential \( \Phi \), both in the point \( P \):

\[
W_0 = U_P + \Phi_P. 
\]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>6378.136 km</td>
</tr>
<tr>
<td>( f^{-1} )</td>
<td>298.275</td>
</tr>
<tr>
<td>( \gamma_e )</td>
<td>( 978032 \cdot 10^5 ) ( \frac{m}{s^2} )</td>
</tr>
<tr>
<td>( \omega )</td>
<td>( 7292115 \cdot 10^{-11} ) rad/s</td>
</tr>
</tbody>
</table>

Table 2.1 Parameters of the international reference ellipsoid model EOS1983, from Lambeck1988, p.17.
$U_P$ can be written as function of the reference potential $U_0$ and the reference potential perturbation at point $Q$, which is the projection of $P$ on the reference ellipsoid, using a first order Taylor expansion:

$$U_P = U_0 + \left( \frac{\partial U}{\partial r} \right)_Q \mathcal{N}. \tag{2.8}$$

$\left( \frac{\partial U}{\partial r} \right)_Q$ is just the negative of the reference gravity at $Q$, $\gamma_Q$ and equation 2.7 becomes:

$$W_0 = U_0 - \gamma_Q \mathcal{N}_P + \Phi_P. \tag{2.9}$$

$\mathcal{N}$ is the height from the reference ellipsoid to the geoid, which is the geoid height or geoid anomaly which is of main interest. Here we see the advantage of choosing the potential of the reference ellipsoid $U_0$ equal to the potential of the geoid $W_0$, and we find Brun’s formula [Heiskanen and Moritz, 1967], p.85:

$$\mathcal{N} = \frac{\Phi_P}{\gamma_Q}. \tag{2.10}$$

$\gamma_Q$ is known since the gravity field of the reference ellipsoid is known. It is a function of the geometric flattening, the reference centrifugal force at the equator, and the reference gravity at the equator. In order to find the geoid height at point $P$, we need to know the perturbation potential $\Phi$ at the same point. Introducing the gravity anomaly:

$$\Delta g = g_P - \gamma_Q. \tag{2.11}$$

in which $\gamma_Q$ is known and $g_P$ is the gravity at the geoid which, so it seems, can not be measured directly because the location of the geoid is not known. However, if assumptions for the density distribution between the topographic surface and the geoid are made, gravity measurements on the surface can be reduced to the geoid. A reduction method consists of three steps [Heiskanen and Moritz, 1967], [Lambeck, 1988]:

- remove the mass between the geoid and point $P'$, this lowers the measure gravity value;
- move the point $P'$ to the geoid, which increases the measured gravity value because the point gets closer to the center of the Earth;
- account for isostatic compensation of topography in the crust. The compensation has a positive value for mountain areas, and a negative for oceans.

For a disc of infinite radius the first step can be approximated by (the Bouguer correction):

$$\delta g_B \simeq -2\pi \kappa \rho h. \tag{2.12}$$

The effect of topography is modeled by summing the attraction of topographic masses in the point of observation. This is the topographic correction $\delta_t$. In mountain areas it can amount up to $1 \cdot 10^{-3} \frac{m}{s^2}$. The second step is labeled free-air correction and can be approximated by:

$$\delta g_{FA} \simeq \frac{\partial g}{\partial r} h. \tag{2.13}$$

and calculated using the ellipsoidal approximation. This yields for the Bouger gravity anomaly (which usually includes the topographic correction) at the observation point $P'$:

$$\Delta g_B = g(P') + \delta g_{FA} + \delta g_B - \gamma_Q. \tag{2.14}$$
The Bouguer anomaly is found to be largely negative in mountain areas, which points to some kind of isostatic correction $\delta_I$ to be applied on the gravity measurements (step 3). The concept of isostasy is sometimes defined as compensation of a surface load by density anomalies below, so that at a certain depth the hydrostatic pressure is equal. Today, nearly all of the Earth is found to be in isostasy. Some of the regions that depart from isostasy were recognized as former glaciated areas [Lambeck, 1988], p. 419.

The isostatic correction in step 3 essentially redistributes the topography masses that are removed in step 1 in the crust. Mountains are compensated by lower density columns at a certain depth in the crust which means the Bouguer correction is too large; the isostatic correction to the gravity anomaly will then be positive. Because ocean water has lower density than the average crust, oceans are compensated by columns with higher than average density and the isostatic correction will be negative. The isostatic anomaly is given by:

$$\Delta g_I = g(P') + \delta g_A + \delta g + \delta I - \gamma_Q.$$  \hspace{1cm} (2.15)

Redistribution of mass in the case of isostatic anomaly, or removing of mass in case of the Bouguer correction, has effect on the gravitational potential so the geoid that is computed in this way will be shifted with respect to the ‘real’ geoid. This (indirect) effect is moderate for the isostatic anomaly, which is therefore of more practical value than the Bouguer anomaly [Heiskanen and Moritz, 1967], p.141. Sometimes just the free-air correction (step 2) is applied to the gravity anomalies, yielding the free-air gravity anomaly.

The gravity anomaly, reduced according to the methods mentioned above, is an observable quantity, however for equation 2.10 we need the perturbation potential. The relation between the perturbation potential and the gravity anomaly is given by the fundamental equation of geodesy [Heiskanen and Moritz, 1967], p.86:

$$\Delta g = -\frac{\partial \Phi}{\partial h} + \frac{1}{\gamma} \frac{\partial \gamma}{\partial h}$$  \hspace{1cm} (2.16)

The gravity anomaly is a perturbation quantity, therefore in this expression the reference ellipsoid can be approximated by a sphere (replacing $h$ by $r$) with relatively small error (spherical approximation, see [Heiskanen and Moritz, 1967], p. 87) which leads to:

$$\Delta g = -\frac{\partial \Phi}{\partial r} - \frac{2}{r} \Phi$$  \hspace{1cm} (2.17)

This equation does not yield the perturbation potential directly; integration is necessary. Equation 2.16 can be seen as a boundary condition on the geoid; because the masses outside the geoid have been reduced to the geoid there are no masses outside the geoid in this model, and $\Delta g$ and $\Phi$ are harmonic functions. From this assumption Stokes derived in 1849 an expression that relates the perturbation potential to a measured quantity (see [Heiskanen and Moritz, 1967], p94 and before). Inserting it in Brun’s equation 2.10 it becomes an expression for the geoid anomaly:

$$\mathcal{N} = \frac{R}{4\pi\gamma} \iint_{\Omega} \Delta g(P') S(\chi, P') \, d\Omega,$$  \hspace{1cm} (2.18)

where $P$ is the computation point and $P'$ the points on the sphere where the gravity anomalies are given. This is an important equation because the geoid anomaly $\mathcal{N}$ can be derived from measured gravity anomalies at points $P'$. The Stokes function $S(\chi, P')$ can be seen as a weighting function depending on the solid angle $\chi$ between point $P$ and $P'$. Formally, gravity anomalies should be given on all points of the sphere. If only a smaller region around the point of interest is given, the geoid anomaly roughly reflects only those wavelengths of the same spatial scale as the region.

An example of a large regional geoid model is the Canadian Gravimetric Geoid Model of 2000 (CGG2000) of the Geodetic Survey Division of Natural Resources Canada [Veronneau, 2000]. The model gives geoid heights which are defined as the height between the reference ellipsoid of GRS80 and the equipotential surface $W_0 = 6263655.8 \times 0.5 \text{ m}$.

In essence, the model follows the approach described in the previous section. Large
datasets of surface, airborne and ship-borne gravity measurements were used to obtain a dense net of gravity anomalies. The error, as follows from error propagation calculations, ranges from 0.2 to 18 cm. However, Veronneau [2000] admits that the accuracy might be too optimistic for some parts of Canada.

A big improvement in geoid anomaly determination is foreseen by the satellite gravity missions GRACE, CHAMP and GOCE. CHAMP, which was launched in July 2000, measures the long-wavelength part of the static gravity field with revolutionary accuracy. GRACE, a configuration of two satellites launched in March 2002 will add to this the observation of the time-variability of the long- and medium wavelength parts, while with GOCE, it is aimed to reach an accuracy in terms of geoid height of 1 cm on a spatial scale of 100 km [Visser et al., 2002].

The question is how do the geoid anomalies computed from modeling in this study relate to existing geoid models? As will be clear from the treatise of the sea level equation, the point of reference for the computations in this study is the center of mass of the Earth, which makes easy comparison with geoid models obtained by satellite mission possible, since these measurements are also referenced to the center of the Earth [Mitrovica and Peltier, 1989].
Chapter 3

Response of a Viscoelastic Planet to a Surface Load

In this chapter we will derive a relation between forcings at the surface of a spherical model of the Earth or an Earth-like planet and the displacements or gravitational potential perturbation at the surface. In this thesis the external load is due to the melting of the Pleistocene ice sheets which started 18,000 years ago and took roughly 10,000 years. In principal the same theory can be used to compute the response to other surface forcings such as atmospheric pressure or ocean loading.

A common theory to compute the response of a visco-elastic planet to surface loads, is the normal mode theory, developed by Peltier and Wu [Peltier, 1974], [Wu, 1978], [Wu and Peltier, 1982]. It is described in section 3.5, starting from the equation of conservation of momentum, Poisson’s equation for the gravitational potential field and the continuity equation for conservation of mass. The variables we want to compute are radial and tangential displacement and the gravitational perturbation potential. At the surface however, the surface load is prescribed in terms of stress, so a relation between stress and strain is required. This relation is given by the rheology, which is the subject of sections 3.2 to 3.4. Sections 3.2 and 3.3 describe the two end members in material behavior, namely purely elastic and purely viscous behavior. Section 3.4 combines those to form a linear visco-elastic rheological model that will be used in the normal mode analysis in section 3.5. A micro-physical approach to rheology of Earth mantle material (supported by extrapolated laboratory experiments) leads to a non-linear description of at least part of the mantle. This approach, and the reasons why linear stress-strain relations are still being used in GIA modeling is discussed in section 3.1. In section 3.6 special attention is paid to the important last step in the normal mode analysis, namely transformation from the Laplace domain to the time domain. Section 3.7 gives a brief account on the validation of the normal mode theory for multi-layer Earth models which raised some questions in the past.

3.1 Introduction to Rheology of Mantle Materials

From a rheological point of view, even the hardest rocks flow, if pressure is high enough or on geological time scales. To express the slowness of the movement the term creep is used, which is defined as ‘slow deformation under constant stress’ [Ranalli, 1995], p.66.

This section describes some of the micro-physical creep mechanisms, which lead to non-linear stress-strain relations. Non-linear rheology equations provide a better description of silicate polycrystals properties in laboratory experiments, while linear behavior is only seen at low-stresses and for materials with small grain sizes [Ranalli, 1995], p.327. Still, in almost all GIA studies a linear rheology is assumed (a justification follows in the section 3.4) therefore the equations in this section (and in appendix D) are not used in the rest of this thesis.
Realistic non-linear creep is represented by a power-law [Ranalli, 1995], p.75:

$$\dot{\varepsilon}_s = A\sigma_s^n.$$

(3.1)

Laboratory results indicate that a value of the stress-exponent $n = 3$ is common for polycrystalline olivine, which is the most abundant material in the upper mantle. Different phases of olivine give stress exponents of 3.0-4.0. Also, in a different temperature and pressure regime another deformation mechanism may take over that increases the value of $n$. A stress-exponent of 1 only occurs for very fine-grained materials. Equation 3.1 in tensor form reads [Ranalli, 1995], p.77:

$$\dot{\varepsilon}_{ij} = A\sigma_{E}^{n-1}\sigma_{ij}'\dot{\varepsilon}_{ij},$$

(3.2)

with a prime still denoting deviatoric components and $\sigma_E$ the effective stress $\left(\frac{1}{3}\sigma_{ij}\sigma_{ij}\right)^{\frac{1}{2}}$. Since realistic mantle conditions are not yet possible to simulate; experimental results need always to be extrapolated to lower strain rates and longer time scales (or equivalently higher temperatures) [Schubert et al., 2001], p.246, [Karato and Wu, 1993] when applied on mantle conditions. Additional information on non-linear rheology is provided by micro-physical models. Many different creep mechanisms exist, but the two principal mechanisms for non-linear creep are dislocation creep and diffusion creep [Peltier, 1974], [Schubert et al., 2001] p. 240. Appendix D gives a qualitative description of these creep mechanisms. For now it is important to know that dislocation creep leads to creep of type power-law whereas diffusion creep leads to a linear Newtonian type of behavior.

Most studies favor dislocation creep for the deformation of the upper mantle, although a transition between dislocation and diffusion creep is likely to occur in the upper mantle. Diffusion creep is dominant for low stress, small grain size, low temperature and high pressure. Water content has little influence on the transition depth, but the geo-thermal gradient has significant influence. Based on the latter, dislocation creep is expected mainly beneath mid-ocean ridges, and diffusion creep in the shallow, relatively cold upper mantle. There is a relation between grain size and stress, for dislocation creep. As stress increases, the grain size will decrease below the transition size below which diffusion creep occurs, which adds the deep upper mantle as possible other location for diffusion creep [Karato and Wu, 1993], [Kohlstedt et al., 1995].

Rocks from the upper mantle which are carried to the surface are of little help in determining which creep mechanism prevails, because their micro-structure reveals only the deformation process of the upward movement of these rocks. Dislocation creep leads to anisotropy in mantle materials which can be seen as direction dependent wave velocity. However, detectable anisotropy occurs only at high temperatures, which, for the upper mantle only occurred during or just after rifting. From 200 to 300 km depth, anisotropy of polarized surface waves and seismic wave anisotropy point to a transition of dislocation to diffusion creep.

Thus, we have diffusion creep in the shallow upper part and in the deep upper mantle and dislocation creep in between. What is the interaction between these layers? Karato and Wu [1993] reason that as stress increases, grain size decreases, possibly below the transition size, so that diffusion creep occurs. Displacements become localized so the layers in the upper mantle can be seen as mechanically decoupled. Materials analogous to perovskites, which are the dominant material in the lower mantle (see 2.1) sometimes deform by power-law creep and sometimes in a more linear fashion [Wu, 1992].

Historically, GIA studies were performed using linear, Newtonian rheology, performing well in predicting geophysical observables such as RSL. However, [Ranalli, 1995], p.219 and p. 223 and [Peltier, 1974] stress that by no means this excludes the possibility that a non-Newtonian rheology might reproduce the same observations, because the relation between the type of rheology and surface observables is not unique. Kaufmann and Royden [1994] state that laboratory experiments provide a good description of deformation of crustal rocks and a power-law formula with stress-exponent $n=3$ provides a better description than a stress-exponent $n=1$.

Wu [1992] performed a simulation with Finite Element modeling of a boxcar ice load on a visco-elastic half-space. Non-linear rheology shows a peripheral bulge which does
not migrate inwards, contrary to observations in east coast Canada and the U.S. (see for example [Lambeck, 1988], fig. 26c and d). The presence of a 84 km lithosphere does not affect this result. In the center of rebound the linear rheological model predicts a curve of uplift vs. time which fits data in Fennoscandia rather well, whereas power-law creep does not give a good fit. It is concluded that power-law creep must be rejected for this ice model. However, this says nothing about loads with different amplitudes, shapes and histories. Effective viscosity depends on the stress, and a different magnitude or loading history can alter the stress situation considerably.

Wu [1993] extended his previous study by using axisymmetric disc or parabolic loads. Still it was found that non-linear rheology resulted in continuous downward motion outside the ice sheet and upward motion inside. Neither increasing the stress exponent, nor applying a parabolic load or a shrinking ice sheet gave the observed behavior just outside the ice sheet. Observations near the center of the ice load could be reproduced with a non-linear channel. However, moderate fits for data elsewhere within the former Laurentide ice sheet were obtained. It was not possible to simultaneously fit data within and just outside the ice sheet.

Now one might argue that apparent linearity might occur in the mantle if a stress state exists in which postglacial rebound stresses are small compared to convective stresses. In that case the effective stress is approximately equal to the convective stress $\sigma_C$ so that the rebound strain rate is [Ranalli, 1995], p.224:

$$\dot{\epsilon}_R = \frac{1}{2} A \sigma_C^{n-1} \sigma_R$$

(3.3)

and the mantle ‘sees’ a linear relation between rebound stress and rebound strain rate even though the mantle obeys a power-law rheology. This effect has been studied by [Wu, 1995], who found that in the center of rebound indeed a non-linear mantle in the presence of a large (100 MPa) ambient stress gives identical results as the linear model. However, in the peripheral region the power-law model with the ambient stress predicts continuous subsidence after removal of the load, whereas linear rheology produces the observed inward migration of the peripheral bulge. Also RSL data around the ice margin cannot be reproduced. The explanation can be found in the stress concentrations near the edge of the ice sheets, which give rise to low effective viscosities and fast flow, regardless the magnitude of the primary stress field. The conclusions of [Ranalli, 1995], p. 225 that although GIA studies are a „… very important tool for analysis of the rheology of the mantle… [they have been] less effective in excluding plausible alternatives”, has proven to be incorrect at least for the simplified loading cases that were considered in the studies discussed above.

### 3.2 Elasticity

The simple Hooke’s law of stress linearly proportional to strain well approximates the elongation of a metal bar when it is stretched. Mantle material is different from a metal as it consists of a mixture of minerals bound together, leaving micro-cracks in between. As a result, mantle rocks can deform irreversibly for even a small applied stress and the simple linear elastic stress-strain relation does not hold. Despite this fact, it is useful to start with elastic behavior before going to viscous behavior or even more complicated rheologies. The generalized form of Hooke’s law for infinitesimal stress and strain is (e.g. [Ranalli, 1995] p. 51, [Christescu, 1989], p.50):

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} \epsilon_{kl}.$$  

(3.4)

Now we consider an infinitesimal cube with faces in the coordinate directions $(x_1, x_2, x_3)$. A component of the stress tensor, $\sigma_{ij}$ acts on a plane normal to the $i$-direction and in the direction $j$. Thus, of the 9 elements of the stress tensor $\sigma$ the diagonal elements $\sigma_{11}, \sigma_{22}, \sigma_{33}$ are the stresses normal to the surfaces of the cube, and the off-diagonal terms
are the shear stresses. Because of moment equilibrium the terms with index $ij$ (i not equal to $j$) are equal to the terms with index $ji$. The tensor is symmetric and the 81 components of the material tensor $C$ reduce to 36. The elastic behavior of an isotropic material (a material in which the properties do not depend on direction) is determined by only two constants, which leads to the following common notation of Hooke’s law [Ranalli, 1995], p.52 and [Christescu, 1989], p. 54:

$$\sigma_{ij} = \lambda \delta_{ij} + 2\mu\epsilon_{ij},$$

(3.5)

with the Lamé parameters $\lambda = C_{12}$ and $\mu = \frac{1}{2}C_{11} - \lambda$. Strain is comprised of two parts: a part that results in a change in volume, and a part that results in a change of shape. We can write the isotropic part of the stress tensor as:

$$\sigma_{ij}^{0} = \frac{1}{3}\sigma_{kk}\delta_{ij},$$

(3.6)

i.e. the mean of the compression and elongation terms which are on the diagonal of the stress tensor. On the strain-side of the equation the isotropic strain is therefore defined as that part of the strain which results in a change of volume:

$$\Delta = \epsilon_{ij}^{0} = \frac{1}{3}\epsilon_{kk}\delta_{ij}.$$  

(3.7)

Cubical dilatation is the change in volume divided by the original volume. The deviatoric part of strain is defined as $\epsilon_{ij}^{d} = \epsilon_{ij} - \epsilon_{ij}^{0}$ and represents a change in shape. The Lamé parameter $\mu$ relates deviatoric strain to deviatoric stress by:

$$\mu = \frac{1}{2}\epsilon_{ij}^{d}.$$

(3.8)

The ratio between isotropic stress and isotropic strain is the incompressibility or bulk modulus $k$:

$$\sigma_{ij}^{0} = ke_{ij}^{0},$$

(3.9)

Since only two elastic parameters are independent $k$ can be written as a function of the Lamé parameters. Two other elasticity constants that are sometimes used are the elasticity (or Young’s) modulus and the Poisson’s ratio.

### 3.3 Viscosity

Elasticity theory well describes the short-term response of Earth materials (e.g. the propagation of a seismic wave through the Earth). However, for long-term behavior the Earth can be considered as purely viscous (e.g. a rotating fluid sphere provides a good fit for the oblate shape of the Earth). Postglacial rebound is an intermediate time-dependent response, where the mantle deforms elastically but also viscous flow of mantle material occurs. Apparently, depending on the time-scale of the loading and unloading, the Earth can be regarded as an elastic or as a viscous medium or as a combination of the two. Upon the introduction of viscosity in the equations time becomes an extra variable that can be dealt with by prescribing strain rate and using appropriate boundary conditions. There are different mechanisms of viscous deformation and therefore different equations prescribing strain rate. The simplest description is that of a Newtonian fluid body for which constant stress produces constant strain rate [Ranalli, 1995], p67:

$$\sigma_{ij} = -p\delta_{ij} + \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl}\epsilon_{kl}.$$  

(3.10)

Note that strain rate $\dot{\epsilon}_{ij}$ is defined as:

$$\dot{\epsilon} = \frac{1}{2} \left( \frac{\partial u_{i}}{\partial x_{j}} + \frac{\partial u_{j}}{\partial x_{i}} \right).$$

(3.11)
3.4 Linear Visco-elastic Stress-strain Relations

The Pleistocene ice sheets were of such great scale that the response is largely governed by the mantle. In the first two sections stress-strain relations were given for a purely elastic and for a purely viscous body. The Earth is known to show both elastic and an-elastic behavior. This section presents combinations of the elastic and viscous stress-strain relations of the previous two sections to form linear visco-elastic stress-strain relations that are commonly used in GIA modeling.

In a one-dimensional linear visco-elastic model, elasticity is characterized by a spring, with rheology equation \( \sigma = 2\mu\varepsilon \), and viscosity is characterized by a dashpot with rheological equation \( \sigma = 2\nu\dot\varepsilon \). Some of the possible combinations of spring and dashpot are depicted in figure 3.4 [Ranalli, 1995] [Karasudhi, 1991].

a) The Kelvin model; dash-pot and spring in parallel, the rheology equation is obtained by summation:

\[
\sigma_{ij} = -p\delta_{ij} + \lambda\Delta\delta_{ij} + 2\nu\dot\varepsilon_{ij},
\]

for an isotropic material. The quantity \( \nu \) is the dynamic or Newtonian viscosity. The bulk viscosity \( \nu^* \) (compare the bulk modulus of equation 3.9), appears in the deviatoric stress [Ranalli, 1995], p. 68:

\[
\sigma'_{ij} = \left( \bar{p} - p \right)\delta_{ij} + \nu^*\dot\theta_{ij} + 2\nu\dot\varepsilon_{ij}.
\]

For geodynamic processes usually it is assumed that \( \nu^* = 0 \), the so-called Stokes-condition. In that case the mean normal pressure \( \bar{p} \) is equal to the hydro-dynamic pressure \( p \) and the dynamic viscosity is seen to be:

\[
\nu = \frac{1}{2}\frac{\sigma_s}{\varepsilon_s},
\]

where the subscript \( s \) denotes the shear component.

