THE MODELLING OF FIBRE METAL LAMINATES BY
THICK SHELL ELEMENTS

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ABSTRACT

In circumstances where impact damage and delamination may restrict the
range of applications of standard composite materials the application of Fibre
Metal Laminates (FML) can significantly improve the performance of struc-
tures and components. FML are characterised by a layered set-up in which
metal layers are connected by prepreg layers. In this paper a thick shell ele-
ment is presented which is used to model the behaviour of fibre metal lami-
nates (FML). The theory is derived from a three-dimensional continuum
based theory. To account for plasticity effects the von Mises and the Hoff-
mann yield criteria have been implemented. The numerical results have been
compared with standard benchmark tests. Furthermore, the element has been
used to model a tensile test of a FML.

1. INTRODUCTION

The use of composite materials in structural components has become increas-
ingly popular. Especially in aerospace applications the high strength/weight
and stiffness/weight ratios of these materials are major advantages. However,
the fact that composite laminates show a less favourable behaviour under im-
pact loading and tend to delaminate restricts the range of applications. One
way to improve the characteristics of composite materials is to combine these
with other materials, such as aluminium. The materials GLARE and ARALL
are fibre metal laminates which consist of aluminium layers joined by K-Glas
and Aramid prepreg layers, respectively.

To gain more insight in the behaviour of fibre metal laminates, proce-
dures are necessary which properly predict the interlaminar reaction, the failure mechanisms of the different layers and the geometrical response. As fibre metal laminates typically occur in thin walled applications, computing of the structural behaviour leads to problems when standard finite elements are used. Solid elements tend to lock in thin applications [1,2] whereas standard degenerated shell elements are not able to represent the change of thickness and model the delamination in a three-dimensional state. To overcome these problems thick shell elements can be used. They are robust in thin applications and represent a three-dimensional geometry and state of stress.

In literature various methods to account for the thickness change in a shell element have been described. Most approaches account for the thickness change via a staggered iterative update procedure which is constructed by exploiting the plane-stress assumption. In contrast, Simo, Rifai and Fox [2] present a method in which the stretch is a truly independent field that is coupled with bending, membrane and transverse shear fields through the constitutive equation.

The approach presented by Büchter, Ramm and Roehl [3] uses the enhanced assumed strain concept to obtain a three-dimensional constitutive equation. Under the assumption of a linearly varying thickness director the displacement is separated into the displacement of the mid-surface and the displacement of the thickness director. In the latter two approaches the thickness variation cannot simply be coupled with other elements. For layered material this leads to problems when a structure is modelled with more than one element in the thickness direction.

In this paper the approach proposed by Parisch [1] is followed to describe the element. The theory has been derived from three-dimensional continuum mechanics and provides a three-dimensional state of stress. It includes stretching in the thickness direction. The theory has been used to develop a sixteen-noded element with three displacement degrees of freedom at each node. With this formulation the element can be coupled with other elements in the thickness direction which is advantageous when modelling interlaminar effects (delamination) in layered structures.

2. GEOMETRICALLY NONLINEAR SHELL FORMULATION

2.1 Strains and Kinematics

The strains in every point of the element are defined by the Green strain tensor:

\[ \gamma = \frac{1}{2} (F^T f - F^T F) . \]  \hspace{1cm} (1)

Hereby \( f \) denotes the deformation tensor in the deformed configuration \( \hat{C} \) and \( F \) in the undeformed configuration \( \hat{0C} \). Both tensors are defined by the metric vectors in the following fashion:

\[ f = g_i \otimes g^j \quad \text{and} \quad F = G_i \otimes G^j \] with \( i, j = 1, 2, 3 \). \hspace{1cm} (2)

