BSc. Report Applied Mathematics and Applied Physics

”The Brachistochrone Problem: From Euler to Quantum”

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Preface

Since high school I had always been intrigued by how the great mathematicians used to work in their time. This sparked the idea to translate and research a handful of publications Euler wrote on the brachistochrone problem, involving friction, in his attempt to find a solution. After researching Euler's works, it only felt logical to compare his findings to more recent solutions on this subject, using more sophisticated and modern methods.
Having looked at the brachistochrones of the past and present, the only thing left to do was to find some kind of brachistochrone from the future. Conveniently, there happens to be a problem, which is an exact analogy to the classical brachistochrone, in quantum mechanics, one of the most promising fields of research. And so it happened: the so-called quantum brachistochrone was to be researched next, quickly followed by a closely related problem, risen from curiosity.

Not only are these subjects fun to examine for the sheer underlying mathematics and physics, but they're also relevant in our everyday lives. While the former topics have already been discussed and essentially solved, they are all the more important, since our financial world revolves around optimization, which is what the brachistochrone does by nature; it optimizes a path. Furthermore, they are important for various branches of physics and, as is clear from the bridge made to the latter topics, their general idea is relevant for newer physics as well. The quantum brachistochrone is paramount in the development of the Quantum Computers and the considered niche problem also finds its relevance there.

While both covered topics share the same encompassing theme, they are pretty different by nature. Because of this, it has been opted that they will be covered in different, self contained parts, resulting in a more natural layout of the thesis.

Part I

The Classical Brachistochrone

Abstract

In this part of the thesis, we look at the brachistochrone problem and in particular, at how none other than Leonhard Euler solved the brachistochrone with friction, through translating his original Latin texts, and comparing it to more modern ways of solving the brachistochrone problem. For the modern way of solving the brachistochrone problem, we derive the differential equations for the case where we consider friction and in vacuum. However, since they are not trivial to solve, we numerically solve the equations employing the shooting method in Python.

We look at four of Euler's texts and it is notable how Euler improves his methods between the first and the later three texts (which kind of belong together). In the first text, Euler's methods are a bit primitive and, of course calculus just being invented, the way he derived differential equations is a bit gimmicky. He however finds an analytical result, which reads

$$s = c \cdot \ln\left(\frac{s - ax - ac + c}{c - ac}\right)$$

where c and a are problem specific parameters. c can be determined and is a kind of friction coefficient, but a derives from the boundary conditions, though since there is no equation to find it with, we also have to use the shooting method here.

In Euler's later texts, he works more rigorously and derives a friction compatible, more generalized Euler-Lagrange equation, to aid him in finding the sought friction brachistochrone curve. Among many results, his most notable are the just mentioned Euler-Lagrange equation, dubbed the General Isoperimetric Theorem by Euler, and the solution to the friction brachistochrone problem:

$$x = \int \frac{vdv}{g - hv^{n+1}\sqrt{1+p^2}}; \;\; y = \int \frac{pvdv}{g - hv^{n+1}\sqrt{1+p^2}}$$

where h is some friction coefficient and g is the gravitational acceleration. Furthermore, p and v can be expressed in each other, such that these equations can be solved.

We choose to compare his first result to the modern solutions found with the shooting method, using the boundary conditions $y(0) = 3,\ y(15) = 0$, and we find that the act lengths are respectively 17.87, 17.73 and 17.88 meters, for the modern true brachistochrone, the modern brachistochrone with friction and Euler's solution, where we see that Euler's result with friction almost equals our result without friction, which is odd, but they are nonetheless close to each other anyway.

Introduction

Before considering the formulations, researched in this thesis, we look back to the year 1697, when none other than Johann Bernoulli derived the brachistochrone curve; given two points in space, A and B, where A is above B, but not directly above B, and only considering the force of gravity, without friction, the brachistochrone curve is the curve that leads from point A to point B in the least amount of time. Using Fermat's principle he concluded that this sought curve has to be part of a cycloid.

Fast forward a couple of years to 1726, when the famous Leonhard Euler proposes the same problem, but including friction this time. He received word from some Hermann, claiming he almost found the solution, but sadly he died before being able to share it with the world. Euler, not wanting that anyone else took credit for what Hermann invented, took the responsibility of finishing what Hermann could not and he succeeded. Dissatisfied with his results, Euler wrote another paper regarding the brachistochrone curves shortly after, to clarify more of the properties of said curves.

I will look at said papers, indexed at $E042$ and $E759 - E761$, and discuss his methods and findings, along with a comparison with more contemporary results, to see how well Euler did, despite not having access to numeric methods or more sophisticated calculus in his time.

The Brachistochrone Problem

Like the title of this thesis suggests, everything researched will revolve around the brachistochrone problem. Since it is so important for the remainder of this thesis, it is useful to give a little recap of what this problem consists of and what its solution is.
Like posed for the first time to Newton in 1696, the problem reads:

"Find the shape of the curve down which a bead sliding from rest and accelerated by gravity will slip (without friction) from one point to another in the least time."

First, a short and elegant way to find sought curve is given, after which we will look at the way Euler solved said problem.
Let us call the begin point A and the end point B. The total time needed to pass through the curve is given by:

$$T_{AB} = \int_A^B \frac{1}{v} ds \tag{1}$$

Since energy must be conserved, the following holds:

$$\frac{1}{2} m v^2 = m g y; \; v = \sqrt{2gy} \tag{2}$$

Using (2) and $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2} dx$, we can rewrite the time integral to

$$T_{AB} = \int_A^B \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} dx = \int_A^B \sqrt{\frac{1 + y'^2}{2gy}} dx \tag{3}$$

Now we need to use the Euler-Lagrange equation, where we recognize the functional $f = \sqrt{\frac{1+y'^2}{2gy}}$. Since f does not explicitly depend on x, we can use the Beltrami identity:

$$f - y' \frac{\partial f}{\partial y'} = C \tag{4}$$

$$\sqrt{\frac{1 + y'^2}{2gy}} - \frac{y'^2}{\sqrt{2gy}\sqrt{1 + y'^2}} = \frac{1}{\sqrt{2gy}\sqrt{1 + y'^2}} = C \tag{5}$$

$$\left(1 + y'^2\right) y = \frac{1}{2gC^2} = k^2 \tag{6}$$

which differential equation is solved by the set of parametric equations:

$$\begin{cases} x(\vartheta) = x_A + \frac{1}{2} k^2 \left(\vartheta - \sin(\vartheta)\right) \\ y(\vartheta) = y_A + \frac{1}{2} k^2 \left(1 - \cos(\vartheta)\right) \end{cases} \tag{7}$$

where ϑ ranges from 0 until some unknown constant and k can be determined from the boundary condition (the end point, B), which yields a set of two equations and two unknown parameters. This is actually simply the equation for a cycloid.
Now that we are equipped with the knowledge for solving the original brachistochrone problem, it is time to consider a generalisation.

The Brachistochrone with Friction

Our chosen generalization, is to consider friction. In general, dynamic friction, or drag, has the following form:

$$F_d = -\frac{1}{2}\rho C_d A_\perp v^2 \tag{8}$$

However, for low speeds, $C_d \propto \frac{1}{v}$ and, in that case, $F_d \propto v$. This consideration, where the friction is linear in velocity, has already been done[1], yielding, around the beginning of the curve

$$\frac{gx}{C} = \frac{1}{3y'^3} + \frac{\Lambda}{y'^4} + \dots \tag{9}$$

where g is the gravitational acceleration and C and Λ are constants, specific to the problem. C depends on the position of the end of the curve, while Λ, among other things, depends on the drag coefficient. This result can then be numerically integrated, to obtain a plot for $y(x)$, such that the sought curve is found.

We will here consider the case where $F_d \propto v^2$, that is, for intermediate to high velocities. Let $F_d = -kv^2$. Like in the original brachistochrone problem, equation 1 still holds. Equation 2, though, becomes

$$\frac{d}{dx}\left(\frac{1}{2}mv^2\right) = mv\frac{dv}{dx} = mg\frac{dy}{dx} - kv^2\frac{ds}{dx} \tag{10}$$

Equivalent to how light moves in the path of least time, also known as Fermat's Principle, and subsequently light obeys Snell's law, we're looking at how a body moves in the least time between two points and we can too evoke Snell's law:

$$\frac{v}{\sin\alpha} = \frac{v}{\cos(\vartheta)} = C,\ v = C\cos(\vartheta) = \frac{C}{\sqrt{1+y'^2}} \tag{11}$$

where ϑ is the angle between the curve and the horizontal axis.

[1] A.S. Parnovsky, Some Generalisations of Brachistochrone Problem[1]

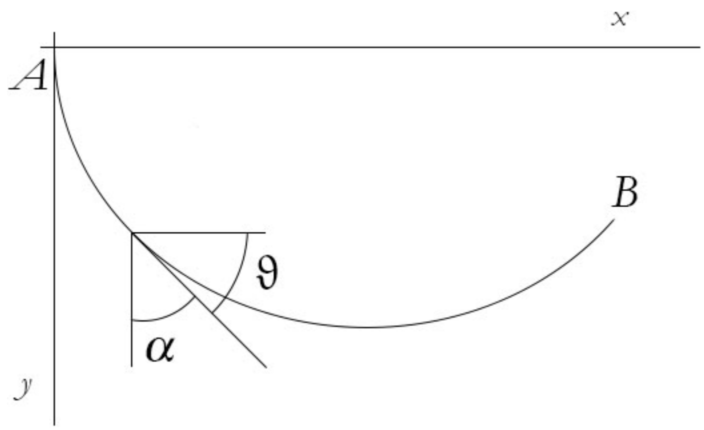


Figure 1: A sketch of the brachistochrone curve, to indicate the used angles.

Combining (10) and (11) yields

$$\frac{mC}{\sqrt{1+y'^2}} \cdot \frac{Cy' \cdot y''}{\sqrt{1+y'^2}\,(1+y'^2)^2} = mgy' - \frac{kC^2}{1+y'^2} \cdot \sqrt{1+y'^2} \tag{12}$$

which reduces to the differential equation

$$C^2 y' y'' = \left(gy' - \frac{kC^2}{m\sqrt{1+y'^2}} \right) \cdot \left(1+y'^2\right)^2 \tag{13}$$

This equation can be integrated numerically.

However, there exists an easier way to determine the sought curve:
Let us call $\tan(\vartheta) = \frac{dy}{dx}$, such that

$$\dot{x} = v\sin(\vartheta);\ \dot{y} = v\cos(\vartheta) \tag{14}$$

And, using the energybalance, equation 10:

$$mvdv = mgdy - kv^2 ds$$

We obtain

$$\dot{v} = \frac{mg\dot{y} - kv^3}{mv} = g\cos\vartheta - \frac{k}{m}v^2 \tag{15}$$

From which we can parametrize sought curve in the parameters v and ϑ. Of course, setting $k = 0$ yields a parametrization for the frictionless brachistochrone. The results will be covered later in the discussion.

Euler's Approach

The main interest in the classical half of this thesis is to look at how Euler derived his equations for the brachistochrone with friction and compare his findings to the more modern result considered in the previous section. Most of Eulers work has not yet been translated, whereas mostly his results are rather known and not necessarily the step by step way he reached them. In order to be able to assess Eulers work, an appendix with self made translations of four texts has been added at the end (Appendix F). These texts will be treated in chronological order. To convey the fashion in which Euler worked, I will stick to his notation.

Euler's First Consideration: The Brachistochrone Property

In the first translated text, indexed at $E042$, Euler describes how he originally proposed this problem, of the brachistochrone in a resistant medium, in 1726, because he found that "the problem requires not to be despised elegance and unique foresight to reach a solution". Someone named Hermann picked up said problem and got close to solving it, but he passed away, so Euler took the responsibility of finishing Hermann's work, so no one else could take the credit.

Euler did this by considering an infinitesimal interval, which he sketched like in Figure 2 [2], below. Let the speed above M be m and below M be n. If both curves LmN and LMN describe a descent in least amount of time, the times for crossing these curves has to be the same: $\frac{LM}{m} + \frac{MN}{n} = \frac{Lm}{m} + \frac{mN}{n}$ are both minimal. If we draw the arcs Mf and mg, this becomes $\frac{mf}{m} = \frac{Mg}{n}$; $\frac{mf}{Mg} = \frac{m}{n}$. The speed of descent depends on the cosine of the incline of the slope (this is Snell's Law, but Euler calls this Huygens' Lemma). However, since n is not constant for every possible curve (which would be the case in vacuum), this Lemma is not readily applicable in our case. We must do a little more work; let the velocity at L be q and the velocity at M be $q + dt$, but at m be $q + dt + dd\theta$ (dt is effectively dq and $dd\theta$ is effectively $ddq = d^2q$; I would also like to take this moment to clarify that Euler writes dd instead of d^2 in differentials), then

$$\frac{mf}{q} = \frac{Mg}{q + dt} + \frac{Mn \cdot dd\theta}{(q + dt)\,(q + dt + dd\theta)} \tag{16}$$

[2]This figure, like every other figure in the Euler section, is basically a cleaner self made copy of images in Euler's original papers[2]. The sources of the images I used are The Euler Archive[3] and a copy of *Leonhardi Euleri Commentationes Analyticae ad calculum variationum pertinentes[4]* from the University of Leiden.

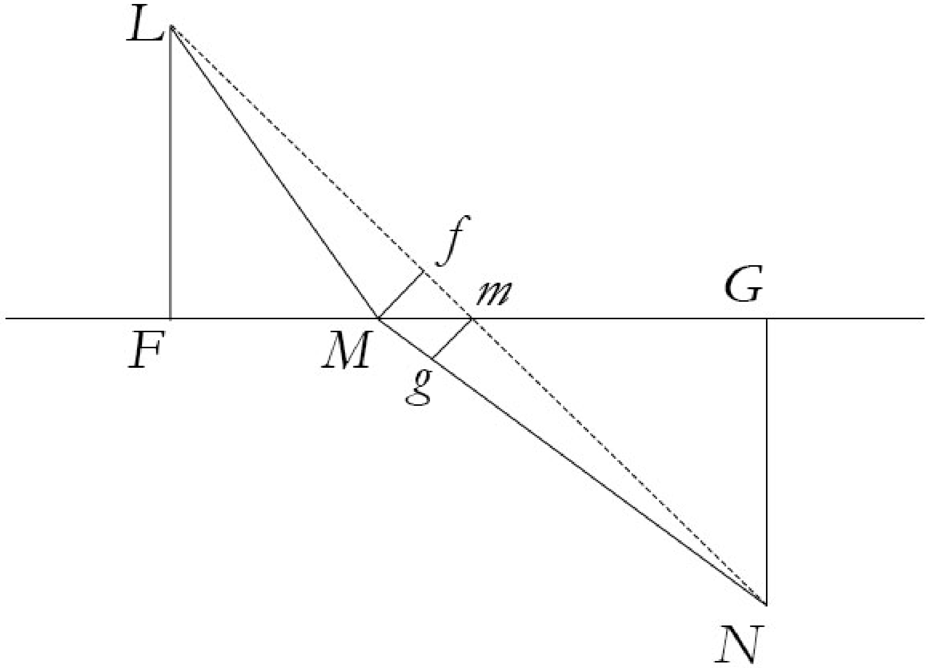


Figure 2: Eulers sketch of an infinitesimal interval, on which he will derive the differential equations for the friction brachistochrone problem.

This can be rewritten, using $mf = \frac{FM \cdot Mm}{LM}$, $Mg = \frac{MG \cdot Mm}{LM}$ and neglecting lower order terms:

$$q\left(\frac{MG}{LM} - \frac{FM}{LM}\right) = \frac{FM \cdot dt}{LM} - \frac{LM \cdot dd\theta}{Mm} \tag{17}$$

To rewrite this into a differential equation, we call $LF = NG = dx$, $FM = dy$, $LM = ds$, $MG = dy + ddy$, $MN = ds + dds$. Bear in mind, here, though, that Euler does not label his axes according to modern conventions; what we call x, he calls y and what we would call $-y$, he calls x. Equation 17 then becomes

$$q\frac{dsddy - dydds}{ds^2} = \frac{dydt}{ds} - \frac{dsdd\theta}{Mm};\ \frac{qdx^2ddy}{ds^3} = \frac{dydt}{ds} - \frac{dsdd\theta}{Mm} \tag{18}$$

where the second equality follows from

$$\sin(\angle mMg) = \frac{ddy}{dds} = \frac{ds}{dy};\ \ dsdds = dyddy \tag{19}$$

and taking dx to be constant, which Euler does all over his paper. Equation 18 then replaces the Huygenian Lemma.

Let us then call the disturbing force field be p, perpendicular to FG, calling the force of gravity unity (such that the force would effectively be $p \cdot m \cdot g$). Let the kinetic energy of whichever to be considered body be c, when the force of friction is equal to the force of gravity and let furthermore the force of friction then be $\frac{v^n}{c^n}$, where v is the kinetic energy, ommitting the prefactor $\frac{m}{2}$ and not the speed, where it is worth emphasizing that v is confusingly not, in fact, the speed. Then, at M, the growth in v will be $pdx - \frac{v^n}{c^n}LM$ and at m, the growth will be $pdx - \frac{v^n}{c^n}Lm$. Euler then concludes that $q = \sqrt{v}$ (the mass, m, has to be assumed 2 as a result, or equivalently, we work in units of $\frac{1}{2}m$) and

$$q + dt = \sqrt{v + pdx - \frac{v^n}{c^n}LM} = \sqrt{v} + \frac{pdx - \frac{v^n}{c^n}LM}{2\sqrt{v}} \tag{20}$$

where the last equality holds from a first order Taylor expansion around v. It follows that

$$dt = \frac{pdx - \frac{v^n}{c^n}LM}{2\sqrt{v}} \tag{21}$$

Lastly

$$q + dt + dd\theta = \sqrt{v} + \frac{pdx - \frac{v^n}{c^n}Lm}{2\sqrt{v}}; \tag{22}$$

From which we can also conclude that

$$dd\theta = \frac{v^n\left(LM - Lm\right)}{2c^n\sqrt{v}} = -\frac{v^n FM \cdot Mm}{2c^n LM\sqrt{v}};\ \frac{dd\theta}{Mm} = -\frac{v^n FM}{2c^n LM\sqrt{v}} \tag{23}$$

Substituting (21) and (23) and multiplying by $2\sqrt{v}$ in equation (18) yields

$$\frac{2vdx^2ddy}{ds^3} = \frac{pdxdy}{ds} - \frac{v^n dy}{c^n} + \frac{v^n dy}{c^n} \text{ or } 2vdxddy = pdyds^2 \tag{24}$$

Every brachistochrone must thus satisfy this property, which we will call the brachistochrone property:

$$v = \frac{pdyds^2}{2dxddy} \tag{25}$$

This property can however be induced in another way:

$$v = \frac{2vdxddy}{ds^3}\frac{pdy}{ds} \tag{26}$$

where $\frac{pdy}{ds} = N$ equals the normal component of the disturbing force field, p. One might also recognize $\frac{ds^3}{dxddy} = r$ as the radius of curvature, and $\frac{2v}{r}$ as the centrifugal force, such that ultimately the brachistochrone property reads:

$$\frac{2v}{r} = N \tag{27}$$

"Every brachistochrone curve posesses this property, where the force, perpendicular on the curve, equals the centrifugal force."

Euler visualizes this in Figure 3:

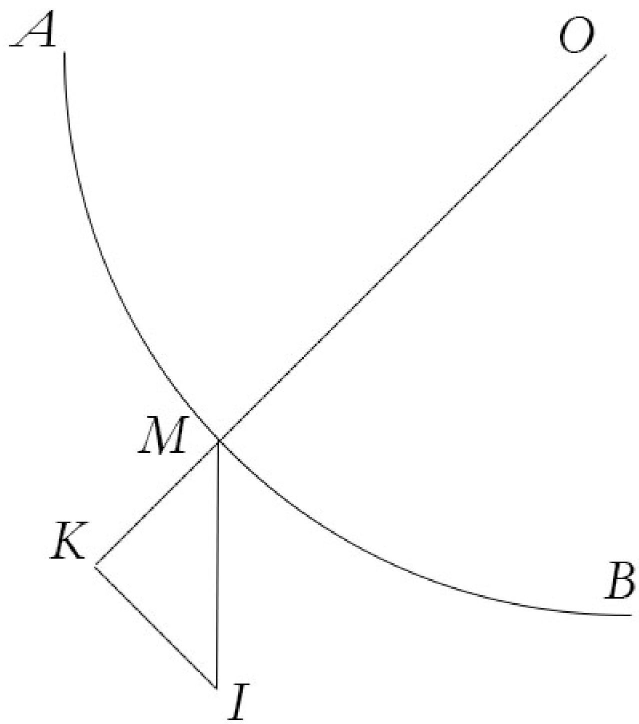


Figure 3: A visualization of the brachistochrone property.

Euler likes to emphasize that the force, denoted in this example by MI here, can be decomposed into two components, a perpendicular one, MK, and a tangential one, KI. MO denotes the radius of curvature and overlaps with the centrifugal force. Of course the centrifugal force and MK will then be equal again. The Huygenian lemma then reads $q \propto \frac{MK}{MI}$ and in this light, equation 27 becomes

$$\frac{MK^2}{MI^2 \cdot MO} \propto MK;\ q \propto \frac{MK}{MI} \propto MI \cdot MO \tag{28}$$

Let the curve always begin in A, from where whichever body descends from rest. Here, the tangent of the curve coincides with the force field. Because of Fermat's principle, $v = 0$ at this point. Armed with (27) and (28), we will look at several cases, of which the first two are simply without friction to illustrate how easily the solution can be found, after which we will consider friction too.

The Brachistochrone in Vacuum with Gravity.

Let us consider the original brachistochrone problem: the fastest descent, as a result of gravity.

Let, like sketched in Figure 4, the applied force field be g, in the same direction as PM, let AP be y, PM be x and AM be s, the brachistochrone curve. Let the sine of the angle that PM makes with the curve, φ, be $\frac{dy}{ds}$ and the radius of curvature is $\frac{ds^3}{dxddy}$. Since these are proportional to each other by (27) and (28), it holds that $\frac{ds^3}{dxddy} = \frac{ady}{ds}$, which can be rewritten to $ds^3 = adxdds$, using $ddy = \frac{dsdds}{dy}$.

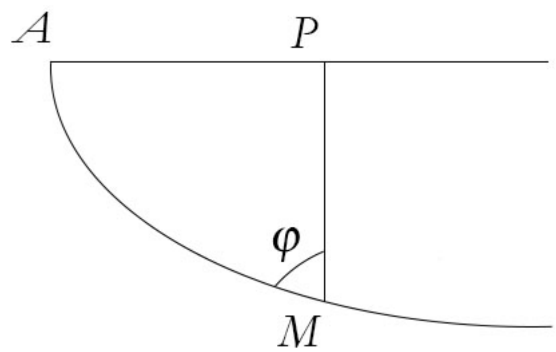


Figure 4: A sketch to illustrate the usefulness of equations (27) and (28)

This equation can be integrated (after division by ds^2) and its integral reads $s = C - \frac{adx}{ds}$. Because $\frac{ds}{dx}(x = 0) = 1$, C must equal a. Integrating once more yields

$$s^2 = 2a\,(s - x) \tag{29}$$

which is the equation for a cycloid, like derived in equations (7), except now formulated in a different way.

After this section, Euler opts to take a look at the vacuum brachistochrone around a center of forces. Albeit interesting, this section strays so much from the general scope of what we are interested in, that I choose not to cover it here; however, it is covered in Appendix A.

Euler's Friction Brachistochrone Solution

Having observed the use of our found brachistochrone property, let us now finally consider the true brachistochrone with friction, in a resistant medium.

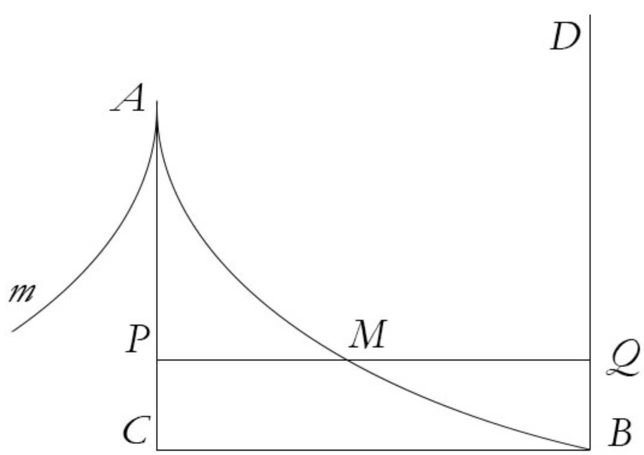


Figure 5: A sketch of the brachistochrone in a resistant medium.

Let the friction, working on whichever considered body, possess the form $\frac{v^n}{c^n}$ for whichever n or c, which depend on the specific problem, like assumed before, and let the force field be g, pointing

downwards. Let lastly AMB be the brachistochrone curve and $AP = x$, $PM = y$, $AM = s$. Then, by conservation of energy $dv = gdx - \frac{v^n ds}{c^n}$. Using the brachistochrone property, we can obtain a differential equation for the friction brachistochrone:
since $v = \frac{gdyds^2}{2dxddy}$ and using $ddy = \frac{dsdds}{dy}$,

$$dv = \frac{gdy^2dds^2 + 2gds^2dds^2 - gdsdy^2d^3s}{2dxdds^2}$$

From which we can eliminate dv:

$$\frac{gdsdy^2d^3s - 3gdy^2dds^2}{2dxdds^2} = \frac{g^n nds^{n+1}dy^{2n}}{2^n c^n dx^n dds^n} \tag{30}$$

For $c \to \infty$, this equation becomes $dsd^3s = 3dds^2$, of which the integral is $adxdds = ds^3$, which is the differential equation for a cycloid, like we saw just before (29).
In order to construct a general equation, let us pose $p = \frac{ds}{dx}$; $dp = \frac{dds}{dx}$; $ddp = \frac{d^3s}{dx}$; etc. and then $dy = dx\sqrt{p^2 - 1}$. v becomes $\frac{gpdx\left(p^2-1\right)}{2dp}$ and (30) becomes

$$pddp - 3dp^2 = \frac{g^{n-1}p^{n+1}dx^n \left(p^2 - 1\right)^{n-1}}{2^{n-1}c^n dp^{n-2}} \tag{31}$$

Let further also $q = \frac{dp}{dx}$ and it follows that $ddp = \frac{-dpdq}{q}$ and equation (31) yet again transforms, after multiplication by np^{-3n-1}:

$$-\frac{np^{-3n}dq - 3np^{-3n-1}qdp}{q^{n-1}} = \frac{ng^{n-1}p^{-2n}\left(p^2 - 1\right)^{n-1}dp}{2^{n-1}c^n} \tag{32}$$

Integrating which, yields:

$$P^{-n} = \frac{1}{p^{3n}q^n} = \frac{ng^{n-1}}{2^{n-1}c^n}\int \frac{\left(p^2 - 1\right)^{n-1}dp}{p^{2n}} \tag{33}$$

It then also holds that $dx = \frac{Pdp}{q^3}$. If we take $n = 1$, such that the friction depends on the squared velocity, we can find our brachistochrone curve with the following integrals:

$$x = \int \frac{Pdp}{p^3}, \quad s = \int \frac{Pdp}{p^2}, \quad y = \frac{Pdp\sqrt{p^2 - 1}}{p^3} \tag{34}$$

Armed with this knowledge we can begin solving for s. Let for example $n = 1$ (such that the friction is proportional to q^2, or to the square of the velocity). We find

$$P^{-1} = \frac{1}{c}\int \frac{dp}{p^2} = \frac{1}{ac} - \frac{1}{cp} = \frac{p - a}{acp}$$

For some constant a, which depends on the problem. While we don't know this a beforehand, we can still numerically find it, as will be done in a later section. Subsequently

$$dx = \frac{acdp}{p^2\left(p - a\right)}; \quad x = b + \frac{c}{p} + \frac{c}{a}l\left(\frac{p - a}{p}\right)$$

where l denotes the natural logarithm (as Euler denoted it).
Because $\frac{ds}{dx}(x=0)=1$, we find $b=-c\left(1+\frac{1}{a}l\left(1-a\right)\right)$ and as a result:

$$x=\frac{c\left(dx-ds\right)}{ds}+\frac{c}{a}l\left(\frac{ds-adx}{ds-ads}\right) \tag{35}$$

This can be conveniently rewritten into

$$acdxdds=ds^3-adxds^2$$

which can be integrated twice to yield

$$s-ax=ac\left(1-\frac{dx}{ds}\right);\;\; s=c\cdot l\left(\frac{s-ax-ac+c}{c-ac}\right) \tag{36}$$

Or

$$e^{\frac{s}{c}}\left(c-ac\right)=s-ax+c-ac$$

The lowest point, or like Euler calls it, the infimum point, of this curve, B, will then be located at $s=a\left(x+c\right)$, where

$$AB=c\cdot l\left(\frac{1}{1-a}\right);\text{ and } AC=\frac{c}{a}\cdot l\left(\frac{1}{1-a}\right)-c \tag{37}$$

We can however rewrite $e^{\frac{s}{c}}$ into a series to obtain:

$$\left(c-ac\right)\left(1+\frac{s}{1!\cdot c}+\frac{s^2}{2!\cdot c^2}+\frac{s^3}{3!\cdot c^3}+\frac{s^4}{4!\cdot c^4}+etc.\right)=s-ax+c-ac$$

which, if $k=\frac{1-a}{a}$, can be rewritten to

$$x=s-\frac{ks^2}{2!\cdot c}-\frac{ks^3}{3!\cdot c^2}-\frac{ks^4}{4!\cdot c^3}-etc. \tag{38}$$

whence we find that $k>0$, because otherwise $x>s$, which can't happen, such that $a=\frac{1}{1+k}$.
Euler also assesses what would happen, were we to use Huygens' Lemma, and what would happen if the curve AC were extended (such that it continued beyond C and before A). However, since this has little to nothing to do with the actual brachistochrone solution, it is not really interesting to discuss here.

Euler's Second Consideration

A while after writing about the friction (and frictionless) brachistochrone for the first time, Euler found that he researched this topic before in various books of his, but his formulae were always too generalized, such that nothing of interest could be learnt from them. Because of this, Euler considered the problem again, but from the "first principles". This chapter will follow what Euler did, according to the translated texts, indexed at $E759$, $E760$ and $E761$, which functionally form a whole.

A Look at the Frictionless Brachistochrone

We're back to considering the frictionless problem, in vacuum, because Euler discovered that there are easy cases that can't be simply solved with added friction. He only considers forces that he calls "absolute"; the force only depends on the place of the moving particle (and, for instance, not on its speed). He divides his dissertation in two parts: 2-dimensional and 3-dimensional.

The Brachistochrone in 2 Dimensions

Let us then first consider only the curves that are only concerned with two dimensions;

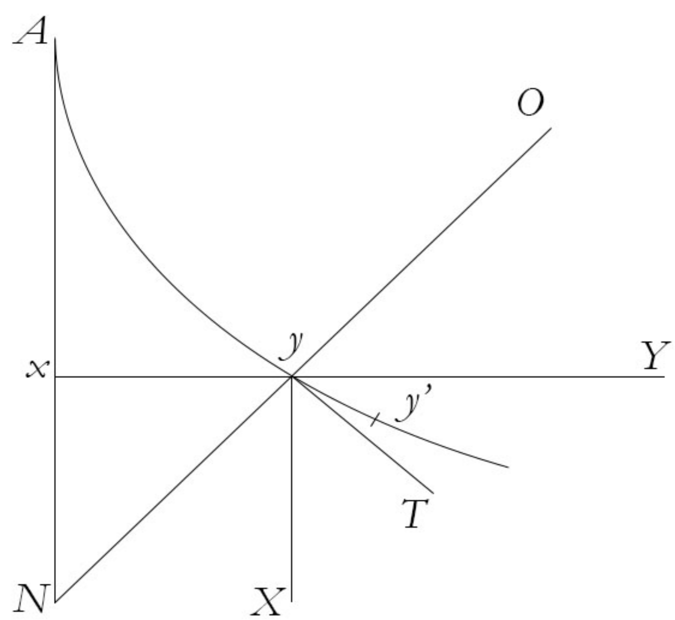


Figure 6: A sketch of the 2-dimensional vacuum brachistochrone, with the disturbing forces.

Let Ay be the brachistochrone curve, originating from A, and let Ax be simply denoted as x and xy as y. If we suppose y' to be really close to y, which is hard to accomplish in a sketch, let us then call $yy' = ds$. Let from now on, unlike before, $p = \frac{dy}{dx}$; $ds = dx\sqrt{1+p^2}$, and subsequently, the radius of curvature, yO, will, like in equation 26, equal $\frac{dx\left(1+p^2\right)^{\frac{3}{2}}}{dp}$.
Having posed the force of gravity unity (i.e. already having divided the forces by the force of gravity), the body is disturbed by some force (or some forces), which can always be decomposed into two perpendicular components $yX = X$ and $yY = Y$ (or $yN = N$ and $yT = T$). N and T can be expressed in the former two;

$$N = \frac{Xdy - Ydx}{ds}; \;\; T = \frac{Xdx + Ydy}{ds}$$

Let furthermore v be the velocity that the body possesses over the segment ds. Euler calls g "the altitude through which a mass falls in the first second" (which corresponds to $\frac{1}{2}g'$, if g' is the acceleration of gravity as we modernly use it) and he uses conservation of energy $vdv = 2gTds$. Integrating this yields

$$v^2 = 4g\int(Xdx + Ydy) \tag{39}$$

We had already assumed that X and Y are only a function of x and y and as such, so is v.
Having established this, let us look back at (1) again. This equation now becomes

$$\int\frac{ds}{v} = \int\frac{dx\sqrt{1+p^2}}{v}$$

which of course has to be minimal for the sought curve.
Euler now evokes the findings in his isoperimetric treatment, some earlier work of his, which he will use a lot throughout this treatment of the brachistochrone curve:

"If whichever integral $\int Vdx$ should either find its minimum or maximum, where V not only depends on x and y, but also on

$$p = \frac{dy}{dx}, \;\; q = \frac{dp}{dx}, \;\; r = \frac{dq}{dx}, \;\; s = \frac{dr}{dx}, \;\; etc.$$

such that

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + etc. \tag{40}$$

Then, the maximum of this equation will take place when

$$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + etc. = 0 \tag{41}$$

which is, mind you, pretty much a generalized Euler-Lagrange equation, which we had already used in the derivation of the vacuum brachistochrone post-Euler, which holds true whenever the integrand is only a function of x, y, and the derivatives of y.

