Dissipation in the Abelian sandpile model

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Dissipation in the Abelian Sandpile Model

Conditions for criticality

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Preface

This thesis is part of my bachelor's programme for the study of Applied Mathematics at the TU Delft.

Even before I began my studies at university, I was interested in probability theory. This has continued during my studies and a final research project in this area of mathematics was certainly my preference. This subject is very interesting to me, since this model is defined by two important things. One is the addition of particles in the system, which is random. The other are the local relaxation rules. The randomness introduced by the addition of particles unleashes a world of uncertainty that can only be captured by rigorous theory.

During the project, my meetings with Dr. F. Redig were very informative and he helped me tackle any issue I ran into while trying to make calculations or formulate proofs. This is why I owe him the biggest thanks. I would also like to thank the other members of the commission: Dr.ir. R. van der Toorn and Drs. E.M. van Elderen, for their interest and time to evaluate my research. I also thank Li Yong Pan for helping me with the layout of the thesis. Finally, I would like to thank my dearest family members and friends for their support during the project.

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Abstract

The Abelian sandpile model was first introduced by Bak, Tang and Wiesenfeld in 1987. Since then, a lot of researchers have studied this model and similar models, all related by the concept of *self-organized criticality*. In this thesis, we study a variant on the classical model where dissipative and anti-dissipative vertices are incorporated in the model. These have an influence on the critical behaviour of the model. We first introduce a definition of criticality in this model and investigate which levels of dissipation are required to guarantee non-critical behaviour. In studying this variant, we encounter random walks and Green's functions.

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Introduction

The sandpile model was first introduced just over 30 years ago, by Per Bak, Chao Tang and Kurt Wiesenfeld in [1]. It was the first model that showed signs of so-called self-organized criticality (SOC). It is a non-equillibrium system, where, in its critical state, local changes can impact the system as a whole. One of the requirements for SOC is the appearance of power-law behaviour without the help of fine-tuning any control parameters. Hence the term *self-organized*. The original authors believed that this model could be used to explain many physical phenomena in our world, because SOC would be something ingrained in nature. If we look at the occurrence and magnitude of earthquakes for example, we can find that they show power-law behaviour.

Because the model is so simple, yet shows such a strange behaviour, it is one of the most studied models of its kind, by both theorists and experimentalists. After the first introduction to the model, many people started examining it. Important papers on the subject are among the most cited papers in the scientific literature.

The model has some simple rules. The paper by Redig [8] contains an alternative interpretation of the model in one dimension. Imagine N easily stressed students sitting next to each other at a rectangular table. Student 1 is sitting at the left end of the table, and student N is sitting at the right end. Then, imagine a professor handing out assignments. He will pick one student at random and give him or her the assignment. The students are fine with assignments as long as they have less assignments than they have neighbours, in this case 2; one on the right and one on the left. Once a student receives a second assignment, they become too stressed, we say they are *unstable*, and distribute their assignments among their neighbours. This distributing is known as *toppling*. For students that are sitting at the end of the table, we can imagine them giving one assignment to their neighbour and throwing the other assignment in the trashcan. Toppling can make a neighbouring student become unstable, whereafter they will topple as well. The toppling of unstable students continues until no student is unstable anymore. The system as a whole is now called *stable*. The entire sequence of topplings as a result of one assignment by the professor is known as an *avalanche*. After stabilizing another assignment is randomly given to a student, and the process continues. In one dimension, the distribution converges to the uniform distribution on configurations where at most one student has no assignment.

We can generalize the model for d dimensions, where every student has $2d$ neighbours, and hence can have at most $2d-1$ assignments before becoming unstable. In these higher-dimensional systems, the model does not converge to a minimally stable configuration. Rather, it will evolve to a critical state, where avalanches of sizes up to the entire system itself can occur.

The students were a nice analogy, but from now on we will refer to them as *sandpiles*, and the assignments as *grains of sand*.

The report is structured as follows. In section 2 we will introduce the original model as proposed by Bak, Tang and Wiesenfeld. Thereafter, in section 3, we will look at a variation on the model, where two other types of sandpiles are incorporated into the system: *sinks*, dissipative vertices where mass is lost upon toppling, and *sources*, anti-dissipative vertices where mass is created upon toppling. Adding these different types of vertices can have a large influence on the behaviour of the model. For example, when too many sites are dissipative, the model may lose criticality, meaning the avalanche sizes do not grow indefinitely. On the other hand, with too many source are anti-dissipative, the model may stay in an unstable state forever. Our goal is to mathematically prove some bound on the level of dissipation for which the system no longer shows critical behaviour. Then we will see if we can prove criticality for systems with both sources and sinks incorporated. Finally, we will focus our attention to the one-dimensional model with only dissipative sites for a more narrow bound.

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The Classical Abelian Sandpile Model

In this section we will introduce the sandpile model as originally introduced by Bak, Tang and Wiesenfeld in [1]. The notation we use is based on Redig (2005) [8] section 3. For the model in d dimensions we consider a simply connected set $V \subseteq \mathbb{Z}^d$. That will be our 'desert'. In this thesis we will only consider a specific form for V, defined by

$$V = [-n, n]^d \cap \mathbb{Z}^d$$

for some $n \in \mathbb{N}$. To make it clear, in this thesis we always say $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. We define the *Toppling matrix* Δ as minus the lattice Laplacian:

$$\begin{aligned}
&\Delta_{x,x} = 2d,\\
&\Delta_{x,y} = -1, \text{ for } x \sim y,\\
&\Delta_{x,y} = 0, \text{ otherwise.}
\end{aligned}$$

We say $x \sim y$ if $|x - y| = 1$. A height configuration η is a map $\eta : V \to \mathbb{N}$, and we denote by $\mathscr{H}$ the set of all height configurations. A height configuration $\eta \in \mathscr{H}$ is called *stable* if for every $x \in V$, $\eta(x) < \Delta_{x,x}$. We denote the set of all stable height configurations by $\Omega = \{\eta \in \mathscr{H} : \eta(x) < \Delta_{x,x} \text{ for all } x \in V\}$. A site $x \in V$ for which $\eta(x) \geq \Delta_{x,x}$ is called an *unstable* site. This site will topple as defined by

$$T_x(\eta)(y) = \eta(y) - \Delta_{x,y} \tag{2.1}$$

This means that the site x will lose $2d$ grains of sand upon toppling and its $2d$ neighbours in V will each gain one grain of sand upon toppling. When x is a boundary site it will have less than $2d$ neighbours. In this case, its neighbours will still gain one grain of sand, after which the excess sand will leave the system, as if falling over the edge of our desert V. The toppling of a site $x \in V$ is called *legal* if the site x is unstable, otherwise it is called *illegal*. We also see that for $x, y \in V$ both unstable sites,

$$T_x T_y(\eta) = \eta - \Delta_{x,\cdot} - \Delta_{y,\cdot} = T_y T_x(\eta) \tag{2.2}$$

This is called the *elementary abelian property* of the Abelian Sandpile Model. It follows directly from the commutativity of addition in $\mathbb{Z}^d$. This property tells us that any finite sequence consisting of the same legal topplings will produce the same end result, independent of the order of toppling. With this information, we can look at the stabilization of a general height configuration $\eta \in \mathscr{H}$ as an operator $\mathscr{S} : \mathscr{H} \to \Omega$, defined by

$$\mathscr{S}(\eta) = T_{x_1} \ldots T_{x_n}(\eta), \tag{2.3}$$

by the requirement that $\mathscr{S}(\eta)$ is stable, and for each $i \in \{1,\dots,n\}$, the toppling at site x_i is legal.
For $\eta \in \mathscr{H}$ and a sequence of legal topplings $T_{x_1},\dots T_{x_n}$ we define the toppling numbers of that particular sequence to be

$$n_x = \sum_{i=1}^{n} I(x_i = x), \tag{2.4}$$

where I is the indicator function. Thus, we are counting how many times the toppling occurs at each site $x \in V$. Since the result of a sequence of legal topplings depends only on the toppling numbers of the sites and not on the order in which they topple, an interesting lemma from the paper by Redig [8] says the following:

Lemma 2.1. *Let $\eta \in \mathscr{H}$ and $T_{x_1}\dots T_{x_n}$ be a sequence of legal topplings such that the resulting configuration is stable. Then the toppling numbers n_x are maximal. This means that for every sequence of legal topplings $T_{y_1}\dots T_{y_m}$ the toppling numbers m_x satisfy $m_x \le n_x$ for all $x \in V$.*

With this Lemma, we can prove that our stabilization operator $\mathscr{S}$ is well-defined. Assume we have $T_{x_1}\dots T_{x_n}$ and $T_{y_1}\dots T_{y_m}$ both sequences of legal topplings such that the resulting configuration is stable. Then both their toppling numbers are maximal, meaning $n_x \le m_x$ and $m_x \le n_x$ for all $x \in V$. This can only be true if $n_x = m_x$ for all $x \in V$. Thus, they both topple exactly the same sites exactly as often, and therefore the resulting stable configuration is the same. Therefore, the resulting configuration $\mathscr{S}$ is independent of the order of toppling. □

