Stellingen behorende bij het proefschrift

The aerodynamics of horizontal axis wind turbine rotors explored with asymptotic expansion methods

G.J.W. van Bussel


[2] De term "tip-correction factor" voor de factor waarmee de gevonden waarde van de geïnduceerde snelheid in het rotorvlak uit de impuls theorie (voor een oneindig aantal rotorbleden) wordt gecorrigeerd is misleidend daar deze bedoeld is om een correctie aan te brengen voor de eindigheid van het aantal bladen, en niet voor de eindigheid van het roterblad.

[3] Het succes van de Prandtl hypothese bij de bepaling van de belastingsverdeling over een vleugel in een uniforme rechte aanslag lijkt een gelijktijdige aanpak voor de windturbine situatie. Daar is de hypothese evenwel niet geldig. (dit proefschrift)

[4] De suggestie die door Van Kuik wordt gewekt dat de vermoeiingsschade van een windturbine voor het overgrote deel wordt veroorzaakt door het concept van een overtrekregeling is onjuist. (stelling 9 bij het proefschrift "On the limitations of Froudes Actuator Disc Concept, G.A.M. van Kuik, 1991"


[8] Indien de prijs die de TU Delft voor windstroom betaalt een afspiegeling is van haar maatschappelijk engagement dan dient voor de toekomst van het duurzame energieonderzoek en -onderwijs aan deze instelling te worden gevreesd.

[9] Concurrentie tussen universitaire instellingen en grote technologische instituten (GTI's) bij het adviseren en ondersteunen van het midden- en kleinbedrijf (MKB) is niet bevorderlijk voor de kwaliteit van de geleverde diensten.


[12] Flexibiliteit is bij de TU Delft vrijwel alleen bekend in de context van onderzoeksthema's en werktijden.


[14] De reclamewereld hangt haar beperkte kennis van meettechniek steeds vaker aan de grote klok.


[16] Proefschriften worden vrijwel nooit meer geschreven.
The aerodynamics of horizontal axis wind turbine rotors explored with asymptotic expansion methods
The aerodynamics of horizontal axis wind turbine rotors explored with asymptotic expansion methods

PROEFSCHRIFT

ter verkrijging van de graad van doctor
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De Wind zeide ik u reeds te voren, is niet anders dan een froom van lucht, even als het water eener froomende rivier. En daar nu de zwaarte en veérkracht derzelve altijd het evenwicht herfild, en dus, alwaar de lucht door warmte of andere oorzaken, ijler is, dan de naburige, daar naar toe tracht te froomen, en zulks ook werkelijk doet, zoodra hare meerdere zwaarte de veérkracht der ijvere lucht overwint; zoo veroorzaakt dit een luchtfroom, welken wij wind noemen. De wind is dan de lucht zelver, en wij voelen die zelfs bij het stille weder, als wij hard loopen.

Behalve dat de Wind noodzakelijk is, als zijnde de lucht zelver (en zonder lucht kan geen schepfel leven) zoo is het nut des winds zeer in het oog loopende. De wind doet onze Zaag-, Water-, Pel- en Pletmolens bewogen; hij stuwt onze rijk geladene schepen over de zeeën, ter bevordering van den koophandel: doch bovenal bevorderen fterke winden zeer de gezondheid, vooral van de ftedelingen.
Zonder dezelve zoude de lucht in de fteden, door vele ademhalingen, verbrandingen, verrotteningen, enz. bedorven, tuschen de huizen blijven hangen, en binnen kort voor de bewoners derzelfe doodelijk zijn. De ftormenten jagen deze bedorvende lucht uit alle ftraten en van alle grachten weg; doen dezelve door de zuivere lucht van het veld, of van afgelegene landen, verwisselen, en bevorderen daardoor niet weinig de gezondheid.

Zoo nuttig en noodzakelijk zijn dan de anders akelige ftoramenten! zoo goed is de lieve GOD, dat Hij de verwoestende natuurverfchijnselen toch ten algemenen nutte, ten zegen van het menschdom, heeft ingerigt!

Johannes Buijs.

Natuurkundig Schoolboek uitgegeven door de Maatschappij tot Nut van 't Algemeen, Gedrukt te Leyden, ter Boekdrukkerij van HERDINGH en DU MORTIER, MDCCXII.
Summary

During the last twenty years the use of wind energy has regained new attention. In contrast to the applications in the past the present use is focusing upon electricity production. This required the development of a totally new generation of windmills, which are nowadays known as wind turbines. The sails of these wind turbines, at present named rotor blades, have to be adequately designed according to modern aerodynamics. Knowledge and experience from aircraft aerodynamics can often be used, although there are also a number of important differences in both applications. This made it necessary to develop a new branch of fluid mechanics called wind turbine aerodynamics.

An important difference with the classical aircraft aerodynamics in the fact that wind turbines are continuously exposed to a strongly fluctuating wind found in the lowest 150 meters of the atmospheric boundary layer. Large fluctuations in wind speed and direction will generate similar variations in aerodynamic loads on the wind turbines. Especially the blades are subject to loads with a specific dynamic character.

In the design of wind turbine rotors these fluctuations in aerodynamic loads have to be taken into account. It is therefore important to develop methods for the prediction of such loads. In the present engineering practice use is still made of global blade element momentum theory methods, which are fundamentally not well equipped to generate details of the load distributions certainly not under dynamic conditions. It is however relatively easy to implement semi empirical corrections to take into account dynamic effects. Since experimental research is very time consuming and laborious use is also made of predictions from elaborated aerodynamic calculation methods. Such methods are however quite complex and still require massive computer effort.

In the present thesis a model is developed based upon acceleration potential theory using matchad asymptotic expansions. The numerical codes implementing the model make it possible to determine blade loads and velocity distributions in the flow about the rotor blades in large detail. Still the computer demands of these codes are so moderate that they can be executed on ordinary personal computers.

In the first chapters the basic equations are presented, and the classical Prandtl approximation for the wing problem is considered again, now in terms of Legendre functions. A comparison with the classical results using harmonic functions is also presented. The wing problem is extended to an infinite formation flight of identical wings, in such a way that mutual interference is taken into account in a correct way. Next the asymptotic acceleration potential solution for a rectangular wing in uniform flow is presented in which the spanwise aerodynamic load distributions are again represented by Legendre functions.
Summary

Then the problem of a rectangular wing in two different rotating situations is treated. The results for the situation where the wing is rotating at a cylindrical surface are compared with those found from the infinite formation flight. For the second situation where the wing rotates in its plane around a point extending outside one of the tips the acceleration potential solution is derived. This rudimental rotor geometry is then extended for real wind turbine rotor blade configurations. Just as in the case of the wing in uniform flow it is necessary to perform an integration of the resulting pressure field in order to determine the coefficients of the Legendre functions. This integration is in fact the application of the kinematic boundary condition, and implies an integration of the pressure gradient along the path of a particle of air travelling to the rotor blade.

The theoretical model developed has been implemented into a number of strongly related computer codes. The code PREDICHAT for the calculation of steady rotor loads is used in two different modes. The simplest implementation is used to compare with other methods and with experimental results. From such global comparisons it is concluded that the quality of the predictions is equal to those of other methods. Calculations with the extended version (PREDICHAT2) reveal a number of effects that have not been noticed before. It is found that the circular inflow of (mainly) the inner part of the rotor blade has a considerable effect upon the lift curve slope of the applied aerfoils.

The PREDICHAT codes are also used as pre-processor for dynamic calculations. These dynamic calculations implemented in the code PREDICDYN refer to situations where variations in e.g. wind speed have a length scale in the order of a rotor radius. Besides that calculations are performed with PREDICDYN for pitch control actions at a fixed rotor speed and for aerodynamic breaking actions, where the rotor speed is slowed down by pitching the blades towards a feathered position. Finally some situations are calculated where the rotor axis is not parallel to the direction of the oncoming undisturbed wind. Predictions with PREDICDYN have also been compared with measurements. A central role in the comparison with experimental results plays the 3 MW Danish research wind turbine at Tjæreborg, having a rotor diameter of 60 meters. Furthermore the results from experiments at a model rotor in the T.U. Delft open jet wind tunnel have been used for comparison.

In conclusion there is a satisfactory agreement between the predictions and the experimental results. Besides that an extensive insight is obtained in the complex flow present at dynamic conditions from the details of the flow calculated with the codes.
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Samenvatting

Curriculum Vitae
2 Notations

$A_n$ far pressure field coefficients
$b$ span of the rotor blade or wing
$C_{Dax}$ axial drag coefficient of the rotor
$C_{Mx}$ axial moment coefficient of the rotor blade
$C_p$ power coefficient of the rotor
$C_T$ power coefficient of the rotor
$c$ chord (distribution) of the rotor blade or wing
$c_d$ aerofoil drag coefficient
$c_l$ aerofoil lift coefficient
$l$ lift (distribution) over the rotor blade or wing
$P_n^m$ associate Legendre function of the first kind
$p$ pressure (field)
$Q_n^m$ associate Legendre function of the second kind
$R$ rotor radius
$r$ radial coordinate depicted in figure 1
$r_b$ local blade coordinate depicted in figure 4
$t$ time
$u, u'$ velocity and velocity-perturbation in x-direction
$V$ velocity vector
$v, v'$ velocity and velocity perturbation in y-direction
$W, W$ undisturbed wind vector resp. velocity (in z-direction)
$w, w'$ velocity and velocity perturbation in z-direction
$x$ coordinate depicted in figure 1
$x_b$ local blade coordinate depicted in figure 2
$y$ coordinate depicted in figure 1
$y_b$ local blade coordinate depicted in figure 2
$z$ coordinate depicted in figure 1
$z_b$ local blade coordinate depicted in figure 2
$\alpha$ angle of attack of the rotor blade
$\alpha_{2d}$ two dimensional angle of attack of the aerofoil
$\alpha_{eff}$ effective angle of attack of the rotor blade
$\alpha_{geo}$ geometric angle of attack of the rotor blade (ignoring induced velocities)
$\alpha_{ind}$ induced angle of attack of the rotor blade
$\gamma$ yaw angle depicted in figure 47
$\zeta$ nondimensional interfering wing coordinate
$\eta$ local elliptical coordinate of the blade depicted in figure 3
$\theta$ prolate spheroidal coordinate of the blade, depicted in figure 5
$\theta_p$ pitch angle of the rotor blade
$\lambda$ tipspeed ratio
$v$ prolate spheroidal coordinate of the blade, depicted in figure 5
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$\rho$</td>
<td>density of the air</td>
</tr>
<tr>
<td>$\phi$</td>
<td>local elliptical coordinate of the blade depicted in figure 5</td>
</tr>
<tr>
<td>$\Phi$</td>
<td>velocity potential</td>
</tr>
<tr>
<td>$\chi$</td>
<td>local blade coordinate depicted in figure 4</td>
</tr>
<tr>
<td>$\psi$</td>
<td>circular cylinder coordinate depicted in figure 1</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>angular velocity of the rotor</td>
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3 Introduction

Wind turbines operate in a very unsteady environment, known as the atmospheric boundary layer. Since they are restricted in height to the first 150 meters of this boundary layer the wind loads experienced by the wind turbine, and especially by the wind turbine rotor, show a specific dynamic character. In the design phase of wind turbines it is therefore necessary to consider also such dynamic loads, apart from steady load calculations. Different theoretical models can be used for the calculation of steady and unsteady aerodynamic loads on wind turbine rotors. In nowadays practice however, with only a few exceptions, all designers make use of blade-element momentum (BEM) approaches to calculate blade loads. Blade Element Momentum theory is however developed for steady situations, and thus all kinds of modifications are necessary. They are usually semi-empirical or derived from more complex theoretical methods. Examples of such more elaborate models are lifting line or lifting surface models. The elaborate models are however too complicated for practical use as a design tool, and unsuitable for implementation into larger computational models for the calculation of blade movements, total loads (including structural and gravitational loads), stresses and fatigue damage of rotor blades etc.

Based upon the acceleration potential theory using matched asymptotic expansions it is possible to derive an aerodynamic model that is reasonable time efficient, but still provides details of (unsteady) load distributions comparable with lifting surface models. Under the assumption of incompressible, inviscid and irrotational flow, the pressure perturbation in the complete flow field is given by a Laplace equation and acts as an acceleration potential function. In such model the rotor blades are represented as discrete surfaces on which a pressure discontinuity is present. The model implies the presence of spanwise and chordwise pressure distributions, which are composed of analytical asymptotic solutions for the Laplace equation. This makes the approach equivalent to a lifting surface model. The present method follows the outlines given by Van Holten [24] for the determination of loads on helicopter blades.

The numerical determination of the approximate solution is restricted to a rather simple integration procedure. As in lifting surface models it is necessary to determine the induced velocities at the rotor blade, in order to fulfil the kinematic boundary condition. For the determination of the induced velocity at one location on the blade in lifting line or lifting surface models it is necessary to perform an integration over the complete vortex surface in the wake. In contrast, the acceleration potential method only requires the integration of the accelerations along the path of the particle of air travelling from far upstream to that specific spot on the rotor blade.

Once the pressure distribution is determined, and thus the aerodynamic loads on the rotor blades are known, it is possible to determine the complete flow about the rotor blade in a similar way by integration of the accelerations experienced by particles of air, which travel
from far upstream to the rotor blade. This makes it possible to assess the assumption of local two dimensional flow adopted in BEM theory.

A further advantage is that the asymptotic acceleration method gives a kind of intrinsic possibility to simplify or elaborate the models in specific areas of interest. In its simplest appearance the model is equivalent to a lifting line model with an axially delinearized wake. With more elaborated codes it is e.g. possible to calculate the near wake and the dynamic loads caused by coherent variations in the windspeed, collective pitching of the blades and rotor speed variations (dynamic inflow) and situations where the wind direction does not coincide with the direction of the rotor axis (yawed flow).

Under dynamic inflow and yawed conditions the (pressure) loads on the blade vary with time. The accelerations in the field therefore also vary with time. So the ultimate velocity of a particle of air arriving at the rotor blade is a function of the load history on the blade ("dynamic inflow"). This velocity then determines the actual load on the blade.

The loads on the rotor blade are therefore not only a function of the actual conditions (which would be the case in a quasi-static approach) but also a function of the load history. For readers familiar with vortex modelling it is probably convenient to translate the dynamic inflow phenomenon into the effect it has in the (vortex) wake structure:

The wake vorticity exists of shed and trailing vorticity, both time dependent. The vorticity is formed at the blade and convected downstream with the local total velocity, which is partly wake induced. Equally, the strength of the trailing and shed vorticity depends on the wake through its effect on the inflow angles. This mutual interaction should be taken into account in a proper dynamic inflow model. With codes developed using asymptotic acceleration potential theory it is, in principle, possible to take into account all kinds of dynamic inflow situations included yawed operation.

The present report is structured according to the following set up:

After an introduction and an overview of the most important coordinate systems used in the report attention is paid to the basic fluid dynamic equations governing the type of flow considered. Furthermore basic solutions for the acceleration potential are related to basic solutions of the more classical velocity potential description.

Next the classical Prandtl approach for a wing in uniform flow is presented together with an equivalent formulation using Legendre functions. The theory is also extended to an infinite in plane staggered formation flight.

Then the situation of a wing in uniform flow based upon the acceleration potential approach is treated. Besides that an asymptotic expansion method is introduced which enables the derivation of approximate solutions for different regions in space. The asymptotic solutions are concatenated to a composite approximate solution valid in the whole field.

The solutions are written in Legendre functions, the natural solutions for problems formulated upon a prolate spheroidal coordinate systems. For the wing in uniform flow a first and a second order approximate solution are derived.

Next a number of situations are treated where wings rotate and mutually interfere. All these situations have some relation with the interference effects present at wind turbine
rotor blades. A comparison will be made with the results of the classical treatment of the loads on a wing in uniform flow and in formation flight.

In chapter 9 the solution of the boundary value problem for a wind turbine blade is introduced. This solution is presented in an analytical form with a series of undetermined coefficients. These coefficients have to be calculated with a numerical procedure. This procedure forms the implementation of the kinematic boundary condition.

The next chapter deals with some special aspects of the pressure field solution, such as the (numerical) determination of local inflow angles, important for the determination of the direction of the loads, as well as the inclusion of (viscous) two-dimensional aerofoil characteristics. Furthermore a method is presented to estimate effects of a second order correction of the solution on the loads.

Then the implementation of the formulas into the numerical codes used for predictions of loads at and flow about rotor blades is treated. Finally the predictions with the code for steady and schematic dynamic inflow situations as well as yawed flow situations are presented and compared with other predictions and/or with measurements.
4 Coordinate systems

In the derivation of the boundary value problems and their solutions for rotor blades and wings and subsequently of the forces acting on them, various coordinate systems have been used. The definitions of these systems are given in this chapter.

In all situations there is a (right handed) cartesian coordinate system, on which the secondary coordinate systems are based.

In the situation of the rectangular wing in uniform flow and in the situation of a formation flight of rectangular wings, as treated in chapter 6 and 7, the basic coordinate system and the wing system will coincide.

For the situation of a rectangular rotating wing discussed in chapter 8 and for the rotor blade situation of chapter 9 and consecutive chapters a second (right handed) cartesian system is used fixed at the rotating wing or rotor blade respectively.

The rotational direction is always chosen in agreement with the positive z-direction, which is also the direction of the undisturbed wind.

The geometry of the rotor blade allows variations in both chord length and pitch angle along its span. The sections of the rotor blade are assumed to have zero thickness, but are allowed to have a camber along the chord.

The direction of the rotor axis is parallel to the wind, except of course for the yawed flow conditions discussed in chapter 12. The rotor blades are assumed to move in a flat plane perpendicular to the rotor shaft. The latter restrictions can be summarised in wind turbine technological terms, that the method is restricted to rotors without tilt and cone angle.

a) As said above the basic \((x, y, z)\) reference system is a cartesian system with its positive \(z\)-axis pointing in the direction of the undisturbed wind velocity \(\mathbf{W}\) (see figure 1).

b) A convenient secondary coordinate system is the circular cylinder coordinate system \((r, \varphi, z)\) associated with the above defined basic reference system is given by (figure 1):

\[
\begin{align*}
x &= r \cos \varphi \\
y &= r \sin \varphi \\
z &= z.
\end{align*}
\]
Figure 1: Basic reference system and associated circular cylinder coordinate system.