Using Eulers isoperimetric theorem and $V = \frac{\sqrt{1+p^2}}{v}$, we find

$$dV = -\frac{dv}{v^2}\sqrt{1+p^2} + \frac{pdp}{v\sqrt{1+p^2}} = -\frac{2g\left(Xdx+Ydy\right)}{v^3}\sqrt{1+p^2} + \frac{pdp}{v\sqrt{1+p^2}}$$

As such we can conclude that

$$M = -\frac{2gX\sqrt{1+p^2}}{v^3}; \ \ N = -\frac{2gY\sqrt{1+p^2}}{v^3}; \ \ P = -\frac{p}{v\sqrt{1+p^2}}$$

where $Q, \ R, \ S, \ etc.$ are all 0. From this we will simply find this brachistochrone equation:

$$N - \frac{dP}{dx} = 0 \tag{42}$$

Rewriting

$$dP = -\frac{dv}{v^2}\cdot\frac{p}{\sqrt{1+p^2}} + \frac{1}{v}d\left(\frac{p}{\sqrt{1+p^2}}\right)$$

Yields, substituting for dv and using our known value for N:

$$\frac{1}{v}d\left(\frac{p}{\sqrt{1+p^2}}\right) = \frac{2g}{v^3\sqrt{1+p^2}}\left(Xdy - Ydx\right) \tag{43}$$

which is the to be solved brachistochrone differential equation.
Euler substitutes

$$\Theta = \frac{Xdy - Ydx}{dx\sqrt{1+p^2}} = \frac{v^2}{2gdx}d\left(\frac{p}{\sqrt{1+p^2}}\right) \tag{44}$$

(which is basically N, the normal force, with another given name) and, since $d\left(\frac{p}{\sqrt{1+p^2}}\right) = \frac{dp}{(1+p^2)^{\frac{3}{2}}}$, (43) becomes

$$\Theta = \frac{v^2}{2gdx}\frac{dp}{(1+p^2)^{\frac{3}{2}}} \tag{45}$$

which, since the radius of curvature was established to be $\frac{dx\left(1+p^2\right)^{\frac{3}{2}}}{dp}$ effectively becomes equation 27, Euler's first significant finding:

$$\Theta = \frac{v^2}{2gr} \tag{46}$$

Having posed r the radius of curvature.

"The normal force equals the centrifugal force"

which is exactly what Euler had already found before.

After this, Euler continues to generalize to three dimensions. However, since this treatment is much like what has been conducted in the foregoing section, it is superfluous to cover it here, but to not neglect this big amount of work Euler has done, this section finds its place in Appendix B.
His results consist of equations B.4, which, like they should, exactly reduce to equation 44.

A Second Look at the Friction Brachistochrone

Euler had already taken a look at this problem a while before, but found his findings to be lackluster and way too general, so that they barely gave insight into the problem. Now Euler wants to take another crack at it.

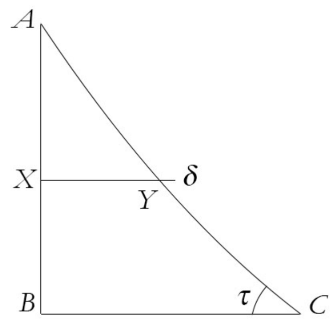


Figure 7: A sketch of a segment of the friction brachistochrone curve with an eventual variation, $Y\delta$, which is used to derive the Generalized Isoperimetric Theorem, and a final angle, τ.

Let AYC then be the brachistochrone curve and let $AX = x$, $XY = y$ and $AY = S$. Let furthermore the speed at Y be v, for which conservation of energy dictates $vdv = gdx - hv^{n+1}ds$ and of course $\int \frac{ds}{v}$ has to obtain a minimum for AYC. If h were 0, which corresponds to vacuum, $v^2 = 2gx$ and every segment of the curve has to be passed in minimal time. However, since v also depends on s, the segments of the curve do not have to be minimal any more and we cannot simply use the isoperimetric theorem.

To remedy this, Euler goes to great lengths to generalize his isoperimetric theorem (which is, again, basically a generalized Euler-Lagrange equation) even more, so that it is readily compatible with the new dependency of the velocity. The derivation itself of this more generalized theorem, does not add anything to the understanding of the solution of our friction brachistochrone, however it is added in Appendix C. The new theorem now reads (using fraktures, a kind of script font, which Euler likes to use, when other alphabets deplete):

Between all curves AYC with begin A and end C, if the integral $\int W dx$ has to be a maximum or minimum, where W depends on x, y, its derivatives, $p = \frac{dy}{dx}$, $q = \frac{dp}{dx}$, $r = \frac{dq}{dx}$, etc. and v in this way:

$$dW = Ldv + Mdx + Ndy + Pdp + etc. \tag{47}$$

Then, having posed $dv = \mathfrak{W}$, v is given by:

$$d\mathfrak{W} = \mathfrak{L}dv + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + etc. \tag{48}$$

Having posed $\Lambda = e^{\int \mathfrak{L}dx''}$ and $\Pi = \int L\Lambda dx$ such that $\Pi(C) = 0$. Let then

$$N' = N - \frac{\Pi\mathfrak{N}}{\lambda};\ P' = P - \frac{\Pi\mathfrak{P}}{\Lambda};\ Q' = Q - \frac{\Pi\mathfrak{Q}}{\Lambda};\ etc. \tag{49}$$

Then the sought curve satisfies

$$N' - \frac{dP'}{dx} + \frac{d^2Q'}{dx^2} - \frac{d^3R'}{dx^3} + etc. = 0 \tag{50}$$

Applying this theorem to our situation, let A be where $x = 0$ and $y = 0$ and let C be where $x = a$ and $y = b$. We of course want to minimize $\int \frac{dx\sqrt{1+p^2}}{v}$ and as such $W = \frac{\sqrt{1+p^2}}{v}$, so that $L = -\frac{\sqrt{1+p^2}}{v}$, $M = N = 0$ and $P = \frac{p}{v\sqrt{1+p^2}}$. Since $dv = \frac{gdx - hv^{n+1}dx\sqrt{1+p^2}}{v}$, $\mathfrak{W} = \frac{g}{v} - hv^n\sqrt{1+p^2}$, whence $\mathfrak{L} = -\frac{g}{v^2} - nhv^{n-1}\sqrt{1+p^2}$, $\mathfrak{M} = \mathfrak{N} = 0$ and lastly $\mathfrak{P} = -\frac{hv^n p}{\sqrt{1+p^2}}$. N' will then of course be 0, but P' is not, such that $\frac{dP'}{dx} = 0$ is our to be solved differential equation, of which the integral is $P' = K$ (for some constant K). After substituting P' and rewriting, we obtain:

$$\Pi = \frac{K\Lambda v\sqrt{1+p^2} - \Lambda p}{hpv^{n+1}} \tag{51}$$

Euler calls

$$\Theta = \frac{K}{v^n}\frac{\sqrt{1+p^2}}{p} - \frac{1}{v^{n+1}}$$

such that

$$d\Pi = L\Lambda dx = \frac{\Theta\Lambda\mathfrak{L}dx}{h} + \frac{\Lambda d\Theta}{h} \text{ or } hLdx = \Theta\mathfrak{L}dx + d\Theta \tag{52}$$

We can rewrite

$$d\Theta = -\frac{nKdv}{v^{n+1}}\frac{\sqrt{1+p^2}}{p} + \frac{K}{v^n}d\left(\frac{\sqrt{1+p^2}}{p}\right) + \frac{(n+1)\,dv}{v^{n+2}}$$

And substitute this, together with $\mathfrak{L}$ and $dx = \frac{vdv}{g - hv^{n+1}\sqrt{1+p^2}}$ into the right equation of (52) which does frankly not add any insight to the problem. However, then p is expressed in v and since $\sqrt{1+p^2}$ is thus also a function of v, so will be x and pretty much everything else, like y, Λ and Π, using the boundary condition $v(x=0) = 0$ and $v(x=a) = K$.

Euler mentions that the superfluously expanded equation, using the abovementioned substitutions does not yield any knowledge of the nature of brachistochrone curves, but using a lot of calculus can reduce the equation to a much simpler one with only four terms:

$$\frac{(n+2)\,dv}{v^2} - \frac{(n+1)\,Kdv\sqrt{1+p^2}}{pv} + K\left(1 - \frac{h}{g}v^{n+1}\sqrt{1+p^2}\right)d\left(\frac{\sqrt{1+p^2}}{p}\right) = 0 \tag{53}$$

which can be reduced even more; let us first call $c = \frac{1}{K}$ and $t = \frac{\sqrt{1+p^2}}{p}$, such that

$$\frac{(n+2)\,cdv}{v^2} - \frac{(n+1)\,tdv}{v} + dt - \frac{h}{g}\frac{v^{n+1}tdt}{\sqrt{t^2-1}} = 0$$

After dividing by v^{n+1} and integrating, the equation assumes this form

$$\frac{t}{v^{n+1}} - \frac{c}{v^{n+2}} - \frac{h}{g}\sqrt{t^2-1} = \Delta$$

For some Δ, depending on the final point C, or, substituting back and again multiplying by v^{n+1}

$$\frac{\sqrt{1+p^2}}{p} - \frac{c}{v} - \frac{h}{g}\frac{v^{n+1}}{p} = \Delta v^{n+1} \tag{54}$$

whence we see that p is defined by v. Since $P' = K$ and $\Pi(C) = 0$, it must hold that $P(C) = \frac{p}{v\sqrt{1+p^2}} = K = \frac{1}{c}$. As a result $c = \frac{v\sqrt{1+\theta^2}}{\theta}$, where θ is $\tan(\tau)$, the tangent of the angle of the incline in C, then $\Delta = -\frac{h}{g\theta}$ and subsequently (54) transforms into

$$\begin{gathered}\frac{\sqrt{1+p^2}}{p} - \frac{c}{v} + \frac{h}{g}v^{n+1}\left(\frac{1}{\theta} - \frac{1}{p}\right) = 0 \\ \text{Or } \sqrt{1+p^2} - \frac{cp}{v} + \frac{h}{g}v^{n+1}\left(\frac{p}{\theta} - 1\right) = 0\end{gathered} \tag{55}$$

where Euler notes that this, that is with the incline of the slope, is a more natural way of expressing a curve, than by its coördinates in space.
Now we have obtained a manageable expression for p in terms of v. From this we can find the coördinates:

$$x = \int \frac{vdv}{g - hv^{n+1}\sqrt{1+p^2}}; \;\; y = \int \frac{pvdv}{g - hv^{n+1}\sqrt{1+p^2}} \tag{56}$$

These equations are then basically a better version of equations 34; the solutions in his previous text.

Having done all this work in the most general situations, we can now look at some examples for different values of h and n.

Example where $h = 0$

Equation 55 simply becomes

$$\sqrt{1+p^2} - \frac{cp}{v} = 0$$

whence $p = \frac{v}{\sqrt{c^2-v^2}}$ and thence we can again elicit an expression for the coördinates:

$$x = \int \frac{vdv}{g}; \;\; y = \int \frac{v^2dv}{g\sqrt{c^2-v^2}} \tag{57}$$

x equals $\frac{v^2}{2g}$, like we're used to and $v = \sqrt{2gx}$, such that

$$y = \int \frac{dx\sqrt{2gx}}{\sqrt{c^2-2gx}} \tag{58}$$

Which is, like it should be, the equation for a cycloid.

Example where $n = -1$

In this case equation 55 reads

$$\sqrt{1+p^2} - \frac{cp}{v} + \frac{h}{g}\left(\frac{p}{\theta} - 1\right) = 0$$

whence

$$v = \frac{cp}{\sqrt{1+p^2} + \frac{h}{g}\left(\frac{p}{\theta} - 1\right)}$$

such that $v(C) = \frac{c\theta}{\sqrt{1+\theta^2}}$, using which we find

$$x = \int \frac{vdv}{g - h\sqrt{1+p^2}};\;\; y = \int \frac{pvdv}{g - h\sqrt{1+p^2}}$$

where v can be substituted for a function of p, however this makes the equations unnecessarily complicated, or like Euler likes to say: it will be superfluous to apply this operation, which I agree with.

Euler's Conclusion

Euler concludes that this problem is not only restricted to friction depending on v to some power, but whatever function V of the velocity. The energy balance dictates

$$vdv = gdx - hVdx\sqrt{1+p^2}$$

And therefore p and v are related as

$$\sqrt{1+p^2} - \frac{cp}{v} + \frac{h}{g}V\left(\frac{p}{\theta} - 1\right) = 0$$

And since

$$dx = \frac{vdv}{g - hV\sqrt{1+p^2}};\;\; dy = \frac{pvdv}{g - hV\sqrt{1+p^2}}$$

Everything else can be determined in the same way as before.

Like in his previous text, $E042$, Euler also again considers the brachistochrone around a center of forces, which is basically the entirety of text $E761$ (albeit now with friction, unlike before). Again, this topic strays a bit too far from the scope of this thesis, but not to neglect Euler's precious work, it will be covered in Appendix D.

Our and Euler's Results

Having discussed how we would tackle the problem in a more contemporary fashion, opposed to Euler's tedious dissertations, it is already obvious which of the two would beat the other in terms of time efficiency. Conversely, for the friction brachistochrone, modern methods do not actually yield a closed form solution, where Euler obviously does a good job in his first text, albeit having made some approximations. In order to assess how close Euler actually got, we first need to devise a plan to quantitatively express both our and his results. We will only compare his result in the first text, but, in the next section, we will discuss what he does in his later texts too.

Like mentioned in the associated sections, the brachistochrone curve (both with and without friction) can be integrated numerically. However, since said curves contain singularities, numerical integration is bound to run into issues, while it is most certainly possible, but it is rather impractical. Besides, the problem contains a constant, C, which can't be determined by whichever equation. Rather, it originates from the boundary conditions (the begin and end points). Since the conditions are not specified on the same boundary, but rather on two different boundaries, we would need to employ the shooting method. We take one of the two boundaries and its respective condition. Subsequently we vary the derivative there and each time integrate the curve numerically, until a curve, which satisfies the condition on the other boundary, is found. To determine C, we also need to shoot, since C can not be determined otherise. This means two dimensional shooting is necessary, which is doable, but harder than needed.

We have also briefly mentioned a second method to solve the contemporary equations; a parametrization with parameters v and ϑ, like described in Figure 1, using equations 14 and 15. Conveniently, the code for plotting this parametrisation has already been written[5] in Python. This is the script that will be used to find the contemporary brachistochrone curves.

Subsequently, Euler simply hands us a clear cut way to calculate the brachistochrone curve, except that he gives $s(s, y)$, through equation 36, instead of $y(x)$. Now, s can be explicitly written using the Lambert W-function, defined as $s = W\left(se^s\right)$. However, like Euler mentioned in equation 38, $y(s)$ can easily be written explicitly in terms of elementary functions. Finding a curve in the form $y(x)$ is really challenging then, since it involves taking the derivative of s with respect to y (also known as p), after which $dx = dy\sqrt{p^2 - 1}$ has to be integrated which is no fun either. We can then simply numerically define our brachistochrone curve.

Taking s in an interval, large enough to pass the end point, and discretizing it in small segments, we can express $y(s)$ on all of those points. Numerically integrating $dx = dy\sqrt{p^2 - 1}$ then gives all the desired x-values, since we can calculate p and dy.

As a result we now both possess ways to find the solutions for both the modern brachistochrone problem and Euler's result. Let us then invent an example to illustrate how these methods fare against each other:

Let $F_g = m \cdot g$ be the force of gravity, like we use it modernly (not like Euler does, with mixed up units), working on a body (a ball for simplicity) with mass $1kg$, to make the force of gravity $9.81N$. Let this ball have no begin speed and no begin energy. Keep in mind that Euler, in his first text, calls $\frac{1}{2}mv^2$ v. The force of friction is, according to Euler $\frac{v^n}{c^n}$, in units of the force of gravity, where c is his kinetic energy when friction equals the force of gravity. I from now on just call his friction force $\frac{v^{2n}}{c^{2n}}$ in order to keep the notation clear; $F_f = -\frac{1}{2}\rho v^2 C_D A_\perp$, where ρ is the density of the medium the body is moving in, C_D is the drag coefficient and $A_\perp$ is the perpendicular area (in this situation, just the area of a circle). To find c, we simply need to equal $F_g + F_f = 0$.

$$c^2 = \frac{2mg}{\rho C_D A_\perp}; \;\; c = \sqrt{\frac{2mg}{\rho C_D A_\perp}} \tag{59}$$

Taking the example a bit further, let $\rho = 1.225\frac{kg}{m^3}$ (such that the body moves through air) and $A = \frac{\pi}{100}$ (the ball has a radius of $10cm$). C_D is a bit harder to arbitrarily choose, because it depends on the velocity. For $Re > 3.5 \cdot 10^5$, we can take $C_D \approx 0.2$, which yields $c = 50.5\frac{m}{s}$, from which we can calculate $Re = \frac{1.225 \cdot 50.5 \cdot 0.2}{1.846 \cdot ^{-5}} \approx 7 \cdot 10^5$, such that our guess is warranted. Euler's c is our c^2, which equals 2549 (which is huge, but we need a huge speed to counteract the tiny friction coefficient and as a result the kinetic energy is even higher). Now, all that remains is finding a. The most straight forward way to accomplish this, is to employ the shooting method; just let a vary from 0 to 1 and it is bound to happen that there is an a, such that the boundary conditions are satisfied.

Let henceforth the boundaries for our example be $y(0) = 3$ and $y(15) = 0$, chosen in this way to ensure that the curve is wide enough, such that it dips below zero for a while, to rise again, which makes it more interesting (otherwise, if the end point were closer to the origin, the solution might never go below zero). Let $F_f = -k \cdot v^2$, with k being the friction coefficient. It then turns out that approximately $k = 0.00384845$. Using the mentioned method for finding a gives Figure 8:

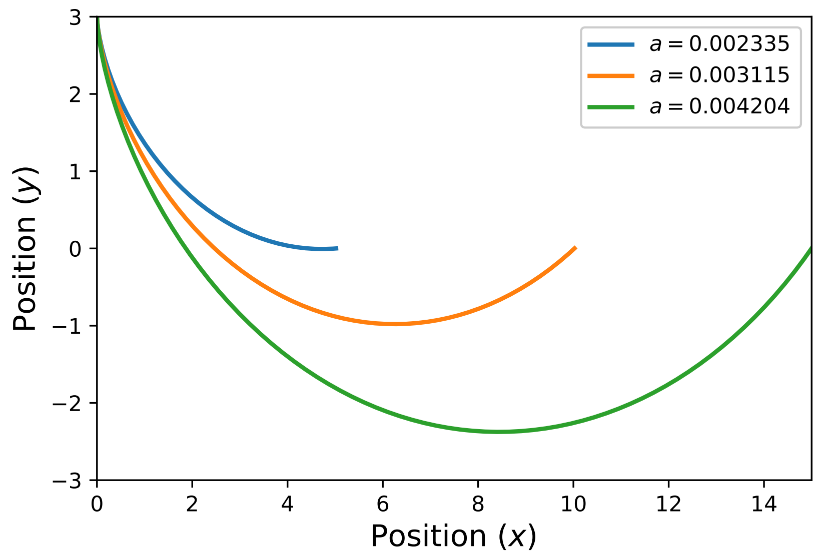


Figure 8: Three different curves, for different values of a. The respective curves intersect the x-axis in $x = 5$, $x = 10$ and $x = 15$.

The found value for a is $a = 0.004202$. For values of a lower than 0 and higher than 1, the curve does not decrease, such that physically, a always has to be between 0 and 1, like Euler mentioned. Comparing this with our findings for the contemporary methods results in Figure 9:

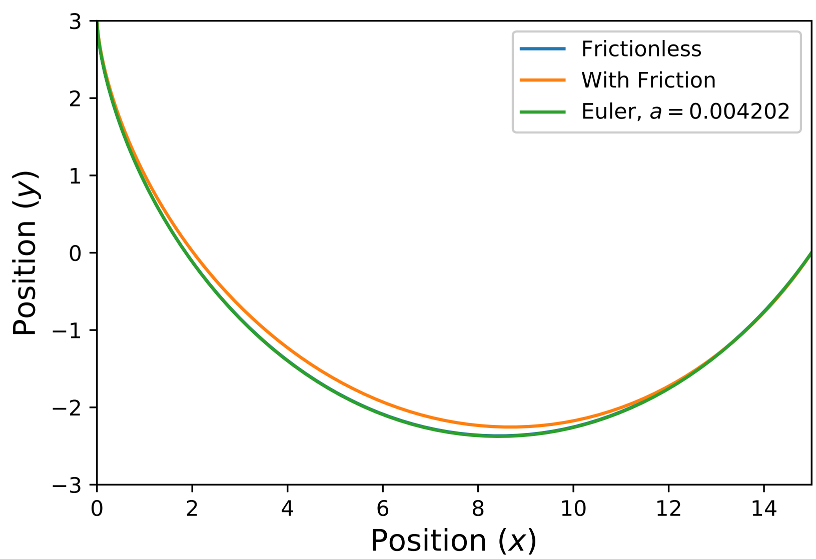


Figure 9: The three sought results for the brachistochrone with and without friction and Euler's solution for the friction brachistochrone, where Euler's result overlaps with our frictionless result, such that they can't be told apart.

There are immediately a couple of things which stand out. Euler's result and our frictionless result pretty much overlap (zooming in 300% reveals that they do not, in fact, completely overlap). The two modern curves also both almost overlap close to the beginning and near the end ($x > 11$). This is due to the friction coefficient being so small. Speaking of the small friction coefficient; the modern curves are surprisingly well separated for such a low coefficient.
The arc lengths of these curves are respectively approximately $17.870m$, $17.733m$ and $17.878m$ for the brachistochrone curves without and with friction, and Euler's solution, which is oddly almost the same as our modern frictionless arc length. All these lengths have been numerically determined, but numerically finding Euler's result provides another result than using equation 37, which is weird; equation 37 gives about $5.73m$ for the total arc length and $AB = 10.73m$, the length until the bottom of the curve, which can clearly not be right.
The easiest explanation for this would be that we determined c and a wrongly. Since a has been numerically determined, given c, I am inclined to think that c is wrong. I am, however, confident that c has been determined right, since the method of calculating it is rather unambiguous.

To create a little bit of a feeling of how the brachistochrone behaves, I consider different friction coefficients;

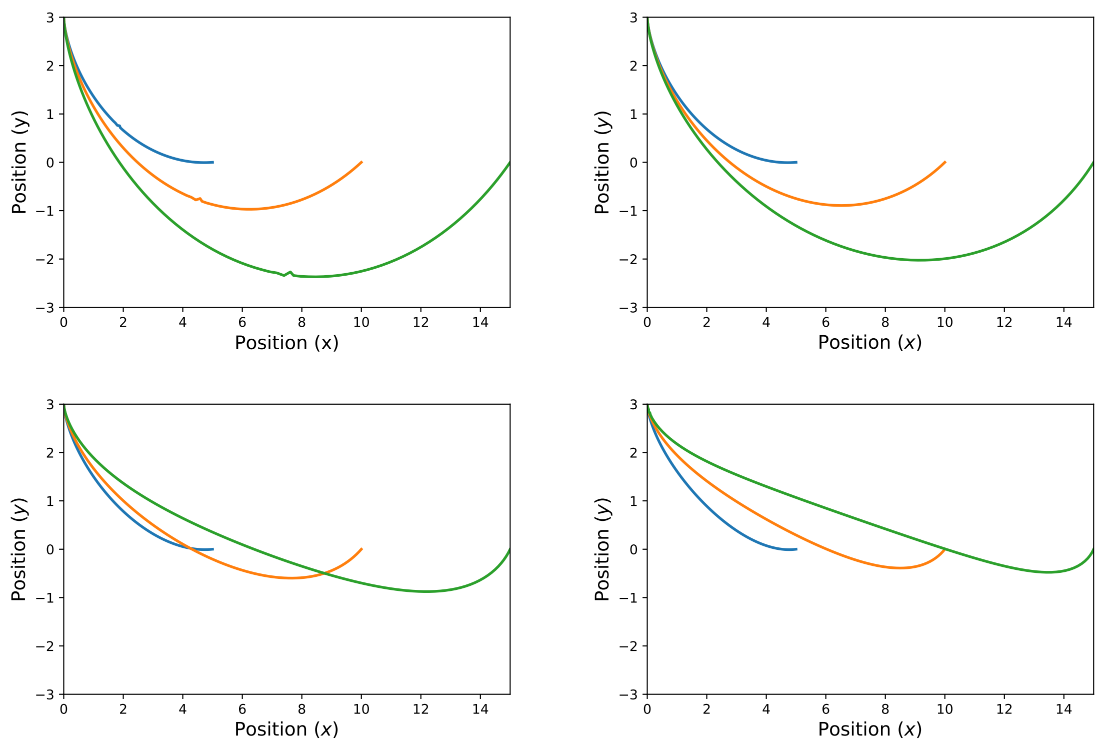


Figure 10: Four different plots for four different friction constants. The top left plot contains the true brachistochrone, without friction. The top right, bottom left and bottom right respectively have $k = 0.01,\ 0.05,\ 0.1$

It is evident that the friction brachistochrone converges to a straight line. Plotting the arc length for different values of k, grants

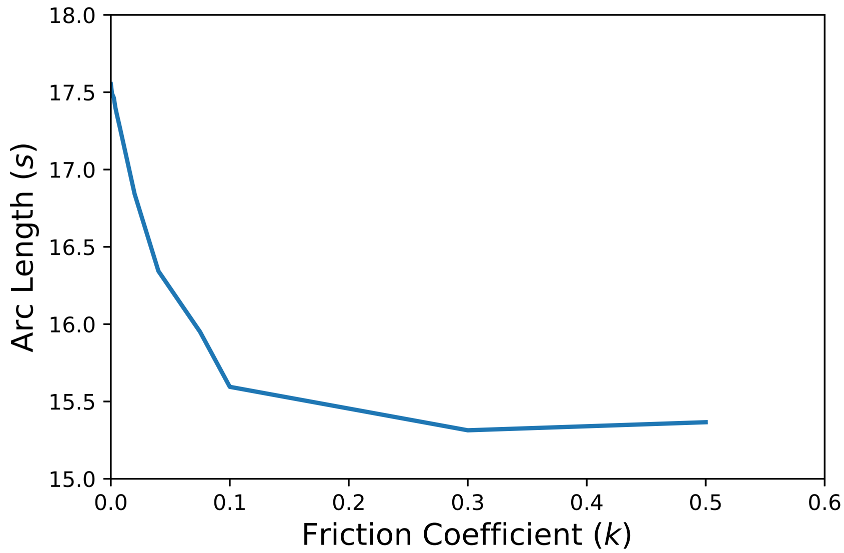


Figure 11: The arc length for different values of k.

Here we see that as $k \to 0$, $s \to 17.87$, the true brachistochrone arc length, and as k grows, the arclength converges to 15.3, the length of the straight line from $(0,3)$ tot $(15,0)$. The behaviour appears to be approximately exponential, but there should be a point when there does not exist a brachistochrone curve anymore, because the friction is too strong, such that it cannot be exponential, because it possesses no asymptote.
In this light, it is still striking that, knowing the behaviour of the friction brachistochrone, Euler managed to find the same result as the frictionless brachistochrone, which is only off by $0.1m$ still, but nonetheless weird. Euler had mentioned himself how for $c \to \infty$ (which is applicable here), the curve moves to the brachistochrone solution. While I am still not sure as to why his error is so big, I first thought that this is due to a combination of numerical inaccuracies of the Python script and that Euler's formula does not work really well with friction coefficients close to 0: applying

$$AC = c\left(\frac{1}{a}\ln\left(\frac{1}{1-a}\right) - 1\right)$$

for very big c and very little a, is bound to be very inaccurate for tiny variations in a, which happens in this case. It is unavoidable that Euler slipped in some approximations on the way and as a result, I think that for very low friction coefficients, due to Euler's approximations, his solution does not work well.

While above mentioned arguments hold true, wrongly determining Euler's confusing constants (using c instead of c^2) in an earlier attempt to plot Euler's results gave me a curve of which the arc length was bigger than $17.88m$, leading me to believe that Euler's solutions "move down" from the true brachistochrone solution with decreasing c (and thus increasing k), like seen in Figure 12, which can not be right:

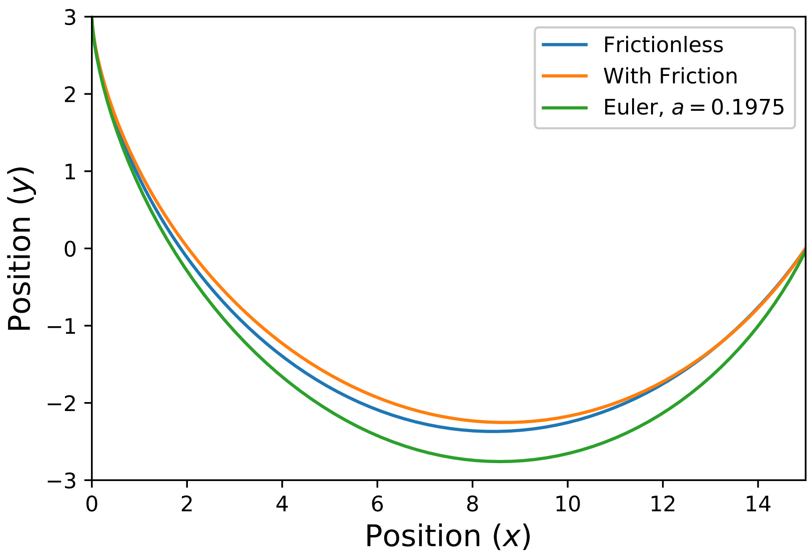


Figure 12: The two modern results, along with Euler's result for $c \approx 50.5$.

Here we clearly see that Euler's solution grants a curve that dips lower than our frictionless solution, which must be wrong. Since none of my arguments provide an explanation for why Euler's solution undershoots the true brachistochrone, and I cannot come up with one, this simply inclines me to believe that either Euler did something wrong (which is unlikely, since Euler was a brilliant individual) or we did something wrong while applying his results, which is strange, since I followed all of his steps.

Comments on Euler's Texts

Not only is it interesting to quantify how well Euler did in his ultimate result, but it is also worth mentioning Euler's quirks and methods, which stand out compared to how modern rigorous proofs and calculus are treated. Let us, though, follow more or less the order in which he works, to maintain structure.

Immediately in the beginning of his first text, it is clear how unconventional Euler's methods are. In Figure 2 and its associated equations. The way how he sketches the curve, discretized into two straight lines, without explicitly telling so, makes for the confusing situation where the reader is not sure what Euler is trying to convey. It only becomes clear when Euler starts to declare the differentials, which in itself is pretty iffy (calling a segment $dy + ddy$ out of the blue without any argument as to why is a bit blunt), but it turns out well, because Euler of course has an unparalleled feeling for how his methods work, combined with tons of experience.

Speaking of those differentials; where Euler introduces dt and $dd\theta$, he of course effectively means dq and d^2q, but opts to write it that way, to emphasize the origin of those differentials (the extra elapsed time and the difference in angle between the two curves). Another breach of our modern conventions is that Euler writes the d^2 differential as dd, while he always denotes d^3 (probably, because otherwise he would run out of space if he would keep writing d's), but it sure does look odd for the first couple of times. Equation 21 funnily yields exactly the same as (10) though, while Euler uses a Taylor approximation and we directly take the derivative of the energy balance. It is however sometimes confusing how Euler does not mention how he approximates certain results (like said Taylor approximation) and how he uses equals signs, from which we really have to guess whether or not his findings are actually exact, or approximate.

Euler likes to repeat a lot and write his results in several, eventually equivalent, ways, which manifests itself in the way he expresses the brachistochrone property in his first text, both with forces and geometrically, only to repeat it again in his second consideration, without having the ease of referring to his first text.

It is also striking how Euler, in his first text, dislikes having derivatives in his expressions, but rather multiplies the differentials such that there are as few fractions as possible, which preference he seems to have dropped in the later texts. The benefit of these differentials in this way, though, is that they allow for easy integration.
Having mentioned his integrals, it comes to mind that he never provides boundaries on which he integrates is sometimes frustrating, because he adds constants to integrals, which are not always trivially

chosen, or well explained for that matter. He does not mention whether he integrates over the whole curve, or only over a segment of it, which makes some derivations a bit ambiguous, until after having read them multiple times.

After the first figure in his first text, though, his figures are a bit better behaving and less ambiguous, along with his notation and the convenient change where he does not confusingly approximate any more. This does not take away from the complexity of his equations, though, which suffers from him starting to work in a thoroughly exact fashion. Take equation 32, a prime example of his intimidating equations. Then, using P, which Euler perspicaciously deduced, Euler defines x, y and s.

Here it becomes apparent why Euler does not attempt to find $y(x)$, but looks for $s(y)$; y is really hard to integrate on itself, and the expressions involve $p = \frac{ds}{dy}$, such that the only way of obtaining $y(x)$ at this point would be to express $y(s)$ and $x(s, y)$ after which he would have to attempt to find a way to substitute y into x and then invert $x(s)$ to resubstitute it into $y(s)$, which would lead to an ungodly expression.

The remainder of Euler's first paper has some great results though, for he reveals some key aspects of the solution, which works rather well for low x, or, equivalently, when the total horizontal displacement is less than the total vertical displacement anyway. His insights in how $0 < a < 1$, along with his equation for $y(s)$ helped tremendously in determining a numerical solution for his equation.

In Euler's later texts, his figures become more sophisticated and formal, using x, y and z as coordinates (and not calling them P or Q and only naming them after), from the start and they feel more polished. He gets to work quicker and actually uses Newton's equations of motion, for a change, along with a more general way to express the forces working on the system, this time around.

Euler immediately evokes his so-called Isoperimetric Theorem, (41), which he does not derive, and, while he explained the conditions with enough detail, for the first reading of his text, it was not at all clear how he came to find this theorem and only afterwards, when linking it to the Euler-Lagrange equations, it became apparent what Euler was on. This was also partly due to the infinite terms he considers (while the Euler-Lagrange equation usually only bother with the first two terms). Of course the equations were not coined as the Euler-Lagrange equations yet. It is hard to actually find the year Euler wrote these texts, but I can confidently say that he considers the Isoperimetric Problem from 1738 to the early 1740s[6], while their correspondence on this subject only starts later, around the 1750s[7]. This time around, again effectively repeating what he derived in his first text, his results and the method to arrive at them, are much neater than before, which makes the repeated derivation worthwhile.