Having defined the toppling numbers in this way, we can now also write the result after the topplings as

$$T_{x_1}\dots T_{x_n}(\eta) = \eta - \Delta n, \tag{2.5}$$

where n is the column vector with elements n_x. Because of the well-definedness of $\mathscr{S}$, if we have a height configuration η and $T_{x_1}\dots T_{x_n}$ and $T_{y_1}\dots T_{y_m}$ two sequences of legal topplings such that the resulting configuration is stable, then the resulting configurations are identical. Furthermore, for all $x \in V$, the toppling numbers are the same for both sequences. It is also important that in the Classical ASM, the toppling numbers from a stabilization are always finite. That is, for every $\eta \in \mathscr{H}$, there exists a finite sequence of sites $x_1,\dots,x_N \in V$ such that

$$\mathscr{S}(\eta) = T_{x_1}\dots T_{x_N}(\eta) = \eta - \Delta n \tag{2.6}$$

is stable. If we define δ_x as the operation of adding one grain of sand at site $x \in V$, we can define the addition operator $a_x : \Omega \to \Omega$ by

$$a_x\eta = \mathscr{S}(\eta + \delta_x) \tag{2.7}$$

The well-definedness of $\mathscr{S}$ also implies that a_x is well-defined and that abelianness holds:

$$a_x a_y \eta = a_y a_x \eta = \mathscr{S}(\eta + \delta_x + \delta_y) \tag{2.8}$$

for every $\eta \in \mathscr{H}$ and every $x, y \in V$. The abelianness of the addition operator is why we call it the *Abelian* Sandpile Model. The classical ASM consists of a seqeuence of stable height configurations. At each time $n \in \mathbb{N}$ a grain of sand is added at a site $x \in V$ and the configuration is stabilized. This process continues infinitely, creating a sequence of stable height configurations. Mathematically, we let $p = p(x)$ be a probability distribution on V, and demand that there is a strictly positive probability to add at each site $x \in V$. This means $p(x) > 0$ for all $x \in V$ and $\sum_{x\in V} p(x) = 1$. Starting from an initial height configuration $\eta_0 \in \Omega$, the configuration η_n at time n is given by the random variable

$$\eta_n = \prod_{i=1}^{n} a_{X_i}\eta_0 \tag{2.9}$$

where $X_i, \dots, X_n$ are i.i.d with distribution p. Equation (2.9) is a Markov process with state space Ω and the Markov transition operator defined on functions $f : \Omega \to \mathbb{R}$ is given by

$$Pf(\eta) = \mathbb{E}(f(\eta_1 | \eta_0 = \eta)) = \sum_{x \in V} p(x) f(a_x \eta) \tag{2.10}$$

For all Markov processes, the states can be divided into two classes, recurrent and transient. A configuration $\eta \in \Omega$ is transient if there is a positive probability that we will never reach η again, and recurrent otherwise. We can prove that the set of transient states is non-empty. Consider for example the finite system

1	0	0

This configuration is transient, because upon adding sand anywhere in the system we will lose the two neighbouring zeroes. The only way to obtain a height of zero is by toppling, but then its neighbour will have a height of one. For any $d > 1$, there are also such examples, therefore the set of transient states is always non-empty.
Because we have a strictly positive probability of adding sand at each site $x \in V$, we know that the maximal configuration, defined by $\eta_{max} = \Delta_{x,x}$ for all $x \in V$, can be reached from every $\eta \in \Omega$. Therefore the set of recurrent configurations, which we will denote by $\mathscr{R}$, is also non-empty.
There is a rather simple algorithm for finding out if a configuration $\eta \in \Omega$ is recurrent or transient, called the *burning algorithm.*

Burning algorithm. *Let* $(V, \eta_V), \eta_V \in \Omega$ *be a desert and its corresponding height configuration. Then remove all sites* $x \in V$ *that satisfy*

$$\eta_x \geq \sum_{y \in V, y \neq x} \left(-\Delta_{x,y}\right) \tag{2.11}$$

Call the remaining sites and their corresponding height configuration (V_1, η_{V_1}). *Then apply the same operation to get* $(V_2, \eta_{V_2}), (V_3, \eta_{V_3}), \dots$ *until* $(V_{n+1}, \eta_{V_{n+1}}) = (V_n, \eta_{V_n})$ *for some* $n \in \mathbb{N}$. *Then* η_V *is recurrent if and only if* $(V_n, \eta_{V_n}) = \emptyset$.

This means η is recurrent if and only if all sites are eventually burnt away. Because we defined Δ as minus the lattice Laplacian we can interpret $\sum_{y \in V, y \neq x} \left(-\Delta_{x,y}\right)$ as the number of sites neighbouring x. For sites on the interior of V this will be equal to $2d$, but for boundary points it will be less than $2d$. An example of this algorithm in $d = 2$ is shown below.

1	1	3
3	2	0
2	2	1

$\longrightarrow$

1	1	
	2	0
	2	1

$\longrightarrow$

	1	
	2	0
		1

$\longrightarrow$

		0

$\longrightarrow$

In $d = 2$ the sites in the corners only have two neighbours. Thus, any corner with height two or three will immediately burn. Similarly, if a site is on the boundary but not a corner, it has three neighbours, and such a site with height three will burn in the first step of the algorithm. That is exactly what we see happen in the first step. After that, we see that the top left corner only has one neighbour left, and with a height of one it will be burned in the second step. This process continues until no more sites can be burned. Because for this configuration, the algorithm eventually burns all sites in V, we can say that this is a recurrent configuration. The burning of a transient configuration in $d = 2$ is shown below.

3	2	1
2	1	0
3	1	1

$\longrightarrow$

	2	1
2	1	0
	1	1

$\longrightarrow$

		1
	1	0
	1	1

$\longrightarrow$

	1	0
	1	1

$\longrightarrow$

	1	0
	1	1

The first steps are similar to the first example, but we see that no new sites are burnt in the last step.

This means that these sites will never be burned, and therefore this configuration is transient. There is another useful criterion that tells us if a configuration η is transient or recurrent.

Definition 2.1. *Let $\eta \in \mathcal{H}$. For $W \subseteq V$, $W \neq \emptyset$, we call the pair (W, η_W) a forbidden subconfiguration (FSC) if for all $x \in W$,*

$$\eta(x) < \sum_{y \in W \setminus x} (-\Delta_{x,y}) \tag{2.12}$$

If for $\eta \in \Omega$ there exists a FSC (W, η_W), then we say that η contains a FSC. A configuration $\eta \in \Omega$ is called allowed if it does not contain forbidden subconfigurations. The set of all stable allowed configurations is denoted by $\mathcal{A}$.

The sites that are unburnt after the burning algorithm finishes form a FSC. We can see that from the second example above.

1	0
1	1

This is the subset of sites that weren't burnt after Dhar's burning algorithm. Indeed, since each site has two neighbours, but all sites have a height of less than two, no sites are burnable. Another example of a forbidden subconfiguration is

0	1
1	2
0	1

This connection between recurrence and allowedness motivates the following theorem.

Theorem 2.1. *A stable configuration $\eta \in \Omega$ is recurrent if and only if it is allowed.*

We will give a sketch of the proof. We know that the maximal configuration, where $\eta_x = 2d-1$ for all $x \in V_n$, is completely burnable. We also know that we cannot create a FSC by adding grains to an allowed configuration. The maximal configuration is recurrent, since it can be reached from every other state, and hence can we reach any other recurrent configuration from the maximal configuration by adding a finite number of grains. Therefore, all the recurrent configurations are allowed. This proves $\mathcal{R} \in \mathcal{A}$. If the reader is interested in a proof for $\mathcal{A} \in \mathcal{R}$, we refer the reader to section 3.1 of Redig's paper [8].

Recall that we denote the set of recurrent configurations by $\mathcal{R}$. Theorem 2.1 says that $\mathcal{R} = \mathcal{A}$. Going back to our example for a transient configuration, we see that

0	0

is a FSC, and therefore each configuration containing this subconfiguration is not allowed. Then, by theorem 2.1, each configuration containing this subconfiguration is a transient configuration. Dhar has shown in [2] that we can find a bijection between rooted spanning trees and recurrent configurations, namely

$$|\mathcal{R}| = \det(\Delta) \tag{2.13}$$

We can also look for the stationary measure of our Markov chain, since restricted to $\mathcal{R}$, our Markov chain is irreducible, we know there exists a stationary measure.

