Mode Enrichment and The Analysis of Through Cracks in Thin Walled Shells under Internal Pressure

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by

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Summary

Basic to the analysis of through-the-thickness cracks in a pressurized fuselage is the calculation of the mixed mode stress intensity factors. A suitable candidate for this task is the mode enrichment method which corresponds to a special version of the p-refinement technique applied to a small region around the crack tip. A complicating factor in the application of this method in this particular situation is the geometrical non-linearity that governs the behavior of the cracks. The necessity to cope with this non-linearity requires several adaptations in the formulation and implementation of the method.

Although more complicated than the so called nodal release method, the mode enrichment method turns out to be an accurate and dependable technique. These characteristics make it an attractive tool for nonlinear crack growth simulations.

1. Introduction

The prediction of the residual strength of a damaged pressurized fuselage must rely on computational methods that can determine the stress intensity factors of through-the-thickness cracks in these shells accurately under loading conditions that closely resemble the conditions that occur in flight (Ref. 1 - 7). The cracks that have to be studied in these simulations are relatively long, i.e. their length compares with the pitch distance of the stiffeners and bulkheads of the shell. A complicating factor in the treatment of these types of cracks is that their behavior is strongly influenced by a geometrically nonlinear effect. This means that the classical methods for stress intensity computations cannot directly be applied to these types of problems, but must be modified to cope with the non-linearity of the situation.

For the computation of the stress intensity factors in pressurized shells, energy based methods prove to be relatively simple and accurate. It is possible to distinguish between two kinds of methods in this category. Methods of the first kind are numerical adaptations of the so called path independent J-integral representation of the energy release rate (Refs. 8 - 9). Special adaptations of this approach for the analysis of cracks in pressurized shells can be found in (Refs. 10 - 11) (for mode I type cracking only). Another method that belongs to the first kind is the node release method (Ref. 12) which, with a slight modification, can be transformed into the so called crack closure inte-
gral (Ref.13). This method is very attractive because it can be used for the analysis of mixed mode cracking, i.e. it produces four energy release rate contributions related to two membrane and two bending modes of crack opening. It also has the advantage it shares with all energy based methods: it does not require special elements to model the structure near the crack tip.

These methods compute the stress intensity factors in an indirect way, i.e. they are extracted from the computed energy release rates. In contrast, the method we want to consider in this paper is to compute the K- factors directly. It is the so called “singular mode enrichment“ or “singular mode superposition“ method (Refs. 13 - 15), which is based on an expansion of the finite element function space by the addition of a set of functions that represent the singular state of stress at the crack tip. The amplitudes of these singular trial functions are the stress intensity factors that appear as unknowns in the formulation of the governing equations (see for additional references Refs. 16 - 20).

There are three principal aspects of the mode injection method that need a close investigation for the applications that we have in mind. The first (i) concerns the capability to deal with mixed mode cracking problems. The second (ii) concerns the nonlinear behavior of the crack and the third (iii) touches the changes that are necessary to adapt the nonlinear solution process to the enrichment process.

(i) mixed mode cracking

Although the mode enrichment method provides a natural framework for the identification of the combination of modes of deformation at the crack tip, one still has to consider what type of shell elements are available for the implementation of the method. Shell elements in use today are either based on the classical Kirchhoff assumptions (Ref. 21), or on kinematic assumptions analogous to those used by Reissner (Refs. 22 - 23). The difference is whether the transverse shear is neglected (the Kirchhoff theory), or retained (Reissner theory). An example of an element of the first kind is described in (Ref. 24). Examples of elements of the second kind can be found in (Refs. 25 - 28). Shell elements of the second kind are very popular because the Reissner assumptions lead to a simplification of the element formulation. The introduction of transverse shear as a mechanism offers a convenient way to circumvent the stringent C1 smoothness requirements that need be fulfilled in the case of the Kirchhoff elements. The description of transverse shear deformation of the shell is thus not the motivation for the construction of these elements.

For convenience, we refer to elements based on the first kind of assumptions as Kirchhoff shell elements and to elements of the second category as Reissner shell elements, even though the description is very imprecise. What is important here is that the difference between the two theories leads to a difference in the definition of the crack opening modes and corresponding stress intensities. Within the context of classical shell theory, the number of independent modes is four. Within the context of a Reissner theory the number of independent modes is five. More precisely, the difference is in the anti-plane modes of cracking. In the Kirchhoff theory, two anti-plane intensity factors can be defined that are connected with a bending and a tearing mode, while in the Reissner theory the number of factors that can be defined is three, including two bending intensity factors and an anti-plane mode factor that is related to the Reissner transverse shear.

Is this distinction important and does it have any implications? It has recently been show that the Kirchhoff intensity factors are determined by the Reissner factors (Ref. 29) (but not th
other way around). This means that the Kirchhoff based factors can be extracted from a discretization that is based on both type of elements. Indeed, this is actually one of the objectives of this paper, i.e. to describe how the Kirchhoff based intensity factors can be computed from shell elements that are based on the Reissner kinematic assumptions.

(ii) **nonlinear crack behavior**

The second problem with our application of the mode enrichment technique is the nonlinear behavior of the crack (Refs. 1-7, 10-12). The nonlinearity implies that the crack tip no longer stays in the plane of the undeformed shell, but translates and rotates when the load is varied. Therefore, the enrichment technique must be carried out on the current orientation of the crack tip and not on the orientation of the undeformed crack as is done in the linear case. Moreover, the stress intensity factors that correspond to the membrane deformation modes are influenced by the nonlinear stretch measures that are used in the formulation. This means that the definition of the stress intensity factors in the nonlinear case is not consistent with the definition that is used in the linear theory unless we adopt a modification.

(iii) **adaptations nonlinear solution process**

The third problem that needs consideration concerns the solution of the nonlinear equations (Refs. 10-12). This issue is of particular interest in this paper because the implementation of the method is carried out within the framework of an existing finite element code. The modifications made necessary by mode enrichment correspond to an expansion of the basic set of equations generated by the underlying finite element mesh. Like the basic set, the additional equations are also nonlinear. The construction of the solution procedure for the resulting expanded set of equations is thus not straightforward, especially if one wants it to be carried out with no changes to the existing path following method. How this difficulty and the others mentioned above can be resolved will be discussed in the following chapters.

2. Description of the formulation

2.1 **Basic principle**

For a shell or a plate, the general form of the linear elasticity solution for the stresses in the immediate surrounding of a sharply edged crack does not depend on the loads and boundary conditions applied to the structure at finite distance of the crack tip. Only the stress intensity factors that are the characteristic parameters in this solution depend on these conditions. See for example (Refs. 30, 31). It is this property that is exploited by the singular mode superposition technique.

The general solution for a crack in a plate or shell in terms of the displacements \( \mathbf{u} = (u, v)^T \) tangential to the mid surface and stresses \( \mathbf{\sigma} = (\sigma_{rr}, \sigma_{\theta \theta}, \sigma_{r\theta})^T \) can be decomposed in two parts:

\[
\mathbf{u} = \mathbf{u}_t(r, \theta, z) + \mathbf{u}_d(r, \theta, z) \\
\mathbf{\sigma} = \mathbf{\sigma}_t(r, \theta, z) + \mathbf{\sigma}_d(r, \theta, z)
\]  

(2.1)

where \( r, \theta \) are the components of a polar coordinate system at the tip of the crack, as is depicted in figure 1, and \( z \) is the distance measured along the normal \( \mathbf{n} \) to the mid surface of the plate. The subscript \( (t) \) refers to the (singular) crack tip solution that is dominant in a close neighborhood of the
crack tip and the subscript \( a \) refers to the regular part of the solution. For example, for a shell in a general state of deformation, the crack tip solutions can be cast in the form (more detailed expressions will be given later):

\[
\begin{align*}
\mathbf{u}_t(r, \theta) &= \sum_{i=1,2} \sqrt{r} \left\{ K_i \Phi_i(\theta) + z k_i \Theta_i(\theta) \right\} \\
\sigma_t(r, \theta) &= \sum_{i=1,2} \frac{1}{\sqrt{r}} \left\{ K_i \Psi_i(\theta) + z k_i Z_i(\theta) \right\}
\end{align*}
\]

\[K_i \ (i = 1,2) = (K_i, K_{ii}); \quad k_i \ (i = 1,2) = (k_1, k_2)\]

where the particular form of the functions \( \Phi_i, \Theta_i, \Psi_i \) and \( Z_i \) depends only on the specific shell theory that is used. The stress intensity factors \( K_i \) and \( K_{ii} \) in these expressions are determined by the load intensity, the boundary conditions, the geometry and physical properties of the structure surrounding the crack. The solutions (2.2) are generally applicable as long as the problem remains within the realm of the theory of linear elasticity.