3.4 Linear Visco-elastic Stress-strain Relations

For the load can be derived, for constant stress \( \sigma_0 \) and initial condition \( \varepsilon = 0 \):

\[
\varepsilon = \frac{\sigma_0}{2\mu_k} \left[ 1 - e^{-\frac{2\mu_k}{\nu_k}} \right].
\]

Instant elastical deformation is slowed down by the dash-pot, after removal of the load the body reaches its initial position after infinite time.

b) The Maxwell model; dash-pot and spring in series. The rheology equation is written in terms of the strain rate [Ranalli, 1995], p. 85:

\[
\dot\varepsilon = \frac{\sigma}{2\mu_M} + \frac{\sigma}{2\nu_M}.
\]

After applying the load first instantaneous elastic displacement takes place, then irreversible viscous deformation. Relaxation (constant strain rate) follows:

\[
\sigma = \sigma_0 e^{-\frac{\varepsilon_M}{\tau_M}}.
\]
The Maxwell relaxation time \( \frac{\mu_1}{\mu + \mu_1} \) is a value characteristic for the relaxation; after such a time had elapsed the stress has decrease to \( \frac{1}{2} \) times its value.

c) Burgers model; Maxwell and Kelvin model combined in series. 2 sets of rigidity and viscosity parameters are required to describe the time behavior.

It can be useful to form a combination of more models, as polycrystalline materials have multiple relaxation times. Such a combination might also be useful to model transient rheologic behavior which lies between instantaneous, elastic and long-term, steady-state, viscous behavior. In a combination of a Kelvin and a Burgers model for example, the 2 viscosity parameters of the Burgers models can account for short-time processes and long-term processes respectively.

There is no general agreement whether transient rheology can be neglected or not. Sabadini et al. [1985a] concludes that it can not. Still transient behavior is neglected in most GIA studies and also in this study.

The simplest rheological model that describes the observed GIA is the Maxwell model. The stress-strain relationship for a Maxwell model in tensor form is [Peltier, 1974]:

\[
\sigma_{ij} + \frac{\mu}{\nu} (\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}) = 2\mu\epsilon_{ij} + \lambda\varepsilon_{ij}. 
\]  

(3.20)

Average values of \( \nu = 1 \cdot 10^{21} \) Pas and \( \mu = 1 \cdot 10^{11} \) Pa, which yields a Maxwell relaxation time of 1 kyear. The relaxation time can be interpreted as the time before which elastic behavior dominates and after which viscous behavior dominates. 1 kyear is found to be in good agreement with observed GIA [Ranalli, 1995], p.222. Laplace transforming the
previous equation gives:

\[(s + \frac{\mu}{\nu})\sigma_{ij}(s) - \frac{1}{3} \frac{\mu}{\nu} \sum_{k=1}^{3} 3\sigma_{kk}(s)\delta_{ij} = 2\mu e_{ij}(s) + \lambda \sum_{k=1}^{3} e_{ij}\delta_{ij}.\]  

(3.21)

This equation can be reordered to give:

\[\sigma_{ij}(s) = \lambda(s) \sum_{k=1}^{3} e_{kk}(s)\delta_{ij} + 2\mu(s)\epsilon_{ij}(s).\]  

(3.22)

The factors between brackets on the right-hand side can be regarded as the Laplace transformed Lamé parameters:

\[\mu(s) = \frac{\mu s}{s + \frac{\mu}{\nu}},\]  

(3.23)

\[\lambda(s) = \frac{\lambda s + \frac{\mu K}{s + \frac{\mu}{\nu}}}{s + \frac{\mu}{\nu}},\]  

(3.24)

and the elastic bulk modulus

\[k = \lambda + \frac{2}{3}\mu.\]  

(3.25)

Equation 3.22 looks just like the Hooke equation for an elastic medium, equation 3.5!.

This is a consequence of the Correspondence Principle, which states that if one solves an elastic problem with \(\lambda\) and \(\mu\) replaced by their Laplace transforms \(\lambda(s)\) \(\mu(s)\), this is equivalent to solving the viscoelastic problem in the Laplace domain [Biot, 1954], [Lee, 1955]. The great advantage is that the elastic field equations can be used to solve the visco-elastic time-dependent response, at the disadvantage of having to transform the Laplace domain solution back to the time domain, which will be discussed in section 3.6.

In this thesis, the Earth will be taken to be incompressible, as in [Wu and Peltier, 1982] section 3.1: the volume does not change in an incompressible medium so the dilatation \(\Delta\) goes to zero and the Lamé parameter \(\lambda\) goes to infinity, but their product remains finite:

\[\lim_{\lambda \to \infty, \Delta \to 0} (\lambda\Delta) = \Pi.\]  

(3.26)

The Laplace transformed constitutive equation becomes:

\[\sigma_{ij}(s) = \Pi(s)\delta_{ij} + 2\mu(s)\epsilon_{ij}(s).\]  

(3.27)

The influence of compressibility is large for the elastic response of the Earth (up to 15% higher vertical deformation if compressibility is included) but small for a homogeneous Earth model [Vermersen et al., 1996b]. The response after infinite time is identical [Wu and Peltier, 1982]. The viscous response is most important for postglacial rebound therefore the effect of compressibility on post-glacial rebound is marginal.

### 3.5 Normal Mode Analysis

In the previous section, the rheological model of a Maxwell visco-elastic Earth has been described. In order to arrive at the response of a multi-layer, spherically symmetric Earth the starting point is the equation of conservation of momentum, Poisson’s equation for the gravity field and the constitutive relations of the previous section. Eventually they lead to a set of coupled differential equations. There are different ways of solving the system of differential equations, the most straightforward of which is numerical integration by propagating the three solutions of the innermost layer outwards and applying the boundary conditions at the surface. The advantage of a numerical solution method over the analytical normal mode analysis is that many different cases, including a non-linear rheology or lateral heterogeneities, can be handled. The disadvantages are that it proves difficult to check the results, and physical insight in the mechanism of relaxation is lost.

Newton’s second law gives the following equation of conservation of linear momentum for a continuum in a fixed reference frame in a Cartesian coordinate frame [Anderson, 1991] p. 101:

\[ \nabla \sigma - \nabla \vec{p} + \rho \vec{X} = \frac{\partial \rho v}{\partial t}. \tag{3.28} \]

where \( X \) is the body force and \( v \) the velocity. The first two terms are surface forces acting on the bounding surfaces of a parcel. The stress tensor is related to the traction \( T \) on a surface of a body by Cauchy’s formula [Ranalli, 1995], p.28:

\[ T = \nabla \sigma. \tag{3.29} \]

The third term in equation 3.28 is a body force which can be gravity or a magnetic force. The time-derivative of velocity on the right-handside is a material (Lagrangian) derivative; it can be neglected because of the slow movement of post-glacial rebound (i.e quasi-static approximation). If we assume that the body force \( \vec{X} \) is conservative, then \( \vec{X} \) can be written as the divergence of a potential \( \phi \):

\[ \nabla \sigma - \nabla \vec{p} = -\rho \nabla \phi. \tag{3.30} \]

This implies that non-conservative forces are left out of the equation, and also that the centrifugal force is not included in the gravity potential [Tromp and Mitrovica, 1999a]. According to Eulerian perturbation theory, we proceed by writing the state variables as the sum of the part before application of the load, and a perturbation quantity:

\[ \rho = \rho_0 + \rho_1 \tag{3.31} \]

\( \rho_1 \) is the density change due to the changing mass distribution inside the Earth.

The potential perturbation can be written as the sum of two parts:

\[ \phi = \phi_0 + \phi_1 + \phi_2. \tag{3.32} \]

- \( \phi_1 \) is the potential of the applied force (for example the ice mass). This mass is located outside the Earth and therefore its potential satisfies Laplace’s equation inside the Earth, and it will not appear in the equations of motion [Farell, 1972]:

\[ \nabla^2 \phi_2 = 0. \tag{3.33} \]

- \( \phi_2 \) is the potential due to internal mass redistribution; it satisfies Poisson’s equation:

\[ \nabla^2 \phi_2 = -4\pi G \rho_1. \tag{3.34} \]

\[ p = p_0 + p_1 \tag{3.35} \]

We try to find an expression for the pressure gradient which appears in equation 3.30. If a block on which the hydrostatical stress \( \rho_0 \) acts is moved by \( u \), the pressure will change with an amount of:

\[ \vec{u} \cdot \nabla \rho_0, \tag{3.36} \]

so that for the pressure gradient it follows that:

\[ \nabla \vec{p} = \nabla \rho_0 + \nabla (\vec{u} \cdot \nabla \rho_0). \tag{3.37} \]
\( \phi_0 \) is the potential as it was before the load was applied and its gradient is the gravity force which causes the hydrological pre-stress:

\[
\nabla \phi_0 = -\rho \nabla \phi_0. \tag{3.38}
\]

Upon substitution in equation 3.30 this term is canceled out by the product on the right hand side of equation 3.30. When products of two perturbation quantities are neglected, on the right hand side the following terms remain:

- the body force \( \rho_0 \nabla (\phi_1 + \phi_2) \);
- the buoyancy force \( \rho_1 \nabla \phi_0 \).

and we have the following form of the linearized equation for the conservation of linear momentum:

\[
\nabla \sigma + \nabla (\vec{u} \cdot \rho \nabla \phi_0) = \rho_0 (\nabla \phi_1 + \nabla \phi_2) + \rho_1 \nabla \phi_0. \tag{3.39}
\]

Recalling the definition of the potential

\[
\nabla \phi_0 = -\rho_0 g \cdot \vec{e}_r, \tag{3.40}
\]

with \( \vec{e}_r \) the unit vector in radial direction and \( \rho_0 \) and \( g \) the reference density and gravity. The density perturbation is prescribed by requiring conservation of mass [Anderson, 1991], p. 96:

\[
\rho_1 = -\nabla (\rho u). \tag{3.41}
\]

3.41. This brings the equation in the form of [Gilbert and Backus, 1968] and [Farell, 1972] which is adopted by [Peltier, 1985]:

\[
\nabla \cdot \sigma - \nabla (\rho u \vec{e}_r) = \rho_0 \nabla (\phi_1 + \phi_2) - g \nabla (\rho u) \cdot \vec{e}_r. \tag{3.42}
\]

This equation, together with Poisson’s equation for the potential perturbation 3.34 and the continuity equation 3.41, form a coupled set of 3 second-order linear differential equations. The stress \( \sigma \) can be written as a function of displacement and dilatation using the Laplace domain form of the constitutive equation 3.22. One simplification can be made because we are solving for a spherically symmetrical body under an axi-symmetrical load. The toroidal motion (twisting of the hemispheres with respect to each other) and therefore the azimuthal dependence can be removed, hence the variables \( u_r, u_\theta \) and \( \phi \) are a function of radius \( r \) and Laplace transform variable \( s \) alone. Expanding the displacement, the potential perturbation and the dilatation in (vector) spherical harmonics (from now on \( \phi \) will be the l-degree coefficient in the spherical harmonic expansion for the potential perturbation) [Farell, 1972]:

\[
\vec{u}(r, \theta, s) = \sum_{l=0}^{\infty} \left( U_l(r, s) P_l(\cos \theta) \vec{e}_r + V_l(r, s) \frac{\partial P_l(\cos \theta)}{\partial \theta} \right), \tag{3.43}
\]

\[
\phi(r, \theta, s) = \sum_{l=0}^{\infty} \phi_l(r, s) P_l(\cos \theta), \tag{3.44}
\]

\[
\Delta = \chi_l P_l(\cos \theta). \tag{3.45}
\]

\( \theta \) denotes the colatitude and \( P_l(\cos \theta) \) are the Legendre polynomials of degree \( l \), given by Rodrigues’ formula [Arfken, 1970], p.554:

\[
P_l(x) = \frac{1}{2^l l!} \left( \frac{d}{dx} \right)^l (x^2 - 1)^l. \tag{3.46}
\]

\( \chi_l \) in equation 3.45 can be written as [Longman, 1963] \( \chi_l = \frac{\partial \phi_0}{\partial r} + \frac{2}{r} U - \frac{(l+1)l}{r} \). Moreover, for an incompressible Earth model the dilatation can be set to zero.

The expansions in equation 3.45 reduce the three second order differential equations to a system of 6 first-order differential equations which can be written in matrix form as:

\[
\frac{d\vec{g}(r, s)}{dr} = A_l(r, s) \cdot \vec{g}(r, s). \tag{3.47}
\]
The vector \( \vec{g} \) not only contains the \( U_i(r, s) \) and \( V_i(r, s) \) and the potential \( \Phi_j(l, s) \), but also the coefficients for tangential stress \( T_{\theta, t} \), radial stress \( T_{r, t} \) and a variable \( Q_l \) related to the potential gradient; or \( \vec{g} = (U_i, V_i, T_{r, t}, T_{\theta, t}, \Phi_j, Q_l)^T \) [Wu and Peltier, 1982]. These stress coefficients are included in the vector \( \vec{g} \) because a minimum of 6 variables are required to solve a system of 6 equations. Radial and tangential stress are useful because boundary conditions for different loading types can easily be formulated at the surface. The variable \( Q_l \) is defined as [Peltier, 1974]:

\[
Q_l = -\frac{\partial \Phi_j}{\partial r} - \frac{1 + \frac{1}{r}}{r} \Phi_j + 4\pi G \rho_0 U_i. \tag{3.48}
\]

The radial stress can be written as function of strain by using the stress-strain relation, equation 3.22. We are solving in the Laplace domain, therefore the visco-elastic equations are equivalent to the elastic equations in the time domain [Wu and Peltier, 1982]:

\[
\sigma_{r, t}(r, \theta, s) = 2\mu \frac{\partial U_i}{\partial r}. \tag{3.49}
\]

Note that here we assume incompressibility, as there is no dilatation part in the equation. The tangential stress equals:

\[
T_{\theta, t}(r, \theta, s) = \mu (\frac{\partial V_i}{\partial r} - \frac{1}{r} \frac{1}{r} V_i + \frac{1}{r} U_i). \tag{3.50}
\]

With the condition that the solutions must be finite at the origin \( r = 0 \), three starting solutions \( T_1, T_2 \) and \( T_3 \) can be found by ordinary differential equation solving methods that are functions of \( r^d \) and therefore regular at the origin. In a numerical integration method the regular solutions can be propagated from the origin to the surface where they are subjected to boundary conditions depending on the type of loading [Peltier, 1974]. Irregular solutions that are functions of \( r^{-l} \) are also found when solving the differential equations, but since they are singular at the origin they can not be used in a numerical propagation from the origin.

Sabadini et al. [1985b] refers to Takeuchi et al. [1962] for the boundary conditions for tidal loading. However, we are interested in a point mass load treated by Longman [1963] and Farell [1972]. The unit point mass \( \gamma \) can be expanded in:

\[
\gamma = \sum_{l=0}^{\infty} \Gamma_l \cos \theta, \tag{3.51}
\]

with

\[
\Gamma_l = \frac{2l + 1}{4\pi r^2 \epsilon} \tag{3.52}
\]

with \( r_\epsilon \) the radius of the Earth. The linearized surface boundary conditions for three elements of the vector \( \vec{g} \) are the following (Farell, 1972):

- the normal stress is equal to the applied load: \( \sigma_{r, t}(r_\epsilon) = -\gamma g \);
- the tangential stress at the surface is zero: \( \sigma_{\theta, t}(r_\epsilon) = 0 \);
- The potential stress at the surface \( Q_l(r_\epsilon) = -4\pi G \gamma g \).

Upon substitution of equation 3.52, the boundary condition vector becomes:

\[
\vec{b}(r_\epsilon, s) = \begin{pmatrix}
0 \\
0 \\
g \frac{2l + 1}{4\pi r^2} \\
0 \\
-4\pi G \gamma g
\end{pmatrix}. \tag{3.53}
\]

For an impulse load with delta function time dependence the boundary conditions do not depend on \( s \), as can be expected.
In the layers outside the core irregular base functions (in which \( r^{-l} \) appears) possess no singularities and therefore they have to be included. The solutions are contained in the fundamental matrix \([\text{Spada et al., 1992}]\), which is valid outside the core:

\[
Y_l(r, s) = \begin{pmatrix}
\frac{b_l^{l+1}}{2(2l+1)^{2l+1}} & \frac{r^{l-1}}{2(2l+1)(l+1)^{2l+1}} & 0 & \frac{r^{l-2}}{2(2l+1)^{2l+1}} & 0 \\
\frac{1}{2(2l+1)(l+1)} & \frac{r^{l-1}}{2(2l+1)(l+1)^{2l+1}} & 0 & \frac{r^{l-2}}{2(2l+1)^{2l+1}} & 0 \\
\frac{1}{2(2l+1)(l+1)} & \frac{r^{l-1}}{2l(2l+1)(l+1)^{2l+1}} & \frac{r^{l-2}}{2(l+1)^{2l+1}} & 0 & 0 \\
0 & \frac{1}{2l(2l+1)(l+1)^{2l+1}} & 0 & \frac{1}{2(l+1)^{2l+1}} & 0 \\
0 & \frac{1}{2(l+1)^{2l+1}} & 0 & 0 & \frac{1}{l+1} \end{pmatrix}
\] (3.54)

The solution vector \( \vec{y}(r, s) \) can be written as a the fundamental matrix \( Y_l(r, s) \) times \( \vec{C}_l(r) \):

\[
\vec{y}(r, s) = Y_l(r, s) \vec{C}_l(r).
\] (3.55)

The same equation 3.55 holds for each layer \( i (i = 1, N) \), assuming that the parameters are constant inside each layer:

\[
\vec{y}^{(i)}(r_i, s) = Y_l^{(i)}(r_i, s) \cdot \vec{C}_l^{(i)}(r_i).
\] (3.56)

The boundary conditions on a boundary between two internal visco-elastic layers are:

- During deformation there will be no slip and cavitation, therefore all the elements in \( \vec{y} \) except \( Q_l \) are continuous. It can be derived that \( Q_l \) is continuous too;
- No material is assumed to cross an internal boundary (which makes the boundaries chemical rather than phase change).

\( \vec{U}_l, \vec{V}_l, \vec{\sigma}_{rl}, \vec{\sigma}_{l\phi}, \vec{\bar{\varphi}}_l \) and \( Q_l \) are now continuous for each inner boundary, therefore each layer \( i \) can be connected with the layer \( i + 1 \) below it by equalizing the conditions:

\[
Y_l^{(i)}(r_i, s) \vec{C}_l^{(i)}(r_i) = Y_l^{(i)}(r_{i+1}, s) \vec{C}_l^{(i+1)}(r_{i+1})
\] (3.57)

Eliminating the unknown \( \vec{C}_l^{(i)}(r_i) \) from the previous equation and extending the result for all the layers gives the following expression for the solution vector at the surface \([\text{Vermeersen, 1993}]\, [\text{Vermeersen et al., 1996a}]\):

\[
\vec{t}_0(r_c, s) = \left( \prod_{i=1}^{N-1} M_l^{(i)}(r_i, s) \right) \left[ Y_l^{(i)}(r_i+1, s) \right]^{-1} \left[ Y_l^{(N)}(r_c, s) \vec{C}_l^{(N)}(r_c) \right],
\] (3.58)

where \( \vec{t}_0(r_c, s) \) is the boundary vector; subscript \( c \) denotes the core and \( M \) is the fundamental matrix with the first, second and fifth row deleted. The inverse of the fundamental matrix can be written as in \([\text{Vermeersen, 1993}]\) and \([\text{Spada et al., 1992}]\) with corrections in \([\text{Vermeersen et al., 1996b}]\):

\[
Y_l^{-1}(r, s) = D_l(r) \tilde{Y}(r, s),
\] (3.59)

with:

\[
Y_l^{-1}(r, s) = \begin{pmatrix}
\frac{q(r)}{\mu(s)} - 2(l + 2) & 2l(l + 2) & -r & \frac{r}{\mu(s)} & 0 \\
\frac{q(r)}{\mu(s)} + 2(l^2 + 3l + 1) & 2(l^2 - 1) & 0 & 0 & 0 \\
4\pi G \rho & 0 & 0 & 0 & -1 \\
\frac{q(r)}{\mu(s)} + 2(l - 1) & 2(l^2 - 1) & -r & \frac{r}{\mu(s)} & 0 \\
4\pi G \rho & 0 & 0 & 0 & 0 \\
\frac{q(r)}{\mu(s)} - 2(l^2 - 3) & -2l(l + 2) & r & \frac{r}{\mu(s)} & 0 \\
4\pi G \rho & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\] (3.60)
and $\mathbf{D}$ a diagonal matrix with diagonal elements:

$$
\text{diag}(\mathbf{D}(r)) = \\
\left( \frac{(l+1)r^{l+1}}{2l+1} \right) \left( \frac{r^{l+1}}{2l+1} \right) \left( \frac{r^{l+1}}{2l+1} \right)
$$

(3.61)

The $N$-th layer in the Earth model is the core, and we can write the solution vector $\mathbf{y}_c^{(N)}$ at the Core Mantle Boundary (CMB) as:

$$
\mathbf{y}_c^{(N)} = \mathbf{Y}_c^{(N)}(r_c, s) \mathbf{C}_c^{(N)}(r_c) = \mathbf{I}_{c,l}(r_c) \cdot \mathbf{C}_c,
$$

(3.62)

with $C_c$ a vector of constants $(C_1, C_2, C_3)^T$ and the interface matrix $\mathbf{I}_{c,l}(r_c)$ a 6x3 matrix specifying the conditions at the CMB. Conditions on this boundary are under debate [Lambeck, 1988], p.461 but agreement exists on the following (the subscripts denote the component of the y-vector) [Sabatini et al., 1985b, [Wu and Peltier, 1982]:

- The gravity is proportional to $r^l$ for $r \leq r_c$ (the other solution $r^{-l}$ is not defined in the center of the Earth.) so: $y_{b,l}^{(N)}(r_c) = C_1 r_c^l$.
- The vertical displacement is not continuous; the CMB should be considered an equipotential surface. $y_{l,l}^{(N)}(r_c) = y_{b,l}^{(N)} + C_3$.
- The core is assumed to be inviscid so the tangential displacement is not restricted and is equal to an unknown constant: $y_{b,l}^{(N)}(r_c) = C_2$.
- The radial stress is continuous and equal to: $y_{b,l}^{(N)}(r_c) = g_c \rho_c C_3$.
- The tangential stress is zero as the fluid core cannot support tangential stress. $y_{b,l}^{(N)}(r_c) = 0$.
- The parameter $Q$ is continuous.