Theorem 2.2. *The Markov process η_n with transition operator*

$$Pf(\eta) = \sum_{x\in V} p(x) f(a_x\eta)$$

has the unique stationary measure

$$\mu(n) = \frac{1}{|\mathscr{R}|}\sum_{\eta\in\mathscr{R}} \delta_\eta$$

This is the uniform distribution on $\mathscr{R}$.

Proof.

$$\begin{aligned}
\sum_{\eta\in\mathscr{R}} Pf(\eta)\mu(\eta) &= \frac{1}{|\mathscr{R}|}\sum_{\eta\in\mathscr{R}}\sum_{x\in V} p(x) f(a_x\eta) \\
&= \frac{1}{|\mathscr{R}|}\sum_{x\in V} p(x)\sum_{\eta\in\mathscr{R}} f(a_x\eta) \\
&= \frac{1}{|\mathscr{R}|}\sum_{x\in V} p(x)\sum_{\eta\in\mathscr{R}} f(\bar\eta), \text{since } a_x \text{ is a bijection} \\
&= \left(\sum_{x\in V} p(x)\right)\frac{1}{|\mathscr{R}|}\sum_{\eta\in\mathscr{R}} f(\bar\eta) \\
&= \frac{1}{|\mathscr{R}|}\sum_{\eta\in\mathscr{R}} f(\bar\eta), \text{since p is a probability distribution on V} \\
&= \sum_{\bar\eta\in\mathscr{R}} f(\bar\eta)\mu(\bar\eta)
\end{aligned}$$

□

Now, rather than defining toppling numbers of sequences of topplings, we will define them as the number of topplings at y as a result of addition at x.

Toppling numbers. *Let $\eta\in\Omega$. Then, for all $x, y \in V_n$. define*

$$n(x, y, \eta) = \sum_{i=1}^{N} I(x_i = y) \tag{2.14}$$

We can now introduce the following formula that is very important in the study of this model.

Dhar's Formula. *For the classical ASM on V_n we have*

$$\mathbb{E}_\mu\left[n(x, y, \eta)\right] = \left(\Delta_{x,y}\right)^{-1} \equiv G(x, y) \tag{2.15}$$

We define $G(x, y)$ here and call it the Green's function.

PROOF. *We know from equation (2.7) that*

$$a_x\eta = \mathscr{S}(\eta+\delta_x) \tag{2.16}$$

To stabilize $\eta+\delta_x$, we need to know how many times each site has toppled. After addition at x, z will have toppled $n(x, z, \eta)$ times. Each time z topples, y loses $\Delta_{y,z}$ grains. This means we can write the height at y after addition on x as

$$a_x\eta(y) = \eta(y) + \delta_{x,y} - \sum_{z\in V_n} \Delta_{y,z} n(x, z, \eta) \tag{2.17}$$

Note that the sum on the right is just the matrix multiplication of Δ *and* n. *Integrating the whole equation with respect to the invariant measure* μ, *we get*

$$\int a_x\eta(y)d\mu = \int \eta(y)d\mu + \int \delta_{x,y}d\mu - \int \Delta n(x,y,\eta)d\mu \tag{2.18}$$

Now we use the property that μ *is invariant on* $\mathscr{R}$, *and that* $\mathscr{R}$ *is closed under the operator* a_x. *This means that* $\int a_x\eta(y)d\mu = \int \eta(y)d\mu$. *Using this and the linearity of the integral we get*

$$\Delta \int n(x,y,\eta)d\mu = \delta_{x,y} \tag{2.19}$$

Since $\int n(x,y,\eta)d\mu = \mathbb{E}_\mu\left[n(x,y,\eta)\right]$, *multiplying both sides by* Δ^{-1} *gives, assuming* Δ^{-1} *exists,*

$$G(x,y) = \mathbb{E}_\mu\left[n(x,y,\eta)\right] = \left(\Delta^{-1}\right)_{x,y} \tag{2.20}$$

□

We can now introduce a definition of criticality for the Sandpile model. We will use the definition given by Redig [8].

Definition 2.2. *A system is called non-critical if the inverse toppling matrix* $\mathbb{E}_\mu\left[n(x,y,\eta)\right] = \left(\Delta_{x,y}\right)^{-1}$ *exists and for all* $x \in V_n$

$$\lim_{n\to\infty} \sum_{y\in V_n} \mathbb{E}_\mu\left[n(x,y,\eta)\right] < \infty \tag{2.21}$$

Otherwise it is critical.

This definition tells us that the model is critical if the expected avalanche size is infinite, as the system size itself grows to infinity, for all $x \in V_n$. If, however, the avalanche size is always finite, the model is non-critical.

2.1. Random walk interpretation of Dhar's formula

Note that $\mathbb{E}_\mu\left[n(x,y,\eta)\right]$ is the expected number of topplings at y upon adding sand at x, averaged over all $\eta \in \Omega$. We can make a connection between the topplings and a random walk $\{X_n\}_{n\geq 0}$ on V_n. To do this, we add a vertex $\{*\}$, called the *root*, to V_n. We can now define the transition probability matrix $P_{x,y}$ for $x,y \in V_n \cup \{*\}$ as

$$P_{x,y} = \begin{cases} \frac{1}{2d}, \text{for } x \sim y, x,y \in V_n \\ \frac{2d-\alpha_{V_n}(x)}{2d}, \text{for } x \in V_n, y = * \\ 1, \text{for } x,y = * \\ 0, \text{otherwise} \end{cases} \tag{2.22}$$

$\alpha_{V_n}(x)$ is the number of neighbours of x that are in V_n. This is equal to $2d$ for all interior vertices but strictly less than $2d$ for vertices on the boundary. We see that the random walk can only travel to the root when it is located on the boundary of V_n. In this way, the root is acting like a global sink for the system. We see that once X_n leaves the system, equivalent to hitting the root, it will stay there. Now define the hitting time of X_n reaching the root as

$$\tau = \inf\{k > 0 : X_k = *\} \tag{2.23}$$

For $x,y \in V_n$, we can then write the expected number of visits to y, starting at x, as

$$g(x,y) \equiv \mathbb{E}_x^{RW}\left[\sum_{i=0}^{\tau} I(X_i = y)\right] \tag{2.24}$$

This sum runs up to τ, since after hitting the root, X_n is trapped and will never reach y again. $I(X_i = y)$ is the indicator function, meaning

$$I(X_i = y) = \begin{cases} 1, \text{ if } X_i = y \\ 0, \text{ if } X_i \neq y \end{cases} \tag{2.25}$$

So in equation (2.24), we are adding one each time the random walk is at y. We now have the following theorem

Theorem 2.3. *The Green's function defined by equation (2.24), which we denote by $g(x, y)$ to prevent confusion with the Green's function $G(x.y)$ defined in equation (2.15), has the property that*

$$\frac{1}{2d} g(x, y) = \left(\Delta^{-1}\right)_{x,y} \tag{2.26}$$

PROOF. *First observe that for $x, y \in V_n$*

$$\Delta_{x,y} = 2d\,(I - P)_{x,y} \tag{2.27}$$

This follows from our definition of Δ and equation (2.22). We now condition equation (2.24) on the first step

$$\mathbb{E}_x^{RW}\left[\sum_{i=0}^{\tau} I(X_i = y)\right] = \sum_{z\in V_n} \mathbb{E}_x^{RW}\left[\sum_{i=1}^{\tau} I\left(X_i = y\right)|X_1 = z\right] \mathbb{P}\left(X_1 = z|X_0 = x\right) + I\left(X_0 = y\right) \tag{2.28}$$

Now see that $\mathbb{P}\left(X_1 = z|X_0 = x\right) = P_{x,z}$. And since our random walk starts at x, $I\left(X_0 = y\right) = I\left(x = y\right) = \delta_{x,y}$, this is shortened notation for the indicator function. Now we have

$$\mathbb{E}_x^{RW}\left[\sum_{i=0}^{\tau} I(X_i = y)\right] = \sum_{z\in V_n} \mathbb{E}_z^{RW}\left[\sum_{i=0}^{\tau} I\left(X_i = y\right)\right] P_{x,z} + \delta_{x,y} \tag{2.29}$$

Using the definition of $g(x, y)$ in equation (2.24), equation (2.29) becomes

$$g(x, y) = \sum_{z\in V_n} P_{x,z}\, g(z, y) + \delta_{x,y} \tag{2.30}$$

The sum on the right side of equation (2.30) is the matrix multiplication of P and G. Subtracting this from both sides and writing everything in matrix form gives

$$(I - P)g = I \tag{2.31}$$

We now make use of equation (2.27) to write this as

$$\frac{1}{2d}\Delta g = I \tag{2.32}$$

Multiplying both sides by Δ^{-1}

$$\frac{1}{2d} g = \Delta^{-1} \tag{2.33}$$

□

We see that, up to a multiplicative constant, the expected lifetime of the random walk that starts at x and is killed upon leaving V_n is equal to the expected number of topplings at y after addition of sand at x.