Figure 2: The rotor blade reference system.
c) On the blade of the wind turbine a blade reference system is defined. The origin of this \((x_b, y_b, z_b)\) reference system is located mid span of the blade. The \(x_b\)-axis points into the direction of the rotational \((\Omega r)\) component of the relative wind felt by the blade, so in the rotor plane, figure 2. The \(y_b\)-axis points into the direction perpendicular to the projection of the rotor blade in the rotor plane, thus in the direction parallel to the rotor axis. The \(z_b\)-axis coincides with the spanwise direction of the blade, and points in radial outward direction. The reference system is thus defined by:

\[
\begin{align*}
    x &= (R - \frac{b}{2} + z_b) \cos \psi + x_b \sin \psi \\
    y &= (R - \frac{b}{2} + z_b) \sin \psi - x_b \cos \psi \\
    z &= -y_b.
\end{align*}
\]

A local cartesian \((x_b, y_b, z_b)\) coordinate system is also used in the situation of the wing in uniform flow. The main axes of this local system coincide with the basic \((x, y, z)\) reference system, is such a way that the \(x_b\)-axis points in the direction of the undisturbed wind \(W\) and the \(z_b\)-axis coincides with the spanwise direction of the wing, see section 7-1, equation (7-4).

d) With the local chord \(c(z_b)\) of the rotor blade an elliptical coordinate system \((\eta, \varphi, z_b)\) is defined, see figure 3:

\[
\begin{align*}
    x_b &= \frac{c(z_b)}{2} \cosh \eta \cos \varphi \\
    y_b &= \frac{c(z_b)}{2} \sinh \eta \sin \varphi \\
    z_b &= z_b.
\end{align*}
\]

At the wing the same coordinate system is used. Since the wing situation is a preliminary for the rotor blade configuration discussed later the choice was made to keep the wing chords constant over the span.
Figure 3: Elliptical coordinate system.

Figure 4: Local circular cylinder coordinate system.
e) A circular cylinder coordinate system related to the wing and rotor blade is also used in the derivation of the resulting pressure field. The axis of this \((r_b, \chi, z_b)\) system coincides with the span-axis of the wing or rotor blade respectively (see figure 4):

\[
\begin{align*}
    x_b &= r_b \cos \chi \\
    y_b &= r_b \sin \chi \\
    z_b &= z_b .
\end{align*}
\]

\[\text{(4-4)}\]

f) A prolate spheroidal coordinate system \((\nu, \theta, \chi)\) is also associated with the wing or rotor blade, which is depicted in figure 5. The ellipsoidal axis coincides with the span of the wing or blade and its definition yields:

\[
\begin{align*}
    x_b &= \frac{b}{2} \sinh \nu \sin \theta \cos \chi \\
    y_b &= \frac{b}{2} \sinh \nu \sin \theta \sin \chi \\
    z_b &= \frac{b}{2} \cosh \nu \cos \theta .
\end{align*}
\]

\[\text{(4-5)}\]

g) In chapter 6 a boundary value problem for interfering wings is discussed. Use is made there of the same basic \((x, y, z)\) reference system with the undisturbed flow velocity \(\mathbf{W}\) pointing in the direction of the positive \(z\)-axis. The wing reference system then equals the \((x, y, z)\) reference system with the span \(b\) of the wing aligned with the \(x\)-axis \((-b/2 \leq x \leq b/2\) ), and the (projected) chord of the wing aligned with the \(z\)-axis. Together with this wing an elliptic cylinder coordinate system is defined with the (longest) axis of the elliptical cylinder equal to the span of the wing. Similar to equation \((4-3)\) it is defined by:

\[
\begin{align*}
    \zeta &= \frac{x}{b/2} = \cosh \eta \cos \varphi \\
    \vartheta &= \frac{y}{b/2} = \sinh \eta \sin \varphi \\
    z &= z .
\end{align*}
\]

\[\text{(4-6)}\]
Figure 5: Prolate spheroidal coordinate system.

Figure 6: Coordinate systems of staggered wings.
h) Together with the wing reference systems defined above use is also made of the reference systems of wings which are staggered in a flat plane with respect to the basic wing located at the origin of the \((x, y, z)\) coordinate system. These wings are also aligned with the \(x\)-axis and have their projected chord of the wing aligned with the \(z\)-axis. The origin of their system, indicated with dashed symbols \((x', y', z')\), is shifted over distances \(x_{sh}\) and \(z_{sh}\) respectively with respect to the basic reference system, see figure 6:

\[
\begin{align*}
x &= x' + x_{sh} \\
y &= y' \\
z &= z' + z_{sh}
\end{align*}
\]  

(4-7)

and its elliptic cylinder coordinate system is also denoted with dashed symbols

\[
\begin{align*}
\zeta' &= \frac{x'}{b/2} = \cosh \eta' \cos \phi' \\
\theta' &= \frac{y'}{b/2} = \sinh \eta' \sin \phi' \\
z' &= z'.
\end{align*}
\]  

(4-8)

i) Finally the staggered wing may experience a rather direct interference with each other when the spanwise stagger \(x_{sh}\) is less than the span \(b\). In such cases use is also made of an elliptical cylinder reference system with the overlapping part of the span coinciding with the longest axis of the ellipsoid:

\[
\begin{align*}
\zeta_{\text{overlap}} &= \frac{2x + x_{sh}}{b - x_{sh}} = \cosh \mu \cos \psi \\
\theta_{\text{overlap}} &= \frac{2y + x_{sh}}{b - x_{sh}} = \sinh \mu \sin \psi \\
z &= z.
\end{align*}
\]  

(4-9)
5 Basic equations and considerations

Introduction

In the present chapter the basic equations will be derived related to the introduction of the velocity potential and the acceleration potential. The velocity potential is most frequently used for the description of inviscid incompressible flows. Especially in low speed aerodynamics the velocity potential description is very often used for calculating loads on lifting surfaces, see e.g. Katz et al [27]. In chapter 6 use will be made of a velocity potential description of the classical wing problem and its extension to a formation flight problem. This is however a preliminary for three dimensional interference of wings, such as present at wind turbine rotors. For treatment of such problems the acceleration potential representation is however preferred. The reason for it is that the modelling effort and the computation time needed using an acceleration potential description are considerably smaller, certainly for complex geometries. This will be elucidated in the second part of the present research, i.e. in the chapters 9-12.

Application of the acceleration potential representation implies the assumption of small velocity perturbations, and a subsequent linearisation of the equations. Furthermore this chapter emphasises the representations of basic solutions of incompressible inviscid flow in both velocity and acceleration potential theory. Not only the three dimensional basic solutions, such as point source and point doublet are treated, but also the two dimensional vortex, source and dipole are presented in both velocity and acceleration potential theory. Finally asymptotic relations between the elliptical and the cylindrical coordinate system around the chord of the blade are presented for large distances from the considered chord. These relations will be used later in the chapters 7, 8 and 9 when asymptotic expansions are derived for approximate solutions of several boundary value problems in terms of acceleration potential theory.

5-1 Basic equations

The two important equations for inviscid flow without external forces are:

\[ \frac{\partial p}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \]  

(5-1)

the continuity equation expressing the conservation of mass, and the Euler equation:
\[
\frac{DV}{Dt} = -\nabla p
\]  \hspace{1cm} (5-2)

for the conservation of momentum.

For an incompressible inviscid flow the continuity equation (5-1) and the Euler equation (5-2) reduce to:

\[
\nabla \cdot \mathbf{V} = 0
\]  \hspace{1cm} (5-3)

and

\[
\frac{DV}{Dt} = \frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{1}{\rho} \nabla p.
\]  \hspace{1cm} (5-4)

The incompressible Euler equation (5-4) expresses the relation between the substantial differential (following the "particles of air") of the velocity field and the gradient of the pressure field.

Once the pressure field is known the velocity of a given particle in the pressure field at a certain time can be determined by integration of the pressure gradient along the path of a particle. When the path of a particle present at time \( t_0 \) at location \( r_0 \) is given by the curve \( C \) the velocity \( V(r_0,t) \) of that particle can be obtained from:

\[
V(\mathbf{r}_0,t) = \int_{-\infty}^{t} \left( \frac{DV}{Dt} \right) dt = -\frac{1}{\rho} \int_{-\infty}^{t} (\nabla p) dt \quad \text{along } C. \]  \hspace{1cm} (5-5)

In general a different, and more direct approach is used in incompressible inviscid aerodynamics for the determination of the velocities. This approach starts with taking the curl of equation (5-4):

\[
\frac{\partial}{\partial t} (\nabla \times \mathbf{V}) + \nabla \times (\mathbf{V} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla \times (\nabla p) = 0
\]  \hspace{1cm} (5-6)

which can be rewritten to:

\[
\frac{\partial}{\partial t} (\nabla \times \mathbf{V}) + (\mathbf{V} \cdot \nabla) \nabla \times \mathbf{V} + (\nabla \times \mathbf{V}) \nabla \times \nabla - (\nabla \times \mathbf{V}) \cdot \nabla \mathbf{V} = 0.
\]  \hspace{1cm} (5-7)

It is evident that a velocity field which is assumed to be irrotational:

\[
\nabla \times \mathbf{V} = 0
\]  \hspace{1cm} (5-8)
Basic equations

does also satisfy equation (5-7) as can be seen by substitution.

It is known, e.g. Lamb [29], that in such irrotational cases the velocity field can be expressed as the gradient of a scalar field $\Phi$:

$$ \nabla \Phi = V.$$  \hspace{1cm} (5-9)

From this equation it can be seen that the scalar field $\Phi$ acts as a velocity potential. When equation (5-9) is substituted into equation (5-3) it can be seen that the scalar field $\Phi$ satisfies the Laplace equation:

$$ \nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0.$$  \hspace{1cm} (5-10)

The velocity potential description of inviscid incompressible and irrotational flow field is most frequently used for obtaining solutions of flow problems. Once the velocity potential is determined, the velocities everywhere in the field can be obtained easily by differentiating the scalar field as can be seen from equation (5-9). This evidently a much easier task than integrating the pressure gradient according to equation (5-5).

Sometimes it is however more convenient to use a different and much less applied approach. It will end up with an integration procedure over the total pressure gradient in a way analogous to equation (5-5).

This different way of describing an irrotational flow field is introduced by substitution of the equation for irrotational flow (5-8) into the incompressible inviscid Euler equation (5-4):

$$ \frac{\partial V}{\partial t} + \frac{1}{2} \nabla (V \cdot V) = -\frac{1}{\rho} \nabla p.$$  \hspace{1cm} (5-11)

When it is assumed that total velocity field $V$ can be written as the sum of a constant uniform velocity field $W$ and a perturbation field $V'$, so $V = W + V'$ then equation (5-11) can be written as:

$$ \frac{\partial V}{\partial t} + (W \cdot \nabla) V + \frac{1}{2} \nabla (V' \cdot V') = -\frac{1}{\rho} \nabla p.$$  \hspace{1cm} (5-12)

where again use is made of the fact that the velocity field $V$ is irrotational.

Rearranging this equation and substituting the expression for the gradient of the velocity potential $\Phi$, equation (5-9), it follows from (5-12):
\[
\left[ \frac{\partial}{\partial t} + (\vec{W} \cdot \nabla) \right] \nabla \Phi = -\nabla \left[ \frac{P}{\rho} + \frac{1}{2} (\vec{V}' \cdot \vec{V}') \right].
\] (5-13)

As can be seen from equation (5-13) the total pressure acts as an acceleration potential (scalar) function for the accelerations found when travelling along a straight path parallel to the undisturbed wind with a fixed convection speed equal to \( \vec{W} \).

Taking the divergence of equation (5-13), substituting the Laplace equation for the velocity potential \( \Phi \) (5-10) into the left hand side results into:
\[
\nabla \left[ \frac{P}{\rho} + \frac{1}{2} (\vec{V}' \cdot \vec{V}') \right] = 0
\] (5-14)

which is the Laplace equation for the total pressure.

Without any loss of generality it can be assumed that the undisturbed wind velocity \( \vec{W} \) is parallel to the \( z \)-axis: \( \vec{W} = (0, 0, W) \). Integrating this acceleration along such straight lines parallel the direction of the undisturbed wind \( \vec{W} \), up to a position \( \vec{r}_0 = (x_0, y_0, z_0) \) at the time \( t_0 \), results into an approximation \( \vec{V}_{ap} \) of the actual velocity \( \vec{V} \) at \( \vec{r}_0 \) at the time \( t_0 \):
\[
\vec{V}_{ap}(\vec{r}_0, t_0) = \frac{1}{W} \int_{-\infty}^{z_0} \left[ \frac{\partial}{\partial t} + (\vec{W} \cdot \nabla) \right] \vec{V} \, dz = -\frac{1}{W} \int_{-\infty}^{z_0} \nabla \left[ \frac{P}{\rho} + \frac{1}{2} (\vec{V}' \cdot \vec{V}') \right] \, dz.
\] (5-15)

The equation shows that the components of the velocity \( \vec{V}_{ap} \) in a certain point at a certain time can be obtained by integration of the right hand side along the undisturbed paths of the particles (parallel to the \( z \)-axis).

Equation (5-15) does not seem very useful in first instant for the calculation of velocities from a given pressure field, since the right hand side of the equation contains the perturbation part of the velocity field as well.

When however numerical schemes are used the velocities found in a given iteration can be used as "known" velocities in a next iteration step for determination of the "new" accelerations and velocities.

Equation (5-15) will not be used directly for the calculation of velocities, but it is presented here for reference of the actual method used in the applied numerical schemes.

From the equation (5-13) an important relation can be derived rather easily:
Under the conditions that the partial derivatives with respect to time \( t \) and space of \( \Phi \) can be interchanged, realising that the velocity vector field \( \vec{W} \) is constant in space and time it is allowed to modify the left hand side of equation (5-13) leading to:
\[
\nabla \left[ \frac{\partial}{\partial t} + (\vec{W} \cdot \nabla) \right] \Phi = -\nabla \left[ \frac{P}{\rho} + \frac{1}{2} (\vec{V}' \cdot \vec{V}') \right]
\] (5-16)
Integration of equation (5-16), substituting (5-9) results into the generalised Bernoulli equation:

$$p \frac{\partial \Phi}{\partial t} + p + \frac{1}{2} \rho (V \cdot V) = p_e + \frac{1}{2} \rho W^2$$  \hspace{1cm} (5-17)

where it has been implicitly assumed that "at infinity" the flow is completely described by the uniform (constant) velocity field $W$. This generalised Bernoulli equation offers a possibility to calculate the pressures, once the velocity field is known (e.g. through a velocity potential description).

5-2 Linear equations for the pressure field

Under the assumption that the perturbation field $V'$ is one order of magnitude smaller than the undisturbed velocity field $W$, the Euler equation (5-12) can be linearised. The third term on the left hand side is small with respect to the other two. Thus linearisation yields:

$$\frac{\partial V}{\partial t} + (W \cdot \nabla)V = -\frac{1}{\rho} \nabla p \ .$$  \hspace{1cm} (5-18)

Taking the divergence of this linerised Euler equation (5-18), using the divergence free property of the fluid, equation (5-3), results into:

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \ .$$  \hspace{1cm} (5-19)

The incompressible Euler equation (5-4) already showed that the pressure, or more accurately the pressure perturbation, is acting as an acceleration potential function. In linearised theory such acceleration potential function satisfies the Laplace equation as is shown in equation (5-19).

Substitution of the gradient of the velocity potential (5-9) into the linearised Euler equation (5-18) yields:

$$\left[ \frac{\partial}{\partial t} + (W \cdot \nabla) \right] \nabla \Phi = -\frac{1}{\rho} \nabla p \ .$$  \hspace{1cm} (5-20)

Equation (5-20) is the linearised equivalent of the equation (5-13).
Performing the integration of the linearised acceleration in the left hand side of (5-20) up to a position \( r_0 = (x_0, y_0, z_0) \) at the time \( t_0 \) results into:

\[
V(r_0 - r_0) = \frac{1}{W} \int_{-\infty}^{z_0} \left[ \frac{\partial}{\partial t} + (W \cdot \nabla) \right] V dz = -\frac{1}{W} \int_{-\infty}^{t_0} \frac{\partial}{\partial t} \left[ \nabla \cdot \frac{P}{\rho} \right] dz
\]

(5-21)

where integration takes place along straight lines parallel to the direction of the undisturbed wind \( W \) with a constant convection velocity \( W \).

The differentiation sequence in the left hand side of the linearised Euler equation (5-20) can also be interchanged such that the gradient is placed in front of the expression:

\[
\nabla \left[ \frac{\partial}{\partial t} + (W \cdot \nabla) \Phi \right] = \nabla \left[ \frac{-P}{\rho} \right].
\]

(5-22)

Integration of equation (5-22) yields a direct relation between the velocity potential \( \Phi \) and the acceleration potential \( p \):

\[
\left[ \frac{\partial}{\partial t} + (W \cdot \nabla) \Phi \right] = \frac{-p}{\rho}.
\]

(5-23)

This relation will be used in section 5-5 to reveal the relations between "standard" velocity potential solutions and the corresponding acceleration potential solutions.

### 5.3 Non-linear equations for the pressure field.

In equation (5-18) a linearised expression was formulated for the pressure gradient in terms of accelerations, i.e. partial derivatives of the velocity field. As will be clarified later it is necessary to develop equations which include some more (non-linear) terms. These will be used in an iterative numerical procedure to calculate velocities in a given pressure field.

In order to apply an efficient numerical scheme a further assumption with respect to the velocity perturbations will be convenient:

Assume that the perturbations in the velocities in \( x, y \) and \( z \) directions are small in directions perpendicular to the undisturbed windspeed \( W \). The velocity perturbations in the direction of the undisturbed wind \( \mathbf{W'} \) may however be larger, so \( \mathbf{V} = \mathbf{W} + \mathbf{W'} + \mathbf{V'} \) where \( |\mathbf{W}| > |\mathbf{W'}| > |\mathbf{V'}| \).

Substitution into the Euler equation (5-4) of this expression for the convection velocity then yields:
\[ \frac{\partial V}{\partial t} + (\mathbf{W} + \mathbf{W}') \cdot \nabla V + (V' \cdot \nabla)V = -\frac{1}{\rho} \nabla p \quad (5-24) \]

When the Euler equation (5-24) is semi linearised such that the third term of the left hand side is ignored the result is:

\[ \frac{\partial V}{\partial t} + (\mathbf{W} + \mathbf{W}') \cdot \nabla V = -\frac{1}{\rho} \nabla p \quad (5-25) \]

Substitution of the velocity potential gradient, equation (5-9), then yields:

\[ \left[ \frac{\partial}{\partial t} + (\mathbf{W} + \mathbf{W}') \cdot \nabla \right] \nabla \phi = -\frac{1}{\rho} \nabla p \quad (5-26) \]

which is the semi linearised version of equation (5-20).

Integration of the semi linearised Euler equation (5-26) yields a relation between the velocity potential \( \Phi \) and the acceleration potential \( p \):

\[ \left[ \frac{\partial}{\partial t} + (\mathbf{W} \cdot \nabla) \right] \phi = \frac{p}{\rho} - \frac{1}{2} \mathbf{W}'^2 \quad (5-27) \]

This equation (5-27) is the semi-linearised version of equation (5-23) and shows a correction with a "dynamic pressure" in the right hand side of the equation.

Integration of the semi linearised Euler equation (5-25) along the z-axis, up to a position \( r_o = (x_o, y_o, z_o) \) at the time \( t_o \), then results into:

\[ V(r_o, t_o) = \int_{-\infty}^{t_o} \left[ \frac{\partial}{\partial t} + (\mathbf{W} + \mathbf{W}') \cdot \nabla \right] V dt = -\int_{-\infty}^{t_o} \frac{\nabla p}{\rho} dt \quad (5-28) \]

In this non-linearised approximation the velocity \( V \) at a certain position at a certain time can thus be obtained by integration of the right hand side of equation (5-28) along the undisturbed paths of the particles (parallel to the z-axis), but with an axially disturbed velocity equal to \( \mathbf{W}' + \mathbf{W} \).