Looking briefly at the results Appendix B, on the three dimensional brachistochrone, his equations, particularly (B.4) are much more elaborate than in his first text, yet they are also much neater. This dissertation also shows how Euler likes to repeat, since he could just as well have done this the first time around and taking $z = 0$ to find his two dimensional solution.

Consequently, Euler derives the General Isoperimetric Theorem, which is pretty much an even more elaborate and general Euler-Lagrange equation, which is actually much needed to solve the problem, since the modern common formulation does not allow for friction this easily. He then proceeds to work the equations through until he finds a closed solution where he can express x and y in terms of p and v (just like we are used to by now), which is exactly the way we have plotted the modern friction brachistochrone, since it is much easier than to actually solve the equations. As usual, Euler then covers a couple of examples to show that his solution reduces to known results, in the proper circumstances, which then also concludes his research.

Conclusions

Our primary interest was to consider the friction brachistochrone in two separate ways; according to Euler and with modern methods. We succeeded in both ways and are as a result in a position to compare Euler's findings to modern methods. Having devised methods for both the contemporary brachistochrone solution and Euler's solution, we looked at the example $y(0) = 3,\ y(15) = 0$. In that setting, the line connecting the boundaries has a length of approximately $15.3m$. The true brachistochrone length is $17.87m$ and the modern friction brachistochrone length is $17.73m$, while Euler's solution gives $17.88m$; almost identical to the frictionless brachistochrone length, which is still pretty close to the friction brachistochrone length. Taking lower values for c, corresponding with higher values of k, gives that Euler's solution is below the frictionless brachistochrone, which can't be true.

Our result was found with the shooting method, while Euler only used calculus (granted he did not have access to computers of course so he had no choice really). While I am not sure how the error in his solutions came to be, it is bound to find its roots in the less sophisticated methods Euler was using and he probably slipped in some approximation somewhere, or due to some error in my calculations. His final equation looks like

$$s = c \cdot \ln\left(\frac{s - ax - ac + c}{c - ac}\right)$$

where c and a are constants, specific to the problem. c can be derived analytically, but a unfortunately also has to be found by shooting.

We have looked at several of Euler's texts and it stands out how Euler grows over time in terms of his notation; his derivations and formulations become much more rigorous and general and his second brachistochrone differential equation looks a lot neater than in his first text. While Euler does a great job in his first text, in his later texts he does not label his infinitesimal intervals in awkward ways and effectively uses his -not yet coined this way- Euler-Lagrange equation. It is almost astounding how Euler starts with the equations of motion, only to arrive at page long equations, which he reduces to some equation with only four terms and this token testifies of Euler's superior insights and feeling for how to handle such problems. There he finds as his primary result that x and y can be expressed in terms of either p or v and he thus finds a set of parametric equations:

$$x = \int \frac{vdv}{g - hv^{n+1}\sqrt{1+p^2}};\ \ y = \int \frac{pvdv}{g - hv^{n+1}\sqrt{1+p^2}}$$

where h is the friction coefficient and g is the gravitational acceleration. p and v can be expressed in each other, such that a closed solution is guaranteed. This result has not been tested against the modern methods, but this might be a good idea for a future research.

Part II

The Quantum Brachistochrone

Abstract

In this part of the thesis we take a look at the quantum brachistochrone, which is analogously formulated to the classical brachistochrone; given some initial state $|\psi_i\rangle$ and some final state $|\psi_f\rangle$, we seek the optimal Hamiltonian to change the initial state in to the final state in the least amount of time. In order to do this, we follow a paper by A. Carlini called *Quantum Brachistochrone*, and define an action which has to be minimized. Using variational rules we derive equations of motion, among the equations to determine the quantum brachistochrone. It turns out that the sought Hamiltonian has to satisfy

$$F = FP + PF \text{ and } \left(\dot{F} + i\left[\tilde{H}, F\right]\right)|\psi\rangle = 0$$

where $P = |\psi\rangle\langle\psi|$, $\tilde{H}$ is the traceless part of H and $F = \sum_{a=1}^{m} \lambda^a \left(\frac{\delta f_a}{\delta H} - \langle\frac{\delta f_a}{\delta H}\rangle P\right)$, where the f_a are the constraints posed on H.

We then choose the constraints in a way that describe relevant problems; we first take $f_1(H) = Tr(\tilde{H}^2/2) - \omega^2$ as the first constraint, which dictates that the strength of the Hamiltonian has to be ω, corresponding with for example a finite magnetic field. The solution in this case reads:

$$\tilde{H} = i\omega\left(|\psi'_f\rangle\langle\psi_i| - |\psi_i\rangle\langle\psi'_f|\right); \quad |\tilde{\psi}(t)\rangle = \cos(\omega t)|\psi_i\rangle + \sin(\omega t)|\psi'_f\rangle; \quad T = \frac{1}{|\omega|}\arccos|\langle\psi_f|\psi_i\rangle|$$

where $|\psi'_f\rangle$ is the orthonormalized final state and T is the optimal brachistochrone time.

Secondly, we choose $f_2(H) = Tr(\tilde{H}\sigma_z)$, such that $\tilde{H}$ contains no σ_z component; $\tilde{H}(t) = \alpha(t)\sigma_x + \beta(t)\sigma_y$. In this case, we choose $|\psi_i\rangle = \psi_{x+}$ and $|\psi_f\rangle = \psi_{x-}$. The derived results are then in this case

$$\tilde{H}(t) = \omega\boldsymbol{\sigma}\cdot\begin{bmatrix}\sin(2\Omega t)\\ \cos(2\Omega t)\\ 0\end{bmatrix}; \quad |\omega|T = \frac{\pi}{2}\sqrt{l^2 - k^2}; \quad \left|\frac{\Omega}{\omega}\right| = \frac{|k|}{\sqrt{l^2 - k^2}}$$

where $\Omega = \lambda_2/\lambda_1$ and $l > |k| \geq 0$, along with a more complicated $|\psi\rangle$, which can be found in equation 93. Only one of these is globally optimal, of course $|\omega|T = \frac{\pi}{2}$; $\Omega = 0$, but the others turn globally optimal if the final state is the same as their first node on the equator of the Bloch sphere.

After this we take a look at a related problem, where we do not possess the needed optimal time, but still want to get as close to our desired final state as possible. We derive a set of equations, which have to be satisfied for the solution to be optimal, but there remain some unknowns, such that we have to employ the shooting method. Luckily we can also use non linear optimization to find the optimal solution, which, lo and behold, happens to be $\tilde{H} = \sigma_y$ for $\omega = 1$, $|\psi_i\rangle = \psi_{x+}$ and $|\psi_f\rangle = \psi_{x-}$, even if

the time is less than the optimal time, which agrees with our findings for the quantum brachistochrone problem.

If we however now choose $|\psi_f\rangle = \psi_{y+}$, the quantum brachistochrone solution reads $T = \frac{\pi}{4}\sqrt{3};\ \ \Omega = \frac{1}{\sqrt{3}}$. Non linear optimization solvers give $\alpha = \sin(2\Omega t)$, but for lower T, they also move sinusoidally from 0 to 1 in the given time T, such that, while we are not fully sure, since we have not proven it, we conjecture

$$\alpha(t) = \sin\left(\frac{\pi}{2}\frac{t}{T}\right)$$

Introduction

We have already covered the classical brachistochrone, with and without friction. Those problems are concerned with geometrical shapes: how can a body move from point A to point B in space, in the shortest time? However, in quantum mechanics one is not interested in where a particle is at a given time, but the probability of it being there, which is determined by its wave function. A natural transition of the brachistochrone formulation to quantum mechanics would then be to wonder how, with a given initial state, $|\psi_A\rangle$, one can "move" to a desired final state $|\psi_B\rangle$, in the least amount of time. While there is no curve, a body traverses in this situation, the analogon we are looking for is the Hamiltonian, which gives rise to the change in quantum mechanical state.

Besides for the sheer mathematical and physical beauty of the problem, this problem is also worth solving, since the results are relevant for modern research. A quantum mechanical system, an qubit in a quantum computer for instance, is prone to noise from the environment. Therefore it is of utmost importance to conduct whichever operation on such qubit in minimal time. Of course this example is not the only use of our problem, but it is a simple and obvious one. The same treatment can be done for, say, spin-1 particles (instead of spin-$\frac{1}{2}$, like the electron), or more advanced systems, such as coupled particles (like a Hydrogen atom), but these are beyond the scope of this thesis. We will only consider the general formulation of the problem, derive a solution and work out an example with the qubit. There are multiple ways to go about this problem, but they are equivalent.

We will here follow the paper *Quantum Brachistochrone* by Carlini *et al*. [8], but we will also take some inspiration from a chapter on a book, or maybe a chapter of a thesis, I found online, from the Tokyo Institute of Technology[9]. The former explains the topic more thorough and step-by-step, and goes a bit further, but the latter is in my humble opinion a little more elegant in its later notations.

Later on, we are going to take a side step and take a quick look at the case where there is not enough time to reach the final state, but rather we want to get as close as possible, which is entirely another problem in itself, but equally as relevant to research.

The Quantum Brachistochrone Problem

Let us first formulate the quantum brachistochrone problem;

Given an initial wave function, $|\psi_i\rangle$, and a final wave function, $|\psi_f\rangle$, find the possibly time-dependent Hamiltonian, $H(t)$, such that $|\psi_i\rangle$ changes into $|\psi_f\rangle$ in the shortest possible time."

Like always, we want to minimize the time required to change from $|\psi_i\rangle$ to $|\psi_f\rangle$, the action is then defined as

$$S(\psi, H\phi, \lambda) = \int dt \left[\frac{\sqrt{\langle\dot{\psi}|\,(1-P)\,|\dot{\psi}\rangle}}{\Delta E} + \Big(i\,\langle\dot{\phi}|\psi\rangle + \langle\phi|H|\psi\rangle + c.c. \Big) + \sum_{a=1}^{m} \lambda^a f_a(H) \right] \tag{60}$$

$P(t) = |\psi(t)\rangle\,\langle\psi(t)|$ is the projection onto the state $|\psi(t)\rangle$ and as usual, $(\Delta E)^2 = \langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2$. However, to make life easier, we take $\hbar$ to be unity.

The first term in the action gives the time duration for the evolution of $|\psi\rangle$, using the Fubini-Study line element $ds^2 = \langle d\psi|\,(1-P)\,|d\psi\rangle$. The second term ensures, through the Lagrange multiplier $|\phi(t)\rangle$ that both the Schrödinger equation is satisfied and the squared norm $|\psi|^2 = 1$ is conserved. Lastly, the third term generates (m) constraints for the Hamiltonian, through the Lagrange multiplier(s) λ^a, which is sometimes necessary; for instance, if we want to change an electron spin and there were no constraints, we could just choose the Hamiltonian (the magnetic field) to be arbitrarily large, such that the total elapsed time would be arbitrarily small, while in practice, there are limitations to the strength of this Hamiltonian. Furthermore, the action is invariant under the $U(1)$ gauge transformation $(|\psi\rangle\,, H, |\phi\rangle\,, \lambda) \;\rightarrow\; (e^{-i\theta}\,|\psi\rangle\,, H+\dot{\theta}, e^{-i\theta}\,|\phi\rangle\,, \lambda)$.

Using the action, S, let us then derive the equations of motion:
Taking variation with respect to $\langle\phi|$, yields, since the first and third term and the complex conjugates do not contain a $\langle\phi|$:

$$\frac{\delta S}{\delta\,\langle\phi|} = i\frac{\delta\,\langle\dot{\phi}|\psi\rangle}{\delta\,\langle\phi|} + \frac{\delta\,\langle\phi|H|\psi\rangle}{\delta\,\langle\phi|} = i\frac{\delta\,\langle\dot{\phi}|\psi\rangle}{\delta\,\langle\phi|} + H\,|\psi\rangle = 0$$

Since

$$\left(\frac{d}{dt}\right)^{\dagger} = -\frac{d}{dt}$$

we have

$$\langle\dot{\phi}| = -\,\langle\phi|\,\frac{d}{dt}$$

and

$$\frac{\delta S}{\delta \langle\phi|} = -i\frac{\delta \langle\phi|\frac{d}{dt}|\psi\rangle}{\delta \langle\phi|} + H|\phi\rangle = -i|\dot{\phi}\rangle + H|\phi\rangle = 0$$

Or

$$i|\dot{\phi}\rangle = H|\phi\rangle \tag{61}$$

which is recognized as the Schrödinger equation (this is why the second term in the action was needed). We will take many more variations like this, but the derivations can be quite lengthy. Henceforth, then, most of the derivation will be done in Appendix E, to keep this section clean, but is mostly necessary to read to be able to understand every equation well.
In particular, equation 61 implies

$$\frac{ds}{dt} = \Delta E \tag{62}$$

Taking the variation with respect to the λ^a yields

$$f_a(H) = 0 \ \ \forall a \tag{63}$$

The most natural way to define these constraints, by the way, is by looking at the traceless part of the Hamiltonian, $\tilde{H} = H - Tr(H)/n$ (where n is the dimension of the Hamiltonian), since we are normally interested in the energy difference between the highest and lowest levels in H, but the constraints can also be imposed on H itself, of course. We will in this thesis opt to formulate the constraints on $\tilde{H}$.
Taking the variation with respect to $\langle\psi|$ yields

$$i\frac{d}{dt}\left(\frac{H - \langle H\rangle}{2(\Delta E)^2}\right)|\psi\rangle - i|\dot{\phi}\rangle + H|\phi\rangle = 0 \tag{64}$$

Lastly, the variation with respect to H yields

$$\frac{\{H,P\} - 2\langle H\rangle P}{2(\Delta E)^2} - \sum_{a=1}^{m}\lambda^a\frac{\delta f_a}{\delta H} - (|\psi\rangle\langle\phi| + |\phi\rangle\langle\psi|) = 0 \tag{65}$$

(where as usual $\langle A\rangle$ denotes the expectation value of the operator A for the state $|\psi\rangle$ and $\{A,B\}$ is the anticommutator $AB + BA$)
We can apply equation 65 to $|\psi\rangle$ to simply obtain

$$|\phi\rangle = \left[\frac{H - \langle H\rangle}{2(\Delta E)^2} - \sum_{a=1}^{m}\lambda_a\frac{\delta f_a}{\delta H} - \langle\phi|\psi\rangle\right]|\psi\rangle \tag{66}$$

While taking the expectation value of (65) provides

$$\langle\psi|\phi\rangle = -\langle\phi|\psi\rangle - \langle\sum_{a=1}^{m}\lambda_a\frac{\delta f_a}{\delta H}\rangle \tag{67}$$

Let us now insert (66) back into (65), using equation 67, and call

$$F = \sum_{a=1}^{m}\lambda^a\left(\frac{\delta f_a}{\delta H} - \langle\frac{\delta f_a}{\delta H}\rangle P\right) \tag{68}$$

To find

$$F = \{F, P\} = FP + PF \tag{69}$$

which guarantees $Tr(F) = \langle F \rangle = 0$, since

$$Tr(F) = Tr(FP + PF) = 2\langle F \rangle = 2\langle FP + PF \rangle = 4\langle F \rangle$$

We can also substitute (66) into (64), which gives us

$$\left(\dot{F} + i\left[\tilde{H}, F\right]\right)|\psi\rangle = 0 \tag{70}$$

These two equations, (69) and (70) can then be integrated to obtain

$$F = UF(0)U^{\dagger} \tag{71}$$

with

$$U[\tilde{H}](t) = \mathcal{T}e^{-i\int_0^t \tilde{H}dt} \tag{72}$$

where $\mathcal{T}$ is the time ordered product.

In general equation 71 can be solved to obtain the desired solution for the quantum brachistochrone. U is Hermitian, since $U = e^{iB}$ for a Hermitian B. Furthermore, $F(0)$ can be determined by solving $F(0) = F(0)P(0) + P(0)F(0)$. It is useful to consider a few cases with specific constraints, the first of which is a finite Hamiltonian strength. Let

$$f_1(H) = Tr(\tilde{H}^2/2) - \omega^2 \tag{73}$$

And the variation with respect to λ trivially grants $Tr\tilde{H}^2 = 2\omega^2$.
Taking its variation with respect to H, though, yields much less trivially

$$\frac{\delta}{\delta H}\lambda Tr(\tilde{H}^2/2) = \lambda\tilde{H} \tag{74}$$

Taking the trace and then the expectation value of 65 gives $\langle\psi|\phi\rangle = -\langle\phi|\psi\rangle$ and, more importantly, that $\langle\tilde{H}\rangle = 0$, which, in turn, implies that

$$\langle H \rangle = Tr(H)/n; \;\; \tilde{H} = H - \langle H \rangle \tag{75}$$

Using $\langle\tilde{H}\rangle = 0$ and (74), F equals $\tilde{H}$ and equation 69 becomes

$$\tilde{H} = \tilde{H}P + P\tilde{H} = \{\tilde{H}, P\} \tag{76}$$

Furthermore, using equation 75;

$$(\Delta E)^2 = \langle\tilde{H}^2\rangle = Tr\tilde{H}^2/2 = \omega^2 \tag{77}$$

Equation 70 moreover takes the form

$$i\frac{d}{dt}\left(\lambda\tilde{H}\right)|\psi\rangle = 0 \tag{78}$$

which implies that $\dot{\lambda} = 0$ (See Section E.9 in the Appendix).
Since $\dot{\lambda} = 0$, we can rewrite equation 78, reducing it to

$$\dot{\tilde{H}} \, |\tilde{\psi}\rangle = 0 \tag{79}$$

where we have defined a new wave function:

$$|\tilde{\psi}\rangle = e^{i \int_0^t dt \langle H \rangle} \, |\psi\rangle \tag{80}$$

which is allowed, since the action is invariant under the $U(1)$ gauge transformation, which changes the phase of the wave function (which still of course satisfies the Schrödinger equation, if the phase is time independent).
Equation 76 now implies that

$$\tilde{H} = i \left(|\dot{\tilde{\psi}}\rangle \langle \tilde{\psi}| - |\tilde{\psi}\rangle \langle \dot{\tilde{\psi}}| \right) \tag{81}$$

Taking the expectation value of (81) reveals that $\langle \dot{\tilde{\psi}} | \tilde{\psi} \rangle = 0$, because we already knew that $\langle \tilde{H} \rangle = 0$.
Equation 79 lastly transforms into

$$\left(1 - \tilde{P}\right) |\ddot{\tilde{\psi}}\rangle = 0 \tag{82}$$

where $\tilde{P} = |\tilde{\psi}\rangle \langle \tilde{\psi}| = P$. Equation 82 here is the geodesic equation on the Bloch sphere.
Combining equations 76 and 81 unravels that $\dot{\tilde{H}} = 0$ (See, again, Section E.12 in the Appendix).

Equation 82 is a second order differential equation, which has the solution

$$|\tilde{\psi}(t)\rangle = \cos(\omega t) \, |\tilde{\psi}(0)\rangle + \frac{\sin(\omega t)}{\omega} \, |\dot{\tilde{\psi}}(0)\rangle \tag{83}$$

We can rewrite the states in terms of the Gram-Schmidt orthonormalized initial state $|\psi_i\rangle = |\tilde{\psi}(0)\rangle$ and final state $|\psi'_f\rangle$, which means that

$$|\psi'_f\rangle = \frac{|\psi_f\rangle - \langle \psi_f | \psi_i \rangle \, |\psi_i\rangle}{\sqrt{1 - |\, \langle \psi_f | \psi_i \rangle \,|^2}}$$

This solution, in case we only know our initial and final state, and not the derivative at $t = 0$ (which is usually the case), can then be rewritten using these orthonormal states:

$$|\tilde{\psi}(t)\rangle = \cos(\omega t) \, |\psi_i\rangle + \sin(\omega t) \, |\psi'_f\rangle \tag{84}$$

Likewise, $\tilde{H}$ can (using $\dot{\tilde{H}} = 0$) be rewritten into:

$$\tilde{H} = i\omega \left(|\psi'_f\rangle \langle \psi_i| - |\psi_i\rangle \langle \psi'_f| \right) \tag{85}$$

And $H(t) = \tilde{H} + \langle H(t) \rangle$, where $\langle H(t) \rangle$ is an abritrary function finding its origin in the gauge degree of freedom.
Finally then, the optimal needed time is

$$T = \frac{1}{|\omega|} \arccos |\, \langle \psi_f | \psi_i \rangle \,| \tag{86}$$

Let us however add a second constraint

$$f_2(H) = Tr(\tilde{H}\sigma_z) \tag{87}$$

(with the consequence that the set of equations after (75) are not true anymore). This constraint means that a magnetic field cannot contain a z component, which effectively dictates that the diagonal entries of $\tilde{H}$ are 0. By the first constraint, the off-diagonal entries need to have absolute value ω.
Using equation 71 and $Tr(F) = 0$, we find

$$F = \lambda_1 \tilde{H} + \lambda_2 \sigma_z = UF(0)U^\dagger \tag{88}$$

This equation also implies that $\dot{\lambda_a} = 0$, on which Appendix section E.18 will elaborate.
Using $i\dot{U} = \tilde{H}U$ (by definition of U), we can solve equation 88 for U and $\tilde{H}$:

$$U = e^{i\Omega t \sigma_z} e^{-i\left[\tilde{H}(0)+\Omega\sigma_z\right]t} \tag{89}$$

$$\tilde{H} = e^{i\Omega t\sigma_z} \tilde{H}(0) e^{-i\Omega t \sigma_z} \tag{90}$$

where $\Omega = \lambda_2/\lambda_1$.
This is the general solution for the quantum brachistochrone problem with these constraints; a qubit must be moved from $|\psi_i\rangle$ to $|\psi_f\rangle$, with magnetic field strength equal to ω and no component in the z-direction. Let us now assume an initial state; $P(0) = \frac{1+\sigma_x}{2}$, such that $|\psi_i\rangle = \psi_{x+} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$. Using equation 69; $F(0) = \{F(0), P(0)\}$ implies, along with the constraints, that $\tilde{H}(0) = -\omega\sigma_y$ (see Appendix section E.20. Using this and equations 89 and 90, we find

$$\langle \boldsymbol{\sigma} \rangle (t) = \begin{bmatrix} \cos(2\Omega t)\cos(2\Omega' t) + \frac{\Omega}{\Omega'}\sin(2\Omega t)\sin(2\Omega' t) \\ -\sin(2\Omega t)\cos(2\Omega' t) + \frac{\Omega}{\Omega'}\cos(2\Omega t)\sin(2\Omega' t) \\ \frac{\omega}{\Omega}\sin(2\Omega' t) \end{bmatrix} \tag{91}$$

$$\tilde{H}(t) = \boldsymbol{\sigma}\cdot\boldsymbol{B}(t); \quad \boldsymbol{B}(t) = \omega\begin{bmatrix}\sin(2\Omega t)\\ \cos(2\Omega t) \\ 0\end{bmatrix} \tag{92}$$

And

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\Omega t}\left(\cos(\Omega' t) - i\frac{\Omega}{\Omega'}\sin(\Omega' t)\right) + e^{i\Omega t}\frac{\omega}{\Omega'}\sin(\Omega' t) \\ e^{-i\Omega t}\left(\cos(\Omega' t) + i\frac{\Omega}{\Omega'}\sin(\Omega' t)\right) - e^{-i\Omega t}\frac{\omega}{\Omega'}\sin(\Omega' t)\end{bmatrix} \tag{93}$$

where $\Omega' = \sqrt{\omega^2+\Omega^2}$.
Consequently, knowing $|\psi\rangle$ and $\tilde{H}$, it is straightforward to calculate ΔE:

$$\Delta E = |\omega|\left(1 - \left(\frac{\Omega}{\Omega'}\sin^2\left(2\Omega' t\right)\right)^2\right)^{\frac{1}{2}} \tag{94}$$

Choosing $|\psi_f\rangle$, we can also determine Ω and T; let us therefore choose the easy final state $P(T) = \frac{1-\sigma_x}{2}$, such that $|\psi_f\rangle = \psi_{x-} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ and $\langle\boldsymbol{\sigma}\rangle(T) = \begin{bmatrix}-1\\0\\0\end{bmatrix}$.

From $\langle \sigma_z \rangle = 0$, we find that $2\Omega' T = l\pi$ for some $l \in \mathbb{N}$ (not in $\mathbb{Z}$, because $\Omega' T \geq 0$), whence $2\Omega T$ also has to be $k\pi$ for some $m \in \mathbb{N}$, such that $\langle \sigma_y \rangle = 0$. In order to make $\langle \sigma_x \rangle = (-1)^{l+k} = -1$, $k + l$ has to be odd and since $\Omega'^2 \geq \Omega^2$, $l > |k| \geq 0$ must hold (l must be bigger than $|k|$, because $l + k$ has to be odd). Using this, we find

$$\omega^2 = \frac{l^2 - k^2}{4T^2} \pi^2$$

and

$$|\omega| T = \frac{\pi}{2} \sqrt{l^2 - k^2}; \quad \left| \frac{\Omega}{\omega} \right| = \frac{|k|}{\sqrt{l^2 - k^2}} \tag{95}$$

A convenient way to express quantum states of spin-$\frac{1}{2}$ particles (such as electrons) is by a point on the Bloch sphere. If we want to project some vector $|\psi\rangle = \begin{bmatrix} x \\ y \end{bmatrix}$, we have to choose ϑ, φ, such that $x = \cos\left(\frac{\vartheta}{2}\right)$ and $y = e^{i\varphi} \sin\left(\frac{\vartheta}{2}\right)$ and we effectively map the state onto a unit sphere, using spherical coordinates. Using this, the first three local optima for the quantum brachistochrone in the considered case ($|\psi(0)\rangle = \psi_{x+}$, $|\psi(T)\rangle = \psi_{x-}$) are given by the following curves:

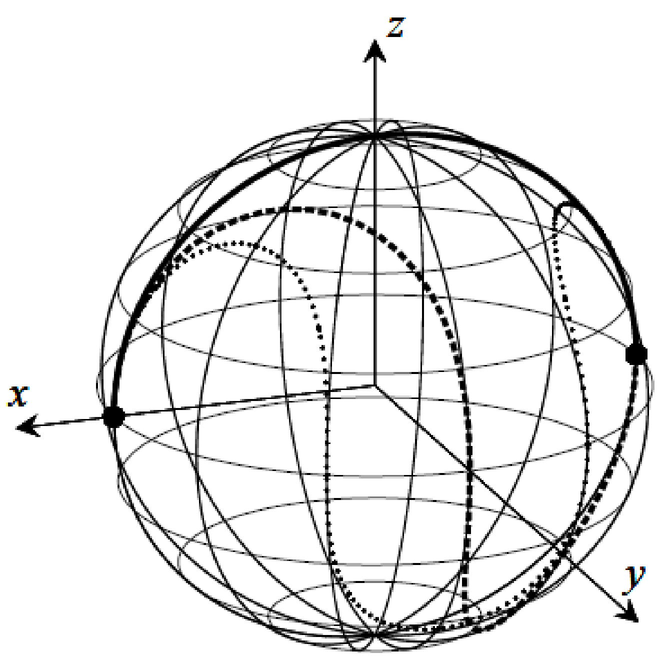


Figure 13: The Bloch Sphere, with initial state in the positive x-direction and final state in the negative x-direction. The three curves are the three first local optima. In order of ascending time are the solid line, the thick dotted line and the thin dotted line. The Image was taken from the paper *Quantum Brachistochrone* by *A. Carlini*[8]

The concrete values for these solutions are then

$$\left(|\omega| T, \left| \frac{\Omega}{\omega} \right| \right) = \left(\frac{\pi}{2}, 0 \right), \quad \left(\frac{\pi}{2}\sqrt{3}, \frac{1}{\sqrt{3}} \right), \quad \left(\frac{\pi}{5}\sqrt{5}, \frac{2}{\sqrt{5}} \right), \ etc. \tag{96}$$

Of course, there is only one optimum: the solid line, a geodesic $\left(\frac{\pi}{2}, 0\right)$. However, one can show that if the final state were in the positive y-direction ($|\psi(T)\rangle = \psi_{y+} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}$), the thick dotted curve would be optimal (since its final state is on one of the its nodes), with the optimal time being half the time needed to pass the whole curve, which coincides with solution $\left(\frac{\pi}{2}\sqrt{3}, \frac{1}{\sqrt{3}}\right)$, except that the optimal time is then halved.

In the same way, every solution for the situation $|\psi_i\rangle = \psi_{x+}$, $|\psi_f\rangle = \psi_{x-}$ also happens to be globally optimal, if the final state is the first node on the equator (on the x, y-plane) of the curve. Trivially, if the final state is in the x, z-plane, the optimal curve is the same geodesic. Moreover, if the final state is not on one of those nodes, life is a little harder, but using the same method for obtaining equation 95, these optimal solutions are obtainable too, of course.

A Related Problem

We have just looked at how one can find the fastest way to move between two given states and we have found the time needed to do this. After having done that, curiosity has led me to think about a much related problem; say, we want to move between two given states, but we do not possess the optimal time, dictated by equation 95, to do so (having chosen an ω in advance of course). What would then be the optimal Hamiltonian? The question is whether or not the solution remains the same as the brachistochrone or not.

Let us formalize the problem;
Let, given a desired final state $|\psi_f\rangle$ and an initial state $|\psi_i\rangle$, $G = |\langle\psi_f|\psi(T)\rangle|^2 = |\langle\psi_f|\mathcal{T}e^{-i\int_0^T H(t)dt}|\psi_i\rangle|^2$ be the measure for how close a final state, after an evolution over a time, T, is to the state $|\psi_f\rangle$. We then search for the Hamiltonian, $H(t)$, which maximizes G:

$$\max_{H(t)} G \tag{97}$$

Let us pose, like in the last example of the brachistochrone problem $Tr(H\sigma_z) = 0$ and $Tr(H^2/2) = \omega^2$ and the constraints dictate that $H(t) = \alpha(t)\sigma_x + \beta(t)\sigma_y$, where α and β are real and where $\alpha^2 + \beta^2 = \omega^2$, such that $\beta = \sqrt{\omega^2 - \alpha^2}$.
We are interested in

$$\frac{dG}{d\alpha} = 0 \ \ \forall t$$

Let us then rewrite

$$\mathcal{T}e^{-i\int_0^T H(t)dt} = \lim_{N\to\infty} \prod_{n=0}^{N} e^{-i\Delta t H_n} \tag{98}$$

where $H_n = H(n\Delta t)$ and $\Delta t = \frac{T}{N}$. Then, discretizing N:

$$\langle\psi_f| \prod_{n=0}^{k+1} e^{-i\Delta t H_n} \frac{d}{d\alpha_k} e^{-\Delta t H_k} \prod_{n=k+1}^{N} e^{-i\Delta t H_n} |\psi_i\rangle = 0 \tag{99}$$

We can in turn approximate

$$e^{-i\Delta t H_n} \approx 1 - i\Delta t H_n = 1 - i\Delta t \alpha_n \sigma_x - i\Delta t\sqrt{\omega^2 - \alpha_n^2}\sigma_y \tag{100}$$

such that

$$\frac{d}{d\alpha_k} e^{-i\Delta t H_k} = -i\Delta t \sigma_x + i\Delta t \frac{\alpha_k}{\sqrt{\omega^2 - \alpha_k^2}} \sigma_y \tag{101}$$

Let then;

$$U_{a,b} = \prod_{n=a}^{b} e^{-i\Delta t H_n}, \text{ and } F_{x,k} = \langle \psi_f | U_{0,k-1} \sigma_x U_{k+1,N} | \psi_i \rangle$$

And let us define F_y and F_z in a similar way.
We can determine the derivative of $F_{x,k}$, taking

$$\begin{aligned}
\frac{F_{k+1} - F_k}{\Delta t} &= \frac{\langle \psi_f | U_{0,k-1}(1 - i\Delta t H_k)\sigma_x U_{k+2,N}|\psi_i\rangle - \langle \psi_f | U_{0,k-1}|\sigma_x(1 - i\Delta t H_{k+1})U_{k+2,N}|\psi_i\rangle}{\Delta t} \\
&\approx \frac{\langle \psi_f | U_{0,k-1}(1 - i\Delta t H_k)\sigma_x U_{k+2,N}|\psi_i\rangle - \langle \psi_f | U_{0,k-1}|\sigma_x(1 - i\Delta t H_k)U_{k+2,N}|\psi_i\rangle}{\Delta t} \\
&\approx \frac{\langle \psi_f | U_{0,k-1}\left(i\Delta t \sigma_x H_k - i\Delta t H_k \sigma_x\right)U_{k+1,N}|\psi_i\rangle}{\Delta t} = \langle \psi_f | U_{0,k-1}\left[\sigma_x, H\right]U_{k+1,N}|\psi_i\rangle
\end{aligned}$$

such that

$$\begin{aligned}
\frac{dF_{x,k}}{dt} = i\langle \psi_f | U_{0,k-1}[H, \sigma_x]U_{k+1,N}|\psi_i\rangle = i\sqrt{\omega^2 - \alpha_k^2}\,\langle \psi_f | U_{0,k-1}[\sigma_y, \sigma_x]U_{k+1,N}|\psi_i\rangle \\
= -2\sqrt{\omega^2 - \alpha_k^2} F_{z,k}
\end{aligned} \tag{102}$$

Similarly it holds that $\frac{dF_{y,k}}{dt} = -i\alpha_k F_{z,k}$.
Moreover, $|\psi_f\rangle$ and $|\psi_i\rangle$ can be decomposed into a linear combination of $|\uparrow\rangle$ and $|\downarrow\rangle$, say;

$$|\psi_i\rangle = \begin{bmatrix} a \\ b \end{bmatrix};\ \ |\psi_f\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$$

such that we can define four times as many variables;

$$F_{x,k} = acF_{x,k}^{\uparrow\uparrow} + bdF_{x,k}^{\downarrow\downarrow} + adF_{x,k}^{\uparrow\downarrow} + bcF_{x,k}^{\downarrow\uparrow} \tag{103}$$

And, again, similar definitions for F_y and F_z.
Since

$$\sigma_x|\uparrow\rangle = |\downarrow\rangle,\ \ \sigma_x|\downarrow\rangle = |\uparrow\rangle,\ \ \sigma_y|\uparrow\rangle = i|\downarrow\rangle\ \ \ \sigma_y|\downarrow\rangle = -i|\uparrow\rangle,\ \ \sigma_z|\uparrow\rangle = |\uparrow\rangle,\ \ \sigma_z|\downarrow\rangle = |\downarrow\rangle \tag{104}$$

we can derive relations between $F_{x,0},\ F_{y,0}$ and $F_{z,0}$ and of course also $F_{x,N},\ F_{x,N}$ and $F_{z,N}$.
Let us define $L = \langle \psi_f | U_{0,N} | \psi_i \rangle$, whence, with equation 99,

$$\frac{dL}{d\alpha_k} = 0, \quad \forall \alpha_k$$

On account of

$$\begin{aligned}
F_{x,0}^{\uparrow\uparrow} = -iF_{y,0}^{\uparrow\uparrow} = F_{z,0}^{\uparrow\downarrow} = L^{\uparrow\downarrow}; \quad F_{x,0}^{\downarrow\downarrow} = iF_{y,0}^{\downarrow\downarrow} = F_{z,0}^{\downarrow\uparrow} = L^{\downarrow\uparrow}; \\
F_{x,0}^{\uparrow\downarrow} = iF_{y,0}^{\uparrow\downarrow} = F_{z,0}^{\uparrow\uparrow} = L^{\uparrow\uparrow}; \quad F_{x,0}^{\downarrow\uparrow} = -iF_{y,0}^{\downarrow\uparrow} = F_{z,0}^{\downarrow\downarrow} = L^{\downarrow\downarrow};
\end{aligned} \tag{105}$$

and four more of these equations, for $t = T$ (or at timestep N), our set of unknowns at $t = 0$ decreases from 12 to 4, just like at $t = T$. Following equation 101, we know α_k, if we solve for

$$F_{x,k}^{fi} = \frac{\alpha_k}{\sqrt{\omega^2 - \alpha^2}} F_{y,k}^{fi}$$

for $t = k\Delta t$, and we can continue this up until $t = T$.
However we still know nothing about the $F_{x,0}$ and the like, but we can employ the shooting method and guess one, to see if the relations we have derived actually hold. If they do, we have ourselves a solution. If they don't, we have to continue shooting.