The metric vectors are evaluated by deriving the position vectors \( x \) of an arbi-
trary point $P$ in the deformed configuration and $X$ in the undeformed configuration with respect to a set of isoparametric coordinates $\Theta^\alpha = (\xi \eta \zeta)$. The position vectors are linked by the displacement vector $u$:

$$u = u^0 + \zeta u^1 + (1 - \zeta^2)u^2,$$

$$X = X^0 + \zeta X^1,$$

$$x = x^0 + \zeta x^1 + (1 - \zeta^2)x^2,$$

$$x = X + u.$$

The quantity $u^0$ denotes the displacement of the mid-surface, $u^1$ the change of the thickness and $u^2$ represents the internal stretching. Accordingly, $X^0$ and $x^0$ represent the middle surface in both configurations. $X^1$ and $x^1$ express the thickness directors $D$ and $d$ in both configurations. Since $u^2$ may not violate the shell kinematics it must be colinear with $d$. The metric vectors contain terms of order higher than one. For deriving the metric matrices these higher order terms have been neglected. The metric matrices then appear as:

$$G_{\alpha\beta} = E_\alpha \cdot E_\beta + \zeta (E_\alpha \cdot D_\beta + E_\beta \cdot D_\alpha),$$

$$G_{\alpha 3} = E_\alpha \cdot D + \zeta D_\alpha \cdot D,$$

$$G_{33} = D \cdot D;$$

for the undeformed configuration $O C$ and for the deformed configuration $C$:

$$g_{\alpha\beta} = e_\alpha \cdot e_\beta + \zeta (e_\alpha \cdot d_\beta + e_\beta \cdot d_\alpha),$$

$$g_{\alpha 3} = e_\alpha \cdot d + \zeta (d_\alpha \cdot d - 2e_\alpha \cdot u^2),$$

$$g_{33} = d \cdot d - \zeta 4d \cdot u^2.$$

Where:

$$E_\alpha = \frac{\partial X^0}{\partial \Theta^\alpha}; \quad e_\alpha = \frac{\partial x^0}{\partial \Theta^\alpha}; \quad \bar{e}_\alpha = e_\alpha + u^2; \quad u^2 = w_3d \quad \text{with} \quad \alpha, \beta = 1, 2.$$

The factor $w_3$ represents the internal stretching in the element. Based on the metric vectors and metric matrices the contravariant metric tensors, the shell shifter and the mixed-variant metric tensor $G^{\alpha\beta}$ can be calculated. By the introduction of the iterative change of the strains $d\gamma = \gamma^{\varepsilon 1} - \gamma^I$ and the decomposition of the strains into a $\zeta$-dependent bending and a $\zeta$-independent membrane part the iterative strain tensor in a global frame equals:

$$d\gamma^\alpha = (d\varepsilon_{\alpha\beta} + \zeta d\rho_{\alpha\beta})E^\alpha \otimes E^\beta + (d\varepsilon_{\alpha 3} + \zeta d\rho_{\alpha 3})E^\alpha \otimes E^3 + (d\varepsilon_{33} + \zeta d\rho_{33})E^3 \otimes E^3.$$

Where the iterative strains appear in the following manner:

$$2d\rho_{\alpha\beta} = e_\beta \cdot d\varepsilon_{\alpha\beta} + d\varepsilon_{\alpha\beta} \cdot d\varepsilon_{\alpha\beta} + d\varepsilon_{\alpha\beta} \cdot d\varepsilon_{\alpha\beta}$$

$$+ e_\beta \cdot d\varepsilon_{\alpha\beta} + d\varepsilon_{\alpha\beta} \cdot d\varepsilon_{\alpha\beta} + d\varepsilon_{\alpha\beta} \cdot d\varepsilon_{\alpha\beta}$$

$$- (e_\alpha \cdot d\varepsilon_{\alpha\beta} + e_\beta \cdot d\varepsilon_{\alpha\beta} + d\varepsilon_{\alpha\beta} \cdot d\varepsilon_{\alpha\beta})\bar{G}_{\alpha\beta},$$