The equation (5-28) will be used in numerical schemes were the velocities found in a given iteration can be used as "known" velocities in a next iteration step for determination of the "new" velocities.
5-4 The velocity potential

In equation (5-9) it was shown that a scalar field $\Phi$ can be defined for an irrotational velocity field such that its gradient is equal to the velocity $\mathbf{V}$. Such a scalar field is therefore often called the velocity potential. It was furthermore shown in equation (5-10) that this scalar field $\Phi$ satisfies the Laplace equation. This way of describing an inviscid incompressible and irrotational flow field is most often used for obtaining solutions of flow problems.

So the Laplace equation for the potential $\Phi$ governs the velocity field. For a specific flow problem the solution of the Laplace equation, i.e. the determination of $\Phi$ will depend on the boundary conditions.

For aerodynamic problems usually one of the boundary conditions is applied at infinity. This condition states that the disturbances of the original flow caused by an aerodynamic active surface have vanished far away from that surface.

Consider for example the type of flow in which the original steady uniform flow $\mathbf{W}$ is perturbed by such aerodynamic surface(s). Far away from the aerodynamic active surface(s), for $r \to \infty$, the perturbations are not "felt" any more thus:

$$\mathbf{V} - \mathbf{W} \quad \text{for} \quad r \to \infty.$$  \hspace{1cm} (5-29)

When $\Phi^*$ is defined as velocity potential of the velocity perturbation field $\mathbf{V}^* = \mathbf{V} - \mathbf{W}$, it also satisfies the Laplace equation:

$$\nabla^2 \Phi^* = 0.$$  \hspace{1cm} (5-30)

Then equation (5-29) can be written as:

$$\nabla \Phi^* \to 0 \quad \text{for} \quad r \to \infty.$$  \hspace{1cm} (5-31)

Another boundary condition usually applied is the normal flow condition at the aerodynamic active surface(s). When the aerodynamic surface, denoted by $S$, moves with a velocity $\mathbf{V}_S$ then this boundary condition states:

$$n \cdot (\mathbf{V}^* - \mathbf{V}_S) = 0 \quad \text{at the surface} \ S.$$  \hspace{1cm} (5-32)

where $n$ is the normal vector of the body at the considered point of the surface.

An alternative formulation, using the velocity potential function $\Phi^*$ is given by:

$$\frac{\partial \Phi^*}{\partial n} = n \cdot (\mathbf{V}_S - \mathbf{W}) \quad \text{at the surface} \ S.$$  \hspace{1cm} (5-33)

A very important property of the Laplace equation is its linearity. This means that any linear combination of solutions of the Laplace equation is a solution itself. This property gives the opportunity to cover a complete aerodynamic active surface with elementary solutions of the Laplace equation. When boundary condition at infinity (5-31) is fulfilled
by all the elementary solutions, it will also be fulfilled by any linear combination of them. The boundary condition (5.32) then determines the required linear combination (i.e. the strength of the elementary solutions). It can furthermore be shown that it is sufficient to determine the distribution of such elementary solutions along the problem boundaries (the aerodynamic active surfaces). It is clear that these properties are very well suited for numerical modelling, and indeed such numerical "panel"-codes (because the complete aerodynamic surface is panelled with basic solutions) have been in development from the beginning of the sixties and have been extensively used over the past twenty years in aerodynamic problems, see Katz et al [27].

The basic solutions used for such modelling comprise the point source velocity field, determined with:

$$\Phi^* = -\frac{\alpha}{4\pi r},$$

(5.34)

and the point doublet velocity field:

$$\Phi^* = \frac{\mu}{4\pi r} 2\pi \delta(-\frac{1}{r}),$$

(5.35)

where \(\delta\) is the direction of the point doublet, see figure 7. Apart from these point singularities use is often made of distributed sources and/or doublets.

Figure 7: Point doublet in \(x\)-direction with corresponding streamlines
It is well known in the modelling of aerodynamic problems that the type of basic solutions used cannot always be restricted to source and/or doublet distributions on the solid aerodynamic surface itself. When the surface generates lift, i.e. when a force is found perpendicular to the direction of \( \mathbf{F} \), then vortex elements have to be included in the linear combination of basic solutions which will form the solution of the boundary value problem (5.30), (5.31) and (5.32). It is also known that the lift on such surfaces is directly related to the circulation. This seems to be a contradiction with the assumption of irrotational flow, unless it is assumed that initially the lift on the surface was not present. As time goes by the solid surface can build up lift (or circulation which is an equivalent statement), but not without generating a vortex wake which incorporates the same circulation with an opposite sign. The closed curves in the fluid, attached to the fluid particles originally surrounding the surface, will then surround the solid surface plus its vortex wake. But this in its turn makes the panelling of the vortex wake inevitable, since this is also an aerodynamic active (though not solid) surface. For complex flows, such as the flow around rotating lifting surfaces, the necessity of wake modelling makes the numerical effort rather complicated as well.

Since vortex elements as such are no solutions of the three dimensional Laplace equation, because they only obey the equations in combination with other vortex elements forming closed systems of constant circulation, they cannot be represented as a "point vortex" solution. A ring vortex however is a solution of the boundary value problem. In a similar way a straight infinite vortex line also satisfies the equations, since it is closed "at infinity".

The infinite vortex line will be discussed below, as a basic solution of the two dimensional velocity field.

Apart from the three dimensional velocity fields, use is often made of two-dimensional flow fields which as well satisfy the Laplace equation. The two-dimensional problem can be formulated very easily by stating that the velocity (perturbation) in one direction, e.g. the \( y \)-direction is zero everywhere in the (three-dimensional) space, thus:

\[
\frac{\partial \Phi^*}{\partial y} = 0 .
\]  

(5-36)

The two dimensional Laplace equation then yields for the perturbation velocity potential \( \Phi^* \):

\[
\nabla^2 \Phi^* = \frac{\partial^2 \Phi^*}{\partial x^2} + \frac{\partial^2 \Phi^*}{\partial z^2} = 0 .
\]  

(5-37)

The infinite vortex line in the three dimensional situation is equivalent to the two dimensional point vortex in a two dimensional field which is given by:

\[
\Phi^* = -\frac{\Gamma}{2\pi} \theta ,
\]  

(5-38)

where \( \theta \) is the azimuthal coordinate.
Two important situations can be distinguished here, depending upon the orientation of the
infinite long vortex line with respect to the oncoming wind. At first the situation where
the vortex line is perpendicular to the undisturbed flow velocity $\mathbf{W}$. When the vortex line
is assumed to be aligned with the $y$-axis, and with $\mathbf{W}$ parallel to the $z$-axis then the
relation $\theta = \arctan(z/x)$ holds, and the vortex line is the representative of a lifting line
exhibiting a force in the direction perpendicular to $\mathbf{W}$.
In the other situation the vortex line is parallel to $\mathbf{W}$. Then the partial derivative of $\Phi^r$ with
respect to $y$ in expression (5-36) must of course be replaced by the derivative with respect
to $z$. In that situation the relation $\theta = \arctan(y/x)$ holds. This is the situation of a trailing
vortex which is aligned with the undisturbed flow and then no forces act on the flow field,
as will be shown later.

Later on in this chapter, in section 5-5, a relation will be derived between velocity
potential solutions and acceleration potential solutions. With this in view, it is convenient
to introduce here the two dimensional velocity potential of a source and of a doublet. In
the three dimensional case such a source or doublet is of course equivalent to an infinite
source or doublet line with constant strength. Its potential functions yield:

$$\Phi^r = \frac{\sigma}{2\pi} \ln r$$  \hspace{1cm} (5-39)

for the source line and:

$$\Phi^* = \frac{\mu}{2\pi r^2} (z'y)$$  \hspace{1cm} (5-40)

for the doublet line, where $g$ is the direction of the doublet in the $x-z$ plane.

The radial distance $r$ in the equations (5-39) and (5-40) has to be taken in the $x-z$ plane,
so:

$$r = \sqrt{x^2 + z^2}$$  \hspace{1cm} (5-41)

5-5 The acceleration potential

In section 5-2 it was shown that in an irrotational velocity field $V$ with the assumption of
small velocity perturbations with respect to a constant uniform velocity field $\mathbf{W}$ the Euler
equation may be linearised. Such linearisation then leads to the conclusion that the
Corresponding pressure (perturbation) field $p$ does satisfy the Laplace equation, (equation
(5-20)). Since the gradient of the pressure field is directly related to the substantial
differential of the velocity field, equation (5-4) which is the acceleration experienced by
particles of air moving through the velocity field the (scalar) pressure field is often called
the acceleration potential. This way of describing an inviscid incompressible and
irrotational flow field is seldom used for obtaining solutions of flow problems, but it has
its advantages for more complex flow problems.

The solution of the Laplace equation for a specific flow problem, i.e. the determination of the pressure perturbation field $p$ will depend on the boundary conditions. The boundary condition "at infinity" is in terms of the acceleration potential expressed as:

$$p \rightarrow 0 \quad \text{for} \quad r \rightarrow \infty . \quad (5-42)$$

The boundary condition stating parallel flow at the aerodynamic active surface(s) $S$ is now formulated by:

$$\frac{\partial p}{\partial n} = 0 \quad \text{at the surface} \quad S . \quad (5-43)$$

Strictly spoken this equation is not sufficient for parallel flow. It needs to be completed by the statement that, for the particles of air travelling over the surface $S$, the velocity at a certain point on $S$ must be parallel to the surface as well. Once this is satisfied for one position on $S$ only such particle is, by equation $(5-43)$, forced to follow the contour of the surface.

It is interesting to see the relation between the (steady) basic point source and point doublet velocity field distributions $(5-34)$ and $(5-35)$ and their corresponding representation in terms of the pressure field $p$. Substitution of the point source velocity potential $(5-34)$ into $(5-23)$ yields:

$$\frac{p}{p} = \frac{a}{4\pi} W \nabla(\frac{1}{r}) . \quad (5-44)$$

The pressure field representation of a (steady) point source velocity field has the character of a "doublet" field, with its direction parallel to the undisturbed flow velocity $W$. Such pressure field is usually called a point dipole or a doublet field.

Substitution of the point doublet velocity potential $(5-35)$ into $(5-23)$ gives:

$$\frac{p}{p} = - \frac{1}{4\pi} (W \nabla s \nabla(\frac{1}{r})) . \quad (5-45)$$

This pressure field represents a pressure quadrupole with its axes in the directions of $W$ and $s$ respectively.

The two dimensional pressure potential equation is given by:

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial z^2} = 0 . \quad (5-46)$$

And the pressure field equivalent of the two dimensional point vortex is obtained by
substitution of the expression for the two dimensional velocity potential of a vortex, equation (5-38), into equation (5-23):

\[ \frac{p}{\rho} = \frac{\Gamma}{2\pi} W \nabla \theta. \]  \hspace{1cm} (5-47)

In the situation of a vortex line perpendicular to the flow, i.e. in y-direction, the azimuthal coordinate \( \theta \) satisfies \( \theta = \arctan(z/x) \) as was already seen in the previous section. Evaluation of equation (5-47) then yields:

\[ \frac{p}{\rho} = \frac{\Gamma}{2\pi} \frac{W \cos \theta}{r} = \frac{\Gamma}{2\pi} \frac{W_x}{r^2} \]  \hspace{1cm} (5-48)

and this is the potential function of a two dimensional pressure dipole field pointing in the \( x \)-direction (perpendicular to the direction of \( \mathbf{W} \)).

In the situation of a vortex line parallel to the flow i.e. in z-direction, (the \( z \)-coordinate in equation (5-43) must then of course be replaced by the \( y \)-coordinate), the azimuthal coordinate \( \theta \) satisfies \( \theta = \arctan(y/x) \) and evaluation of equation (5-44) does show that the pressure field vanishes!! Thus no pressure difference, and therefore no forces are present.

In the acceleration potential theory the most important singularity used to model a lifting line or a lifting surface is thus a pressure dipole distribution. It is the representation of the bound vortex distribution used in the velocity potential approach. The dipole representation in acceleration potential theory has the advantage that there is no need to model the wake of the lifting surface, since vorticity in the wake is not related to any pressure difference.

Substitution of two dimensional velocity potential of a source (5-39) into (5-23) yields its pressure field equivalent:

\[ \frac{p}{\rho} = -\frac{\alpha}{2\pi} \frac{(W \cdot r)}{r^2} = -\frac{\alpha}{2\pi} \frac{W_z}{r^2}. \]  \hspace{1cm} (5-49)

By comparison with equation (5-48) it can be seen easily that this is a pressure dipole (or doublet) line with its direction along the \( z \)-axis, thus parallel to the direction of \( \mathbf{W} \).

This kind of pressure distribution will not play an important role in the research presented here. It is solely derived to make a comparison possible with the basic source distribution as a solution of the velocity potential equation. In situations where flow separation should be modelled it is however necessary to implement pressure distributions according to equation (5-49).

When the direction of the velocity doublet in the expression for the velocity potential (5-40) is taken parallel to the \( x \)-axis and then substituted into (5-23) its pressure equivalent is obtained which yields:
\[ \frac{p}{\rho} = \frac{\mu W_{xz}}{\pi r^4} . \] (5-50)

This pressure field represents a two dimensional pressure quadrupole distribution with its orthogonal directions pointing along the \( x \)- and \( z \)-axis respectively. Thus one of the directions is perpendicular to the undisturbed oncoming velocity \( W \) and the other is parallel to it.

With such pressure field it is possible to represent pressure variations due to the curvature of the aerofoil used in a wing (the aerofoil camber), or curvature of the flow in the region of the aerofoil. This will be further elucidated in chapter 7 where it is seen that both the two dimensional pressure dipole and the quadrupole distributions will appear in the acceleration potential solution for a wing in uniform flow.

5-6 Asymptotic considerations concerning the coordinate systems

Asymptotic expansion techniques will be used later in finding solutions of Laplace equations. In these asymptotic techniques use is made of the special coordinate systems defined in chapter 4. Especially the relation between the elliptic coordinate system (4-3) and the circular cylinder coordinate system (4-4) around the (local) chord of the wing or rotor blade is important.

From (4-4), substituting (4-3) it follows:

\[ r_b = \sqrt{x_b^2 + y_b^2} = \]
\[ = \frac{c(z_b)}{2} \sqrt{\cosh^2 \eta - \sin^2 \phi} = \]
\[ = \frac{c(z_b)}{2} \sqrt{\sinh^2 \eta + \cos^2 \phi} . \] (5-51)

For large \( r_b \), which means large with respect to the local chord \( c \) and thus of the order of the span \( b \), the following asymptotic expansion can be derived:

\[ \frac{r_b}{c/2} = \cosh \eta = \sinh \eta = \frac{1}{2} e^n \]
\[ \text{for } \frac{r_b}{c/2} \rightarrow 1 . \] (5-52)

The asymptotic equivalence of equation (5-52) is valid up to the order \( O(A^2) \) with respect to the leading term, where the aspect ratio of the wing written as \( A = b/c \).

With (5-52) it follows from the comparison of the elliptical and the circular cylinder coordinate systems:
\[ \varphi = \chi \quad \text{for} \quad \frac{r_b}{c/2} > 1. \quad (5-53) \]

again with an asymptotic accuracy up to the order \( O(A^{-2}) \).

A third asymptotic approximation proved to be very useful in the modelling of loads on wings. It is especially applicable in the case in which the wing is represented by vortex models. In the far field behind the wing, assuming that the flow field is steady, the perturbation velocities and accelerations will be dominated by the distribution of vorticity in the wake. When on its turn it is assumed that the wake and its vorticity distribution far behind the wing will be settled, i.e. is not any more dependent upon its distance behind the wing, the flow field can be approximated by a two-dimensional field. The vorticity positioned at the location of the wake will introduce velocity perturbations and accelerations only in a plane perpendicular to the wake, thus yielding its two dimensional character. The velocity perturbations originating from the bound vorticity at the wing are thus neglected. Such an approximation is known as a Trefftz-plane approach, see e.g. Katz et al [27].
6 Classical wing theory and its application to an infinite formation flight

Introduction

In this chapter a classical problem will be discussed. It concerns the determination of the load distribution over a wing in uniform flow. For the treatment of this problem the concept of the Trefftz-plane, see e.g. Katz et al [27], has proven to be very adequate. Also in the situation of wings which are part of an infinite staggered formation flight the concept of the Trefftz-plane can be applied and leads to useful results. In the present chapter this approach will be elucidated and applied for the calculation of the loads in such formation flight. For the representation of the load distribution over the wing use will be made of both Fourier function series and Legendre function series. The first representation is part of the classical treatment of the problem. The latter representation is also introduced here since Legendre functions are "natural solutions" of the two dimensional Laplace equation in elliptical coordinates which are used in the problem, and besides that because they will form the core of the solutions obtained in the full three dimensional (acceleration potential) description.

It is thus a preliminary for the treatment in chapter 8 of interfering rotating wing problems, for which the need of a full three dimensional description is inevitable.

In the section 6-5 of this chapter some numerical results will be presented for both the classical wing problem and the interfering wing problem. Results are presented for both Fourier series representation and Legendre function representation which makes mutual comparison possible.

Finally the limitations are discussed of the Trefftz-plane modelling used in this chapter. It will be concluded that for more detailed information concerning the flow about a wing in uniform flow a real three dimensional problem needs to be solved, which will subsequently take place in the next chapter 7.

6-1 Classical wing theory

Consider first the situation of one rectangular wing in a uniform steady flow. In its simplest representation the wing is modelled by a bound vortex representing the lifting line approximation of the wing. The wing is modelled by a bound vortex representing the lifting line approximation of the wing. Since bound vorticity has to vanish at the wing tips, trailing vorticity is shed into the flow. A rudimental lifting line representation of such wing in velocity potential theory makes use of a horseshoe vortex. Then the bound vortex is of constant strength and equal to the strength of the trailing vortex shed at both tips. A
more realistic representation implies a bound vortex of varying strength along the span together with a sheet of trailing vorticity which is shed into the flow. The surface on which this trailing vorticity is located is, in linearised theory, assumed to be a straight plane from the wing towards infinity, in the direction of the undisturbed flow \( \mathbf{W} \), figure 8. Far downstream of the wing the flow becomes equal in all planes perpendicular to the (straight) vortex sheet. In such Trefftz-planes the flow satisfies the two dimensional Laplace equation for the velocity potential \( \Phi \):

\[
\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0
\]

(6-1)

where the \( x-y \) plane is perpendicular to the direction undisturbed flow \( \mathbf{W} \) (pointing in the \( z \)-direction). The vortex wake in the three dimensional situation is represented in the Trefftz plane by a vorticity distribution along a line segment (in linearised theory).

It is convenient to introduce an elliptical coordinate system \((\eta, \varphi, z)\) with the longest axis of the ellipses equal to the bound vortex segment, figure 9. This coordinate system is defined by the equations (4-6). In the Trefftz planes (planes perpendicular to the \( z \)-axis at large values of \( z \)) this coordinate system becomes a two dimensional elliptical coordinate system with the longest axis of the ellipses coinciding with the location of the trailing vorticity.