It is now noted that the function \( \mathbf{u}_t \) varies with \( \sqrt{r} \) while the lowest order terms of \( \mathbf{u}_a \) are proportional to \( r \). For the stresses a similar qualification holds. The crack tip solutions \( \sigma_t \) are proportional to \( (\sqrt{r})^{-1} \) while the leading term in \( \sigma_a \) is a constant. Consequently, very close to the crack tip the solution of the crack problem is represented by \( \mathbf{u}_t \) and \( \sigma_t \) while away from the crack the solution field is represented by the composition (2.1). At the crack tip \( r = 0 \), all derivatives of \( \mathbf{u}_t \) with respect to \( r \) are singular, while away from the crack tip the solution is smooth with derivatives that are continuous in the entire remaining domain of the structure.

As noted in (Refs. 13 - 17), the separation of the general solution into two parts: i.e. into a local singular field plus a regular global field can be carried over into a finite element solution scheme in an almost natural way. The first step in this approach is to construct a representation of the global field \( \mathbf{u}_a \) in terms of a regular finite element distribution. The second step is to add (locally) the functions \( \mathbf{u}_t \) (2.2) to this field so that the completion of the solution is obtained. The result of such a discretization is that the solution is represented in terms of a nodal displacement set \( \mathbf{d} \) belonging to the finite element distribution and a set \( \mathbf{k} \) of intensity factors (mode amplitudes) that are coupled with the local singular field (2.2). In this way the stress intensity factors appear as an additional set of computational freedoms. In other words, this technique produces the stress intensity factor as an integrated part of the solution set.

### 2.2 The crack opening modes

In this paper we will focus on through cracks of a type that occur in the outer skin of the fuselage of an airliner. The cracks (with length \( 2a \)) are relatively long, i.e. the ratio:

\[
\alpha = \frac{2a}{\sqrt{hR}} \sqrt{\frac{1}{1 - \nu^2}}
\]

\[\text{(2.3)}\]

varies between \( \alpha = 1 - 30 \), where \( h \) = the thickness and \( R \) = the radius of the shell. This corresponds to cracks of a length varying between the width of a stiffener and a multiple of bulkhead
pitches. It will further be assumed that the state of stress at the crack tip can be approximated by a state of plane stress in the case of classical shell theory (the Kirchhoff assumptions) or by a combination of plane stress with

![Figure 1 Basic Conventions](image)

a simple state of transverse shear in the case of a through-the-thickness shear deformable shell theory (the Reissner assumptions). In either case, these theories lead to a gross simplification of the actual three dimensional state of stress at the crack tip, but this difficulty will be ignored in this paper.

Under these particular restrictions, the state of deformation at the crack tip can be seen to be composed of a set of distinct deformation modes (see figure 2). In the classical Kirchhoff theory the number of independent modes that can be defined is four; two membrane modes $I_m$ and $II_m$, a bending mode $I_b$ and a tearing mode $II_b$ (which could be called the effective tearing mode (see appendix 1)). The transverse shear mode $III_T$ associated with a transverse shear stress distribution through the thickness of the shell does not exist in this theory.

While the loading associated with the first three modes consists of the membrane stress resultants $T_n$, $T_t$ and the bending stress resultant $M_t$ respectively, the loading associated with the tearing mode is given by $Q^* = Q - \frac{\partial M_n}{\partial s}$, the so called effective transverse shear force. Alternatively, this load can be replaced by a statically equivalent effective twisting moment $M^* = M_n - M_n^*$ with
$M^* = \int Q ds$ (s = arclength of the generator along the imaginary cut through the shell). For a illustration of these concepts refer to the appendix A1.

Figure 2 Crack opening modes for a shell

In contrast, in the Reissner theory, the number of independent modes is five with the associate loads as pictured in figure 2. In this case, the resultants $M_n$ and $Q$ and corresponding modes are no longer related to each other but defined by two independent states of stress: the through the thickness varying in plane shear stresses and the through the thickness varying transverse shear. The mode $II_b$ can now be identified with the tearing mode, while the mode $III_T$ is a transverse shear mode.

In the following we will denote the stress intensity factors that correspond to $I_m$ and $II_m$ (in both theories) by $K_1$ and $K_{11}$. The stress intensities that belong to the modes $I_b$ and $II_b$ in the Kirchhoff theory will be denoted by $k_1$ and $k_2$. If we refer to a Reissner type of theory, then the
stress intensities pertaining to I, II, and III will be denoted by $K_{B1}$ and $K_{B2}$ and $K_{III}$ respectively.

Although we will focus exclusively on the computation of the intensity factors defined in the Kirchhoff theory, it will be done here via elements that are based on assumptions that correspond to the Reissner assumptions. This is the reason that we have to discuss mode injection in the context of both theories.

![Diagram of internal stress resultants along an imaginary cut of the shell]

**Figure 3** Internal stress resultants along an imaginary cut of the shell

In what follows we present solutions that are derived for plates, not shells. The difference plays no role in our considerations because the mode enrichment is carried out in a very small region $\Omega_0$ around the crack tip (figure 1). This is expressed by $\delta/R << 1$, where $\delta$ is the largest distance measured between the crack tip to the outer boundary of the enrichment domain. Consequently, it is permissible to neglect the effect of the curvature on the singular crack tip solutions.
2.3 Specification of the crack tip solutions for a plate

(i) The crack tip solutions for the Kirchhoff plate

It is clear that if mode injection is applied to shell elements based on the Kirchhoff assumptions, the injection process should also be based on these assumptions. In the context of classical plate theory, the solution of the problem of a plane through crack can be written in the form (Refs. 31, 29):

\[ u = r^{1/2} K^T \Phi_u(\theta) - z \frac{\partial w(r, \theta)}{\partial x} \]  
\[ v = r^{1/2} K^T \Phi_v(\theta) - z \frac{\partial w(r, \theta)}{\partial y} \]  
\[ w = r^{3/2} k^T \Phi_w(\theta) \]  

(2.4a)

(2.4b)

(2.4c)

where the vector functions \( \Phi_u = (\Phi_{1u}, \Phi_{2u})^T \); \( \Phi_v = (\Phi_{1v}, \Phi_{2v})^T \); \( \Phi_w = (\Phi_{1w}, \Phi_{2w})^T \) are specified below. The symbols \( K \) and \( k \) represent here the stress intensity factors \( K_i, K_{II} \) for the membrane part of the solution and \( k_1, k_2 \) for the bending and tearing part of the solution, so that:

\[ K = (K_i, K_{II})^T; k = (k_1, k_2)^T \]

(2.5)

The polar coordinate system that is used here to describe the functions is defined in Figure 1. The displacement components \( u, v \) lie along the directions of the in plane Cartesian coordinates \((x, y)\) while \( w \) is the displacement component of the reference plane of the plate in the direction \( z \) perpendicular to \((x, y)\).