$$- (e_\alpha \cdot d\varepsilon_{\alpha\beta} + e_\beta \cdot d\varepsilon_{\alpha\beta} + d\varepsilon_{\alpha\beta} \cdot d\varepsilon_{\alpha\beta})\bar{G}_{\alpha\beta}.$$
\[ 2d \rho_{a3} = \mathbf{d}_a \cdot \mathbf{d} u_1^* + \mathbf{d} \cdot \mathbf{d} u_{a3} + \mathbf{d} u_{a3} \cdot \mathbf{d} u^1, \quad (14) \]
\[ 2d \rho_{33} = -8w_3 \mathbf{d} \cdot \mathbf{d} u_1^* - 4d \cdot dw_3 - 4dw_3 \mathbf{d} \cdot \mathbf{d} u_1^* - 4w_3 \mathbf{d} u_1^* \cdot \mathbf{d} u^1. \quad (15) \]
\[ 2d \varepsilon_{a3} = \mathbf{e}_a \cdot \mathbf{d} u_{a3} + \mathbf{e}_a \cdot \mathbf{d} u_{a3}^0 + \mathbf{d} u_{a3} \cdot \mathbf{d} u_1^0, \quad (16) \]
\[ 2d \varepsilon_{33} = 2\mathbf{d} \cdot \mathbf{d} u_1^* + 2d \mathbf{d} u_1^* \cdot \mathbf{d} u^1. \quad (17) \]
\[ 2d \varepsilon_{33} = 2d \cdot \mathbf{d} u_1^* + 2d \mathbf{d} u_1^* \cdot \mathbf{d} u^1, \quad (18) \]

According to eqn.(12) the components of the strain tensor \( \varepsilon_{a\beta}, \rho_{a\beta} \) are referred to the in general non-orthogonal triplet \( \mathbf{E}' \) spanned at the material point \( P \) in the undeformed reference configuration. However, for composite materials the constitutive relation is set up conveniently in a local frame \( \mathbf{m}_j \) provided by the material directions of the individual layer. Denoting the components of the strain tensor when referred to the materials frame as \( \gamma_{ij} \) we have the transformation:

\[ \gamma_{ij} = (\mathbf{e}_i + \xi \mathbf{e}_i)\mathbf{m}_j, \quad \text{with} \quad \mathbf{e}_i = (\mathbf{E}_i \cdot \mathbf{m}_i) \]

Using vector-matrix notation \( d \gamma_{ij} \) can be decomposed into the linear part \( d \gamma^L \) and the nonlinear part \( d \gamma^N \). The linear contribution is expressed:

\[ d \gamma^L = \mathbf{H}^0 d\mathbf{v} + \mathbf{H}^1 d\mathbf{v}_A + \mathbf{H}^2 d\mathbf{v}_N, \quad (19) \]

with:

\[ d \gamma^L = (d\gamma_{11}, d\gamma_{22}, d\gamma_{33}, d\gamma_{12}, d\gamma_{23}, d\gamma_{31})^T \quad \text{and} \quad d\mathbf{v} = (d\mathbf{u}^0, d\mathbf{u}^1, dw_3)^T. \]

This theory has been applied to derive the formulation of a 16-noded element (Fig. 1) with four additional internal degrees of freedom.

![Element geometry and nodes](image)

Figure 1. Element geometry and nodes

The displacements \( d\mathbf{u}^0, d\mathbf{u}^1 \) and \( dw_3 \) are expressed in terms of nodal displacements and their corresponding shape functions. The element displacement vector is therefore arranged in this manner:

\[ d\mathbf{u}^T = (d\mathbf{u}_1^*, \ldots, d\mathbf{u}_{16}^*, \ldots, d\mathbf{u}_1^1, \ldots, d\mathbf{u}_{16}^1, dw_3^1, \ldots, dw_3^6), \quad (20) \]

which for \( d\mathbf{v} \) leads to:

\[
\begin{bmatrix}
  d\mathbf{u}^0 \\
  d\mathbf{u}^1 \\
  dw_3
\end{bmatrix}
= \begin{bmatrix}
\text{diag}(\Pi^0) & 0 \\
\text{diag}(\Pi^1) & 0 \\
0^T & \Pi^v
\end{bmatrix}
\begin{bmatrix}
 d\mathbf{u} = A \mathbf{d} \mathbf{u}
\end{bmatrix},
\quad (21)
\]
The vectors $\Pi'$ are defined by:

$$\Pi^0 = \psi_2(\psi_1, \ldots, \psi_8), \quad \Pi^1 = \psi_2(\psi_1, \ldots, -\psi_8, \psi_1, \ldots, \psi_8),$$

$$\Pi'' = (\phi_1, \phi_2, \phi_3, \phi_4).$$

Where $\psi_i$ represent the 8 shape functions of a standard 8-noded shell element and $\phi_i$ the 4 isoparametric shape functions of a 4-noded standard shell element. Substitution of eqn.(21) into eqn.(19) leads to:

$$d\gamma^L = (H_0A + H_1A_\alpha + H_2A_{\alpha})d\theta = B_Ld\theta.$$  \hspace{1cm} (22)