While this method is not yet completely finalized and entirely worked through, there's an easier, non-theoretical way of obtaining the desired Hamiltonian. Let us discretize G into N parts, like we did before. G thus effectively depends on $N+1$ variables, α_0 up until α_N. Using non linear optimization solvers in, say, Matlab, we can simply find the optimal α_k, such that G is as close to 1 as possible, given the constraints $-1 \leq \alpha_k \leq 1$. Effectively though, this is a more elaborate and readily available multi dimensional shooting method, so we are not doing anything new.

Results

Equation 100, also known as the Lie-Trotter approximation, holds up really well, since the error is in order Δt^2. Because of this we can warrant taking N pretty low (less than 100) and still get an alright approximate result.

Taking the same example as in the quantum brachistochrone is barely exciting, since it yields $\alpha_k = 0 \ \forall k$: the solution is a geodesic in the x, z-plane after all, such that the σ_x component is zero. Let us rather take $|\psi_i\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $|\psi_f\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$. Like derived in the quantum brachistochrone section, the optimal time to make this transition is $T = \frac{\pi}{2}\sqrt{3}$. Let us then take half this time and see how close we can get, for different amounts of timesteps:

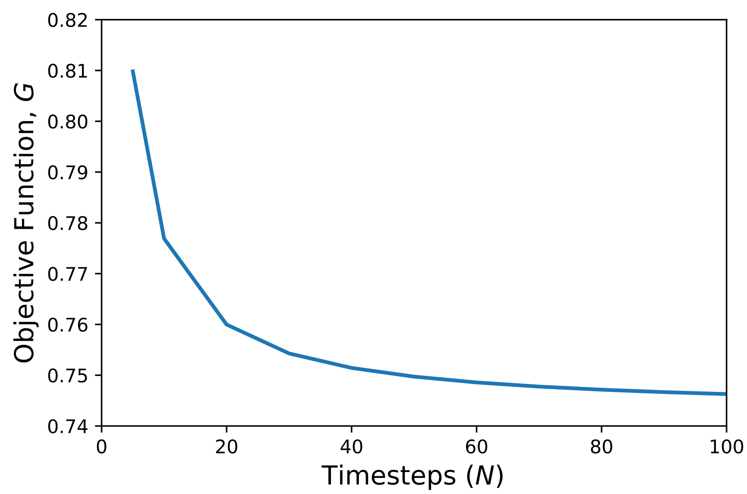


Figure 14: The Objective Function, G, on the final iteration, plotted against the amount of timesteps used in the optimization, taking the values 5, 10, 20 all the way up to 100 in steps of 10.

This figure illustrates how well the approximation does its job; for $N = 5$, the optimization overshoots about 0.06, whereas for increasing N, the objective function seems to converge to about 0.746, which does not seem that strange, considering the system only has half the time to move from a state that has an overlap of 0.5 ($|\langle\psi_f|\psi_i\rangle| = 0.5$), to a state that has an overlap of 1; the inner product remains well in the middle.

However, the obvious disadvantage of taking N large is that the optimization takes a while. For

$N = 5$, the process took less than a second, while for $N = 100$, it took about eight minutes, using the Interior point method. Different methods, like the SQL method, were tested too, but none of them seemed to significantly improve the results, while they also did not improve the run time, so I just stuck with the Interior point method.

Next, it felt important to also look at how well the approximations of the Hamiltonian did, comparing low N to high N, resulting in

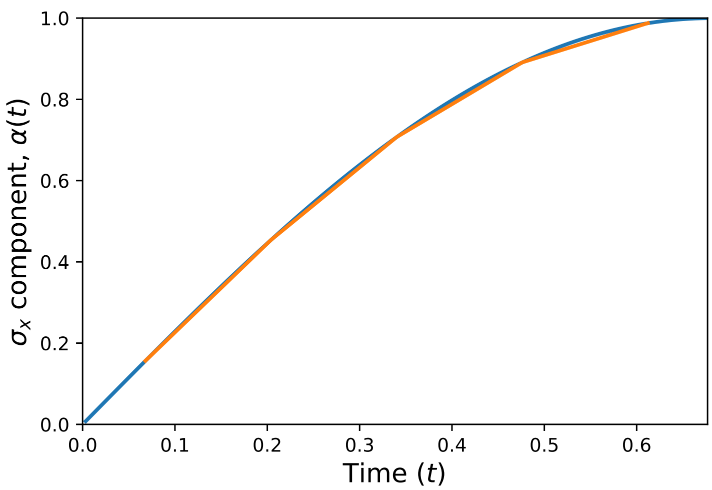
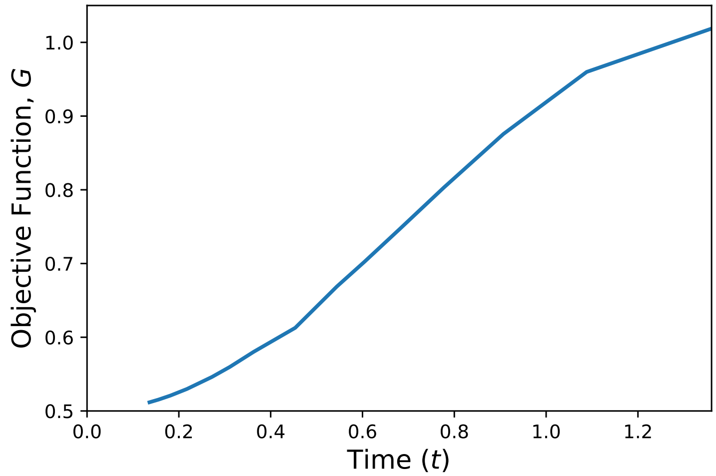


Figure 15: A plot of the time against α, the x-component of the Hamiltonian. The blue line represents the optimization for $N = 100$, while the yellow line represents $N = 5$.

It is striking how well the yellow "curve" (albeit not really a curve, since it connects five points), follows the blue curve. The blue curve notably resembles a sine, which leads us to the somewhat surprising conclusion that the solution does not appear to be the same as the brachistochrone solution. On the contrary, $\alpha(t)$ always seems to go from 0 to 1, which indicates that the state initially, at $t = 0$ moves on the geodesic from $|\psi_{x+}\rangle$ to $|\psi_{x-}\rangle$, only to curve away and finally return to the equator connecting $|\psi_{x+}\rangle$ and $|\psi_{y+}\rangle$, such that $\psi(T)$ is located on the equator.

Lastly, it is interesting how the solution changes, when the available time is changed from the optimal time all the way to zero:



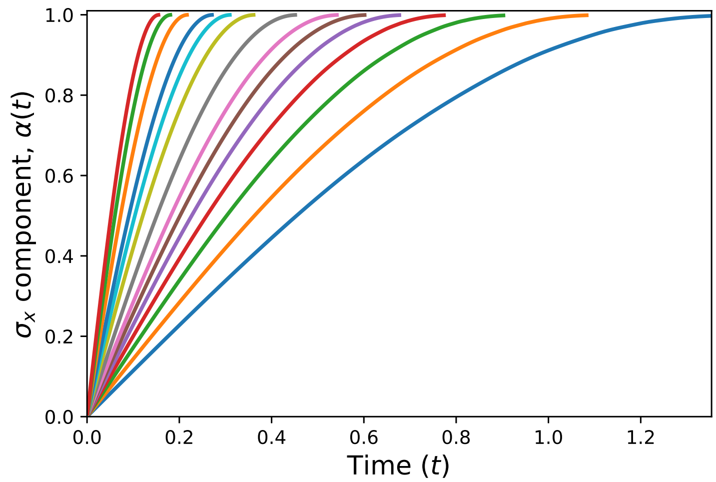


Figure 16: The solutions, found for different end times, ranging from $T = \frac{\pi}{40}\sqrt{3}$ to $T = \frac{\pi}{4}\sqrt{3}$ (where the denominator assumes the intermediate values $35, 30, 25, 20, 17.5, 15, 12.5, 10, 9, 8, 7, 6, 5$). The used value for N was 100 in all of these optimizations. The upper figure represents the Objective function, G, at time T and the figure below plots α for the different end times T.

Here we see that the inner product behaves well like we expected; for the optimal time, it goes to 1 and for $T \to 0$, it remains 0.5, as seen in the upper image in Figure 16. Furthermore, the solution for the Hamiltonian remains the quarter of the period of a sine, but narrower every time.

Conclusions

We have looked at the quantum brachistochrone problem and derived the solutions that dictate the time evolution of the initial state. If we take the Hamiltonian constraints to be

$$f_1(H) = Tr(\tilde{H}^2/2) - \omega^2; \;\; f_2(H) = Tr(\tilde{H}\sigma_z)$$

$\tilde{H}$ has no diagonal entries and we can write $\tilde{H}(t) = \alpha(t)\sigma_x + \beta(t)\sigma_y$, since it contains no σ_z component and the off diagonal entries have to have absolute value ω. The solution for $|\psi_i\rangle = \psi_{x+}$ and $|\psi_f\rangle = \psi_{x-}$ is then

$$\tilde{H}(t) = \omega\boldsymbol{\sigma} \cdot \begin{bmatrix} \sin(2\Omega t) \\ \cos(2\Omega t) \\ 0 \end{bmatrix}; \;\; |\omega|T = \frac{\pi}{2}\sqrt{l^2 - k^2}; \;\; \left|\frac{\Omega}{\omega}\right| = \frac{|k|}{\sqrt{l^2 - k^2}}$$

where $\Omega = \lambda_2/\lambda_1$, $l > |k| \geq 0$ and which is globally optimal for $l = 1, \; k = 0$. If we choose $|\psi_f\rangle = \psi_{y+}$, however, the solution remains the same, except for $|\omega|T = \frac{\pi}{4}\sqrt{3}$; $|\frac{\Omega}{\omega}| = \frac{1}{\sqrt{3}}$.

In the setting of the related problem, where the time is insufficient to reach the final state, we found with non linear optimization that $\alpha(t) = 0$ if we choose $|\psi_f\rangle = \psi_{x-}$, but we find that it bears a sinusoidal form if $|\psi_f\rangle = \psi_{y+}$. Were the time enough, then the solution is the same as mentioned above, but if not, the optimal solution is not only sinusoidal, but also goes from 0 to 1 (and does not stop before 1). This means that the final state is on the equator of the Bloch sphere, in the x, y-plane. This inclines me to conjecture that

$$\alpha(t) = \sin\left(\frac{\pi}{2}\frac{t}{T}\right)$$

such that for $t = T$, $\alpha = 1$.

However, to be able to tell with certainty that the final state is on the equator, more research has to be conducted. Along with this, the analytic way of describing this problem, using $\frac{dL}{d\alpha_k} = 0 \;\; \forall k$ should also be explored more, in order to derive this result in a rigorous mathematical way in future research.

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Bibliography

[1] A.S. Parnovsky, *Some Generalisations of Brachistochrone Problem*, Lyceum N0. 145 for Natural Sciences, Sh. Rustaveli 46, Kiev- 23, 252023, Ukraine.

[2] Leonhard Euler, *E42 - De linea celerrimi descensus in medio quocunque resistente*, *E759 - Investigatio accuratior circa brachystochronas*, *E760 - De vera brachystochrona seu linea celerrimi descensus in medio resistente* and *De brachystochrona in medio resistente, dum corpus ad centrum virium utunque attrahitur*

[3] Dominic Klyve, Lee Stemkoski and Erik Tou, The Euler Archive.
`http://eulerarchive.maa.org/`

[4] Leonhard Euler, *Leonhardi Euleri Commentationes Analyticae ad calculum variationum pertinentes, Opera Omnia: Series 1, Volume 25*, pp. 41-53 and pp. 314 - 342, Leiden University Library.

[5] Trent Okeson, APMonitor, *Brachistochrone Optimal Control*.
`http://apmonitor.com/wiki/index.php/Apps/BrachistochroneProblem`

[6] Craig G. Fraser, *Isoperimetric Problems in the Variational Calculus of Euler and Lagrange*, Institute for the History and Philosophy of Science and Technology, Victoria College, University of Toronto, Toronto, Ontario, Canada M5S 1K7.

[7] Leonhard Euler and Joseph Louis Lagrange, *Correspondance de Lagrange avec Euler*, National Library of France.

[8] Alberto Carlini, Akio Hosoya, Tatsuhiko Koike and Yosuke Okudaira, *Quantum Brachistochrone*, Department of Physics, Tokyo Instutute of Technology, Tokyo, Japan and Department of Physics, Keio University, Yokohama, Japan, November 4, 2005.

[9] *Chapter 3: Quantum brachistochrone*, Tokyo Institute of Technology Library. http://tdl.libra.titech.ac.jp/hkshi/xc/contents/pdf/300063861/5

Appendix A

Euler's Brachistochrone in Vacuum around a Center of Forces.

In this section, Euler considers a variation of the original problem, where the forces attract to some point in space, C, like sketched in Figure A.1, say; if the distance between C and the body is R, then the force will be R^m. Let $CA = a$, $CM = y$ and $CT = z$. AM will then be a segment of the brachistochrone curve. C pulls on M with a force y^m. The radius of curvature is $-\frac{ydy}{dz}$ (since AM is convex) and the sine of the angle that the curve makes with CM is then $\frac{z}{y}$. Like for the previous example, now $A\frac{z}{y} = \frac{y^{m+1}dy}{dz}$ or rather $Azdz = y^{m+2}dy$, of which the integral is $C + Az^2 = y^{m+1}$ (C and A are constants, not points in space, here). Using the boundary condition; $z(y = a) = 0$ results in $C = a^{m+3}$ and consequently (A arbitrarily chosen negative):

$$Az^2 = a^{m+3} - y^{m+3} \tag{A.1}$$

which is the brachistochrone equation, for a center of forces.

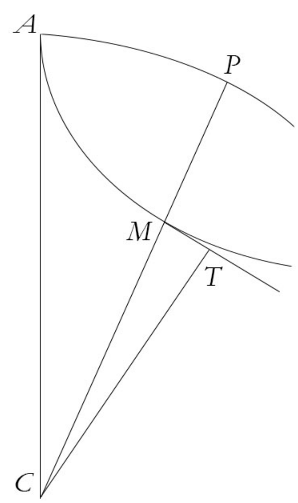


Figure A.1: Another sketch, used to consider a case with a center of forces.

Appendix B

The Brachistochrone in 3 Dimensions

Having looked at the two dimensional brachistochrone, Euler, for completeness, takes a look at the three dimensional case, which is effectively a more generalized solution, basically when there are two or more disrupting forces that do not point in the same plane.

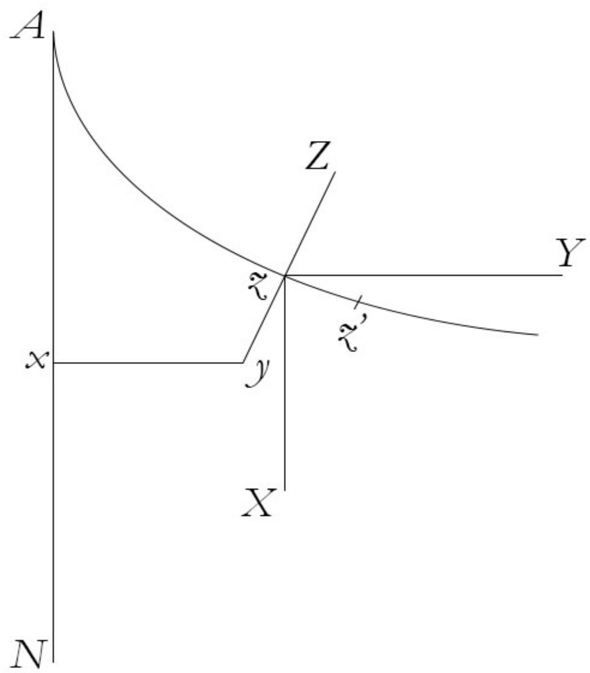


Figure B.1: A sketch of the 3-dimensional vacuum brachistochrone, with the disturbing forces.

Let, analogously to the two dimensional case, Az be the brachistochrone curve. Let furthermore Ax be denoted as x, $xy = y$ and $yz = z$. zz' corresponds with $ds = \sqrt{dx^2 + dy^2 + dz^2}$ and lastly, like before, the disrupting forces can be decomposed into the three coordinate directions: $zX = X$; $zY = Y$; $zZ = Z$, which are again only functions of x, y and z.
The equations of motion read

$$\frac{d^2x}{dt^2} = 2gX; \ \frac{d^2y}{dt^2} = 2gY; \ \frac{d^2z}{dt^2} = 2gZ$$

where g is again half the gravitational acceleration. Integrating these equations, after adding them up, yields

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = \frac{ds^2}{dt^2} = v^2 = 4g \int (Xdx + Ydy + Zdz) \tag{B.1}$$

whence

$$vdv = 2g\,(Xdx + Ydy + Zdz)$$

From the equations of motion, we also find:

$$\frac{yd^2x - xd^2y}{dt^2} = 2g\,(yX - xY)\,; \ \frac{ydx - xdt}{dt} = 2g \int (yX - xY)\,dt$$

Since $\frac{ds}{dt} = v$ we can write $\frac{ydx - xdy}{ds} = \frac{2g}{v} \int (Xy - Yx)\,\frac{ds}{v}$. In the same way:

$$\frac{zdx - xdz}{ds} = \frac{2g}{v} \int (Xz - Zx)\,\frac{ds}{v}; \quad \frac{zdy - ydz}{ds} = \frac{2g}{v} \int (Yz - Zy)\,\frac{ds}{v}$$

An obvious approach for further analysis would of course again be Euler's isoperimetric theorem, if it weren't only suited for two variables. However we can simply consider two projections of our sought curve and apply the isoperimetric theorem on both of them.

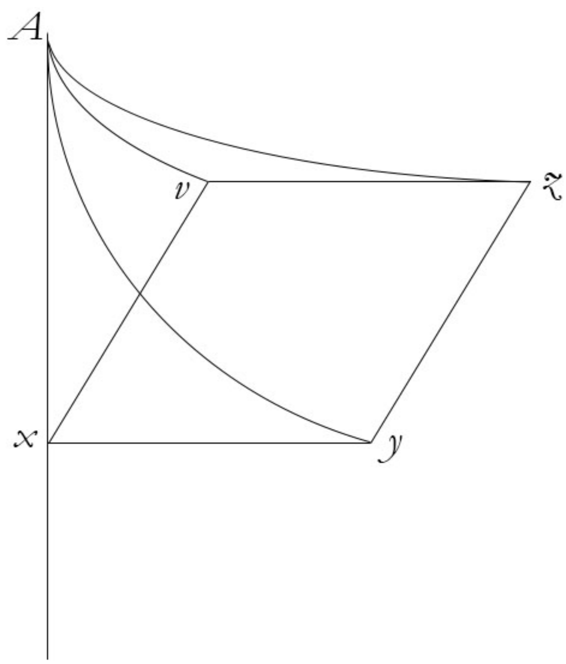


Figure B.2: A sketch of the sought brachistochrone curve and its two projections.

The curve can be projected on Ay and the curve Av, which can be joined together to form Az. Let like before $p = \frac{dy}{dx}$ and let now (contrary to the formulation of the isoperimetric theorem) $q = \frac{dz}{dx}$. ds reads $dx\sqrt{1 + p^2 + q^2}$. For both projections the integral $\int \frac{ds}{v} = \int \frac{dx\sqrt{1+p^2+q^2}}{v}$ has to be minimal. Let like before v only depend on x, y and z so that we can use the isoperimetric theorem. Let us first consider

Ay and the integrand will bear this form $d\left(\frac{\sqrt{1+p^2+q^2}}{v}\right) = Mdx + Ndy + Pdp$ in order to be able to establish $Ndx - dP = 0$ again. Since v does not explicitly depend on p, $P = \frac{p}{v\sqrt{1+p^2+q^2}}$. Because we're only considering Ay, we can assume that v only depends on y here and as such $d\left(\frac{1}{v}\right) = -\frac{dv}{v^2}$. We can substitute dv using (B.1) to find $N = -\frac{2gY}{v^3}\sqrt{1+p^2+q^2}$ yielding

$$\frac{2gY dx}{v^3}\sqrt{1+p^2+q^2} + d\left(\frac{p}{v\sqrt{1+p^2+q^2}}\right) = 0 \tag{B.2}$$

Similarly, for Av this equation reads

$$\frac{2gZ dx}{v^3}\sqrt{1+p^2+q^2} + d\left(\frac{q}{v\sqrt{1+p^2+q^2}}\right) = 0 \tag{B.3}$$

(B.2) and (B.3) are then the equations required to achieve our three dimensional brachistochrone solution. We can elaborate by determining

$$d\left(\frac{p}{v\sqrt{1+p^2+q^2}}\right) = -\frac{dv}{v^2}\frac{p}{\sqrt{1+p^2+q^2}} + \frac{1}{v}d\left(\frac{p}{\sqrt{1+p^2+q^2}}\right)$$

We can also again substitute dv, multiply by $\frac{v^3}{2g}$ and use $dy = pdx,\ dz = qdx$ to obtain

$$\begin{aligned}\frac{Y\left(1+q^2\right) - pX - pqZ}{\sqrt{1+p^2+q^2}} + \frac{v^2}{2gdx}d\left(\frac{p}{\sqrt{1+p^2+q^2}}\right) = 0\\ \frac{Z\left(1+p^2\right) - qX - pqY}{\sqrt{1+p^2+q^2}} + \frac{v^2}{2gdx}d\left(\frac{q}{\sqrt{1+p^2+q^2}}\right) = 0\end{aligned} \tag{B.4}$$

which equations, if we ommit the third dimension z (and therefore also q), simply become

$$\frac{Xp - Y}{\sqrt{1+p^2}} = \frac{v^2}{2g}d\left(\frac{p}{\sqrt{1+p^2}}\right) \tag{B.5}$$

which is effectively the same as equation (41)*.
Now that we have taken a keener look at the original brachistochrone, we're armed to face the brachistochrone in a resistant medium; the friction brachistochrone.

Appendix C

The Generalized Isoperimetric Theorem

Euler wants to consider a speed that depends on the traversed arc length, but his Isoperimetric Theorem does not yet allow for that. In order to be able to consider this situation, Euler first needs to generalize his Isoperimetric theorem, by considering that, if $\int V dx$ needs to be maximized or minimized, V may depend on x, y, $p = \frac{dy}{dx}$ and higher order derivatives, and finally also v. Let then

$$dV = Ldv + Mdx + Ndy + Pdp + Qdq + etc.$$

and $dv = \mathfrak{B}dx$, where $\mathfrak{B}$ is again some function of v and $x, y, p, etc.$ and thence

$$d\mathfrak{B} = \mathfrak{L}dv + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + etc.$$

The calculus of variation was taking shape in Euler's time, so, while he of course does his best in coming up with ways to solve said problem, his methods are rather primitive. Euler provides a pretty insightful sketch, namely, Figure 7, where it is clear how his method works:
Let AYC be the brachistochrone and $AX = X$; $XY = Y$; $AY = S$. Euler proposes to add a little variation: $Y\delta$, which he calls δy in true variational fashion (the point to which the variation is drawn from Y is δ). He also adds that δx should be 0 such that variations are only in the y-direction. It is also, again, worth mentioning here, that Euler still does not follow the modern conventional axis labels, like in Euler's first translated text.
Let us then denote $\delta y = \omega$ and, since Euler recognizes that $\frac{\delta dy}{dx} = \frac{d\delta y}{dx}$, $\delta p = \frac{d\omega}{dx}$, $\delta q = \frac{dd\omega}{dx^2}$, $etc.$, where δ always denotes the variation of the respective variable. We want to vary $\int V dx$ to find its minimum and it conveniently holds that $\delta \int V dx = \int \delta V dx$ and Euler mentions that variations can be taken like differentials, such that

$$\begin{aligned} \delta V &= L\delta v + M\delta x + N\delta y + P\delta p + Q\delta q + etc. \\ &= L\delta v + N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + etc. \end{aligned} \tag{C.1}$$

whence

$$\delta \int V dx = \int L dx \left(\delta v + N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + etc. \right) \tag{C.2}$$

Since we called $dv = \mathfrak{B}dx$, it holds that

$$\delta\mathfrak{B} = \mathfrak{L}\delta v + \mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + etc. \tag{C.3}$$

And consequently

$$\delta v = \int dx \left(\mathfrak{L}\delta v + \mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + etc. \right) \tag{C.4}$$

Euler calls $\delta v = u$ and finds, after taking differentials,

$$du - \mathfrak{L}udx = \mathfrak{N}\omega dx + \mathfrak{P}d\omega + \frac{\mathfrak{Q}dd\omega}{dx} + etc. \tag{C.5}$$

Writing $e^{\int \mathfrak{L}dx} = \Lambda$, we can create an integrating factor for the left hand side of equation C.5, since $\frac{d\Lambda}{\Lambda} = \mathfrak{L}dx$. Dividing on both sides by Λ yields

$$\begin{aligned} \frac{u}{\Lambda} &= \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}\mathfrak{d}\mathfrak{d}\omega}{dx^2} + etc. \right) \\ \delta v &= \Lambda \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc. \right) \end{aligned} \tag{C.6}$$

whence the first term of the right hand side of equation C.2 reads

$$\int L\Lambda dx \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc. \right)$$

which is a nested integral.
Let subsequently $L\Lambda dx = d\Pi$ and, using integration by parts, we find

$$\int d\Pi \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + etc. \right) = \Pi \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + etc. \right) - \int \frac{\Pi dx}{\Lambda} \left(\mathfrak{N}\omega + etc. \right) \tag{C.7}$$

Let the end point be such that $AB = a$. The integration constant for $\Pi = \int L\Lambda dx$ can arbitrarily be chosen, but it is useful to choose it such that it vanishes at the end. Doing so results in $\Pi \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + etc. \right)$ vanishing over the whole curve. All that remains is

$$\int L dx \delta v = - \int \frac{\Pi dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc. \right) \tag{C.8}$$

Having found this, our sought variation $\delta \int V dx$, equation C.2, becomes

$$- \int \frac{\Pi dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc. \right) + \int dx \left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + etc. \right) \tag{C.9}$$

To clean up this unrefined notation, we Euler poses

$$N' = N - \frac{\Pi\mathfrak{N}}{\Lambda}; \quad P' = P - \frac{\Pi\mathfrak{P}}{\Lambda}; \quad Q' = Q - \frac{\Pi\mathfrak{Q}}{\Lambda}; etc. \tag{C.10}$$

such that equation C.9 becomes

$$\delta \int V dx = \int dx \left(N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + etc. \right) \tag{C.11}$$

which should equal zero, for the whole curve, since we want to minimize the integral.
Let us then integrate by parts to obtain

$$\int P'd\omega = P'\omega - \int \omega dP'; \;\; \int Q'dd\omega = Q'd\omega - \omega dQ' + \int \omega ddQ';$$

$$\int R'd^3\omega = R'dd\omega - d\omega dR' + \omega ddR' - \int \omega d^3R'; \;\; etc.$$

Since there can be no variation on the boundaries, A and C, ω is zero there. As a result, only the last term, the integral, of these equations do not vanish over the entire curve.
For that reason

$$\delta \int V dx = \int \omega dx \left(N' - \frac{dP'}{dx} + \frac{ddQ'}{dx^2} - \frac{d^3R'}{dx^3} + etc. \right) \tag{C.12}$$

Of which the value, again, has to be zero, such that the final equation, which constitutes the general isoperimetric treatment, reads

$$N' - \frac{dP'}{dx} + \frac{ddQ'}{dx^2} - \frac{d^3R'}{dx^3} + etc. = 0 \tag{C.13}$$

All remaining integration constants can then be discovered by posing that $y(x=0)=0, \; y(x=a)=b$ and finally $v(x=0)$ should also be given.

Appendix D

Euler's Friction Brachistochrone around a Center of Forces

Last but not least, Euler also takes a look at the situation where there's some point with an attracting force field.

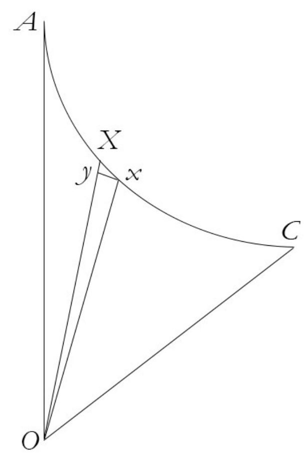


Figure 10. A sketch of the brachistochrone involving a centre of forces.

Let O be the centre of forces of which the attracting force at a distance x is X, depending on x. Let the velocity of the moving body like usual be v. The friction is V, depending on v. Let $OA = a$, $OC = c$ and $\angle OAC = b$. Let $OX = x$ and $\angle AOX = y$. Let furthermore $AX = s$, $Xx = ds$, $Xy = dx$ and $XOx = dy$. Then $xy = xdy$ (bear in mind that X is both the attracting force, and a point on the curve, like x is both a length and a point on the curve). Then

$$Xx = ds = \sqrt{dx^2 + x^2dy^2}$$

Like usual, $p = \frac{dy}{dx}$, then $ds = -dx\sqrt{1+p^2x^2}$. The force in the direction of motion then reads $-\frac{Xdx}{ds} - V$ and we find $vdv = -Xdx - Vds$, after which, substituting for ds, we obtain an expression for dv:

$$vdv = dx\left(V\sqrt{1+p^2x^2} - X\right)$$

As always, we want to minimize $\int \frac{dx\sqrt{1+p^2x^2}}{v}$, so we will use the general isoperimetric theorem again. $W = \frac{\sqrt{1+p^2x^2}}{v}$ and thus $\mathfrak{W} = \frac{V\sqrt{1+p^2x^2}-X}{v}$. M and $\mathfrak{M}$ are unnecessary to be acquired, since they don't affect the solution, but $L = -\frac{1+p^2x^2}{v^2}$, $N = 0$ and $P = \frac{px^2}{v\sqrt{1+p^2x^2}}$.

$$\mathfrak{L} = -\frac{V\sqrt{1+p^2x^2}-X}{v^2} + \frac{V'\sqrt{1+p^2x^2}}{v}$$

where $V' = \frac{dV}{dv}$. Subsequently $\mathfrak{N} = 0$ and $\mathfrak{P} = \frac{Vpx^2}{v\sqrt{1+p^2x^2}}$. Like previous time, $\frac{dP'}{dx} = 0$ and therefore $P' = P - \frac{\Pi\mathfrak{P}}{\Lambda} = K$, such that again $P(C) = K$ and $K = \frac{px^2}{v\sqrt{1+p^2x^2}}$ there.
Then

$$\Pi = \frac{\Lambda\,(P-K)}{\mathfrak{P}} = \frac{\Lambda}{V} - \frac{K\Lambda v\sqrt{1+p^2x^2}}{Vpx^2}$$

If we then substitute $\omega = \sqrt{1+p^2x^2}$ and $t = \frac{\sqrt{1+p^2x^2}}{px^2}$, we can divide by Λ to obtain

$$\frac{\omega dx}{v^2} + \frac{\mathfrak{L}dx}{V} - \frac{dV}{V^2} - \frac{K\mathfrak{L}tvdx}{V} - Kx\frac{(vdt+tdv)}{V} + \frac{KtvdV}{V^2} = 0 \tag{D.1}$$

With $\mathfrak{L} = -\frac{V\omega+X}{v^2} + \frac{V'\omega}{v}$. Since $vdv = dx\,(V\omega - X)$, $dx = \frac{vdv}{(V\omega-X)}$ and we can substitute for dx. Doing so, along with multiplication by $V\omega - X$ and writing $V'dv = dV$, (60)* becomes, after neglecting the terms that mutually vanish

$$\frac{X}{V}\left(\frac{dV}{V} + \frac{dv}{v}\right) - Kv\omega dt + \frac{KvXdt}{V} - \frac{KvtXdV}{V^2} = 0 \tag{D.2}$$

Dividing this equation by CvX yields

$$\frac{1}{CVv}d\left(l\left(Vv\right)\right) - \frac{\omega dt}{X} + \frac{dt}{V} - \frac{tdV}{V^2} = 0 \tag{D.3}$$

(remember that l denotes the natural logarithm), which actually is integrable. This integral then reads

$$-\frac{1}{CVv} + \frac{t}{V} - \int\frac{\omega dt}{X} = \Delta \tag{D.4}$$

where Δ is a constant depending on the problem. $\int \frac{\omega dt}{X}$ can lastly be calculated, since it only depends on p and x. v can be expressed in x and p, like established before and on this account the solution is expressed in x and p.