In such coordinate system equation (6-1) yields:

\[
\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial \eta^2} + \frac{\partial^2 \Phi}{\partial \varphi^2} = 0
\]

(6-2)

By separation of variables a solution of (6-2) can be obtained which satisfies the boundary condition at infinity:

\[
\nabla \Phi = 0 \quad \text{for} \quad \eta \to \infty
\]

(6-3)

and the symmetry conditions with respect to the \( x-z \) plane:

\[
\Phi(\eta, \varphi) = \sum_{n=1}^{\infty} \left( d_n e^{-\kappa \eta} \sin n \varphi \right).
\]

(6-4)

The induced velocity in the \( y \)-direction follows from equation (6-4) by taking the partial derivative in that direction:

\[
\nu_{\text{ind}}(\varphi) = \left[ \frac{\partial \Phi}{\partial y} \right]_{\eta=0} = -\frac{1}{b} \sum_{n=1}^{\infty} n d_n \sin n \varphi.
\]

(6-5)

This induced velocity causes a deformation of the vorticity sheet (the vortex wake). As can be seen from (6-5) the first term (with coefficient \( d_n \)) introduces a constant \( \nu_{\text{ind}} \) (the downwash). The second and further terms describe the variation of the downwash along the span.
Figure 8: Vortex representation of lifting line with corresponding trailing vorticity sheet in linearised theory

Figure 9: Elliptical coordinate system for the wing and the trailing vorticity in the Trefftz plane (linearised theory)
In our assumptions the trailing vorticity sheet was straight. So the effect of the higher order terms $d_n$ with $n \geq 2$ on the position of the vorticity sheet is not taken into account in the linearised theory adopted here. However this induced velocity distribution influences the lift distribution over the wing, because it changes the direction of the oncoming flow. It causes the wing to operate with an effective angle of attack which is smaller than the geometrical angle.

Equation (6-5) yields the induced velocity far downstream of the wing (where the flow has become two-dimensional). This implies that seen from such Trefftz-plane, the vorticity sheets stretches itself from minus infinity towards infinity. At the wing (in linearized theory again) only a half infinite vorticity sheet is seen (the vortex wake), which therefore will introduce only half the induced velocity:

$$ [v_{\text{ind}}(\psi)]_{z=0} = -\frac{1}{b \sin \phi} \sum_{n=1}^{\infty} n d_n \sin n\phi . \quad (6-6) $$

When the approximation $v_{\text{ind}}/W$ is used for the induced angle of attack it follows for the effective angle of attack:

$$ \alpha_{\text{geo}} = \alpha_{\text{eff}} + \alpha_{\text{ind}} = \alpha_{\text{geo}} + \frac{1}{b W \sin \phi} \sum_{n=1}^{\infty} n d_n \sin n\phi . \quad (6-7) $$

6-2 Representations of the lift coefficient distribution

Differentiation of the potential function (6-4) in $x$-direction yields the velocities parallel to the vorticity sheet. The velocity difference over the vorticity sheet is a direct measure for the magnitude of the trailing vorticity in the wake, and this trailing vorticity is directly related to the bound vorticity at the wing which represents the load. The latter can be obtained by integration of the trailing vorticity.

With the relation $\Gamma = 1/2 \, Wc \, C_l$ the bound vorticity $\Gamma$ is directly related to the lift coefficient $C_l$ on a wing with a chord $c$. In then turns out, see Van Bussel [8], that by subsequent differentiation of $\Phi$, and integration along the trailing vorticity sheet the lift coefficient distribution is expressed in terms of the following harmonic series:

$$ C_l(\psi) = \sum_{n=1}^{\infty} d_n^* \sin \phi \quad (6-8) $$

where $d_n^* = 4 \, d_n/Wc$.

Then equation (6-7) for the geometrical angle of attack in relation to the effective angle and the induced angle yields:
\[ \alpha_{geo} = \alpha_{eff} + \alpha_{ind} = \frac{1}{2\pi} \sum_{n=1}^{\infty} d_n^* \sin \varphi + \frac{c}{4b \sin \varphi} \sum_{n=1}^{\infty} n d_n^* \sin \varphi \] (6-9)

where use is made of the well known relation:

\[ C_i = 2\pi \sin \alpha_{eff} \approx 2\pi \alpha_{eff} . \] (6-10)

Equation (6-9) together with the assumption made in equation (6-8) is known as the classical Prandtl equation in terms of the elliptic coordinate \( \varphi \), see e.g. Schlichting et al [34].

An alternative description of the lift coefficient distribution makes use of Legendre functions. Although it is not necessary to introduce this type of representation for the formation flight problem treated in the coming sections of this chapter, it will ease the introduction of the load representation on rotor blades in the acceleration potential modelling, and is therefore performed here.

It can be shown, see Van Bussel [8], that the equivalent of equation (6-8) in term of Legendre functions yields:

\[ C_i(\zeta) = \left(1 - \zeta^2\right) \sum_{n=1}^{\infty} f_n^* P_n'(\zeta) \] (6-11)

where \( \zeta = \frac{x}{b/2} \) and \( P_n'(\zeta) = \frac{dP_n(\zeta)}{d\zeta} \).

Here \( P_n \) denotes the Legendre polynomial of the \( n \)'th degree, see appendix A and figure 10.

Formally the lift coefficient distributions \( C_i \) of equation (6-11) and equation (6-8) should be distinguished from each other since their independent variable is different. However by choosing a different variable name for the independent variable the discrimination is evident. The same holds for the induced velocity distributions of equation (6-6) and the following expressions (6-12) and (6-13) in terms of Legendre functions.

Differentiation of the lift coefficient distribution in spanwise direction yields the trailing vorticity strength. The induced velocity originating from a trailing vortex line is inversely proportional to the distance. Thus the relation between lift coefficient distribution and induced velocity distribution at the lifting line is given by:

\[ \frac{v_{ind}(\zeta)}{W} = -\frac{1}{4\pi} \frac{c}{b} \int_{-1}^{1} \frac{dC_i(s)}{ds} \frac{1}{(\zeta - s)} ds . \] (6-12)
Substitution of equation (6-11), making use of the relations between the Legendre functions of the first kind $P_n$ and of the second kind $Q_n$, see appendix A, results into:

$$\frac{v_{ind}(\zeta)}{W} = \frac{1}{2\pi b} c \sum_{n=1}^{\infty} n(n+1)f_n^*Q_n(\zeta). \quad (6-13)$$

Figure 11 shows some examples of induced velocity distributions according to expression (6-13).

In terms of Legendre functions the Prandtl equation is thus given by:

$$\rho_{geo} = \frac{(1-\zeta^2)}{2\pi} \sum_{n=1}^{\infty} f_n^*P_n(\zeta)' - \frac{1}{2\pi b} c \sum_{n=1}^{\infty} n(n+1)f_n^*Q_n(\zeta). \quad (6-14)$$

Comparison of equation (6-9) with (6-14) shows the equivalence of the use of Legendre functions with the classical approach using harmonic functions.
6-3 The interfering wing problem in terms of Fourier series

Suppose the wing to be a member of an infinite array of wings in a plane staggered formation, see figure 12. In such a situation all the wings are located in a flat plane. The position of a wing with respect to its predecessor is staggered, which means that there is a shift in both the lateral direction and the direction of the undisturbed flow. All the wings in such infinite formation are oriented parallel, and shifted over the same distance $x_{sh}$ in lateral direction, and $z_{sh}$ in flow wise direction.

The oncoming flow to a given wing has then been influenced by the infinitely stretched array of wings in front of the considered one. Assume that the distance between the wings in the direction of the undisturbed flow $z_{sh}$ is large with respect to their chord $c$, and finally assume that the amount of shift in lateral direction exceeds half the span ($x_{sh}>b/2$).

Then it seems to be reasonable to state that the character of the lift distribution (not its magnitude !!) over a wing is principally affected by its preceding wing. In order to determine this character it is thus sufficient to take into account the velocity perturbation caused by this predecessor, which is of course identical to the perturbation caused by the wing itself ("felt" by its follower).
First assume that the amount of lateral shift exceeds the span ($x_{ab} > b$). Then the trailing vorticity left by a wing will not "hit" its successor. In that case no singularity in the velocity distribution will be experienced within its span, so the lift distributions are not expected to show any irregular behaviour and can thus still be represented by either (6-8) or (6-11).

The induced velocity far behind one wing along and beside its trailing vorticity wake is given by:

$$\frac{V_{ind}}{W} = \begin{cases} \frac{1}{W} \left[ \frac{d\Phi}{dy} \right]_{\eta=0} = -\frac{1}{2b} \sum_{n=1}^{-\infty} \left( -1 \right)^n n d_n e^{-n\eta} & x < -\frac{b}{2} \\ \frac{1}{W} \left[ \frac{d\Phi}{dy} \right]_{\eta=0} = -\frac{1}{2b} \sum_{n=1}^{\infty} n d_n \sin n\varphi & -\frac{b}{2} < x < \frac{b}{2} \\ \frac{1}{W} \left[ \frac{d\Phi}{dy} \right]_{\eta=0} = \frac{1}{2b} \sum_{n=1}^{\infty} n d_n e^{-n\eta} & x > \frac{b}{2} \end{cases}$$

(6-15)

As a matter of notation the variables $\varphi, \eta$, and $x$ are replaced by their primed equivalences $\varphi', \eta'$, and $x'$ when it concerns the induced velocity due to the predecessor of the considered wing.
When the total induced velocity at the lifting line, originating from one preceding wing (equation (6-15)) and from the self induced velocity of the wing itself (equation (6-6)) is substituted into the Prandtl equation for the case $x_{sh} > b$ it results into:

$$
\alpha_{geo} = \frac{1}{2\pi} \sum_{n=1}^{\infty} d_n^* \sin n\varphi + \frac{c}{4b \sin \varphi} \sum_{n=1}^{\infty} nd_n^* \sin n\varphi + \frac{1}{2} \frac{c}{b \sinh \eta} \sum_{n=1}^{\infty} nd_n^* e^{-n\zeta}.
$$

(6-16)

Now assume that the lateral shift is such that the trailing vorticity sheet of the predecessor hits the following wing on one side $b > x_{sh} > b/2$. In such case the trailing vorticity is, in general, accompanied by a singular velocity field that "hits" the follower. This implies that the lift distribution over the wing cannot a priori be assumed regular (i.e. differentiable over the span). This implies that the representation assumed in equation (6-8) is no longer valid. However assuming such distribution on the preceding wing may help to determine an improved representation on the follower. The induced velocity distribution originating from the predecessor exhibits a singularity for $x = (b/2 - x_{sh})$.

Assuming an extra lift distribution over only the interfering part of the considered wing:

$$
C_{1,extra}(\psi) = \sum_{n=1}^{\infty} e_n^* \sin n\psi
$$

(6-17)

where $\psi = \arccos \frac{2x-x_{sh}}{b-x_{sh}}$,

gives rise to an extra self induced velocity having the same character as the induced velocity distribution of the preceding wing.

With an extra condition for the coefficients $e_n^*$ of this extra lift coefficient distribution can be chosen such that the singular character of the induced velocity caused by the preceding wing is exactly annihilated by this extra self induced velocity. According to Van Bussel [8] this condition yields:

$$
\sum_{n=1}^{\infty} nd_n^* + \frac{1}{2} \sqrt{\frac{b}{b-x_{sh}}} \sum_{n=1}^{\infty} (-1)^n ne_n^* = 0.
$$

(6-18)

The result is then that, under the assumption that the total lift distribution is given by:

$$
C(\psi) = \sum_{n=1}^{\infty} d_n^* \sin n\varphi + \sum_{n=1}^{\infty} e_n^* \sin n\psi \left[ \frac{b - 2\psi}{2} \right]
$$

(6-19)

with coefficients satisfying equation (6-18), the singular behaviour of the induced velocity at the wing is suppressed.
In equation (6-19) the symbol \( \mathbf{1} \) is used. It is defined as the unit function over the interval given in the index, and equals 0 outside the given interval.

Finally the equation determining the coefficients \( d_n^* \) and \( e_n^* \) can be formulated analogous to the classical Prandtl equation (6-9) derived in the beginning of his chapter:

\[
\alpha_{geo} = \frac{1}{2\pi} \sum_{n=1}^{\infty} d_n^* \sin n\phi + \frac{c}{4b} \sum_{n=1}^{\infty} n d_n^* \sin n\phi + \\
+ \frac{1}{2} \frac{1}{b \sinh \eta'} \sum_{n=1}^{\infty} n d_n^* e^{-n\eta'} + \frac{1}{4} \frac{c}{(b-x_{sh}) \sinh \eta} \sum_{n=1}^{\infty} (-1)^n n e_n^* e^{-n\mu} \tag{6-20}
\]

for \( -\frac{b}{2} < x < x_{sh} \frac{b}{2} \)

and

\[
\alpha_{geo} = \frac{1}{2\pi} \sum_{n=1}^{\infty} d_n^* \sin n\phi + \frac{1}{2\pi} \sum_{n=1}^{\infty} e_n^* \sin n\psi + \frac{c}{4b} \sum_{n=1}^{\infty} n d_n^* \sin n\phi + \\
+ \frac{1}{2} \frac{1}{b \sin \phi'} \sum_{n=1}^{\infty} n d_n^* \sin n\phi' + \frac{1}{4} \frac{c}{(b-x_{sh}) \sin \psi} \sum_{n=1}^{\infty} n e_n^* \sin n\psi \tag{6-21}
\]

for \( x_{sh} \frac{b}{2} < x < \frac{b}{2} \).

6-4 The interfering wing problem in terms of Legendre series

The equivalent expression of equation (6-21) for the situation where the lift coefficient distribution is represented by Legendre functions (see equation (6-11)) yields:

\[
C_L(x) = (1-\zeta^2) \sum_{n=1}^{\infty} f_n^* P_n(\zeta) + (1-\sigma^2) \sum_{n=1}^{\infty} g_n^* P_n(\sigma) \frac{1}{[x_{sh} \frac{b}{2} \frac{b}{2}]} \tag{6-22}
\]

with \( \zeta = \frac{x}{b/2} \) and \( \sigma = \frac{2x-x_{sh}}{b-x_{sh}} \).

The condition to be fulfilled by the coefficients \( f^* \) and \( g^* \) is, according to Van Bussell [8] given by:

\[
\sum_{n=1}^{\infty} n(n+1) f_n^* = \frac{b}{2(b-x_{sh})} \sum_{n=1}^{\infty} n(n+1) g_n^*(-1)^n \tag{6-23}
\]
The equivalent expression to (6-20) and (6-21) determining the coefficients $f_n^*$ and $g_n^*$ then yields:

$$
\alpha_{geo} = \frac{(1-\zeta^2)}{2\pi} \sum_{n=1}^{\infty} f_n^* P_n'(\zeta) + \frac{(1-\sigma^2)}{2\pi} \sum_{n=1}^{\infty} g_n^* P_n'(\sigma) \left[ \frac{1}{z a} \frac{b}{2} \frac{b}{2} \right] + \\
\frac{1}{2\pi} \frac{c}{b} \sum_{n=1}^{\infty} n(n+1) f_n^* Q_n(\zeta) - \frac{1}{2\pi} \frac{c}{b} \sum_{n=1}^{\infty} n(n+1) f_n^* Q_n(\zeta') + \\
- \frac{1}{2\pi} \frac{c}{(b-x_{um})} \sum_{n=1}^{\infty} n(n+1) g_n^* Q_n(\sigma) .
$$

(6-24)

6-5 Numerical results for the loads on interfering wings

With the extension of the classical Prandtl equation for interfering wings presented in the preceding sections, it is possible to obtain numerical results for the load distributions and the induced velocity distributions.

As a reference the situation of the rectangular wing in a uniform steady flow will be used.

Figure 13 shows the calculated load distribution over wings with varying aspect ratio according to the standard Prandtl equation using a Fourier series approximation, and the result for the Legendre function approximation.

Apart from the behaviour very near to the tip of the wing it can be seen that the results are equivalent.

The differences close to the tip are evidently caused by the different behaviour of the representative spanwise distributions. The derivative of the spanwise function in the Fourier series representation shows an infinite value, whereas the Legendre representation leads to a finite value of $dC_f/dx$ at the tip.

Figure 14 shows the results of the interfering wings calculations for an overlap of $0.3b$ and for the synchronous situation. It can be seen that also in this situation the general behaviour of the two different representations is quite similar.

In the synchronous situation the influence of the up wash from the preceding wing is evidently present by the asymmetric increase of load over the symmetric wing, yielding the largest increased of load on the most interfering part of the wing.

The overlapping situation shows an even larger increase of the load up to the overlap position. The trailing vorticity of the preceding wing causes a steep gradient in the load distribution leading to a decreased load on the outer right part of the wing. The discontinuity in the gradient of the load at the overlap point is related to a vorticity shed in the wake that counteracts the vorticity generated by the left tip of the preceding wing.
Figure 13: Comparison of load distributions for Fourier and Legendre function representations in uniform flow

Figure 14: Comparison of load distributions for Fourier and Legendre function representations in staggered flow
6-6 A more detailed description of loads and velocities at a wing

For a more detailed description of the load distribution and the induced velocity distribution over isolated and interfering wings the Trefftz-plane approach can no longer be applied. In these cases it is required to solve of a real three dimensional boundary value problem.

The situation of interest is still the situation of one rectangular wing in a uniform steady flow. The span of the wing is perpendicular to the direction of the undisturbed flow \( \mathbf{W} \). In its most common representation wings are modelled by bound vorticity. Its simplest representation is the lifting line approximation of the wing, already discussed above. Since the bound vorticity has to vanish at the wing tips, a sheet of trailing vorticity is shed into the flow. The position of the surface on which this trailing vorticity is located is, in linearized theory, determined by the undisturbed flow velocity \( \mathbf{W} \).

Only in its most simple representation, the situation where the trailing vorticity sheet is assumed to be a straight surface, which shape is "prescribed" by the undisturbed flow velocity \( \mathbf{W} \), a simple analysis will give useful result, see above. An more elaborate model will allow the vorticity sheet to deform under the effect of the "induced" velocities. In such cases it is necessary, in terms of a vorticity representation, to model not only the wing but also the emanating vortex sheet with its (rather) complex shape. A three dimensional analytical potential function can, in general, not be found and thus a numerical approach will be needed.

In such situations a pressure field representation is very suitable. As was seen in section 5-5 the equivalent of the bound vortex line in terms of a pressure field is a line distribution of pressure dipoles. Furthermore it was concluded that trailing vorticity, vorticity travelling with the flow, has no representation in a pressure field description. The reason for it is that pressure singularities can only occur at solid surfaces such as the wing itself and not at surfaces consisting of particles of air. Thus the modelling of the wake is not necessary in an acceleration potential representation. Under the assumptions of small velocity perturbations (consistent with the above described linearized model of the vortex wake) it was shown in chapter 5 that the pressure field satisfies the Laplace equation. This equation, together with suitable boundary conditions, will form the basis for the solutions of the more advanced wing models treated in the next chapter.
7 Acceleration potential wing theory using matched asymptotic expansions

Introduction

In the preceding chapter it was concluded that a Trefftz-plane approximation is no longer suitable when more detailed information is needed concerning the loads and the velocity distribution at the wing. Also in situations where the interest is focused on wings interfering in a rotating configuration more elaborate methods have to be applied.

