Memorandum M-363

STATE TRANSITION MATRIX COMPUTATIONS FOR
KALMAN FILTER ORBIT DETERMINATION FROM LASER
RANGE OBSERVATIONS

A. Kamp
J.J.P. van Hulzen

(paper presented at the 30th IAF congress,
Munich, September 1979)

Delft - The Netherlands
February 1980
ABSTRACT

At the Department of Aerospace Engineering of Delft University of Technology the Kalman filter technique for satellite orbit determination has been investigated. For this technique the transition matrix needs to be computed for each integration step of the covariance matrix. A numerical method for computing the transition matrix has been used successfully, but at the cost of much computing time. Therefore an investigation was started into the applicability of analytical expressions for the transition matrix. These have been developed and successfully implemented in the Kalman filter program, giving considerable savings in computing time.

Keywords - orbit determination
- Kalman filter
- transition matrix
- partial derivatives
- perturbations
- laser measurements
INTRODUCTION

Satellite orbit computation has been a major research topic at the Dept. of Aerospace Engineering of Delft University of Technology since 1970. The recent studies concentrate on highly accurate orbit determinations of geodetic satellites on the basis of the laser range observations obtained at the Satellite Observatory at Kootwijk (The Netherlands). One of the goals of these studies is to investigate the possibilities to supply this station with operational ephemerides, derived from their own measurements.

In general, several methods can be used to obtain estimated orbits from observations of positions and/or velocities of satellites. The method most commonly used is the least-squares method. R.E. Kalman developed another method which is also a minimum variance estimator. This Kalman filter processes the observations sequentially, in contrast with the least-squares method which processes all observations simultaneously (batch). The sequential character of the Kalman filter opens possibilities for real-time use of the method, for instance in control systems.

Both methods, the batch as well as the sequential one, make it possible to determine the orbital parameters that describe the satellite motion. Besides it is possible to estimate parameters that describe the gravitational model (the zonal and tesseral harmonic coefficients), or the atmospheric drag (the ballistic parameter, the atmospheric density). In geodetic purposes the methods are used to compute geocentric coordinates of the observatories.

In both methods the so-called state transition matrix plays a vital role. Some methods of computing the state transition matrix are investigated and tested for use in the Kalman filter with simulated observations.

KALMAN FILTER PRINCIPLE

In least-squares estimation techniques all observation data are gathered before processing starts. Then an orbit is fitted through these observations which is optimal in the least-squares sense. In general an iterative procedure, for instance differential correction, is used to find this optimal orbit.
In contrast, the Kalman filter is a sequential estimation method. As soon as an observation is available it is used to update a predicted state. The updated state estimate is then available and might be used for instance to direct subsequent measurements in real-time systems. The next observation will be used to update the latest state estimate. Thus a recursive algorithm is used in which the observations one after another update the state estimate. To start the process an initial state estimate must be used.

The information of the processed measurements is thus stored in the state estimate. For optimal estimation the subsequent measurements should be weighed in the estimation process in accordance with the number and the accuracy of the processed measurements. This kind of information gives an indication on the accuracy of the latest state estimate. Therefore, it can be represented (assuming a linear system with Gaussian distributions) by a covariance matrix of the estimated state.

The Kalman filter process can be divided into two steps:
- an observation update step in which a predicted state and its covariance matrix, known at the time of observation, are updated with the observation using a minimum variance estimation method. For this step the accuracy of the observation, represented in a covariance matrix of an observation vector, is also needed. The results of this step are an updated state estimate with its corresponding covariance matrix.
- a time update step in which the state estimate and its covariance matrix are integrated to give a predicted state and covariance matrix at the time of the next observation. This step will also be necessary if a state prediction is not known at the exact time of the first observation.

The assumption of linear systems with Gaussian distributions deserves some special attention. The assumption of Gaussian distributions is one commonly used to simplify computations. However, linear equations are fairly uncommon in the description of satellite dynamics. Under the influence of non-linear equations a Gaussian distribution will not stay Gaussian. Higher order quasi-moments will be introduced in the distribution. These quasi-moments can not be represented in a covariance matrix. So, the use of covariance matrices is restricted to Gaussian distributions in linear systems. In fact Kalman developed his filter theory for linear systems.

If the Kalman filter technique is to be used in non-linear systems in favour of its sequential character and the simple expressions involved, the equations having an effect on the covariance matrix have to be
linearized. Thus in the observation update step linearized relations between the state vector and the vector of observations are needed. In the time update step linearized equations of motion are needed for the integration of the covariance matrix. For the state integration the non-linear equations of motion can be used. In this paper our attention will focus on the computation of the transition matrix, a linearized description of the motion of a satellite. Using this matrix the covariance matrix can be integrated by simple matrix-multiplications.