The matrix becomes:

$$
\mathbf{I}_c(r_c, s) = \\
\left( \begin{array}{ccc}
-\frac{r_c^{l+1}}{A_c} & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & \rho_c A_c r_c \\
0 & 0 & 0 \\
1 & 0 & 0 \\
2(l+1)r_c^{l-1} & 0 & 3A_c \\
\end{array} \right),
$$

(3.63)

with $A_c = \frac{4}{3} \pi G \rho_c$. Only $\rho_c, G, r_c$ and $l$ are contained in the matrix, other parameters in the core need not be specified. For a situation of no loading, the constrained parameters will all be zero at the surface. Therefore equation 3.58 becomes, using equation 3.62 [Vermeersen et al., 1996a]:

$$
\left( \prod_{i=1}^{N-1} \mathbf{M}_i^{(i)}(r_i, s) \left[ \mathbf{Y}_i^{(i)} \right]^{-1}(r_{i+1}, s) \right) \mathbf{I}_{c,l}(r_c) \mathbf{C}_c = 0.
$$

(3.64)

This equation has solutions only if:

$$
\text{Det} \left( \prod_{i=1}^{N-1} \mathbf{N}_i^{(i)}(r_i, s) \left[ \mathbf{Y}_i^{(i)} \right]^{-1}(r_{i+1}, s) \right) \mathbf{Y}_i^{(N)}(r_c, s) = 0.
$$

(3.65)

This is the so-called secular equation ([Vermeersen et al., 1996b] with corrections in [Vermeersen and Sabadini, 1997]). Its roots are interpreted as the inverse relaxation times belonging to a particular mode which may emerge due to a density discontinuity or change in Maxwell relaxation time across a layer boundary. Taking the limit $s \to 0$ we see from equation 3.23 that $\mu(s) \to 0$. This is expected: for long relaxation times the Earth behaves more and more as a fluid, which has no rigidity. For each harmonic degree the secular equation gives the same number of modes. In order to retrieve the complete response the
Strength of each mode should also be known. To find the solution kernels at the roots \( s_j \) we eliminate the matrix \( C_i^{(N)}(r, c) \) from equation 3.58:

\[
\tilde{C}_i^{(N)}(r, c) = \left[ \prod_{i=1}^{N-1} M_i^{(i)}(r, s) \left[ Y_i^{(i)} \right]^{-1} (r_{i+1}, s) Y_i^{(N)}(r, c, s) \right]^{-1} \tilde{h}_i(r, c). \quad (3.66)
\]

The \( U_i, V_i \) and \( \phi_i \) at the surface remain to be computed by projecting the conditions in the mantle \( C_i(r) \) on the first, second and fifth row of the fundamental matrix \( N_i(r, s) \):

\[
\begin{pmatrix} U_i \\ V_i \\ -\phi_i \end{pmatrix} (r_e, s) = \left( \prod_{i=1}^{N-1} N_i^{(i)}(r, s) \left[ Y_i^{(i)} \right]^{-1} (r_{i+1}, s) \right) I_{e,f}(r_e) \tilde{C}_e. \quad (3.67)
\]

which is the same as equation 3.58 with \( M \) replaced with \( N \) and using equation 3.62. Combining the two previous equation yields a direct expression for the unconstrained parameters of the solution vector:

\[
\begin{pmatrix} U_i \\ V_i \\ -\phi_i \end{pmatrix} (r_e, s) = \left( \prod_{i=1}^{N-1} N_i^{(i)}(r, s) \left[ Y_i^{(i)} \right]^{-1} (r_{i+1}, s) \right) I_{e,f}(r_e) \cdot \left[ \prod_{i=1}^{N-1} M_i^{(i)}(r, s) \left[ Y_i^{(i)} \right]^{-1} (r_{i+1}, s) I_{e,f}(r_c) \right]^{-1} \tilde{h}_i(r, s). \quad (3.68)
\]

Earth models for a finite number of layers can be solved. The parameters that for each layer need to be given are those that appear in the momentum equation and the constitutive equation, the continuity equation and the Poisson equation: rigidity, density and viscosity. These parameters and the depth of the layers form the Earth model.

### 3.6 Transformation to Time Domain

Equation 3.68 provides the spectral solution in the Laplace domain. As [Peltier, 1974] noted, it is advisable to first perform an inverse Laplace domain before summing over all Legendre degrees (equation 3.45). Transforming the Laplace domain solutions to the time domain is not a trivial task. [Wu, 1978] developed an elegant method for the transformation based on complex contour integration, as described below.

From the secular equation 3.65 follow a set of \( m \) \( s \)-values, the poles (appendix A describes how the zeros of the secular equation are found by using a bisection algorithm). In the general case of an analytical function in the complex plane which is analytical everywhere except in a singularity \( z = z_0 \) the function can be written as a Laurent series [Arfken, 1970], p.320:

\[
f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad (3.69)
\]

where \( z_0 \) are the poles of the integrand. If all poles are assumed to be of first order (see section 3.7 for a remark on this) the Laurent series of a solution to the surface loading problem becomes (in vector form):

\[
\tilde{n}_i(s) = \tilde{n}_i^{(i)} + \sum_{j=1}^{m} \frac{\tilde{f}_j}{s + \delta_j}. \quad (3.70)
\]

Positive roots are found for example when density inversions exist in a multi-layer Earth model [Vermeersen and Sabadini, 1997], but here the poles of the secular equation are assumed to lie all on the negative real axis, which agrees with stable exponential relaxation. Therefore the Laurent series can be Laplace inverted to give the solution in the space domain:

\[
\tilde{n}_i(t) = \tilde{n}_i^{(i)} \delta(t) + \sum_{j=1}^{m} \tilde{f}_j e^{-\delta_j t}. \quad (3.71)
\]
To obtain these expressions use has been made of closed contour integration over the Bromwich path. The residues \( \mathcal{R}^n_i \) are given by [Wu, 1978] for the original reference and [Vermeersen et al., 1996b] for the notation that is used here:

\[
\mathcal{R}^n_i = \left( \frac{B_M I_{c,1}(r_e) \cdot (B_N I_{c,1}(r_e))^{-1}, \det(B_M I_{c,1}(r_e))}{\frac{d}{ds} \det(B_M I_{c,1}(r_e))} \right)_{s = s_i}, \tag{3.72}
\]

in which has been written for shorter notation:

\[
B_M = M^{(i)}_i Y^{(i)}_i (r_{i+1}, s) \tag{3.73}
\]

\[
B_N = N^{(i)}_i Y^{(i)}_i (r_{i+1}, s). \tag{3.74}
\]

The solutions in the time domain are usually expressed in dimensionless Love numbers as [Wu and Peltier, 1982]:

\[
\begin{bmatrix}
U_i(r, s) \\
V_i(r, s) \\
\Phi_{2, f}(r, s)
\end{bmatrix} = \Phi_{2, n}(r) \begin{bmatrix}
h_i(r, s)/g_0 \\
\ell_i(r, s)/g_0 \\
-k_i(r, s)
\end{bmatrix}.
\]

The potential of the unit mass is [Farell, 1972]:

\[
\phi_2(\mathbf{r}) = \int \frac{G \gamma d^3 r'}{|r' - r|^3}. \tag{3.75}
\]

At the surface where \( r = r_e \), the potential turns out to be:

\[
\phi_{2, n}(r_e) = \frac{r_e \theta_e}{M_e}. \tag{3.76}
\]

with \( M_e \) the mass of the Earth, and \( g_e \) the surface gravity. If all the parameters are taken dimensionless in the computation, the visco-elastic Love numbers are given by equation 3.71. The first term has the same delta-function form as the load and therefore represents the elastic response. The second term is a summation of \( m \) viscous terms:

\[
h_i(t) = h_i^E \delta(t) + \sum_{j=1}^{m} v_i^j e^{s_i^j t}. \tag{3.77}
\]

\[
k_i(t) = k_i^E \delta(t) + \sum_{j=1}^{m} v_i^j e^{s_i^j t}. \tag{3.78}
\]

High angular degree Love numbers correspond to small scale features, and moreover they are found to be insensitive to the deep interior of the Earth [Vermeersen and Sabadini, 1997].

### 3.7 Comments on the Normal Mode Analysis

The normal mode theory which is laid out in the previous two sections is the common method to compute the response of a visco-elastic Earth to surface loads. This section contains some extensions and supposed problems.

Han and Wahr [1995] found, using numerical codes, that each internal boundary introduces three new modes to the modal spectrum: a pair of visco-elastic or transient modes and a buoyancy mode. An analytical proof for a 2 layer non-self-gravitating incompressible Earth model is given in [Wu and Ni, 1996], who show that a density discontinuity in a viscous Earth model gives at most two eigenvalues. It is important to note however, that modes arise in the first place because of viscosity of the Maxwell rheological model. It is also shown that a different Maxwell relaxation time on each side of the boundary introduces two eigenmodes simultaneously; they can not be attributed to either the shear modulus or the viscosity. Already for a 2-layer model the expressions, generated with
use of a software package, are so long that they require the simplification of non self-gravitation in order to be able to present the results. However, the assumption of self-gravitation has no effect on the number of modes.

With the matrix propagation method, the Love-numbers can be computed for a multi-layer Earth model. In the model the density, rigidity and viscosity are kept constant within each layer, while in the real Earth these parameters vary continuously with depth. Thus, it seems the real Earth is best approximated with as much layers as possible, to reflect the continuous behavior. This poses computational difficulties as each new layer requires extra computation time. With a layering profile according to the PREM-model of Dziewonski and Anderson [1981] transient modes of short relaxation time become densely packed in a small interval [Han and Wahr, 1995] where they are difficult to separate. Han and Wahr [1995] also predict the emergence of a dense spectrum of long-term buoyancy modes but for low s-values convergence problems arise. The system of 6 equations turns into a system of 2 equations for a fluid which is numerically unstable [Wu and Ni, 1996]. However, Vermeersen et al. [1996b] show that it is better to use a smaller number of layers, and find all the modes, than use more layers and run the risk of missing some modes.

One supposed short-coming claimed by Fang and Hager [1995] arises from the definition of the Laplace transformed Lamé parameters:

\[
\lambda(s, r) = \frac{\lambda s + \mu r}{s + \frac{4\pi}{\rho}}, \tag{3.79}
\]

and

\[
\mu(s, r) = \frac{\mu s + \lambda r}{s + \frac{4\pi}{\rho}}, \tag{3.80}
\]

which become singular if the denominator equals zero, thus if \( s = -\frac{4\pi}{\rho} \). For N layers there exist N singular points, and when \( N \rightarrow \infty \) the singular points fill the entire interval \([s_{\text{min}}, s_{\text{max}}]\). A computational problem arises because root-finding procedures require an interval around a root to be able to detect it. However, since it is known how many roots are to be found, it will be clear whenever a root is not detected. In most cases all modes will be found for one degree, but for the next higher degree some are missing. Because Love numbers vary smoothly with degree, a small s-domain can be identified in which the ‘missing’ mode should be located. The root-finding procedure can be executed with higher accuracy in that domain. In most cases the missing root will carry a negligible strength, as can be verified for transient modes with the fluid Love number test (see appendix A).

Another claim of [Fang and Hager, 1995] was that a cluster of singular points (which they dubbed a ‘continuous pole’) may have a non-zero contribution. To investigate this possible contribution they defined an isolation function as:

\[
I_w(x) = \lim_{t \to 0} \frac{1}{2\pi i} \oint_{C(x)} w(z)dz, \tag{3.81}
\]

with \( \epsilon \) the height of the integration path along the imaginary axis. If \( w(z) \) has a pole at \( x_0 \) the isolation function is equal to the residue of the singular point, and therefore will reveal whatever signal will be in the cluster of singular points. Vermeersen and Sabadini [1997] have shown that using the isolation function is nothing more than an alternative for transforming the solutions to the time domain and therefore non-modal signals found by Fang and Hager [1995] do not result from singular points but rather are the results of numerical errors or the number of layers being insufficient. Vermeersen and Sabadini [1997] refute their claim that singularities carry any strength at all, arguing that singular points pose no problems in an analytical model. For a 2 layer model the singularities can simply be removed by multiplying the secular equation by factors containing the Maxwell relaxation times \( \tau_1 \) and \( \tau_2 \): \((s + \tau_1)^2(s + \tau_2)^2\). The secular function becomes continuous everywhere and still has the same roots. Generalizing the results for the 2 layer model, it seems that in a N-layer model the singularities can be removed by multiplying by \( \Pi_{i=1}^{N} (s + \tau_i)^2 \) [Wu and Ni, 1996].
Vermeersen and Sabadini [1997] leave the possibility open for higher order poles to exist. If they do, the Laurent series approximation of the analytical secular function (equation 3.69) in the complex domain does not hold. However, if higher order poles would emerge, residues would become infinite, and this has never been observed.
Chapter 4

Sea Level Adjustment

In the previous chapter, expressions for potential perturbation and horizontal and vertical displacement in the form of dimensionless Love numbers were presented. The main purpose of this chapter is to present the method for computing geoid anomalies as a result of GIA. The relation between ice melt and geoid anomalies is governed by the so-called ‘Sea Level Equation’ (SLE); the derivation of this equation will be given in 4.1, where it will be clear that solving the sea level equation also gives the geoid anomalies. Section 4.2 gives a spectral approach for solving the sea level equation. The method used in this thesis is the pseudo-spectral method, which is treated in 4.3. The chapter concludes with a description of a finite element method that is used for comparison with the pseudo-spectral sea level code.

4.1 Sea Level Equation

In this section we derive an expression for the sea level equation, which is required for computing any geophysical observable. We will see that substituting in the expression for the Relative Sea Level (RSL) different combinations of Love numbers yields different parameters such as the geoid anomaly and the free-air gravity anomaly.

The sea level is nothing more than the shape of an equipotential surface of the gravitational potential. Inclusion of the rotational potential is not trivial. In this thesis the effect of rotation is neglected, (see [Milne and Mitrovica, 1998] for inclusion of rotation). In section 2.4, the perturbation potential was assumed to be unknown and it was shown that it can be derived from surface gravity measurements. Here we will model the potential perturbation resulting from a certain surface load, namely the change in ice sheet height. Computed sea levels and geoid anomalies are therefore a consequence of ice growing and melting alone. We must keep in mind however that real sea level data and geoid data are influenced by other phenomena such as tectonic movement.

The perturbation potential can be calculated from a space-time convolution of the surface load $L(\theta, \psi, t)$ and the gravitational potential perturbation Greens function $\phi(\theta, \psi, t')$ [Mitrovica and Peltier, 1991]:

$$\Phi(\theta, \psi, t) = \int_{-\infty}^{t} \int_{\Omega} \int r_{\theta}^2 L(\theta', \psi', t') \phi(\chi, t - t') d\Omega' d\rho' , \quad (4.1)$$

in which $r_e$ is the radius of the Earth. The factor $r_{\theta}^2$, not written in [Mitrovica and Peltier, 1991], appears as the integration is over the full solid angle, in accordance with [Milne and Mitrovica, 1998] and [Milne et al., 1999]. $\theta$ and $\psi$ are latitude and longitude of the point on the sphere where the potential is to be computed; the point with prime moves along all the points on the sphere where the load acts. In the same way the solid Earth radial displacement can be calculated using the appropriate Greens function:

$$R(\theta, \psi, t) = \int_{-\infty}^{t} \int_{\Omega} \int r_{\theta}^2 L(\theta', \psi', t') \Gamma(\gamma, t - t') d\Omega' d\rho' . \quad (4.2)$$
The gravitational potential perturbation Greens function can be separated in a part resulting from the load itself and a part resulting from the mass redistribution inside the Earth:

$$\phi(\chi, t) = \phi^L(\chi, t) + \phi^{MR}(\chi, t)$$  \hfill (4.3)

The load part can be expanded in spherical harmonics as [Farell, 1972]:

$$\phi^L(\chi, t) = \frac{r^2 \theta}{M_E} \sum_{l=0}^{\infty} P_l(\cos \chi) \delta(t),$$  \hfill (4.4)

in which $M_E$ is the mass of the Earth.

A Greens function of a system of differential equations is defined as the delta function response of the system. Recalling the definition of Love numbers in equation 3.6 we see that the load Love numbers are the dimensionless coefficients in the spherical harmonic expansion for the Greens function. With the expression for the radial displacement Love number $k_l$ in equation 3.77 and the potential perturbation Love number $k_l$ in equation 3.78 and equation 3.76 we can write for the Greens functions $\phi(\chi, t)$ and $\Gamma(\chi, t)$ [Mitrovica and Peltier, 1991] [Milne et al., 1999]:

$$\phi(\chi, t) = \frac{r^2 \theta}{M_E} \sum_{l=0}^{\infty} \left( k_l^F \delta(t) + \sum_{k=1}^{K_l} r_k^l e^{-i k l t} \right) P_l(\cos \chi)$$  

$$+ \frac{r^2 \theta}{M_E} \sum_{l=0}^{\infty} P_l(\cos \chi) \delta(t)$$  \hfill (4.5)

$$\Gamma(\chi, t) = \frac{r^2 \theta}{M_E} \sum_{l=0}^{\infty} \left( h_l^F \delta(t) + \sum_{k=1}^{K_l} r_k^l e^{-i k l t} \right) P_l(\cos \chi)$$

with $P_l(\cos \chi)$ the Legendre polynomials as function of the solid angle $\chi$.

The expressions for the radial displacement and gravitational potential perturbation Greens functions are the building blocks for quantities that are more relevant in a geophysical sense. The general form of the Greens function for such a quantity can be written as:

$$X(\chi, t) = \sum_{l=0}^{\infty} (X_l^F \delta(t) + \sum_{k=1}^{K_l} X_k^l e^{-i k l t} P_l(\cos \chi))$$  \hfill (4.6)

Subsequently will be discussed the geoid anomaly, the free-air gravity anomaly and the relative sea level.

The geoid anomaly, which we continue to denote by $N$, due to ice melting is (compare Brun’s equation 2.10):

$$N = \frac{\phi(\theta, \psi, t)}{g} + \mathcal{G}(t)$$  \hfill (4.7)

In this equation $\mathcal{G}(t)$ is the mass conservation term, included to express the fact that the sea surface (i.e. the geoid) may shift to a new equipotential surface. If melt water is added to the ocean the ocean surface is still an equipotential surface, but a new one which is higher than the unperturbed situation in which it was assumed to be equal to the geoid. Mass conservation during the transition of ice to water or vice versa is assured by the term $\mathcal{G}(t)$, which also includes the change in potential due to solid Earth displacement. It is given as:

$$\mathcal{G}(t) = -\frac{M(t)}{\rho w A_0} - \frac{1}{A_0} \int_{\Omega} \mathcal{N}(\theta, \psi, t) - R(\theta, \psi, t)$$  \hfill (4.8)

The geoid anomaly Greens function becomes:

$$N = \frac{r^2 \theta}{M_E} \sum_{l=0}^{\infty} \left( k_l^F \delta(t) + 1 + \sum_{k=1}^{K_l} r_k^l e^{-i k l t} \right) P_l(\cos \chi) + \mathcal{G}(t)$$  \hfill (4.9)

Mitrovica and Peltier [1989] give the correct expression for the $l$-th degree coefficient in the spherical harmonic expansion of the free-air gravity anomaly (before people had been
computing the gravity anomaly at the perturbed surface instead of the anomaly reduced to the geoid:
\[ g^{FA}(s) = \frac{g}{M_E} (I + 2 - (I - 1)k(s)) \]  \hspace{1cm} (4.10)

The free-air gravity anomaly can be computed with equation 4.6 with \( X_k^F = \frac{g(I+2)(-1)^kF_k}{M_E} \) and \( X'_k = \frac{g(I+1)F_k}{M_E} \).

If the geoid defines the top of a column of water in the ocean, then the bottom is the solid ocean surface. The relative sea level is then given by the difference between geoid height \( \mathcal{N} \) and the ocean floor displacement \( \mathcal{R} \):
\[ S(\theta, \phi, t) = \mathcal{N}(\theta, \phi, t) - \mathcal{R}(\theta, \phi, t) \]  \hspace{1cm} (4.11)

Solving this equation yields the difference between the geoid anomaly and the radial surface displacement on the entire Earth. In order to get relative sea level over ocean areas only one simply multiplies with an ocean function which is defined as:
\[ C(\theta, \phi) = \begin{cases} 1 & \text{in the oceans} \\ 0 & \text{elsewhere} \end{cases} \]  \hspace{1cm} (4.12)

Knowing that the average sea level rise since deglaciation is 110 m, [Tushingham and Peltier, 1991] it is likely that after deglaciation some coastal areas were flooded that were dry before. In this study however, the shorelines are assumed to be constant throughout the rebound process.

For GIA the surface load consists of the ice, with density \( \rho_i \) taken to be \( 920 \frac{kg}{m^3} \), and the water mass with density \( \rho_w \) of \( 1000 \frac{kg}{m^3} \):
\[ L(\theta', \phi', t') = \rho_i L(\theta', \phi', t') + \rho_w S(\theta', \phi', t'). \]  \hspace{1cm} (4.13)

In this equation \( I \) is the height of the ice that is added or removed at time \( t' \) and \( S \) is the incremental sea level height (with respect to the RSL at time \( t=0 \)). Using equations 4.1 and in equation 4.11 to 4.12 into equation 4.11 yields the following form of the sea level equation for a Maxwell viscoelastic non-rotating Earth model:
\[ S(\theta, \phi, t) = C(\theta, \psi, t) \left( \int_{-\infty}^{t} \int_{\Omega} r_0^2 (\rho_i I(\theta', \psi', t') + \rho_w S(\theta', \psi', t')) \right) \]
\[ \times \left[ \frac{\phi(x, t - t')}{g} - \Gamma(x, t - t') \right] d\Omega' dt' + g(t) \]  \hspace{1cm} (4.14)

The equation is an integral equation that can only be solved iteratively because the sea level is present on the left hand side and on the right hand side of the equation. A solution to the equation is gravitationally self-consistent since it obeys external mass conservation and it is consistent with solid Earth displacement.

A remark must be made about the elastic terms which are still present in equation 4.6, 4.6 and 4.9 whereas in [Mitrovica and Peltier, 1989] they are not. If both growth and decay of an ice sheet are modeled and the unperturbed state of the Earth is the reference state, an elastic depression rebounds as soon as the load is removed and seems to have zero net effect. If however only the deglaciation phase is modeled, the elastic response will have to be present in the signal. Moreover, for a self-consistent relative sea level computation it is required to take the elastic response in account, because an elastic ocean floor displacement leads to a different solid Earth displacement and ocean load which over time induces contributions to gravitational potential and radial displacement.

To be able to handle the time integral, the ice and ocean load are generally modeled as a series of \( N \) Heaviside step load increments [Wu and Peltier, 1983a], [Mitrovica and Peltier, 1991]:
\[ I(\theta, \psi, t) = \sum_{n=0}^{N} \delta I^n(\theta, \psi) H(t - t_n) \]
\[ S(\theta, \psi, t) = \sum_{n=0}^{N} \delta S^n(\theta, \psi) H(t - t_n). \]  \hspace{1cm} (4.15)
It is now possible to derive an analytical solution of the time integral. The elastic Love number of the RSL incorporates the elastic contributions from three effects:

\[ E_t = 1 + k_t^E - h_t^F. \] (4.16)

where \( k_t^E \) is the elastic Love number for gravitational potential, \( h_t^E \) the elastic Love number for radial displacement and the factor 1 comes from the load itself. Substituting the radial displacement and gravitational potential Greens functions together with equation 4.13 into the sea level equation 4.14 yields for the elastic part:

\[
\begin{aligned}
C(\theta, \psi) \left( G(t) + \int \int_{\Omega} r_\theta^2 \left( \left\{ p_1 \delta I^\nu(\theta', \psi') + p_2 \delta S^\nu(\theta', \psi') \right\} \right) \right) \sum_{l=0}^{\infty} \left( \frac{1}{M_e} \sum_{l=0}^{\infty} (1 + k_l^E - h_l^E) \right) P_l(\cos(\chi)) \int d\Omega \\
\end{aligned}
\] (4.17)

The viscous part of the SLE contains functions of time that can be difficult to integrate. However, because of the Heaviside function, \( \int_{t_0}^{t_\infty} \) in equation 4.14 can be replaced by \( \int_{t_0}^{t_\infty} \) and primitive functions can easily be found. The ice and sea load times the Heaviside function give the respective ice and sea level increments. The viscous part of the sea level equation becomes:

\[
\begin{aligned}
C(\theta, \psi) \sum_{n=0}^{\infty} H(t - t_n) \int \int_{\Omega} r_\theta^2 \left( \left\{ p_1 \delta I^\nu(\theta', \psi') + p_2 \delta S^\nu(\theta', \psi') \right\} \right) \sum_{l=0}^{\infty} \frac{\alpha}{M_e} \sum_{k=1}^{K} r_k^L - r_k^E \left[ 1 - e^{-s_k(t-t_n)} \right] P_l(\cos(\chi)) \int d\Omega \\
\end{aligned}
\]

The factor that contains all the information on the viscous behavior of a specific Earth model is labeled \( \beta \) by [Mitrovica and Peltier, 1991]:

\[ \beta(t, t_n) = \sum_{k=1}^{K} \frac{r_k^L - r_k^E}{s_k^L} \left[ 1 - e^{-s_k(t-t_n)} \right] \] (4.18)

\( \beta \) is written here as function of the difference \( (t - t_n) \) because in this way it is used in the sea level codes. It is used later to demonstrate some of the characteristics of Earth models with a LVZ. Note that we can solve for the geoid anomaly instead of the RSL if we remove from the elastic Love number for RSL the elastic Love number for radial displacement \( h_t^E \), and if we remove the residues for the visco-elastic Love number for radial displacement.