Corollary. *Since $g(x,y)$ is a scalar multiple of $\left(\Delta_{x,y}\right)^{-1}$, Dhar's formula tells us that an equivalent definition for non-criticality is*

$$\lim_{n\to\infty}\sum_{y\in V_n} g(x,y) < \infty \tag{2.34}$$

We now state and prove the following theorem.

Theorem 2.4. *The classical ASM, where Δ is defined to be the inverse Laplacian, is critical for all $d \geq 1$.*

PROOF. *Recall from equation (2.24) that*

$$g(x,y) = \mathbb{E}_x^{RW}\left[\sum_{k=o}^{\tau} I\left(X_k = y\right)\right] \tag{2.35}$$

Plugging this in equation (2.21) gives

$$\lim_{n\to\infty}\sum_{y\in V_n}\mathbb{E}_x^{RW}\left[\sum_{k=o}^{\tau} I\left(X_k = y\right)\right] \tag{2.36}$$

Now we should be careful, since the results we got for finite volume may not hold anymore if we go towards infinite volume, as we will see. In [7] by Novak, it was proven that for $d \leq 2$, the random walk on Z^d is recurrent. This means that the random walk will visit every point an infinite number of times with probability 1. In this case, $g(x,y)$ from equation (2.35) is already infinite. Hence, we know the classical ASM is critical for $d \leq 2$. For $d \geq 3$, the random walk on Z^d is transient, meaning the random walk will always have a finite number of visits to a point. To continue, using the linearity of the expectation, we can take the sum into the expectation and swap the sums, so we obtain

$$\lim_{n\to\infty}\mathbb{E}_x^{RW}\left[\sum_{k=o}^{\tau}\sum_{y\in V_n} I\left(X_k = y\right)\right] \tag{2.37}$$

Now we know that $X_k \in V_n$. Therefore, if we take the sum over all $y \in V_n$, one of these will be equal to X_k. This means that

$$\sum_{y\in V_n} I(X_k = y) = 1 \tag{2.38}$$

Therefore the sum always equals one, reducing the equation to

$$\lim_{n\to\infty}\mathbb{E}_x^{RW}\left[\sum_{k=o}^{\tau} 1\right] \tag{2.39}$$

Now we know that as n goes to infinity, our domain becomes infinitely large, and the probability of leaving this domain goes to zero. This means that

$$\lim_{n\to\infty}\tau = \infty \tag{2.40}$$

Using equation (2.40), we get

$$\lim_{n\to\infty}\mathbb{E}_x^{RW}\left[\sum_{k=o}^{\tau} 1\right] = \mathbb{E}_x^{RW}\left[\sum_{k=o}^{\infty} 1\right] = \infty \tag{2.41}$$

□

The connection between the random walk and the expected number of toppling numbers made proving this much easier. Another interesting fact about the Green's function was proven in Section 4.3 of the book by Lawler and Limic (2010) [6]. It turns out that, for $x, y \to \infty$,

$$g(x, y) \sim \frac{1}{|x-y|^{d-2}}$$

Here we see the occurrence of power-law behaviour, another indication of the existence of self-organized criticality. In the next section we will introduce a variation on the classical model, where criticality is not always guaranteed anymore.

3

Extended Sandpile Model

3.1. Introducing sources and sinks

We will introduce different sites to our model. Define $V_n = [-n, n] \cap \mathbb{Z}^d$ our finite lattice. We denote by $D_n \subset V_n$ the set of *dissipative* or *sink* sites, and let $S_n \subset V_n$ be the set of *anti-dissipative* or *source* sites. $R_n \subset V_n$ will be the set of regular sites that we have seen in the previous section. Then we can write $V_n = D_n \cup S_n \cup R_n$ as a union of disjoint sets. Because we are going to have V_n grow infinitely large when talking about criticality, define $D, S, R \subset \mathbb{Z}^d$ such that $D_n = V_n \cap D$, and analogously for S and R. We can then define our toppling matrix $\Delta_{x,y}^{D_n,S_n}$ as

$$\Delta_{x,y}^{D_n,S_n} = \begin{cases} -1, \text{for } x, y \in V_n, |x-y| = 1 \\ 2d, \text{for } x = y, x \in R_n \\ 2d+1, \text{for } x = y, x \in D_n \\ 2d-1, \text{for } x = y, x \in S_n \end{cases} \tag{3.1}$$

We can interpret these rules in the following way. Where the regular sites we introduced in the last section topple with requirement $\eta_x \geq 2d$, *dissipative* sites topple when $\eta_x \geq 2d+1$ and, upon toppling, they give 1 grain of sand to each of their neighbours, and the 1 grain left over is sent to an invisible sink. Conversely, *source* sites topple when $\eta_x \geq 2d-1$. They also give 1 grain of sand to each of their neighbours, and to do so they create 1 grain of sand from nothing.

One important thing we need to think about when we add different sites to our model is *stability*. In the Classical ASM we know that the stabilization always consists of a finite number of topplings. Everytime a sink site topples, 1 grain of sand is removed from the system. This means the total number of topplings will still be finite. If we incorporate source sites, they can be problematic, as stability can be lost. We will only concern ourselves with sink sites for now, and add source sites afterwards.

Without source sites, our original definitions for the stabilization operator $\mathscr{S}$ and addition operator a_x are still well-defined. The markov process is unchanged and we can describe the recurent configurations as

$$\mathscr{R}^{D_n} = \{\eta + \nu, \eta \in \mathscr{R}^{\emptyset}, \nu_x \in \{0,1\} \forall x \in D_n \cap \{x : \eta_x = 2d-1\}\} \tag{3.2}$$

This notation says we can create all recurrent configurations for the model with dissipative sites from the recurrent configurations for the classical ASM. For every site that is at maximal height for a regular site, $(2d-1)$, if it is a sink site we can add one more grains of sand on top, giving us a different configuration that can obviously be reached from other recurrent configurations.

3.2. Connection to random walk

Just like in Chapter 2, we can make the connection between the expected number of topplings and a random walk. This random walk is called upon leaving V_n, equivalent to hitting the root, $*$. Recall our transition probability matrix $P_{x,y}$

$$P_{x,y} = \begin{cases} \frac{1}{2d}, \text{for } x \sim y, x, y \in V_n \\ \frac{2d-\alpha_{V_n}(x)}{2d}, \text{for } x \in V_n, y = * \\ 1, \text{for } x, y = * \\ 0, \text{otherwise} \end{cases} \tag{3.3}$$

We have shown in chapter 2 that when $D_n = \emptyset$, the Green's function is given by

$$g(x,y) = \mathbb{E}_x^{RW}\left[\sum_{i=0}^{\tau} I(X_i = y)\right] \tag{3.4}$$

We will now let $D_n \neq \emptyset$, and find the Green's function according to the model.

Theorem 3.1. *The ASM extended with dissipation, where* Δ *is defined as equation* (3.1)*, is governed by the Green's function*

$$g(x,y) = \mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{l_k(D_n)} I(X_k = y)\right) \tag{3.5}$$

where $l_k(D_n)$ *is defined to be the number of visits by* X_k *to* D_n *at time* k*. In other words, the number of time the random walk is at a dissipative site.*

More formally, we say

$$l_k(D_n) = \sum_{i=0}^{k} I(X_i \in D_n) \tag{3.6}$$

PROOF. *We know from equation* (3.1) *that*

$$\Delta_{x,x} = (2d+1)I(x \in D_n) + (2d)I(x \notin D_n) \tag{3.7}$$

Remind also that if $z \sim x$ *then* $P(x,z) = 1/2d = -\Delta_{x,z}/2d$*. We claim that*

$$\Delta^{-1}_{x,y} = \frac{1}{2d} g(x,y) \tag{3.8}$$

Now we first split out the contribution from time $k = 0$ *in* $G(x,y)$

$$\begin{aligned}
g(x,y) &= \mathbb{E}_x^{RW}\left(\left(\frac{2d}{2d+1}\right)^{I(X_0 \in D_n)}\left(I(X_0 = y) + \sum_{k=1}^{\tau}\left(\frac{2d}{2d+1}\right)^{\sum_{i=1}^{k} I(X_i \in D_n)} I(X_k = y)\right)\right) \\
&= \left(\frac{2d}{2d+1}\right)\delta_{x,y} I(x \in D_n) + \delta_{x,y} I(x \notin D_n) \\
&+ \left(\frac{2d}{2d+1}\right) I(x \in D_n)\mathbb{E}_x^{RW}\left(\sum_{k=1}^{\tau}\left(\frac{2d}{2d+1}\right)^{\sum_{i=1}^{k} I(X_i \in D_n)} I(X_k = y)\right) \\
&+ I(x \notin D_n)\mathbb{E}_x^{RW}\left(\sum_{k=1}^{\tau}\left(\frac{2d}{2d+1}\right)^{\sum_{i=1}^{k} I(X_i \in D_n)} I(X_k = y)\right) \\
&= \left(\frac{2d}{2d+1}\right)\delta_{x,y} I(x \in D_n) + \delta_{x,y} I(x \notin D_n) \\
&+ \left(\frac{2d}{2d+1}\right) I(x \in D_n) \sum_{z \sim y} \frac{-\Delta_{x,z}}{2d} g(z,y) \\
&+ I(x \notin D_n) \sum_{z \sim y} \frac{-\Delta_{x,z}}{2d} g(z,y)
\end{aligned}$$