THE STATE TRANSITION MATRIX

A state transition matrix describes the transition of the state vector at epoch to the state vector at time t.

\[ x_t = \phi x_0 \]  

(1)

where \( x \) denotes the state vector and \( \phi \) the state transition matrix that exists of thirty-six partial derivatives of the state vector at time t with respect to the state vector at epoch. This equation is exact for linear systems and can be used as an approximation for non-linear systems. Using this equation the covariance matrix can be integrated with equation (2) to:

\[ P_t = \phi P_0 \phi^T \]  

(2)

where \( P \) denotes the covariance matrix and the superscript \( T \) the transpose of a matrix.

For the computation of the state transition matrix several methods are available. A method commonly used in aircraft dynamics uses the linearized differential equations:

\[ \dot{x}_t = M x_t \]  

(3)

Combination of the two equations yields the variational equations of the elements of the transition matrix

\[ \dot{\phi} = M \phi \]  

(4)

The solution of this equation, under the assumption of a constant matrix \( M \) is the Taylor series expansion
\[ \vartheta = I + \eta \Delta t + \frac{H^2 (\Delta \kappa)}{2!} + \frac{H^3 (\Delta \xi)}{3!} + \ldots \] 

(5)

When used for highly non-linear equations of motion, such as in satellite dynamics, this solution is accurate for very short time intervals only. For time intervals of more than a few seconds the computation of \( \vartheta \) should be done in several steps. Step sizes of one second were found to be adequate, which makes this method useful for short observation time intervals. For the integration over longer time intervals, in the order of a quarter of an hour, this method becomes time consuming and therefore inefficient.

For time intervals of this order of magnitude the direct integration of the variational equations, simultaneously with the state vector seems to be appropriate. A higher-order integration method is required but the method is less time consuming than the numerous evaluations of the polynomial expansion (5). The gain in computing time is expected to be a factor three to four. For increasing time intervals the rise in computing time of this method makes a still less time consuming method desirable. Such a solution can be found in an analytical method, that needs a fraction of the computing time required by the numerical integration, but which may be less accurate.

The analytical method is based on the use of analytical expressions for the elements of the state transition matrix. Unfortunately, it is impossible to find simple expressions for the analytic derivatives that include a sophisticated perturbation theory in the description of the orbital motion. An approximation of the real transition matrix can be found when its elements are calculated for the unperturbed motion. However, its application is limited in time: for time intervals of a day and more, perturbations affect the matrix considerably and should be included in the analytical expressions for the elements of the state transition matrix. Because of the considerable savings of computing time for time intervals of half an hour or more, the analytical method will be focussed on.

The calculation of the analytical transition matrix

The analytical expressions for the elements of the state transition matrix are obtained by the application of the chain-rule to the relations that exist between the state vector and the classical orbital elements at any instant, and the relations between the orbital elements at time \( t \) and
those at epoch. As the state vector is determined by the six osculating orbital elements $\mathbf{r}^{\text{osc}}$, the matrix is calculated as follows:

$$\frac{\partial \mathbf{r}_t}{\partial \mathbf{r}_0} = \frac{\partial \mathbf{r}_t^{\text{osc}}}{\partial \mathbf{r}_0^{\text{osc}}} \cdot \frac{\partial \mathbf{r}_0^{\text{osc}}}{\partial \mathbf{r}_0}$$  \hspace{1cm} (6)

Herein the matrices $\partial \mathbf{r}_t^{\text{osc}}/\partial \mathbf{r}_0^{\text{osc}}$ and $\partial \mathbf{r}_0^{\text{osc}}/\partial \mathbf{r}_0$ can easily be derived from the geometrical expressions that describe the Keplerian orbit relative to the geocentric inertial reference system. In the latter matrix singularities may appear in some of its elements because of vanishing denominators for circular or equatorial orbits. In practice, the theoretical singularities do not lead to serious difficulties. Due to the long word length of our computer (the IBM 370/158) the use of classical orbital elements is "restricted" to orbits for which the excentricity $> 10^{-10}$ and the inclination deviates more than $10^{-10}$ from 0 and $\pi$ radians.

The matrix $\partial \mathbf{r}_t^{\text{osc}}/\partial \mathbf{r}_0^{\text{osc}}$ describes the variation of the osculating orbital elements with time. For convenience this matrix is often approximated by the matrix of the unperturbed motion, i.e. the identity matrix, because all orbital elements are constant for such an orbit. In the next Section it will be shown that the quality of the unperturbed state transition matrix may become poor for time intervals of a day and more. For more accurate values of the transition matrix, perturbations must then be included.