### 4.2 Spectral Approach

[Mitrovica and Peltier, 1991] present two ways of obtaining a gravitationally self-consistent solution of the sea level equation 4.14: a full-spectral and a pseudo-spectral method. This section until equation 4.23 derives the spectral sea level equation which is valid for both. The pseudo-spectral method in the next section continues from equation 4.23. The advantages of a spectral method compared to a finite element or finite disc method include [Laprise, 1981], p.1:

- no problems occur if load point and computation point are identical;
- the mass conservation property is simply handled;
- spectral coefficients is the natural representation form for constructing a power spectral density function and for filtering.

A single valued function on a sphere (e.g. global sea level, the ocean function) \( \zeta(\theta, \phi) \) can be expanded into a series of spherical harmonics as [Hobson, 1931]:

\[ \zeta(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Q_m \tilde{S}_{lm}(\theta, \phi). \] (4.19)
where $\theta$ denotes colatitude running from 0 to $\pi$, and $\psi$ denotes longitude from 0 to $2\pi$. The spherical harmonics are solutions to the angular part of the Laplace equation in spherical coordinates, and therefore they are suitable as solutions of the sea level equation. The harmonics $\tilde{Y}_{lm}$ are related to the associated Legendre polynomials [Lambeck, 1988], with order $m$ and degree $l$ by:

$$\tilde{Y}_{lm} = \sqrt{\frac{2l+1}{(l-m)!}} P_{lm}(\cos \theta) e^{im\phi},$$

(4.20)

in which the normalization is chosen so that the integral over the sphere gives: [Mitrovica and Peltier, 1991]

$$\int_0^{2\pi} \int_0^\pi Y_{lm}(\theta, \psi) Y_{lm}^*(\theta, \psi) \sin \theta d\theta d\psi = 4\pi \delta_{mm'} \delta_{ll'}. $$

(4.21)

(Another common definition of the normalization does not have the 4$\pi$ factor and sometimes a factor $(-1)^m$ is put before the square root, see e.g. [Arfken, 1970], p.571). All the spatial dependent quantities in the sea level equation 4.14 (the ocean function, the ice load increment and the total ice load, the sea level increment and the total sea level variation) can be transformed to the spectral domain. A more detailed explanation of transformation to and from the spectral domain can be found in B. In the sea level equation 4.14 integrals of field quantities multiplied with a Legendre polynomial in the Greens function appear. Because of the properties of the spherical harmonics these factors can be rewritten as:

$$\int_\Omega \int_\Omega \zeta(\theta, \psi) P(\cos \chi) d\Omega' = \frac{4\pi r_\varepsilon^3}{2l+1} \sum_{m=-l}^l \zeta_m Y_{lm}(\theta, \psi).$$

(4.22)


$$\sum_{l,m} s_{lm}(t) Y_{lm}(\theta, \psi) = \sum_{r,s} C_{rs} Y_{rs}(\theta, \psi) \sum_{l,m} \left\{ G(t) + \sum_{l,m} E_l T_l \left( \rho_l I_{lm}(t_j) + \rho_w S_{lm}(t_j-1) \right) Y_{lm}(\theta, \psi) \right\} + \sum_{l,m} T_l \sum_{n=0}^N \left( \rho_l \delta I_{lm}^n + \rho_w \delta S_{lm}^n \right) \beta(l, t - t_n) H(t - t_n) Y_{lm}(\theta, \psi).$$

(4.23)

in which $C_{rs}, I_{lm}, S_{lm}, \delta I_{lm}$ and $\delta S_{lm}$ represent the spherical harmonic coefficients of the ocean function, the total ice load, the total sea load, the ice load increment and the sea load increment, respectively. $t_n$ is the time at which the Heaviside melting takes place and $t_j$ is the time at which the response is to be calculated. A shorter notation for the double summation has been used:

$$\sum_{l,m} = \sum_{l=0}^\infty \sum_{m=-l}^l$$

(4.24)

Note that all dimensional parameters ($r_\varepsilon^2$ of equation 4.14, $\frac{4\pi}{2l+1}$ of equation 4.22 and $\frac{3r_\varepsilon^3}{M_\varepsilon}$ of equation 4.6) are packed in $T_l$:

$$T_l = \frac{4\pi r_\varepsilon^3}{M_\varepsilon (2l+1)}$$

(4.25)

The values of the average Earth radius $r_\varepsilon$ and the Earth mass $M_\varepsilon$ are, respectively 6371 km and 5.976 $\cdot 10^{24}$ kg.

The advantage of the spectral formulation becomes clear as the convolution of the ice and sea loads with the Greens function in the spatial domain is replaced by a simple multiplication in the spectral domain. With the finite disc method the spatial convolution leads to matrices containing Greens functions coefficients which have to be computed again for each disc size and distance from the center of the disc to the computation point,
see appendix C. Additionally, the error made in approximating the sea level is uniform and depends on the truncation \( L \).

Equation 4.23 is not a recipe for direct computation because the spectral coefficients for the sea level still appear on the left hand side and on the right hand side. We proceed by writing [Mitrovica and Peltier, 1991]:

\[
\delta S_{lm}(t_j) = S_{lm}(t_j) - S_{lm}(t_{j-1}) \tag{4.26}
\]

The sea level equation has to be solved only to find the increment \( \delta S_{lm}^j \) at time \( t_j \). Putting equation 4.26 in equation 4.23 yields:

\[
\sum_{l,m} \delta S_{lm}^j Y_{lm} = - \sum_{l,m} S_{lm}(t_{j-1}) Y_{lm} + \left\{ \sum_{r,s} C_{rs} Y_{rs} \right\} \cdot \left( G(t_j) + \sum_{l,m} E_l T_l \left( \rho_i I_{lm}(t_j) + \rho_\omega S_{lm}(t_{j-1}) \right) + \rho_\omega \delta S_{lm}^j \right) Y_{lm} + \sum_{l,m} T_l \sum_{n=0}^{j-1} \beta(l, t_n, t_j) \left( \rho_d I_{lm}^n + \rho_\omega \delta S_{lm}^n \right) Y_{lm} \tag{4.27}
\]

A procedure is applied which is known as the Galerkin procedure: multiply with a smooth arbitrary test function, in this case \( Y^\ast_{pq}(\theta, \phi) \), and integrate over the domain, in this case the Earth’s surface. We get an infinite set of linear algebraic equations, which has to be truncated. The main computation effort of solving these equations goes into generating a matrix which consists of coefficients \( M^{1^\text{st}}_{pq} \):

\[
M^{1^\text{st}}_{pq} = \sum_{r,s} C_{rs} \int_\Omega Y^\ast_{pq} Y_{rs} Y_{lm} \sin \theta \, d\theta \, d\psi.
\tag{4.28}
\]

In order to compute these matrix elements, the order and degree are truncated at a certain degree. Mitrovica and Peltier [1991] find that for a truncation degree which is 3 higher than the previous, the computational effort for generating the matrix of equation 4.28 doubles, so in practice computation is limited to around degree 30. This is not enough to examine large RSL or geoid variation over a small distance, e.g. near continental margins or ice sheet boundaries, as a LVZ is expected to have. In the next section we see that the nuisance of projecting on the ocean area is solved by performing the projection in the space domain.

### 4.3 Pseudo-spectral Approach

The pseudo-spectral method is derived from circulation models in atmospheric sciences where they have been in use since the seventies [Laprise, 1981]. The derivation starts with the spectral sea level equation in the form of equation 4.27. Rewriting it in a more compact form:

\[
\sum_{l,m} \delta S_{lm}^j Y_{lm} = - \sum_{l,m} S_{lm}(t_{j-1}) Y_{lm} + \left\{ \sum_{r,s} C_{rs} Y_{rs} \right\} \cdot \left( \sum_{p,q} R_{pq}(t_j) Y_{pq} + G(t_j) \right), \tag{4.29}
\]

with

\[
R_{lm}(t_j) = E_l T_l \left( \rho_i I_{lm}(t_j) + \rho_\omega S_{lm}(t_{j-1}) \right) + \rho_\omega \delta S_{lm}^j + T_l \sum_{n=0}^{j-1} \beta(l, t_n, t_j) \left( \rho_d I_{lm}^n + \rho_\omega \delta S_{lm}^n \right). \tag{4.30}
\]

The second term on the right hand side of this equation represents the sea level change up to time \( t_j \). If we assume that the \( i \)th estimate for the sea level increment at time \( t_j \), \( \left[ \delta S_{lm}^j \right]^i \), exists, we can plug this value into equation 4.30. Equation 4.29 then provides an estimate for the sea level increment:

\[
\sum_{l,m} \left[ \delta S_{lm}^j \right]^{i+1} Y_{lm} = - \sum_{l,m} S_{lm}(t_{j-1}) Y_{lm} + \left\{ \sum_{r,s} C_{rs} Y_{rs} \right\} \cdot \left( \sum_{p,q} \left[ R_{pq}(t_j) \right]^i Y_{pq} + G(t_j) \right). \tag{4.31}
\]
The computation of the coefficients $R_{lm}$ at time $(t_j)$ is straightforward once the input parameters (ice and sea load increments, total ice and sea load) are expanded in spherical harmonics, but the product of the summations over $p$ and $q$, and $r$ and $s$ seems awkward. But not if we see both sums as transformation of the respective variables, which leaves a multiplication in the space domain! The second summation gives:

$$R^j(\theta, \psi, t_j) = \sum_{p,q} [R_{pq}(t_j)]^j Y_{pq}. \quad (4.32)$$

The first sum does the same for the ocean function so that the multiplication leaves the spatial form of $R$ projected on the ocean function:

$$RO^j(\theta, \psi, t_j) = C(\theta, \psi) R^j(\theta, \psi, t_j). \quad (4.33)$$

Because the sea level equation is in spectral form, $RO$ has to be transformed to the spectral domain again. Thus we have replaced the term $\sum_{r,s} C_{rs} Y_{rs} \cdot \sum_{p,q} [R_{pq}(t_j)]^j Y_{pq}$ by

$$\sum_{l,m}[RO_{lm}(t_j)]^j Y_{lm}$$

and in this way circumvented the difficulties which arose in the computation of 4.28 in the full spectral method. The pseudo-spectral sea level equation becomes:

$$\sum_{l,m} \delta S^j_{lm} Y_{lm} = - \sum_{l,m} S^j_{lm}(t_{j-1}) Y_{lm} + \left\{ \sum_{l,m}[RO_{lm}(t_j)]^j Y_{lm} + \sum_{r,s} C_{rs} Y_{rs} G(t_j) \right\}. \quad (4.34)$$

Because of the orthogonality property of the spherical harmonics it follows that the previous equation is also valid for each degree and order:

$$\delta S^j_{lm} = S^j_{lm}(t_{j-1}) + RO_{lm}(t_j) + G(t_j) C_{lm}. \quad (4.35)$$

When computing the term $RO_{lm}(t_j)$ we need a first guess for the sea level increment, for which we take the eustatic sea level increment. From mass conservation of the load it follows that

$$-\rho \delta J_{00}(t_j) + \rho_w \delta S_{00}(t_j) = 0. \quad (4.36)$$

(Note that mass conservation removes the term with zero order and degree from the sea level equation). The eustatic sea level increment reads [Mitrovica and Peltier, 1991]:

$$\delta S^{j-1}_{lm} = \frac{\rho_4 \pi r^2}{\rho_w A_w} \delta J_{00}(t_j) C_{lm}. \quad (4.37)$$

Multiplication with the expansion of the ocean function:

$$\delta S^{j-1}_{lm}(t_j) = C_{lm} \delta S^{j-1}. \quad (4.38)$$

Note that for the pseudo-spectral method the ocean function is needed in both expanded as spatial form for equation 4.33.

The entire right hand side of equation 4.35 is known, except for the last term. It can be computed by stating that mass must be conserved [Mitrovica and Peltier, 1991]:

$$[G(t_j)]^j = \left( \frac{\rho}{\rho_w} f_{00}(t_j) + [RO_{00}(t_j)]^j \right) (C_{00})^{-1}. \quad (4.39)$$

(Compare this to equation 4.8). The input parameters ice load increments in meters and ocean functions are assumed to be transformed to the spectral domain (this expansion is termed global spherical harmonic analysis, see B). A new approximation $[\delta S_{lm}]^{j+1}$ is computed and the entire process is repeated until a specified level of convergence is reached. The pseudo-spectral method can be programmed efficiently because a difficult time convolution is avoided. However, because the computation is in the time domain it is not possible to use the Laplace form of the Greens functions and multiply with the Laplace form of the load to obtain the response to forcings with more complicated time dependence than equation 4.15 (see for example [Vermeersen, 1993] who applies a linear mountain growth for simulation of Earth rotational characteristics).
4.4 Finite Element Method

A completely different method for solving the Sea Level Equation was presented in [Wu, 2002] and [Wu, 2003] as the Coupled Poisson Finite Element method. This method does not follow the Love number approach. Instead, a grid of 18 layers and 360 elements in colatitude direction is constructed. The surface load is expressed in terms of forces at the grid points, which can be related to the displacements at the nodes. The expression that relates the two is the Principle of Virtual Work, which states that the work done by the external forces (surface loads) at the nodes is equal to the work done by the stresses and distributed forces (body forces, initial stress) expressed in terms of the virtual displacements [Zienkiewicz, 1977]. Time-dependent deformation is handled by separating the strain in an elastic part and a creep part:

$$\varepsilon = \varepsilon_e + \varepsilon_c$$  \hspace{1cm} (4.40)

with

$$\varepsilon = \mathbf{D}^{-1}\sigma,$$  \hspace{1cm} (4.41)

where $\mathbf{D}$ contains combinations of elastic parameters such as the Lamé parameters or the elasticity modulus and Poisson’s ratio. The creep part can be written in the same way, with a matrix $\mathbf{D}$ now containing differential or integral operators so that the stress strain-rate relations are those pertaining to the linear models discussed in 3.4. It is also possible to incorporate a non-linear rheology (see [Wu, 1992]). Note that the strains can always be written as function of displacements by using a matrix of linear (differential) operators.

Provided enough boundary conditions and forces are given, the displacements at the nodes can be solved, and the nodal values are linearly interpolated to give the values at any desired point. From the displacements at the nodes, strain and gravitational potential are calculated. Using these, the sea level equation is solved iteratively. The dominating mechanism there is solid Earth deformation; only in the far field oceans the subsidence of the ocean surface is more important, the cause of which lies in the term $\Delta \Phi(t)/g$ that is, the change in potential which ensures mass conservation. More precisely it is the difference of $\Delta \Phi^{MR}(t)/g$ and $\Delta \Phi^{F}(t)/g$. A physical explanation for why water should move away from the far field oceans is that ocean water fills the forebulge which is subsiding. Mitrovica and Peltier [1991] have called this phenomenon ocean syphoning.
Chapter 5
Model Input

Glacial Isostatic Adjustment (GIA) studies provide a unique possibility to determine the mantle viscosity. Viscosity profiles which have been obtained so far from GIA inversion studies and also rotational data, are discussed in 5.1. This puts in perspective the choice of mantle viscosity values in this study. The rheological profile of the lithosphere is varied, in order to simulate the effect of a Low Viscosity Zone (LVZ) on GIA. Diverse geophysical evidence exists for the presence of such a shallow low-viscosity zone, an overview of which is given in section 5.2. Based on this, a selection is made in section 5.3 of Earth models with a LVZ at varying depth, and with varying thickness and viscosity, to suit the purpose of this study.

Besides Earth models, the input required for GIA study is a model of the formation and retreat of ice sheets. To this extent, GIA studies can be used to derive ice sheet chronology for a fixed viscosity profile. It may seem strange that the same type of simulations are used to constrain both mantle rheology and ice sheet thicknesses. Section 5.4 elaborates on this, explaining the laborious trial-and-error process of deriving a model of the Pleistocene ice sheets from all kinds of rebound data. The focus is on the ICE3G model of [Tushingham and Peltier, 1991] which is the main ice model in this study. For completeness, the chapter concludes with a brief section on the ocean function, which describes the ocean-land distribution.

5.1 Viscosity Inferences

In section 2.1 it was explained how composition, density and rigidity can be deduced from seismic wave data. For GIA, the relevant rheological parameter is the viscosity, which does not affect wave propagation in a direct manner at all. Experiments have been carried out on polycrystalline samples under high pressure and temperature to measure stress and strain rate, and thus determine the viscosity. However, realistic mantle conditions involve several orders of magnitude higher hydrostatic pressure and lower strain rates, therefore experimental results need to be extrapolated before they apply to mantle conditions. In section 3.4 it was argued that GIA studies indicate that a linear rheology in the mantle best predicts observed uplift around former glaciated areas. In this section it will be explained how viscosity is inferred from observations of relative sea level, gravity and polar wander.

Observations of postglacial rebound led Haskell [1935] to the classical constraint of $1 \cdot 10^{21}$ Pas for mantle viscosity. Presently, debate on mantle viscosity concentrates on the supposed increase which occurs at the upper-lower mantle transition at 670 km depth. The reason that a viscosity increase is expected there, is the transition from olivine to perovskite and magnesio-wüstite, as mentioned in section 2.1. Magnesiowüstite is more densely packed and therefore has a higher creep activation energy, resulting in a higher micro-physical viscosity of at least one order (see equation D.6). However, if mass exchange across the boundary is limited, the upper and lower mantle form separate convection systems. In that case a relatively large ($> 500^\circ$ C) temperature difference has to
be bridged by a small layer in between the convection cells, the effect of which is large enough to almost cancel the effect of the density increase on viscosity.

GIA simulations have to deal with the problem of having to fit simultaneously ice chronology and a viscosity profile. Tushingham and Peltier [1991] proclaim step-by-step iteration: estimate a viscosity profile based on previous Earth models and subsequently try to find ice sheet histories so that RSL or gravity data are reproduced as good as possible. Then keep the ice model constant and adjust viscosity values until deviations are minimized. Gravity anomalies are useful because they provide additional constrains in the range of the longer wavelengths. In some studies the problem of an unknown viscosity profile and marginally constrained ice distribution is tackled by investigating which type of data is insensitive to either viscosity or ice sheet chronology, and which is therefore most suitable for deriving the other. Examples are [Wu and Peltier, 1983a], [Mitrovica and Forte, 1996]:

- The larger an ice sheet, the more sensitive it is to lower mantle structure compared to upper mantle structure;
- Sites on the edge of a former ice sheet are more sensitive to lower mantle structure;
- The shape of a RSL-time curve since the last glacial maximum (LGM) is largely determined by the mantle viscosity, but its amplitude is determined by local ice height;
- Relaxation times are relatively insensitive to local ice sheet geometry.

Elaborating on the last point, Mitrovica and Forte [1996] assume the following exponential form for the RSL at time $t$:

$$RSL_0(t) = A_I e^{-\tau t} - 1$$

(5.1)

in which $t$ is time and $\tau$ is the relaxation time. The elastic uplift immediately after removal of the ice load is removed from the data by means of a time window. The time window must be such that eustatic sea level due to global ice melt change does not yet dominate the local uplift signal close to the edge of the former ice complexes.

Some studies addressing the issue of mantle viscosity increase at the 670 km discontinuity will be discussed shortly.

Wu and Peltier [1983a] employ a database of free-air gravity data averaged on a 1x2 degree grid. Maximum anomalies are 15-20 mgal in Fennoscandia and 25-30 mgal over North-America. It is found that an Earth model with a moderate viscosity increase of a factor 2 with respect to the upper mantle viscosity of $1 \cdot 10^{21}$ Pas satisfies the free-air gravity data in Canada. They reject a lower mantle viscosity greater than $1 \cdot 10^{22}$ Pas because it is incompatible with the observed Fennoscandian maximum anomaly and the observed inward migration of the peripheral bulge.

A uniform mantle viscosity is found by Sabadini et al. [1985b], computing the polar wander and non-tidal variation in the length of day as a result from ice age cycles only. A second branch of viscosity values which are also allowed by the rotation data is found, in the order of $1 \cdot 10^{20}$ Pas. This ambiguity in possible viscosity profiles is characteristic for inferences from rotational data. However, when polar wander is computed for a number of glacial cycles, only the $1 \cdot 10^{22}$ Pas comes close to predicting the observed 8 degree polar wander in the last 65 million years. This conclusion was somewhat modified by Wu and Peltier [1983b] who note that as far as polar wander is concerned, mantle viscosity can be ‘traded off’ against the amount of disequilibrium at the onset of glaciation. Their preferred profile consists of an upper mantle viscosity of $1 \cdot 10^{21}$ Pas and a lower mantle viscosity of 3 times this value.

A different view than the nearly uniform mantle originates from simulating mantle convection (see the introduction of [Mitrovica and Forte, 1996]): a marked increase in viscosity at the 670 km seismic discontinuity is preferred. Later, GIA studies tend to contribute to this conclusion but the two different views of moderate increase vs. large increase have existed side by side for the past decades. Part of the explanation of the divergence in published results lies in modeling factors such as [Lambeck et al., 1998a]:

- lateral heterogeneities which are not being modeled;
- incorrect modeling of the ice history;
5.2 Low Viscosity Zones

The term Low Viscosity Zone is encountered throughout literature in different meanings. Sometimes it refers to the entire zone of relatively low viscosity (~ $10^{20}$ Pas) below the crust up to 200 kilometer depth [Schubert et al., 2001], p.228. In this study it is used to denote a thin (10 to 20 kilometer thickness) zone of low viscosity located in the elastic lithosphere. Evidence for the presence is pluriformal: simulated material properties, circumstantial seismic evidence and inferences of crustal rheology from deformation processes. This section lists some of the publications on the existence of a LVZ. This provides the basis for the variations in depth, thickness and viscosity of the LVZ for use in the geoid anomaly computations.