The derivation of the nonlinear part $d\gamma^{NL}$ yields to a sum of different matrices representing the shape functions, their derivatives and transformations into the material system. The results will be used to evaluate the nonlinear contribution to the stiffness matrix. For detailed information the reader is referred to [1,7].

2.2 The Stiffness-Matrix

The derivation of the stiffness matrices is based on the weak form of the equilibrium equations:

$$\delta W_{ex} + \delta W_{in} = 0.$$  \hspace{1cm} (23)

For the total Lagrangian approach and the formulation with incremental quantities the weak form can be rewritten by using the stiffness matrix $D$ and the 2nd Piola-Kirchhoff stress $\sigma$:

$$\sum_{i=1}^{n_t} \int \delta (d\gamma^L)^T D (d\gamma^L) dV_0 + \sum_{i=1}^{n_t} \int \delta (d\gamma^{NL})^T \sigma dV_0 = $$

$$\int (d\mu)^T f_{ex} - \int \delta (d\gamma^L)^T \sigma dV_0,$$  \hspace{1cm} (24)

where the integration in thickness direction is decomposed into a number $n_t$ of subintegrations which corresponds to the number of layers in the element. For each layer the constitutive relation is set up in the local frame of coordinates with layer-dependent material parameters. By carrying out the integration two matrices $K_L$ and $K_{NL}$ appear. The components of the stiffness matrix can be written as:

$$K_L = \sum_{i=1}^{n_t} \int B_L^T D_i B_L \sqrt{\det G} d\xi d\eta d\zeta,$$

and

$$K_{NL} = \sum_{i=1}^{n_t} \int \omega^{\alpha\beta} (\Omega^T \Omega + \Omega^T \Omega + \Omega^T \Omega) + \omega^{33} \Omega^T \Omega + \omega^{22} (\Omega^T \Omega + \Omega^T \Omega)$$

$$+ \zeta \omega^{\alpha\beta} (\Omega^T \Omega + \Omega^T \Omega - C_{\alpha\beta} \Omega^{00} - \Omega^{00} C_{\alpha\beta})$$

$$- \zeta \omega^{33} (4w_3 \Omega^T \Omega + 2\Omega^T \Omega + 2\Omega^T \Omega)$$

$$+ \zeta \omega^{\alpha\beta} (\Omega^T \Omega + \Omega^T \Omega) \sqrt{\det G} d\xi d\eta d\zeta.$$  

The matrices $\Omega$ and $C_i$ and the factor $\omega^U$ are defined as:
\[ \Omega^\phi = \begin{bmatrix} \text{diag}(\Pi^\phi) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Omega^{\Pi^\phi} = [0, 0, 0, \Pi^{\Pi^\phi}] \] and \[ C_a = C_a^1 + C_a^2 \Omega^{\Pi^\phi}, \]

\[ \omega^\phi = E_a^1 E_b^1 t^1 t_1^1 s_1^1. \]

The final equation for each iteration \( j \):

\[(K_L + K_{NL})d\mathbf{u}^{j+1} = \mathbf{f}_a - \mathbf{f}_n,\]

will be solved for the change of the displacement \( d\mathbf{u}^{j+1} \) with:

\[ \Delta\mathbf{u}^{j+1} = \Delta\mathbf{u}^j + d\mathbf{u}^{j+1}. \] (25)

Therefore the internal degrees of freedom \( dw \) are condensed at element level which yields to the following element stiffness matrix:

\[ K^{j}_e = K^{j}_w - K^{j}_w K^{j-1}_w K^{j}_w, \] (26)

and the vector of external forces:

\[ \mathbf{f}^{j}_e = \mathbf{f}^{j}_a - (\mathbf{f}^{j}_n - K^{j}_w K^{j-1}_w \mathbf{f}^{j}_w). \] (27)