The substitution of the right-hand side of \( w \) in (2.4c) results in:

\[ u = r^{1/2} K^T \Theta_u(\theta) - z \frac{3}{2} r^{1/2} k^T \Theta_u(\theta) \]
\[ v = r^{1/2} K^T \Theta_v(\theta) - z \frac{3}{2} r^{1/2} k^T \Theta_v(\theta) \]
\[ w = r^{3/2} k^T \Phi_w(\theta) \]

(2.6)

where the functions \( \Theta_u, \Theta_v \) are defined by:

\[ \Theta_u = \Phi_w(\theta) \cos \theta - \frac{2}{3} \Phi'_w(\theta) \sin \theta \]
\[ \Theta_v = \Phi_w(\theta) \sin \theta + \frac{2}{3} \Phi'_w(\theta) \cos \theta \]
\[ \Phi'_w(\theta) = \frac{d}{d\theta} \{ \Phi_w(\theta) \} \]

\[ \Phi_u(\theta) = c_1 \left( \frac{\cos \frac{\theta}{2} \left( 1 - \frac{1}{1 + \nu} \right) + \sin^2 \frac{\theta}{2}}{1 + \nu} \right), \sin \frac{\theta}{2} \left( \frac{2}{1 + \nu} + \cos^2 \frac{\theta}{2} \right) \right)^T \]
\[ \Phi_v(\theta) = c_1 \left( \sin \frac{\theta}{2} \left( \frac{2}{1 + \nu} - \cos^2 \frac{\theta}{2} \right), \cos \frac{\theta}{2} \left( \frac{1 - \nu}{1 + \nu} + \sin^2 \frac{\theta}{2} \right) \right)^T \]
\[ \Phi_w(\theta) = c_2 \left( \frac{1}{3} \frac{1 - \nu}{1 - \nu} \cos \frac{3\theta}{2} - \cos \frac{\theta}{2} \left( \frac{1}{3} \frac{1 - \nu}{1 - \nu} \sin \frac{3\theta}{2} - \sin \frac{\theta}{2} \right) \right)^T \]

(2.8a)

(2.8b)

(2.8c)
\[ c_1 = \frac{2(1 + v)}{E \sqrt{2\pi}} \quad c_2 = \frac{\sqrt{2}}{Eh} \frac{1 - v^2}{3 + v} \]  

(2.8d)

The form of the membrane part of the solution can be found in (Refs. 30, 31). The bending part was first derived by Williams (Ref. 32), see also (Refs. 29, 33, 34). The particular specification that is given above was taken from (Ref. 29).

(i) The crack tip solutions for the Reissner plate

If the available elements are based on the kinematic assumptions that correspond to the Reissner assumptions, it is advisable to make the injection process conform to these assumptions. Most shell elements of this kind are capable of describing simple shear, i.e., the shear distribution is assumed to be constant through the thickness (which is physically impossible), as opposed to more refined assumptions that yield a parabolic solution (which is physically possible). The solutions presented here refer to the simple constant shear theory.

If the stress intensity factors \(K_i, K_{ii}, K_{bi}, K_{bii}, K_{iii}\) are collected as:

\[ \mathbf{K} = (K_i, K_{ii})^T ; \mathbf{k} = (K_{bi}, K_{bii}, K_{iii})^T \]  

(2.9)

these solutions read:

\[ u = r^{1/2} \mathbf{K}^T \Phi_u(\theta) - z \frac{3}{2} r^{1/2} \mathbf{k}^T \Phi_0(\theta) \]
\[ v = r^{1/2} \mathbf{K}^T \Phi_v(\theta) - z \frac{3}{2} r^{1/2} \mathbf{k}^T \Phi_0(\theta) \]  

(2.10)

\[ w = r^{1/2} \mathbf{k}^T \Phi_w(\theta) \]

\[ \Phi_u(\theta) = c_1 \left( \cos \frac{\theta}{2} \left( \frac{1 - v}{1 + v} + \sin^2 \frac{\theta}{2} \right), \sin \frac{\theta}{2} \left( \frac{2}{1 + v} + \cos^2 \frac{\theta}{2} \right), 0 \right)^T \]  

(2.11a)

\[ \Phi_v(\theta) = c_1 \left( \sin \frac{\theta}{2} \left( \frac{2}{1 + v} - \cos^2 \frac{\theta}{2} \right), \cos \frac{\theta}{2} \left( - \frac{1 - v}{1 + v} + \sin^2 \frac{\theta}{2} \right), 0 \right)^T \]  

(2.11b)

\[ \Phi_w(\theta) = c_2 \left( F_1(r^{3/2}, \theta), F_2(r^{3/2}, \theta), (\sin \frac{\theta}{2}) \right)^T \]  

(2.11c)

\[ \Phi_0(\theta) = c_3 \left( \cos \frac{\theta}{2} \left( \frac{1 - v}{1 + v} + \sin^2 \frac{\theta}{2} \right), \sin \frac{\theta}{2} \left( \frac{2}{1 + v} + \cos^2 \frac{\theta}{2} \right) \right)^T \]  

(2.12d)

\[ \Phi_0(\theta) = c_3 \left( \sin \frac{\theta}{2} \left( \frac{2}{1 + v} - \cos^2 \frac{\theta}{2} \right), \cos \frac{\theta}{2} \left( - \frac{1 - v}{1 + v} + \sin^2 \frac{\theta}{2} \right) \right)^T \]  

(2.12e)

\[ c_1 = \frac{2(1 + v)}{E \sqrt{2\pi}} ; c_2 = \frac{\sqrt{2}}{E} \frac{2(1 + v)}{E} ; c_3 = \frac{2\sqrt{2}}{3h} \frac{2(1 + v)}{Eh} \]  

(2.12f)
3. Aspects of the Implementation

3.1 The computation of the Kirchhoff intensity factors from transverse shear deformable elements

Our interest is primarily focused on the Kirchhoff stress intensity factors, because these factors are more universally applicable and because it is doubtful that the refinement that Reissner theory presents can be exploited in crack growth simulations. However, the elements that are considered in this paper belong to the class that are based on kinematic assumptions analogous to those employed in the Reissner theory (Refs. 26, 28). This means, as was already mentioned, that in the case the injection process can better be based on the functions defined by equations (2.10 - 2.12) rather than those given by (2.6 - 2.8). How do we compute the Kirchhoff factors in that case?

Hui and Zhender (Ref. 29) recently showed that a relationship exists between the Kirchhoff stress intensity factors on the one hand and the Reissner factors on the other. In their study, the compared the Kirchhoff theory with a Reissner type of plate theory where the transverse shear stresses are assumed to be parabolic through the plate thickness. In that case the relation between the factors is:

\[
\begin{align*}
    k_1^2 &= \frac{3 + \nu}{1 + \nu} K_{B1}^2 \\
    k_2^2 &= \frac{3 + \nu}{1 + \nu} \left[ K_{B2}^2 + K_{III}^2 \frac{8(1 + \nu)}{5} \right]
\end{align*}
\]  

(3.1a)

This result shows that in the case of transverse shear deformable elements, the calculation of \( k_1 \) and \( k_2 \) can be based on \( K_{B1} \), \( K_{B2} \) and \( K_{III} \) using (2.9 - 2.12). Indeed, this is the approach followed in this paper, but with a slight modification.

As mentioned, the shear deformable shell elements that are used here for this study of the injection method are variants of the elements described in (Ref. 26, 28). They are 4, 8 and 16-node isoparametric elements. In contrast to the plate theory used by Hui and Zhender, the through-the-thickness transverse shear distribution represented in these elements is linear. As a result of this simplification, the relations (2.13) must be changed to:

\[
\begin{align*}
    k_1^2 &= \frac{3 + \nu}{1 + \nu} K_{B1}^2 \\
    k_2^2 &= \frac{3 + \nu}{1 + \nu} \left[ K_{B2}^2 + K_{III}^2 \frac{3(1 + \nu)}{\pi} \right]
\end{align*}
\]  

(3.1b)

The derivation is given in (Ref. 35) and sketched in the appendix A2 of the present paper.

The enrichment process technique described here is thus based on the superposition of the functions (2.9 - 2.12). The factors \( k_1 \) and \( k_2 \) are then computed from relations (3.1b).
3.2 Corotational formulation of the superposition technique

The singular set of functions is taken from the linear elasticity solution for an infinite plate with a crack under general loading conditions. It is defined with respect to the orientation of the crack geometry which in the linear theory does not appreciably change during the loading phase. When the problem is geometrically nonlinear and we deal with a long crack, the crack front may undergo a finite displacement and rotation under the given load. In that case, the definition of the set of shape functions $u_i$ should be with respect to the rotated geometry. Consequently, in the nonlinear case it is not correct to simply add the classical crack tip solutions to the global solution. This can only be done after these special solutions are modified appropriately. The following development is included to make this issue clear. (Notice, that we will restrict the discussion to the membrane part of the solution, because the discussion that includes the bending part is completely analogous.)