A region where low-viscosity zones may occur is on the boundary between an oceanic and a continental plate. The oceanic lithosphere is formed at mid-oceanic ridges and pushed towards the continental lithosphere. When they meet, the heavier oceanic plate dives under the continental plate. The continental plate is bended and the edge pulled downwards, forming an oceanic trench. The material of the oceanic lithosphere melts when it gets to the hot upper mantle. Some of this heated material finds its way through the crust and erupts at the surface in the form of volcanos. S-wave dispersion and characteristic P-wave arrival times provide a clear indication of low-velocity zones that are found to be eminent features of subduction [Fowler, 1990].
To simulate the effect of the asthenosphere on postglacial rebound a relatively small ice load, covering an area only a few hundred kilometers wide. Assuming channel flow, lag between uplift away from the ice load and the glaciated center give information about the viscosity. Studies of this nature give a viscosity of $10^{20}$ Pas beneath ocean islands and $2 \cdot 10^{20}$ Pas at continental margins in the far-field. For our study these areas are not particularly interesting, because in this area the tectonic forces will be dominating any GIA effects on the geoid.

A study of upwelling of mantle material in a rift near the Galapagos islands cite Schubert and Hey 1986 finds values as low as $10^{17}$ or $10^{18}$ Pas for the underlying layer. The volcanic material which comes to the surface naturally has a much lower viscosity than the bulk of the mantle. However, the conclusions also hold for the asthenosphere as there is a maximum viscosity for which the underlying layer can support flow of volcanic material in the crack.

Arvidsson [1996] gives some clues about the presence of a LVZ. He compares recent Earthquakes to the location of postglacial fault scars in Scandinavia which were created shortly after deglaciation 9,000 years ago. The deepest of those earthquakes originated from a depth of 34 to 37 kilometer. Since it is known that intra-plate Earthquakes are absent in zones of crustal weakness, the lack of Earthquakes below 40 kilometers can be interpreted as evidence for the presence of a ductile layer, as do Klemann and Wolf [1999]. The same argument can be used for the geophysical setting in New Madrid where seismicity is absent below a depth of 16 kilometer [Rydelek and Pollitz, 1994]. A local high geotherm is thought to be the cause for the ductile layer. Rydelek and Pollitz [1996] use horizontal and vertical motion as detected by GPS measurements to constrain a lower crustal viscosity of $3 \cdot 10^{19}$ Pas based on simulated horizontal separation along the Krafla rift across Iceland.

Kaufmann and Royden [1994] give in their introduction account of a LVZ at places where the lower crust is hot, referring to a number of numerical studies which estimate the lower crustal viscosity to be $10^{18}$ to $10^{21}$ Pas. Their study concerns uplift and tilting of lake sediments in the eastern Mojave desert, California, due to crustal extension a million years ago. Using micro-physical stress-strain relations it is derived that for a particular choice of thermal gradient a zone of about 10 kilometers in which ductile flow occurs is expected. Flow is directed outward of regions of thick upper crust towards thin crustal regions. Small wavelength features are smoothed out quickly, but long-wavelength features can exist for millions of years. It is found that the current rate of tilting puts an upper bound of $10^{19}$ Pas on the viscosity of the ductile zone in the observed region.

Ter Voorde et al. [1998] investigate whether crust and mantle are decoupled in response to extensional faulting. Decoupling depends on the rate of extension and on viscosity. The viscosity is given by (conform the linear dashpot model, see section 3.4):

$$\nu = \frac{\sigma}{2\varepsilon}$$  

A function of viscosity vs. depth for various stress level can be obtained for an average temperature distribution. In this way a rather high viscosity for the lower crust is found in the order of $10^{21}$ to $10^{25}$ Pas.

Ranalli and Murphy [1987] looks at the rheology from a material point of view. Viscous deformation is assumed to occur by power law creep, cf. equation D.6, with strong dependence on temperature. He chooses the crust to be composed of quartz/granite, overlying an olivine mantle. For particular tectonic provinces, it is derived that continental lithosphere has one or more ductile layers sandwiched by brittle layers. The models are supported by seismicity confined to the uppermost 20 kilometer in continental zones of plate convergence. Zones of low seismic velocity are usually associated with a change in elastic parameters but might also be an indication of layered composition which can be identified with viscous zones.

More recently, Kohlstedt et al. [1995] have compiled a strength-depth diagram using constitutive equations describing the main deformation processes in the lithosphere. The strength profiles mark a transition from brittle to ductile deformation at a depth of 15 km under continents, and at a depth of 40 km in oceanic lithosphere. Such ductile zones are stated by the authors to be important for large-scale deformation effects.
From the previous publications it is concluded that a typical viscosity of a LVZ is $10^{18}$ to $10^{19}$ Pas. The depth range is anything between 15 to 150 km and thicknesses of 15-20 km. In the next section we present the earth models that are used for GIA studies. GIA models incorporating a ductile crustal layer are few. Thoma et al. [2000] model the ablation of the Icelandic ice sheet and calculate the effect for various lithospheric thicknesses. They deduce an asthenosphere viscosity of $7 \cdot 10^{18}$ to $3 \cdot 10^{18}$ Pas. Wu [1997] investigates the effect of a 25 and 50 km thick ductile crustal layer on post-seismic stress. The studies of DiDonato et al. [2000a], Klemann and Wolf [1999] and Vermeersen [2002] will be discussed together with results of this study in chapter 6.

5.3 Earth Models

In this thesis the effect of a LVZ on the GIA induced geoid anomaly is investigated by varying the viscosity, the depth and the thickness of the LVZ. The objective is to do a sensitivity analysis, therefore parameter variation is allowed to be larger than realistic. The geoid anomaly perturbation of a LVZ is defined as the geoid anomaly resulting from a reference Earth model subtracted from the geoid anomaly resulting from an Earth model with a LVZ. In order to allow to investigate the geoid anomaly perturbation solely due to the change of an intra-crustal layer from elastic to visco-elastic, the reference model has the same number of lithospheric layers but they are elastic.

Concerning the reference Earth model with which the Earth model with LVZ is to be compared, the right choice is a balance between representing the main features which are expected to influence GIA, and at the same time keeping the model simple enough to be able to distinguish the responses of the LVZ. In principal, the more layers, the more accurate the Love numbers will be. However, Vermeersen and Sabadini [1997] show that increasing the layer number beyond five does not give significant improvement, while finding all the roots of the secular equation becomes a nuisance. Also, if all modes are found for a model with many layers, attributing them to a layer boundary will be nearly impossible. Furthermore, it does not make sense to refine the mantle since the effort concentrates on differences in the lithosphere. Lambeck et al. [1998a] even states that for the Fennoscandian ice sheet, three layers are sufficient for predicting sea levels and shoreline evolution.

Wu [1997] addresses the problem of representing a natural density gradient by an inevitable discontinuous change at a layer boundary in a finite layer model. An artificial density change gives rise to an extra mode which increases the rebound speed. On the other hand, trying to fit the entire density change in one single discontinuity also leads to buoyancy forces that are too large. Considering this, it is decided to represent the major seismic discontinuities at 420 and 670 km depth by a layer boundary. Together with three layers in the lithosphere (elastic-viscous-elastic) and one layer to represent the core this makes 7 layers. Figure 5.3 schematically shows those 7 layers. As for the reference model with a fully elastic lithosphere, it is relatively unimportant where the elastic layers in the lithosphere are located. It is chosen to henceforth denote the model with the extra layer of 10 km thickness at a depth of 20 km as model 7REF.
The elastic parameters of the Earth models used in this study are all derived from the Preliminary Reference Earth Model of [Dziewonski and Anderson, 1981]. Despite it being labeled ‘preliminary’ it has been widely accepted for use in geodynamic modeling. On each layer boundary the elastic parameters change discontinuously; inside a layer the variation is assumed to be polynomial. An Earth model is highly predetermined by the placement of discontinuities, in PREM these are at: 19 km (the Mohorovicic discontinuity), 220, 400 km, 670 km and 2891 km, and the inner-outer core boundary at 5150 km. There are two possibilities for deriving a model of less layers out of the 111 layers of PREM: by volume-averaging or by keeping the discontinuities the same value as in PREM. A volume averaged model is found to be in closer agreement with models with more layers and continuum models and it is therefore the preferred option [Vermeersen and Sabadini, 1999].

The volume-averaging procedure can be described as follows. The value of, for example, the rigidity of a layer which is made up of N layers in the PREM model is the sum of the contribution of each PREM-layer in the range 1 to N, weighted according to the radial distance

$$\mu(1, j) = \sum_{j=1}^{N} \left( \frac{R(j)^3 - R(j+1)^3}{R(1)^3 - R(13)^3} \cdot \mu_0(1) \right)$$

(5.3)

This weighting factor is independent of the harmonic degree. In reality however, lower degree responses are determined more by the lower parts of the mantle. An accurate approach requires a weighting factor to be formulated which is degree dependent [Vermeersen and Sabadini, 1997]. This is not pursued here, because probably reduction of the model to a few layers has more impact than degree dependence of the weighting factor. The elastic parameters of model 7REF, which are obtained form the PREM values by volume averaging, can be found in table 5.1

<table>
<thead>
<tr>
<th>layer</th>
<th>depth range [km]</th>
<th>$\mu_0$ [GPa]</th>
<th>$\mu$ [GPa]</th>
<th>$\nu$ [Pas]</th>
</tr>
</thead>
<tbody>
<tr>
<td>lithosphere</td>
<td>0-20</td>
<td>2411</td>
<td>30.7</td>
<td>1.0 $\cdot$ 10^{20}</td>
</tr>
<tr>
<td>lithosphere</td>
<td>20-30</td>
<td>3204</td>
<td>59.4</td>
<td>1.0 $\cdot$ 10^{20}</td>
</tr>
<tr>
<td>lithosphere</td>
<td>30-115</td>
<td>3376</td>
<td>67.6</td>
<td>1.0 $\cdot$ 10^{20}</td>
</tr>
<tr>
<td>upper mantle</td>
<td>115-400</td>
<td>3438</td>
<td>72.9</td>
<td>5.0 $\cdot$ 10^{20}</td>
</tr>
<tr>
<td>upper mantle</td>
<td>400-670</td>
<td>3870</td>
<td>108.0</td>
<td>5.0 $\cdot$ 10^{21}</td>
</tr>
<tr>
<td>lower mantle</td>
<td>670-2891</td>
<td>4891</td>
<td>221.0</td>
<td>5.0 $\cdot$ 10^{21}</td>
</tr>
<tr>
<td>core</td>
<td>2891-6371</td>
<td>10925</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 5.1 Reference 7-layer Earth model 7REF with a lithosphere of 115 km thickness and volume-averaged parameters obtained from PREM.

A problem arises because the boundaries of the PREM model do not necessarily coincide with the boundaries of the depths at which the LVZ is placed. However, as can be seen in table IV of [Dziewonski and Anderson, 1981], the variation in density and rigidity for the lower half of the lithosphere is very small so a layer boundary in between two PREM boundaries can be addressed to one of the two PREM boundaries. There is also a density inversion in PREM (the density decreases with depth between 24.4 and 220.0 km), which leads to positive roots of the secular equation and unstable responses in the normal mode analysis [Wu and Ni, 1996]. Vermeersen and Mitrovica [2000] find that for a visco-elastic layered Earth the strengths of these unstable modes are comparable to that of the 400 km depth discontinuity, and a minimum characteristic time scale (analogue to the relaxation time for stable modes) of 1 $\cdot$ 10^6 for degree 50. Still, unstable modes are undesired for the GIA modeling in this thesis and for this reason the density inversion is removed.

The Earth models are labeled with the parameters of the LVZ. For example, in Md20t10v18 the LVZ is placed at a depth of 20 km with a thickness of 10 km and a viscosity of 1 $\cdot$ 10^{18} Pas. In other models in the series, the layers are placed at depths of 40 and 60 km. Viscosity is varied from 1 $\cdot$ 10^{17} Pas to 1 $\cdot$ 10^{18} Pas. Finally, the influence of a variation in thickness is investigated by a model series with the same depth dependence as models Mt10, but now for a thickness of 20 km. See table 5.2.
5.4 Ice Models

This section describes how ice models can be constructed with the use of geological and sea level data. In particular are discussed, the ICE3G model and a parabolic shaped ice dome used in this thesis.

The presence of an ice sheet leaves marked imprints in the topography, such as glacial tills, striations and moraines. However, these remains show the boundaries of the ice sheets at a certain time; it still remains to derive what was the distribution of the ice within these boundaries. For that sea level data is indispensable, although there exist ice models that describe the maximum ice sheet thickness based on geological data [Hughes, 1987]. Sea level at the time of deglaciation can be constrained by the following observations [Lambeck et al., 1998a]:

- Dating of isolated basins. When the sea level retreats it leaves behind several basins which become isolated from the sea. Composition and age of sediments at the basins tell the time at which the basin became isolated.
- Varve records. A varve is a pair of sediment beds in a lake at the edges of a glacier, representing an annual cycle.
- An imprint of a shoreline when relative sea level was constant for a long time span.
- (submerged) peat deposits or samples of marine organisms. It is known at which height above sea level certain plants and organisms occurs. Dating of the samples provides only an upper limit on the sea level; as the plants and organisms might have been moved by waves or wind.

For dating the samples deposits or sediments, the most common method is the C-14 method. Recently, C14 time is found to deviate from actual, calendar, time by about 15%. The question is if this affects the conclusions drawn from rheology or ice inferences from rebound modeling, as time enters the models in the evolution of the ice sheet as well as in the viscosity. The last glacial maximum occurred in fact not 18,000 years but 21,000 years ago. Systematic errors are be introduced only if the deviation of C-14 time from calendar time is non-linear. However, Lambeck [1998] finds that conclusions regarding are unaltered if calendar time is used instead of radiocarbon time. Thus, we can safely continue to use the radiocarbon time scale, when we keep in mind that the mantle viscosity is defined with respect to this time scale.

Currently, the total eustatic sea level change since the LGM is believed to be 120-130 m [Lambeck, 1988], p.532. Solid Earth deformation is weak at sites far from the former ice sheets so that the RSL change there is almost entirely due to the eustatic sea level change. From the estimation for the eustatic sea level rise in the far field the ice mass $M_I$ is calculated.

Closer to the former ice sheets, the sea level data are sensitive to the shape and history of the ice sheet but depend also heavily on the local coastal geometry and the rheology. Since both the glaciation history and the viscosity are imperfectly known one can proceed by keeping the viscosity profile fixed and modifying the ice history until it fits RSL data.

<table>
<thead>
<tr>
<th>Name</th>
<th>Depth [km]</th>
<th>Thickness [km]</th>
<th>Viscosity [Pas]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Md20t10v18</td>
<td>20</td>
<td>10</td>
<td>$1 \cdot 10^{18}$</td>
</tr>
<tr>
<td>Md40t10v18</td>
<td>40</td>
<td>10</td>
<td>$1 \cdot 10^{18}$</td>
</tr>
<tr>
<td>Md60t10v18</td>
<td>60</td>
<td>10</td>
<td>$1 \cdot 10^{18}$</td>
</tr>
<tr>
<td>Md20t20v18</td>
<td>20</td>
<td>20</td>
<td>$1 \cdot 10^{18}$</td>
</tr>
<tr>
<td>Md40t20v18</td>
<td>40</td>
<td>20</td>
<td>$1 \cdot 10^{18}$</td>
</tr>
<tr>
<td>Md60t20v18</td>
<td>60</td>
<td>20</td>
<td>$1 \cdot 10^{18}$</td>
</tr>
<tr>
<td>Md20t10v17</td>
<td>20</td>
<td>10</td>
<td>$1 \cdot 10^{17}$</td>
</tr>
<tr>
<td>Md20t10v18</td>
<td>20</td>
<td>10</td>
<td>$1 \cdot 10^{18}$</td>
</tr>
<tr>
<td>Md20t10v19</td>
<td>20</td>
<td>10</td>
<td>$1 \cdot 10^{10}$</td>
</tr>
</tbody>
</table>

Table 5.2 Models used for studying the effect of depth, thickness and viscosity of a LVZ.
Once a glaciation history is established the viscosity can be estimated from this and the process can be started over until ideally a state of convergence is reached. [Wu and Peltier, 1982]. This is another example of how the paradox addressed in the introduction of this chapter (how can observations of GIA simultaneously solve ice models and viscosity profiles) can be tackled: by clever picking those sites which are for particular sea level data insensitive to either rheology or ice geometry, and where tectonic effects are minor.

ICE3G
Tushingham and Peltier [1991] have constructed the ICE3G model which is widely used in GIA studies (for example [Vermeersen et al., 1997], [DiDonato et al., 2000], [Milne and Mitrovica, 1998]). The viscosity was fixed at the following profile, on which agreement existed at the time: a lithosphere of 120 km, an upper mantle viscosity of \(1 \cdot 10^{21}\) Pas and a lower mantle viscosity of \(2 \cdot 10^{21}\) Pas. This simple model was based upon forward modeling with a previous ice model. The mantle viscosities that underlie the glaciation histories severely bias the ice model when it is used in studies constraining viscosity. The Finite Disc Method (FDM) which the authors used to solve the sea level equation is discussed in appendix C. Assumptions and simplifications that have been made in constructing the ICE3G model are the following:

- The system is in isostasy prior to 18 kyears before present. [Wu and Peltier, 1983a] give some justification that indeed 18,000 years ago the ice was in isostasy.
- Time discretization in steps of 1,000 years; processes of time scale of 1 kyear and smaller cannot be reflected.
- Melting is assumed to take place instantaneously (i.e. the step function discretization in the FDM), not linearly in the 1,000 year interval. A realistic sawtooth melting profile was found to only marginally increase the accuracy.
- The ocean area is constant throughout the deglaciation period. In the ICE4G model of Peltier [1994] a time-dependent-ocean are is included.
- Spatial discretization with circular elements. The effect of using ice discs rather than a more realistic parabolic shape is that the maximum depression does not occur below the center of the disc but below the edge. Compared to its predecessor ICE2, of Wu and Peltier [1983a] the overlap and gaps are minimized and the grid refined for more detailed representation of the coastal structure.

At certain time and given the melting history, the predicted sea level of a disc element is compared to the locally observed sea level. Both the thickness of the ice sheet and the instant of melting can be adjusted. The new value for the sea level is again compared to the observed value and the process is repeated until satisfactory fits are obtained. ICE3G gives a total eustatic sea level rise of 115 m, which is somewhat lower than present estimates. Some ice may still be missing in the model; thermal expansion of the oceans can play a role in this.

ICE3G is given as ice increments at each of the 1 kyear time steps for each of the 808 discs of the model, prescribed by latitude and longitude coordinates and radius. To arrive at the Gaussian grid as required for the spectral decomposition method used in this thesis (or any other grid) the following routine is performed: Determine for each point in the Gaussian grid if the spherical angle between the that point and any of the discs is smaller than the disc radius. If this is the case then the point is assigned the ice history of the disc. The disc elements in ICE3G are constructed in such as way that overlap does not occur, but points can lie in the between discs. Melting was at its maximum for the Laurentide ice sheet at 9,000 years BP, melting of the Fennoscandian ice sheet has its maximum before that.

Alternative ice models
Lambeck et al. [1998a] use an extensive database of RSL data for Scandinavia to arrive at more detailed constrains for the structure of the ice sheets than available before. In a large part of Scandinavia, sea level has dropped since the LGM, leaving traces of shorelines above the current sea level. An important data source are the isolated basins mentioned
in the beginning of this section, which formed when topographic depressions became separated from sea during land uplift. Due to error sources such as an unknown tidal range at the time the basin was formed and sediments from other areas that mixed with those in the basin, the maximum RSL height error is estimated at 15 m for the oldest basins, while for recent basins a minimum height error of 1 or 2 m is assumed. The model of Lambeck et al. [1998a] for the Fennoscandian ice complex differs from previous models by the following:

- The maximum ice thickness is lowered to 2000 m. ICE3G gives 2200 m which is in agreement, however in ICE4G [Peltier, 1994] the maximum thickness is 3000 m.
- A less rapid increase in ice height going from the boundary to the center, than that of quasi-parabolic ice profiles which are often used.

In Milne et al. [2001] the ice model of Lambeck et al. [1998a] has been used to describe the Fennoscandian glaciation besides ICE3G for the other ice sheets. In this study the ice model has not been used.

As will be shown in chapter 6, and is also visible in figure 3 of Milne et al., 1999, the ICE3G model gives artificial disc features in North-America. This is not surprising considering the disc shapes of the ICE3G model and the large ice height increments. The shape of the ice sheet representing the Laurentide ice mass needs to have a more natural shape in order to not distort the geoid anomaly signal of a LVZ. Filtering out the high degree terms in a grid representation of ICE3G could do the job. This should be investigated in future work.

Following Wu, 1992, Mitrovica and Forte, 1996, Thoma et al., 2000 and DiDonato et al., 2000b we construct a single ice dome which has elliptical horizontal cross-section and parabolic vertical cross-section, henceforth denote as PAR. The location and size are chosen to be equal to DiDonato et al., 2000b: the center point is located in Hudson Bay at 271 degree east-longitude and 31 degree colatitude (and not latitude as is stated in the paper) and semi-major axis and semi-minor axis of the elliptical cross-section are 17.5 and 14.0 degree, respectively. The ice height at maximum is not known beforehand but can be calculated as we require the volume enclose by the paraboloid to be equal to the volume of ice in the Laurentide ice sheet. The equation that makes up the surface of the elliptic paraboloid is:

\[ -c \left( \frac{(x-x_c)^2}{a^2} + \frac{(y-y_c)^2}{b^2} \right) + c = 0. \]  

(5.4)

with \( a \) and \( b \) denoting the semi-major and minor axes and subscript \( c \) denoting coordinates of the center point. The resulting shape is drawn in figure 5.2.

Note that the paraboloid is curved due the spherical surface of the Earth if \( x \) and \( y \) are taken to be latitude and longitude, so the true shape is no longer that of a paraboloid. Still we calculate the volume enclosed by the paraboloid and the Earth’s surface (V) by integrating the height of the parabola over the elliptic ground surface:

\[ V = \int_{-a}^{a} \int_{-\sqrt{b^2-a^2 y^2}}^{\sqrt{b^2-a^2 y^2}} c \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} \right) \, dy \, dx. \]

(5.5)

Using the transformations \( u = \frac{x}{a} \) and \( v = \frac{y}{b} \) the integral can be evaluated to give the simple expression:

\[ V = \frac{\pi}{2} abc. \]

(5.6)

According to Tushingham and Peltier, 1991 the total volume of the Laurentide ice sheet is \( 2.1 \cdot 10^{17} \) m\(^3\) which makes \( c \) equal to 4413.3 m. Considering the maximum ice heights of the Laurentide ice sheet of 3500 m [Tushingham and Peltier, 1991] and the fact that the ICE3G model is spread over a larger area, this is a reasonable value.

As will be shown in chapter 6, modeling of one or more earlier ice age cycles is important for considering the effect of a LVZ. Oxygen isotope data, from which the existence of ice age cycles can be derived, is less decisive for older deep sea sediments, therefore little is known of the precise history of earlier cycles. Spectral analysis of the isotope signal shows a dominant period of 100,000 years, which divides in a grow phase of 90,000
Figure 5.2 Graph depicting the function which represents the parabolic ice model labeled PAR. The center point is at 270 degree longitude and 31 degree latitude. The semi-major axis is 17.5 degree, the semi-minor axis is 14 degree. Maximum height is 4413 m. The ellipse that is also drawn is the cross section of the parabolic surface with the z=0 plane.

years and a melt phase of 10,000 years [Wu and Peltier, 1983b]. Usually the precycles are considered to be of the same shape as the last one. Although there exists evidence that the amount of continental ice varied from cycle to cycle, and that the cycle frequency increases towards the end of the Pleistocene [Sabadini et al., 1985b].