Figure 2. Geometry and finite element lay-out of a cylindrical shell

2.3 Numerical examples

The example concerns a layered shell roof subjected to a concentrated load. One quarter of a cylindrical shell structure (Fig. 2) is modelled with 9 thick shell elements. A concentrated force of 1.0 kN is applied at the center of the structure. An anisotropic configuration is analysed with 3 different stacking sequences: [90/0/90]_R, [0/90/0]_R and [+45/-45]_R. In the latter case the whole structure has been modelled. The structure is assumed to be composed of an anisotropic material for which the parameters are listed in Tab. 1. The straight boundaries are clamped in one case and hinged in the other. The curved boundaries remain free. In Fig. 3 the results for all clamped configurations are presented whereas Fig. 4 shows the results for the hinged cases. The calculations with the thick shell elements show a similar response as reported for the standard elements. Since results for other benchmark tests are also satisfactory [5-7], it is concluded that thick shell elements are sufficient to compute the behaviour of fibre metal laminates. For the calculations carried out in Reference [7] the number of iterations is approximately equal as reported in literature [6]. It is observed that the convergence behaviour depends on the ratio thickness to element length. For thin applications (t/L = 0.001) the number of iterations tends to increase slightly.
Table 1: Material parameters for the cylindrical shell

<table>
<thead>
<tr>
<th>$E_{11}$ [N/mm$^2$]</th>
<th>$E_{22}, E_{33}$ [N/mm$^2$]</th>
<th>$G_{ij}$ [N/mm$^2$]</th>
<th>$\nu_{12}, \nu_{23}$</th>
<th>$\nu_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.35e+03</td>
<td>1.0e+03</td>
<td>6.6e+02</td>
<td>0.25</td>
<td>0.45</td>
</tr>
</tbody>
</table>

Figure 3. Load-deflection curve of the clamped cylindrical shell roofs assuming geometric nonlinearity.

Figure 4. Load-deflection curves of the hinged cylindrical shell roofs assuming geometric nonlinearity.

3. PHYSICALLY NONLINEAR BEHAVIOUR

3.1 Formulation of the Plastic Relations

The advantage of this kind of element formulation is that standard 3D constitutive models can be applied. To account for the physical nonlinear behaviour the Hoffmann yield criterion and the von Mises yield criterion have been applied for anisotropic and isotropic plasticity, respectively. The yield function $\Phi(\sigma)$ equals:
\[ \Phi(\sigma) = \frac{1}{2} \sigma^T P_e \sigma + p_e \sigma, \]  
(28)
for the Hoffmann criterion [9], and:
\[ \Phi(\sigma) = \sqrt{\frac{3}{2}} \sigma^T P \sigma - \bar{\sigma}, \]  
(29)
for the von Mises yield criterion [8]. In eqns. (28) and (29) \( P_e, p_e \) and \( P \) are projection matrices and vectors, defined in Reference [8,9]. Assuming small strains the total strain rate can be decomposed into an elastic and a plastic part. By applying an associative flow rule the elastic and plastic components equal:
\[ \dot{\gamma}^{el} = D^{-1} \sigma, \]  
(30)
\[ \dot{\gamma}^{pl} = \dot{\lambda} \frac{\partial \Phi}{\partial \sigma}. \]  
(31)