As a consequence,

$$\begin{aligned} & (2d+1)I(x\in D_n)g(x,y) \\ = \quad & 2d\delta_{x,y}I(x\in D_n)+I(x\in D_n)\sum_{z\sim x}-\Delta_{x,z}g(z,y) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} & (2d)I(x\notin D_n)g(x,y) \\ = \quad & 2d\delta_{x,y}I(x\notin D_n)+I(x\notin D_n)\sum_{z\sim x}-\Delta_{x,z}g(z,y) \end{aligned} \tag{3.10}$$

Adding up equations (3.9) *and* (3.10), *using equation* (3.7) *yields*

$$\Delta(x,x)g(x,y)=2d\delta_{x,y}+\sum_{z\sim y}-\Delta_{x,z}g(z,y)$$

which gives

$$\sum_{z\in V_n}\Delta_{x,z}g(z,y)=2d\delta_{x,y}$$

showing that

$$\Delta^{-1}=(2d)^{-1}g$$

as desired □

3.3. Criticality

It is important to note that Dhar's formula still applies, given stability of the system. We know that adding dissipation will never corrupt this property, so we can still apply Dhar's formula. Then all definitions of toppling numbers and criticality still apply for this variation of the model. We will now show that if the system is made up completely of dissipative sites, criticality is lost.

Theorem 3.2. *The system consisting of only dissipative sites is non-critical.*

PROOF. *From equation* (3.5) *we know that*

$$g(x,y)=\mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{l_k(D_n)}I(X_k=y)\right) \tag{3.11}$$

If all sites are dissipative, then $l_k(D_n)=k$. Then

$$\begin{aligned} \lim_{n\to\infty}\sum_{y\in V_n}g(x,y) &= \lim_{n\to\infty}\sum_{y\in V_n}\mathbb{E}_x^{RW}\left[\sum_{k=o}^{\tau}\left(\frac{2d}{2d+1}\right)^{l_k(D_n)}I\left(X_k=y\right)\right] && (3.12)\\ &= \lim_{n\to\infty}\sum_{y\in V_n}\mathbb{E}_x^{RW}\left[\sum_{k=o}^{\tau}\left(\frac{2d}{2d+1}\right)^{k}I\left(X_k=y\right)\right] && (3.13)\\ &= \lim_{n\to\infty}\mathbb{E}_x^{RW}\left[\sum_{k=o}^{\tau}\left(\frac{2d}{2d+1}\right)^{k}\sum_{y\in V_n}I\left(X_k=y\right)\right] && (3.14)\\ &= \mathbb{E}_x^{RW}\left[\sum_{k=o}^{\infty}\left(\frac{2d}{2d+1}\right)^{k}\right] && (3.15)\\ &= \frac{1}{1-\frac{2d}{2d+1}} && (3.16)\\ &= 2d+1 && (3.17) \end{aligned}$$

□

This tells us that somewhere between zero and full dissipation lies the *border* of criticality. It has been proven in [9] by Redig, Ruszel and Saada, that if the distance between sink sites is uniformly bounded, the system is also non-critical. Therefore, the density of sink sites in V_n must decrease to 0 as n goes to infinity.

3.4. Adding source sites

We have seen what source sites are in the previous section. If we add source sites in the system, stability is no longer guaranteed. This is because mass can build up inside the system, never reaching a stable configuration. We will find and prove the Green's function for this model, assuming the system is stable, meaning there are always a finite number of topplings required to stabilize.

Theorem 3.3. *The ASM extended with dissipation and anti-dissipation, where Δ is defined as equation (3.1), is governed by the Green's function*

$$g(x,y) = \mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{l_k(D_n)}\left(\frac{2d}{2d-1}\right)^{l_k(S_n)} I(X_k = y)\right) \tag{3.18}$$

where $l_k(D_n) = \sum_{i=0}^{k} I(X_i \in D_n)$ is the number of visits to D_n up to time k. Likewise, $l_k(S_n)$ is the number of visits to S_n up to time k, and $X_k, k \geq 1$ is the discrete random walk killed upon exiting V_n (at time τ).

PROOF. *The proof follows the same structure as the proof of Theorem 3.3. We know from equation (3.1) that*

$$\Delta_{x,x} = (2d+1)I(x \in D_n) + (2d-1)I(x \in S_n) + (2d)I(x \in R_n) \tag{3.19}$$

Remind also that if $z \sim x$ then $P(x,z) = 1/2d = -\Delta_{x,z}/2d$. Again we first split out the contribution from time $k = 0$ in $G(x,y)$

$$\begin{aligned}
g(x,y) &= \mathbb{E}_x^{RW}\left(\left(\frac{2d}{2d+1}\right)^{I(X_0\in D_n)}\left(\frac{2d}{2d-1}\right)^{I(X_0\in S_n)}\left(I(X_0=y)+\sum_{k=1}^{\tau}\left(\frac{2d}{2d+1}\right)^{\sum_{i=1}^{k} I(X_i\in D_n)}\left(\frac{2d}{2d-1}\right)^{\sum_{i=1}^{k} I(X_i\in S_n)} I(X_k=y)\right)\right)\\
&= \left(\frac{2d}{2d+1}\right)\delta_{x,y} I(x\in D_n) + \left(\frac{2d}{2d-1}\right)\delta_{x,y} I(x\in S_n) + \delta_{x,y} I(x\in R_n)\\
&+ \left(\frac{2d}{2d+1}\right) I(x\in D_n)\mathbb{E}_x^{RW}\left(\sum_{k=1}^{\tau}\left(\frac{2d}{2d+1}\right)^{\sum_{i=1}^{k} I(X_i\in D_n)}\left(\frac{2d}{2d-1}\right)^{\sum_{i=1}^{k} I(X_i\in S_n)} I(X_k=y)\right)\\
&+ \left(\frac{2d}{2d-1}\right) I(x\in S_n)\mathbb{E}_x^{RW}\left(\sum_{k=1}^{\tau}\left(\frac{2d}{2d+1}\right)^{\sum_{i=1}^{k} I(X_i\in D_n)}\left(\frac{2d}{2d-1}\right)^{\sum_{i=1}^{k} I(X_i\in S_n)} I(X_k=y)\right)\\
&+ I(x\in R_n)\mathbb{E}_x^{RW}\left(\sum_{k=1}^{\tau}\left(\frac{2d}{2d+1}\right)^{\sum_{i=1}^{k} I(X_i\in D_n)}\left(\frac{2d}{2d-1}\right)^{\sum_{i=1}^{k} I(X_i\in S_n)} I(X_k=y)\right)\\
&= \left(\frac{2d}{2d+1}\right)\delta_{x,y} I(x\in D_n) + \left(\frac{2d}{2d-1}\right)\delta_{x,y} I(x\in S_n) + \delta_{x,y} I(x\in R_n)\\
&+ \left(\frac{2d}{2d+1}\right) I(x\in D_n)\sum_{z\sim y}\frac{-\Delta_{x,z}}{2d} g(z,y)\\
&+ \left(\frac{2d}{2d-1}\right) I(x\in S_n)\sum_{z\sim y}\frac{-\Delta_{x,z}}{2d} g(z,y)\\
&+ I(x\in R_n)\sum_{z\sim y}\frac{-\Delta_{x,z}}{2d} g(z,y)
\end{aligned}$$

As a consequence,

$$\begin{aligned} & (2d+1)I(x \in D_n)g(x,y) \\ = \; & 2d\delta_{x,y}I(x \in D_n) + I(x \in D_n)\sum_{z \sim x} -\Delta_{x,z}g(z,y) \end{aligned} \tag{3.20}$$

$$\begin{aligned} & (2d-1)I(x \in S_n)g(x,y) \\ = \; & 2d\delta_{x,y}I(x \in S_n) + I(x \in S_n)\sum_{z \sim x} -\Delta_{x,z}g(z,y) \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} & (2d)I(x \in R_n)g(x,y) \\ = \; & 2d\delta_{x,y}I(x \in R_n) + I(x \in R_n)\sum_{z \sim x} -\Delta_{x,z}g(z,y) \end{aligned} \tag{3.22}$$

Adding up equation (3.20), (3.21) *and* (3.22), *using equation* (3.19) *yields*

$$\Delta(x,x)g(x,y) = 2d\delta_{x,y} + \sum_{z \sim y} -\Delta_{x,z}g(z,y)$$

which gives

$$\sum_{z \in V_n} \Delta_{x,z}g(z,y) = 2d\delta_{x,y}$$

showing that

$$\Delta^{-1} = (2d)^{-1}g$$

as desired □

If we let $S_n = \emptyset$ in equation (3.18), we get equation (3.5) back, so this is the general form for mixed systems with both sources and sinks. We can now prove criticality for systems with both sources and sinks incorporated.