In that case it means that the time dependence of the osculating orbital elements will have to be considered. Initially it will be sufficient to include the dominating perturbations in the transition matrix. These perturbations are the first-order secular perturbations due to the oblateness of the earth and for orbits with perigee altitudes less than 300 km, also due to atmospheric drag.

Analytical expressions for the partial derivatives of the secularly perturbed motion can easily be derived from, for example, the first-order Kozai theory, and a simple atmospheric drag theory. These theories describe the secular perturbations as a function of the initial secular orbital elements $\mathbf{r}_0^{\text{sec}}$. As an illustrative example the secular perturbation of the argument of perigee $\omega$ is given by equation (7)

$$\omega_t = \omega_0 + \frac{3}{2} J_2 \frac{R^2}{\mathbf{r}_0^{\text{sec}}} n_o (2 - \frac{5}{2} \sin^2 \lambda) (t - t_o)$$  \hspace{1cm} (7)
where $J_2$ is the first-order zonal harmonic coefficient, $R_e$ is the radius of the earth, $n$ is the mean angular motion, $P$ is the semilatus rectum, and $i$ the inclination.

In this way the matrix $\frac{\partial \mathbf{P}_t^{osc}}{\partial \mathbf{P}_o^{osc}}$ in equation (6) is approximated by the matrix of derivatives of the secular elements $\frac{\partial \mathbf{P}_t^{sec}}{\partial \mathbf{P}_o^{sec}}$. The effects of short- and long-periodic perturbations, present in the matrix $\frac{\partial \mathbf{P}_t^{osc}}{\partial \mathbf{P}_o^{osc}}$, are neglected because of the complicated expressions involved and the minor effects on the state transition matrix.

Numerical evaluation

From the curves of the derivatives as a function of time, conclusions can be drawn with reference to the derivatives:

- For geodetic satellites, that orbit at an altitude of more than 700 km, the elements of the state transition matrix are negligibly affected by airdrag, even if a time interval of a day or more is considered (Fig. 1)

- The oblateness of the earth has a noticeable influence on the transition matrix. Inclusion of first-order secular perturbations is not compulsory for time intervals of less than a few orbital revolutions (Fig. 2), but it is for a time interval of a day or more (Fig. 1)

- Fig. 1 shows that the effects of the oblateness on the transition matrix can be split up into a small change of amplitude and a phase difference. The latter is caused partly by the secular perturbation of the mean anomaly. This perturbation causes the partial derivatives to be evaluated at different points in the perturbed and unperturbed orbit respectively, although both evaluations are at the same time. Only a part of this phase difference can be nullified by the use of the same mean motion in the expressions for both the perturbed and the unperturbed derivatives. By the reduction of the phase difference the unperturbed state transition matrix is a reasonable approximation for the perturbed matrix over longer time intervals (Fig. 3).
RESULTS OF THE KALMAN FILTER SIMULATIONS

The performances of the Kalman filter have been tested with simulated observations. For these simulations a couple of passes of GEOS-3 over Kootwijk were used for which passes also real observations are available which will be used in future tests with the Kalman filter. The orbit of the satellite during this period is accurately computed taking into account perturbations due to the first five zonal harmonics ($J_2$ to $J_6$) and the first sectorial harmonic ($J_{2,2}$) in the earth gravitational potential. From this computed orbit simulated Kootwijk observations are generated and contaminated with noise. The simulated observations are laser measurements with range standard deviations of 1 meter and azimuth and elevation standard deviations of one milliradian. For the chosen orbit the angular standard deviations correspond to position deviations of about 1 km.

The simulated observations and their noise statistics are used as input for the Kalman filter. The resulting estimates can be compared with the computed orbit which forms the basis of the simulated observations. From the differences conclusions can be drawn with reference to the performances of the Kalman filter.

The results of these calculations are given in a number of Figures in which the deviations of the estimates from the simulated orbit are given together with their corresponding standard deviations as computed and used in the Kalman filter. These data are given after each observation update step for two successive passes. The Figures give the cross-track, radial and along-track position and velocity components of the deviations.

The results obtained using the polynomial expansion up to and including the third degree for the transition matrix are shown in Fig. 4. It was found that an integration stepsize of one second for the transition matrix is necessary to obtain these results. The total computing time for filtering the two successive passes using the polynomial expansion was over five minutes. The inclusion of perturbations in the computation of the transition matrix showed negligible effects over these passes.

The name "filter" for the Kalman filter is illustrated by Fig. 4. During the first pass the position estimate is not yet very accurate. The observations, contaminated with noise give large corrections to this state estimate. As the state estimate becomes more accurate after several observations have been filtered, the high frequencies of the observation noise are filtered out of the state estimates. As laser range
measurements are used, primarily giving information on the satellite's position, this effect is especially obvious in the position components of the state estimate.