The basic problem of computational plasticity is the integration of eqn. (30) and eqn. (31). Here a single-point integration rule has been adopted to integrate the plastic strain rate, which leads to:
\[ \Delta \gamma = \Delta \gamma^{el} + \Delta \gamma^{pl}, \]  
(32)
\[ \Delta \gamma^{el} = D^{-1} \Delta \sigma, \]  
(33)
\[ \Delta \gamma^{pl} = \Delta \dot{\lambda} \frac{\partial \Phi}{\partial \sigma} \bigg|_{t + \alpha \Delta t} \]  
(34)
with \( t \) and \( t + \alpha \Delta t \) denoting the beginning and the end of a loading step, respectively, and \( 0 \leq \alpha \leq 1 \). By combining eqn. (32) - (34) the stress increment can be derived as:
\[ \Delta \sigma = D \Delta \gamma - \Delta \dot{\lambda} D \frac{\partial \Phi}{\partial \sigma} \bigg|_{t + \alpha \Delta t}. \]  
(35)
The term \( D \Delta \gamma \) is the incremental stress which is used to calculate the trial stress \( \sigma_t = \sigma_0 + D \Delta \gamma \). If \( \sigma_t \) does not satisfy the yield condition \( \Phi(\sigma) \leq 0 \) a corrector is computed to return the stress to the yield surface. The resulting new stress equals:
\[ \sigma_n = \sigma_0 + \Delta \sigma = \sigma_t - \Delta \dot{\lambda} D \frac{\partial \Phi}{\partial \sigma} \bigg|_{t + \alpha \Delta t}. \]  
(36)
For a fully implicit Euler backward method the gradient is evaluated at the end of the load step (\( \alpha = 1 \)). For the von Mises yield criterion it can be expressed as:
\[ \frac{\partial \Phi}{\partial \sigma} = \frac{3}{2} \sqrt[2]{P} \sigma + \frac{3}{2} \sqrt[2]{P} \sigma, \]  
(37)
After substituting the gradient into eqn. (36) and some algebra we arrive at:
\[ \sigma_n = (I + \frac{3 \Delta \lambda}{2 \sigma} P^{-1} \sigma), \]  
(38)
Since the stress $\sigma$ has to satisfy eqn.(29) the yield function $\Phi(\sigma)$ can be formulated as function of $\Delta \lambda$. The equation $\Phi(\Delta \lambda) = 0$ is solved by using a local Newton-Raphson method:
\[
\Delta \lambda^{k+1} = \Delta \lambda^k - \left. \frac{\Phi}{d\Phi/d\Delta \lambda} \right|_{\Delta \lambda^k} ,
\]
with [8]:
\[
\frac{d\Phi}{d\Delta \lambda} = \frac{-9}{8\tilde{\sigma}[\Phi(\Delta \lambda) + \tilde{\sigma}]} \tilde{\sigma}^T \left[ PD(A^{-2})PA^{-1} + A^{-T}PA^{-2}DP \right] \sigma .
\]
In case of the Hoffmann yield criterion the gradient of the yield function is computed according:
\[
\frac{\partial \Phi}{\partial \sigma} = P_a \sigma + P_a ,
\]
which leads to a new stress $\sigma_n$:
\[
\sigma_n = \sigma_0 + \Delta \sigma = (D^{-1} + \Delta \lambda P_a)^{-1} (\gamma^d + \Delta \gamma - \Delta \lambda P_a) .
\]
Substituting eqn.(41) into eqn.(28) the yield function is again reformulated as function of $\Delta \lambda$. For Hoffmann plasticity the gradient $d\Phi/d\Delta \lambda$ equals [9]:
\[
\frac{d\Phi(\Delta \lambda)}{d\Delta \lambda} = -(P_a \sigma_n + P_a)(D^{-1} + \Delta \lambda P_a)^{-1} \times \left[ (D^{-1} + \Delta \lambda P_a)^{-1} P_a (\gamma^d + \Delta \gamma - \Delta \lambda P_a) + P_a \right] .
\]

### 3.2 Consistent Tangent Operator

In nonlinear finite element analysis a global iterative procedure has to be carried out for every load step. When using the Newton-Raphson method a tangent stiffness operator must be derived by consistent linearization of the incremental quantities as expressed by eqn.(33) and eqn.(34). The result reads:
\[
\dot{\gamma} = D^{-1} \dot{\sigma} + \Delta \lambda^j \frac{\partial \Phi}{\partial \sigma^2} \dot{\sigma}^j + \Delta \lambda \frac{\partial \Phi}{\partial \sigma} ,
\]
with the superscript $j$ referring to the iteration number. When finite steps are considered the second term of the right-hand side becomes an important contribution to the stiffness relation. After some manipulations and in combination with the consistency condition $\Phi = 0$ eqn.(43) can be rewritten as:
\[
\dot{\sigma} = \left[ H^{-1} \left( \frac{\partial \Phi}{\partial \sigma} \right) \left( \frac{\partial \Phi}{\partial \sigma} \right)^T \right] H^{-1} \dot{\gamma} = D_{\text{cons}} \dot{\gamma} ,
\]
with
\[ H^{-1} = D^{-1} + \Delta \lambda \frac{\partial^2 \Phi}{\partial \sigma^2} \]