Usually a velocity potential method is used, where, in its simplest representation, a wing is modelled by a bound vortex: its "lifting line" approximation. This however implicates the modelling of the vorticity wake emanating from the wing. For a single wing in a uniform flow such modelling effort is rather straightforward. Whenever such wing becomes part of an interfering wing problem the modelling effort, and the programming effort for the computational program becomes quite considerable. Therefore the alternative pressure field approach is used here, in which the pressure (perturbation) field acts as an acceleration potential function for the flow around the wing. By integration of the pressure field along the trajectories of particles of air velocities can be obtained for any position in the field. And the pressure distribution over the wing directly determines the load distribution. Modelling of a complex vortex wake is therefore no longer necessary.

As a preliminary to the situation of interfering wings, and finally to the wind turbine rotor situation, the situation of a rectangular wing in a uniform steady flow is considered again, but now within the framework of (linear) acceleration potential theory.
It was seen in section 5-5 the equivalent of the vortex line in terms of a pressure field is a line distribution of pressure dipoles. They will form part of the approximate solution of the Laplace equation, determined using a matched asymptotic expansion technique.
With the application of matched asymptotic expansions in the acceleration potential theory the classical results will appear again in the first order (linear) approximation. The extension to a treatment of problems including nonlinear effects is then quite straightforward and can be applied whenever it is found inevitable. The method follows the ideas of Timman and Van Holten as presented in Van Holten [25].

7-1 The wing in uniform flow modelled in acceleration potential theory

In this section the problem is formulated regarding the determination of the loads on a rectangular wing perpendicular to an undisturbed steady flow with velocity \( W \).
The problem will be formulated as a pressure field description, which will turn out to be very suitable. The reason is that pressure singularities are only occurring at the wing itself, and thus the modelling of the wake is not necessary. Under the assumptions of small velocity perturbations it was shown in chapter 5 that the pressure field acts as a potential function for the accelerations in the field.

The Laplace equation for the pressure perturbation field $p$, or in other words the acceleration potential, is given by:

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 .$$  \hspace{1cm} (7-1)

This equation was derived under the small velocity perturbation assumption. The linearised Euler equation under those assumptions is given by:

$$(\mathbf{W} \cdot \nabla) \mathbf{V} = -\frac{1}{\rho} \nabla p .$$  \hspace{1cm} (7-2)

The first boundary condition for the flow around the wing is found by the requirement that at infinity the pressure perturbations must vanish, so:

$$p \rightarrow 0 \quad \text{for} \quad r \rightarrow \infty ,$$  \hspace{1cm} (7-3)

where $r = \sqrt{x^2 + y^2 + z^2}$.

From this point onwards the local $(x_b, y_b, z_b)$ will be used, because it facilitates the translation of results obtained in this chapter to situations treated later.

The relation between the basic $(x, y, z)$ system and the local $(x_b, y_b, z_b)$ system is simple for the isolated wing considered here:

$$x_b = z$$
$$y_b = -y$$
$$z_b = x .$$  \hspace{1cm} (7-4)

In the local $(x_b, y_b, z_b)$ coordinate system the rectangular wing surface is given by:

Here $c$ and $b$ denote the chord and the span of the considered wing. The function $\theta$ indicates the pitch angle of the blade and the function $y_c$ expresses the camber of the wing. The minus sign in front of the $\theta$ originates from the fact that the chordwise coordinate $x_b$ is positive in the direction of the flow and thus corresponds to a "nose-up" orientation. The undisturbed flow is equal to $\mathbf{W}$, and points into the $z$ direction, which is equal to the $x_b$ direction.
\[
y_b(x_b z_b) = -\theta(z_b) x_b + y_c(x_b z_b),
\]

\[
\text{where } |x_b| \leq \frac{c}{2}, \quad |z_b| \leq \frac{b}{2}
\]

\[
\text{and } y_c = 0 \text{ for } |x_b| = \frac{c}{2}.
\]

(7-5)

The kinematic boundary condition requires the particles of air moving over the wing to follow a path which is tangential to the wing surface. This gives rise to two more boundary conditions. It is assumed that it is justified to apply these boundary conditions on the projection of the wing surface on the \((x_b, z_b)\) plane. This is in accordance with linearised theory when the values of the surface function \(y_b(x_b z_b)\) are small with respect to the chord length \(c\).

Then the two kinematic boundary conditions can be formulated as follows:

- At first the accelerations of these particles should fit the curvature of the wing.
  
  Equation (7-5) shows that the particles of air travelling over the wing surface from leading edge to trailing edge (in \(x_b\) direction), will experience accelerations into the \(y_b\) direction. These accelerations are imposed by the pitch angle function \(\theta\) and the camber function \(y_c\).
  
  The total velocity in linearised theory points into the \(x_b\) direction. With use of the linearised Euler equation (7-2), this requirement leads to:
  
  \[
  \frac{1}{\rho W^2} \frac{\partial \rho}{\partial x_b} = -\frac{\partial y_c(x_b z_b)}{\partial x_b^2}.
  \]

  (7-6)

- The leading edge singularity at the wing should be such that just aft of it a tangential flow (with respect to the wing surface) is established. This means that the velocity of the particles of air after the leading edge are such that their paths follow the wing surface given by (7-5). However, with equation (7-6) it is already arranged such that the accelerations are in accordance with the wing curvature, it is therefore sufficient to require the velocities at the wing surface to be in accordance with the wing geometry at one station along the path. A convenient choice is then the mid chord line. So:

  \[
p \rightarrow \infty \text{ at } x_b = \frac{c}{2}, \quad y_b = 0
  \]

  (7-7)

  in such a way that \(\frac{v_b}{W} = 0(z_b) + [\frac{\partial y_c(x_b z_b)}{\partial x_b}]_{x_b=0}\).

For the boundary value problem (7-1), (7-3), (7-6) and (7-7) a first order and a second order approximate solution \((p^{(1)}\text{ and } p^{(2)})\) will be derived with the use of asymptotic expansion techniques:
Often the span of a lifting surface is large with respect to the chord (e.g. an aircraft wing or a wind turbine rotor blade). At some distance of such a surface the experienced velocity perturbations or accelerations will be approximately the same as those felt by a line on which the load is concentrated. The approximation in which the load on the actual surface is replaced by a load concentrated on a line is often referred to as a lifting line approach. When the load is represented by pressure differences concentrated on a line, such lifting line approximation will be equivalent to a pressure dipole line at the location of the wing, as was already concluded in section 5-5. The far field term will then consist of a lifting line expression.

As said above such approximation is only valid at some distance of the lifting surface, i.e. in the far field. Very close to the surface, in the near field, the experienced accelerations will be dominated by the chordwise load distribution. The velocity perturbations or accelerations in the near field, staying away from the tip regions, will be approximately the same as those experienced from a two dimensional lifting surface with the same local load distribution. So in the near field it may be expected that the velocity perturbations and accelerations can be approximated by those caused by two dimensional load distributions.

In the method that will be elaborated in the following sections both asymptotic approximations are combined with the presence of a common field term. This third term is used to eliminate the far field (lifting line) effects in the vicinity of the lifting surface, as well as the near field effects far away from the aerodynamic active surface. In such a way both terms are matched with this common field term yielding an expression which is valid throughout the whole field.

### 7-2 The approximate near field solution.

The first step in the asymptotic expansion technique, see e.g. Van Dyke [17], treats the various terms in the Laplace equation (7-1) in a different way. This is done under the assumption that, close to the wing and staying away from the tip regions, the accelerations in spanwise ($z_b$) directions will be an order of magnitude smaller than the accelerations in chordwise ($x_b$), and in $y_b$ direction. Of course this is valid only in the situation of slender wings, having an aspect ratio $A >> 1$ (typically $A \geq 5$). Rewritten into pressure variations of about equal magnitude equation (7-1) yields:

\[
\nabla^2 p = \frac{\partial^2 p}{\partial (\frac{x_b}{c})^2} + \frac{\partial^2 p}{\partial (\frac{y_b}{c})^2} + \frac{1}{A^2} \frac{\partial^2 p}{\partial (\frac{z_b}{b/2})^2} = 0
\]

(7-8)

where $A = \frac{b}{c}$, the aspect ratio of the wing.
Equation (7-8) can be rewritten into:

$$\frac{\partial^2 p}{\partial (\frac{x_b}{c})^2} + \frac{\partial^2 p}{\partial (\frac{y_b}{c})^2} = -\frac{1}{A^2} \frac{\partial^2 p}{\partial (\frac{z_b}{b})^2}.$$  \hspace{1cm} (7-9)

In the asymptotic approach it is assumed that in the solution of the boundary value problem close to the wing can be written as an asymptotic expansion like:

$$p = p_{\text{two-dim}} + \frac{1}{A} p_1 + \frac{1}{A^2} p_2 + \frac{1}{A^3} p_3 + \ldots$$ \hspace{1cm} (7-10)

$$\text{for } A \to \infty.$$

Equation (7-10) describes the way in which the pressure field becomes two dimensional for increasing value of the aspect ratio $A$. It must be stated here that a priori it is not known that the asymptotic expansion can indeed be expressed according to the power series used in (7-10). In fact it will turn out later that an expression with $A^2 \ln A$ has to be included. For the time being, not having obtained any knowledge yet concerning the asymptotic behaviour of the near field solution, it is sufficient to state here that such term is of course of the order $O(A^{-1})$, and therefore formally included in (7-10).

Substitution of the asymptotic series (7-10) into (7-9) yields:

$$\frac{\partial^2 p_{\text{two-dim}}}{\partial (\frac{x_b}{c})^2} + \frac{\partial^2 p_{\text{two-dim}}}{\partial (\frac{y_b}{c})^2} = 0 \text{ for } A \to \infty,$$ \hspace{1cm} (7-11)

$$\frac{\partial^2 p_1}{\partial (\frac{x_b}{c})^2} + \frac{\partial^2 p_1}{\partial (\frac{y_b}{c})^2} = 0 \text{ for } A \to \infty,$$ \hspace{1cm} (7-12)

$$\frac{\partial^2 p_2}{\partial (\frac{x_b}{c})^2} + \frac{\partial^2 p_2}{\partial (\frac{y_b}{c})^2} = -\frac{\partial^2 p_{\text{two-dim}}}{\partial (\frac{z_b}{b})^2}.$$ \hspace{1cm} (7-13)

So up to the order $O(A^{-2})$ it can be seen that in the near field de solution of the form (7-10) satisfies a two dimensional Laplace equation.

Suppose a solution $p^{(i)}$ up to the order $O(A^{-2})$ of the complete boundary value problem is found using asymptotic expansion techniques. Then, by definition, this solution $p^{(i)}$
satisfies the three dimensional Laplace equation up to the order \( O(A^{-2}) \). But close to the wing, this solution \( p^{(l)} \) also has to satisfy the two dimensional Laplace equation:

\[
\frac{\partial p^{(l)}_{x_b}}{\partial (x_b/cf^2)^2} + \frac{\partial p^{(l)}_{y_b}}{\partial (y_b/cf^2)^2} = 0,
\]

of course again up to the order \( O(A^{-2}) \).

In the near field such a solution \( p^{(l)} \) is then equivalent to a solution \( p_{\text{near}} \), satisfying equation (7-14) throughout the whole field.

In the following paragraphs the "near field" solution \( p_{\text{near}} \) will be determined. Apart from equation (7-14) this solution \( p_{\text{near}} \) must also satisfy the "near field boundary conditions" (7-6) and (7-7).

When equation (7-14) is written in terms of the \((\varphi, \eta)\) elliptical coordinate system defined in chapter 4 by the equations (4-3) it does maintain the structure of a two dimensional Laplace equation. General solutions can then be easily obtained by the method of separation of variables, see e.g. Courant et al [15].

Analogous to Van Holten [24] such general solution of the near field boundary value problem (7-14), (7-6) and (7-7) for the wing in terms of elliptical coordinates can be expressed as:

\[
\frac{p_{\text{near}}}{1/2 \rho W^2} = -\frac{C_l(z_b/cf^2)}{\pi} \frac{\sin \varphi}{\cos \eta + \cos \varphi} + \frac{1}{\pi} \sum_{n=1}^\infty a_n(z_b/cf^2) e^{-n \eta \sin \varphi} \]

\[
+ \frac{1}{\pi} \sum_{n=1}^\infty b_n(z_b/cf^2) \cosh n \eta \sin \varphi.
\]

The second and the third term in (7-15) result from the solution obtained by separation of variables. The reason for introducing expressions containing \( e^{-\eta} \) and \( \cosh \eta \) and not using \( e^\eta \) and/or \( \sinh \eta \) will become clear later. Of course they are implicitly present in the general expression (7-15).

The boundary condition (7-7) requires a singularity at the leading edge. This is taken care of by the first term in (7-15), which represent the load distribution over a flat plate aerofoil. It is the pressure field equivalent of the well known Birnbaum circulation distribution, see Schlichting et al [34]. This distribution is such that the Kutta condition requiring a smooth flow at the trailing edge is satisfied.

The kinematic boundary condition requiring tangential flow at the flat plate surface is fulfilled with a proper choice of the coefficient \( C_l \). This coefficient \( C_l \) then equals the lift coefficient, at least for a flat plate aerofoil. It should be emphasised that the second and
third term of equation (7-15) also contribute to the load on the wing and therefore they will in general also contribute to the lift.

The boundary conditions (7-6) and (7-7) should be satisfied on the wing surface. In linearised theory it is however allowed to transform the boundary conditions to the projection of the wing surface on the \( y_b = 0 \) plane. Taking this into account the substitution of (7-15) into (7-6) leads to:

\[
\sum_{n=1}^{\infty} n a_n \left( \frac{z_b}{c/2} \right) \frac{\sin(n \varphi)}{\sin \varphi} = \pi c \frac{\partial^2 y_c(x_b z_b)}{\partial x_b^2} \tag{7-16}
\]

on the wing.

In can thus be seen that the first and the third term of (7-15) yield an acceleration equal to 0 at the (projection of the) wing surface.

In the special case that the wing does not have camber, i.e. \( y_c(x_b z_b) = 0 \), this boundary condition results into \( a_n = 0 \), \( n = 1, 2, 3, \ldots \).

The situations considered in later chapters will be restricted to wings and rotor blades without camber, or with a simple parabolic camber function. For the latter cases this implies that the coefficients \( a_n \) will all be zero except for the first one, so \( a_1 \neq 0; a_n = 0 \) for \( n = 2, 3, 4, \ldots \).

The third boundary condition (7-19) is not necessarily fulfilled by \( p_{near} \), since this is a condition imposed upon \( p^{(i)} \) in the far field! For that reason it is not (yet) possible to rule out the \( b_n \) terms in the equation (7-15) !!

### 7-3 The approximate far field solution.

The investigation of the behaviour of \( p^{(i)} \) in the far field is the second step in the matched asymptotic expansion procedure. In the far field, staying away from the wing, its chord may considered to have shrunk into one single line carrying the load. This is legitimate since errors of the order \( A^2 = (c/b)^2 \) are neglected. On this line the solution must be antisymmetric with respect to the plane in which the wing is located.

The first order solution \( p^{(i)} \) is thus, in the far field, equivalent to the pressure field \( p_{far} \) that satisfies, apart from the three dimensional Laplace equation and the pressure perturbation decay condition (7-2) the boundary condition:

\[
p_{far} \text{ singular at } x_b = 0, y_b = 0 \text{ and antisymmetric with respect to the } x_b, z_b \text{ plane.} \tag{7-17}
\]
When the equation (7-1) is written in terms of prolate spheroidal coordinates according to (4-5), a general solution can be obtained using the method of separation of variables, see e.g. Courant et al [15]. Such a "natural" general solution will consist of Legendre functions since they are the solutions of the (ordinary) differential equations obtained after the process of separation of variables. The general solution satisfying the equations (7-1), (7-3) and (7-17) then yields:

\[
\frac{P_{\text{far}}}{\frac{1}{2} \rho W^2} = \sum_{m=1}^{\infty} \sum_{n=1}^{m} A_{mn} P_n^m(\cos \theta) \ Q_n^m(\cosh \nu) \sin(m \chi) .
\]  

(7-18)

The functions \( P_n^m \) and \( Q_n^m \) in equation (7-18) represent the associate Legendre functions of the first and second kind respectively, see appendix A.

At first sight it seems that the summation \( P_{\text{near}} + P_{\text{far}} \) is a competent candidate for the solution \( P^{(i)} \) valid in the complete field. When taking a closer look it becomes clear that this is not the case! One of the requirements is evidently that the contribution of \( P_{\text{near}} \) in the far field should be neglected (at least to the first order) with respect to \( P_{\text{far}} \) and vice versa.

However, according to equation (7-17), \( P_{\text{far}} \) exhibits a singular behaviour at the mid chord line, and can therefore certainly not be neglected with respect to \( P_{\text{near}} \).

So a matching procedure has to be applied in order to obtain a solution valid in the complete field. The construction of such a solution based upon the near and far field expressions obtained so far will take place in the next section of this chapter.

7-4 The common field expression and the first order solution.

In the previous section it was concluded that the solution \( P^{(i)} \), valid in the complete field, cannot be obtained with a simple addition of the two solutions \( P_{\text{near}} \) and \( P_{\text{far}} \).

In Van Dyke [17] an approach is followed in which the near and far field expressions are expanded, and a procedure is suggested in which such inner and outer expansions should be matched.

In terms of Van Dyke this leads to a modification of the "inner solution" \( P_{\text{near}} \) and of the "outer solution" \( P_{\text{far}} \), in such a way that addition of both does yield a complete solution valid in the complete field. That would be a more or less direct way to find a representation for the first order solution.

In stead of finding such a direct representation it is aimed here to determine a composite solution which uses a common field correction term. Such a common field term can be determined whenever there is a domain of overlap for both asymptotic expansions.

The procedure applied here follows the method developed by Lagerstrom et al [28].

Asymptotic expansions of the near and far field solutions are obtained on the basis of characteristic length scales. For the near field solution the characteristic length scale is the
chord $c$, whereas the characteristic length for the far field solution is the span $b$.

The asymptotic expansion for the near field will be obtained under the assumption that the nondimensional local distance to the wing $r_j/(c/2)$ is increasing. In a similar way the asymptotic expansion for the far field expression is obtained for the condition that the distance to the wing $r_j/(b/2)$ is decreasing.

The domain of overlap of both expansions is then the region where the distance to the wing $r_j$ is small with respect to the span, but large with respect to the chord.

According to a lemma presented by Lagerstrom [28] both expansions have to match on that domain of overlap up to the required order of accuracy. The order of accuracy is of course equal to the (least) order of accuracy of the near and far field expansions. Such a matched expression is then designated the common pressure field expression $p_{\text{common}}$.

A further (physical) interpretation of such a common pressure field will be given later.

It is evident that $p_{\text{common}}$ shows the same undesired unbounded behaviour of $p_{\text{near}}$ far away from the wing, as well as it inherits the singularities at the wing introduced by $p_{\text{far}}$. By the way the common field expression is obtained an even more formal statement can be made concerning the properties of $p_{\text{common}}$:
- in the near field it equals $p_{\text{near}}$, to the required order of accuracy
- in the far field it equals $p_{\text{far}}$, to the required order of accuracy

Thus the addition of $p_{\text{near}}$ and $p_{\text{far}}$, and subsequent subtraction of $p_{\text{common}}$ is a competent candidate for a first order solution $p^{(i)}$:

$$ p^{(i)} = p_{\text{near}} + p_{\text{far}} - p_{\text{common}} \quad (7-19) $$

Such a solution will still be equal to $p_{\text{near}}$ close to the wing, and equal to $p_{\text{far}}$ at large distances from the wing, up to and including the order $O(A^{-1})$.