The first implementation of the analytical method used the initial state to compute the matrix $\partial P_o / \partial x_o$. The matrix $\partial x_t / \partial P_o$ was set equal to the identity matrix, thus assuming a two-body motion. The matrix $\partial x_t / \partial P_t$ was computed using the state at time $t$ available from the Kalman filter. This state was integrated with a perturbation model that includes the $J_2$ to $J_6$ and $J_{2,2}$ effects. The results of the simulation given in Fig. 5 indicate that the estimates diverge from the real state. This divergence is caused by an incorrect computation of the transition matrix. This results in computed standard deviations, which are too small, thus incorrectly indicating an accurate state estimate. Subsequent observations will then give only small corrections to this estimate, which are not enough to correct the real deviations from the state.

The divergence of Fig. 5 is probably caused by the effect that Rice (1967) detected in an analytical research on the computation of transition matrices for a gravitational model with $J_2$-perturbations only. Rice showed that secular terms may disappear from the transition matrix, resulting in increasing errors, if perturbation models are mixed. In our case, the matrix $\partial x_t / \partial x_o$ is computed using an advanced perturbation model, while for the matrix $\partial P_t / \partial P_o$ a two-body motion is assumed.

When short-periodic perturbations are removed from the orbital elements, used to compute the matrices $\partial P_o / \partial x_o$ and $\partial x_t / \partial P_t$, the divergence disappears. So, by removing the main perturbations, which for these short time intervals are the short-periodic perturbations, the divergence is suppressed.

The remedy Rice suggested, i.e. to prevent mixed use of perturbation models in the transition matrix also gives good results. In this case the state vector is integrated twice: once with a complete perturbation model and once with the same perturbation model as used for the matrix $\partial P_t / \partial P_o$. This second state integration is used only to compute the matrix $\partial x_t / \partial P_t$, while the first one is used in the actual estimation process. The results of such a simulation are given in Fig. 6 showing that a correct implementation of the analytical formulae gives good results.

Comparison of the numerical and analytical results shows slight differences. From these differences no conclusions can be drawn with reference to
of the two methods would be better from the accuracy point of view. Concerning the computing time, the CPU-time for the Kalman filter using analytical transition matrix computation, is a factor ten smaller than for the filter using the Taylor series expansion.

CONCLUSIONS

An analytical method for computing the state transition matrix was developed and tested successfully in a Kalman filter. Mixed use of perturbation models in the computation of the transition matrix was found to cause divergence of the Kalman filter, but can be avoided easily. The analytical method implies a considerable saving of computing time. The computing time and the accuracy of the position estimates generated by the Kalman filter opens possibilities for real-time use of the filter technique in order to direct laser measurements.

Until now, the Kalman filter has worked in simulated conditions only. When real observations will be processed the problem of modelling errors will occur, which also can cause divergence of the Kalman filter. Attacking this problem will be the next step in the development of the Kalman filter technique at our Department.

LITERATURE

D.H. Rice  
Initial orbital elements

\[ a = 6800 \text{ km} \]
\[ e = 0.001 \]
\[ i = 97^\circ \]
\[ \omega = 0^\circ \]
\[ \Omega = 0^\circ \]
\[ M_0 = 0^\circ \]

---

**Fig. 1** Derivative \( \frac{\partial z}{\partial y_0} \) after a time interval of one day.
Initial orbital elements

\[ a = 6800 \text{ km} \]
\[ e = 0.001 \]
\[ i = 40^\circ \]
\[ \omega = 30^\circ \]
\[ \Omega = 70^\circ \]
\[ M_0 = 50^\circ \]

**Fig. 2.** Derivative \( \frac{\partial y}{\partial x} \), after a time interval of one orbital revolution.
Initial orbital elements

\[ a = 6800 \text{ km} \]
\[ e = 0.001 \]
\[ i = 40^\circ \]
\[ \omega = 30^\circ \]
\[ \Omega = 70^\circ \]
\[ M_0 = 50^\circ \]

\( \bar{n} \) = mean motion

perturbed by \( J_2 \)

\[ \frac{\partial z}{\partial x_0} \]

unperturbed \( \bar{n} \)

perturbed \( \bar{n} \)

unperturbed \( n_0 \)

Time (min.)

Fig. 3 Derivative \( \frac{\partial z}{\partial x_0} \) after a time interval of a week.
Fig. 4 Results of the Kalman filter using transition matrix computation by means of polynomial expansions.
Fig. 4 Continued
Fig. 4 Continued
Fig. 5 Results of the Kalman filter using analytical transition matrix computation with mixed perturbation models.
Fig. 5 Continued
Fig. 5 Continued
Fig. 6 Results of the Kalman filter using analytical transition matrix computation avoiding mixed perturbation models.
Fig. 6 Continued