For the Hoffmann plasticity this leads to [9]:

\[ D_{\text{con.Hoff}} = (D^{-1} + \Delta \lambda P_a)^{-1} \]

\[ = \frac{(D^{-1} + \Delta \lambda P_a)^{-1}(P_a \sigma + p_a)(P_a \sigma + p_a)^T(D^{-1} + \Delta \lambda P_a)^{-1}}{(P_a \sigma + p_a)^T(D^{-1} + \Delta \lambda P_a)^{-1}(P_a \sigma + p_a)} \]

and for von Mises plasticity [8]:

\[ H^{-1} = D^{-1} + \Delta \lambda \left( \frac{\sqrt{2}}{2} \right)^2 \frac{\sigma^T P \sigma P - P \sigma \sigma^T P}{(P \sigma)^{3/2}} \]

with the gradient \( (\partial \Phi / \partial \sigma) \) defined in eqn.(37).

### 3.3 Numerical examples

For verification of the implementation of the yield criteria in connection with the thick shell element the perforated strip proposed by Ramm [10] is used. The model consists of a 20mm × 36mm plate with a central hole (d=10mm). To model the structure a configuration with 132 thick shell elements has been used to model one quarter of the structure. The plate has been loaded with a force of 100N/mm² at the shorter side of the plate. This load equals an edge load of 100N/mm applied for a standard shell element with a thickness of 1.0 mm. In this example the plasticity has been assumed to be governed by the von Mises yield criterion whereas the structure behaves geometrically linear.

In the linear regime a Young-modulus of \( E_s = 70000 \text{N/mm}^2 \) has been chosen. In this case hardening has been assumed with \( E_s = 2240 \text{N/mm}^2 \) after reaching the yield strength \( \tau = 243 \text{N/mm}^2 \). For the derivation of the tangential stiffness matrix incorporating hardening the reader is referred to [12]. The force has been applied using an arc-length control procedure [11].

![Figure 5. Load-deflection curve for the perforated strip problem using the von Mises yield criterion.](image-url)
The results obtained agree well with the solution obtained by Ramm[10] and Schellekens[11], Fig. 5. To verify the Hoffmann yield criterion one quarter of the same structure has been modelled using 132 standard solid elements, standard shell elements or thick shell elements. T300-5208 epoxy has been chosen [11]. The results of the analysis is displayed in Fig. 6.

Figure 6. Perforated strip computed with different element types for the Hoffmann yield criterion.

3.4 Experimental Comparison

To investigate whether thick shell elements are suitable for computation of the material behaviour of fibre metal laminates a GLARE tensile test has been calculated. The test has been done at the Production and Materials Laboratory of the Faculty of Aeronautical Engineering of Delft University of Technology in the framework of a research project on fibre metal laminates. The geometry of the specimen and the material properties are given in [13,14]. The material is produced at a temperature of T=393 K which causes initial

Figure 7. Finite element simulation of a GLARE specimen in a tensile test.
stresses at the test temperature of $T=293$ K. This effect has been accounted for by deriving the load vector for temperature loads. During the calculations the material is assumed to react geometrically linear. The aluminium layer is governed by the von Mises yield criterion. In one case hardening has been assumed. In Fig. 7 it is clear that the simulation incorporating hardening agrees well with the experimental result.

4. CONCLUSION

It has been demonstrated that thick shell elements are well suited for the calculation of the behaviour of fibre metal laminates. The results agree well with those obtained in standard benchmark tests. Since the element represents a three-dimensional state of stress it differs from the behaviour of standard shell elements for physically nonlinear applications. In addition the comparison with experimental data leads to the conclusion that thick shell elements are well suited for practical applications.

REFERENCES