The shell mid-surface in the undeformed state $B_0$ is described by the mapping:

$$X = y(\xi, 0)$$  \hspace{1cm} (3.2a)

where $\xi$ denotes the set of intrinsic shell coordinates $\xi = (\xi_1, \xi_2)$. After deformation to state $B$, the shell mid-surface is described by

$$x = y(\xi, \eta)$$  \hspace{1cm} (3.2b)

where $\eta$ is the parameter that represents the intensity of the deformation, loading or the time, whichever is appropriate. (The inclusion of the evolution parameter in this discussion is purely for the sake of clarity of exposition.)

The difference between the two states is the displacement (vector) field

$$u = u(\xi, \eta) = y(\xi, \eta) - y(\xi, 0)$$  \hspace{1cm} (3.3)

The displacement $u$ in the mode superposition method consists of the global field $u_a$ furnished by the finite element discretization representation:

$$u_a = u_a(\xi, \eta) = d(\eta)^TN(\xi)$$  \hspace{1cm} (3.4a)

and the singular crack tip solution:

$$u_t = u_t(\xi, \eta) = K(\eta)^T\Theta(\xi)$$  \hspace{1cm} (3.4b)

where as before $K$ denotes the set of the membrane stress intensity factors:

$$K = (K_I, K_{II})^T$$  \hspace{1cm} (3.5)

In these expressions, $N$ and $\Theta$ denote the shape functions for the global and local field respectively, while $d$ represents the set of nodal freedoms of the global field $u_a$.  

11
Figure 5 Corotational formulation

The dimension of the vectors of nodal variables $\mathbf{d}$ and $\mathbf{K}$ differs usually by several orders of magnitude, i.e. for our applications, $\mathbf{K} \in \mathbb{R}_2$ and $\mathbf{d} \in \mathbb{R}_N$ where $N >> 2$. As discussed before the global field $\mathbf{u}_g$ extends itself over the whole domain of the structure while the singular field $\mathbf{u}_t$ is non-zero only in a small domain $\Omega_0$ around the crack tip.

Although the singular crack tip shape function $\mathbf{u}_t$ must be constructed on the basis of the classical crack tip solutions (2.6 to 2.12), it must be formulated in a way that takes the crack tip rotation into account during the deformation process. If the classical shape functions related to the membrane part of the solution are denoted by $\mathbf{v}(\xi)$, the function $\mathbf{u}_t$ can be approximated by:

$$\mathbf{u}_t = R(\xi_T) \mathbf{v}(\xi) = K \mathbf{R}(\xi_T) \Theta(\xi)$$

(3.6)

as long as the region $\Omega_0$ in which $\mathbf{u}_t$ is defined is small. The matrix $R(\xi_T)$ represents the rotation of the crack tip ($\xi_T = \text{location of the crack tip}$). Note that the rotation matrix is dependent on the solution $\mathbf{u}_d(\eta)$, so that one could write:

$$\mathbf{u}_t = R(\xi_T, \eta) \mathbf{v}(\xi, \eta) = K(\eta)R(\xi_T, \eta) \Theta(\xi)$$

(3.7)
It is noted that the configuration in the deformed state $B$ can be seen to be attained in two stages, the first stage by the displacement $u_a$, the second by the addition of the crack tip solution $u_t$. In this way we identify an intermediate state $B'$ described by

$$x' = X + u_a = y(\xi, 0) + u_a(\xi, \eta)$$  \hspace{1cm} (3.8)

The base vectors that belong to the undeformed state are defined by

$$A_i = \frac{\partial}{\partial \xi^i} y(\xi, 0)$$  \hspace{1cm} (3.9)

In the intermediate state $B'$ the base vectors are given by:

$$b_i = \frac{\partial}{\partial \xi^i} y'(\xi, \eta) = A_i + \frac{\partial}{\partial \xi^i} (u_a(\xi, \eta)) = A_i + u_{a,i}$$  \hspace{1cm} (3.10)

Note that a comma followed by an index (i): ( ,i) denotes partial differentiation with respect to $\xi_i$.

In our formulation we use the Green-Lagrange strains, which we can write in the form:

$$\gamma_{ij} = \frac{1}{2} \left[ (A_i)^T u_j + (u_{i,j})^T A_j + (u_{i,j})^T u_{j,i} \right]$$  \hspace{1cm} (3.11)

Substitution of the composite field $u_a + u_t$ gives:

$$\gamma_{ij}(u_a + u_t) = \frac{1}{2} \left[ (A_i)^T u_{a,j} + (u_{a,i})^T A_j + (u_{a,i})^T u_{b_{nj}} + (A_i + u_{a,i})^T R v_{ij} + (R v_{i})^T (A_j + u_{a,j}) + (R v_{i})^T (R v_{j}) \right]$$

$$\gamma_{ij}(u_a + u_t) = \gamma_{ij}(u_a) + \frac{1}{2} \left[ (A_i + u_{a,i})^T R v_{ij} + (R v_{i})^T (A_j + u_{a,j}) + (v_{i,j})^T (v_{i,j}) \right]$$

$$\gamma_{ij}(u_a + u_t) = \gamma_{ij}(u_a) + \frac{1}{2} \left[ (b_i)^T R v_{j} + (R v_{i})^T b_j + (v_{i,j})^T (v_{i,j}) \right]$$  \hspace{1cm} (3.12)

We can write this result in the form:

$$\gamma_{ij}(u_a + u_t) = \gamma_{ij}(u_a) + \gamma^*_{ij}(u_a, R v)$$

$$\gamma^*_{ij}(u_a, R v) = \frac{1}{2} \left[ (b_i)^T R v_{j} + (R v_{i})^T b_j + (v_{i,j})^T (v_{i,j}) \right]$$  \hspace{1cm} (3.13)

where the second part of $\gamma$ describes the increase of the deformation in going from $B'$ to $B$.

It follows from the foregoing that $\gamma^*_{ij}$ is only locally defined, i.e. in $\Omega_0$ of the crack tip while everywhere outside $\Omega_0$, $\gamma^*_{ij} = 0$. Thus as long as the initial curvature and the deformation $\gamma_{ij}(u_a)$ remain small in $\Omega_0$, and this is a reasonable assumption, the changes of $b_i$ in this region will also be small. Therefore, we can replace the base vectors $b_i(\xi)$ belonging to state $B'$ in (3.13) by the base vectors $b_i = b_i(\xi_T)$ that are attached to a point determined by $\xi_T$ in the region $\Omega_0$ (for which the node at the crack tip presents itself as the natural candidate).
Let $\bm{v}$ be specified by:

$$
\bm{v} = v_k(\xi_0) \mathbf{E}_k = v_k(\xi_0) A_k(\xi_T) \tag{3.14}
$$

where we assume that the base vectors at $\xi_T$ belonging to the undeformed state $\mathbf{E}_k = A_k(\xi_T)$ are unit vectors and mutually orthogonal. Substitution in (3.13) yields:

$$
\gamma^*_{ij}(\mathbf{u}_a, \mathbf{Rv}) = \frac{1}{2} \left[ (\beta_i)^T \mathbf{R} \mathbf{E}_k v_{kji} + v_{kii} (\mathbf{R} \mathbf{E}_k)^T \beta_j + v_{kji} v_{kij} \right] \tag{3.15}
$$