Theorem 3.4. *Define a system with only sinks and sources, meaning $V_n = D_n \cup S_n$. If we distribute the sources and sinks such that*

$$\lim_{k \to \infty} \frac{\lim_{n \to \infty} \mathbb{E}_x^{RW}[l_k(S_n)]}{k} = \lim_{k \to \infty} \frac{\mathbb{E}_x^{RW}[l_k(S)]}{k} = \frac{1}{2} \tag{3.23}$$

then one of the following is true

1. *The system is not stabilizable*

2. *The system is critical*

The condition in equation (3.23) says that the expected number of times the random walk is on a source site after k steps, is equal to $k/2$, as the system grows infinitely large. This is not a trivial condition. If we were to place sinks all around the origin such that half the area is covered, and fill the rest of the space around it with sources, it is not necessarily true that this holds. One example of distributing sources and sinks to achieve this is to put them in a checkerboard formation. Then on even and odd steps it will alternate between sources and sinks, guaranteeing the limit as $k \to \infty$. A different approach is to define each vertex to be either a source or a sink, each with probability 1/2, independently.

PROOF. *In section 3 of the paper by Jongbloed [5], it is shown that certain configurations of sources result in a system that is not stabilizable. Assuming we have a configuration that does stabilize, we will prove that it is critical. From* $V_n = D_n \cap S_n$*, we know* $l_k(D_n) = k - l_k(S_n)$*. This means*

$$\sum_{y\in V_n} g(x,y) = \sum_{y\in V_n} \mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{l_k(D_n)}\left(\frac{2d}{2d-1}\right)^{l_k(S_n)} I(X_k=y)\right) \tag{3.24}$$

$$= \sum_{y\in V_n} \mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{k-l_k(S_n)}\left(\frac{2d}{2d-1}\right)^{l_k(S_n)} I(X_k=y)\right) \tag{3.25}$$

$$= \sum_{y\in V_n} \mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{k}\left(\frac{2d+1}{2d}\right)^{l_k(S_n)}\left(\frac{2d}{2d-1}\right)^{l_k(S_n)} I(X_k=y)\right) \tag{3.26}$$

$$= \sum_{y\in V_n} \mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{k}\left(\frac{2d+1}{2d-1}\right)^{l_k(S_n)} I(X_k=y)\right) \tag{3.27}$$

Now observe again that $\sum_{y\in V_n} I(X_k=y)=1$*, and we can use linearity of the expectation to take the sum into the expectation. this means that*

$$\sum_{y\in V_n} g(x,y) = \sum_{y\in V_n} \mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{k}\left(\frac{2d+1}{2d-1}\right)^{l_k(S_n)} I(X_k=y)\right) \tag{3.28}$$

$$= \mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{k}\left(\frac{2d+1}{2d-1}\right)^{l_k(S_n)}\right) \tag{3.29}$$

Because $\left(\frac{2d+1}{2d-1}\right)^x$ *is a convex function, we can employ Jensen's inequality with* $x = l_k(S_n)$ *to get*

$$\lim_{n\to\infty}\sum_{y\in V_n} g(x,y) = \lim_{n\to\infty}\sum_{k=0}^{\tau}\mathbb{E}_x^{RW}\left[\left(\frac{2d}{2d+1}\right)^{k}\left(\frac{2d+1}{2d-1}\right)^{l_k(S_n)}\right] \tag{3.30}$$

$$\geq \lim_{n\to\infty}\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{k}\left(\frac{2d+1}{2d-1}\right)^{\mathbb{E}_x^{RW}[l_k(S_n)]} \tag{3.31}$$

Now we know that $\tau\to\infty$ *and* $S_n \to S$ *as* $n\to\infty$*. So we can write this as*

$$\lim_{n\to\infty}\sum_{y\in V_n} g(x,y) \geq \sum_{k=0}^{\infty}\left(\frac{2d}{2d+1}\right)^{k}\left(\frac{2d+1}{2d-1}\right)^{\mathbb{E}_x^{RW}[l_k(S)]} \tag{3.32}$$

Now we can use the root test to prove divergence and thus criticality.

$$\lim_{k\to\infty}\left[\left(\frac{2d}{2d+1}\right)^{k}\left(\frac{2d+1}{2d-1}\right)^{\mathbb{E}_x^{RW}[l_k(S)]}\right]^{\frac{1}{k}} = \lim_{k\to\infty}\left(\frac{2d}{2d+1}\right)\left(\frac{2d+1}{2d-1}\right)^{\frac{\mathbb{E}_x^{RW}[l_k(S)]}{k}} \tag{3.33}$$

Now we use the condition in equation (3.23) to write this as

$$\left(\frac{2d}{2d+1}\right)\sqrt{\frac{2d+1}{2d-1}} = \sqrt{\left(\frac{4d^2}{(2d+1)^2}\right)\left(\frac{2d+1}{2d-1}\right)} \tag{3.34}$$

$$= \sqrt{\frac{4d^2}{4d^2-1}} \tag{3.35}$$

Which is greater than 1 for all $d \geq 1$*. That proves that the system is critical.* □

We can in fact generalize the result we just obtained. If we alter the condition in equation (3.23) to

$$\lim_{k\to\infty} \frac{\mathbb{E}_x^{RW}[l_k(S)]}{k} = \frac{1}{N} \tag{3.36}$$

Then, we can use the same derivations as above to obtain (3.32). Applying the root test now gives

$$\lim_{k\to\infty} \left[\left(\frac{2d}{2d+1}\right)^k \left(\frac{2d+1}{2d-1}\right)^{\mathbb{E}_x^{RW}[l_k(S)]} \right]^{\frac{1}{k}} = \lim_{k\to\infty} \left(\frac{2d}{2d+1}\right)\left(\frac{2d+1}{2d-1}\right)^{\frac{\mathbb{E}_x^{RW}[l_k(S)]}{k}} \tag{3.37}$$

$$= \left(\frac{2d}{2d+1}\right)\left(\frac{2d+1}{2d-1}\right)^{\frac{1}{N}} \tag{3.38}$$

$$= \left[\left(\frac{2d}{2d+1}\right)^N \left(\frac{2d+1}{2d-1}\right) \right]^{\frac{1}{N}} \tag{3.39}$$

$$= \left(\frac{4d^2}{4d^2-1}\right)\left(\frac{2d}{2d+1}\right)^{N-2} \tag{3.40}$$

$$\tag{3.41}$$

The system is critical if the quantity in equation (3.40) is greater than 1. Since the first term is always greater than 1, this is true if and only if

$$\left(\frac{2d}{2d+1}\right)^{N-2} > \frac{4d^2-1}{4d^2} \tag{3.42}$$

If we pick some $d \geq 1$, then we can solve this equation to get a critical value for N. This result is in line with our previous calculations. If we let $N = 2$, we get condition (3.40) back. The left hand site is 1 and this is greater than the right hand site, regardless of our choice for d. On the other hand, if we let $N \to \infty$, then the left hand site will converge to 0 and so we will never have a critical model. But if $N \to \infty$, then the system is completely made up of dissipative sites, and we have already proven that this will indeed never be critical.