Appendix E

Derivations for the Quantum Brachistochrone

E.1 Equation 62

Let us start from the definition of ds:

$$\left(\frac{ds}{dt}\right)^2 = \langle\dot{\psi}|(1-P)|\dot{\psi}\rangle$$

Using the Schrödinger equation, (61), we find

$$\langle\dot{\psi}|(1-P)|\dot{\psi}\rangle = \langle\psi|iH(1-P)(-iH)|\psi\rangle = \langle\psi|H(1-P)H|\psi\rangle$$

Denoting $\langle\psi|A|\psi\rangle = \langle A\rangle$, this becomes

$$\langle H^2\rangle - \langle HPH\rangle = \langle H^2\rangle - \langle H\rangle\langle H\rangle = \langle H^2\rangle - \langle H\rangle^2 = (\Delta E)^2$$

(Since $\langle\psi|HPH|\psi\rangle = \langle\psi|H|\psi\rangle\langle\psi|H|\psi\rangle$).
We can then safely conclude that

$$\frac{ds}{dt} = \Delta E$$

E.2 Equation 64

Since the third term in the action does not contain H, we only need to consider the first two terms. Let us first look at the first term (which is by far the hardest of the two):

$$\frac{\delta\left(\frac{\sqrt{\langle\dot{\psi}|(1-P)|\dot{\psi}\rangle}}{\Delta E}\right)}{\delta\langle\psi|} = \frac{\delta\left(\sqrt{\frac{A}{B}}\right)}{\delta\langle\psi|}$$

where we define $A = \langle\dot{\psi}|(1-P)|\dot{\psi}\rangle\,,\quad B = (\Delta E)^2$. Varying both A and B yields (if $A(\langle\psi+\delta\psi|) = A+\delta A;\ B(\langle\psi+\delta\psi|) = B+\delta B$) the Taylor expansion

$$\frac{A+\delta A}{B+\delta B} = \frac{A}{B} + \frac{\delta A}{B} - \frac{A\delta B}{B^2} + \mathcal{O}(\delta\psi^2)$$

We can then conclude that

$$\delta\sqrt{\frac{A}{B}} = \frac{1}{2}\sqrt{\frac{B}{A}}\left(\frac{\delta A}{B} - \frac{A\delta B}{B^2}\right) = \frac{1}{2}\sqrt{\frac{1}{AB}}\delta A - \frac{1}{2}\sqrt{\frac{A}{B}}\frac{1}{B}\delta B$$

(Again a Taylor approximation).
Knowing what A and B are, we can calculate δA and δB:

$$\delta A = \langle \delta\dot{\phi}|(1-P)|\phi\rangle - \langle \delta\psi|\dot{\psi}\rangle \langle \dot{\psi}|\psi\rangle + \mathcal{O}(\delta\psi^2)$$

$$\delta B = \langle \delta\psi|H^2|\psi\rangle - 2\langle \delta\psi|H|\psi\rangle \langle H\rangle + \mathcal{O}(\delta\psi^2)$$

As a result

$$\frac{\delta\left(\sqrt{\frac{A}{B}}\right)}{\delta\langle\psi|} = \frac{1}{2}\left(\sqrt{\frac{1}{AB}}\left(\langle\frac{d}{dt}|(1-P)\dot{\psi}\rangle + |\dot{\psi}\rangle\langle\dot{\psi}|\psi\rangle\right) - \sqrt{\frac{A}{B}}\frac{1}{B}\left(H^2|\psi\rangle - 2H|\psi\rangle\langle H\rangle\right)\right)$$

After having taken the variation, we can take $A = B = (\Delta E)^2$ to find

$$\frac{\left(-\frac{d}{dt}(1-P)|\dot{\psi}\rangle + |\dot{\psi}\rangle\langle\dot{\psi}|\psi\rangle\right) - \left(H^2|\psi\rangle - 2H|\psi\rangle\langle H\rangle\right)}{2(\Delta E)^2}$$

After which we can apply the Schrödinger equation, (61), while taking ΔE constant, to achieve

$$i\frac{d}{dt}\left(\frac{(1-P)H|\psi\rangle}{2(\Delta E)^2}\right) - \frac{\langle H\rangle H|\psi\rangle + \left(H^2|\psi\rangle - 2\langle H\rangle H|\psi\rangle\right)}{2(\Delta E)^2}$$
$$= i\frac{d}{dt}\left(\frac{H-\langle H\rangle}{2(\Delta E)^2}\right)|\psi\rangle + i\left(\frac{H-\langle H\rangle}{2(\Delta E)^2}\right)|\dot{\psi}\rangle - \frac{\left(H^2|\psi\rangle - \langle H\rangle H|\psi\rangle\right)}{2(\Delta E)^2}$$

Conveniently though, the last two terms cancel, since

$$i(H-\langle H\rangle)H|\dot{\psi}\rangle = H^2|\psi\rangle - \langle H\rangle H|\psi\rangle$$

such that the only term that contributes to the variation of the first term of the action is

$$i\frac{d}{dt}\left(\frac{H-\langle H\rangle}{2(\Delta E)^2}\right)|\psi\rangle$$

Let us now look at the second term:

$$\delta(i\langle\dot{\phi}|\psi\rangle + \langle\phi|H|\psi\rangle + c.c.) = -i\langle\delta\psi|\dot{\phi}\rangle + \langle\delta\psi|H|\phi\rangle$$

where only the complex conjugate contributes to this variation, since only there $\langle\psi|$ appears. This variational derivative is simply:

$$\frac{\delta(i\langle\dot{\phi}|\psi\rangle + \langle\phi|H|\psi\rangle + c.c.)}{\delta|\psi\rangle} = -i|\dot{\phi}\rangle + H|\phi\rangle$$

such that the total variation with respect to $\langle\psi|$ yields

$$i\frac{d}{dt}\left(\frac{H-\langle H\rangle}{2(\Delta E)^2}\right)|\psi\rangle - i|\dot{\phi}\rangle + H|\phi\rangle = 0$$

E.3 Equation 65

Here, we need to consider all three terms in the action, since every term contains H. The variation of the first term works analogous to how it was found for equation 64:
Let us call $A = \langle\dot{\psi}|(1-P)|\dot{\psi}\rangle\,;\;\; B = (\Delta E)^2$. Again, then, the variation becomes

$$\delta\sqrt{\frac{A}{B}} = -\frac{1}{2}\sqrt{\frac{A}{B}}\frac{1}{B}\delta B$$

where there is no δA, since A does not contain H (explicitly). δB is straight forward to determine:

$$\delta B = \delta\left(\langle H^2\rangle - \langle H\rangle^2\right) = \langle H\delta H + \delta H H\rangle - 2\langle H\rangle\langle\delta H\rangle$$

To further compute this variation, we need to expand in an arbitrary basis χ:

$$\langle\delta H\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n}\delta\langle\psi|\chi_i\rangle\langle\chi_i|\delta H|\chi_j\rangle\langle\chi_j|\psi\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n}\delta\langle\psi|\chi_i\rangle\,\delta H_{ij}\langle\chi_j|\psi\rangle$$

such that

$$\frac{\langle\delta H\rangle}{\delta H_{ij}} = \langle\chi_j\rangle\,\psi\,\langle\psi|\chi_i\rangle = P_{ji}$$

So we can write

$$\frac{\langle\delta H\rangle}{\delta H} = P$$

Similarly we can find

$$\langle H\delta H + \delta H H\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle H\psi|\chi_i\rangle\langle\chi_i|\delta H|\chi_j\rangle\langle\chi_j|\psi\rangle + \sum_{p=1}^{n}\sum_{q=1}^{n}\langle\psi|\chi_p\rangle\langle\chi_p|\delta H|\chi_q\rangle\langle\chi_q|\psi\rangle$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\delta H_{ij}\langle\chi_j|\psi\rangle\langle\psi|H|\chi_i\rangle + \sum_{p=1}^{n}\sum_{q=1}^{n}\delta H_{pq}\langle\chi_q|\psi\rangle\langle\psi|\chi_p\rangle$$

which, like before, implies

$$\frac{\langle H\delta H + \delta H H\rangle}{\delta H} = HP + PH = \{H, P\}$$

such that

$$\delta\sqrt{\frac{A}{B}} = \frac{2\langle H\rangle P - \{H, P\}}{2(\Delta E)^2}$$

which concludes the variation of the first term.
For the second term, we need to take a look at

$$\frac{\langle\phi|\delta H|\psi\rangle + \langle\psi|\delta H|\phi\rangle}{\delta H}$$

We can again expand in a basis, such that

$$\langle\phi|H|\psi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n}\delta H_{ij}\langle\chi_j|\psi\rangle\langle\phi|\chi_i\rangle$$

which again implies

$$\frac{\langle\phi|\delta H|\psi\rangle}{\delta H} = |\psi\rangle\langle\phi| \text{ and analogously } \frac{\langle\psi|\delta H|\phi\rangle}{\delta H} = |\phi\rangle\langle\psi|$$

The variation for the third term is simply

$$\sum_{a=1}^{m} \lambda_a \frac{\delta f_a}{\delta H}$$

such that adding all the variations, we find:

$$|\psi\rangle\langle\phi| + |\phi\rangle\langle\psi| + \sum_{a=1}^{m} \lambda_a \frac{\delta f_a}{\delta H} + \frac{2\langle H\rangle P - \{H,P\}}{2(\Delta E)^2}$$

E.4 Equation 69

Let us call, for brevity, $M = \sum_{a=1}^{m} \lambda_a \frac{\delta f_a}{\delta H}$. Inserting (66) back into (65)

$$\frac{\{H,P\} - 2\langle H\rangle P}{2(\Delta E)^2} - M - P\left[\frac{H - \langle H\rangle}{2(\Delta E)^2} - M - \langle\psi|\phi\rangle\right] - \left[\frac{H - \langle H\rangle}{2(\Delta E)^2} - M - \langle\phi|\psi\rangle\right] P$$

where $M^\dagger = M$, since f_a can be denotes as power series of H, such that they are hermitian.
This then reduces to

$$-M + PM + P\langle\psi|\phi\rangle + MP + \langle\phi|\psi\rangle P = -M + PM + MP - \langle M\rangle P$$

$$M = \{M,P\} - \langle M\rangle P$$

where the $\langle M\rangle$ originates from equation 67. Since $\langle M\rangle$ commutes with P ($\langle M\rangle$ is a constant) and $P^2 = P$, we can subtract on both sides by $\langle M\rangle P$ to achieve

$$F = M - \langle M\rangle P = \{M - \langle M\rangle P, P\} = \{F,P\}$$

E.5 Equation 70

We can insert (66) into (64):

$$i\frac{d}{dt}\left(\frac{H - \langle H\rangle}{2(\Delta E)^2}\right)|\psi\rangle - i\frac{d}{dt}\left[\frac{H - \langle H\rangle}{2(\Delta E)^2} - \sum_{a=1}^{m} \lambda_a \frac{\delta f_a}{\delta H} - \langle\phi|\psi\rangle\right]|\psi\rangle$$

$$-i\left[\frac{H - \langle H\rangle}{2(\Delta E)^2} - \sum_{a=1}^{m} \lambda_a \frac{\delta f_a}{\delta H} - \langle\phi|\psi\rangle\right]|\dot\psi\rangle + H\left[\frac{H - \langle H\rangle}{2(\Delta E)^2} - \sum_{a=1}^{m} \lambda_a \frac{\delta f_a}{\delta H} - \langle\phi|\psi\rangle\right]|\psi\rangle = 0$$

The first term vanishes with part of the second term and the fractions $\frac{H-\langle H\rangle}{2(\Delta E)^2}$ also vanish, upon using Schrödinger's equation.

$$i\dot M|\psi\rangle + i\frac{d}{dt}\left(\langle\psi|\phi\rangle\right)|\psi\rangle + MH|\psi\rangle + \langle\psi|\phi\rangle H|\psi\rangle - HM|\psi\rangle - H\langle\psi|\phi\rangle|\psi\rangle = 0$$

We can fill in equation (64) to obtain

$$\frac{d}{dt}(\langle\psi|\phi\rangle)\,|\psi\rangle = i\,\langle\psi|H|\phi\rangle - i\,\langle\psi|H|\phi\rangle - i\frac{\langle\frac{d}{dt}(H-\langle H\rangle)\rangle}{2(\Delta E)^2} = -i\frac{\langle\dot{H}-\langle\dot{H}\rangle\rangle}{2(\Delta E)^2} = 0$$

where the last step holds true, because of Ehrenfest's Theorem:

$$\frac{d}{dt}\langle A\rangle = -i\,[A,H] + \langle\frac{dA}{dt}\rangle$$

Thence

$$i\dot{M}\,|\psi\rangle + MH\,|\psi\rangle - HM\,|\psi\rangle = 0$$

Let us now call $H = \tilde{H} + \tilde{G}$, where $\tilde{H}$ is the traceless part of H ($\tilde{H} = H - Tr(H)/n$), such that $\tilde{G}$ is a constant diagonal matrix. M and $\tilde{G}$ commute, such that the equation can be reduced to

$$\left(\dot{M} - iM\tilde{H} + i\tilde{H}M\right)|\psi\rangle = 0$$

Lastly,

$$\frac{d}{dt}(\langle M\rangle\,P) = \langle\dot{M}\rangle\,P - i\,\langle[M,H]\rangle\,P + \langle M\rangle\,\dot{P}$$

M and H commute and the last term can be rewritten, such that

$$\langle M\rangle\,\dot{P} = -i\,\langle M\rangle\,HP + i\,\langle M\rangle\,PH$$

And we can add and subtract $\frac{d}{dt}(\langle M\rangle\,P)\,|\psi\rangle$ (to effectively add zero) to obtain

$$\left(\dot{M} - \frac{d}{dt}\left(\langle M\rangle\,P\right) + i\left[\tilde{H}, M - \langle M\rangle\,P\right]\right)|\psi\rangle = \left(\dot{F} + i\left[\tilde{H},F\right]\right)|\psi\rangle = 0$$

E.6 Equation 71

Solving

$$\dot{F} = iF\tilde{H} - i\tilde{H}F$$

Requires the solution to be of the form

$$UAU^\dagger$$

where U is defined like in equation 72: $U = \mathcal{T}e^{-i\int_0^t \tilde{H}dt}$, since $\dot{U} = -i\tilde{H}U = -iU\tilde{H}$. A should be $F(0)$, since $F(0) = U(0)AU^\dagger(0) = A = F(0)$ ($U(0) = e^0 = 1$).

E.7 Equation 74

$$\frac{\lambda}{2}\left(Tr\tilde{H}\delta\tilde{H} + Tr\delta\tilde{H}\tilde{H}\right) = \frac{\lambda}{2}\sum_{i=1}^{n}\left[\tilde{H}_i^T\delta\tilde{H}_i + \delta\tilde{H}_i^T\tilde{H}_i\right] = \frac{\lambda}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\tilde{H}_{ji}\delta\tilde{H}_{ij} + \delta\tilde{H}_{ij}\tilde{H}_{ji}\right]$$

$$\frac{\lambda}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[2\delta\tilde{H}_{ij}\tilde{H}_{ji}\right]$$

which implies that

$$\frac{\delta}{\delta H}\lambda Tr(\tilde{H}^2/2) = \lambda\tilde{H}$$

E.8 Equation 77

Using Equation 75:

$$\langle \tilde{H}^2 \rangle = \langle \tilde{H}^2 - H\langle H \rangle - H\langle H \rangle - \langle H \rangle^2 \rangle = \langle H^2 \rangle - 2\langle H \rangle^2 + \langle H \rangle^2 = \langle H^2 \rangle - \langle H \rangle^2 = (\Delta E)^2$$

Not only does this hold, but also

$$\langle \tilde{H}^2 \rangle = Tr\left(\tilde{H}^2 P\right)$$

The trace is invariant under cyclic permutations, such that

$$Tr\left(\tilde{H}^2 P\right) = Tr\left(\tilde{H} P \tilde{H}\right) = Tr\left(\tilde{H}^2 - \tilde{H} P \tilde{H}\right) = Tr\left(\tilde{H}^2\right) - Tr\left(\tilde{H} P \tilde{H}\right)$$

whence we can conclude that

$$Tr\left(\tilde{H} P \tilde{H}\right) = Tr(\tilde{H}^2/2) = \omega^2$$

E.9 Derivation of $\dot{\lambda} = 0$

Let us multiply equation 78 by $\langle \psi | H$ to get

$$\dot{\lambda}\langle H\tilde{H} \rangle + \lambda \langle H\dot{\tilde{H}} \rangle = \dot{\lambda}(\Delta E)^2 + \lambda \langle H\dot{\tilde{H}} \rangle = 0$$

Because $\langle H\tilde{H} \rangle = \langle H^2 \rangle - \langle H\langle H \rangle \rangle = \langle H^2 \rangle - \langle H \rangle^2 = (\Delta E)^2$ and thence

$$\dot{\lambda} = -\lambda \frac{\langle H\dot{\tilde{H}} \rangle}{(\Delta E)^2}$$

It conveniently holds, though, that $\langle H\dot{\tilde{H}} \rangle = \frac{1}{2}\frac{d}{dt}\langle H\tilde{H} \rangle$ because

$$\frac{d}{dt}\langle H\tilde{H} \rangle = \langle \dot{\psi}|H\tilde{H}|\psi \rangle + \langle \psi|H\tilde{H}|\dot{\psi} \rangle + \langle \dot{H}\tilde{H} \rangle + \langle H\dot{\tilde{H}} \rangle = i\langle H\tilde{H}H \rangle - i\langle H^2\tilde{H} \rangle + \langle \dot{H}\tilde{H} \rangle + \langle H\dot{\tilde{H}} \rangle$$

which, because $\left[H, \tilde{H}\right] = H^2 - H\langle H \rangle - H^2 + \langle H \rangle H = 0$, reduces to

$$\langle \dot{H}\tilde{H} \rangle + \langle H\dot{\tilde{H}} \rangle = \langle \dot{H}(H - \langle H \rangle) \rangle + \langle H(\dot{H} - \langle \dot{H} \rangle) \rangle = \langle \dot{H}H \rangle - \langle \dot{H} \rangle \langle H \rangle + \langle H\dot{H} \rangle - \langle H \rangle \langle \dot{H} \rangle$$

$$= \frac{d}{dt}\langle H^2 \rangle - \langle H \rangle^2 = \frac{d}{dt}(\Delta E)^2 = 0$$

(Since we had just confirmed that the energy difference is constant)
Clearly our claim that $\langle H\dot{\tilde{H}} \rangle = \frac{1}{2}\frac{d}{dt}\langle H\tilde{H} \rangle$ holds by the way, since $\dot{H}$ commutes with H (because $\frac{d}{dt}H^2 = 2\dot{H}H = \dot{H}H + H\dot{H}$).

E.10 Equation 81

Since $|\tilde{\psi}\rangle$ satisfies Schrödinger's equation (it is merely $|\psi\rangle$ with an extra phase factor), it holds that

$$\tilde{H}\,|\tilde{\psi}\rangle = i\,|\dot{\tilde{\psi}}\rangle$$

And $|\psi\rangle = e^{-i\int_0^t dt\langle H\rangle}\,|\tilde{\psi}\rangle$, such that

$$\tilde{H} = \tilde{H}P + P\tilde{H} = \tilde{H}e^{-i\int_0^t dt\langle H\rangle}\,|\tilde{\psi}\rangle\,\langle\tilde{\psi}|\,e^{i\int_0^t dt\langle H\rangle} + e^{-i\int_0^t dt\langle H\rangle}\,|\tilde{\psi}\rangle\,\langle\tilde{\psi}|\,e^{i\int_0^t dt\langle H\rangle}\tilde{H}$$

$$= \tilde{H}\,|\tilde{\psi}\rangle\,\langle\tilde{\psi}| - |\tilde{\psi}\rangle\,\langle\tilde{\psi}|\,\tilde{H} = i\left(|\dot{\tilde{\psi}}\rangle\,\langle\tilde{\psi}| - |\tilde{\psi}\rangle\,\langle\dot{\tilde{\psi}}|\right)$$

where the exponentials vanish, since they are mere constants that mutually multiply to 1.

E.11 Equation 82

Having a gander at equations 79 and 81, we obtain:

$$\dot{\tilde{H}}\,|\tilde{\psi}\rangle = i\frac{d}{dt}\left(|\dot{\tilde{\psi}}\rangle\,\langle\tilde{\psi}| - |\tilde{\psi}\rangle\,\langle\dot{\tilde{\psi}}|\right)|\tilde{\psi}\rangle = i\left(|\ddot{\tilde{\psi}}\rangle\,\langle\tilde{\psi}| + |\dot{\tilde{\psi}}\rangle\,\langle\dot{\tilde{\psi}}| - |\dot{\tilde{\psi}}\rangle\,\langle\dot{\tilde{\psi}}| - |\tilde{\psi}\rangle\,\langle\ddot{\tilde{\psi}}|\right)|\tilde{\psi}\rangle$$

$$= i\,|\ddot{\tilde{\psi}}\rangle + |\dot{\tilde{\psi}}\rangle\,\langle\tilde{H}\rangle - |\dot{\tilde{\psi}}\rangle\,\langle\tilde{H}\rangle - i\,|\tilde{\psi}\rangle\,\langle\ddot{\tilde{\psi}}|\tilde{\psi}\rangle = i\left(|\ddot{\tilde{\psi}}\rangle - |\tilde{\psi}\rangle\,\langle\ddot{\tilde{\psi}}|\tilde{\psi}\rangle\right)$$

Furthermore

$$|\tilde{\psi}\rangle\,\langle\ddot{\tilde{\psi}}|\tilde{\psi}\rangle = -\,|\tilde{\psi}\rangle\,\langle\tilde{\psi}|H^2|\tilde{\psi}\rangle = |\tilde{\psi}\rangle\,\langle\tilde{\psi}|\ddot{\tilde{\psi}}\rangle = \tilde{P}\,|\ddot{\tilde{\psi}}\rangle$$

such that we finally find

$$\dot{\tilde{H}}\,|\tilde{\psi}\rangle = \left(1 - \tilde{P}\right)|\ddot{\tilde{\psi}}\rangle = 0$$

E.12 Derivation of $\dot{\tilde{H}} = 0$

Taking the derivative of equation 81, gives

$$-i\dot{\tilde{H}} = |\ddot{\tilde{\psi}}\rangle\,\langle\tilde{\psi}| - |\tilde{\psi}\rangle\,\langle\ddot{\tilde{\psi}}|$$

Like conducted in section E.11 above.
This expression conveniently equals

$$P\tilde{H}^2 - \tilde{H}^2P = -i\dot{\tilde{H}}$$

We can now use equation 76 to rewrite this into

$$-i\dot{\tilde{H}} = \tilde{H}^2 - \tilde{H}P\tilde{H} - \tilde{H}^2 + \tilde{H}P\tilde{H} = 0$$

And as a result clearly $\dot{\tilde{H}} = 0$.

E.13 Equation 83

To solve equation 82 it is first necessary to simplify it a little bit:

$$(1-P)\,|\ddot{\tilde{\psi}}\rangle = 0; \quad |\ddot{\tilde{\psi}}\rangle + |\tilde{\psi}\rangle \langle \tilde{H}^2\rangle = 0$$

Using equation 77 gifts us $\langle \tilde{H}^2\rangle = \omega^2$, so that we have our to be solved differential equation

$$|\ddot{\tilde{\psi}}\rangle + \omega^2 \,|\tilde{\psi}\rangle = 0$$

We can judiciously guess its solution to bear the form

$$|\tilde{\psi}(t)\rangle = A\cos(\omega t) + B\sin(\omega t)$$

Using the boundary conditions $|\tilde{\psi}(0)\rangle$ and $|\dot{\tilde{\psi}}(0)\rangle$, we find

$$A = |\tilde{\psi}(0)\rangle \text{ and } B = \frac{|\dot{\tilde{\psi}}(0)\rangle}{\omega}$$

such that

$$|\tilde{\psi}(t)\rangle = \cos(\omega t)\,|\tilde{\psi}(0)\rangle + \frac{\sin(\omega t)}{\omega}\,|\dot{\tilde{\psi}}(0)\rangle$$

E.14 Equation 84

We want to write $|\tilde{\psi}\rangle$ and $\tilde{H}$ in terms of the orthonormalized states $|\tilde{\psi}(0)\rangle = |\psi_i\rangle$ and $|\psi'_f\rangle$. The former is already orthonormal, but the latter is not and we wish to express it in the actual final state $|\psi_f\rangle$.
It holds that

$$|\psi'_f\rangle = \frac{|\psi_f\rangle - \langle\psi_f|\psi_i\rangle\,|\psi_i\rangle}{\sqrt{1-|\,\langle\psi_f|\psi_i\rangle\,|^2}}$$

Since we already know that $\langle\psi_i|\dot{\psi}_i\rangle = 0$ and $\langle\psi_i|\psi'_f\rangle = 0$ (since they are orthonormal), it must hold that

$$|\psi'_f\rangle = C\,|\dot{\tilde{\psi}}(0)\rangle$$

Thus

$$|\tilde{\psi}(t)\rangle = \cos(\omega t)\,|\psi_i\rangle + \frac{\sin(\omega t)}{C\omega}\,|\psi'_f\rangle$$

Using that its norm has to be conserved;

$$1 = \cos^2(\omega t) + \frac{\sin^2(\omega t)}{C^2\omega^2}$$

Implying $C = \frac{1}{\omega}$.

E.15 Equation 85

We had just derived in section E.12 that $\dot{\tilde{H}} = 0$, which means that $\tilde{H}(t) = \tilde{H}(0)$. Thence we find

$$\tilde{H} = \tilde{H}(0) = i\left(|\dot{\tilde{\psi}}(0)\rangle\,\langle\psi_i| - |\psi_i\rangle\,\langle\dot{\tilde{\psi}}(0)|\right)$$

Using $|\dot{\tilde{\psi}}(0)\rangle = \omega\cos(\omega t)\,|\psi'_f\rangle$, this becomes

$$\tilde{H} = i\omega\left(|\psi'_f\rangle\,\langle\psi_i| - |\psi_i\rangle\,\langle\psi'_f|\right)$$

E.16 Equation 86

Simply requiring that $|\tilde{\psi}(T)\rangle$ equals $|\psi_f\rangle$ gives

$$\langle\psi_f|\tilde{\psi}(T)\rangle = 1 = \cos(\omega T)\,\langle\psi_f|\psi_i\rangle + \sin(\omega T)\,\langle\psi_f|\psi_f'\rangle$$

Or

$$\cos(\omega T)\,\langle\psi_f|\psi_i\rangle + \sin(\omega T)\frac{\langle\psi_f|\psi_f\rangle - \langle\psi_f|\psi_i\rangle^2}{\sqrt{1-|\,\langle\psi_f|\psi_i\rangle\,|^2}} = 1$$

Of which

$$T = \frac{1}{|\omega|}\arccos|\,\langle\psi_f|\psi_i\rangle\,|$$

is a solution, since substituting it gives

$$\langle\psi_f|\psi_i\rangle^2 + \left(\sqrt{1-|\,\langle\psi_f|\psi_i\rangle\,|^2}\right) = 1$$

E.17 Equation 88

The Trace of both $\tilde{H}$ as σ_z is zero. We already know that $Tr(F) = 0$. P is two-dimensional, where $|\psi\rangle = \begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}$, such that $Tr(P) = |\psi_1|^2 + |\psi_2|^2 = 1$ (because $|\psi\rangle$ is normalized). This grants that

$$Tr\left(\lambda_1\,\langle\tilde{H}\rangle\,P + \lambda_2\,\langle\sigma_z\rangle\,P\right) = \left(\lambda_1\,\langle\tilde{H}\rangle + \lambda_2\,\langle\sigma_z\rangle\right)\cdot 1 = 0$$

As a result

$$F = \lambda_1\tilde{H} + \lambda_2\sigma_z = UF(0)U^\dagger$$

E.18 Derivation of $\dot{\lambda}_i = 0$

It holds that

$$\lambda_1(t)\tilde{H}(t) = \lambda_2(t)\sigma_z(t) = UF(0)U^\dagger = U\left(\lambda_1(0)\tilde{H}(0) + \lambda_2(0)\sigma_z(0)\right)U^\dagger$$

$$= \lambda_1(0)\tilde{H}(t) + \lambda_2(0)\sigma_z(t)$$

such that necessarily $\lambda_1(0) = \lambda_1(t);\;\; \lambda_2(0) = \lambda_2(t)$.

E.19 Equations 89 and 90

By definition of $U = \mathcal{T}e^{-i\int_0^t \tilde{H}(t)dt}$, $\dot{U} = -i\tilde{H}U$. Multiplying both sides of equation 88 gives

$$\lambda_1\tilde{H}U = i\lambda_1\dot{U} = UF(0) - \lambda_2\sigma_z U$$

Calling $\Omega = \frac{\lambda_2}{\lambda_1}$, this equation reads

$$\dot{U} = i\Omega\sigma_z U - iUF(0)/\lambda_1$$

Of which the solution is

$$U = e^{\int_0^t i\Omega\sigma_z dt} e^{-i\int_0^t iF(0)/\lambda_1 dt} = e^{\int_0^t i\Omega\sigma_z dt} e^{-i\int_0^t \left[\lambda_1 \tilde{H}(0) + \lambda_2\sigma_z\right]/\lambda_1 dt}$$

Since λ_i are constant, along with σ_z and $\tilde{H}(0)$, this equation becomes

$$U = e^{i\Omega\sigma_z t} e^{-i\left[\tilde{H}(0)+\Omega\sigma_z\right]t}$$

Since $\dot{U} = -i\tilde{H}U$; $\tilde{H} = i\dot{U}U^\dagger$, using $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$, we can calculate $\tilde{H}$:

$$\dot{U} = i\Omega\sigma_z U - ie^{i\Omega\sigma_z t}\left[\tilde{H}(0) + \Omega\sigma_z\right] e^{-i\left[\tilde{H}(0)+\Omega\sigma_z\right]t}$$

$$= i\Omega\sigma_z U - ie^{i\Omega\sigma_z t}\tilde{H}(0)e^{-i\left[\tilde{H}(0)+\Omega\sigma_z\right]t} - ie^{i\Omega\sigma_z t}\Omega\sigma_z e^{-i\left[\tilde{H}(0)+\Omega\sigma_z\right]t}$$

$$= i\Omega\sigma_z U - i\Omega\sigma_z U - ie^{i\Omega\sigma_z t}\tilde{H}(0)e^{-i\left[\tilde{H}(0)+\Omega\sigma_z\right]t} = -ie^{i\Omega\sigma_z t}\tilde{H}(0)e^{-i\left[\tilde{H}(0)+\Omega\sigma_z\right]t}$$

Subsequently;

$$\tilde{H} = i\dot{U}U^\dagger = e^{i\Omega\sigma_z t}\tilde{H}(0)e^{-i\left[\tilde{H}(0)+\Omega\sigma_z\right]t}e^{i\left[\tilde{H}(0)+\Omega\sigma_z\right]t}e^{-i\Omega\sigma_z t} = e^{i\Omega\sigma_z t}\tilde{H}(0)e^{-i\Omega\sigma_z t}$$

E.20 Derivation of $\tilde{H}(0) = i\omega\sigma_y$

Let $F(0)$ be $\begin{matrix} a & b \\ c & d \end{matrix}$, and it has already been established that $P(0) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Solving equation 69 at $t = 0$ then reads

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2a+b+c & a+2b+d \\ a+2c+d & b+c+2d \end{bmatrix}$$

which can only be true if $a + d = 0$ and $b + c = 0$. The former is surely true, since $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The latter implies $\tilde{h}_{12}(0) = -\tilde{h}_{21}(0)$ and since we had already established that $|\tilde{h}_{12}| = |\tilde{h}_{21}| = \omega$, we find $\tilde{h}_{12}(0) = \pm i\omega$.
Since the choice is trivial (we can just turn around the magnetic field), we can choose $\tilde{H}(0) = -i\omega\sigma_y$.