Note with respect to the melting history that the cross section of the paraboloid does not change with time as in Klemann and Wolf [1999], only the height diminishes.

Here we extend ICE3G with the Laurentide ice sheet replaced by PAR with precycles to form a total of 7 cycles. Since it is not possible in the pseudo-spectral sea level code to use any other form of loading than Heaviside loading we choose to divide the maximum height as given by ICE3G or PAR in 10 steps of 9,000 years, for each of the growth periods. The deglaciation period is also divided in 10 steps which yields a step-size of 1,000 years. Because more recent ice history has the greater influence on present day geoid anomalies the last step of 9,000 is subdivided in smaller steps of 1,000 years. Figure 5.4 shows the last melting cycle, with refined last ice accumulation step and melting according to ICE3G. The last deglaciation period is formed by PAR completely melting away in equal-sized increments in 10,000 years for North-America, or by ICE3G for the ice sheets other than the Laurentide. Although is ICE3G small melting continues for the Antarctic ice sheet up to present, all melting is assumed to cease at 4,000 years before present.

5.5 Ocean function

The ocean function assigns for each node a values of 1 when it belongs to the oceans or a value 0 when it is on land. As noted in section 4.3 the ocean function is required in spherical harmonic expansion as well as in spatial form. The ocean function up to degree 128 was supplied together with an early version of the pseudo-spectral sea level code by Anna Maria Marotta of the University of Milan. The ocean function up to degree and order 256 was obtained from a land mask file which is present in binary form in the
5.5 Ocean function

Figure 5.3 Schematic drawing of the last of the seven cycles in the ice loading history which is used in chapter 6.

Generic Mapping Tools (GMT) software package. The resolution can be specified as well as grid-spaces in x and y direction. It was chosen to make lakes, ponds or islands part of the continental area. A topography file cut off at zero meters height can also be used as ocean function, but has the disadvantage that land which lies below the sea level will be recognized as sea.

Since the method used for transformation of the spatial to the spherical domain uses Gaussian quadrature for analysis of the spherical harmonic coefficients, (see appendix B), the ocean function needs to be supplied on a grid with Gaussian colatitudes, which are the zeros of the Legendre polynomial of order equal to the maximum degree. Longitudinal grid spacings can be chosen equal in the GMT mask file and in the Gaussian grid. Interpolation therefore can be a simple comparison along a meridian as to which point in the mask grid is nearest to the Gaussian point. The value of the node in the mask file (0 or 1) is assigned to the Gaussian node. Although it seems that some weighted average between neighboring mask-points (resulting in values between 0 and 1) gives a smoother ocean function and therefore better description of the sea-land boundary, this is found to give erroneous results. Errors in the ocean function are at most half the grid-spacing.
This chapter presents the results of Glacial Isostatic Adjustment modeling (GIA), using the input models discussed in chapter 5 and 4. Our final goal is to investigate the effect of a Low Viscosity Zone (LVZ) on the GIA induced geoid anomaly. The geoid anomaly is computed with the pseudo-spectral sea level method, using the modal responses computed with the normal mode analysis. Before the results can be presented, some tests are to be performed at different stages of the computation. They are explained in the first section.

The set of 9 input Earth models, which is presented in section 5.3, is formed by constructing three models spanning limit cases in depth, thickness and viscosity of a LVZ. Computation of the impulse response takes place in the spectral domain and some interesting conclusions with respect to the input Earth models can already be drawn from the modal responses. Section 6.2 is dedicated to the analysis of the impulse responses. After that we proceed to compute geoid anomaly perturbations for a very simplified Earth model before turning to the more realistic Earth and ice models. Finally, section 6.4 is devoted to a preliminary comparison of the results with geoid data.

6.1 Testing of the Fortran Codes

In order to find the visco-elastic Love numbers, Fortran codes are used that were developed by L.L.A. Vermeersen of Delft University of Technology. A gridding bisection algorithm finds the roots of the secular equation (see appendix A). Since the secular equation is given analytically, the roots can be obtained with any desired precision up to the computer precision. The codes were originally written for computation of rotational behavior of the Earth and as such they have been benchmarked with other (numerical) codes [Vermeersen and Sabadini, 1997].

It is a time-consuming process to search the entire interval of possible roots for each harmonic degree. A faster routine exists, also developed by L.L.A. Vermeersen, which searches for each degree in an interval close to the root found for the previous degree. The Love numbers can not be computed correctly if not all the roots are found by the root-finding algorithm. Depending on the specific Earth model, a root can be located outside the domain in which the bisection algorithm searches or roots can be lumped together so that they are both in one step of the bisection algorithm. For example, the transient modes with short relaxation times are difficult to discern, although they can safely be ignored because of their negligible strength. A check for the presence of all relevant modes is to plot the fluid, or infinite time, Love number. The fluid Love number is given by letting $s$ approach 0 (which is equivalent to letting $t$ got to infinity) in equation 3.78 [Vermeersen and Sabadini, 1997]:

$$k_i^{\infty}(s) = k_i^{F}(s) - \sum_{j-1}^{m} \frac{r_j^{s}}{s_j}$$  \hspace{1cm} (6.1)
If modes with non-negligible strength are not found by the codes a discontinuity in the plot of the fluid k-Love number versus the degree will appear. For those particular degrees where a jump in the graph occurs, the search needs to be refined. A better test is to let all layers be visco-elastic and see if the fluid Love numbers approach 1 in the limit of infinite time. Vermeersen and Sabadini [1997] show that the values of 1 should be reached with very high precision.

**pseudo-spectral sea-level codes.** Since their publication in 1991 [Mitrovica and Peltier, 1991], the pseudo-spectral sea level codes have been employed in [Mitrovica et al., 1994a], [Mitrovica et al., 1994b], [Mitrovica and Forte, 1996]. Results by another group have been published [DiDonato et al., 2000b], [DiDonato et al., 2000a], and extensions have been made [Milne and Mitrovica, 1998], [Milne et al., 1999]. The pseudo-spectral sea-level codes used in this thesis were originally obtained from A.M. Marotta of the University of Milan. However, they did not give satisfactory results therefore modifications and improvements were made by the author, which made comparison with other results a necessity.

During a sabbatical at the section of AstroDynamics and Satellite Systems of the Delft University of Technology, P.P. Wu of the University of Calgary proposed to do a comparison between his sea level codes based on a Coupled Poison Finite Element method and the pseudo-spectral sea level codes. The results can be found in [Wu and Van der Wal, 2003] and will be discussed in greater detail here. Subsequently the ice model, the Earth model, the ocean function and the ice equivalent sea level are discussed below. The finite element method employed by Wu is discussed shortly in section 4.4, further reference can be found in [Wu, 2003].

The ice model, labeled DISC, is a simple disc-shaped cap with a radius of 15 degrees, which was put on the north pole of a schematic Earth at \( t = 0 \). Outside the ice sheet the Earth was assumed to be covered with water (see drawing in figure 6.1). The ice sheet is selected so that it exerts a pressure of 24 MPa on the surface. The ice height can be calculated with the following approximation:

\[
p = \rho gh.
\]

With an ice density of 1000 \( \frac{kg}{m^3} \) (In the other calculations in this thesis the density will be 990 \( \frac{kg}{m^3} \)) and a surface gravitational acceleration of 9.90 \( \frac{m}{s^2} \) this yields an ice height of 2446 m.

The ocean function is obtained by putting all the values for each point outside the disc equal to 1, and the value for a point inside the disc equal to zero. It is best to choose the grid so that the 15 degree colatitude is right in between two points. However, because of the requirement of the use of a Gaussian grid due to the choice of global spherical harmonic synthesis and analysis, the choice in the grid is limited and we have to accept the misplacement of the disc. The finite element method gives results on an equidistant grid. For visual comparison between the two methods the difference between the equidistant and Gaussian can be ignored.

![Figure 6.1 Schematic depicting the ice load model DISC and the corresponding Earth outside the disc, used in a comparison of the pseudo-spectral sea level codes with Finite Element codes.](image)
6.1 Testing of the Fortran Codes

The loading history is taken to be a simple Heaviside function, instantly loading the Earth at \( t = 0 \). The water to form the ice cap is extracted from the ocean, therefore at \( t = 0 \) there will be a significant eustatic sea level drop. The Earth is modeled as a three-layered sphere with incompressible Maxwell rheology with elastic parameters given in table 6.1. The volume of ice in the disc \( V \) can be calculated analytically by the following integral:

\[
V = 2\pi \int_0^{\theta_1} \int_R^{R+b} r \, d\theta \cdot r \sin \theta \, dr.
\]  

With \( \theta_1 = 15^\circ \) and the ice height \( h \) as given before this makes \( 2.1268 \cdot 10^{16} \) m\(^3\) of ice. If this is spread out equally over the oceans (the part of the sphere outside the disc) this results in a eustatic sea level rise of about 43.5 m, neglecting products of small quantities.

![Figure 6.2](image)

**Figure 6.2** a) Radial displacement and b) geoid anomaly, computed with DISC ice model put on the Earth at \( t = 0 \), results shown for a finite element method by P.P. Wu and for the pseudo-spectral sea level method.

Figures 6.2 and 6.3 show the radial displacement, geoid anomaly and Relative Sea Level (RSL), calculated by the Coupled Laplace Finite Element Method (shown by the crosses) and the pseudo-spectral method (the colored lines). The RSL is plotted for the entire Earth in figure 6.3 since a projection on the ocean surface leads to large Gibbs effects near the edge of the ice sheet which would distort the picture. The results show remarkable good agreement, considering the fact they are from two completely different methods.

After 16 kyears, isostatic equilibrium is nearly reached, when the displacement in the center of the ice load is more than 500 m. The radial displacement clearly shows the inward migration of the peripheral bulge which is characteristic for linear rheologies (see the discussion of linear versus non-linear rheology in section 3.4). At \( t=0 \) the geoid anomaly is composed of a part due to the potential of the load itself and part due to the
water which is attracted to it. The Earth tries to restore the equilibrium; as time passes the ice sinks, thereby removing the anomalies. The geoid anomaly due to the existence of the peripheral bulge is still visible after 16 kyears.

The results for the test-case give confidence in the results of the pseudo-spectral sea level codes for more realistically modeled ice loads. A final check for these ice models is to compute the eustatic sea level rise at each 1,000 year time-step. In figure 6.4 the total eustatic sea level rise is depicted for ICE3G. The value agrees with the 110 m that Tushingham and Peltier [1991] give. Note that because of the truncation of the spherical harmonic expansion (here at 128), ‘ringing’ occurs near coastlines.

The accuracy of the pseudo-spectral sea level equation can be defined as the change with respect to the previous iteration [Mitrovica and Peltier, 1991]:

$$\zeta^{i+1} = \sum_{l,m} \frac{||\delta S_{l,m}^{j+1}|| - ||\delta S_{l,m}^{j}||}{||\delta S_{l,m}^{j}||}$$

(6.4)

Mitrovica and Peltier [1991] claim to get an accuracy for the relative sea level $\zeta < 10^{-4}$ with 3 or 4 iterations. The number of iterations has a direct effect on the computation time for relative sea level computations up to degree 256, because each iteration requires a transformation to and from the space domain which are the most time-consuming steps. A computation of the self-consistent sea level at all 18 time steps of deglaciation takes more than 1 hour CPU time at an IBM RS/6000 workstation so it is useful to limit the number of iteration at each time step.

The relative sea level itself however, is not of major interest in this study; it is only through loading of the oceans that it influences the present day geoid anomaly. Previous geoid anomaly computations in [Vermeersen, 2002] also use a eustatic approximation for the sea level increments at each time step. For the computations of the geoid anomaly the sea levels changes are approximated by the eustatic changes and the geoid itself is computed in one iteration therefore computation time is not an issue. It is worthwhile however, to investigate the difference between computing the geoid anomaly using eu-

Figure 6.3  Same as figure 6.2, but now the Relative Sea Level. The sea level is computed over the entire Earth because projecting the sea level on the ocean area introduces unwanted Gibbs effects.
static sea level changes and the self-consistent sea levels. This has been done for the DISC ice model in figure 6.5 up to 16 kyears after removal of the ice. Note that the main results in this thesis are perturbations on the geoid anomaly. This figure therefore shows the difference in geoid anomaly perturbation between using a eustatic approximation for the sea level and using the RSL computed visco-elastically. Earth model Md60t20v18 has been used for this case, because it is expected that any effects that occur are higher for this model than for other Earth models. The geoid anomaly perturbations for this Earth model are plotted in figure 6.18. Figure 6.5 should be considered a limiting case; the large differences near the center of the ice sheets are primarily a result of the sharp edge. Away from the disc, the differences do not exceed 10 cm. The large difference at $75^\circ$ is expected to diminish when also the ice accumulation phase is taken into account. The more time has elapsed the more time steps with different values for eustatic and self-consistent sea level and the greater the difference in geoid anomaly perturbation.

Figure 6.4 Total eustatic sea level rise for ICE3G, computed up to degree and order 128.

Figure 6.5 Geoid anomaly perturbation computed with self-consistent sea level minus the geoid anomaly computed with the eustatic sea level increments at each time step. Ice model is DISC; Earth model is M60t20v18.
6.2 Impulse Response of Earth models in the Spectral Domain

The codes for computing the Love numbers deliver the roots of the secular equation from small to large for a certain degree. For the next higher degree the s-values and also the corresponding residues change little compared to the previous degree. This means that when the modes and amplitudes are plotted against degree, smooth lines appear which can be labeled as the impulse response or mode of a certain layer or, better, transition between layers with different Maxwell relaxation time. Relaxation times and k-Love number residues are plotted for model 3REF used in section 6.1 in figure 6.6. For such a simple model it is clear by visual inspection which line comes which layer boundary, but for Earth models using different layers, identification becomes difficult. This is one of the reasons that the Earth models in this study have been limited to 5 layers and all the modes can be identified by visual inspection or comparison to other publications.

There exists a formal method for mode identification presented by [Peltier, 1976]. The roots of the secular equation are the eigenvalues of the fundamental matrix in equation 3.54. The corresponding eigenfunctions are used to form an expression for the shear energy distribution. When plotted against radial distance, it can be seen that shear energy for a certain mode is confined to a certain region. The shear energy kernel computed for a certain eigenvalue or mode can thus be attributed to a region in the Earth. Another method to identify the spectral responses of layer boundaries is described in [Tromp and Mitrovica, 1999a].

![Figure 6.6 a. Relaxation times for Earth model 3REF used for the test case with the disc ice load. b) Corresponding k-Love number amplitudes](image)

The C0 mode of the Core Mantle Boundary corresponds to the line with largest relaxation time and least strength for high degrees. The mantle mode has the highest strength and therefore governs most of the response. A crossing of modes occurs at degree 5 in figure 6.6a. Actually it is not clear whether the modes actually cross, or that one mode is pushed downward. It could be that the entire upper branch is one mode and the lower mode another. In general, mode identification for low degrees (< 10) is debatable.

Model 7REF excites for each harmonic degree 9 harmonic modes, according to the rules of thumb given by [Han and Wahr, 1995]. A LVZ introduces two extra visco-elastic to elastic boundaries which increases the number of modes to 13. At the end of section 3.5
it was stated that higher degrees are most sensitive to lithospheric layering. Also modes 
that originate from shallower layers decay slower than mantle modes [Han and Wahr, 1995]. 
We therefore expect the LVZ to manifest in extra modes which have shorter relaxation 
times and strengths which are increasingly important for higher degree Love numbers. 

Figure 6.8 compares two by two the two modes of the extremes in the models, with a 
LVZ of 10 km thick, that of 20 km thick and the LVZ with different viscosity. Note that 
the graphs show relaxation times which are equal to the inverse of the roots of the secular 
equation in the Laplace domain. Clearly visible are the two extra visco-elastic modes re-
sulting from the new layer. Following Klemann and Wolf [1999] and DiDonato et al. [2000a] 
they are labeled MC and L.C. Klemann and Wolf [1999] use shear energy distribution to 
characterize the MC mode as coming from the lower crust. The MC mode has a long 
relaxation time for low degrees, but falls of for higher degrees (100) to values below 1 
kyear. The LC mode relaxes in a considerable shorter time, from 10 to 1000 years, which 
means this mode will not be visible in the present day geoid anomaly signal. 

The effect of depth on the MC mode, as visible in figures 6.8a and 6.8b, is to make 
the mode ‘steeper’, leading to an increase of the relaxation time of this mode for degrees 
roughly up to 70, 80 and a decrease after. As the MC mode gets closer to the M0 mode, 
it ‘pushes’ the latter downward. This can also be seen in figures 6.8c and 6.8d, for the 
model with a thicker LVZ. The effect of a lower viscosity is comparable to making the 
LVZ thicker, at least for the shallow LVZ: both decrease the relaxation time. This can be 
understood by viewing the low-viscosity layer as a channel. The lower the viscosity or 
the thicker, the faster mantle material can be transformed to restore equilibrium. 

Looking at the amplitudes in 6.9 we see that the MC mode is dominant for wave-
lengths from degree 50, 60 onwards. For a deeper LVZ, the lines belonging to the MC 
and M0 modes pair off (the intermediate model with a LVZ at 40 km depth, not plotted 
here, clearly shows that the two line are approaching each other. This can be interpreted 
as that for change in potential at the surface, the low-viscosity zone is seen as part of the 
mantle. Even when the lines seem to join, there are still two fluid amplitudes which are 
to be summed for the total response. 

Furthermore, the resemblance between figures 6.9c and 6.9e is remarkable confirming 
the idea of the previous section that a lower viscosity and a thicker layer have the same
Figure 6.8 Relaxation times for Earth models (numbering row by row, from top to bottom) a) Md20t10v18, b) Md60t10v18, c) Md20t20v18, d) Md60t20v18, e) Md20t10v17, f) Md20t10v19
6.2 Impulse Response of Earth models in the Spectral Domain

Figure 6.9 a) Modal strengths for Earth models a) Md20t10v18, b) Md60t10v18, c) Md20t20v18, d) Md60t20v18, e) Md20t10v17, f) Md20t10v19
effect. As also noted by Klemann and Wolf [1999] when two modes cross in the relaxation time diagrams, the amplitudes show strong, discontinuous interactions.

The viscous effects enter the pseudo-spectral sea level equation through $\beta$ of equation 4.18. $\beta$ is equal to the visco-elastic k-Love number of equation 3.78, without the elastic part. The root of the secular equation determines how fast the $\beta$ curve becomes horizontal, the residues divided by the amplitudes determine the asymptotical value. From a display of $\beta$ vs. time we expect to learn about the effect of a LVZ over time. Figure 6.10 shows beta computed for degrees 2, 16, 32, 64, 128, 256, for models 7REF and the models with variation in depth of the LVZ with thickness 10 km and viscosity of $10^{19}$. In 6.11 the LVZ has a thickness of 20 km and in figure 6.12 the models with a LVZ with viscosity of $10^{17}$ to $10^{19}$ are plotted. It can be seen that for degrees 2 and 16 the influence of a LVZ in the lithosphere is not noticeable. Figure 6.9 confirms that for degrees lower than 20 the M0 mode strength is not modified by the presence of a LVZ as the MC mode has too little strength.

From figures 6.10 and 6.11 we see that for degree 32 holds: the deeper the LVZ lies, the more it increases the visco-elastic Love number. At degree 64 the response depends on the thickness of the layer (depth is increased simultaneously if the LVZ is made thicker) and for degrees 128 in both figures the shallower LVZ gives the greatest response. This finding agrees with the MC mode becoming steeper in figure 6.8b. High degree Love numbers are insensitive to the lower part of the lithosphere and they only ‘see’ the top 20 kilometer. This is also visible in figure 6.8, where the MC mode has a relaxation time of about 1 kyyear. From figure 6.9 it can be verified that for model Md20t10v18, the MC mode is dominant for degrees greater than 50. The effect of thickness on $\beta$ is clearest in figure 6.11d for degree 64, where the deeper layers models response is much higher for the 10 km thick layers. A general rule for all degrees can be deduced from figure 6.12: the lower the viscosity the faster the mode relaxes and the higher the amplitude.

The characteristics of the visco-elastic Love number can not be applied directly on the geophysical observables because of the factor $T$ of equation 4.25 performs filtering of almost two orders of magnitude over the span of degrees up to 256. Figure 6.13 shows $T$ vs. degree on a logarithmic scale.

6.3 Geoid Anomaly Perturbations

A few papers have been published on the results of incorporating a ductile layer in the lithosphere. Klemann and Wolf [1999] use a homogeneously layered, incompressible Maxwell viscoelastic half-space subjected to surface loading. The Earth model used by them is given in table 6.2. They find only a marginal effect of the viscous lower crust on the radial displacement rate. The reason for this is that displacement rate is actually a displacement difference divided by a time interval, and this difference is not affected much by a LVZ. In fact this is the reason why it is interesting to look at geoid anomalies: they are an integrated signal which reflects also the contribution of visco-elastic modes over a long time-span.

The study of Vermeersen, 2002] used an Earth model that is given in table 6.3; elastic parameters are according to PREM. Since in that paper use has been made of numerical codes instead of the semi-analytical, the PREM layering can be used. Here we use less layers and the best approximation to the elastic profile of PREM is by volume averaging

table 6.2 Homogeneously layered half-space Earth model used by Klemann and Wolf (1999). Note that half-space implies that the mantle continues to infinity.

<table>
<thead>
<tr>
<th>depth range[km]</th>
<th>$\rho_{\text{mg}}$</th>
<th>$\mu$[GPa]</th>
<th>$\nu$[Pas]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-35</td>
<td>2900</td>
<td>64.0</td>
<td>$\infty$</td>
</tr>
<tr>
<td>35-45</td>
<td>2900</td>
<td>64.0</td>
<td>$1 \cdot 10^{17}$</td>
</tr>
<tr>
<td>45-130</td>
<td>3380</td>
<td>64.0</td>
<td>$\infty$</td>
</tr>
<tr>
<td>130-$\infty$</td>
<td>3380</td>
<td>145.0</td>
<td>$1 \cdot 10^{21}$</td>
</tr>
</tbody>
</table>
Figure 6.10 Visco-elastic influence factor in pseudo-spectral sea level equation for models with variation of depth of LVZ (thickness is 10 km, viscosity is $1 \cdot 10^{18}$ Pas).
Figure 6.11  Visco-elastic influence factor in pseudo-spectral sea level equation for models with variation of depth of LVZ (thickness is 20 km, viscosity is $1 \cdot 10^{18}$ Pas).
6.3 Geoid Anomaly Perturbations

Figure 6.12 Visco-elastic influence factor in pseudo-spectral sea level equation for models with variation of viscosity of LVZ (thickness is 10 km, depth is 20 km).
Figure 6.13  dimensional parameter $T$, specified in equation 4.25

the layers of PREM that correspond to a layer in the Earth model of table 6.3. Table 6.3 lists the PREM layers that correspond to the layers in the Earth model 5DID.