The base vectors $\beta_i$ are obtained from the base vectors $\mathbf{E}_i$ by a rotation, which we identify with $\mathbf{R}$, and a small deformation caused by the field $\mathbf{u}_a$. It is now natural and consistent with our previous assumptions to neglect the deformation of the triad $\mathbf{E}_i$ and replace $\beta_i$ by rotated triad $\mathbf{R} \mathbf{E}_i$. The additional strains $\gamma^*_{ij}$ that arise can then be simplified to the expressions:

$$
\gamma^*_{ij}(\mathbf{u}_a, \mathbf{u}_t) = \gamma^*_{ij}(\mathbf{u}_a, \mathbf{Rv}) \approx \gamma_{ij}(\mathbf{v}) = \frac{1}{2} \left[ v_{kji} + v_{kii} + v_{kji} v_{kij} \right] \tag{3.16}
$$

Thus, the addition of the corotated classical crack tip solutions $\mathbf{Rv}$ to the global representation $\mathbf{u}_a$ produces a strain increment $\gamma^*$ that can be computed by a direct evaluation of the Green Lagrange functional expressions $\gamma(\mathbf{u})$ in terms of the local (not rotated) representation $\mathbf{v}$ of this crack tip displacement field given by (3.14). It is seen that the two fields $\mathbf{u}_t$ and $\mathbf{u}_a$ are not coupled in the strain measures, but only in the specific elastic energy $\mathcal{E} = \frac{1}{2} \gamma \mathbf{C} \gamma = \frac{1}{2} \gamma_{ij} C_{ijkl} \gamma_{kh}$ at least to an acceptable degree of approximation.

Finally, there is still one minor difficulty that needs clarification. This concerns the nonlinear terms that are present in the expression (3.11). The classical solution for the stress singularity is obtained from the linearized equations of elasticity. This is different from our formulation where nonlinear Green-Lagrange strains are used to account for the expected geometric nonlinear effect. Consequently, if the stress intensity factors are based on a formulation that includes these nonlinear strain measures, the results will not agree with values obtained by the linearized theory. This effect is already present in the case of a centrally cracked flat plate in tension. In that case, the (classical) theory based on linearized strains predicts stress intensity factors that are proportional to the intensity of the load, while, if the nonlinear terms in the strains are retained, this proportionality no longer exists. Although the difference is small for small load intensities, it will become more and more pronounced when the load intensity increases. Consequently, if we want to keep the definition of the stress intensity factors in line with what is used in the linear theory, it is necessary to linearize the strain measures in the terms of the equations that determine the stress intensity factors which are produced by the enriched elements.

The conclusion of the previous considerations can thus be summarized as follows:

a) The composition of the displacement fields $\mathbf{u}_a$ and $\mathbf{u}_t$ cannot be carried out without taking into account the global rotation of the crack tip (see equation (3.7)).
b) On the other hand, as far as the strains are concerned, the composition of the two fields corresponds to a simple addition of the strains induced by the two fields separately, see equation (3.13) and (3.16).

c) If one wants to be consistent with the classical definition of the stress intensity factors, it is necessary to linearize the global strain measures in the crack tip elements at the moment $K$ is calculated.

3.3 The solution of the governing equations

The resulting nonlinear equations that arise from the foregoing definitions can be written as:

$$ F(d, k, \lambda) = 0 $$

(3.17)

where the dimension of the nonlinear vector function $F$, the deformation vector $d$ and the stress intensity factors $k$ are given by: $F \in \mathbb{R}^{N+6}$; $d \in \mathbb{R}^N$; $k = (K_i, K_{II}, K_{B1}, K_{B2}, K_{III})^T \in \mathbb{R}^5$. The parameter $\lambda$ represents the intensity of the loading by the definition $B = \lambda B_0$ ($B$ = the loading, $B_0$ is the base load). Notice that we introduced a slight change in the notation for the set of stress intensity factors: $k$ denotes now the set of 5 stress intensity factors which differs from the definition (2.9) given in section 2.2.

We can decompose this set of equations in the form:

$$ f(d, \lambda) + f^*(d, k, \lambda) = 0 $$

(3.18a)

$$ h^*(k, \lambda) = 0 $$

(3.18b)

where $f, f^* \in \mathbb{R}^N$; $h^* \in \mathbb{R}^5$

where the first term in the first equation corresponds to the contribution of the finite element discretization scheme. The three other terms in this expression arise when we apply the injection scheme described in the previous chapters. Notice that from a formal point of view, we could see the equations as being derived from the requirement that the total potential energy of the structure with its loads must be stationary. The contributions $f$, $f^*$, $h$ and $h^*$ are then computed from:

$$ f = \frac{\partial P}{\partial d}; \quad f^* = \frac{\partial \tilde{P}}{\partial d}; \quad h^* = \frac{\partial \tilde{P}^*}{\partial k} $$

(3.19a)

where the potential energy is given in the two parts:

$$ P_{\text{total}} = P(d, \lambda) + P^*(d, k, \lambda) $$

(3.19b)

The first term in (3.19b) refers to the global part of the formulation, the second part $P^*$ is due to the mode enrichment process.

The nonlinear equations (3.17) are solved for a range of values of the loading depending on the particular requirements of the analysis of the problem at hand. This solution process is carried out step by step by variation of the load parameter $\lambda$, or more generally, by variation of a generalized path parameter $\eta$. It is characteristic for these types of procedures that each new step along the
loading path is carried out by a predictor-corrector-process. For descriptions of such procedures refer to (Refs. 36, 37).

To cope with the extended set of equations, two approaches can be followed. The first (i) is to let the solution algorithm be autonomous. This is another way of saying that the extended set of equations is formed outside the solution algorithm so that the equations are integrally solved. In contrast, in the second approach (ii), the solution of the extended system is carried out in two steps. The first step (a) deals with the solution of the primitive system. When that is successful, the solution of the extended system is carried out in the second step (b) in terms of an additional and closing set of operations. The closing set of operations can be executed outside or inside the existing solution algorithm.

The difficulty with the first approach (i) is that the bandwidth of the system's stiffness will be compromised if the extra freedoms are not mixed into the array of primitive degrees of freedom. Consequently a mixing scheme has to be devised, and this we did not consider worthy of a pursuit. We thus preferred the second approach (ii). It is a fairly straightforward method and relatively simple to install.

The solution of the corrector equations in each new step of the step by step solution algorithm furnishes sequentially a set of corrections $\Delta \sigma^i : i = 1, 2, 3,...$ to the predicted solution $\sigma^0$. This system of equations corresponds to a linearization of (3.17). What particular form of linearization is used is immaterial at this point.

The governing equations on which the operations must be carried out are first written as:

$$\mathbf{H}(\mathbf{x}, \eta) = 0$$

(3.20)

$$\mathbf{x} = \begin{pmatrix} \mathbf{d} \\ \mathbf{k} \end{pmatrix} \in \mathbb{R}^{N+4+1} \quad \mathbf{H} = \begin{pmatrix} \mathbf{F}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}, \eta) \end{pmatrix} \in \mathbb{R}^{N+4+1}$$

The extra equation that is introduced here is a device to introduce the path parameter $\eta$ that measures the distance between the solution points along the solution path. The successive solutions obtained along the solution path defined by (3.20) are denoted by: $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, ..., \mathbf{x}_{j-1}$. They are determined by the parameter values $\eta_1, \eta_2, \eta_3, ..., \eta_{j-1}$. For the computation of the next point $\mathbf{x}_j$ for $\eta = \eta_j$, one starts with the prediction $\sigma^0_j$ derived from the information contained in the sequence of previous points $\mathbf{x}_{j-1}, \mathbf{x}_{j-2}, ...$. The predictor is used as the starting value in the corrector process (we use here Newton’s method as an example):

$$\mathbf{H}_x(\sigma^{h-1}_j, \eta_k) \Delta \sigma^h_j = -\mathbf{H}(\sigma^{h-1}_j; \eta_k)$$

(3.21a)

$$\sigma^h_j = \sigma^{h-1}_j + \Delta \sigma^h_j \quad h = 1, 2, 3, ...$$

(3.21b)

The configurations $\sigma^h_j \ j - 1, 2, 3, 4, ...$ signify successive approximations to the actual solution $\mathbf{x}$ while $\Delta \sigma^h_j$ stands for successive corrections to these approximations. The system matrix of equations (3.21) is given by:
\[ H_{X} = \begin{pmatrix} F_{d} & f_{k}^{*} & f_{\lambda} \\ h_{d}^{*} & h_{k}^{*} & 0 \\ g_{d} & 0 & g_{\lambda} \end{pmatrix} \]

(3.22a)

\[ F_{d} = f_{d} + f_{d}^{*} \]

(3.22b)

(The notation \( f_{d} \) means here that \( f_{d} = \{ \frac{\partial f_{i}}{\partial d_{j}} \} \) (i, j = 1, 2, 3, ..., N) is the whole set of partial derivatives of the vector \( f \) with respect to the set of freedoms \( d \), \( f_{k}^{*} = \{ \frac{\partial f_{i}}{\partial \alpha \lambda} \} \); i = 1, 2, 3, ..., N; \( \alpha = 1, 2, 3, 4 \). etc. etc.) Thus the dimensions are: \( F_{d} \in \mathbb{R}^{N \times N} \); \( f_{d}^{*} \in \mathbb{R}^{N \times 5} \); \( h_{d}^{*} \in \mathbb{R}^{5 \times N} \); \( h_{k}^{*} \in \mathbb{R}^{5 \times 5} \).