The common pressure field expression is thus obtained by observing the asymptotic behaviour of both $p_{\text{near}}$ and $p_{\text{far}}$:
- The near field solution $p_{\text{near}}$ asymptotically expanded for increasing distances from the wing $r_j/(c/2)$ satisfies up to the order $O(A^{-2})$:

$$ \frac{p_{\text{near}}}{1/2 \rho W^2} = -\frac{C}{\pi} \frac{z_b}{bf} \frac{c/2}{r_b} \sin \chi + \frac{1}{\pi} a \left( \frac{z_b}{bf} \right) \frac{c/2}{r_b} \sin \chi $$

$$ + \frac{1}{\pi} \sum_{n=1}^{\infty} b_n \left( \frac{z_b}{bf} \right) \left( \frac{r_b}{c/2} \right)^n \sin n \chi \quad \text{for} \quad \frac{r_b}{c/2} \gg 1 \quad (7-20) $$

- The far field solution $p_{\text{far}}$ asymptotically expanded for decreasing distances from the wing $r_j/(b/2)$ satisfies up to the order $O(A^{-2})$:
\[
\frac{P_{\text{far}}}{\frac{1}{2} \rho W^2} = b f_2 \sin \chi \sqrt{1 - \left(\frac{z_b}{b f_2}\right)^2} \sum_{n=1}^{\infty} A_{1n} P_n^1\left(\frac{z_b}{b f_2}\right) + \\
+ \left(\frac{b f_2^2}{r_b}\right)^2 \sin 2 \chi \sqrt{1 - \left(\frac{z_b}{b f_2}\right)^2} \sum_{n=1}^{\infty} A_{2n} P_n^2\left(\frac{z_b}{b f_2}\right) + \ldots \text{ for } \frac{r_b}{b f_2} < 1.
\] (7-21)

The matching condition according to Lagerstrom et al [28] requires that in the domain of overlap both expansions coincide:

\[
P_{\text{near}} = P_{\text{common}} = P_{\text{far}} \quad \text{for } A^{-1} < \frac{r_b}{b f_2} < 1.
\] (7-22)

Equating terms of equal power in \( r_b \) shows that the terms with \( A_{nn} \) in (7-21) must vanish for \( m > 1 \). For the same reason the coefficients \( b_n \) in the expression (7-20) must be equal to 0.

Due to the matching condition it is at this stage possible to refine the general expression for the far pressure field presented in the previous section:

\[
\frac{P_{\text{far}}}{\frac{1}{2} \rho W^2} = \frac{1}{\pi} \sum_{n=1}^{\infty} A_n P_n^1(\cos \theta) \left(\frac{\cosh v}{\cosh v}\right) \sin \chi \quad \text{where } \frac{A_n}{\pi} = A_{1n}.
\] (7-23)

Two expressions for the common field are now obtained, according to the matching condition (7-21):

- From the asymptotic expression for the near field pressure distribution:

\[
\frac{P_{\text{common}}}{\frac{1}{2} \rho W^2} = \frac{C(s)}{\pi} \frac{b f_2^2}{r_b} \sin \chi + \frac{1}{\pi} \frac{a_1(s)}{b f_2} \frac{c f_2}{r_b} \sin \chi;
\] (7-24)

- From the asymptotic expression for the far field pressure distribution:

\[
\frac{P_{\text{common}}}{\frac{1}{2} \rho W^2} = -\frac{\sin \chi}{\pi} \frac{b f_2}{r_b} \sqrt{1 - \left(\frac{z_b}{b f_2}\right)^2} \sum_{n=1}^{\infty} A_n P_n^1\left(\frac{z_b}{b f_2}\right) \quad \text{where } A_n = A_{1n}.
\] (7-25)

When comparing (7-24) and (7-25) the following relation between \( C_n \), the camber coefficient \( a_i \), and the coefficients \( A_n \) is found:
\[ C(\frac{z_b}{b/2}) = A \sqrt{1 - \frac{z_b^2}{b^2}} \sum_{n=1}^{\infty} A_n P_n^1(\frac{z_b}{b/2}) + a_1(\frac{z_b}{b/2}). \] (7-26)

Taking a closer look to the expression for the common field shows that it has a two dimensional dipole behaviour (see e.g. equation (5-48)). Since the expression has a maximum for \( \chi = \pi/2 \) the dipole points into the direction of the y_b-axis. The dipole strength however is not constant but varies according to the variation in load over the span.

By substitution it can be seen that \( p_{\text{common}} \) is the solution of the boundary value problem for the two-dimensional differential equation (7-14) satisfying the boundary condition (7-3) at infinity, and the degenerated boundary condition (7-17) at the lifting line representing the wing.

Thus the common field term \( p_{\text{common}} \) has a physical meaning! It is the solution of a two-dimensional lifting line problem and therefore a "real" pressure field and not merely a mathematical matching expression!

The complete pressure field is now obtained according to equation (7-19):

\[ \frac{p^{(l)}}{2 \rho W^2} = -\frac{C(\frac{z_b}{b/2})}{\pi} \frac{\sin \varphi}{\cosh \eta + \cos \varphi} + \frac{1}{\pi} a_1(\frac{z_b}{c/2}) e^{-\eta \sin \varphi} + \]

\[ + \frac{C(\frac{z_b}{b/2})}{\pi} \frac{c_f}{r_b} \frac{\sin \chi}{\cos \theta} \frac{1}{r_b} \frac{1}{\pi} a_1(\frac{z_b}{b/2}) c_f P_n^1(\cos \theta) Q_n^1(\cosh \nu) \] (7-27)

and is a solution of the boundary value problem up to the order \( O(A^{-2}) \).

Comparing the above expression (7-27) with the asymptotic expansion close to the wing for increasing aspect ratio presented in (7-10), it can be seen that the first two expressions of (7-27) represent the term \( p_{\text{two-dim}} \) whereas the latter three terms represent the term \( A^{-1} p_1 \).

This becomes clear when it is realised that the third and fourth term in equation (7-27) together forming the common field expression, are close to the wing equal to the fifth term, which is the far field expression. Equal means here of course equal to the order \( O(A^{-1}) \).

Returning now to the boundary value problem presented in section 7-1 it must be concluded that there is still a boundary condition left to fulfil: the tangential flow condition of equation (7-7).

By the choice of the near field expression the singularity at the leading edge is already
present. The coefficients $A_n$, and with them the spanwise lift coefficient distribution $C_t$ should now be determined such that the kinematic boundary condition for the velocity at the mid chord line is satisfied.

With the linearised Euler equation (7-2) this tangential flow condition results in the following integro-differential formulation:

$$
-\frac{1}{1/2 \rho W^2} \int_{-\infty}^{0} \frac{\partial p^{(1)}(x_p y_p z_p t)}{\partial y/R} d(x/R) = 2\theta(z_p) - 2 \frac{\partial y_c(x_p z_p)}{\partial x_p} \tag{7-28}
$$

for a particle of air arriving at the midchordline at $t=0$.

The evaluation of this integral can be carried out numerically. The principle of the procedure is such that first an initial choice is made concerning the magnitude of the coefficients $A_n$. Then the pressure field is completely determined and its gradient can be integrated according to the left hand side of (7-28). The result of the integration of the accelerations experienced along the path of the particle of air is a velocity which should be such that the tangential flow condition is fulfilled. From the difference between the (nondimensional) velocity in $y_p$ direction obtained from the integration and the tangential flow requirement represented by the right hand side of (7-28) a set of coefficients $A_n$ is estimated. For the situation of the wind turbine rotor blade considered in chapter 9 and 10 further details of the numerical procedure will be presented.

According to the linearised Euler equation the integration of equation (7-28) takes place along straight paths which are travelled with a constant speed $W$. Thus:

$$
x_p(t) = z(t) = Wt, \quad y_p(t) = y(t) = 0, \quad z_p(t) = x(t) = z_{p0}. \tag{7-29}
$$

7-5 Higher order solutions.

If terms of order $A^2$ are included it is no longer possible to neglect the right hand side of equation (7-8). The derivation of the higher order solution will be restricted to the situation of an uncambered wing.

It will turn out that the asymptotic series for the near field expression assumed in (7-10) has to be extended with an extra term. The higher order solution derived in this section takes into account terms of this intermediate order $A^2 \ln A$.

The two dimensional part of the solution $p^{(1)}$ obtained in equation (7-15) is in that case equal to:
\[ P_{two-dim} = \frac{1}{\frac{1}{2} \rho W^2} \left( \frac{C_k(z_b/b)}{b/2} \right) \frac{\sin \varphi}{\pi \sinh \eta + \cos \varphi} . \] (7-30)

So the partial differential equation for the near field term accurate up to and including the order \( O(A^2) \) yields:

\[ \frac{\partial^2 p}{\partial (\frac{x_b}{c/2})^2} + \frac{\partial^2 p}{\partial (\frac{y_b}{c/2})^2} = -\frac{\partial^2 P_{two-dim}}{\partial (\frac{z_b}{b/2})^2} \]

\[ = \frac{\partial^2}{\partial (\frac{z_b}{b/2})^2} \left[ \frac{C_k(z_b/b)}{b/2} \frac{\sin \varphi}{\pi \sinh \eta + \cos \varphi} \right] \] (7-31)

together with the near field boundary conditions (7-6) and (7-7).

A particular solution of (7-31) is, according to Van Holten [24] given by:

\[ \frac{P_{near}}{\frac{1}{2} \rho W^2} = \frac{1}{\pi A^2} \frac{d^2 C_k(z_b/b)}{b/2} \left( \frac{1}{2} \eta \sinh \eta \sin \varphi + \frac{1}{8} \sin 2\varphi \right) \] (7-32)

which can be checked by substitution.

The general near field solution for an uncambered wing is then given by:

\[ \frac{P_{near}}{\frac{1}{2} \rho W^2} = -\frac{C_k(z_b/b)}{b/2} \frac{\sin \varphi}{\pi \cosh \eta + \cos \varphi} + \frac{1}{\pi} \sum_{n=1}^{\infty} b_n(z_b/b) \cosh \eta \sin n \varphi \]

\[ + \frac{1}{\pi A^2} \frac{d^2 C_k(z_b/b)}{b/2} \left( \frac{1}{2} \eta \sinh \eta \sin \varphi + \frac{1}{8} \sin 2\varphi \right) . \] (7-33)

The partial derivative of the particular solution (7-32) with respect to the \( y_b \) direction is equal to 0 on the projection of the wing on the \( y_b \) plane, so the near field solution (7-33) still satisfies boundary condition (7-16) for the uncambered wing.
The near field solution $p_{\text{near}}$ asymptotically expanded for increasing distances from the wing $r_b/(c/2)$ including terms of to the order $O(A^{-2})$ results into:

$$\frac{p_{\text{near}}}{\frac{1}{2} \rho W^2} = -\frac{1}{\pi} C_1 \frac{z_b}{b/2} \frac{c/2}{r_b} \sin \chi + \frac{1}{2\pi} C_1 \frac{z_b}{b/2} \frac{c/2}{r_b} \sin 2\chi + \frac{1}{\pi} b_1 \frac{z_b}{b/2} \frac{r_b}{c/2} \sin \chi + \frac{1}{\pi} b_2 \frac{z_b}{b/2} \frac{r_b}{c/2} \sin 2\chi + \ldots +$$

$$+ \frac{1}{2\pi A^2} \frac{d^2C_1(z_b/b/2)}{d^2(z_b/b/2)} \frac{c/2}{r_b} \ln \left( \frac{r_b}{c/2} \right) \sin \chi + \frac{1}{8\pi A^2} \frac{d^2C_1(z_b/b/2)}{d^2(z_b/b/2)} \frac{c/2}{r_b} \sin 2\chi$$

(7-34)

for $\frac{r_b}{c/2} > 1$.

Rewriting expression (7-34) from the stretched nondimensional coordinate $r_b/(c/2)$ to the ordinary nondimensional coordinate $r_b/(b/2)$ shows that, apart from the power series terms with the $b_n$ coefficients, a term of magnitude $A^{-1}\ln A$ has entered the asymptotic expansion. The expansion of the far field expression will also show terms of this magnitude. They result from the asymptotic behaviour of the associate Legendre functions of the second kind $Q_n^m$ used in the far field expressions.

The higher order far field expressions derived by Van Holten [24] are written in terms of Bessel functions. Finally he also arrives with far field expansions including such logarithmic terms, but the present representation using Legendre functions turns out to be much more straightforward.

Matching with the far field expansion should of course be applied before something can be said about its presence in the composite approximate solution. At this point it however becomes clear that it probably necessary to modify the a priori assumption concerning asymptotic behaviour presented in (7-10) such that terms with $\ln A$ are included.

In order to determine the terms that will remain into the expression (7-34) it must now be matched to the far field expansion.

The far pressure field should now be written as:

$$\frac{p_{\text{far}}}{\frac{1}{2} \rho W^2} = \frac{1}{\pi} \sum_{m-1}^{2} \sum_{n-1}^{\infty} A_{mn} P_n^m(\cos \theta) Q_n^m(\cosh \nu) \sin (m\chi)$$

(7-35)

and asymptotic expansion of the far field solution $p_{\text{far}}$ for decreasing distances from the wing $r_b/(b/2)$ up to the order $O(A^{-3})$ yields:
\[
\frac{P_{\text{far}}}{\frac{1}{2} \rho W^2} = -\frac{\sin \chi}{\pi} \left( \frac{b/2}{r_b} \right) \sqrt{\frac{1}{2} \sum_{n=1}^{\infty} A_{1n} P_n^1 \left( \frac{z_b}{b/2} \right)} + \\
+ \frac{2 \sin 2 \chi}{\pi} \left( \frac{b/2}{r_b} \right) \left( 1 - \left( \frac{z_b}{b/2} \right)^2 \right) \sum_{n=1}^{\infty} A_{2n} P_n^2 \left( \frac{z_b}{b/2} \right) + \\
- \frac{\sin \chi}{2\pi} \left( \frac{r_b}{b/2} \right) \ln \left( \frac{r_b}{b/2} \right) \left( 1 - \left( \frac{z_b}{b/2} \right)^2 \right) \sum_{n=1}^{\infty} A_{1n} P_n^1 \left( \frac{z_b}{b/2} \right) n(n+1) + \\
\left( \frac{\sin 2 \chi}{\pi} \sum_{n=1}^{\infty} A_{2n} P_n^2 \left( \frac{z_b}{b/2} \right) \left( 1 - \left( \frac{z_b}{b/2} \right)^2 \right) - n(n+1) \right) + \\
\left( \frac{\sin \chi}{2\pi} \left( \frac{r_b}{b/2} \right) \left( 1 - \left( \frac{z_b}{b/2} \right)^2 \right) \sum_{n=1}^{\infty} A_{1n} P_n^1 \left( \frac{z_b}{b/2} \right) n(n+1) \ln 2 - \sum_{m=1}^{n} \frac{m(m-1)}{(n+1-m)} \right) \\
\text{for } r_b < \frac{b/2}{b/2} < 1.
\]

Application of the matching condition implies the comparison of the terms of equal order in the equations (7-34) and (7-36) on the domain of overlap. Up to the order to which the far field is expanded now it can be seen that the near field terms with the coefficients \( b_n \) for \( n > 1 \) do not have their equivalent in the far field expansion (7-36) So they must be equal to 0: \( b_n = 0; n = 2, 3, 4, \ldots \)

Further comparison leads to the following set of equations that have to be fulfilled:

\[
C_{111} \left( \frac{z_b}{b/2} \right) = A \sqrt{\frac{1}{2} \sum_{n=1}^{\infty} A_{1n} P_n^1 \left( \frac{z_b}{b/2} \right)}, \tag{7-37}
\]

\[
C_{222} \left( \frac{z_b}{b/2} \right) = 4A^2 \left( \frac{1 - \left( \frac{z_b}{b/2} \right)^2}{b/2} \right) \sum_{n=1}^{\infty} A_{2n} P_n^2 \left( \frac{z_b}{b/2} \right), \tag{7-38}
\]
\[
\frac{d^2C_i(xz)}{dz^2} = -A \frac{1}{\sqrt{1-\left(\frac{z}{b}\right)^2}} \sum_{n=1}^{\infty} A_{1n} P_n^1\left(\frac{z}{b}\right) n(n+1) , \quad (7-39)
\]

\[
\frac{d^2C_i(xz)}{dz^2} = 8A^2 \sum_{n=1}^{\infty} A_{2n} P_n^2\left(\frac{z}{b}\right) \left(1-\left(\frac{z}{b}\right)^2\right) - n(n+1) . \quad (7-40)
\]

By substitution of the expressions (7-37) and (7-38) for the function \( C_i \) into the equations (7-39) and (7-40), using the relations for \( P_n^1 \) and \( P_n^2 \) derived in appendix A, it can be verified that the above relations indeed hold.

From the equations (7-37) and (7-38) a relation is found between the coefficients \( A_{1n} \) and \( A_{2n} \).

Finally the coefficient \( b_1 \) should satisfy:

\[
b_1\left(\frac{z}{b}\right) = \frac{1}{2A} \sum_{n=1}^{\infty} A_{1n} P_n^1\left(\frac{z}{b}\right) n(n+1) \ln 2 - \sum_{m=1}^{n} \frac{m(m-1)}{n+1-m} \]

\[
+ \frac{\ln A}{2A} \sum_{n=1}^{\infty} A_{1n} P_n^1\left(\frac{z}{b}\right) n(n+1) . \quad (7-41)
\]

In this expression it can be seen that the term with \( A^3 \ln A \) is remanent.

Since the magnitude of the coefficients \( A_{1n} \) is of the order \( O(A^{-3}) \) the term with the coefficient \( b_1 \) in the near field expression incorporates terms of both the order \( A^{-3} \ln A \) and \( A^{-2} \).

With the conclusion that terms \( b_n \) for which \( n > 1 \) must be ruled out of the near field expansion the common field takes the form:
\[
\frac{P_{\text{common}}}{\frac{1}{2} \rho W^2} = -\frac{1}{\pi} C_l\left(\frac{z_b}{b/2}\right) \frac{c/2}{r_b} \sin \chi + \frac{1}{2\pi} C_l\left(\frac{z_b}{b/2}\right) \left(\frac{c/2}{r_b}\right)^2 \sin 2\chi + \\
\frac{1}{\pi} b_1\left(\frac{z_b}{b/2}\right) \frac{r_b}{c/2} \sin \chi + \\
\frac{1}{2\pi A^2} \frac{d^2 C_l\left(\frac{z_b}{b/2}\right)}{d\left(\frac{z_b}{b/2}\right)^2} \frac{r_b}{c/2} \ln \left(\frac{r_b}{c/2}\right) \sin \chi + \frac{1}{8\pi A^2} \frac{d^2 C_l\left(\frac{z_b}{b/2}\right)}{d\left(\frac{z_b}{b/2}\right)^2} \sin 2\chi.
\] (7-42)

This common field is now characterised as the combination of a two dimensional (pressure) dipole field together with a two dimensional quadrupole field (the first two terms in equation (7-42)). This can be seen when the definition of the circular cylinder coordinate system is substituted. The \(\sin \chi\) term introduces an \(y_b\) in the numerator and a \(r_b\) in the denominator. Similarly the \(\sin 2\chi\) term can be replaced by \(2x_b y_b / (r_b^3)\). Comparison with the equations (5-49) and (5-50) then shows the presence of both a dipole and a quadrupole field in the expression (7-42). The other terms in the equation have a regular behaviour everywhere in the field and correspond to regular (three dimensional) distortions of the flow field.