E.21 Equations 91, 92 and 93

Since $\Omega\sigma_z$ is diagonal, $e^{-i\Omega\sigma_z t}$ is straight forward to compute:

$$e^{-i\Omega\sigma_z t} = \begin{bmatrix} e^{-i\Omega t} & 0 \\ 0 & e^{i\Omega t} \end{bmatrix}; \quad e^{i\Omega\sigma_z t} = \begin{bmatrix} e^{i\Omega t} & 0 \\ 0 & e^{-i\Omega t} \end{bmatrix}$$

For 2 by 2 matrices, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the matrix exponential reads:

$$e^A = \frac{1}{\Delta}\begin{bmatrix} e^{\frac{a+d}{2}}\left[\Delta\cosh\left(\frac{\Delta}{2}\right) + (a-d)\sinh\left(\frac{\Delta}{2}\right)\right] & 2be^{\frac{a+d}{2}}\sinh\left(\frac{\Delta}{2}\right) \\ 2ce^{\frac{a+d}{2}}\sinh\left(\frac{\Delta}{2}\right) & e^{\frac{a+d}{2}}\left[\Delta\cosh\left(\frac{\Delta}{2}\right) + (d-a)\sinh\left(\frac{\Delta}{2}\right)\right] \end{bmatrix}$$

where $\Delta = \sqrt{(a-d)^2 + 4bc}$.
$\tilde{H}$ then reads

$$e^{i\Omega\sigma_z t}\tilde{H}(0)e^{-i\Omega\sigma_z t} = \begin{bmatrix} e^{i\Omega t} & 0 \\ 0 & e^{-i\Omega t} \end{bmatrix} \begin{bmatrix} i\omega & 0 \\ 0 & -i\omega \end{bmatrix} \begin{bmatrix} e^{-i\Omega t} & 0 \\ 0 & e^{i\Omega t} \end{bmatrix} = \begin{bmatrix} 0 & i\omega e^{2i\Omega t} \\ -i\omega e^{-2i\Omega t} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & i\omega\cos(2\Omega t) - \omega\sin(2\Omega t) \\ -i\omega\cos(2\Omega t) - \omega\sin(2\Omega t) & 0 \end{bmatrix}$$

$$= \omega\sin(2\Omega t)\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + \omega\cos(2\Omega t)\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -\omega\left(\sin(2\Omega t)\sigma_x + \cos(2\Omega t)\sigma_y\right)$$

such that

$$\tilde{H}(t) = -\boldsymbol{\sigma}\cdot\boldsymbol{B}(t); \;\; \boldsymbol{B}(t) = \begin{bmatrix} \sin(2\Omega t) \\ \cos(2\Omega t) \\ 0 \end{bmatrix}$$

Now comes the tedious part; $-i\left[\tilde{H}(0) + \Omega\sigma_z\right]t = \begin{bmatrix} -i\Omega t & \omega t \\ -\omega t & i\Omega t \end{bmatrix}$ and thence,

$$e^{-i\left[\tilde{H}(0)+\Omega\sigma_z\right]t} = \begin{bmatrix} \cosh(i\Omega' t) - \frac{\Omega}{\Omega'}\sinh(i\Omega' t) & -i\frac{\omega}{\Omega'}\sinh(i\Omega' t) \\ i\frac{\omega}{\Omega'}\sinh(i\Omega' t) & \cosh(i\Omega' t) + \frac{\Omega}{\Omega'}\sinh(i\Omega' t) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\Omega' t) - i\frac{\Omega}{\Omega'}\sin(\Omega' t) & \frac{\omega}{\Omega'}\sin(\Omega' t) \\ -\frac{\omega}{\Omega'}\sin(\Omega' t) & \cos(\Omega' t) + i\frac{\Omega}{\Omega'}\sin(\Omega' t) \end{bmatrix}$$

where $\Omega' = \sqrt{\Omega^2 + \omega^2}$.
Now we need to calculate U:

$$U = e^{i\Omega\sigma_z t}e^{-i\left[\tilde{H}(0)+\Omega\sigma_z\right]t} = \begin{bmatrix} e^{i\Omega t}\left(\cos(\Omega' t) - i\frac{\Omega}{\Omega'}\sin(\Omega' t)\right) & e^{i\Omega t}\frac{\omega}{\Omega'}\sin(\Omega' t) \\ -e^{-i\Omega t}\frac{\omega}{\Omega'}\sin(\Omega' t) & e^{-i\Omega t}\left(\cos(\Omega' t) + i\frac{\Omega}{\Omega'}\sin(\Omega' t)\right) \end{bmatrix}$$

whence

$$|\psi(t)\rangle = U|\psi(0)\rangle = \frac{1}{\sqrt{2}}U\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} e^{i\Omega t}\left(\cos(\Omega' t) - i\frac{\Omega}{\Omega'}\sin(\Omega' t)\right) + e^{i\Omega t}\frac{\omega}{\Omega'}\sin(\Omega' t) \\ e^{-i\Omega t}\left(\cos(\Omega' t) + i\frac{\Omega}{\Omega'}\sin(\Omega' t)\right) - e^{-i\Omega t}\frac{\omega}{\Omega'}\sin(\Omega' t) \end{bmatrix}$$

where the squared norm, $|\psi|^2 = 1$, is conserved, since:

$$\frac{1}{2}\left(\cos(\Omega' t) - i\frac{\Omega}{\Omega'}\sin(\Omega' t) + \frac{\omega}{\Omega'}\sin(\Omega' t)\right)\left(\cos(\Omega' t) + i\frac{\Omega}{\Omega'}\sin(\Omega' t) + \frac{\omega}{\Omega'}\sin(\Omega' t)\right)$$

$$+\frac{1}{2}\left(\cos(\Omega' t) + i\frac{\Omega}{\Omega'}\sin(\Omega' t) - \frac{\omega}{\Omega'}\sin(\Omega' t)\right)\left(\cos(\Omega' t) - i\frac{\Omega}{\Omega'}\sin(\Omega' t) - \frac{\omega}{\Omega'}\sin(\Omega' t)\right)$$

$$= \frac{1}{2}\cos^2(\Omega' t) + \frac{\omega}{\Omega'}\sin(\Omega' t)\cos(\Omega' t) + \frac{1}{2}\frac{\omega^2}{\Omega'^2}\sin^2(\Omega' t) + \frac{1}{2}\frac{\Omega^2}{\Omega'^2}\sin^2(\Omega' t)$$

$$+\frac{1}{2}\cos^2(\Omega' t) - \frac{\omega}{\Omega'}\sin(\Omega' t)\cos(\Omega' t) + \frac{1}{2}\frac{\omega^2}{\Omega'^2}\sin^2(\Omega' t) + \frac{1}{2}\frac{\Omega^2}{\Omega'^2}\sin^2(\Omega' t) = 1$$

We can then calculate $\langle\sigma_x\rangle$ in the following way:

$$\langle\sigma_x\rangle = \frac{1}{2}e^{2i\Omega t}\left[\cos^2(\Omega' t) - 2i\frac{\Omega}{\Omega'}\sin(\Omega' t)\cos(\Omega' t) - \frac{\Omega^2+\omega^2}{\Omega'^2}\sin^2(\Omega' t)\right]$$

$$+\frac{1}{2}e^{-2i\Omega t}\left[\cos^2(\Omega' t) + 2i\frac{\Omega}{\Omega'}\sin(\Omega' t)\cos(\Omega' t) - \frac{\Omega^2+\omega^2}{\Omega'^2}\sin^2(\Omega' t)\right]$$

(which has already been simplified a couple of times for the sake of brevity)

$$= \frac{1}{2}e^{2i\Omega t}\left[\cos(2\Omega' t) - i\frac{\Omega}{\Omega'}\sin(2\Omega' t)\right] + \frac{1}{2}e^{-2i\Omega t}\left[\cos(2\Omega' t) + i\frac{\Omega}{\Omega'}\sin(2\Omega' t)\right]$$

$$= \cos(2\Omega t) cos(2\Omega' t) + \frac{\Omega}{\Omega'}\sin(2\Omega t)\sin(2\Omega' t)$$

Analogously:

$$\langle\sigma_y\rangle = \frac{i}{2}e^{-2i\Omega t}\left[\cos(2\Omega' t) + i\frac{\Omega}{\Omega'}\sin(2\Omega' t)\right] - \frac{i}{2}e^{2i\Omega t}\left[\cos(2\Omega' t) - i\frac{\Omega}{\Omega'}\sin(2\Omega' t)\right]$$

$$= \frac{\Omega}{\Omega'}\cos(2\Omega t)\sin(2\Omega' t) - \sin(2\Omega t)\cos(2\Omega' t)$$

And lastly,

$$\langle\sigma_z\rangle = \frac{1}{2}\left(\cos(\Omega' t) + \frac{\omega}{\Omega'}\sin(\Omega' t)\right)^2 + \frac{\Omega^2}{\Omega'^2}\sin^2(\Omega' t) - \frac{1}{2}\left(\cos(\Omega' t) - \frac{\omega}{\Omega'}\sin(\Omega' t)\right)^2 - \frac{\Omega^2}{\Omega'^2}\sin^2(\Omega' t)$$

$$= 2\frac{\omega}{\Omega'}\sin(\Omega' t)\cos(\Omega' t) = \frac{\omega}{\Omega'}\sin(2\Omega' t)$$

such that:

$$\langle\boldsymbol{\sigma}\rangle = \begin{bmatrix} \cos(2\Omega t)\cos(2\Omega' t) + \frac{\Omega}{\Omega'}\sin(2\Omega t)\sin(2\Omega' t) \\ \frac{\Omega}{\Omega'}\cos(2\Omega t)\sin(2\Omega' t) - \sin(2\Omega t)\cos(2\Omega' t) \\ \frac{\omega}{\Omega'}\sin(2\Omega' t) \end{bmatrix}$$

E.22 Equation 94

Since $\tilde{H} = H - \tilde{G}$, where $\tilde{G}$ is a constant diagonal matrix, it holds that

$$(\Delta E)^2 = \langle H^2\rangle - \langle H\rangle^2 = \langle\tilde{H}^2\rangle - \langle\tilde{H}\rangle^2$$

Using equations 92 and 93, we find:

$$\langle\tilde{H}\rangle = -\omega\sin(2\Omega t)\cos(2\Omega t)\cos(2\Omega' t) - \frac{\Omega\omega}{\Omega'}\sin^2(2\Omega t)\sin(2\Omega' t)$$

$$-\frac{\Omega\omega}{\Omega'}\cos^2(2\Omega t)\sin(2\Omega' t) + \omega\sin(2\Omega t)\cos(2\Omega t)\cos(2\Omega' t) = -\frac{\omega\Omega}{\Omega'}\sin(2\Omega' t)$$

Because $\tilde{H}^2 = \omega^2\sin^2(2\Omega t)I + \omega^2\cos^2(2\Omega t)I = \omega^2 I$, $\langle\tilde{H}^2\rangle$ is simply $\omega^2|\psi|^2 = \omega^2$.

Consequently,

$$(\Delta E)^2 = \omega^2 - \left(-\frac{\omega\Omega}{\Omega'}\sin(2\Omega' t)\right)^2 = \omega^2\left(1 - \frac{\Omega^2}{\Omega'^2}\sin^2(2\Omega' t)\right);$$

$$\Delta E = |\omega|\left(1 - \left(\frac{\Omega}{\Omega'}\sin(2\Omega' t)\right)^2\right)^{\frac{1}{2}}$$

E.23 Equation 95

Because $2\Omega' T = l\pi$, $2\Omega T = k\pi$ and $\omega^2 = \Omega'^2 - \Omega^2$, we find

$$\omega^2 = \left(\frac{l\pi}{2T}\right)^2 - \left(\frac{k\pi}{2T^2}\right)^2 = \frac{l^2 - k^2}{4T^2}\pi^2$$

Multiplying by T^2 on both sides and taking the square root gives

$$|\omega| T = \frac{\pi}{2}\sqrt{l^2 - k^2}$$

Because $\Omega = \frac{k\pi}{2T}$ and with the equation for $|\omega| T$ just above here:

$$\left|\frac{\Omega}{\omega}\right| = \frac{\frac{|k|\pi}{2T}}{\frac{\pi}{2T}\sqrt{l^2 - k^2}} = \frac{|k|}{\sqrt{l^2 - k^2}}$$

Appendix F

Translation of Euler's Texts

Text E042

On

The line of the fastest descent

in whichever resistant medium

By the author

L. Euler.

§. 1. The curves, to be exposed to some certain motion in vacuum, are found without much work. The same curves in a resistant medium, are not only found with much more work; but they also require more skill and caution. It also repeatedly occurs, that many problems in the hypothesis for a resistant medium either utterly obstruct a solution, or only allow a solution for particular cases. The problem about tautochrones, of which I rather very much have doubts about whether hypotheses for friction can be solved, for other frictions, that are more than simple and a ratio of multiplied speeds, is of this kind.

§. 2. It also reaches out here to the problem of the brachistochrone lines or the fastest descents, which is proposed by the celebrated Johannes Bernoulli in his hypothesis for empty geometries, soon after he encountered multiple different solutions, which one may see in the *Actis Lipsiensibus, Transact. Angl. Comment. Parisinis*, and many more other books. However, I proposed the same to be solved problem of the hypothesis for a resistant media first in *Actis Lipf. A. 1726.*[1], and it doesn't leave anyone uncertain, both for its not to be despised elegance, and its unique foresight, which is necessary to use in its solution.

§. 3. Moreover, after I had proposed this problem, the celebrated Hermans deemed it a worthy problem, of which he included the solution in the dissertation on various motions in *Tom. II. Comment.*[2]. He very carefully examined a copy of the matters sufficiently, however, which he investigated in that dissertation, as it was seen by the most perspicacious man, that that problem, which had only been mentioned by few, did not allow a solution, and he carefully examined the found solution. From this it was argued, that the curves, assigned by this problem, are neither convenient, nor do they possess the brachistochrone

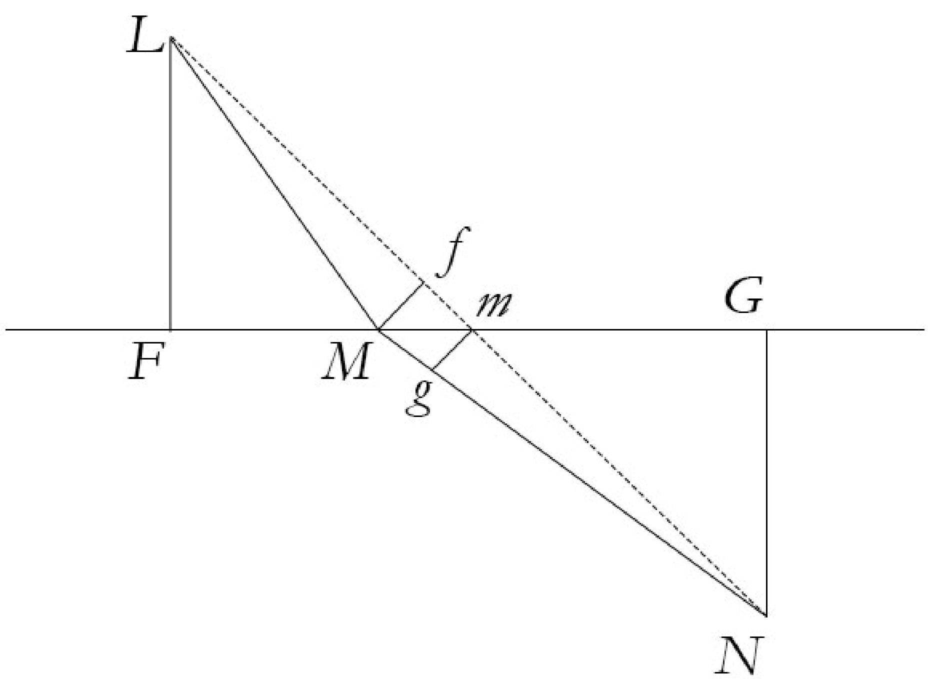
[1]Consult the dissertation E 1 of this edition, vol. II, 6

[2]Consult the commentaries of the academy of sciences of Saint Petersburg 2 (1727), 1729, p. 139, and also the preface of the preceding volume

property. I have also advised the man with good memory about this matter by letter, and I have sent him my solution, that disagreed with his, so that he searched for the cause of the discrepancy, about which he responded me, both that he began to doubt his solution, how the troubles, that were about to be resolved, were first, and that he wanted to perfect his amendment for its remarkable soundness, which we would already certainly have had too, if death didn't intervene.

§. 4. Because he himself would do it, if he stayed alive, I therefore don't judge anyone about to reluctantly report, if I will do the same and I will yet correct his solution. I don't only not settle this injustice, but I do also believe myself to be obliged to do it, in order that in the future others, which remarkable men are responsible for the diminishing of his fading fame and reputation, don't accidentally take his fame. I will however show both how much foresight is required to be applied in order to avoid errors of this kind, and each one will forgive those errors, rather easily made by the deceased man, and also not find fault with my intention, to which end I decided to resolve the problem, proposed by me, with a genuine method.

§. 5. A particular sought lemma, to which we have to attend in the solution to his problem, is from the nature of maxima and minima, through which the direction of two contiguous segments of a sought curve is determined, over which a body would move, to be descended in less time, than whichever other segments, posed within the very same boundaries. Such a proposition, demonstrated by Huygens, is considered, and Hermans used it in his solution: as soon will be clear though, it yielded more, than was necessary, and not enough attention was paid to the restriction, that this proposition requires. On that account I will yet bring forth both the Huygenian Lemma and another utility, wider, than in whichever case in an appropriate medium.



§. 6. Let it therefore be necessary to define a point, M, on the straight line FG from which at given ends, L and M, the drawn lines LM and MN are traversed by a descending body in the fastest time: let moreover the speed of the body above FG be m and below be M on FG. After posing this, $\frac{LM}{m} + \frac{MN}{n}$ then consequently has to be minimal, because the time is assigned to LMN by this quantity. Although, the letter is already used, m is to be chosen close to point M, and by drawing Lm and mN, LMN and LmN are to be passed during equal times. Thus, henceforth $\frac{LM}{m} + \frac{MN}{n} = \frac{Lm}{m} + \frac{mN}{n}$ will hold, from which, by having drawn the arcs Mf and mg with L and N as centres, this equation holds, $\frac{mf}{m} = \frac{Mg}{n}$, or this analogy, $mf : Mg = m : n$. Truly, mf relates to Mg as the cosine of the angle LMF relates to the cosine of the angle GMN. Therefore the cosine of the angles, that the two segments should establish with the line FG, are proportional to the speeds, with which those segments are passed. That is the Huygenian lemma, which Hermans used to reach his solution to the problem.

§. 7. Where it's however seen, how widely this lemma is accessible and in which cases it can be evoked, attention must be had for that, what this lemma is used for; I freed every segment below the line FG from their assumed speed n. On which account, if the bodies in all these here segments, wherever the point M is assumed, don't possess this same speed, this lemma is incorrectly applied, and leads to the wrong solution. That however happens in a resistant medium, and it was used so, although the celebrated Hermans, after he had used it abundantly in this lemma on the discovery of the brachistochrone in vacuum, for resistant media was tempted by this very lemma in a proper way.

§. 8. In vacuum a matter also still must be built upon in this way, in order that the line FG is everywhere perpendicular to the direction of the disturbing force field. Truly, when this, which is required, holds, and the body itself, descending to any point on the line FG from L, always gains an increase in speed, in order that thus single segments, situated within FG, are traversed with equal speed. Therefore, the curve in these cases, naturally in vacuum, will be a brachistochrone, if the speed of the body on whichever segment, would have been proportional to the sines of the angles, that this segment establishes in the direction of the disturbing force field. On this account, the curve of the fastest descent in vacuum will be able to have been discovered with the help of this rule, whichever law of the disturbing force field will exist.

§. 9. From this it's already seen plentily that the given rule for finding a brachistochrone in a resistant medium can't be adapted. For indeed the growths in speed, that the body in descent from L to the points of the line FG acquires, aren't mutually equal, if not only the line FG were perpendicular to the direction of the disturbing force field; but in addition they go down by the incline of the segments that will be passed, as is easily evident from the nature of friction. For these cases it's furthermore necessary for a peculiar lemma to be established, in which the speeds through the lower segments are ordained to be variables, in which the point M on FG is accepted for diverse loci.

§. 10. Then consequently, with like beforehand both the points M and m, being assumed close, and the segments LM, MN and also Lm and mN being drawn, let the speed through the segments LM and Lm is q, the speed through $MN = q + dt$, but on the segment mN it's $q + dt + dd\theta$. The growth of speed, acquired through LM, is of course called dt, and the growth, that is acquired through Lm, is called $dt + dd\theta$. For therefore the time through LMN becomes a minimum, it's necessary that it becomes equal in time to LmN. From this it's obtained that $\frac{LM}{q} + \frac{MN}{q+dt} = \frac{Lm}{q} + \frac{mN}{q+dt+dd\theta}$, but from this it comes forth that $\frac{mf}{q} = \frac{Mg}{q+dt} + \frac{Mndd\theta}{(q+dt)(q+dt+dd\theta)}$ or $\left(q^2 + 2qdt + dt^2 + qdd\theta + dtdd\theta\right) mf = \left(q^2 + qdt + qdd\theta\right) Mg +$

$qmNdd\theta$. It truly holds that $mf = \frac{FM \cdot Mm}{LM}$ and $Mg = \frac{MG \cdot Mm}{LM}$. By having substituted those and neglected what had to be neglected $q\left(\frac{MG}{LM} - \frac{FM}{LM}\right) = \frac{FMdt}{LM} - \frac{LMdd\theta}{Mm}$ arises. Because $dd\theta$ is always in this way determined by Mm, in order that it is of the form $Z \cdot Mm$, it won't involve other quantities, if they won't depend on the point M.

§. 11. If the segments LF and NG are set to be equal, and they're called dx, and also FM becomes dy, $LM = ds$, MG will be $dy + ddy$ and $MN = ds + dds$. With these substitutions, the aforementioned formula passes over to $\frac{qdsddy - qdydds}{ds^2} = \frac{dydt}{ds} - \frac{dsdd\theta}{Mm}$, or because $dsdds = dyddy$, with dx being fixed constant, to $\frac{qdx^2 ddy}{ds^3} = \frac{dydt}{ds} - \frac{dsdd\theta}{Mm}$. And this is the lemma, which, instead of the Huygenian, we should use to to find brachistochrones in a resistant medium.

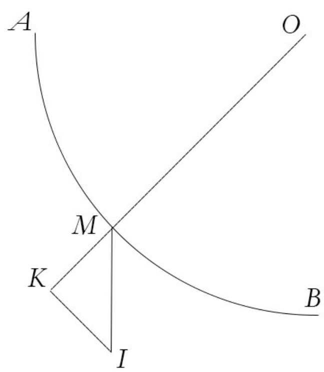
§. 12. Let there now be whichever disturbing force field, and its direction specifically, as before, perpendicular to the line FG. Let the force field $= p$, posing the force of gravity $= 1$, be evoked, driving a body, describing the segment LM or Lm. Let it further resist the medium in whichever multiplied ratio of the speeds, of which the exponent is $2n$, and this friction therefore maintains itself, as it is equal to the force of gravity, 1, if the speed of the body had size c. Let the speed of the body in L already be as much, as is acquired by sliding the weight through the height, v. By having posed these things, the force of friction, which retards the motion of the body, that proceeds from L through FG, will be $= \frac{v^n}{c^n}$.

§. 13. The body descends because of the force field p, either through LM or through Lm and receives the same increase in speed, because FG is situated perpendicular to the direction of the force field. The size of v also will take on an increase $= pdx$. Moreover, the friction will thus delay the body, descending through LM, so that in a decrease of size of v is $= \frac{v^n}{c^n}LM$. But if the body is placed to move through Lm, the decrease of the size of v will be $= \frac{v^n}{c^n}Lm$. By which means the speed, with which the segments LM and Lm are passed over, is owed to the size of v; in particular the speed through MN to the size $v + pdx - \frac{v^n}{c^n}LM$ and the speed through mN to the size $v + pdx - \frac{v^n}{c^n}Lm$.

§. 14. After having compared this with our lemma, we have $q = \sqrt{v}$, $q + dt = \sqrt{v + pdx - \frac{v^n}{c^n}LM} = \sqrt{v} + \frac{pdx - \frac{v^n}{c^n}LM}{2\sqrt{v}}$, from which $dt = \frac{pdx - \frac{v^n}{c^n}LM}{2\sqrt{v}}$. And also $q + dt + dd\theta = \sqrt{v + pdx - \frac{v^n}{c^n}Lm} = \sqrt{v} + \frac{pdx - \frac{v^n}{c^n}Lm}{2\sqrt{v}}$. From this thus becomes $dd\theta = \frac{v^n(LM - Lm)}{2c^n\sqrt{v}} = -\frac{v^n FM \cdot Mm}{2c^n LM\sqrt{v}}$, consequently $\frac{dd\theta}{Mm} = -\frac{v^n dy}{2c^n ds\sqrt{v}}$. The following equation will then consequently come up from multiplying everything by $2\sqrt{v}$, $\frac{2vdx^2 ddy}{ds^3} = \frac{pdxdy}{ds} - \frac{v^n dy}{c^n} + \frac{v^n dy}{c^n}$ or $2vdxddy = pdyds^2$. A brachistochrone curve should therefore have this here property, which is $v = \frac{pdyds^2}{2dxddy}$, from which it will be easy to discover it.

§. 15. Because the boundaries, in which the friction $\frac{v^n}{c^n}$ begins, mutually cancel each other, this most extensive lemma is attainable and this lemma can be adapted to whichever friction, without any change. Then this is the universal property of all brachistochrones, both in vacuum, as in whichever resistant medium. To the end that it's possible to remember that lemma easier by memory, though, we induce it in another form.

§. 16. If the found equation, $2vdxddy = pdyds^2$ is divided by ds^3, it transforms into this $\frac{2vdxddy}{ds^3} = \frac{pdy}{ds}$, in which solution the disturbing force field p gives rise to the risen perpendicular force $\frac{pdy}{ds}$. In the other portion, $\frac{2vdxddy}{ds^3}$, $-\frac{ds^3}{dxddy}$ means the radius of curvature of the curve LMN, following the region extended from F. However, because the curve is convex towards F, the radius of the origin will be directed to the opposing intersection G, and for this reason it has a negative value. Its length will thus be $\frac{ds^3}{dxddy}$. Therefore, by posing the radius of the origin $= r$, and the perpendicular force $= N$ this equation will be had $\frac{2v}{r} = N$. Moreover, $\frac{2v}{r}$ marks the centrifugal force, by which a body, to which extent it can't advance in a straight line, pursues a curve, on which it moves. On account of this matter every brachistochrone has that property, so that the perpendicular force is equal to the centrifugal force.

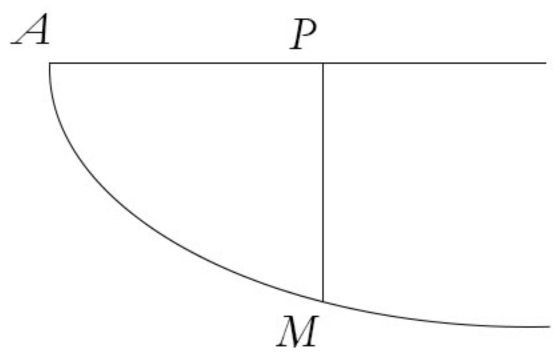


§. 17. It must however be noted that every body, that moves either in vacuum or in a resistant medium because of any disturbing force over the concave part of a certain curve AMB, that the curve pursues by two forces, naturally the force, perpendicular to the original disturbing force field, and its centrifugal force. Let there be a force field along MI, disturbing a body on M; this is usually decomposed in two others, MK and KI, of which the direction of MK is perpendicular to the curve and therefore this force is called perpendicular, and of the other, KI, the direction along the tangent of the curve and is called tangential. Therefore it's evident that the body only presses on the curve with the perpendicular component. In addition, the curve AMB is pressed in M by the centrifugal force, which maintains itself through the the gravitational force, so that its size generates the speed v, towards half the radius of curvature, MO.

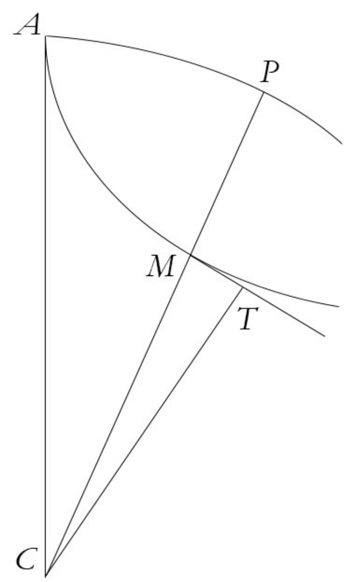
§. 18. If thus the curve AMB were in this way compared, either in vacuum or in whichever resistant medium, as both forces on the body, that descends over the curve, by which the curve is naturally pressed perpendicular and centrifugal, will be mutually equal, the curve will always be a brachistochrone, or the body descends over it from A to M in less time, than passing over whichever other line through A and M. Therefore this equality between the perpendicular force and centrifugal force is the true and universal law of all brachistochrone curves, and its benefit in whichever hypothesis for both a disturbing force field and friction will be that brachistochrone curves are easily determined.

§. 19. Because according to the Huygenian Theorem the speed in vacuum has to be proportional to the sine of the angle, that the curve establishes with the direction of the force field, i.e. to $\frac{MK}{MI}$ itself, $\frac{MK^2}{MI^2 \cdot MO}$ will be proportional to MK itself or $\frac{MK}{MI}$ to $MI \cdot MO$ itself. Therefore all brachistochrones in vacuum possess this property, that the sine of the angle, which the direction of the force field makes with the curve, is everywhere proportional to the radius of curvature and jointly the disturbing force field. Hence with help of this rule without determining the speed all brachistochrones in vacuum can be found.

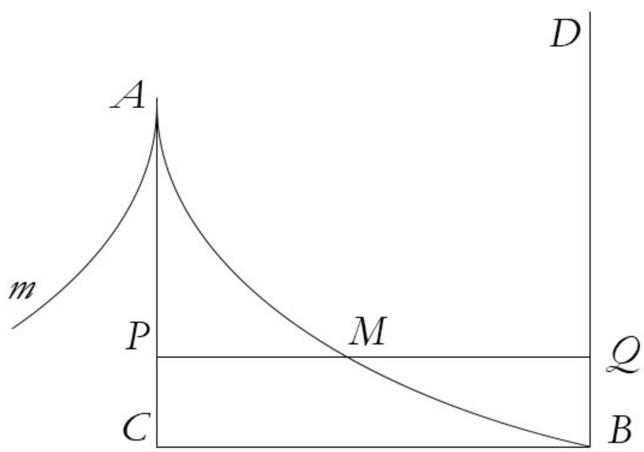
§. 20. The beginning of the curve is however always in A, on which all descents should happen from rest, the locus, on which the tangent of the curve coincides with the direction of the force field. Let in this locus in vacuum the speed itself of the body be equal to 0, because the angle of the curve, with the same direction as the force field, also becomes 0. In a resistant medium, however, that beginning of motion doesn't differ from vacuum, and on account of this matter in this case too, the tangent of the beginning of the curve should coincide with the direction of the force field. Truly, this reasoning must be held in the addition of constant quantities, when we integrate the differentio-differential equation of the brachistochrone, and we should work it out, so that the curve has a given beginning and moves through a given point.



§. 21. Let's illustrate the rule, given in §. 19. for finding brachistochrones in vacuum with examples, and let the disturbing force field be constant $= g$, its direction vertical along PM. Let the sought brachistochrone truly be AM and let the abscissae on the horizontal line, passing through the beginning of the curve, be AP. These things being done, let $AP = y$, $PM = x$, $AM = s$ and the sine of the angle, that PM makes with the curve, be $= \frac{dy}{ds}$, and the radius of curvature $= \frac{ds^3}{dxddy}$, posing dx a constant, which should be proportional to that $\frac{dy}{ds}$ due to the constant force field. Therefore $\frac{ds^3}{dxddy}$ becomes $\frac{ady}{ds}$ or because $ddy = \frac{dsdds}{dy}$ in this way $ds^3 = adxdds$. By dividing by ds^2 and integrating, $s = C - \frac{adx}{ds}$ comes forth. Because by making $s = 0$, dx should become ds, C will be a and for that reason $sds = ads - adx$, which further yields $s^2 = 2as - 2ax$, the equation for a cycloid, after integrating, just as agreed upon.



§. **22.** Let onwards C be the center of the forces, that attracts in whichever multiplied ratio of distances, of which the exponent is m. Let the curve AM be a brachistochrone for a body that is moving in vacuum. Let be declared that $CA = a$, $CM = y$ and that the perpendicular CT on the tangent MT, drawn from C, $= z$. The force in M along the line MC, pulling the body, will thus be as y^m, the sine of the angle of the curve with that direction will be $= \frac{z}{y}$, and the radius of curvature will be $-\frac{ydy}{dz}$. Therefore, since the rule must be enforced, $\frac{z}{y}$ will be as $\frac{y^{m+1}dy}{dz}$ or $Azdz = y^{m+2}dy$, of which the integral is $C + Az^2 = y^{m+3}$. Because, if $y = a$, $z = 0$ will hold, C will be a^{m+3}, and consequently $Az^2 = a^{m+3} - y^{m+3}$, with A arbitrarily being taken negative. And this equation involves all brachistochrones, which exists around a center of forces.



§. **23.** Let's however go back to a medium that resists in whichever multiplied ratio of the speeds, of which the exponent is $2n$. Let the disturbing force field be posed constant, specifically $= g$, and having a vertical direction everywhere, parallel to that AP. Let AMB be the to be found curve of the fastest descent, on which we pose $AP = x$, $PM = y$ and $AM = s$. Let the speed on M further be owed to the size v, whereby the friction in M will be $= \frac{v^n}{c^n}$. Hence from the disturbance of the force field and the effect of the friction will simultaneously, $dv = gdx - \frac{v^n ds}{c^n}$ will be had. The brachistochronism truly yields $2vdxddy = gdyds^2$, having posed dx constant (§. 14.). From these connected equations, the equation for the sought brachistochrone curve will advance, after getting rid of the letter v.

§. 24. Because dx will be constant, $ddy = \frac{dsdds}{dy}$ and therefore $v = \frac{gdsdy^2}{2dxdds}$. Thus $dv = \frac{gdy^2dds^2+2gds^2dds^2-gdsdy^2d^3s}{2dxdds^2}$. By substituting these values in the equation $dv = gdx - \frac{v^n ds}{c^n}$, $\frac{gdsdy^2d^3s-3gdy^2dds^2}{2dxdds^2} = \frac{g^n nds^{n+1}dy^{2n}}{2^n c^n dx^n dds^n}$ will be had, or $dsd^3s - 3dds^2 = \frac{g^{n-1}ds^{n+1}dy^{2n-2}}{2^{n-1}c^n dx^{n-1}dds^{n-2}}$. This equation, if the resistant medium is an exotic infinite or it transforms into vacuum, in which case $c = \infty$ holds, transforms into $dsd^3s = 3dds^2$, of which the integral is $adxdds = ds^3$. We showed this in §. 21., which constitutes a cycloid.

§. 25. However, to construct a general equation, I pose $ds = pdx$, so that $dds = dpdx$ holds and $d^3s = dxddp$. Hence $dy = dx\sqrt{p^2-1}$ and $v = \frac{gpdx(p^2-1)}{2dp}$ will hold. However, that equation will transform into this $pddp - 3dp^2 = \frac{g^{n-1}p^{n+1}dx^n(p^2-1)^{n-1}}{2^{n-1}c^n dp^{n-2}}$. Further $dx = qdp$ is posed, and $ddp = -\frac{dpdq}{q}$ will hold. By substituting this, $-\frac{pdq-3qdp}{q^{n+1}} = \frac{g^{n-1}p^{n+1}(p^2-1)^{n-1}dp}{2^{n-1}c^n}$ will come forth. This equation is multiplied by np^{-3n-1}; by having done this,

$$-\frac{np^{-3n}dq - 3np^{-3n-1}qdp}{q^{n-1}} = \frac{ng^{n-1}p^{-2n}\left(p^2-1\right)^{n-1}dp}{2^{n-1}c^n}$$

will be had. Of this, the integral is

$$\frac{2^{n-1}c^n}{ng^{n-1}p^{3n}q^n} = \int \frac{\left(p^2-1\right)^{n-1}dp}{p^{2n}}$$

Let for the grace of brevity be $\frac{ng^{n-1}}{2^{n-1}c^n}\int\frac{(p^2-1)^{n-1}}{p^{2n}} = P^{-n}$, which quantity, if the integration doesn't succeed, can always be showed by allowing quadratures. Having posed this, $p^3q = P$ will thus hold, and on account of $q = \frac{dx}{dp}$, dx becomes $\frac{Pdp}{p^3}$. Consequently $x = \int\frac{Pdp}{p^3}$, $s = \int\frac{Pdp}{p^2}$ and $y = \int\frac{Pdp\sqrt{p^2-1}}{p^3}$. Therefore the brachistochrone hypothesis in whichever resistant medium will be possible to be constructed in this way.