If there is no enrichment applied, the system \( F_{d} \) reduces to \( f_{d} \) which is the stiffness matrix of the "primitive system" i.e. the equation set that belongs to the regular finite element discretization scheme. The difference between \( F_{d} \) and \( f_{d} \) is thus \( f_{d}^{*} \) which is a matrix with very few non-zero elements, i.e. only those that concern the freedoms in the domain \( \Omega_{0} \) around the crack tip. To keep the changes to the existing solution algorithm as small as possible, we can neglect this contribution to the systems matrix. We claim that this simplification will not appreciably alter the convergence characteristics of the iterative process embodied by (3.21 - 3.22) (a supposition that is confirmed by the test results described in Chapter 4). Notice that, in addition to the regular parts of the equations, the enrichment process requires the factoring of the residual components \( f_{d}^{*} \) and \( h_{d}^{*} \) and the matrices \( f_{k}^{*} \), \( h_{d}^{*} \) and \( h_{k}^{*} \); these factors must be produced by a separate module in the code.

A practical way to solve the extended set of equations (3.21) is to use block Gaussian elimination with the stiffness matrix \( f_{d} \) as pivot. The advantage of this method is not only in the transparency of the necessary code changes. It also allows the enrichment to be added to the solution process only in the last stages of the iterations, which is especially useful in cases where we let the crack grow through the structure, and where the enrichment process must be relocated to the new crack tip of the newly formed extension of the crack. For the details of the block elimination operation we refer to (Ref.35).

4. Test Results and Conclusion
4.1 Finite Element System B2000.

The method described in the previous sections was implemented in the B2000 system that is operational in the department of Structures and Materials of the Faculty of Aerospace Engineering of the Technical University of Delft. The B2000 system is a database oriented software package (Refs. 38 - 40) that is designed for the development and application of numerical methods in solid mechanics. The plate and shell finite elements of this package that were used for the development of the mode enrichment method are based on the elements described in (Refs. 26, 28). They consist of four, eight and nine noded isoparametric shell finite elements that are suitable for the description of small deformations under large displacements and large rotations.
4.2 The Choice of the Domain $\Omega_0$

The enrichment process is carried out in a small region around the crack tip, but the question how this should be carried out so that the best possible results are obtained has not yet been addressed.

The variational procedure employed to formulate the governing equations requires the composite field (2.1) to be at least $C_0$ continuous at the boundary $\partial \Omega_0$ of the domain $\Omega_0$ where the superposition of these functions is carried out. It is proper, therefore, although not strictly necessary, to introduce an artificial device to enforce this continuity condition. To accomplish this we used a weight function as multiplier for the modes (2.10 - 2.12). This (scalar) function $= X(r, \theta)$ has the property that at the crack tip $X = X(0, 0) = 1$ while at the boundary $\partial \Omega_0$, $X = 0$. In between the variation of $X$ in the domain $\Omega_0$ is linear. The most convenient way to fix the area $\Omega_0$ is to let it coincide with patch of elements of underlying finite element mesh of the global field $u_a$. The weight function is then of the form portrayed in figure 4.

How far $\Omega_0$ should be extended over the shell surface around the crack? Our numerical experiments showed that it is not necessary to extend the enrichment zone $\Omega_0$ over a distance further than a couple of times the wall thickness. This means that in all practical cases, not more than two layers of elements are needed to be used as enrichment zone. Note that this includes the transition zone described above. It also turned out from our experiments that the accuracy of the membrane intensity factors $K_\text{II}$ and $K_{\text{III}}$ is not strongly influenced by the size of the domain $\Omega_0$ and thus the size of the elements immediately bordering the crack tip; an effect that was also observed by Heyliger and Kriz (Ref.20). On the other hand, with the elements used here, comparable accuracy of the anti-plane terms ($\mathcal{O}(2\%)$ relative to other solutions), can only be obtained if the domain $\Omega_0$ is shrunk to a dimension that corresponds to elements with a width of the order of magnitude of the 1/2 of the thickness of the plate or shell. Is there a convincing explanation for this behavior?

The necessity to take such a fine rosette of elements for the calculation of the anti-plane intensity factors must be attributed to the circumstance that the elements we use here try to obtain the solutions according to the Reissner plate theory. The solution of the anti-plane stress distribution around a crack in the Reissner formulation differs from that obtained in the Kirchhoff theory by terms that are only present in a small boundary layer around the crack edges (Ref. 29), because it is precisely this boundary solution that the Reissner solution tries to correct. Outside this layer, which is of the order of the thickness of the shell, the two solutions are virtually identical. Consequently, to extract the anti-plane stress intensity factors according to the Reissner theory, it is necessary to confine the extraction region $\Omega_0$ to this boundary layer zone as was argued by Hui and Zhender (Ref. 29) and as is confirmed by our numerical experiments.

The evaluation of the integrals involving the elements in the region $\Omega_0$ is carried out with a numerical differentiation scheme that differs from that used for the elements outside this region. The elements that are not enriched are integrated by a low order $2\times2$ or a $3\times3$ scheme. But this is not sufficient for the enriched elements. To obtain acceptable accuracy in the calculation of the intensity factors the order of integration should be increased (see also (Refs. 18 - 20)). A $10\times10$ Gaussian scheme turns out to be satisfactory for all the applications that we considered so that this was adopted as the standard scheme.
4.3 Calibration Tests on Classical Problems

The enrichment method was first tested on some classical problems of which we mention:
1.) The centrally cracked plate with symmetrical loading to compare the mode enrichment solution with classical analytical results, 2.) the plate with a slanted crack to test the mixed mode capability of the method, and also 3.) the problem of a centrally cracked plate loaded by edge moments and edge shear forces. In all of these experiments the agreement with existing analytical solutions or solutions computed by other methods was excellent.

To put the corotational formulation of the enrichment process to the test, we loaded the centrally cracked plate up to a certain load level and then, under constant load, turned the plate by an arbitrary finite angle. The calculated intensity factors in this experiment were indifferent to the rotations, thus confirming the correctness of the chosen approach. More extensive details about these numerical experiments can be found in (Ref. 35).

4.4 Geometrical nonlinear shell problem

To demonstrate the capabilities of the mode enrichment technique we focus here on one example that represents the problems that are encountered in fuselage shells. It is a cracked cylindrical shell segment previously analyzed in (Refs. 41, 42 and 10). The model consists of a cylindrical shell under internal pressure with an $R/h$ ratio that compares to what is used for an actual fuselage shell. The shell has a large number of identical straight cracks aligned along the axis of the cylinder. The cracks are distributed over the shell in an even and periodic fashion (see figure 6). In that case, the deformation of the shell is also periodic and that means that the crack behavior can be studied by taking a small segment of the shell with boundary conditions that match the periodicity conditions. The particular dimensions, boundary conditions and loading of the shell segment are also indicated in the figure 6.