And so the composite field is given by:

\[
\frac{P^{(2)}}{\frac{1}{2} \rho W^2} = -\frac{1}{\pi} C_l\left(\frac{z_b}{b/2}\right) \frac{\sin \varphi}{\sinh \eta + \cos \varphi} + \frac{1}{\pi} C_l\left(\frac{z_b}{b/2}\right) \frac{c/2}{r_b} \sin \chi + \\
\frac{1}{\pi} \sin \chi \sum_{n=1}^{\infty} A_n P_n^1(\cos \theta) Q_n^1(\cosh \nu) + \\
\frac{1}{\pi} b_1\left(\frac{z_b}{b/2}\right) \left(\cosh \varphi \sin \varphi - \frac{r_b}{c/2} \sin \chi\right) - \frac{1}{2\pi} C_l\left(\frac{z_b}{b/2}\right) \left(\frac{c/2}{r_b}\right)^2 \sin 2\chi + \\
\frac{1}{2\pi A^2} \frac{d^2 C_l^{(1)}\left(\frac{z_b}{b/2}\right)}{d\left(\frac{z_b}{b/2}\right)^2} \left(\eta \sinh \eta \sin \varphi + \frac{1}{4} \sin 2\varphi - \frac{r_b}{c/2} \frac{\ln \left(\frac{r_b}{c/2}\right)}{c/2} \sin \chi - \frac{1}{4} \sin 2\chi\right) + \\
\frac{1}{\pi} \sum_{n=1}^{\infty} A_{2n} P_n^2(\cos \theta) Q_n^2(\cosh \nu) \sin 2\chi.
\] (7-43)

The terms in equation (7-43) have been organised such that the first three terms represent the first order solution, equivalent to equation (7-27). The second and the third term taken together represent the additional terms of the order \(O(A^4)\), as was already concluded in the previous section 7-4. The fourth term inhibits an additional term of the order \(O(A^2 \ln A)\).
as was concluded in (7-41) and the rest of the terms represent the higher order approximation in the expansion up to and including terms of the order \( O(\lambda^2) \).

Thus in a higher order expression the far and common field expression found in the first order approximation are corrected to fit the higher order requirements.

As already noticed from the common field expression, the fourth term in expression (7-43) represents a two dimensional quadrupole with varying strength. The last term is the corresponding three dimensional quadrupole distribution.

The fourth term of equation (7-43) shows that the diverging term with \( r_b \) is counteracted by a diverging term originating from the higher order near field expression.

When the definition of the circular cylinder coordinate system (4-4) is substituted into this term with \( b_1 \) it follows:

\[
b_1 \left( \frac{z_b}{b/2} \right) \left( \cosh \eta \sin \varphi - \frac{r_b}{c/2} \sin \chi \right) = b_1 \left( \frac{z_b}{b/2} \right) e^{-\eta} \sin \varphi .
\]

(7-44)

The fourth term of equation (7-43) represents thus a pressure field term vanishing at infinity with exactly the same properties as the pressure term for simple parabolic camber of the aerofoil. (the \( a_1 \) term in section 7-4)! Physically this is equivalent to a curvature in the outer flow, which is counteracted by an extra pressure distribution on the blade in order to satisfy the boundary conditions.

In a similar way it can be seen that the other terms of the common field expression corresponding to regular distortions of the flow field are corrected in such a way that the boundary condition at infinity is satisfied.

The lift on the wing can be determined by taking the difference of the pressure between the upper and the lower wing surface. With the near field coordinates this means a difference between the pressure field for a positive and a negative value of \( \varphi \) (for \( \eta = 0 \)).

It can than be seen that the \( b_1 \) term is contributing to the lift on the wing.

Since the occurrence of the term \( b_1 \) in (7-43) represents the curvature of the "outer flow", it shows that in the higher order approximation it is no longer valid to use two dimensional aerofoil characteristics in a three dimensional situation. Thus the hypothesis used in Prandtl's lifting line theory that the induced velocity can be assumed constant over the wing, is restricted to a first order solution only!

It is thus shown that even in the situation of an uncambered aerofoil a virtual camber term enters the solution in the second order matched asymptotic approximation.

The presence of such three dimensional effects reflected in the addition of near field terms was pointed out already by Van Holten [24]. With the present representation using Legendre functions these terms can be quantified in a convenient way by their explicit representation in the Legendre function coefficients \( A_{1n} \) as is shown above.
8 Loads and interference effects at a rotating wing

Introduction

When a single wing is rotating around a fixed axis in space, several rotational- and interference effects influencing the load on the wing can be observed. In the present chapter the solutions will be derived for rectangular untwisted wings in two different interference situations. As a preliminary however for partial span pitch controlled wind turbine rotor blades the situation of a rectangular wing with a discontinuous step in pitch angle will be treated.

At first the problem for the situation of a wing rotating around an axis parallel to its span will be treated. After that the situation of a wing rotating around an axis perpendicular to the span and the chord will be elucidated.

The first situation was deduced from an innovative wind turbine configuration studied in the past by a group of people including the author at Delft University of Technology, the tipvane turbine, see e.g. Van Bussel et al [10]. For the present purpose it is used to compare acceleration potential results with the results from more classical theory presented in chapter 6 for the infinite formation flight of wings. Some of these comparisons, within the framework of a representation of the lift distributions with Legendre functions, will be presented.

The second situation discussed in the present chapter is, of course, the rotating wing as a rudimentary single bladed wind turbine geometry. In that case the wing is rotating around an axis perpendicular to the span and the chord. The rotational axis is assumed to be located beside one of the wing tips, and at a large distance from the wing. Apart from the situation of a rectangular untwisted wing, the situation of a wing with a discontinuous step in twist angle will be treated. As said above this is a preliminary for the determination of loads on wind turbine blades with partial span pitch control. It will be seen that the approach to this problem has a large similarity with the earlier discussed treatment of the interference of a wing rotating around an axis parallel to its span.

8-1 The two different rotating situations considered

The starting point is the situation of one rectangular wing rotating with a uniform angular speed $\Omega$ in a uniform steady flow. The rotational axis coincides with the z-axis and points into the direction of the undisturbed flow $\mathbf{w}$. In its most common representation the wing is modelled by bound vorticity. In its simplest representation of a bound vortex line it is often referred to as the lifting line approximation of the wing. Since the bound vorticity
has to vanish at the wing tips, a sheet of trailing vorticity is shed into the flow. The position of the surface on which this trailing vorticity is located is, in linearized theory, determined by the undisturbed flow velocity \( \mathbf{W} \) and the rotational speed \( \Omega \). Since the position of the wing is no longer steady in the basic (inertial) reference frame \((x, y, z)\) the vortex sheet emanating from the wing will have a helical shape.

In the first case considered here, where the wing rotates around an axis parallel to the span, this helical surface will deform to a "flat" spiral located at a cylinder with a radius \( R \) equal to the distance between the \( z \)-axis and the wing, figure 15.

When the pitch of the flat spiral decreases it can be seen from figure 15 that the influence of the trailing vorticity on the downstream tip is increasing. In terms of the acceleration potential theory as presented in the previous chapter this is equivalent to the statement that particles of air travelling to the downstream tip of the wing more severely experience the presence of the pressure jump at the (upstream tip of the) wing one revolution before arrival at the wing. For very small pitch angles of the flat spiral a situation will be present which has a large similarity with the interfering wing problem in the infinite formation flight described in chapter 6.

![Figure 15: Wing rotating at cylindrical surface with trailing vorticity sheet](image)

In the second case the vorticity will be located on a screw type surface with inner and outer radius equal to the corresponding distances of the wing tips to the \( z \)-axis, figure 16. The assumption is made that the wing is still rectangular and untwisted. Furthermore it is assumed that the distance of the inboard tip to the rotational direction is large. Since beside of that only one wing rotating this situation can be described as a rudimental wind
turbine geometry.
For a small pitch of the screw type helical wake depicted in figure 16 again some interference can be expected. The helical surface will come close to the wing and therefore introduce extra induced velocities. These induced velocities add up to the regular induced velocities behind a wing in uniform flow and in total they can be considered as the reaction of the flow on the wing load. In contrast to a non rotating wing the force applied by the wing on the flow is in this case directed against the direction of the oncoming wind.
The acceleration potential equivalent is that particles of air travelling towards the rotating wing will experience the passage of the wing during previous revolutions prior to arrival. The pressure distribution on the wing will introduce extra accelerations which have the global tendency to slow down the speed of the particle on its way to the wing.

Even in these two relatively simple situations with a helical surface having a shape "prescribed" by the undisturbed flow velocity \( \mathbf{W} \) and the rotational speed \( \Omega \) it is not so easy to determine the load distribution on the wing. Goldstein [19] describes an analytical method for the second case. In general however an analytical potential function can not be found and thus a numerical approach will be needed.

In such situations a pressure field model is very suitable. The reason is that pressure singularities are only occurring at the wing itself, and thus the modelling of the wake is not necessary. Therefore the Laplace equation for the pressure perturbation, together with suitable boundary conditions, will form the basis for the solutions of the two problems treated in this chapter.

![Figure 16: Wing rotating in a plane with trailing vorticity sheet](image)
8-2 The wing rotating at a cylindrical surface of a large radius (a tipvane geometry).

In this section the problem is discussed of the determination of the loads on a rectangular wing rotating at a large radius $R$ (large with respect to the span of the blade), with a large rotational frequency $\Omega$ (such that $\Omega R$ is large with respect to the undisturbed flow velocity $W$. The rotational axis points into the direction of $W$ and the span of the blade is approximately parallel to $W$, in such a way that it is perpendicular to the relative undisturbed velocity $W + \Omega \times R$, see figure 15.

The problem has a large similarity with the problem treated in chapter 6. This becomes clear when the cylindrical surface at which the wing rotates is given a circumferential coordinate $\Psi_R$.

Figure 17: Rotating wing depicted in the circumferential $z_b$-$\Psi_R$ plane showing the over synchronous interference situation.
When the situation is depicted in the \( z_r - \psi_r \) plane it can be seen that the particle of air, prior to arriving at the wing, passes sideways along the wing in every previous revolution, while experiencing the presence of the pressure distribution on that wing, figure 17. Therefore there is an interference effect present which will modify the load distribution over the wing rotation on a cylindrical surface with respect to a single (non rotating) wing in uniform flow. It will be shown that the solutions found in this way exhibit a large analogy with the solutions found in the sections 6-3 and 6-4.

As in the previous chapter an approximate first order solution \( p \), will be derived for this boundary value problem with the use of asymptotic expansion techniques.

When the assumption is made that the chord of the considered wing is small with respect to the radius at which it is rotating then the boundary value problem is, to a first order, a rather straightforward extension of the wing problem discussed in chapter 6 as will be shown.

The relation between the basic \((x, y, z)\) system and the local \((x_b, y_b, z_b)\) system for the rotating wing considered here is given by:

\[
\begin{align*}
x &= R \cos \psi + x_b \sin \psi - y_b \cos \psi \\
y &= R \sin \psi - x_b \cos \psi - y_b \sin \psi \\
z &= -z_b.
\end{align*}
\] (8-1)

In this case the undisturbed flow is, in linearized theory, equal to \( \Omega \times R \), and points into the \( x_b \) direction.

The kinematic boundary condition requires the particles of air moving over the wing to follow a path which is tangential to the wing surface. In the present situation the wing moves with respect to the inertial \((x, y, z)\) coordinate system. So the velocities and accelerations of such particles must be derived in the inertial coordinate system as well. This has been carried out in appendix B. After the derivation of the general expressions, velocities and accelerations a position of the wing \( \psi = \pi/2 \) was chosen.

The kinematic boundary condition is now fulfilled under the following requirements:
- The accelerations of the particles moving over the wing should fit its curvature.
- The leading edge singularity at the wing should be such that just aft of it a tangential flow (with respect to the wing surface) is established.

At first the accelerations of the particles moving at the wing surface are derived. These accelerations should be such that the particles move exactly along the wings curvature. When it is assumed that this curvature is such that \( x/R, y/R \) and the first and second partial derivatives of the surface function \( y_b(x_b, z_b) \) are small with respect to 1, then the following linearised expression for the acceleration of such particles in the inertial \((x, y, z)\) reference system can be derived from equation (B-19):
\[
\frac{dv}{dt} = \left[ (1 - \frac{y_b}{R}) - \frac{\partial y_b}{\partial x_b} R \right] \Omega^2 R + \\
2(1 - \frac{\partial y_b}{\partial x_b} R) \Omega u + 2 \frac{\partial y_b}{\partial x_b} R \Omega w + \\
- \frac{\partial^2 y_b}{\partial x_b^2} - \frac{\partial^2 y_b}{\partial x_b \partial z_b} uw - \frac{\partial^2 y_b}{\partial z_b^2} w^2 + \\
+ \left( \frac{\partial y_b}{\partial x_b} - \frac{\partial y_b}{\partial z_b} \right) \Omega R - \frac{\partial y_b}{\partial x_b} \frac{du}{dt} + \frac{\partial y_b}{\partial z_b} \frac{dw}{dt}.
\]  

(8-2)

When it is furthermore assumed that the rotational speed is constant; that the \( u \) and \( w \) component of the wind can be written as \( u = u' \); \( w = W + w' \), where \( u' \) and \( w' \) are small with respect to the undisturbed wind speed \( W \); that the rotational speed \( \Omega R \) is large with respect to \( W \) and that the accelerations \( \frac{du}{dt} \) and \( \frac{dw}{dt} \) are small with respect to \( \Omega^2 R \) then equation (8-2) can be further linearised. But even then a term with the unknown acceleration \( \Omega u' \) remains in the expression. This means that (even small) velocities in the chordwise direction of the wing cause significant accelerations perpendicular to it. This effect is well known as a Coriolis acceleration. Only in the situation that the curvature of the wing in chordwise direction satisfies

\[
\frac{\partial^2 y_b}{\partial x_b^2} = 1
\]  

(8-3)

the term with \( \Omega u' \) vanishes.

For the surface function \( y_b(x_b, z_b) \) this implies that it must satisfy:

\[
y_b(x_b, z_b) = -\theta(z_b)x_b - \frac{c^2}{8R} + \frac{1}{2R} x_b^2 + y_{c^*}(x_b, z_b)
\]  

(8-4)

where the first and second partial derivatives of the second camber function \( y_{c^*}(x_b, z_b) \) are small with respect to 1. Equation (8-3) implies that the second partial derivative of \( y_b(x_b, z_b) \) with respect to \( x_b \) is no longer small with respect to 1 !!

The physical interpretation of equation (8-3) is that the Coriolis acceleration is small only if the curvature of the wing closely corresponds to the curvature of the circular track at which it rotates. This can be seen by an asymptotic expansion of the circular curvature:

\[
\sqrt{R^2 - x_b^2} - \sqrt{R^2 - (c/2)^2} = \Omega R \left( \frac{1}{2} \left( \frac{x_b}{R} \right)^2 - \frac{1}{8} \left( \frac{c}{R} \right)^2 + O((\frac{c}{R})^4) \right).
\]  

(8-5)

So the expression (8-4) contains the best parabolic approximation of the actual circular track on which the wing is rotating. Perfect adaptation is of course obtained when the camber part of the surface function \( y_{c^*}(x_b, z_b) \) satisfies:
\[ y_C(x_b z_b) = \sqrt{R^2 - (c/2)^2} - \sqrt{R^2 - x_b^2} + y_C(x_b z_b) \]  

but in the asymptotic approximation considered here such camber function is equivalent to the one given in equation (8-4).

With the above mentioned assumptions concerning the order of magnitude of the second order derivatives applied to the surface function \( y_C(x_b, z_b) \) equation (8-2) can be further linearised and, with substitution of (8-4) be rewritten to:

\[ \frac{dv}{dt} = -\frac{\partial^2 y_C}{\partial x_b^2} (\Omega R)^2 + 2 \frac{\partial^2 y_C}{\partial x_b \partial z_b} \Omega RW - \frac{\partial^2 y_C}{\partial z_b^2} W^2. \]  

Equation (8-7) shows that the particles of air following the blade surface will experience accelerations into the \( y \) direction (which is equal to the \( -y_b \) direction for the specific position chosen) dictated by the camber function \( y_C \). This camber function represents the deviation of the total camber of the wing from the circular curvature of the track at which it rotates.

The \( y \)-component of the linearised Euler equation yields:

\[ \frac{\partial v}{\partial t} + W \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y}. \]  

Substitution of (8-7) into (8-8) leads to:

\[ \frac{1}{\rho(\Omega R)^2} \frac{\partial p}{\partial y_b} = -\frac{\partial^2 y_C}{\partial x_b^2} + 2 \frac{\partial^2 y_C}{\partial x_b \partial z_b} \Omega R - \frac{\partial^2 y_C}{\partial z_b^2} \frac{W^2}{(\Omega R)^2}. \]  

With this equation (8-9) it is assured that the accelerations of the particles at the rotating wing fit its curvature.

When furthermore a flow tangential to the wing surface is established just after the leading edge singularity the kinematic boundary conditions are completely fulfilled. A tangential flow at the wing means that the velocity of the particles of air after the leading edge are such that their paths follow the wing surface given by (8-4). However, with equation (8-9) it is already arranged that the accelerations are in accordance with the wing curvature. It is therefore sufficient to require the velocities at the wing surface to be in accordance with the wing geometry at one station along the path. A convenient choice is then the mid chord line. So:
\[ p \downarrow -\infty \text{ at } x_b = -\frac{c}{2}, \; y_b \rightarrow 0 \text{ in such a way that} \]
\[ \frac{v_b}{\Omega R} = \Theta(z_b) - \frac{\partial y_c}{\partial x_b} \]
\[ \text{at the midchord line.} \tag{8-10} \]

The solution of this boundary value problem, formulated in the equations (7-1) through (7-3), (8-9) and (8-10) can then directly be derived analogous to chapter 7.

With the restriction to a first order solution for a wing with a camber equal to its circular track (i.e. \( y_c(x, y, z_b) = 0 \)) the complete pressure field can be written down completely equivalent to equation (7-27):

\[
\frac{1}{2} \rho (\Omega R)^2 \frac{P^{(1)}}{P^{(0)}} = -\frac{C_L(z_b)}{b/2} \frac{\sin \varphi}{\pi} \frac{\cos \eta + \cos \varphi}{r_b} + \frac{C_L(z_b)}{b/2} \frac{c/2 \sin \chi}{\pi} \frac{\sum_{n=1}^{\infty} A_n P_n^{(1)}(\cos \vartheta) Q_n^1(\cosh \eta)}{\pi} \tag{8-11}
\]

with the following relation between \( C_l \) and the coefficients \( A_n \):

\[
C_L(z_b) = A \sqrt{1 - \left(\frac{z_b}{b/2}\right)^2 \sum_{n=1}^{\infty} A_n P_n^{(1)}(z_b)} \tag{8-12}
\]

By the choice of the near field expression the singularity at the leading edge is already present. The coefficients \( A_n \) of the lift coefficient \( C_l \) distribution still have to be determined such that the kinematic boundary condition for the velocity at the mid chord line (8-10) is fulfilled.