§. 26. If the friction of the medium is as a square of the speed, $n = 1$ will hold, and for that reason $P^{-1} = \frac{1}{c}\int\frac{dp}{p^2} = \frac{1}{ac} - \frac{1}{cp} = \frac{p-a}{acp}$. Hence P becomes $\frac{acp}{p-a}$; and $p^3q = \frac{ac}{p-a} = \frac{p^2dx}{dp}$, or $dx = \frac{acdp}{p^2(p-a)}$, of which the integral is $x = b + \frac{c}{p} + \frac{c}{a}l\frac{p-a}{p} = b + \frac{cdx}{ds} + \frac{c}{a}l\frac{ds-adx}{ds}$ [l denotes the natural logarithm]. In this equation, because, after making $x = 0$, ds should be dx, becomes $b = -c - \frac{c}{a}l\,(1-a)$. For the sought curve this equation will thus be had $x = \frac{c(dx-ds)}{ds} + \frac{c}{a}l\frac{ds-adx}{ds-ads}$. Or if the equation is wanted free of logarithms; this differentio-differential, $acdxdds = ds^3 - adxds^2$, having posed dx constant. After this is arranged in another way, it transforms into $\frac{acdxdds}{ds^2} = ds - adx$, of which the integral is $s - ax = ac - \frac{acdx}{ds}$ or $sds - axds = acds - acdx$. Integrating this yields

$$s = cl\frac{s-ax-ac+c}{c-ac}$$

or

$$e^{\frac{s}{c}}\left(c-ac\right)=s-ax+c-ac$$

The infimum point of this curve, B, will be there, where $s=a\left(x+c\right)$ holds. Therefore in this case AB will be $cl\frac{1}{1-a}$ and $AC=\frac{c}{a}l\frac{1}{1-a}-c$.

§. 27. If however the Huygenian Theorem was used, just as in that suitable case, we would thence immediately have had this equation $v=\frac{ady^2}{ds^2}$. Hence $dv=\frac{2adx^2ddy}{ds^3}=gdx-\frac{a^n dy^{2n}}{c^n ds^{2n-1}}$ or $2adx^2ddy=gdxds^3-\frac{a^n dy^{2n}}{c^n ds^{2n-4}}$. This, after having posed $ds=pdx$, transforms into $\frac{2apdp}{\sqrt{p^2-1}}=gpx^2dx-\frac{a^n dx\left(p^2-1\right)^n}{c^n p^{2n-2}}$, which is already separated by itself, and therefore it can be constructed. If $n=1$ is posed, so that the brachistochrone for a medium, resistant in a ratio of squared speeds, appears , $2acdx^2ddy=cgdxds^3-ady^2ds^2$ will hold or $2acdx^2dds=cgdxdyds^2-ady^3ds$. This equation, if it furthermore rests on a simpler lemma, is yet much more ordered and intricate, than our found brachistochrone; it is usually often a criterium of truth by itself, especially if the more painstaking calculus deduces it.

§. 28. Where it moreover appears, along what figure our brachistochrone in a resistant, with the square of the speed, medium must be had, we sum this equation $e^{\frac{s}{c}}\left(c-ac\right)=s-ax+c-ac$. This, after converting $e^{\frac{s}{c}}$ into a series, transforms into

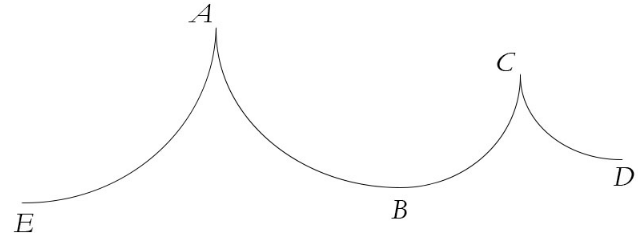
$$\left(c-ac\right)\left(1+\frac{s}{c}+\frac{s^2}{1\cdot 2\cdot c^2}+\frac{s^3}{1\cdot 2\cdot 3\cdot c^3}+\frac{s^4}{1\cdot 2\cdot 3\cdot 4\cdot c^4}+etc.\right)=s-ax+c-ac$$

from which, after posing $\frac{1-a}{a}=k$, this equation is discovered

$$x=s-\frac{ks^2}{1\cdot 2\cdot c}-\frac{ks^3}{1\cdot 2\cdot 3\cdot c^2}-\frac{ks^4}{1\cdot 2\cdot 3\cdot 4\cdot c^3}-etc.$$

It's thus perceived that k should necessarily be a positive number, because otherwise x is made $>s$, which can't happen; thus $a=\frac{1}{1+k}$ will hold. From this series, because it exceedingly easily converges for whichever value of that s, the answer for that x will be found. Besides, it is understood that this curve beyond A is continued in Am, which is similar to that AM.

§. 29. In what way the curve beyond B verily is extended, I investigate in this reasoning. By having drawn a vertical axis BD from B and applying MQ in it, let $BQ=PC=u$, and the arc $BM=t$. Having posed this, $s=cl\frac{1}{1-a}-t$ will hold, and $x=\frac{c}{a}l\frac{1}{1-a}-c-u$, by substituting which, this equation emerges $ce^{-\frac{t}{c}}=au-t+c$ or this differential, $tdt-audt=acdu$. Through the series $au=\frac{t^2}{1\cdot 2\cdot c}-\frac{t^3}{1\cdot 2\cdot 3\cdot c^2}+\frac{t^4}{1\cdot 2\cdot 3\cdot 4\cdot c^3}-etc.$ will verily be had. This equation directly corresponds with that, which I invented in *A. 1729.* for the tautochrone in that rising hypothesis on friction. Therefore another relation of the permitted curve beyond the axis BD will be the tautochrone, pertaining to the descent.



Thus the brachistochrone curve will have the form of this kind $EABCD$, provided with infinite cusps $A, Cetc.$, of which some are higher than A, some are lower than C. The branches from each of two parts of each one cusp are mutually equal and similar. The elevation of higher cusps is $\frac{c}{a}l\frac{1}{1-a} - c$, and of lower ones in particular $c - \frac{c}{a}l(1+a)$. Those higher branches specifically AB or AE are $= cl\frac{1}{1-a}$ and the length of the more depressed CB, CD is $= cl(1+a)$.That convenience among the tautochrone and brachistochrone is certainly deserved to be inspected beyond vacuum and too especially in this hypothesis on friction, and the investigation rests, what similar analogy perhaps maintains the locus in the remaining hypotheses of friction? That, which renders the most difficult invention of the tautochrones the easiest.

Text E759

A more accurate research

About

Brachistochrones

By the author

L. Euler.

§. 1. A principle of this kind, that I taught about these curves in book II of my Mechanicae[3], rests upon, what can't be allowed in a resistant medium. Thereafter, I tried to obtain the same argument from the first principles of Maxima and Minima in my isoperimetric treatment; to such a great degree are they, which I conducted there on the brachistochrone in a resistant medium, truly involved in the excessively generalised analytic formulae, such that thence barely anyone is able to pick out a true nature of those curves. On that account I decided to expand this same argument in a bigger study here and to derive it clearly and perspicuously from the first principles.

§. 2. From this principle I have consequently thereafter derived all brachistochrones in resistant media too. However, after a more fruitful isoperimetric theory was researched, I soon discovered that, which that principle in a resistant medium could not allow, nor did any of those things, which I investigated, studying all my mechanics work, yet concentrate on this defect, that I myself however happily corrected in my treatment of the isoperimetric problems, and I demonstrated determining the true brachistochrones for whichever resistant medium so much.

§. 3. That error, which I frankly admit, is meanwhile still not so enormous, that it can not only in a certain manner not be excused, but also united with truth, if only the state of the question is only altered briefly. Because if among all curves, over which a descending body acquires the same speed (of which the amount in any case still is infinite), and that the body is allowed to be led over from a higher end to a lower one, but only between them, intelligibly the one is sought, over which the body arrives from the highest point all the way to the lowest in the shortest time, then all brachistochrones, assigned by me and derived from the told principle, will be agreeing with the truth.

§. 4. Where it becomes however clearer, under which condition this principle has its place, and when it fails, I decide to develop a more accurate complete theory of brachistochrones. Since I observed, that the forces of this kind, to which this principle by no means can be adapted, can yet be presented, if also only

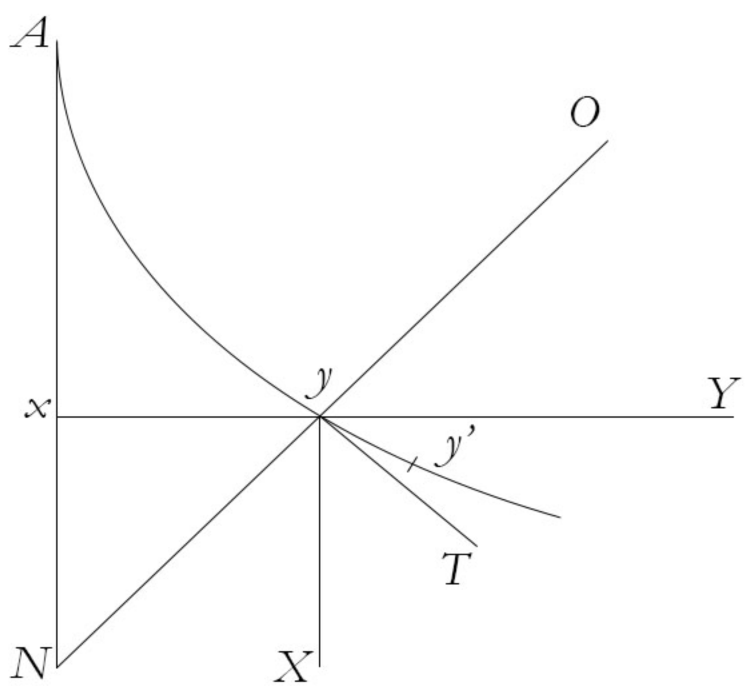
[3]Opera Omnia, II vol 2 p.170 §376, p.332 §673. Consult in addition the dissertation E042 of this volume, p.41

the motion in vacuum is considered; on this account I will draw your attention away from all resistance here, since this argument is already sufficiently and courteously studied in my little isoperimetric work. On this account, I will not observe other forces, beyond such, that I called absolute, of which the action depends on a single place, on which the body moves, nor does its speed bring anything to the disturbing forces.

§. 5. This treatment is besides however divided in two parts, so that every motion of the body is either solved in the same plane, or it moves out of the same plane. Because for this distinction the method of finding brachistochrones should certainly be extended to different situations, although in the former case two coordinates, to introduce the calculus, suffice, in the latter inevitable case three coordinates are required, which new case is so much straight forward; neither do the brachistochrones, which don't continue in the same plane, come anyone to mind to investigate, as much as I indeed remember; on this account I will propose the treatment following this differential sequence bipartite.

I. On the Brachistochrones

Existing in the same plane.



§. 6. Therefore, it is also necessary that all these disturbing forces, existing in the same plane, are, what I will however most generally consider. Let us therefore pose the motion of a body in the same plane, enclosed in the figure, and let Ay be the curve, over which the body moves, after it departed from the point A, which curve we refer to the axis Ax and we call the both coordinates $Ax = x$ and $xy = y$, let us call yy' in particular ds, such that thus, having posed $dy = pdx$, $ds = dx\sqrt{1 + pp}$ holds; whence if yO were the radius of curvature of the curve, it will consistently hold that $yO = \frac{dx(1+pp)^{\frac{3}{2}}}{dp}$. The body is already disturbed by whichever forces in y, and it's always allowed to decompose them into both yX and yY, although they have the same directions as the coordinates. Let us therefore call these forces $yX = X$

and $yY = Y$, and because the action of those forces is assumed to depend on the unique locus of the body, y, it's as much allowed to consider that these letters X and Y are whichever functions of both coordinates x and y. I then consider those forces, which appear, if the true motoric forces are divided by the mass of the body and therefore are expressed by absolute numbers, already as much to be accelerative, having denoted the accelerative force of natural gravity, of which it's possible to compare all other forces, unit.

§. 7. When, while the body descends over the curve Ay, it then consequently sustains the action of the two forces $yX = X$ and $yY = Y$ in that place y, these forces unbind according to the direction of motion, or into the tangent yT and the direction normal to it yN, and the tangential force is found to be $yT = \frac{Xdx+Ydy}{ds}$, while the other, the normal force is surely $yN = \frac{Xdy-Ydx}{ds}$, by the former of which that motion of the body, proceding through the segment yy', will be accelerated, but the other normal force gives rise to the pressure, that the body exerts on the curve, if it's applied to the mass of the body, which, if the mass of the body is denoted by M, will be $\frac{M(Xdy-Ydx)}{ds}$, to which thus, according to the principle, which I established above, the centrifugal force, born from the curvature, of the body should be equal for brachistochrones.

§. 8. Let us now denote the speed, with which the body traverses the segment yy', with the letter v, which expresses the space, which will be traversed by this speed in a common second; and where we refer everything to the to be measured determinates, let g denote the altitude through which the mass firstly falls for a common second, and from the principles of motion it holds that $vdv = 2gTds$, if accordingly T denotes the tangential force, which was $\frac{Xdx+Ydy}{ds}$, from which this equation follows: $vdv = 2g\,(Xdx + Ydy)$; whence the determination of the speed depends on the integration of this formula, because it holds that $vv = 4g \int (Xdx + Ydy)$.

§. 9. Because if the letters X and Y will already be such functions of x, y, that this formula permits integration, which happens, as is the case, if $\frac{dX}{dy} = \frac{dY}{dx}$ will hold; then the speed of the body, v, will be a function directly determined by both variables x and y, and therefore it will depend on the sole location of the body, y. But if however this condition doesn't take place, then the speed will furthermore not depend on the sole location y, but besides involve the whole track, of the already traversed curve Ay, according to the values, which the formula receives through the whole traversed curve Ay by Xdx and Ydy; hence these two cases, to very carefully be mutually distinguished by each other, occur, such that naturally the integration formula $Xdx + Ydy$ is wide ranging. Soon the principle, told above, will however be accessible to take place in the sole former case, but it can surely by no means be evoked to be used in the other case.

§. 10. Because the tiny amount of time, in which the segment of the curve $yy' = ds = dx\sqrt{1+pp}$ is traversed, truly is so, such that the time through the curve Ay turns out to be minimal, or such that that curve is a true brachistochrone, it is necessary that the integral formula $\int \frac{ds}{v} = \int \frac{dx\sqrt{1+pp}}{v}$ obtains its minimal value between all curves, that are able to lead from the point A to the point y. In my isoperimetric treatment I however showed, if whichever integral formula $\int Vdx$ should be either the maximum or the minimum, where V does not only depend on both those coordinates x and y in whichever way, but also on the relation between the differentials of those, and of which ordinate $dy = pdx$ holds, like already posed , as we already did, further $dp = qdx, dq = rdx, dr = sdx, etc.$ and it will hold that

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + etc.$$

when for the case of the maximum or minimum this equation always takes place:

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + etc. = 0$$

which equation then only thus takes place, when V will be a function of the quantities $x, y, p, q, r, etc.$, that is, when its value only depends on the sole point y and the segment of the curve in that locus. When however the function V furthermore involves whichever integral formulae, then, too, the ends, hence depending on that equation, should be added, in which case all the calculus demands most large digressions, which I will however not undertake here, but I will only stick to the equation, provided here.

§. 11. Hence it's then consequently apparent, that that equation of maximum or minimum can not take place, if the speed v isn't a function, determined by both x and y, or if the formula $\int (Xdx + Ydy)$ actually admits integration, which case I will therefore consider more accurately here. Because then consequently for our brachistochrones $\int Vdx$ should become $\int \frac{dx\sqrt{1+pp}}{v}$, and thus $V = \frac{\sqrt{1+pp}}{v}$, dV will be $-\frac{dv}{vv}\sqrt{1+pp} + \frac{pdp}{v\sqrt{1+pp}}$, where instead of v it is thus necessary to substitute its value by x and y. Above, however, we had this equation: $vdv = 2g\,(Xdx + Ydy)$, whence dv is $\frac{2g}{v}\,(Xdx + Ydy)$, and just like that v is partly expressed by x and partly by y; on that account, if this value is substituted and a comparison is done with the general form, told above: $dV = Mdx + Ndy + Pdp + Qdq + etc.$ becomes

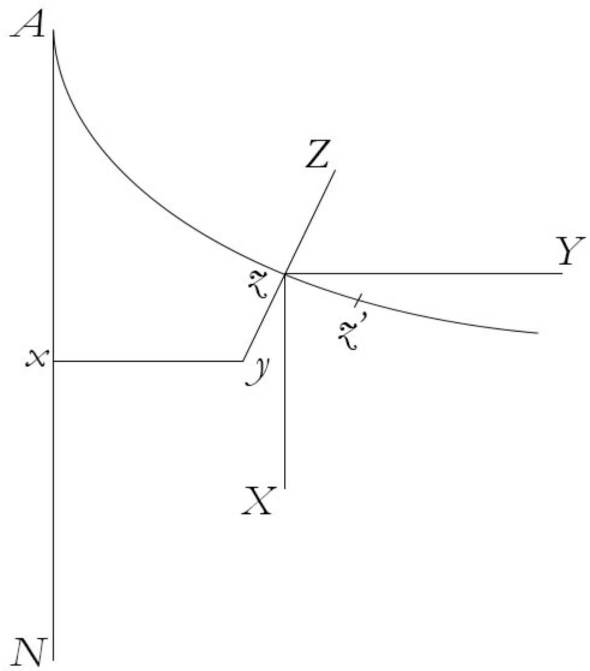
$$M = -\frac{2gX\sqrt{1+pp}}{v^3}; N = -\frac{2gY\sqrt{1+pp}}{v^3}; P = -\frac{p}{v\sqrt{1+pp}}; Q = 0; R = 0; etc.$$

and just like that we will now have this simple equation for the brachistochrone: $N - \frac{dP}{dx} = 0$, or $Ndx = dP$, such that the value of that P should therefore already again be differentiated. dP will moreover be $-\frac{dv}{vv} \cdot \frac{p}{\sqrt{1+pp}} + \frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp}}$, and for that reason $dP = -\frac{2g(Xdx+Ydy)}{v^3} \cdot \frac{p}{\sqrt{1+pp}} + \frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp}}$, to which expression the quantity $Ndx = -\frac{2gYdx\sqrt{1+pp}}{v^3}$ should be equal, from which equation further is acquired, that will hold: $\frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp}} = \frac{2gXdx}{v^3} \cdot \frac{p}{\sqrt{1+pp}} - \frac{2gYdx}{v^3\sqrt{1+pp}}$ or $\frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp}} = \frac{2g}{v^3\sqrt{1+pp}}\,(Xdy - Ydx)$.

§. 12. Moveover we invented above, that the normal force, to be born from the disturbing forces and pressing along yN, is $\frac{Xdy-Ydx}{ds}$, which equation of ours, if it's called Θ, such that $\Theta = \frac{Xdy-Ydx}{dx\sqrt{1+pp}}$, will be discovered $\frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp}} = \frac{2g\Theta dx}{v^3}$, and thus Θ will be $\frac{vv}{2gdx} d \cdot \frac{p}{\sqrt{1+pp}}$. Truly, it holds $d \cdot \frac{p}{\sqrt{1+pp}} = \frac{dp}{(1+pp)^{\frac{3}{2}}}$, and thus Θ will become $\frac{vv}{2gdx} \cdot \frac{dp}{(1+pp)^{\frac{3}{2}}}$. We saw however further that the radius of curvature in the point y is $\frac{dx(1+pp)^{\frac{3}{2}}}{dp}$, which, if it's called r, will make $\Theta = \frac{vv}{2gr}$. It is moreover further true that this formula $\frac{vv}{2gr}$ expresses the centrifugal force, with which the curve in the point y is pressed by a body, descending along that curvature, which force we thus now observe to be equal to the normal force Θ, whenever the formula $\int (Xdx + Ydy)$ permits integration, contrary to that the equation for a brachistochrone must otherwise surely very much have itself, and of which the determination requires most intricate calculi. Conveniently however, it comes with experience, whenever a body is disturbed by real forces, of which kind gravity is, and whichever centripetal forces and however many, disturbing according to whichever functions of distance, such that the formula $\int (Xdx + Ydy)$ permits integration and for that reason the principle established above actually takes place. Only imaginary forces are certainly excluded, which indeed can find whichever place in the nature of matters.

II. On the Brachistochrones

Not existing in the same plane.



§. 13. This case occurs, when forces, by which a body is simultaneously disturbed, won't be situated in the same plane. Let henceforth the curve Az be the sought brachistochrone, over which a body will begin to be moved from the point A. Let us therefore determine whichever its point z by the three coordinates, which are $Ax = x; xy = y; yz = z$; let a segment of the curve verily be called $zz' = ds$; such that so ds^2 is $dx^2 + dy^2 + dz^2$. Moreover, the disturbing forces, whenever they will be compared, are decomposed in the same three fixed directions and are called $zX = X; zY = Y; zZ = Z$; which quantities thus can be whichever functions of the three variables x, y, z.

§. 14. To already define the motion of the curve, let's define the whole matter from the first principles of motion, and, after posing a segment of time $= dt$, the determination of the motion of the body is contained in these three formulae:

$$1\ ^\circ)\ \frac{ddx}{dt^2} = 2gX;\ 2\ ^\circ)\ \frac{ddy}{dt^2} = 2gY;\ 3\ ^\circ)\ \frac{ddz}{dt^2} = 2gZ;$$

where g again describes the altitude of the fall of a mass in the first common second since we want to express the time t in common seconds. Now, the first of these equations multiplied by dx, the second by dy, the third by dz and integrated, they yield:

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = 4g \int (Xdx + Ydy + Zdz)$$

which equation, on account of $dx^2+dy^2+dz^2=ds^2$, is reduced to this:

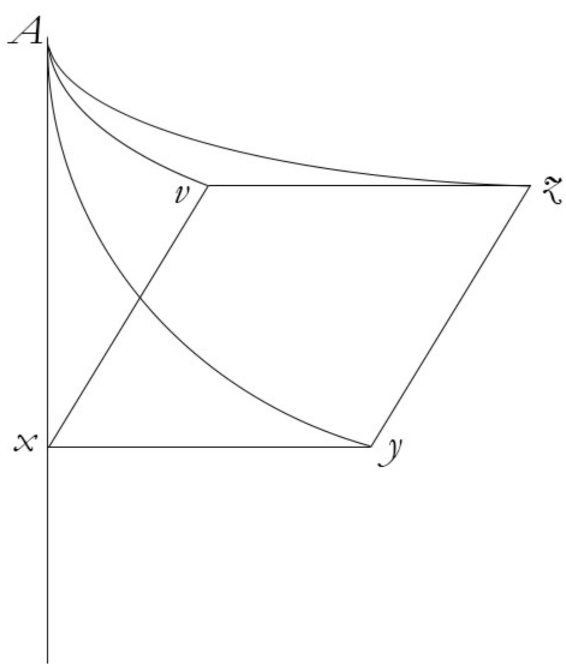
$$\frac{ds^2}{dt^2}=4g\int(Xdx+Ydy+Zdz)$$

Hence, because $\frac{ds}{dt}$ expresses the speed, with which the body traverses the segment zz', if it is posed $=v$, we will have this determination for it: $vv=4g\int(Xdx+Ydy+Zdz)$, whence it follows that this will hold:

$$vdv=2g\,(Xdx+Ydy+Zdz)$$

§. 15. Moreover, from these differentio-differential formulae it will too follow that this integrand derives: $\frac{yddx-xddy}{dt^2}=2g\,(yX-xY)$ of which the integral will be $\frac{ydx-xdy}{dt}=2g\int(yX-xY)\,dt$. Because we just discovered $\frac{ds^2}{dt^2}=vv$, we write $\frac{ds^2}{vv}$ instead of dt^2, and it holds that $\frac{ydx-xdy}{ds}=\frac{2g}{v}\int(Xy-Yx)\,\frac{ds}{v}$. In the same way we will learn that $\frac{zdx-xdz}{ds}=\frac{2g}{v}\int(Xz-Zx)\,\frac{ds}{v}$, and lastly $\frac{zdy-ydz}{ds}=\frac{2g}{v}\int(Yz-Zy)\,\frac{ds}{v}$. And it will help to have noted these formulae in the following one.

§. 16. Already having invented the speed of a body, such relation between the three coordinates x, y and z must be investigated, that the time, in which the arc of the curve Az is traversed, becomes minimal for all. In this matter thus we must return to the isoperimetric method. But this method is verily accommodated to only two variables, like how I truly managed it; meanwhile this question, too is yet able to be reduced to the case of two variables, since we call those to help, which are taught from the projections of the curves, that aren't situated in the same plane.



§. 17. Let us then consequently consider the projection of our curve Az, made in the plane of the table, which is Ay, of which thus the nature is expressed by an equation between both variables x and y, for which we state $dy=pdx$ and it will hold that an element of this projection $=dx\sqrt{1+pp}$. Let in a similar way Av be the projection of our curve, constructed in the plane, normal to the table, above the axis Ax, of which the nature is expressed by an equation between both variables $Ax = x$ and $xv = yz = z$, for which we pose $dz = qdx$, such that an element of this projection is $dx\sqrt{1+qq}$. Moreover, it is evident that an element of the true curve Az will be $= ds = dx\sqrt{1+pp+qq}$. Let us call the prior projection Ay 'lying', and the other Av 'upright'.

§. 18. It is however manifest that, if both these projections will be found, from joining them the same curve Az can most easily be determined. Because the abscissa $Ax = x$ is truly common to both projections, if we erect yz, itself being equal to xv, perpendicular from the point y, the point z will be in this same sought curve. Yet, one of those projections doesn't accomplish a matter by any means, because as the lying projection is able to meet together with infinitely many curves, so does the upright one.

§. 19. Having noted this well, the whole question of the minimum sought will be thus established bipartite. Let us first of course observe the upright projection as given, and between all curves, to which the same upright projection responds, we seek the one, in which the integral formula $\int \frac{ds}{v}$ obtains the minimum value, that which by only two coordinates will be possible to be provided. Because truly the upright projection Axv is observed as given, it will be possible to consider it, applied to its z, as a function of the abscissa x, and in the same way the quantity $q = \frac{dx}{dz}$ too will be a function of that x, and if we apply the isoperimetric precept to this case, we will discover that one, for which the formula $\int \frac{ds}{v}$ obtains a minimum value, between all curves, having the same upright projection.

§. 20. In the same way, the lying projection Axy will be considered as noted, and between all curves that have this projection in common, the one, for which the same formula $\int \frac{ds}{v}$ obtains a minimum value, is sought by the same method of maxima and minima, and now in this investigation, both y and $p = \frac{dy}{dx}$ can be had for functions of x, such as thus only both remaining x and z should already be counted as variables again, and the calculus by the same precept and before will be possible to be procured, if we just write z instead of y and q instead of p.

§. 21. But if in this way already we invented a curve of minimum both between all curves having the same upright projection, and between all curves having the same lying one, since for the former a certain equation came forth between x and y, for the other surely an equation between x and z, these two determinations, taken together, will provide a true brachistochrone, between all intelligibly possible curves.

§. 22. According to that precept it will already be easy to pick out brachistochrones, or those curves, in which the formula $\int \frac{dx\sqrt{1+pp+qq}}{v}$ assumes a minimum value. Moreover, like before, it is necessary, that v is a function determined by the variables x, y, z, that which is unable to occur, if the formula $\int (Xdx + Ydy + Zdz) = \frac{vv}{4g}$ does not allow integration; on that account we here treat only those cases. Then hence consequently vdv will be $2g\,(Xdx + Ydy + Zdz)$, and for that reason $dv = \frac{2g}{v}\,(Xdx + Ydy + Zdz)$. Let us thus first observe the upright projection as given, such that therefore both z and q are functions of only x; whence if we pose

$$d \cdot \frac{\sqrt{1+pp+qq}}{v} = Mdx + Ndy + Pdp$$

The equation for the sought curve will be $Ndx - dP = 0$, where it conveniently occurs, that the quantity M does not enter in that equation.

§. 23. Since we therefore do not engage in the quantity M, only two variables come in the computation in this differentiation, of course y and p, since z and q are had for functions of x, and the differentials of these are contained in the portion Mdx, which we're allowed to remove. By these means it is necessary that the values of the letters N and P are sought by differentiation, and since the quantity p does not enter in the speed v, for the portion Pdp, whence at once $P = \frac{p}{v\sqrt{1+pp+qq}}$ appears.

§. 24. Then consequently the variable v rests, which is possible to be considered as a function of only that y, and just like that for our present use dv will be $\frac{2gYdy}{v}$, and for that reason $d \cdot \frac{1}{v} = -\frac{2gYdy}{v^3}$, and thus N will be $-\frac{2gY}{v^3} \cdot \sqrt{1+pp+qq}$. Hence the sought equation is thus elicited:

$$+\frac{2gYdx}{v^3}\sqrt{1+pp+qq} + d \cdot \frac{p}{v\sqrt{1+pp+qq}} = 0$$

§. 25. In a similar way, if we assume the lying projection for known, such that y and p are already functions of only that x, the equation discovered before is transfered to this case, if only the letters y and z and likewise p and q are mutually permuted. In this way this equation comes forth:

$$\frac{2gZdx}{v^3}\sqrt{1+pp+qq} + d \cdot \frac{q}{v\sqrt{1+pp+qq}} = 0$$

which equation, connected with the previous, will determine the same sought brachistochrone, seeing that its determination requires two equations, for that reason, because both remaining y and z should be defined by the abscissa x anywhere.

§. 26. Behold, thus the resolution to our problem is contained in these two equations:

$$\frac{2gYdx}{v^3} \cdot \sqrt{1+pp+qq} + d \cdot \frac{p}{v\sqrt{1+pp+qq}} = 0$$

$$\frac{2gZdx}{v^3} \cdot \sqrt{1+pp+qq} + d \cdot \frac{q}{v\sqrt{1+pp+qq}} = 0$$

where all quantities for the variables must be already intelligibly had. Moreover, it fits that the posterior formulae here evolved somewhat more with help of this reduction:

$$d \cdot \frac{p}{v\sqrt{1+pp+qq}} = -\frac{dv}{vv} \cdot \frac{p}{\sqrt{1+pp+qq}} + \frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp+qq}}$$

Now moreover on account of $dv = \frac{2g(Xdx+Ydy+Zdz)}{v}$ $\frac{dv}{vv}$ will be $+\frac{2g(Xdx+Ydy+Zdz)}{v^3}$, and hence our two equations assume the following forms.

$$\frac{2gYdx}{v^3}\sqrt{1+pp+qq} - \frac{2g\,(Xdx+Ydy+Zdz)}{v^3} \cdot \frac{p}{\sqrt{1+pp+qq}} + \frac{1}{v} d \cdot \frac{p}{\sqrt{1+pp+qq}} = 0$$

$$\frac{2gZdx}{v^3}\sqrt{1+pp+qq} - \frac{2g\,(Xdx+Ydy+Zdz)}{v^3} \cdot \frac{q}{\sqrt{1+pp+qq}} + \frac{1}{v} d \cdot \frac{q}{\sqrt{1+pp+qq}} = 0$$

These equations are multiplied by $\frac{v^3}{2g}$ and the parts, reduced prior to the denominator $\sqrt{1+pp+qq}$, are restored in the following way:

$$\frac{(Y(1+qq)-pX)\,dx-pZdz}{\sqrt{1+pp+qq}}+\frac{vv}{2g}d\cdot\frac{p}{\sqrt{1+pp+qq}}=0$$

$$\frac{(Z(1+pp)-qX)\,dx-qYdy}{\sqrt{1+pp+qq}}+\frac{vv}{2g}d\cdot\frac{q}{\sqrt{1+pp+qq}}=0$$

which equations onwards, on account of $dy=pdx$ and $dz=qdx$, will in this way be transformed:

$$\frac{Y(1+qq)-pX-pqZ}{\sqrt{1+pp+qq}}+\frac{vv}{2gdx}d\cdot\frac{p}{\sqrt{1+pp+qq}}=0$$

$$\frac{Z(1+pp)-qX-pqY}{\sqrt{1+pp+qq}}+\frac{vv}{2gdx}d\cdot\frac{q}{\sqrt{1+pp+qq}}=0$$

Because, if we delete the terms, containing z and q, here, the equation, invented for the preceding case, appears from the prior evident equation, from it surely produces:

$$\frac{Xp-Y}{\sqrt{1+pp}}=\frac{vv}{2g}d\cdot\frac{p}{\sqrt{1+pp}}$$

which equation excellently convenes with the above discovered; the posterior equation verily intelligibly disappears in this case.

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Text E760

On true Brachistochrones

Or

Lines of the fastest descent

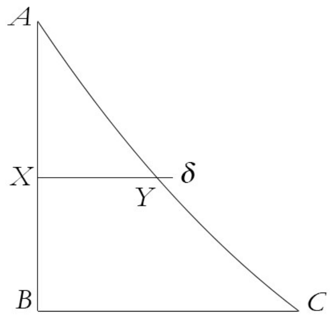
In a resistant medium

By the author

L. Euler.

§. 1. A principle of this kind, that I taught about these curves in book II[4] of my Mechanicae, rests upon, what can't be allowed in a resistant medium. Thereafter, I tried to obtain the same argument from the first principles of maxima and minima in my isoperimetric treatment; to such a greater degree are they, which I conducted there on the brachistochrone in a resistant medium, truly involved in the excessively generalised analytic formulae, such that thence barely anyone is able to pick out a true nature of those curves. On that account I decided to expand this same argument in a bigger study here and to derive it clearly and perspicuously from the first principles.

[4]Consult the note on page 314 [which is the footnote in the previous text, E759]



§. 2. Here, let us finally consider whichever AYC, related to the vertical axis AB, over which a body, beginning to slip from A, descends in a resistant medium according to whichever multiplied ratio of the speed. The abscissa is already called $AX = x$ for whichever point Y of the curve, the attached {ordinal?} $XY = y$ and the arc of the curve $AY = s$. Let moreover the speed in Y be v, of which the quantity is thus expressed by such equation: $vdv = gdx - hv^{n+1}ds$, which is in this way compared, like it is able to be integrated in general in the cases $n = -1$ and $n = +1$. Meanwhile yet, having thence defined the value of that v, the element of time will be $\frac{ds}{v}$, of which thus the integral should obtain the property of the minimum, since the curve AYC will be a brachistochrone.