3.5. Connection to trapped random walk

An important step in proving criticality was to exploit Dhar's formula, giving us the equivalence between the expected avalanche size and the expected number of visits of a random walk that is killed upon leaving the boundary. If we only consider dissipative sites, so $S_n = \emptyset$, then we know from theorem 3.2 that

$$g(x,y) = \mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau} \left(\frac{2d}{2d+1}\right)^{l_k(D_n)} I(X_k = y) \right) \tag{3.43}$$

The only way dissipative sites are involved here is in $l_k(D_n)$. We see that each time a dissipative site is reached, we multiply by $2d/(2d+1)$. This is explained by the fact that there are $2d$ options for the random walk, but the mass can be $2d+1$. We can define another random walk. This random walk will still be killed upon leaving V_n, but it will also have a probability to be killed when reaching a dissipative site. Since the height can be $2d+1$, but there are only $2d$ neighbours, the transition probabilities add up to $2d/(2d+1)$. This leaves $1/(2d+1)$ as the natural choice for the killing probability. We now claim and prove

$$\tilde{g}(x,y) = \mathbb{E}_x^{TRW}\left(\sum_{k=0}^{\tau} I(X_k = y) \right) \tag{3.44}$$

Again, we first condition the expectation on the first step

$$\mathbb{E}_x^{TRW}\left(\sum_{k=0}^{\tau} I(X_k = y) \right) = \sum_{z\in V_n} \mathbb{E}_x^{TRW}\left(\sum_{k=1}^{\tau} I(X_k = y) | X_1 = z \right) \mathbb{P}(X_1 = z | X_0 = x) + \delta_{x,y} \tag{3.45}$$

We know from the killing probability of $1/(2d+1)$, that

$$\mathbb{P}(X_1 = z | X_0 = x) = \begin{cases} \frac{1}{2d+1}, \text{ for } z \sim x, x \in D_n \\ \frac{1}{2d}, \text{ for } z \sim x, x \notin D_n \end{cases} \tag{3.46}$$

This means that

$$\mathbb{E}_x^{TRW}\left(\sum_{k=0}^{\tau} I(X_k = y)\right) = I(x \in D_n)\frac{1}{2d+1}\sum_{z\sim x}\mathbb{E}_x^{TRW}\left(\sum_{k=1}^{\tau} I(X_k = y)|X_1 = z\right) \tag{3.47}$$

$$+ I(x \notin D_n)\frac{1}{2d}\sum_{z\sim x}\mathbb{E}_x^{TRW}\left(\sum_{k=1}^{\tau} I(X_k = y)|X_1 = z\right) + \delta_{x,y} \tag{3.48}$$

We use the identity

$$\frac{1}{2d+1} = \frac{2d}{2d+1}\cdot\frac{1}{2d} \tag{3.49}$$

Then we can write

$$\mathbb{E}_x^{TRW}\left(\sum_{k=0}^{\tau} I(X_k = y)\right) = \left(\frac{2d}{2d+1}\right)^{I(x\in D_n)}\sum_{z\sim x}\mathbb{E}_x^{TRW}\left(\sum_{k=1}^{\tau} I(X_k = y)|X_1 = z\right) + \delta_{x,y} \tag{3.50}$$

Now we use the fact that the random walk is memoryless, meaning

$$\mathbb{E}_x^{TRW}\left(\sum_{k=1}^{\tau} I(X_k = y)|X_1 = z\right) = \mathbb{E}_z^{TRW}\left(\sum_{k=0}^{\tau} I(X_k = y)\right) \tag{3.51}$$

With this we obtain

$$\mathbb{E}_x^{TRW}\left(\sum_{k=0}^{\tau} I(X_k = y)\right) = \left(\frac{2d}{2d+1}\right)^{I(x\in D_n)}\frac{1}{2d}\sum_{z\sim x}\mathbb{E}_z^{TRW}\left(\sum_{k=0}^{\tau} I(X_k = y)\right) + \delta_{x,y} \tag{3.52}$$

The factor of

$$\left(\frac{2d}{2d+1}\right)^{I(x\in D_n)} \tag{3.53}$$

is familiar, since we defined

$$l_k(D_n) = \sum_{i=0}^{k} I(X_i \in D_n) \tag{3.54}$$

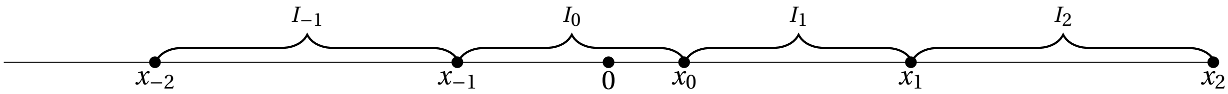
If we iterate the process of conditioning on each step, we end up with

$$\tilde{g}(x,y) = \mathbb{E}_x^{TRW}\left(\sum_{k=0}^{\tau} I(X_k = y)\right) = \mathbb{E}_x^{RW}\left(\sum_{k=0}^{\tau}\left(\frac{2d}{2d+1}\right)^{l_k(D_n)} I(X_k = y)\right) = g(x,y) \tag{3.55}$$

3.6. One-dimensional dissipative model

Now that we have this interpretation of the trapped random walk, we will focus on the one-dimensional model to try to prove a bound on the level of dissipation upon which criticality is lost. Rather than writing it like in equation (3.44), we will interpret it in the following way.

In $d = 1$, V_n is simply the one-dimensional lattice. We can see dissipative sites, or traps, as the vertices $\{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$, and define $I_z = [x_{z-1}, x_z]$ as the intervals between two consecutive traps. To introduce an order, we assume that $\{0\} \in I_0$, meaning $x_{-1} \leq 0$ and $x_0 \geq 0$. A visual representation is given below.



Then each time the random walk passes one of these points, it gets killed with a probability of $p = 1/(2d+1) = 1/3$, since $d = 1$. This means the amount of sinks it will pass before being killed, N, is given by the geometric distribution with parameter p, denoted by $N \sim \text{Geo}(p)$. Now we can define another random walk process ξ_n, which denotes the intervals we traverse. This means that we start in I_{ξ_1}, then after hitting a trap and surviving, we travel to I_{ξ_2}. We can see this as a "macro" random walk, since ξ_n is determined by the smaller steps of X_n. The total lifetime of the random walk, T, is now given by

$$T = \sum_{i=1}^{N} Y_i$$

where Y_i is the total time the random walk was in interval I_{ξ_i}. Conditioning on N, We can then write the expected lifetime of the random walk as

$$\mathbb{E}[T] = \mathbb{E}\left[\sum_{i=1}^{N} Y_i\right] = \sum_{n=1}^{\infty}\left(\mathbb{P}(N = n)\sum_{i=1}^{n}\mathbb{E}[Y_i]\right) \tag{3.56}$$

Criticality of the system is now equivalent to the expected lifetime of the random walk being infinite, and conversely, a finite value for the expected lifetime means that the system is non-critical. It has been proven that a finite number of sinks will not suffice to make the system non-critical. Contrariwise, if there is a uniform bound on the distances between sinks, it will be non-critical. So, we are looking for some way of arranging sinks such that we will have an infinite number of them, but distributed sparsely enough to allow for critical behaviour to occur.

There are two ways we can arrange the dissipative sites. The first we will cover is to place them using a deterministic function. The other possibility is to distribute the sink sites randomly, as we will see.

3.6.1. Deterministically placed dissipative sites

We want to find an upper bound for $\mathbb{E}[T]$, to make claims about criticality. Let us assume we start in I_0. We will consider the worst-case scenario, meaning the largest possible time the random walk can survive. Therefore we assume the walk always move to the bigger interval, to make the survival time longer. We assume that I_0 is the smallest interval, and without loss of generality, we also assume the intervals are symmetric around I_0. Then, in the worst-case scenario, we will just assume that the walk always goes to the right. This means that $\xi_i = i$. After hitting n sink sites, the biggest interval we can find ourselves in is I_n. In Lawler Limic (2010) [6], it was proven that the expected time it takes for a simple random walk X_n that starts at $\bar{x}$ with $a < \bar{x} < b$ to reach either $x = a$ or $x = b$ is given by

$$(b - \bar{x})(\bar{x} - a) \tag{3.57}$$

For $a < \bar{x} < b$, this is always less than $(b-a)^2$. Therefore we have an upper bound for $\mathbb{E}[Y_i]$.

$$\mathbb{E}[Y_i] = \mathbb{E}[\text{Time in } I_{\xi_i}] = \mathbb{E}[\text{Time in } I_i] < |I_i|^2 \tag{3.58}$$

Where we used that $\xi_i = i$. Using equation (3.58), equation (3.56) becomes

$$\mathbb{E}[T] = \sum_{n=1}^{\infty}\left(\mathbb{P}(N=n)\sum_{i=1}^{n}\mathbb{E}[Y_i]\right) \tag{3.59}$$

$$< \sum_{n=1}^{\infty}\left(\mathbb{P}(N=n)\sum_{i=1}^{n}|I_i|^2\right) \tag{3.60}$$

$$< \sum_{n=1}^{\infty}\left(\mathbb{P}(N=n)\, n|I_n|^2\right) \tag{3.61}$$

Where we used that the intervals are strictly increasing in size. We now define $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(n) = |I_n|^2 \tag{3.62}$$

we can write equation (3.61) as

$$\mathbb{E}[T] < \sum_{n=1}^{\infty}\left(\mathbb{P}(N=n)\, n\phi(n)\right) \tag{3.63}$$

Since N is geometrically distributed with parameter $1/3$, we can write $\mathbb{P}(N=n)$ explicitly,

$$\mathbb{P}(N=n) = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \tag{3.64}$$

This gives us

$$\mathbb{E}[T] < \frac{1}{3}\sum_{n=1}^{\infty}\left(\left(\frac{2}{3}\right)^{n-1} n\phi(n)\right) \tag{3.65}$$