<table>
<thead>
<tr>
<th>layer</th>
<th>depth [km]</th>
<th>viscosity</th>
</tr>
</thead>
<tbody>
<tr>
<td>upper crust</td>
<td>0-25</td>
<td>$\infty$</td>
</tr>
<tr>
<td>lower crust</td>
<td>25-40</td>
<td>$1 \cdot 10^{18}$</td>
</tr>
<tr>
<td>lithosphere</td>
<td>40-80</td>
<td>$\infty$</td>
</tr>
<tr>
<td>upper mantle</td>
<td>80-670</td>
<td>$5 \cdot 10^{20}$</td>
</tr>
<tr>
<td>lower mantle</td>
<td>67-2891</td>
<td>$5 \cdot 10^{21}$</td>
</tr>
<tr>
<td>core</td>
<td>2891-6371</td>
<td>$1 \cdot 10^{21}$</td>
</tr>
</tbody>
</table>

Table 6.3 Elastic parameters of the Earth model 5DID used by DiDonato et al (2000a) and Vermeersen (2002).

It was found that the introduction of a LVZ in an Earth model, with ice model PAR leads to geoid anomaly perturbations up to 1 m near the ice loads, and a value of 1 or 2 decimeter in the far field. We define the geoid anomaly perturbation (sometimes in this report referred to only as perturbances), as the geoid anomaly with LVZ minus the geoid anomaly without LVZ. Thus, it is that part of the GIA induced geoid anomaly that has to be added to the geoid anomaly computed for a fully elastic lithosphere. If geoid anomalies are negative (which is the case for the former glaciated regions) a positive perturbation implies that the geoid anomaly for a model with LVZ is less negative and therefore is closer to isostasy than the model with elastic lithosphere.

Before turning to more realistic ice models, the simple disc model of section 6.1 can be used to obtain a first impression of the effect of the Earth models of section 5.3, with variation in depth, thickness and viscosity of a LVZ. Where $\beta$ gave insight in the time behavior the disc loading case is expected to show spatial characteristics. Instead of a positive surface load by increased ice mass as in section 6.1, we will simulate signals due
6.3 Geoid Anomaly Perturbations

<table>
<thead>
<tr>
<th>layer</th>
<th>radius [km]</th>
<th>layers in PREM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6371-6346.55</td>
<td>1-12</td>
</tr>
<tr>
<td>2</td>
<td>6346-6331</td>
<td>13-14</td>
</tr>
<tr>
<td>3</td>
<td>6331-6291</td>
<td>15-19</td>
</tr>
</tbody>
</table>

Table 6.4 Layers of PREM that make up a single layer in the 5DID model.

to melting of the ice sheets, which started 18,000 years ago and continued up to 5,000 years before present. The melt water from the disc is redistributed over the Earth in a self-consistent way, and the geoid is computed using the true RSL at each time step.

The plot of the geoid anomaly resulting from removing the ice cap at t = 0, figure 6.14, shows that relaxation proceeds quite different from the three layer reference model of section 6.1 (Compare with figure 6.3). Apart from the different signs, the RSL shows isostasy already after 4 years in figure 6.3, while in figure 6.14 it is just half-way. Also the collapse of the peripheral bulge is greater for the 5REF model than for the 3-layer model. A comparison between the L0 and M0 modes of models 5REF and 3REF shown in figure 6.15 confirms that model 5REF has highest relaxation times around degree 20, where the amplitude is high (see figure 6.7) and where least filtering by the factor $T$ takes place.

In figures 6.17, 6.18 and 6.19, the geoid anomalies perturbations are plotted for the disc load. An ice cap which is removed instantly is not a realistic representation of true ice melting. However, it suffices to deduce trends which can be verified using more realistic Earth models. Here we already see that the GIA induced geoid anomaly is negative in former glaciated areas. A negative geoid anomaly perturbation in those areas denotes a less negative geoid anomaly for the Earth model with the LVZ. This is not surprising: the viscous layer can relax much more than the small uplift released by a pure elastic layer. Note, that the relaxation of the viscous layer implies that it was compressed during the accumulation phase of the ice sheet, but for the moment we compute the anomalies relative to the situation at the time of melting.

The radial displacement perturbations are included in figure 6.16 for comparison. It can be noticed that the geoid anomaly perturbation is most of all a function of time after (un)loading. In figures 6.17 to 6.19 we see that at $t = 1$ the models with shallow, thick, and least viscous LVZ give positive perturbations in the former glaciated area, which is a sign of faster relaxation. Indeed, the relaxation diagrams showed that for low degrees

Figure 6.14 Geoid anomaly for model 5REF and DISC load, removed at $t=0$. 
these models have smaller relaxation times and we know that the spherical harmonic representation of the ice sheet peaks in degrees 5-10. The graphs of $\beta$ vs. time are too close to verify if $\beta$ is indeed higher for shallow, thick, less viscous LVZ models.

With respect to the spatial distribution, we see that relaxation proceeds quickest close to 75 degrees colatitude, indicating material flow from the sides. Further southwards the perturbations are negative due to the collapse of the peripheral bulge. At $t = 4$ the situation is unchanged except that now the deeper LVZ models have overtaken the shallow LVZ models. At $t = 16$ we see that all models except for model Md60t20v18, have zero perturbation at the north pole, and slightly negative perturbations more south and a big increase as the edge is approached. Relaxation in the mantle is nearly completed as shown in figure 6.14. This feature we expect to see also in the geoid anomalies for a realistic Earth, although we have to keep in mind that for the DISC model Gibbs effects occur because of the sharp edge of the ice load and the ocean-water boundary, which happen to be at the same location.

In principle, it is possible to compute the self-consistent visco-elastic sea level at each time step and then compute the geoid anomalies with these sea level increments as has been done to obtain figures 6.17 to 6.19 also for realistic ocean and ice models. However, the sea level that is computed using the ICE3G is in error near glaciated areas, especially near the Hudson Bay where it give large negative values. This can be seen in figure 6.20. The phenomenon can be explained as follows: the sea level is computed as the geoid anomaly minus the radial displacement. The load consists of the ice increments and sea level increments. When the Laurentide ice sheet starts to melt, being a large negative ice load, the geoid anomaly drops dramatically locally. For the moment consider the situation 18,000 years ago, just after melting of part of the ice sheet when radial displacement has not yet started. There will be a large negative sea level over the Hudson Bay, forming a negative load. In reality sea water will fill the depression left by the retreating ice, leading to a counteracting force on the rebounding surface. The negative sea level over Hudson Bay is in the codes compensated by term which states mass conservation, leading to an overestimation of sea level rise in other areas. Note that the problem of ice being defined over part of the ocean does not occur in section 6.1.

In reality, the Hudson bay should be considered land until the ice covering it has

Figure 6.15 M0 and L0 modes of models 5REF and 3REF.
6.3 Geoid Anomaly Perturbations

Figure 6.16  Radial displacement perturbation for models with LVZ at varying depth. Thickness and viscosity are fixed at 10 km and $1 \cdot 10^{18}$ Pas, respectively.

Figure 6.17  Geoid anomaly perturbation for models with LVZ at varying depth. Thickness and viscosity are fixed at 10 km and $1 \cdot 10^{18}$ Pas, respectively.

Figure 6.18  Geoid anomaly perturbation for models with LVZ at varying depth. Thickness and viscosity are fixed at 20 km and $1 \cdot 10^{18}$ Pas, respectively.
melted and then water should be allowed to poor in. Since we choose to use the eustatic sea level at all time steps to compute the geoid anomalies we make the same error but it it markedly reduced. The eustatic sea water level during the 10,000 years of deglaciation rises to about 100 m, which means increments do not exceed 10 m. Compared to thickness of ice removed in each step, this erroneous water load presents a small factor. However, in future work on geoid anomaly perturbations in the Hudson Bay, it is advisable to use a time-dependent ocean, given the complicated land-water interaction there. An assessment of the concept of sea water occupying former glaciated depressions can be found in [Milne et al., 1999] and [Peltier, 1998] where it is dubbed ‘near field hydro-isostasy’ and ‘implicit ice’ respectively.

If we use the ICE3G ice model for only the the melting period, we find the geoid anomaly of figure 6.21. The maximum values of around 25 m are as expected, but also disc-shaped patterns appear, much alike the discs of the ICE3G model. These patterns are not realistic features of the ICE3G model; rather they are artefacts introduced by the limited ability to model the ice sheets. The same features are also visible in figure 3 of [Milne et al., 1999]. The patterns are less visible in the geoid anomaly over Scandinavia in figure 6.21.

In the perturbation on the geoid anomaly due to a LVZ the ICE3G fine-scale topography will be even more present, blurring the effect of a LVZ. Therefore the paraboloid ice model PAR, presented in section 5.4 is used to represent the Laurentide ice sheet.
With PAR the total geoid anomaly in North-America is somewhat less than that with the ICE3G method as can be seen in figure 6.22.

The area around Hudson Bay is chosen as the subject of our study, as there the chances of the presence of a LVZ are higher than in the old continental plate of which Scandinavia is part. From figures 6.23 to 6.25 can be seen that the amplitudes are within the values found with the DISC model. Also we see the characteristic pattern of negative perturbances near the center of the former ice sheet and larger positive perturbances closer to the edge, which agrees with figures 6.17 to 6.19 for a time of unloading somewhere between 4 and 16 kyears before present. It can be seen that the effect of the PAR ice model is that the negative perturbations become more negative, and the positive perturbations are decreased, in agreement with the shape of the parabolic surface.

Summarizing, it can be said that amplitudes agree well with those found in Vermeersen [2002] although the shapes in our graphs are more remnant of the elliptic cross-section of the ice sheet. The effect of depth for a 10 kilometer thick LVZ can be seen as increasing the perturbations. For a 20 km thick layer the negative perturbations in the center are enlarged but the positive perturbations near the margin are lowered and outside the former ice margin negative perturbations are found. The effect of a thicker and less viscous LVZ
Figure 6.23 Geoid anomaly perturbation computed with ice model PAR for deglaciation only. a) model Md20t10v18. b) model Md60t10v18.

Figure 6.24 Geoid anomaly perturbation computed with ice model PAR for deglaciation only. a) model Md20t20v18. b) model Md60t20v18.

Figure 6.25 Geoid anomaly perturbation computed with ice model PAR for deglaciation only. a) model Md20t10v19. b) model Md20t10v17.
6.4 Comparison with Geoid Models

is to decrease the area of positive perturbations. We can confirm this observation looking back at the DISC load figures 6.17 to 6.19. Furthermore, it is interesting to see that for a deeper LVZ a patch of negative perturbations occurs near the coast of the Hudson Bay. Apparently the water load is of such spatial scale that a deeper LVZ is excited more. These two features consider the shape of the signal, and they may be better detectable in a geoid model than absolute values of perturbations.

So far we have only simulated glacial melting from the LGM onward, which implies the assumption that the Earth is in isostatic equilibrium prior to melting. Also, all the computed quantities are relative to the situation at the onset of the melting. The total sea level rise will be around 110 m, forming a huge extra load on the oceans. The accumulated ice decrements will add in each time step to form negative loads over the former glaciated land masses. In this sense one can understand that a viscous layer in an otherwise elastic lithosphere gives much more relaxation than if the same layer would be elastic. Elastic layers are compressed only a few percent; at the moment of unloading they instantly deform back to their old shape. The ductile material is compressed significantly. If only the melting is taken into account it might be as if the viscous layers relax to an original shape which was larger than if the same layer would be elastic, forgetting the fact that the layer was also compressed more when the surface load developed.

These are all arguments in favor of modeling the growing of the ice too. Moreover it was shown that the extra modes introduced by the LVZ have long relaxation times for small to medium wavelengths. However, the Earth at the onset of an ice age might not be in isostatic equilibrium either and is therefore also not ideal as reference situation. A compromise is by to use enough growth-decay cycles so that the influence of the reference shape is brought back far enough in time.

The pre-cycles have been modeled as described in 5.4. The way the pseudo-spectral method is coded, makes it impossible to use linear or more difficult growth and decay scenarios for the ice sheets, therefore the cycles have been modeled with step increments. For deglaciation only, it is in principal possible to compute first the self-consistent visco-elastic sea level at each time step from \( t = -18 \) to \( t = 0 \), and then compute the geoid anomaly at \( t = 0 \) with the visco-elastic sea levels (this has been done to produce figures 6.17 to 6.19. To compute the visco-elastic sea level for each of the steps in the ice age cycles would require an immense amount of work.

In figure 6.26 we compare the geoid anomaly for the Earth models of Vermeersen [2002] and DiDonato et al. [2000a] for only the melting period and for 7 precycles, both using the PAR ice model. Figures 6.27 to figure 6.29 show the same series of models as in figures 6.23 to 6.25. It is clear that using a series of precycles completely alters the signal. In general it can be said that (with the exception of model Md20t10v18) roughly the signs are reversed and the amplitudes are lowered to 70 centimeter as was expected when also the ice accumulation phase was taken into account. It is more difficult to discover trends in the signal with regard to thickness, depth and viscosity. A thick LVZ manifests in negative perturbations towards the edge which are increased with depth, while the positive perturbations at the center become smaller. For the 10 km thick layer we see the reverse: negative perturbations at the center which become positive for a deeper layer. A lower viscosity is seen to produce positive followed by negative perturbations close to the edge. The difference between a thick deep layer with a shallow, thin layer has the highest chance of being detected in a comparison with a geoid model. Also the thin zone of positive and negative perturbations resulting from a lower viscous LVZ is characteristic.

6.4 Comparison with Geoid Models

[Mitrovica and Peltier, 1989] show that the geoid from which tectonic effects are removed is well correlated with geoid anomalies predicted by postglacial uplift studies. This conclusion is confirmed by a glance at figure 6.30, which shows the CGG2000 geoid model, obtained from the Geodetic Survey Division of Natural Resources Canada where a large geoid low can be seen in Canada.
Figure 6.26 Geoid anomaly perturbation of the Earth models of Vermeersen (2002) and DiDonato et al. (2000b) using PAR ice model. a) 7 ice age cycles. b) only melting.

Figure 6.27 Geoid anomaly computed with combined ICE3G and PAR in 7 cycles of loading and unloading. Maximum degree of 256. a) model Md20t10v18. b) model Md60t10v18.

Figure 6.28 Geoid anomaly computed with combined ICE3G and PAR in 7 cycles of loading and unloading. Maximum degree of 256. a) model Md20t20v18. b) model Md60t20v18.
Since the geoid anomaly perturbation due to a LVZ is such a weak signal, it can only be expected to be detectable in the geoid signal after filtering. When looking at higher degree terms only, the problem of different reference shapes for the geoid anomaly, being the ellipsoid in the case of the CGG2000 model, and a sphere relative to which geoid anomalies are computed here, becomes irrelevant. The interval where the geoid signal is relatively uncontaminated by non-GIA signals is not known beforehand. [Mitrovica and Peltier, 1989] argue that the coefficients in the degree range 10-22 are dominated to a large part by GIA. However, from figures 6.10, 6.11 and 6.12 for example, it was concluded that for such low degrees the influence of a LVZ is not noticeable. In figure 6.31 the degree variance of the CGG2000 and the geoid anomalies perturbation signal for model Md60t20v18 are plotted. The LVZ geoid perturbation signal can be seen to have several orders smaller energy content than the CGG2000 geoid. LVZ perturbation was expected to manifest in high degree terms, however for degrees higher than 30 the degree variance decreases rapidly.

One case of filtering the geoid signal is given here for coefficient range of 30-80. The choice is a somewhat arbitrary compromise: at low degrees the LVZ does not manifest itself but at higher degrees the noise becomes increasingly important. From figure 6.32 can be seen that the two figures show no correlation at all and also the filtered CGG2000 geoid amplitudes are an order of magnitude higher. This does not prove that the LVZ signal is completely absent in the CGG2000 geoid. It is meant to show that care must be taken in picking the coefficient range and that more advanced forms of filtering should be used. Moreover, it is advisable to do a formal correlation study (see for example [Mitrovica and Peltier, 1989]).

Even after good filtering there are still other geophysical effects that can have a sig-
significant effect on the geoid. For higher degrees, hydrology can be expected to cause considerable geoid anomaly signal. In fact, it is suggested that the current satellite gravity missions are able to detect ground water movements on a limited spatial scale.

From the recordings of the test mass accelerations aboard GOCE to a centimeter precision geoid is a long chain. Perturbative forces on the satellite, instrument noise and time-variability of the gravity field for example are to be taken into account during data reduction of the GOCE raw measurements. Figure 3 of [Visser et al., 2002] gives the predicted geoid recovery as a function of the harmonic degree. It can be seen in that figure that the GOCE geoid accuracy is in principal high enough for LVZ geoid effects to be detected.

![Figure 6.31](image1.jpg)  
Figure 6.31 Degree variance of a) CGG2000 b) Geoid anomaly perturbations for model Md60t20v18.

![Figure 6.32](image2.jpg)  
Figure 6.32 Geoid heights after filtering for the coefficients between 30 and 80 for a) CGG and b) Geoid anomaly perturbations computed by model Md60t20v18.
Conclusions and Recommendations

Conclusions of the parameter study of a Low Viscosity Zone in Glacial Isostatic Adjustment modeling are presented, and issues for further improvement are discussed.

The progress made in this study includes the computation of the Earth’s response using a semi-analytical normal mode method, which can be verified to give high precision results and allows to investigate the response of the individual layer boundaries to be investigated individually. A parameter study has been carried out, with Earth models that differ in location, thickness and viscosity of a shallow low-viscosity zone in the lithosphere. Based on existing literature, it is concluded that a layer with a viscosity of $10^{18}$ to $10^{19}$ Pas is likely to exist in the lithosphere at depths from 15 to 100 km.

The global ICE3G ice model constituting of changing height of disc elements is shown to be not satisfactory for the purpose of this study. Therefore a single parabolic ice dome is constructed to simulate the ice complex in North-America. Studies are performed of the spectral response of the LVZ, of GIA induced geoid anomalies following deglaciation only for a schematic Earth, and for geoid anomalies using more realistic models. Also the effect of including earlier ice age cycles in the model has been investigated. Finally a very preliminary comparison with other geoid models has been performed.

The impulse responses in the spectral domain show the appearance of two modes due to two extra viscoelastic-elastic boundaries, of which one has too short a relaxation time to play a role in GIA. The effect of increasing thickness and lowering viscosity of the LVZ on the other mode is a decrease in relaxation time, while the effect of increasing the depth of a low-viscosity zone is degree dependent: longer relaxation times for low degrees and shorter for high degrees. Increasing depth and thickness, and lower viscosity are seen to increase the amplitude, but at high degrees the shallowest layers have the highest amplitude.

Using realistic Earth and ice models we have confirmed the results of Vermeersen [2002] about the order of magnitude of the geoid anomaly perturbations. Modeling deglaciation only, gives a characteristic pattern in the perturbation signal of negative perturbations in the center of the former ice sheet and positive towards the edge. It is shown that this pattern emerges already 4 kyears after deglaciation, therefore the conclusions are hardly influenced by the time of deglaciation. Increasing the depth of a 10 kilometer LVZ gives higher amplitudes, while a thicker or less viscous layer increases the amplitudes of positive perturbations. Amplitude differences will be difficult to detect in comparisons with geoid data, therefore spatial characteristics are of more interest. It is found that a deeper LVZ introduces negative perturbations that are correlated to the land-water area near the Hudson Bay. Second, the effect of a thick or viscous layer (thickness and viscosity cannot be constrained separately based on GIA geoid data) is to make the area near the former ice boundary, where the positive perturbations occurred, smaller.

The simulations of GIA using 7 glacial cycles show that the precycles are important when modeling the perturbation geoid perturbation. In fact, the results can be seen as having opposite sign with respect to the results based on deglaciation only and also the
amplitudes are lowered to 70 centimeter. Trends regarding LVZ parameters are however difficult to deduce. A 20 km layer has positive perturbations at the center which are lowered with depth of the LVZ. A 10 km thick, 20 km deep LVZ causes negative perturbations in the center which become positive as depth is increased. The difference between a shallow, thin layer and a deep thick layer has the most chance of being detected in geoid data.

The perturbations in the geoid height introduced by the presence of a 20 km thick, 60 deep LVZ with a viscosity of $10^{18}$ Pas are plotted for degree range 30-80 on a map of Canada, next to degree range 30-80 for the geoid model CGG2000. It is meant to show if this type of comparison is feasible, but it is shown to be of little use, as the signal seems to be dominated by noise or other artificial high degree components introduced during computation.

When addressing the validity of the results we differentiate in modeling errors, input model errors and computation errors. Each of these is treated subsequently.

An important simplification is made in the ice-land interaction as we have assumed constant coastline throughout. Certainly in the Hudson Bay the inflow of water will have significant effect. Furthermore, the Earth is reduced to a radially symmetric, non-rotating sphere. Lateral heterogeneities from the radially symmetric model can be quite large. The assumption of eustatic sea level change during deglaciation introduced limited, gradual changing errors in the geoid anomaly perturbation, not exceeding 10 cm. Only for the extreme case of a large ice mass at the ocean-land boundary, the difference between the geoid anomaly computed with the eustatic sea levels and that computed with self-consistent sea levels totals up to 50 cm after 16 kyears.

It is difficult to quantify errors in input ice and viscosity models such as those resulting from, for example, uncertainty in dating of sealevel indicators. The viscosity profile is constrained more and more as better data sets become available, but there is no agreement yet. Uncertainties in the upper 100 km of the Earth may be significant for the geoid anomaly perturbations, but representation of the ice history is more relevant to the geoid signal. The simulation results for the model PAR and ICE3G show that the shape of the models is reflected in the geoid as well as in the perturbation signal. The time of deglaciation has no major influence on the conclusions.

Computational errors are determined by the truncation degree of the spectral method. By computing the results up to degree and order 256, we have ensured that the computation error is small enough for all geoid computations.

Some improvements should be made in future work in the modeling part. The ice sheet model representing the ice sheet can be constructed to include also a change in horizontal cross section during melting. For a situation as in the Hudson Bay, with sea enclosed by land, which was completely covered with ice it is furthermore necessary to include the effect of water inflow upon removal of the ice. Also it is advisable to investigate how the results depend on rheological parameters that are kept fixed in this study, such as the lithospheric thickness. The parameter variation expressed in our input Earth models can be extended to include two low-viscosity zones or a low-viscosity zone at or below the bottom of the lithosphere.

Concerning the comparison with geoid models from the GOCE satellite mission or regional geoid models, it is necessary to determine more precisely the locations of low-viscosity zones and the specific layering at these regions. The perturbation signal for these areas should be compared with geoid heights from which other effects such as ground water movement or tectonics have been removed as much as possible. More study has to be done to find out which degree interval is best for comparison. Advanced types of filtering can be used to remove noise effects before correlation with the geoid is investigated.
This appendix describes in more detail how the Love numbers are computed in the fortran codes developed by L.L.A. Vermeersen. Visco-elastic properties of the Earth models are set in the codes in the subroutine LAYERING. The core is modeled by a density value and the conditions at the Core Mantle Boundary (CMB). In the program an average procedure can be invoked to obtain volume-averaged PREM values in the case that not all of the PREM layers are used. The gravity of each layer is calculated with the following formula, from [Vermeersen et al., 1996b]):

\[ g_N(R_k) = \frac{4\pi G}{3} \rho_0 R_0 \]  

(A.1)

All parameters are non-dimensionalized with respect to the upper layer density \( \rho \), the Lamé constant \( \mu \) and the Earth’s radius \( R \) as follows:

\[
\begin{align*}
\rho I &= \frac{\rho_0(I)}{\rho_0(0)} \\
R_f &= \frac{R_0(f)}{R_0(0)} \\
\mu(I) &= \frac{\mu_0(I)}{\mu_0(0)} \\
\nu(I) &= \frac{\nu_0(I)}{\nu_0(0) \text{kyears}}
\end{align*}
\]

The roots of the secular equation 3.65 form a visco-elastic response spectrum with the characteristic response time being the inverse of the roots. The roots are computed by a bisection algorithm in combination with the Newton-Raphson method, which is the only numerical part of the program. A typical domain of roots (non-dimensionalized through division by kyears) ranges from \( 10^{-17} \) to \( 10^{-10} \) and is divided into smaller intervals. In one such an interval, subroutine FUNZIONE is called to compute the determinant for the first \( s \)-value in the interval and also for the value \( s + \text{DELT}\). If the two determinants differ in sign then a root of the secular equation must lie in between the two \( s \)-values; the interval is divided in two and the procedure repeated to obtain a higher accuracy. Grid size, grid bounds and accuracy can be set in the program. It is assumed that all modes lie on the negative real axis. The domain of possible \( s \)-values has to be adjusted too. Elastic love numbers can be calculated quite easily in the program, by setting the viscosity several order of magnitude higher than the surrounding layers. It has been verified that the corresponding relaxation time approaches zero.