There is one extra provision built in this model that brings it closer to the actual fuselage under pressure. To mimic the influence of the bulkheads that are present in a real fuselage, the cylinder is periodically restrained in the sense that at the stations $x = k \times L$ ($k = 1,2,3,4,5$...) the shell is forced to keep its cylindrical shape. In the small model (Figure 6) this is done by restraining the boundary 3 to have uniform normal displacements.

The state of stress at the crack tip of a crack in a cylinder under pressure is different from that of a similar crack in an infinitely large flat plate of the same thickness under the same far field loading conditions. The difference concerns the intensity as well as the composition of the stress field. In the case of the plate there is only mode $I_m$ membrane stress at the crack tip when the loading is symmetrical with respect to the crack. In the case of the cylinder there are also bending stresses (mode $I_b$).

Due to the curvature of the shell, the intensity of the mode $I_m$ stress mode is higher than that of the plate. This is expressed by the so called bulge factor:

$$b_{\text{bulge}} = \frac{K_{\text{curved}}}{K_{\text{flat}}}$$

(4.1)
internal pressure $p = \lambda \cdot 0.057 \text{ N/mm}^2$

boundary conditions

1. $\phi_z = 0$
2. $u = 0$
3. $\phi_y = 0$
4. $w = \text{uniform}$
5. $u = 0$
6. $\phi_x = 0$
7. $\phi_z = 0$
8. $\phi_x = 0$
9. $\phi_y = 0$
10. $\phi_z = 0$

material properties

- Young's modulus $E = 66250.0 \text{ N/mm}^2$
- Poisson's ratio $\nu = 0.3$

Figure 6 Pressurized cylinder with cracks
The stress intensity $K_{I_{\text{flat}}}$ is a linear function of the intensity of the applied load $\lambda$, i.e. $K_{I_{\text{flat}}} = \lambda K_0$. But in the case of the cylinder, this linear relationship has disappeared and $K_{I_{\text{curved}}} = F(\lambda)$ where $F$ is a nonlinear function of the load $\lambda$.

The mesh that was employed is given in figure 7. It is created by a mesh generator that will be described elsewhere. Figure 8 and 9 summarize the results that were obtained for a 200 mm crack, which for this model corresponds to a value of the aspect ratio $\alpha = 4$. The bulge factor as function of the load is presented in figure 8 and compared with known solutions from the literature. It is seen that the bulge factor is a monotonically decreasing function of the load intensity $\lambda$. Results from analytical formula's and finite element calculations (Refs. 42 - 45) for this case agree well with the results obtained with the mode enrichment technique.

A result that is new for this problem is the behavior of the mode I_b crack intensity $k_1$ pictured in figure 9. In this case there were no comparable results available. The values obtained by the mode enrichment method are therefore checked by the node release/crack closure method (Ref. 12) that we also implemented but do not discuss in this paper. As can be seen, the agreement is exemplary.
Figure 8 Bulge factor results

Figure 9 Results Mode I bending intensity factor $k_1$
4.5 Conclusion

The mode enrichment technique in the form that was first described by Benzley (Ref. 17) for linear problems can also be used for geometric nonlinear problems as it was shown in this paper. The implementation of this method is more involved than the node release/crack closure method (Ref. 12), but this is the only disadvantage. The attractive points of the method are:

-(i) It produces accurate predictions for the stress intensity factors in mixed mode cracking,

-(ii) it can be used in build up structures without restrictions.

The method is also very useful for the calculation of the so called T-stress that plays a role in many crack growth predictors proposed in the literature. However, this topic is outside the scope of this paper. It will be addressed at another place.

References


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Appendix
A1 An interpretation of the “effective” tearing mode in classical plate theory

Consider an elastic plate of arbitrary shape (Figure A1) which contains an edge crack of
length $s_0$. The plate is properly supported and loaded by a set of forces $\mathbf{F}$ subject to the restriction
that the crack opens up when the load intensity is increased. Notice that the load intensity can be
defined as $\lambda: \mathbf{F} = \lambda \mathbf{F}_0$ where $\mathbf{F}_0$ is a unit or nominal force system that determines the particular mode
of loading. For more details on the concepts introduced here refer to (Refs. 21, 46, 47) or the references therein.

The deformation of the crack at a certain value of the load can be seen as the end point of a
deformation path that is represented by the function $\mathbf{d} = \mathbf{G}(\lambda)$ which represents the response of the
structure to a gradual increase of the intensity of the load. The deformation path $\mathbf{d} = \mathbf{G}(\lambda)$ is thus
considered to be a smooth function of the load factor $\lambda$. Because the structure is elastic and because we assume conservative loading conditions, the crack opening at $\lambda = \lambda_e$ can also be attained by an
alternative loading process. In this alternative loading program, the plate is loaded in two stages. In
the first stage (l), the crack is first kept closed during the time the loading $\mathbf{F}$ is applied. For this operation two appropriate sets of opposing edge forces and moments $\{-\mathbf{S}^-, -\mathbf{S}^+\}$ along the two
 crack faces are necessary to keep the crack closed. The first part of the deformation process of the
alternative loading program is thus equivalent to the loading of the plate with two force systems: $\mathbf{F}$ and $\{-\mathbf{S}^-, -\mathbf{S}^+\}$. It leads to an different deformation path $\mathbf{d}^* = \mathbf{G}^*(\lambda_e)$ and a different state of defor-

![Figure A1 Plate with edge crack](image-url)
In the second stage (II) of the alternative loading program, we release the edge loads that were necessary to keep the crack closed. The deformation of the crack can then be studied by focusing on the loading process by which the state $G^*(\lambda_c)$ is transformed to the state $G(\lambda_c)$ by application of the load $\{S', S^+\}$. We will now look at the action of this additional load system.

As mentioned in chapter 2, along an imaginary cut of the plate we can define three internal distributed forces and two distributed moments (line loads and line moments) (Refs. 46, 47). We refer to Figure 3 for the conventions. To keep the crack closed during the application of the external forces $F$, we need two force and moment distributions $T^- = \{-T_t(s), -T_n(s), -Q(s), -M_n(s), -M_t(s)\}$ and $T^+ = \{-T_t(s), -T_n(s), -Q(s), -M_n(s), -M_t(s)\}^+$ along each crack face. The distributions are statically opposing, i.e., $T_1(s) = -T_1^+(2s_0 - s)$, etc., where we measure $s$ from $(-)$ to $(+)$ along the crack edge in a counter clockwise fashion.

To open the crack from state $G^*(\lambda_c)$ we need to apply the distributions $T^{*+} = \{T_t(s), T_n(s), Q(s) \text{ and } M_n(s), M_t(s)\}$ and $T^{*+} = \{T_t(s), T_n(s), Q(s) \text{ and } M_n(s), M_t(s)\}^+$. The rate of work that is done on the crack edge during the application of this additional force system is then given by:

$$A^* = \int \{T_n \dot{u}_n + T_t \dot{u}_t + M_n \dot{\beta}_n + M_t \dot{\beta}_t + Q \dot{w} \} ds \quad (A1.1)$$

where the integral is taken along the crack edge $\Gamma$, $\dot{u}_n$, $\dot{u}_t$ are the velocities measured in the plane of the plate along $e_n$ and $e_t$, $\dot{w}$ is the velocity along $n$ perpendicular to this plane and $\dot{\beta}_n$, $\dot{\beta}_t$ are the angular velocities of the edge measured along directions $e_n$ and $e_t$. In the Reissner theory, the five kinematic components $\{\dot{u}_n, \dot{u}_t, \dot{w}, \dot{\beta}_n, \dot{\beta}_t\}$ are independent of each other. But in the Kirchhoff theory, $\dot{w}$ and $\dot{\beta}_n$ are coupled and thus not independent. To show this and explain its consequences, it is sufficient to restrict the discussion to linear plate theory. In that case, the angular velocities are defined by the rotation of the normal to the mid surface of the plate. They are given by:

$$\dot{\beta}_t = \dot{w}_n = \frac{\partial \dot{w}}{\partial n}; \quad \dot{\beta}_n = \dot{w}_s = \frac{\partial \dot{w}}{\partial s} \quad (A1.2)$$

where the differentiation's are carried out with respect to the directions $n$ and $t$ respectively.