We know that if we let $\phi(n)$ be any polynomial in n, the coefficients in our sum explode, as a consequence of the exponential decay of the geometric distribution. Perhaps an exponential function will be a good choice. So let

$$\phi(n) = e^{\alpha n} \tag{3.66}$$

A nice way of rewriting exponential functions is

$$x^{n-1} = e^{ln(x^{n-1})} = e^{(n-1)ln(x)} = \frac{1}{x}e^{nln(x)} \tag{3.67}$$

Using equation (3.67) with $x = \frac{2}{3}$, and equation (3.66), equation (3.65) becomes

$$\mathbb{E}[T] < \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{3}{2}e^{nln(\frac{2}{3})}ne^{\alpha n}\right) \tag{3.68}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(ne^{nln(\frac{2}{3})+\alpha n}\right) \tag{3.69}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(ne^{n(ln(\frac{2}{3})+\alpha)}\right) \tag{3.70}$$

which goes to infinity if and only if

$$ln\left(\frac{2}{3}\right) + \alpha \geq 0 \tag{3.71}$$

Therefore, a system with dissipative sites distributed via equation (3.62), where $\phi(n)$ is defined by equation (3.66) is *critical* if $\alpha \geq -ln(2/3)$. If $\alpha \leq -ln(2/3)$, we cannot say that the model is non-critical, since we used an overestimation. We do know that $\alpha > 0$, so for some $0 < \alpha < -ln(2/3)$, the model should be non-critical. As a note, the decimal value of $-ln(2/3)$ is approximately 0.4055.

3.6.2. Randomly placed dissipative sites

If we assume the dissipative sites are placed on the lattice randomly, we cannot say that I_n is the biggest interval anymore. We can however say things about their expectation. The first thing we will assume is that the intervals are again symmetric around I_0, meaning

$$\mathbb{E}[|I_{-i}|] = \mathbb{E}[|I_i|], \text{ for all } i \in \mathbb{N} \tag{3.72}$$

The other thing we assume is that their moments are increasing, meaning

$$\mathbb{E}\left[|I_i|^n\right] \geq \mathbb{E}\left[|I_k|^n\right], \text{ for all } k \in [-i, i] \text{ and all } n \in \mathbb{N} \tag{3.73}$$

Then, without loss of generality, we can say that $\xi_i = i$ is the worst-case, and thus a good idea to use for calculating an upper bound. Then we still have

$$\mathbb{E}[T] = \sum_{n=1}^{\infty}\left(\mathbb{P}(N=n)\sum_{i=1}^{n}\mathbb{E}[Y_i]\right) \tag{3.74}$$

$$< \sum_{n=1}^{\infty}\left(\mathbb{P}(N=n)\sum_{i=1}^{n}\mathbb{E}\left[|I_i|^2\right]\right) \tag{3.75}$$

We define $|I_i|^2 = Z_i$, where Z_i is a random variable that may depend on i. Our argument now is that

$$\sum_{i=1}^{n}\mathbb{E}\left[|I_i|^2\right] = \sum_{i=1}^{n}\mathbb{E}[Z_i] \leq n\mathbb{E}[\max_{1\leq i\leq n} Z_i] \tag{3.76}$$

We now want to give a bound for the expectation of the maximum of Z_i. First, observe that $e^{\lambda x}$, with $\lambda > 0$, is a convex function, now plug in $\max_{1\leq i\leq n} Z_i$ for x and use Jensen's inequality. This gives us

$$e^{\lambda\mathbb{E}[\max_{1\leq i\leq n} Z_i]} \leq \mathbb{E}\left[e^{\lambda \max_{1\leq i\leq n} Z_i}\right] \leq \mathbb{E}\left[\sum_{i=1}^{n} e^{\lambda Z_i}\right] \tag{3.77}$$

where the second inequality comes from the fact that the sum over all $1 \leq i \leq n$ must be at least equal to the maximum, since all the terms are positive. Isolating $\mathbb{E}[\max_{1\leq i\leq n} Z_i]$ gives us

$$\mathbb{E}[\max_{1\leq i\leq n} Z_i] \leq \frac{1}{\lambda}\ln\left(\sum_{i=1}^{n}\mathbb{E}\left[e^{\lambda Z_i}\right]\right) \tag{3.78}$$

Now we need to choose our Z_i a priori to see if we can calculate the *border* of criticality. Again, the Z_i's cannot be uniformly bounded, since in that case we will have non-criticality for certain. As an example, we consider Z_i's uniformly distributed on the interval $[1, \phi(i)]$, where $\phi(i)$ has to be some unbounded increasing function of i. Then, since each Z_i is less than $\phi(i)$, and $\phi(i)$ is an increasing function, we know that

$$\mathbb{E}[\max_{1\leq i\leq n} Z_i] \leq \phi(n) \tag{3.79}$$

Combining equation (3.79) and equations (3.75) and (3.76), we obtain

$$\mathbb{E}[T] < \sum_{n=1}^{\infty}\left(\mathbb{P}(N=n)\, n\phi(n)\right) \tag{3.80}$$

Note that this is of the same form as equation (3.63). Therefore we know that if we let $Z_i \sim U[1, \phi(i)]$, where $\phi(i)$ is a polynomial in i, the system will be non-critical. The only way to obtain criticality is if we have at least exponential growth. So let $Z_i \sim U\left[1, e^{i\alpha}\right]$. Then, using the same calculations as in the previous section, we can conclude that the system is critical if $\alpha \geq -ln(2/3)$, and non-critical otherwise.

In this example we did not utilize equation (3.78). During calculations of the distributions we tried for Z_i, we always ran into an issue. Nevertheless, we can say something about criticality. If we combine equations (3.75), (3.76) and (3.78) we get

$$\mathbb{E}[T] < \sum_{n=1}^{\infty}\left(\mathbb{P}(N=n)\,\frac{n}{\lambda}\ln\left(\sum_{i=1}^{n}\mathbb{E}\left[e^{\lambda Z_i}\right]\right)\right) \tag{3.81}$$

We can write out $\mathbb{P}(N=n)$ to get

$$\mathbb{E}[T] < \frac{1}{3}\sum_{n=1}^{\infty}\left(\left(\frac{2}{3}\right)^{n-1}\frac{n}{\lambda}\ln\left(\sum_{i=1}^{n}\mathbb{E}\left[e^{\lambda Z_i}\right]\right)\right) \tag{3.82}$$

So to obtain criticality one will have to find a random distribution Z_i such that

$$\ln\left(\sum_{i=1}^{n}\mathbb{E}\left[e^{\lambda Z_i}\right]\right) \geq \left(\frac{3}{2}\right)^{n-1} \tag{3.83}$$

4

Conclusions

In this report we have started by examining the abelian sandpile model, as first introduced by Bak, Tang and Wiesenfeld. Starting with this definition of the model, we have seen how Dhar's formula relates the expected avalanche size to the inverse of the toppling matrix Δ, which we called the Green's function. We then defined a model to be critical if the expected avalanche resulting from one grain of sand somewhere in the system will be infinitely large, as the system itself becomes infinitely large. If the expected avalanche is always of finite size, then the system is non-critical.

We then gave another interpretation of the Green's function. It turns out that, up to a multiplicative constant, the expected avalanche size after adding a grain of sand at the site x is equal to the expected number of visits to y of a random walk that starts at x, and that is killed upon leaving V_n. With this second interpretation we have proved that the classical model is critical, and the completely dissipative model is non-critical.

Then we introduced sources to the system. Where dissipation can lose criticality, anti-dissipation can retrieve it. We proved that if a system has equally distributed sources and sinks and no regular sites, the system is critical. This in a way told us that source sites are more powerful than sink sites, in their war against each other.

Thereafter we gave a third interpretation of the Green's function, this time of a trapped random walk. One that is killed not only upon leaving the domain V_n, but when hitting a sink site too, with a certain probability. With this new insight, we zoomed in on the one-dimensional model and managed to prove some bound on the level of dissipation where criticality is lost. We found that the distance between sink sites should be exponentially increasing following $e^{\alpha n}$ and the critical value for the exponent α above which the model is critical is equal to $-ln(2/3) \approx 0.4055$. If the dissipative sites are placed randomly, we found the same results for uniformly distributed interval sizes. No other distribution has been found that allows us to calculate an exact bound, but we do have the formula that guarantees criticality if

$$\ln\left(\sum_{i=1}^{n} \mathbb{E}\left[e^{\lambda Z_i}\right]\right) \geq \left(\frac{3}{2}\right)^{n-1} \tag{4.1}$$

We finish with some discussion on improving our results. Further research is required to find different possible distributions of sink sites that are able to touch on this *border* of criticality. Perhaps estimates can be found for higher dimensions, but another approach will be required. Also, by using an upper bound we have found a *sufficient* condition for criticality. We could look to prove some lower bound for the expected time of the random walk to find a *necessary* condition for criticality.

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