Output of the program are the roots, the residues in the Laurent series expression for the Love numbers. Love numbers for horizontal (l) and vertical (h) deformation and gravity potential (k) can be computed, both for tidal and surface load.
Appendix B

Global Spherical Harmonic Analysis and Synthesis

An essential part of the pseudo-spectral method for solving the sea level equation is the transformation of a function \( f(\theta, \phi) \) into spherical harmonics. Conform [Smeets, 1994] and [Colombo, 1981] we will call this procedure analysis. To transform discrete data on the sphere to spherical harmonics the Gauss-Neumann method is employed, which requires the data to be given on a Gaussian grid. Transforming from the spectral domain to the space domain, synthesis, is more straightforward. It is the subject of section B.3, after which analysis is discussed in section B.4. Spectral harmonic decomposition is intimately related to Fourier transforms, some properties of which are transferred to spectral harmonics. Moreover, Fourier transforms will appear in both synthesis and analysis. Therefore the section B.1 features the Discrete Fourier Transform, and section B.2 explains the Fast Fourier Transform which is used in practice to compute the discrete Fourier Transform.

B.1 Discrete Fourier Transform

The Fourier transform is used in the part of the code that transforms the fields in the sea level equation to spectral coefficients. As such it is merely a computational tool, but since the expansion into spherical harmonics has some similarities with the Fourier transform, the subject deserves some attention. The Fourier Transform is a different representation of a function which is usually a time-dependent. In this report the functions are functions on the sphere and therefore we will consider a space-dependent function \( h(x) \). The continuous Fourier transform in one dimension is defined as:

\[
H(f) = \int_{-\infty}^{\infty} h(x) e^{2\pi ifx} dx
\]

and the inverse Fourier transform as:

\[
h(x) = \int_{-\infty}^{\infty} H(f) e^{-2\pi ifx} df
\]

A function \( h(x) \) which is real, becomes a complex function \( H(f) \) after transformation. The dimension of the frequency \( f \) is inverse wavelength, but the name frequency will still be used in the remainder of this section. One of the advantages of a Fourier representation is that information about the frequency content of the signal can easily be obtained. For example, assuming the wavelengths vary from 0 to \( \infty \) (the frequency will also), the one-sided power spectral density gives the energy distribution over the frequency [Press et al., 1993], p.493:

\[
P_h(f) = |H(f)|^2 + |H(-f)|^2 \quad \text{for } 0 \leq f < \infty
\]

The energy contained in an inverse wavelength interval is the integral of this function over the interval.
The spatial functions that we consider (the ocean function and ice load histories) are not given as analytical functions on the sphere but rather their values on a finite number of points are given. This sampling of a continuous function has some important implications on the Fourier transform. Let $\Delta$ denote the sampling interval, then the maximum frequency that can be retrieved from samples of a continuous function is the Nyquist frequency:

$$f_c = \frac{1}{2\Delta}$$

or half the sampling frequency. If the signal $h(x)$ contains any frequency higher than this it will be aliased, or falsely translated, into the range $(-f_c, f_c)$. If the Fourier transform approaches zero at the boundaries of the Nyquist interval then the part of the signal present in frequencies above the Nyquist frequency is negligible.

With this in mind we proceed and take N consecutive samples along $x$:

$$h_k \equiv h(x_k), \quad x_k \equiv k\Delta, \quad k = 0, 1, 2, \ldots, N-1$$

The frequencies will all lie in the range $(-f_c, f_c)$. The Fourier transforms can be estimated at the sampling points:

$$f_n = \frac{n}{N\Delta}, \quad n = -\frac{N}{2}, -\frac{N}{2} + 1, \ldots, \frac{N}{2}$$

The Fourier transform of a continuous function can now be approximated in the case of a sampled function as [Press et al., 1993]:

$$H(f_n) = \sum_{k=0}^{N-1} h_k e^{2\pi i f_n k} \Delta$$

Using equations B.5 and B.6 the expression for the discrete Fourier transform of the N points $h_k$ becomes:

$$H_n = \sum_{k=0}^{N-1} h_k e^{\frac{2\pi i k n}{N}}$$

Notice that while $k$ runs from 0 to $N-1$, $n$ varies between $-\frac{N}{2}$ and $\frac{N}{2}$. Because the expression of $H_n$ is periodic in $n$ we can also vary $n$ from 0 to $N-1$ and remember that in fact the frequency is related to $n$ by equation B.6. The inverse discrete Fourier transform is given by [Press et al., 1993], p.497:

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-\frac{2\pi i n k}{N}}$$

### B.2 Fast Fourier Transform

To efficiently compute discrete Fourier transforms algorithms exist that reduce the number of computations from $O(N^2)$ to $O(N \log_2 N)$. Those algorithms are for that reason called Fast Fourier Transforms (FFT). One derivation of an algorithm by Danielson and Lanczos ([Press et al., 1993]) divides the discrete Fourier Transform as the sum of two others containing the even numbers and the odd numbers of the original N points:

$$\sum_{j=0}^{N-1} e^{\frac{2\pi i j k}{N}} f_j = \sum_{j=0}^{N-1} e^{\frac{2\pi i j k}{N}} f_{2j} + \sum_{j=0}^{N-1} e^{\frac{2\pi i j (k+1)}{N}} f_{2j+1}$$

which can be written as:

$$F_k = F_k^e + W^k F_k^o,$$

with $W \equiv e^{2\pi i N}$ and $F_k$ the $k$th component of the Fourier transform and $F_k^e$ and $F_k^o$ respectively the even and odd components of $F_k$, with length $\frac{N}{2}$. The same can be done
for each of the two terms in B.11 so we obtain 4 terms of length $\frac{N}{2}$. If $N$ is a power of 2, then the process can be continued to arrive at all transforms of length 1, which is just the input number itself $f_i$. The question is which input number; remember $n$ was running from $-\frac{N}{2}$ to $\frac{N}{2}$. It turns out that the sequence of even and odd is the reversed binary value of $n$ if you read 'even' as 0 and 'odd' as 1.

The structure of the FFT algorithm is as follows: take the data which is of course stored in a computer in binary values, and sort it into bit-reversed order. Combine adjacent value of $n$ if you read 'even' as 0 and 'odd' as 1. The process can be continued to arrive at all transforms of length 1, which is just the input number itself. The question is which input number; remember $n$ was running from $-\frac{N}{2}$ to $\frac{N}{2}$. It turns out that the sequence of even and odd is the reversed binary value of $n$ if you read 'even' as 0 and 'odd' as 1.

The structure of the FFT algorithm is as follows: take the data which is of course stored in a computer in binary values, and sort it into bit-reversed order. Combine adjacent numbers to form $\frac{N}{2}$ transforms of length 2, then combine two adjacent numbers to form $\frac{N}{4}$ transforms of length 4, until the final transform is obtained.

### B.3 Synthesis

Because of orthogonality of the Legendre polynomials, each function analytical on a sphere can be expanded into harmonic coefficients $C_{lm}$ and $S_{lm}$ as [Hobson, 1931]:

$$
\begin{align*}
\frac{C_{lm}}{S_{lm}} & = \frac{1}{4\pi} \int_\sigma f(\theta, \psi) \tilde{Y}_{lm}(\theta, \psi) \, d\sigma. \\
&B.12
\end{align*}
$$

in which a bar denotes normalization so that the integral over the sphere is equal to:

$$
\int_0^{2\pi} \int_0^\pi Y_{lm}(\theta, \psi) Y_{lm}^*(\theta, \psi) \sin \theta d\theta d\psi = 4\pi \delta_{ll'} \delta_{mm'}.
$$

(\text{B.13})

which gives for the normalization factor [Heiskanen and Moritz, 1967]:

$$
N_{lm} = \sqrt{\frac{(2 - \delta_{0m})(2l + 1)}{(2l + 1)!}}.
$$

(\text{B.14})

with

$$
\delta_{0m} = \begin{cases} 
1, & m = 0 \\
0, & m \neq 0
\end{cases}
$$

The $4\pi$ in the integral over the sphere is sometimes incorporated in the normalization constant. The surface harmonics $\tilde{Y}_{lm}$ are related to the normalized Legendre polynomials by:

$$
\tilde{Y}_{lm}(\theta, \psi) = \tilde{P}_{lm}(\cos \theta) \begin{bmatrix} \cos m\lambda \\ \sin m\lambda \end{bmatrix}
$$

(\text{B.16})

To retrieve the function $f(\theta, \psi)$ from the coefficients: [Lambeck, 1988], [Hobson, 1931]:

$$
f(\theta, \psi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \tilde{P}_{lm}(\cos \theta) (C_{lm} \cos m\psi + S_{lm} \sin m\psi)
$$

(\text{B.17})

Note that in [Mitrovica and Peltier, 1991] the second summation runs from $-l$ to $l$, while in [Colombo, 1981] and [Lambeck, 1988] the summation runs from 0 to $l$.

Colombo [Colombo, 1981] explains that a spherical harmonic expansion corresponds to a subset of a 2-D Fourier expansion in the domain $-\pi \leq \theta \leq \pi, 0 \leq \psi \leq 2\pi$. If the data is sampled with fixed spacing $\frac{\Delta}{\pi}$ all the Fourier coefficients $n$ and $m$ smaller than $L$ can be recovered. However if $n$ and $m$ exceed $L$, coefficients can be found that fit the data which are not the true coefficients because higher frequency terms are aliased in the data. Since spherical harmonics are a subset of the Fourier transform the same holds as we replace $\infty$ by $L$ in the summations.

The summations in B.17 can be reordered ([Colombo, 1981]):

$$
\sum_{l=0}^{L} \sum_{m=-l}^{l} \to \sum_{m=0}^{L} \sum_{l=m}^{L}
$$
so that latitude and longitude information can be treated independently and synthesis of coefficients to give the functions at the nodes of a grid can be performed in two steps:

\[
\begin{align*}
A_m(\theta_k) \\
B_m(\theta_k)
\end{align*}
\] 

\[= \sum_{l=-m}^{N} \tilde{P}_{lm}(\cos \theta_k) \left[ \frac{C_{lm}}{S_{lm}} \right] \]  

(B.19)

and

\[
f(\theta_i, \psi_j) = \sum_{m=0}^{L} A_m(\theta_i) \cos m\psi_j + B_m(\theta_i) \sin m\psi_j
\]

(B.20)

The second step is identical to taking the FFT of coefficients \(A_m\) and \(B_m\). As noted in the section B.2, the FFT is a fast procedure if the number of samples is a power of 2. If this is not the case it might be more efficient to use a less-efficient FFT (for example the mix-radix FFT) [Colombo, 1981].

In the pseudo-spectral sea level codes the harmonics are complex numbers defined by [Balmino, 1994]:

\[
\bar{Y}_{lm}(\theta, \psi) = \tilde{P}_{lm}(\cos \theta) e^{i m \lambda}
\]

(B.21)

which requires some tricks in the programming but essentially the \(C_{lm}\) and the \(S_{lm}\) coefficients can be seen as the real and imaginary part of the complex numbers in the codes respectively.

### B.4 Analysis

The discrete form of B.19 is not as straightforward as for synthesis. The first step can be written as [Sneeuw, 1994]:

\[
\begin{align*}
A_m(\theta_i) \\
B_m(\theta_i)
\end{align*}
\] 

\[= \frac{1}{L(1 + \delta_{m0} + \delta_{mL})} \sum_{j=0}^{2L-1} f(\theta_i, \lambda_j) \left[ \frac{\cos(m\psi_j)}{\sin(m\psi_j)} \right] \]  

(B.22)

but the second part is problematic because the sampled harmonics are no longer orthogonal. The Fourier base functions do maintain their orthogonality property after discretization, so the Fourier coefficients can be computed exact (up to the Nyquist frequency) by:

\[
a_m^\alpha = H \sum_{k=0}^{2L-1} \left[ \frac{\cos(mk\Delta \psi)}{\sin(mk\Delta \psi)} f(h\psi) \right] d\psi
\]

(B.23)

The sampled harmonics only then have the orthogonality property if they are properly weighted with weights \(w_i\) [Sneeuw, 1994]:

\[
\begin{align*}
C_{lm} \\
S_{lm}
\end{align*}
\] 

\[= \frac{1}{4} (1 + \delta_{m0}) \sum_{i=1}^{L} \sum_{j=0}^{L} w_i \tilde{P}_{lm}(\cos \theta_i) A_m(\theta_i) B_m(\theta_i) \]  

(B.24)

In the continuous case orthogonality is maintained. A comparison with the continuous case suggests that the weights be some kind of sine function. The nineteenth century mathematician Gauss derived that there exist weights for which equation B.24 is exact, and that they must fulfill the quadrature formulas [Sneeuw, 1994]:

\[
\sum_{i=1}^{N} w_i x_i^n = \int_{-1}^{1} x^n dx = \left\{ \begin{array}{ll}
\frac{2}{m+1}, & n \text{ even} \\
0, & n \text{ odd}
\end{array} \right.
\]

(B.25)

Moreover he showed that if for the latitudinal grid-spacing the zeros of the \(l + 1\)th order Legendre polynomials are chosen orthogonality is maintained so that follows [Sneeuw, 1994]:

\[
\sum_{i=1}^{N} w_i \tilde{P}_{lm}(x_i) \tilde{P}_{lm}(x_i) = 2(2 - \delta_{m0}) \delta_l \delta_l
\]

(B.26)
the number of parallels need not be higher than $L + 1$. The quadratic formula leads to Gaussian quadrature, which is a special case of numerical integration of functions. The x-values are denoted abscissas. In general the quadrature formula for numerical integration is [Press et al., 1993], p.143:

$$\int_a^b W(x)f(x) \, dx \sim \sum_{j=1}^N w_j f(x_j)$$  \hspace{1cm} (B.27)

The $N$ abscissas with the weighting function $W(x)$ are exactly the roots of the polynomial which is orthogonal on the same interval and weighting function $W(x)$. In our case, the weighting function $W(x)$ is 1 (since there is no other function before the Legendre polynomials in B.19) and the x are equal to $\cos \theta_i$ so that the integral boundaries 0 to $\pi$ become −1 to 1. The orthogonal polynomial is the $N$th degree Legendre polynomial and the quadrature is referred to as Gauss-Legendre quadrature. Corresponding weights $w_j$ can be calculated by:

$$w_j = \frac{2}{(1 - x_j^2)(\frac{dP_N(x_j)}{dx})^2}$$  \hspace{1cm} (B.28)

Tabulated values for the abscissas and the weights can be found in [Stroud and Secrest, 1966].

The Gauss-Legendre quadrature method solves the orthogonality problem of the spherical harmonics only for $L + 1$ parallels, at the cost of having to work on a grid which is not equi-angular, which means that the ocean function and the ice loads need also to be given on the Gaussian grids and therefore some kind of interpolation must be performed.
Appendix C

Finite Disc Method

This section outlines a finite element method for solving the sea level equation, which can be found in [Wu and Peltier, 1983a] and [Tushingham and Peltier, 1991]. The method employs the Love numbers resulting from the normal mode method of section 3.5 to obtain Greens functions for potential perturbation and radial displacement. Apart from a discretization in time (as in equation 4.15), a spatial discretization is used to allow computation of the integrals over ocean and glaciated areas. To facilitate the computation the ice elements are replaced by discs with equal area. Along the coasts one expects the most sharp changes, therefore the coastal elements will be smaller there.

To find the sea level rise at an element $i$ the following steps have to be taken:

- Decide for each element $j$ whether it is an ocean element or a land element. Multiply by the ocean water density, $\rho_w$, if it is an ocean element, or by the ice density, $\rho_i$, if it is an ice element.
- Multiply by the elastic interaction matrix $M_{ij}^{E}$ to get the elastic sea level increment at element $i$. $GF^{E}$ is the elastic Greens function, which depends on the distance between $r$ and $r'$.
- Multiply the sea level increment at time $t_i$, $\delta S_{ij}$, with $\rho_w$ and the viscous interaction matrix $M_{ij}^{H,V}$ to get the visco-elastic Greens function for a Heaviside load at time $t_j$. (Here, the Greens function has to be seen as an influence function).
- Multiply the ice load increment at time $t_i$, $\delta I_{ij}$, with $\rho_i$ and the viscous interaction matrix $G_{ij}^{H,V}$ to get the elastic Greens function.
- Compute the eustatic sea level rise $G(t)$.

The sea level equation in discrete form becomes [Wu and Peltier, 1983a]:

$$S_{ip} = \rho_u G_{ij}^{E} S_{jp} + \rho G_{ij}^{E} I_{jp} + \rho_u G_{ij}^{H,V} \delta S_{j} + \rho G_{ij}^{H,V} \delta I_{j} - \frac{\rho_i}{\rho_w A_0} E_{ij} I_{jp} +$$

$$- \frac{1}{A_0} \left\{ E_{kj} \rho_u G_{kj}^{E} S_{jp} + \rho G_{kj}^{E} I_{jp} + \rho_u G_{kj}^{H,V} \delta S_{j} + \rho G_{kj}^{H,V} \delta I_{j} \right\},$$

(C.1)

with $S_{ip}$ and $I_{ip}$ the ice and sea level change at element $i$ at time $t_{ip}$, respectively, and $\delta S_{ip} = S_{ip} - s_{i(p-1)}$ and $\delta I_{ip} = I_{ip} - I_{i(p-1)}$. $j$ involves the summation over the ocean elements or over ice elements, depending on whether ice or ocean load is considered.

The eustatic sea level acts as a first guess to the unknown $S_{ip}$:

$$S_{ip}^{(1)} = \rho \frac{A_j I_{jp}}{\rho_w A_0},$$

(C.2)

with $A_j$ denoting the area of the $j$th element. Left- and right hand side differ by a residue $S_{ip}^{R}$ which is used to correct the eustatic sea level and obtain a second iterate. The input for
the computation is the glaciation load history $I_n$, and the Greens functions $G_i^{E}$ and $G_i^{H,V}$. The interaction matrices are the bottleneck, as they not only depend on the distance of the element from the ice load, but also on the radius of the disc. For each location and radius a new interaction matrix should be computed, or at least be calculated for a number of radii and locations only to obtain the other elements by interpolation [Wu and Peltier, 1983a].

Errors specifically for the finite disc way of solving the SLE originate from the spatial discretization. First, glacial features smaller than the disc size will not be represented. Second, since equivalent circular discs were used, inevitably points exist which are part of two discs, or points that are in the gap between the discs.
Appendix D

Microphysical Approach to Creep Mechanisms

The mechanisms of dislocation and diffusion creep can be explained from the presence of defects in the crystal lattice which divide in:

- point defect, a vacancy in the crystal lattice;
- dislocations, defects along a line in the lattice; They can emerge during the growth of a crystal but also multiplicate at stress concentrations.
- planar defects, two-dimensional defects at, for example, sub-grain boundaries.

**Diffusion creep.** Vacancies are distributed evenly over the lattice by the natural process of diffusion. The diffusion flux is proportional to concentration gradients:

\[ J_i = D_{ij} \frac{\partial C_j}{\partial x_j}. \] (D.1)

The diffusion coefficient \( D \) has dimension \( \frac{[m^2]}{[s]} \). It depends on the grid-size of the lattice, the vacancy density, and the probability that one of the neighbor atoms jumps into the vacancy. The diffusion-coefficient \( D_{SD} \) can be written as [Ranalli, 1995], p.287, [Schubert et al., 2001], p.240:

\[ D_{SD} = D_0 e^{-\frac{E+pV}{RT}}. \] (D.2)

The activation energy \( E \) is the energy needed for a vacancy to jump to an adjacent place in the lattice. The term \( pV \), where \( V \) is the activation volume, is the effect of pressure \( p \) obstructing vacancy migration; \( R \) is the gas constant and \( T \) the temperature. For oxides the diffusion coefficient varies from \( 10^{15} \) to \( 10^{15} \text{ m}^2 \). Diffusion can directly control creep if diffusion takes place at the grain boundaries. It can be derived that the strain rate obeys [Ranalli, 1995], p.321:

\[ \dot{\epsilon} \sim \frac{D^* p \Omega}{kT d^c} \left( \frac{\sigma}{\nu} \right). \] (D.3)

Here \( \sigma \) is the deviatoric part of the stress which is linearly related to strain rate.

**Dislocation creep.** See figure D.1.

Two types of dislocation creep mechanisms can occur: dislocation slip and dislocation climb. In the former the dislocation line moves along the slip plane by breaking inter-atomic bonds. Climb takes place when a neighboring atom adds to the dislocation leaving a vacancy in the lattice. The velocity \( \nu \) with which the vacancies move is controlled by diffusion inside the grain boundaries. Once this velocity is known, the strain rate can be calculated using Orowan’s equation [Ranalli, 1995], p.300:

\[ \dot{\epsilon}_{ij} = \rho_m \nu b \delta_{ij}, \] (D.4)

where \( \rho_m \) is the density of movable dislocations, \( \vec{b} \) is the so-called Burgers vector which represents the direction and magnitude of slip. The stress-dependency is hidden in the
density: the larger the applied stress, the higher the dislocation density. The expression for strain rate using the previous equation and the expression for the self-diffusion coefficient $D$ becomes [Ranalli, 1995], p.316:

$$
\dot{\varepsilon}_{ij} \sim \frac{D_{ij} b_{ij}}{kT} \left( \frac{\sigma_{ij}}{\mu} \right)^n.
$$

(D.5)

In general both diffusion creep and dislocation creep will take place. A creep relation that incorporates both can be formulated as [Schubert et al., 2001], p.243, [Karato and Wu, 1993]:

$$
\dot{\varepsilon} = A \left( \frac{\sigma}{\mu} \right)^n \left( \frac{b}{h} \right)^m e^{-\frac{E_g \varepsilon}{RT}}.
$$

(D.6)

in which $A$ a pre-exponential factor. The shear modulus $\mu$ is about 80 GPa, the Burgers vector is roughly 0.5 nm, $n$ is the stress exponent and $m$ is the grain-size exponent For diffusion creep $n = 1$ and $m = 2.5$, and for dislocation creep $n = 3.5$ and $m = 0$. Note that the stress-strain relations are not given in tensor notation because here we are primarily interested in qualitative analysis. See [Ranalli, 1995], p.326 for the tensor notation of equations D.3 and D.6.)

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**Figure D.1** Dislocation climb. From: Schubert et al. (2002).
Colombo, O.L.A (1981), Numerical Methods for Harmonic Analysis on the Sphere, 310, The Ohio State University, Department of Geodetic Science and Surveying.


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