§. 3. If a motion is produced in vacuum, in which case h were 0 and $vv = 2gx$, because the speed in Y depends on only its altitude, it is evident, that the whole curve AYC turns out to be a brachistochrone, and also that single parts of this AY should be traversed in minimal time; in a resistant medium the matter yet very much has itself otherwise, where the speed no longer depends on the locus of the point Y, but simultaneously involves the whole preceding arc AY; whence it can happen that the time through the whole arc AYC becomes minimal, and also if the time through the arc AY were not minimal, it could naturally happen that a considerably larger speed, which in such size produces a shorter time through the following arc YC, were generated in Y by the descent through the arc AY; on this account our problem for the resistant medium should be proposed in this way:

Between all curves, which are allowed to be drawn from the point A all the way through C, the one is sought, over which a body, beginning a descent from A, arrives at the end C the quickest.

§. 4. With this, moreover, this investigation more widely extends the problem a lot more general, which, I will contemplate, is not restricted to only brachistochrones, because the solution further not only not becomes harder, but also more extended to be reduced to analytic formulae; on this account the following problem convenes to be procured before all:

General Problem.

Between all curves, that can be drawn from a given point A to a given point C; investigate that one, in which this integral formula: $\int V dx$ obtains a maximum or minimum value, where the letter V, besides the coordinates of x and y and the differentials of whichever ordinal, also involves the quantity v, which is determined by whichever differential equation.

Solution.

§. 5. Because the function V is also assumed to imply differentials of whichever ordinal, let us pose, having accustomed to the custom, $dy = pdx; dp = qdx; dq = rdx; etc.$ in this way, such that V besides the quantities $x, y, p, q, r, etc.$ already also involves that quantity v; whence its differential in this way will have the form:

$$dV = Ldv + Mdx + Ndy + Pdp + Qdq + etc.$$

Moreover, the quantity v is expressed by this differential equation: $dv = \mathfrak{V}dx$; where $\mathfrak{V}$ is whichever function of that v, with the quantities, pertaining to the curve, $x, y, p, q, r, etc.$ Wherefore its differential will have such form:

$$d\mathfrak{V} = \mathfrak{L}dv + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + etc.$$

§. 6. Let us use the desired method from the calculus of variation, with which the maximum or minimum value of the integral formula $\int Vdx$ can be procured, which we in the end bestow, having applied $XY = y$ as the minimal increment $Y\delta$, which we indicate by δy, such that in this way δy is the variation of that y; it is verily needed that no variation is granted to the other coordinate x, such that in this way $\delta x = 0$. To what extent thus the remaining quantities depend on the varied y, so far do they too receive certain variations, which is necessary to obtain before all.

§. 7. Let us pose with grace of brevity that the variation $\delta y = \omega$, and because $p = \frac{dy}{dx}$, δp will be $\frac{\delta dy}{dx}$. It is moreover demonstrated that $\delta dy = d\delta y = d\omega$, whence $\delta p = \frac{d\omega}{dx}$. In a similar way, because $q = \frac{dp}{dx}$, δq will be $\frac{\delta dp}{dx} = \frac{d\delta p}{dx} = \frac{dd\omega}{dx^2}$. It is equally manifest that δr will be $\frac{d^3\omega}{dx^3}$; $etc.$ Here of course everywhere the letter δ prefixed to which quantities denotes its variation born from the variation of that y.

§. 8. Having posed these, let us investigate the variation of that proposed integral formula $\int Vdx$, which will thus be $= \delta \int Vdx$. From the calculus of variation however it consists that $\delta \int Vdx = \int \delta V dx$ will hold, and because it is allowed to take the variations by the same rule, by which differentials are indicated, it will hold:

$$\delta V = L\delta v + M\delta x + N\delta y + P\delta p + Q\delta q + etc.$$

where the term $M\delta x$ vanishes; and if instead of $\delta y, \delta p, \delta q, \delta r, etc.$ the values of the just invented are written, we will have:

$$\delta V = L\delta v + N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + etc.$$

Hence the variation of the proposed integral formula will thus be:

$$\delta \int Vdx = \int Ldx \left(\delta v + N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + etc. \right)$$

or

$$\delta \int Vdx = \int L\delta v dx + \int N\omega dx + \int Pd\omega + \int \frac{Qdd\omega}{dx} + etc.$$

The whole matter reverts thus to this, such that the value of the first portion $\int L\delta v dx$ is obtained with all care.

§. 9. From §. 5. $v = \int \mathfrak{B}dx$ follows, hence it will hold that $\delta v = \delta \int \mathfrak{B}dx = \int \delta\mathfrak{B}dx$; for this reason, because $d\mathfrak{B} = \mathfrak{L}dv + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + etc.$ will in a similar way be:

$$\delta\mathfrak{B} = \mathfrak{L}\delta v + \mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + etc.$$

this is:

$$\delta\mathfrak{B} = \mathfrak{L}\delta v + \mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + etc.$$

consequently we will have:

$$\delta v = \int dx \left(\mathfrak{L}\delta v + \mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + etc. \right)$$

from which equation the value of that δv is now possible to be derived.

§. 10. To this end, where the calculus is more raised, let us pose $\delta v = u$, and with the <u>assumed</u> differentials it will hold that:

$$du = \mathfrak{L}udx + \mathfrak{N}\omega dx + \mathfrak{P}d\omega + \frac{\mathfrak{Q}dd\omega}{dx} + etc.$$

which equation is represented in this way:

$$du - \mathfrak{L}udx = \mathfrak{N}\omega dx + \mathfrak{P}d\omega + \frac{\mathfrak{Q}dd\omega}{dx} + etc.$$

which, such that it is rendered integrable, is multiplied by $e^{-\int \mathfrak{L}dx}$, instead of which for the grace of brevity we write $\frac{1}{\Lambda}$, such that in this way $\Lambda = e^{\int \mathfrak{L}dx}$ holds, and for that reason $\frac{d\Lambda}{\Lambda} = \mathfrak{L}dx$. Then consequently the integral equation will be:

$$\frac{u}{\Lambda} = \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc. \right)$$

and in this way we obtained the sought quantity δv, which will be:

$$\delta v = \Lambda \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc. \right)$$

§. 11. Now we will therefore have for the first term of the formula, by which the variation $\delta \int V dx$ is expressed:

$$\int L\Lambda dx \int \frac{dx}{\Lambda} \left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc. \right)$$

where after the integration sign $\int$ further another is involved, whence it will need to be attacked in it, such that all are revoked to simple integration.

§. 12. Let us to this end state $L\Lambda dx = d\Pi$, and it will be

$$\int d\Pi \int \frac{dx}{\Lambda}\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + etc.\right) = \Pi \int \frac{dx}{\Lambda}(\mathfrak{N}\omega + etc.) - \int \frac{\Pi dx}{\Lambda}(\mathfrak{N}\omega + etc.)$$

Because Π is already $\int L\Lambda dx$, the constant, to be attached to this integral, is abandoned by our arbitrarity; whence this constant is determined in this way, such that for the whole curve AYC, where x becomes $AB = a$, this quantity Π vanishes, surely, having agreed to this, the prior part $\Pi \int \frac{dx}{\Lambda}(\mathfrak{N}\omega + etc.)$ for the whole curve, to which it is necessary to devise a calculus, spontaneously vanishes, since that integral formula, otherwise joined to nothing, cannot be reduced. Therefore, having taken the integral $\int L\Lambda dx = \Pi$ in this way, such that, having posed $x = a$, it vanishes, it will hold:

$$\int L dx \delta v = -\int \frac{\Pi dx}{\Lambda}\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc.\right)$$

§. 13. This sought variation $\delta \int V dx$, having already invented the value, will be expressed in the following way:

$$-\int \frac{\Pi dx}{\Lambda}\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + etc.\right) + \int dx\left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + etc.\right)$$

which expression, to be posed for the grace of brevity:

$$N - \frac{\Pi\mathfrak{N}}{\Lambda} = N'; P - \frac{\Pi\mathfrak{P}}{\Lambda} = P'; Q - \frac{\Pi\mathfrak{Q}}{\Lambda} = Q'; etc.$$

is reduced to this plentily simple form:

$$\delta \int V dx = \int dx\left(N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + etc.\right)$$

of which thus this value, through the whole curve AYC, is extended until $x = a$; it should be equal to zero.

§. 14. Let be denoted that $\int P'd\omega = P'\omega - \int \omega dP'$ holds, by which this formula is further reduced; thereafter $\int Q'dd\omega = Q'd\omega - \int d\omega dQ'$. Yet it verily holds that $\int d\omega dQ' = \omega dQ' - \int \omega ddQ'$, and for that reason $\int Q'dd\omega = Q'd\omega - \omega dQ' + \int \omega ddQ'$. In the same way it will hold that

$$\int R'd^3\omega = R'dd\omega - d\omega dR' + \omega ddR' - \int \omega d^3R'$$

and so on in this way; where, because in the final end C no variation ω is applied, it is allowed to onwards neglect absolute boundary terms, and for that reason we will have

$$\delta \int V dx = \int \omega dx\left(N' - \frac{dP'}{dx} + \frac{ddQ'}{dx^2} - \frac{d^3R'}{dx^3} + etc\right)$$

of which thus the value through the whole curve, extended from A to C, should be equal to zero, whenever the variations ω are accepted.

§. 15. It is however evident, that this cannot happen otherwise, if it won't hold that $N' = N - \frac{\Pi\mathfrak{N}}{\Lambda}$; $P' = P - \frac{\Pi\mathfrak{P}}{\Lambda}$; $etc.$ Then it will verily hold that $\Lambda = e^{\int \mathfrak{L}dx}$ and $\Pi = \int L\Lambda dx$, which integral should in this way be taken, such that it vanishes, having posed $x = a$. Besides, it is necessary that all proceeding constants are verily defined by integration in this way, such that it is satisfied by all circumstances, that is, such that, having assumed that $x = 0$, y, too, becomes 0; thereafter, having assumed that $x = a$, y becomes $BC = b$. Moreover, a certain given value for the quantity v, for the case $x = 0$, should be granted.

An Application
For Brachistochrones in a Resistant Medium

§. 16. Because the time of descent through the arc AY is $\int \frac{ds}{v}$, on account of $ds = dx\sqrt{1+pp}$, the integral formula, extended from the end A, where $x = 0$, until the end C, where $x = a$ and $y = b$, and to be reduced to a minimum, will be $\int \frac{dx\sqrt{1+pp}}{v}$ and for that reason $V = \frac{\sqrt{1+pp}}{v}$, which formula, because it contains only two variables v and p, will be $L = -\frac{\sqrt{1+pp}}{v}, M = 0, N = 0$, but $P = \frac{p}{v\sqrt{1+pp}}$. Next, because $dv = \frac{gdx - hv^{n+1}dx\sqrt{1+pp}}{v}$, $\mathfrak{B}$ will be $\frac{g}{v} - hv^n\sqrt{1+pp}$; whence further $\mathfrak{L} = -\frac{g}{vv} - nhv^{n-1}\sqrt{1+pp}; \mathfrak{M} = 0; \mathfrak{N} = 0$; but $\mathfrak{P} = -\frac{hv^n p}{\sqrt{1+pp}}$. From these values $\frac{d\Lambda}{\Lambda}$ will already first be $\mathfrak{L}dx$; then verily $\Pi = \int L\Lambda dx$

§. 17. Having invented this, firstly N' will be 0; $P' = P - \frac{\Pi\mathfrak{P}}{\Lambda}$; wherefore the equation for the sought curve will be $N' - \frac{dP'}{dx} = 0$, or $\frac{dP'}{dx} = 0$, whence immediately by integrating $P' = C$ is obtained; having thus substituted the values for P and $\mathfrak{P}$, this equation for the curve appears:

$$\frac{p}{v\sqrt{1+pp}} + \frac{h\Pi v^n p}{\Lambda\sqrt{1+pp}} = C$$

From this equation we then elicit the value of Π, for which we surely gave an integral formula, and it will hold that:

$$\Pi = \frac{C\Lambda v\sqrt{1+pp} - \Lambda p}{hpv^{n+1}}$$

Let us here pose for the grace of brevity $\frac{C}{v^n} \cdot \frac{\sqrt{1+pp}}{p} - \frac{1}{v^{n+1}} = \Theta$, such that $\Pi = \frac{\Lambda\Theta}{h}$, and on account of $d\Lambda = \Lambda\mathfrak{L}dx$, it will hold that:

$$d\Pi = L\Lambda dx = \frac{\Theta\Lambda\mathfrak{L}dx}{h} + \frac{\Lambda d\Theta}{h}$$

Which equation, divided by Λ, will be $hLdx = \Theta\mathfrak{L}dx + d\Theta$. It verily holds that:

$$d\Theta = -\frac{nCdv}{v^{n+1}} \cdot \frac{\sqrt{1+pp}}{p} + \frac{C}{v^n}d \cdot \frac{\sqrt{1+pp}}{p} + \frac{(n+1)\,dv}{v^{n+2}}$$

Whence our equation will be:

$$-\frac{hdx\sqrt{1+pp}}{vv} = \frac{C\mathfrak{L}dx}{v^n} \cdot \frac{\sqrt{1+pp}}{p} - \frac{\mathfrak{L}dx}{v^{n+1}} - \frac{nCdv}{v^{n+1}} \cdot \frac{\sqrt{1+pp}}{p} + \frac{(n+1)\,dv}{v^{n+2}} + \frac{C}{v^n} \cdot d \cdot \frac{\sqrt{1+pp}}{p}$$

where $\mathfrak{L} = -\frac{g}{vv} - nhv^{n-1}dx\sqrt{1+pp}$ enters.

§. **18.** This equation, already freed from the integral formula, moreover contains three variables, of course p and v with the differential dx, and from this the segment dx can be easily removed. Because $vdv = gdx - hv^{n+1}dx\sqrt{1+pp}$, dx will be $\frac{vdv}{g-hv^{n+1}\sqrt{1+pp}}$, which value is substituted, the equation, only containing two variables, is obtained. To this end, let us resolve in our equation all terms containing the segment dx to the same side, and it will hold that:

$$\frac{\mathfrak{L}dx}{v^{n+1}} - \frac{dx\sqrt{1+pp}}{vv}\left(h + \frac{C\mathfrak{L}}{pv^{n-2}}\right) = \frac{(n+1)\,dv}{v^{n+2}} - \frac{nCdv}{v^{n+1}} \cdot \frac{\sqrt{1+pp}}{p} + \frac{C}{v^n} \cdot d \cdot \frac{\sqrt{1+pp}}{p}$$

Because, if we want to substitute their values instead of dx and $\mathfrak{L}$, a very complicated equation arises, as appointing this will be superfluous. It is meanwhile yet evident that the future differential equation between p and v is first degree, whence we can rightly postulate its, so to speak, allowed resolution in such a laborious matter.

§. **19.** Because then consequently the quantity p is given by v in this equation, and on account of integration, a new constant quantity is engaged in, all remaining things, which pertain to the solution, will be allowed to easily be procured. Because firstly $\sqrt{1+pp}$ is a certain function of that v, the quantity x will also be allowed to be defined by v with help of the equation $dx = \frac{vdv}{q-hv^{n+1}\sqrt{1+pp}}$, whence again a new constant is introduced, it is allowed to define which in this way, such that, having assumed that $v = 0$, x becomes 0. Thereafter verily also $\int \mathfrak{L}dx$ will be determined by only v, and hence further that value of the letter Π from the equation $\Pi = \frac{C\Lambda v\sqrt{1+pp}-\Lambda p}{hpv^{n+1}}$; where the constant C should be determined in this way, such that, having posed $x = a$, that value vanishes, because thus, if we assume that in the case that $x = a$, v becomes C, it happens in this way; and just like that, having defined all constants duly, that construction of the curve toils moreover without trouble. Because x and p are already given by v, the attached y will also be possible to be ascribed to v on account of $y = \int pdx$, and in these determinations we should acquiesce in the so sublime investigation, to what extent of course a general solution, that extends to all values of the exponent n, is desired

Supplement
in which the nature of the brachistochrones in a resistant medium is determined more accurately.

§. **20.** Although the last differential equation between both variables p and v, to which our method of Maxima and Minima leads, will in this way be regarded complex, such that barely anything about knowing the nature of those curves is thenceforth seen to be able to be concluded: the following plenily convenient equation appears, having built the calculus duly:

$$0 = \frac{(n+2)\,dv}{vv} - \frac{(n+1)\,Cdv\sqrt{1+pp}}{pv} + C\left(1 - \frac{h}{g}v^{n+1}\sqrt{1+pp}\right) d \cdot \frac{\sqrt{1+pp}}{p}$$

which only consists of four terms, and by no means difficultly can it be reduced to a simpler form.

§. **21.** Let us truly first state $C = \frac{1}{c}$ and $\frac{\sqrt{1+pp}}{p} = t$, whence it holds that $p = \frac{1}{\sqrt{tt-1}}$ and $\sqrt{1+pp} = \frac{t}{\sqrt{tt-1}}$, having substituted which values, this equation appears:

$$\frac{(n+2)\,cdv}{vv} - \frac{(n+1)\,tdv}{v} + dt - \frac{h}{g} \cdot \frac{v^{n+1}tdt}{\sqrt{tt-1}} = 0$$

Where it is immediately exposed that both middle terms $dt - \frac{(n+1)tdv}{v}$ are rendered integrable, if they are divided by v^{n+1}, because the integral surely advances $= \frac{t}{v^{n+1}}$. Further on, however, the first and last term spontaneously admit integration, such that the complete integral of this equation becomes in this way:

$$\frac{t}{v^{n+1}} - \frac{c}{v^{n+2}} - \frac{h}{g}\sqrt{tt-1} = \Delta$$

which equation, having restored the values $t = \frac{\sqrt{1+pp}}{p}$ and $\sqrt{tt-1} = \frac{1}{p}$, multiplying by v^{n+1}, assumes this form:

$$\frac{\sqrt{1+pp}}{p} - \frac{c}{v} - \frac{h}{g} \cdot \frac{v^{n+1}}{p} = \Delta v^{n+1}$$

whence thus the value of that p is defined by v with only extraction of the square root.

§. 22. It will moreover before all help here to have noted that the constant Δ is defined from the locus of the last point C, where the descent is terminated. Because truly in this end $\Pi = 0$ should hold, and the method of Maxima and Minima immediately had satisfied this equation: $P - \frac{\Pi \mathfrak{P}}{\Lambda} = C$, it is evident that the quantity Π can not vanish, if not in that place, where $P = C$ holds. Moreover $P = \frac{p}{v\sqrt{1+pp}}$ held, and because we now posed $C = \frac{1}{c}$, this will happen, where c is $\frac{v\sqrt{1+pp}}{p}$. In this case however our invented equation will provide the value $\Delta = -\frac{h}{gp}$, where p expresses the tangent of the angle, with which the curve declines from the present vertical; on which account, if we want, that this incline in the point C is equal to a given angle α, of which the tangent is θ, Δ will be $-\frac{h}{g\theta}$, having thus substituted which value, our equation will be thoroughly determined, and becomes

$$\frac{\sqrt{1+pp}}{p} - \frac{c}{v} + \frac{h}{g}v^{n+1}\left(\frac{1}{\theta} - \frac{1}{p}\right) = 0$$

or

$$\sqrt{1+pp} - \frac{cp}{v} + \frac{h}{g}v^{n+1}\left(\frac{p}{\theta} - 1\right) = 0$$

This determination, however, of the edge point C by a given incline of the curve to the present vertical is regarded much more accommodated to a matter of nature, than if we want to define that point by the abscissa $x = a$ and $y = b$.

§. 23. Since then consequently the quantity p is equal to a so much algebraic function of that v, hence the construction of the curve will be able to be established sufficiently conveniently. Because $dx = \frac{vdv}{g - hv^{n+1}\sqrt{1+pp}}$ holds, dy will be $\frac{pvdv}{g - hv^{n+1}\sqrt{1+pp}}$ and each formula should in this way be integrated, such that, having posed $v = 0$, that which occurs in that beginning A, the integrals vanish, and in this way both coordinates x and y are obtained for that point of this curve, where the speed of the body is v. Of course it will hold that $x = \int \frac{vdv}{g - hv^{n+1}\sqrt{1+pp}}$ and $y = \int \frac{pvdv}{g - hv^{n+1}\sqrt{1+pp}}$, and this curve, continued until there, where p is θ, will be a true brachistochrone, over which the body descends in the shortest time from A to C.

Evolution of the case, in which $h = 0$,
or the resistance vanishes.

§. 24. In this case our equation then consequently shortens to this most simple form: $\sqrt{1+pp}-\frac{cp}{v}=0$, to which the equation $P-C=0$ corresponds; whence it is exposed, that it is allowed for the curve that the point Y can be assumed for the final end, such that in this way all portions of this curve, beginning from the beginning A, rejoice with the brachistochronism property, which, as is agreed upon, is the distinguished property of the brachistochrone, already long ago invented for vacuum.

§. 25. Because therefore here h is 0, p will be $\frac{v}{\sqrt{cc-vv}}$ and both coordinates are expressed in this way: $x=\int\frac{vdv}{g}$ and $y=\int\frac{vvdv}{g\sqrt{cc-vv}}$. Thence consequently x will be $\frac{vv}{2g}$, whence in turn $v=\sqrt{2gx}$, which value, substituted in the other formula, yields $y=\int\frac{dx\sqrt{2gx}}{\sqrt{cc-2gx}}$, which equation is manifest for the Cycloid, of which the cusp happens in that beginning A and describes a revolution of a circle over the horizontal straight.

Evolution of the case, in which $n=-1$,

or the resistance is the same everywhere.

§. 26. In this case our equation between p and v assumes thus this form:

$$\sqrt{1+pp}-\frac{cp}{v}+\frac{h}{g}\left(\frac{p}{\theta}-1\right)=0$$

from which equation $v=\frac{cp}{\sqrt{1+pp}+\frac{b}{g}\left(\frac{p}{\theta}-1\right)}$ is derived. Whence, having assumed $p=\theta$, the speed in the last end C will be $v=\frac{c\theta}{\sqrt{1+\theta\theta}}$. Moreover, the coordinates are now expressed by v in this way, such that $x=\int\frac{vdv}{g-h\sqrt{1+pp}}$ holds and $y=\int\frac{pvdv}{g-h\sqrt{1+pp}}$, which, if instead of v the discovered value is substituted, are found to be expressed by p. It will be however superfluous to establish this operation.

§. 27. This curve will thus be a brachistochrone in a medium, of which the friction is constant, and does not depend on the speed, or, how *Newton* describes such friction, she is proportional to the momentum of times.

Conclusion

§. 28. If we carefully examine the equation between p and v invented here more accurately, we discover, that she can be extended much more widely, such that not only a certain friction is proportional to a power of the velocity v, but it so much follows a ratio of whichever function of that v such that, having assumed V for that function of the speed v, we have this equation for the motion of the body in this way:

$$vdv=gdx-hVdx\sqrt{1+pp}$$

Because truly in our integral equation the exponent n does not occur, if not in the exponent of that v, it is hence not allowed to conclude safely, that there is need of nothing else, except as in our formulae instead of v^{n+1} V is writtenw. In this way the equation between p and v will therefore now have itself in this way:

$$\sqrt{1+pp}-\frac{cp}{v}+\frac{h}{g}V\left(\frac{p}{\theta}-1\right)=0$$

Whence, because $dx=\frac{vdv}{g-hV\sqrt{1+pp}}$ holds, dy will be $\frac{pvdv}{g-hV\sqrt{1+pp}}$, all remaining things are determined in the same way, as before.

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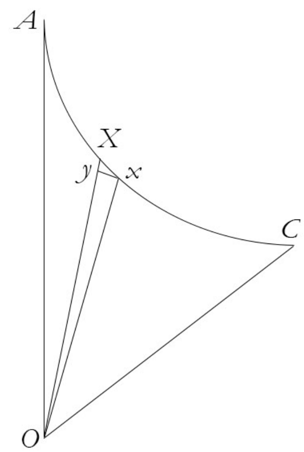
Text E761

On the brachistochrone

In a resistant medium
while a body is attracted to a centre of forces
in one way or another
By the author
L. Euler.



§. 1. Let O be the centre of forces, of which the attraction at a distance $= x$ is X, whichever function of that x; then, if the speed of the body will be $= v$, the friction force $= V$ is verily contrary to motion, whichever function of that v. Let the curve AXC already be the sought brachistochrone, over which a descending body traverses in the smallest time from A to C, since the descent in A begins from rest. Truly nothing hinders, that a certain speed is bestowed to it already in A. Let for the beginning of this curve A the distance $OA = a$ be posed and for the end C the distance $OC = c$ and the angle $AOC = b$. Yet verily for whichever its point X, let us pose its distance $OX = x$ and the angle $AOX = y$; and it is manifest that the curve is equally determined by the relation between x and y and by the equation between the orthogonal coordinates. Let moreover the arc $AX = s$ be posed and its element $Xx = ds$, and the drawn straight Ox and, drawn perpendicular from x to X Xy, Xy will be dx, and on account of the angle $XOx = dy$ it will hold that $xy = xdy$, whence the element is

$$Xx = ds = \sqrt{dx^2 + xxdy^2}$$

hence, if we pose $dy = pdx$, ds will be $-dx\sqrt{1 + ppxx}$.

§. 2. Because now the body is disturbed in X in the direction XO, with force $= X$, hence for the direction of the motion Xx the force appears $X \cdot \frac{Xy}{Xx} = -\frac{Xdx}{ds}$; the friction force however, having posed the speed of the body in $X = v$, is $= V$, whence the body will accelerate by the force $= -\frac{Xdx}{ds} - V$, which, drawn in the element of space ds, yields an increment of the square of the speed, whence thus $vdv = -Xdx - Vds$ will hold, and hence on account of $ds = -dx\sqrt{1+ppxx}$ becomes

$$vdv = dx\left(V\sqrt{1+ppxx} - X\right)$$

which equation expresses the relation between the speed v and the quantities particularly pertaining to the curve. Because the tiny amount of time through $Xx = ds$ is consequently $\frac{ds}{v} = -\frac{dx\sqrt{1+ppxx}}{v}$, between all curves, drawn from A to C, the one is sought, for which the value of this integral formula $\int \frac{dx\sqrt{1+ppxx}}{v}$ becomes minimum for all.

§. 3. It will help here before all to have observed, if the end C is accepted on that straight AO, that brachistochrone should coincide with that there straight, for the motion of which thus, on account of $y' = 0$ and for that reason also $p = 0$, this equation is born: $vdv = dx\,(V - X)$, which, because it can in general by no means be resolved, much less will it be possible to be postulated, that in general the determination of the motion for the brachistochrone AC is thoroughly obtained, but clearly it will have to be judged with us to be accomplished, if only we can pick out the differential equation between the three variables x, y, v, by which surely, with the connected formula: $vdv = dx\left(V\sqrt{1+ppxx} - X\right)$, it is understood that it is in itself possible that the speed v is eliminated and that for that reason the equation between both variables x and y can be obtained.

§. 4. Because consequently between all curves AG the one should be sought, for which the value of this integral formula $\int \frac{dx\sqrt{1+ppxx}}{v}$ is minimal, it will be needed to revert to the general isoperimetric problem, solved in a preceding dissertation. But, because the circumstances here are a bit varied, the decision will be to transfer the solution, invented there, within the form of a theorem here, which, if in this way, we will have:

The general isoperimetric Theorem.

§. 5. *If between all curves, which can be drawn from the point A to C, the one is sought, in which the value of the integral formula $\int Wdx$ is a maximum or a minimum, where W, besides both variables x and y and their differentials $\frac{dy}{dx} = p$; $\frac{dp}{dx} = q$; $\frac{dq}{dx} = r$; etc. furthermore involves the variable v, such that in this way holds:*

$$dW = Ldv + Mdx + Ndy + Pdp + etc.$$

then the quantity v is verily given by the differential equation in this way, such that, having posed $dv = \mathfrak{W}$, it holds:

$$d\mathfrak{W} = \mathfrak{L}dv + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + etc.$$

having posed this, $\Lambda = e^{\int \mathfrak{L}dx}$ is sought, and hence further the quantity $\Pi = \int L\Lambda dx$, which integral is in this way taken on, such that it vanishes for the end C, or, because it returns to itself, that end C is established there, where $\Pi = 0$ holds, having discovered which, it's assumed that

$$N' = N - \frac{\Pi\mathfrak{N}}{\Lambda}; P' = P - \frac{\Pi\mathfrak{P}}{\Lambda}; Q' = Q - \frac{\Pi\mathfrak{Q}}{\Lambda}; etc.$$

from this for the nature of the sought curve this equation is deduced:

$$0 = N' - \frac{dP'}{dx} + \frac{ddQ'}{dx^2} - \frac{d^3R'}{dx^3} + etc.$$

where the element dx is assumed constant.

§. 6. For our case then W is $\frac{\sqrt{1+ppxx}}{v}$ and $\mathfrak{W} = \frac{V\sqrt{1+ppxx}-X}{v}$, which formulae only involve three variables, of course v, x and p; and since the letters M and $\mathfrak{M}$ don't ingress in the final equation, it is also not necessary to evolve them too. Hence from the prior formula it will hold that $L = -\frac{\sqrt{1+ppxx}}{vv}$, $N = 0, P = \frac{pxx}{v\sqrt{1+ppxx}}$. From the other formula it verily holds that:

$$\mathfrak{L} = -\frac{V\sqrt{1+ppxx}+X}{vv} + \frac{V'\sqrt{1+ppxx}}{v}$$

having of course posed $dV = V'dv$; then verily $\mathfrak{N} = 0$ and $\mathfrak{P} = \frac{Vpxx}{v\sqrt{1+ppxx}}$ will hold, having invented which, our final equation will be $\frac{dP'}{dx} = 0$, and for that reason $P' = C$, that is $C = P - \frac{\Pi\mathfrak{P}}{\Lambda}$. Thence it is exposed, that the quantity Π vanishes, where $P = C$. Therefore the end C of the brachistochrone should be established here, where $C = \frac{pxx}{v\sqrt{1+ppxx}}$.

§. 7. Because now $\Lambda = e^{\int \mathfrak{L}dx}$ it will hold that $\frac{d\Lambda}{\Lambda} = \mathfrak{L}dx$, thus $d\Lambda = \Lambda\mathfrak{L}dx$. Hence we will moreover further have $\Pi = \int L\Lambda dx$. Therefore, because from the final equation

$$\Pi = \frac{\Lambda P}{\mathfrak{P}} - \frac{C\Lambda}{\mathfrak{P}}, \text{ that is } \Pi = \frac{\Lambda}{V} - \frac{C\Lambda v\sqrt{1+ppxx}}{Vpxx}$$

is made, let us for the grace of brevity state $\sqrt{1+ppxx} = \omega$ and $\frac{\sqrt{1+ppxx}}{pxx} = t$, such that in this way $t = \frac{\omega}{x\sqrt{\omega\omega-1}}$. Let us now differentiate the discovered equation, and because $d\Pi = L\Lambda dx$ and $d\Lambda = \Lambda\mathfrak{L}dx$, having done this substitution, the whole equation will be able to be divided by Λ, and therefore it wasn't necessary to determinate its integral value. By now substituting the found values for L and $\mathfrak{L}$ thus, we arrive at this equation:

$$0 = \frac{\omega dx}{vv} + \frac{\mathfrak{L}dx}{V} - \frac{dV}{VV} - \frac{C\mathfrak{L}tvdx}{V} - Cx\frac{(vdt+tdv)}{V} + \frac{CtvdV}{VV}$$

where $\mathfrak{L} = -\frac{V\omega+X}{vv} + \frac{V'\omega}{v}$.

§. 8. Now, because $vdv = dx\,(V\omega - X)$, it will hold that $dx = \frac{vdv}{(V\omega-X)}$, which value we substitute in our equation instead of dx, let us everywhere write vdv for dx, verily multiply the remaining terms by $V\omega - X$ and write dV instead of $V'dv$, having done which, the equation will assume the following form:

$$0 = \frac{\omega dv}{v} + \frac{\omega dV}{V} - \frac{Cv\omega tdV}{V} + \frac{V\omega - X}{VV}\left(CvtdV - CVvdt - dV - \frac{Vdv}{v}\right)$$

§. 9. Because this equation is not insufficiently complex, let us first evolve only those terms, in which the constant C does not occur, and they will be discovered

$$\frac{\omega dv}{v} + \frac{\omega dV}{V} - \frac{\omega dV}{V} + \frac{XdV}{VV} - \frac{\omega dv}{v} + \frac{Xdv}{Vv}, \text{ or } \frac{X}{V}\left(\frac{dV}{V} + \frac{dv}{v}\right)$$

And the terms, containing the constant C, are verily

$$-\frac{Cv\omega tdV}{V} + Cv\omega dt + \frac{Cv\omega tdV}{V} + \frac{CXvdt}{V} - \frac{CXtvdV}{VV}$$

or, having deleted the terms that destroy themselves

$$-\frac{CvtXdV}{VV}+\frac{CvXdt}{V}-v\omega dt$$

wherefore the whole equation will have itself in this way:

$$\frac{X}{V}\left(\frac{dV}{V}+\frac{dv}{v}\right)-Cv\omega dt+\frac{CvXdt}{V}-\frac{CvtXdV}{VV}=0$$

§. 10. But if this equation is already divided by CvX, it will appear in this form:

$$\frac{1}{CVv}d\cdot lVv-\frac{\omega dt}{X}+\frac{dt}{V}-\frac{tdV}{VV}=0$$

of which equation the first portion as much as the following admit integration. Having consequently assumed the integral, it will be $-\frac{1}{CVv}+\frac{t}{V}-\int\frac{\omega dt}{X}=\Delta$, where in the summation sign only both variables p and x are involved, because $\omega=\sqrt{1+ppxx}$ and $t=\frac{\sqrt{1+ppxx}}{pxx}$, and besides X is a function of that x. On this account, the third variable v, with its given function V, must be assessed to be determined by this equation; but if these values were substituted in the equation $vdv\left(V\sqrt{1+ppxx}-X\right)$, an equation, involving only both variables x and p, or x and y, by which thus the nature of the sought brachistochrone curve is expressed; and nothing further can be postulated for the solution of this problem. Moreover, this invented curve should establish the end of descent C there, where, like we observed, $P=C$, or where $C=\frac{pxx}{v\sqrt{1+ppxx}}$.

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