According to this definition the velocity distribution $\dot{w}(s)$ completely determines the derivative $\frac{\partial \dot{w}}{\partial s} = \dot{\beta}_n$ so that it follows that $\dot{\beta}_n$ and $\dot{w}(s)$ are coupled. Notice that this does not count for $\dot{\beta}_t$ and $\dot{w}(s)$.

We now introduce these definitions (A1.2) in (A1.1):

$$A^* = \int \{T_n \dot{u}_n + T_t \dot{u}_t + M_n \dot{w}_s + M_t \dot{w}_n + Q \dot{w} \} ds \quad (A1.3)$$

and invoke an transformation of the term $M_n \dot{w}_s$ under the integral sign. With $M_n \dot{w}_s = [M_n \dot{w}]_s - M_{n,s} \dot{w}$ and integration by parts of this term $[M_n \dot{w}]_s$ we get:

$$A^* = \int \{T_n \dot{u}_n + T_t \dot{u}_t + M_n \dot{w}_n + [Q - M_{n,s} \dot{w}] \} ds + [M_n(s) \dot{w}(s)]^+ \quad (A1.3)$$

which can be written as:
\[ A^* = \int^+ \{ T_n \dot{u}_n + T_t \dot{u}_t + M_t \dot{w}_n + Q^s \dot{w} \} ds + N^+ \dot{w}(s^+) - N^- \dot{w}(s^-) \]  \hspace{1cm} (A1.4a)

\[ Q^s = [Q - M_{n,s}] ; N^+ = M_n(s^+) ; N^- = M_n(s^-) \]  \hspace{1cm} (A1.4b)

The rate of work done by the edge forces and moments is thus written in a form whereby the original system of forces and moments is replaced by a statically equivalent set of forces and moments that are now conjugate to four independent kinematic variables as opposed to the five constrained variables before the transformation. This set no longer contains the twisting moment \( M_t \), while the transverse shear force distribution \( Q(s) \) is replaced by the so-called effective shear force \( Q^s = Q - M_{n,s} \). Two edge point loads appear in this equivalent set: \( N^+ = M_n(s^+) \) and \( N^- = M_n(s^-) \). These point loads act on the beginning \( s^- \) and the end point \( s^+ \) of the contour \( \Gamma \) that is defined by the intersection of the fracture surface and the plate mid surface. Notice that when the crack is inside the domain of the plate, the contour \( \Gamma \) is closed so that in that case the point loads \( N^+ \) and \( N^- \) cancel each other.

![Diagram with force and moment labels](image)

Figure A2 The effective forces for the tearing mode

The load distribution of the crack as given by \( Q^s = Q - M_{n,s} \), \( N^+ \) and \( N^- \) clearly gives rise to a tearing mode of deformation as is illustrated in figure (). With an analogous transformation of expression (A1.1), it can also be shown that the loading system for the tearing mode as pictured in Figure A2 can be replaced by a statically equivalent system that consists of distributed twisting moments along the edges. This follows from the assumption that we can replace \( Q \) in (A1.1) by:

\[ Q = M_{n,s}^\wedge \]  \hspace{1cm} (A1.5a)

which follows from the introduction of a twisting moment:
\[ M_{n}(s) = \int_{s}^{t} Q(\sigma) d\sigma + C (= \text{constant}) \]  \hspace{1cm} (A1.5^b)

This results in:

\[ A^* = \int \left\{ T_{n} \dot{u}_{n} + T_{t} \dot{u}_{t} + [M_{n} - M_{n}^*] \dot{w}_{n} + M_{n} \dot{w}_{n} \right\} ds + [M_{n}(s) \dot{w}(s)]^+ \]  \hspace{1cm} (A1.6)

or with the introduction of the effective twisting moment:

\[ M_{n}^* (s) = M_{n}(s) - M_{n}^* (s) \]  \hspace{1cm} (A1.7)

\[ A^* = \int \left\{ T_{n} \dot{u}_{n} + T_{t} \dot{u}_{t} + M_{n}^* \dot{w}_{n} + M_{n} \dot{w}_{n} \right\} ds + [M_{n}(s) \dot{w}(s)]^+ \]  \hspace{1cm} (A1.8)

Notice that we can call the crack intensity \( k_2 \) that belongs to this tearing mode the effective tearing stress intensity.

A2 Determination of the relation between \( k_1, k_2 \) and \( K_{B1}, K_{B2}, K_{III} \)

The relations between \( k_1, k_2 \) and \( K_{B1}, K_{B2}, K_{III} \) can be determined with help of the crack closure integral:

\[ G_e = \frac{1}{2} \int_{-h/2}^{h/2} \frac{\partial}{\partial a} \int_{-h/2}^{h/2} \frac{1}{2} \sigma_y(x) \cdot \nu(\delta a - x) dz dx \]  \hspace{1cm} (A2.1)

\( G_e \) = the energy release rate
\( \nu \) = the crack opening displacement
\( \delta a \) = the increment of crack extension
\( h \) = plate thickness
\( \sigma_y \) = the normal stress defined on the plane parallel to the fracture surface

The crack opening displacement for mode B1 bending (in the framework of the kinematic assumptions for the shear deformable finite element) is:

\[ v = \frac{K_{B1}}{\mu} \frac{2z}{h} \frac{r_1^{1/2} \sin \frac{\theta}{2} (\frac{2}{1+\nu} - \cos^2 \frac{\theta}{2})}{\left( \frac{2}{1+\nu} \right)} \]  \hspace{1cm} (A2.2^a)

where

\[ \mu = \frac{E}{2(1+\nu)} \]

\[ \nu(\theta = 180^\circ) = \frac{K_{B1}}{\mu} \frac{2z}{h} \frac{r_1^{1/2}}{(2z)} \frac{2}{1+\nu} \]  \hspace{1cm} (A2.2^b)

For \( \sigma_y \) we can derive:

\[ \sigma_y = K_{B1} \frac{2z}{h} \frac{1}{2z} \frac{1/2 \cos \frac{\theta}{2} (1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2})}{(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2})} \]  \hspace{1cm} (A2.3^a)
\[ \sigma_y(\theta = 0) = K_{BI} \frac{2z}{h} \left( \frac{1}{2\pi} \right)^{1/2} \]  

(A2.3b)

Substitution of (A2.2b) and (A2.3b) into (A2.1) and carrying out the integration's yields:

\[ G_e(K_{BI}) = \frac{1}{3} K_{BI}^2 \frac{\pi}{E} \]  

(A2.4)

We can also do this for the mode B2 bending term and the transverse shear term III. This yields:

\[ G_e(K_{BI}) = \frac{1}{3} K_{B2}^2 \frac{\pi}{E} \]  

(A2.5)

\[ G_e(K_{III}) = K_{III}^2 \frac{1 + \nu}{E} \]  

(A2.6)

This result can be compared with similar results for the Kirchhoff factors \( G_e(k_1) \) and \( G_e(k_2) \) that were derived by Hui and Zehnder (Ref. 29).

\[ G_e(k_1) = \frac{1 + \nu}{3(3 + \nu)} k_1^2 \frac{\pi}{E} \]  

(A2.7)

\[ G_e(k_2) = \frac{1 + \nu}{3(3 + \nu)} k_2^2 \frac{\pi}{E} \]  

The energy release rates in both cases are related by:

\[ G_e(k_1) = G_e(K_{BI}) \]  

(A2.8)

\[ G_e(k_2) = G_e(K_{B2}) + G_e(K_{III}) \]

and this allows us to derive relations (3.1b) given in section 3.1:

\[ k_1^2 = \frac{3 + \nu}{1 + \nu} K_{BI}^2 \]  

(3.1b)

\[ k_2^2 = \frac{3 + \nu}{1 + \nu} \left[ K_{B2}^2 + K_{III}^2 \frac{3(1 + \nu)}{\pi} \right] \]

For a more detailed derivation of these relations see (Ref. 35).