With the \( y \)-component of the linearised Euler equation, equation (8-8), it follows for the final boundary condition:

\[
-\frac{1}{2\rho (\Omega R)^2} \int_{-\infty}^{0} \frac{\partial p^{(1)}(x,y,z,t)}{\partial y} \Omega dt = 2\Theta(z_b) - 2 \frac{\partial y_c}{\partial x_b} \tag{8-13}
\]

for a particle of air arriving at the midchord line at \( t=0 \).

The evaluation of this integral will be carried out numerically. Note that the partial derivative of the pressure field \( p^{(1)} \) must be taken in the \( y \) direction, which is not equivalent to the \( y_b \) direction for \( t \neq 0 \)!
The integration of the pressure gradient takes place along the linearised path of the considered particle of air. Assume that the particle of air arrives at \( t = 0 \) at the location \((0, R, z_0)\) on the wing then this linearised path is given by:

\[
x(t) = 0, \quad y(t) = R, \quad z(t) = Wt + z_0
\]

which is in accordance with the linearised Euler equation (8-12).

Written in local \((x_b, y_b, z_b)\) coordinates the path is given by:

\[
x_b(t) = R \sin \Omega t, \quad y_b(t) = R (\cos \Omega t - 1), \quad z_b(t) = -W t - z_0.
\]

It has thus been shown that even in the first order approximate solution of the boundary value problem the effect of curvature in the track of the wing has to be taken into account. It enters the problem through a thorough derivation of the kinematic boundary conditions as has been shown.

Alternatively it can be noticed from the derivation of the general first order solution for this case that a wing without camber rotating on a curved surface will obtain a kind of virtual camber. This is illustrated in the appearance of camber terms in the general solution of the problem. This virtual camber is of an entirely different nature than the higher order camber effect derived in section 7-5. The latter is caused by the local curvature of the flow at the wing under its own load, while the present virtual camber effect is the result of an initial (geometric) curvature in the flow.

In the (circumferential) \( z_b - \psi_R \) plane it can be seen that the particle of air, prior to arriving at the wing, passes sideways along the wing in every previous revolution, thus experiencing the presence of the pressure distribution on that wing, figure 17.

This interference effect will modify the load distribution over the wing rotation on a cylindrical surface with respect to a single (non rotating) wing in uniform flow. A situation similar to the interfering wing problem described in chapter 6.

As long as the particles of air pass sideways of the wing during previous revolutions the general expression for the solution (8-11) will be valid. The pressure gradient caused by the passages sideways of the wing prior to arrival at \( t = 0 \) are implicitly taken into account in the integration procedure given in equation (8-14). When however a very high rotational speed is assumed, the particles of air will pass over the wing in previous revolutions before there final arrival at the mid chord line. Such over synchronous operation is established in linear theory when

\[
\frac{\Omega R}{W} > \frac{2\pi R}{b}.
\]

The approximate solution \( p^{(0)} \) expressed in (8-11) is then extended to the solution \( p^{(0)} + p^{(3)}_{ovl} \), where \( p^{(3)}_{ovl} \) is a solution with exactly the same character as \( p^{(0)} \), but representing a pressure distribution on the "overlapping region" of the wing.

This region has a span equal to
and is situated on the downwind side (positive z coordinate) of the wing. With the local coordinates of the overlapping region indicated by the subscript \( \text{ovl} \), the expression for \( P^{(o)}_{\text{ovl}} \) yields:

\[
\frac{p^{(1)}_{\text{ovl}}}{\frac{1}{2} \rho (\Omega R)^2} = -\frac{C_{\text{L,ovl}}(\zeta)}{\pi} \frac{\sin \phi_{\text{ovl}}}{\cosh \eta_{\text{ovl}}} + \frac{C_{\text{L,ovl}}(\zeta) \, c f_2 \sin \chi_{\text{ovl}}}{r_b} + \frac{\sin \chi_{\text{ovl}}}{\pi} \sum_{n=1}^{\infty} B_n P_n^1(\cos \theta_{\text{ovl}}) \, Q_n^1(\cosh \nu_{\text{ovl}}) \]

where

\[
\zeta = \frac{z_b + \pi W}{b_{\text{ovl}/2}} \quad \text{and} \quad C_{\text{L,ovl}}(\zeta) = \frac{b_{\text{ovl}}}{c} \sqrt{1-\zeta^2} \sum_{n=1}^{\infty} B_n P_n^1(\zeta) .
\]

The relation between the coefficients \( A_n \) of the load distribution over the total wing, and \( B_n \) of the load distribution on the overlapping part is given by:

\[
\sum_{n=1}^{\infty} n(n+1)A_n + \frac{1}{2} \sum_{n=1}^{\infty} n(n+1)B_n = 0 .
\] (8-20)

With (8-20) it is assured that the velocity distribution along the mid chord line is non singular.

It was shown by the author [9] that this solution exhibits a large analogy with the solution found in the sections 6-3 and 6-4. The reason is of course that the problems exhibit a large analogy as well. The analogy will become clear when figure 17, showing the unrolled cylindrical surface at which the wing rotates, is compared with figure 12 depicting the infinite array of plane staggered wings. The wings in the infinite array of wings described in chapter 6 are equivalent to the "footprints" of the rotating wing during consecutive revolutions.
8-3 Numerical results for the loads on a wing and a tipvane

The equations derived in the previous section have been implemented into a numerical code, Van Bussel [9]. When the radius at which the wing is rotating is taken very large the situation of a wing in uniform flow is simulated. When also the rotational speed $\Omega$ is taken very large the situation of a formation flight of wings in a plane staggered configuration can be approximated. With the numerical code results are then obtained which should resemble the results obtained from the Trefftz-plane calculations discussed in section 6-5.

Figure 18 shows the calculated load distribution over wings with varying interference effects according to the (extended) Prandtl equation using a Legendre function representation of the lift coefficient. Thus according to the method described in the sections 6-4 and 6-5 for the plane staggered formation of wings. The result for one rectangular wing in uniform flow is also depicted for reference. Apart from the uniform flow case, the results for 0.25$b$ lateral spacing, synchronous operation and 0.3$b$ overlap are compared with the method described in the previous section.

![Figure 18: Comparison of load distributions for Trefftz plane calculations and for linearised 3-d calculations at different interference distances (Legendre function representation).](image)
A lateral spacing of 0.25b means that the (sideways) distance between the tip of the wing and the linearised vortex wake of its predecessor (or the wake of the wing itself originating from the previous revolution) equals 0.25b. Or, in terms of the acceleration potential theory, when the path of the particle of air, one revolution prior to arrival at the downwind wing tip, approached the upwind wing tip to a closest distance equal to 0.25b. Synchronous operation is established when the wake(s) are perfectly aligned in sideways direction (the particle arriving at the downwind wing tip passed exactly over the upwind wing tip during the previous revolution).

An overlap region is established when the rotational speed is increased with respect to synchronous operation (the particles passes over the upwind part of the wing one revolution prior to arrival at the downwind wing tip). In the numerical scheme to fulfil the final boundary condition (8-13) an extra pressure field on the overlapping region as given in (8-18) has then to be added. The choice $B_n = 0$ for $n \geq 2$ is made, and $B_1$ is taken such that the condition (8-20) is fulfilled.

As can be seen the results obtained from the method described in chapter 6 compare very well with the results from the method based upon the equations of the present chapter. The differences between the results from the two different methods are considered to be small, taken into account the totally different ways of modelling used. The calculated load in the overlap situation shows an increase of the load up to the overlap position, when compared with synchronous operation. The trailing vorticity of the preceding wing causes a steep gradient in the load distribution leading to a decreased load on the outer right part of the wing. The discontinuity in the gradient of the load at the overlap point is related to a vorticity shed in the wake that counteracts the vorticity generated by the left tip of the preceding wing.

Translated to acceleration potential theory the operation in overlap mode means that the accelerations "experienced" by a particle of air in the revolution prior to final arrival at the wing, are dominated by the passage over the wings surface. Since such passage requires tangential flow (the kinematic boundary condition) only additional accelerations experienced in the final revolution will perturb the ideal (zero lift) flow direction. Thus the overlap region becomes ineffective in terms of carrying load. The total load at synchronous operation has thus been shifted towards the non-overlapping area. The acceleration at the overlap position is such that it annihilates the acceleration experienced at the upwind tip during the previous revolution.

The numerical code described above was used within the tipvane research program to predict load distributions and induced velocity distributions around such wings rotating at a cylindrical surface. A global theory for the tipvane as well as a first duct model were described by Van Holten [23]. The results obtained with the methods described in the sections 6-4 and 6-5 and the method described in the previous section 8-2 were used for a more detailed theoretical description of the working principle of the tipvane wind turbine. It was shown that, in the situation of an infinite array of wings, or in the situation of an infinitely stretched cylindrical wake, the induced drag of the wing reduces to zero, see
Van Bussel [8], [9] and [10]. This means that the combination of induced velocity distribution (determining the direction of the local flow) and the load distribution (determining the magnitude of the local force on the wing) is such that the resulting force on the wing is perpendicular to the undisturbed relative flow. For the situation of a wing rotating at a circular cylinder a considerable reduction of induced drag was also measured in wind tunnel experiments, see e.g. Van Holten [26]. This means that it is possible to rotate a wing in a cylinder surface almost without additional parasitic drag.

When the load on the rotating wing is directed radially inward, then the flow will diffuse radially outward in the wake, as a reaction. Through the continuity equation it can then be shown that the flow before the rotating wing has to contract. Thus a kind of dynamic diffuser is created. With such a dynamic diffuser it is possible to create a considerable increase in mass flow through the cylinder at which the wing rotates. This mass flow augmentation can then be used to augment the power output of a wind turbine placed inside the cylinder. With the tipvane rotor, where a wing (called a "tipvane") was mounted in a T-tail fashion at each tip of a conventional horizontal axis wind turbine, it was demonstrated that mass flow augmentation was indeed obtained, Van Holten [26]. In theory a considerable increase in power seemed feasible (in the order of doubling the power output).

In open air circumstances however the occurrence of extra parasitic drag, i.e. due to the blade-tipvane connection, together with the unsteady inflow conditions (a constantly varying wind speed and direction) prevented the proof of the concept in natural conditions. No power augmentation was demonstrated in the open air over the year 1984, which resulted in a "no go" decision for the continuation of the tipvane project in the beginning of 1985. However a considerable reduction in induced drag of the tipvanes was demonstrated in the open air since the power of the tipvane rotor equalled the power of the rotor without tipvanes, see e.g. Bruining [6]. Van Holten et al [26] and Van Bussel et al [10] give more details concerning the aerodynamic research performed within this tipvane program.

8-4 The wing rotating in a plane (a rudimentary wind turbine geometry).

The second problem discussed in this chapter concerns the determination of the loads on a rectangular wing rotating in a plane perpendicular to the undisturbed flow \( \mathbf{W} \), see figure 16. The span of the wing is aligned radially, with its most outboard wing tip with at a radius \( R \). The span \( b \) of the wing is small with respect to the radius \( R \) and the wing rotates at a large rotational frequency \( \Omega \). This means that both \( \Omega R \) and \( \Omega (R-b) \) are large with respect to the undisturbed flow velocity \( \mathbf{W} \). So the span of the blade is approximately perpendicular to the relative undisturbed velocity \( \mathbf{W} - \Omega \times (R-b/2+z_4) \), see figure 16.
As in the previous section an approximate first order pressure field solution \( p^{(1)} \) will be derived for this boundary value problem with the use of asymptotic expansion techniques. Therefore the assumption is made that the chord of the considered wing is very small with respect to the radius \( R \). Then the boundary value problem is again, to a first order, equivalent to the problem discussed in chapter 7.

The relation between the basic \((x, y, z)\) system and the local \((x_b, y_b, z_b)\) system for the wing rotating in a plane perpendicular to \( \mathbf{W} \) is given by:

\[
\begin{align*}
x &= (R - \frac{b}{2} + z_b) \cos \psi + x_b \sin \psi \\
y &= (R - \frac{b}{2} + z_b) \sin \psi - x_b \cos \psi \\
z &= -y_b.
\end{align*}
\]  

(8-21)

The undisturbed flow is, in linearized theory, equal to \( \mathbf{W} + \mathbf{\Omega} \times (R - b/2 + z_b) \) and points into the \( x_b \) direction.

In a similar way as discussed in the previous section a relation must be established between the velocities and accelerations of particles of air must be derived in wing reference system and in the inertial coordinate system.

This has been carried out in appendix B. After the derivation of the general expressions for the velocities and accelerations a position of the wing \( \psi = \pi/2 \) was chosen.

In order to fulfil the kinematic boundary conditions the accelerations and the velocities of the particles of air have to satisfy a number of requirements which will be discussed below.

At first the accelerations of the particles moving over the wing should fit its curvature. When it is assumed that the curvature of the wing is such that \( x_b/R, y_b/R \) and the first and second partial derivatives of the surface function \( y_b(x_b, z_b) \) are small with respect to 1, than the following linearised expression for the acceleration of such particles in the inertial \((x, y, z)\) reference system can be derived from equation (B-24):
\[
\frac{d\omega}{dt} = (1 - \frac{bf}{2} + \frac{z_b}{R}) \left[ \frac{\partial y_b}{\partial z_b} - (1 - \frac{bf}{2} + \frac{z_b}{R}) \frac{\partial^2 y_b}{\partial x_b^2} \right] \Omega^2 R + \\
+ 2 \left[ \frac{\partial y_b}{\partial z_b} - (1 - \frac{bf}{2} + \frac{z_b}{R}) \frac{\partial^2 y_b}{\partial x_b^2} \right] \Omega u + \\
+ 2 \left[ \frac{-\partial y_b}{\partial x_b} - (1 - \frac{bf}{2} + \frac{z_b}{R}) \frac{\partial^2 y_b}{\partial x_b \partial z_b} \right] \Omega v + \\
\frac{\partial^2 y_b u^2}{\partial x_b^2} - 2 \frac{\partial y_b}{\partial x_b} \frac{\partial y_b}{\partial z_b} \partial v - \frac{\partial^2 y_b v^2}{\partial x_b^2} + \\
+ (1 - \frac{bf}{2} + \frac{z_b}{R}) \frac{\partial y_b}{\partial z_b} \Omega R - \frac{\partial y_b}{\partial x_b} \frac{du}{dt} - \frac{\partial y_b}{\partial x_b} \frac{dv}{dt}.
\] (8.22)

When it is furthermore assumed that the rotational speed is constant; that the \( u, v \) and \( w \) component of the wind can be written as \( u = u'; v = v'; w = W + w' \), where \( u', v' \) and \( w' \) are small with respect to the undisturbed wind speed \( W \); that the rotational speed \( \Omega R \) is large with respect to \( W \) and that the accelerations \( du/dt \) and \( dv/dt \) are small with respect to \( \Omega^2 R \) then equation (8.22) can be further linearised and rewritten to:

\[
\frac{d\omega}{dt} = (1 - \frac{bf}{2} + \frac{z_b}{R}) \left[ \frac{\partial y_b}{\partial z_b} - (1 - \frac{bf}{2} + \frac{z_b}{R}) \frac{\partial^2 y_b}{\partial x_b^2} \right] \Omega^2 R + \\
\frac{-x_b R \partial^2 (x_b z_b)}{\partial z_b} + \frac{\partial y_c(x_b z_b)}{\partial z_b} + \\
- (1 - \frac{bf}{2} + \frac{z_b}{R}) \frac{\partial^2 y_c(x_b z_b)}{\partial x_b^2} \Omega^2 R.
\] (8.23)

Equation (8.23) shows that the particles of air following the blade surface will experience accelerations into the \( z \) direction (which is equal to the \( y \) direction for the specific position chosen) dictated by the spanwise variation of the pitch angle and the spanwise and chordwise variations of the camber function \( y_c \). Substitution of (8.23) into the \( z \)-component of the linearised Euler equation leads to:

\[
\frac{1}{\rho(\Omega R^2)} \frac{\partial \rho}{\partial y_b} = (1 - \frac{bf}{2} + \frac{z_b}{R}) \left[ \frac{-x_b R \partial^2 \theta}{\partial z_b \partial z_b} + \frac{\partial y_c}{\partial z_b} - (1 - \frac{bf}{2} + \frac{z_b}{R}) \frac{\partial^2 y_c}{\partial x_b^2} \right].
\] (8.24)

To complete the kinematic boundary conditions it is required that the leading edge singularity at the wing should be such that just aft of it a tangential flow (with respect to the wing surface) is established. This means that the velocity of the particles of air after the leading edge are such that their paths follow the wing surface given by (7.5). However, with equation (8.24) it is already arranged such that the accelerations are in accordance with the wing curvature, it is therefore sufficient to require the velocities at the wing surface to be in accordance with the wing geometry at one station along the path. As
before the mid chord line is chosen. So:

\[
p \xrightarrow{1-\infty} \quad x_b = -\frac{c}{2}, \quad y_b \to 0 \quad \text{in such a way that}
\]
\[
\frac{w}{\Omega R} = \left(1 - \frac{bf}{R} + \frac{z_b}{R}\right) \left(\theta(z_b) - \frac{\partial y_c}{\partial x_b}\right) \quad \text{at the mid chord line}.
\]

If the problem is restricted to an uncambered and untwisted wing the boundary conditions (8-24) and (8-25) reduce to:

\[
\frac{1}{\rho(\Omega R)^2} \frac{\partial p}{\partial y_b} = 0 \quad \text{at the wing}
\]

and

\[
p \xrightarrow{1-\infty} \quad x_b = -\frac{c}{2}, \quad y_b \to 0 \quad \text{in such a way that}
\]
\[
\frac{w}{\Omega R} = \left(1 - \frac{bf}{R} + \frac{z_b}{R}\right)\theta(z_b) \quad \text{at the mid chord line}.
\]

The solution of the boundary value problem (7-1) through (7-3), (8-26) and (8-27) can then directly be taken from chapter 7. As in the previous section a restriction is made to a first order approximation of the solution. The complete pressure field is thus obtained completely equivalent to equation (8-27):

\[
\frac{P^{(1)}}{\rho(\Omega R)^2} = -\frac{C_f(z_b)}{bf/2} \frac{\sin \psi}{\pi \cosh \eta + \cos \phi} + \frac{C_f(z_b)}{bf/2} \frac{cf}{2} \frac{\sin \chi}{r_b} \]

\[+ \frac{\sin \chi}{\pi} \sum_{n=1}^{\infty} A_n P_n^1(\cos \theta) Q_n^1(\cosh \nu).\]

And the following relation between \(C_f\) and the coefficients \(A_n\) is found:

\[
C_f(z_b) = A \sqrt{1 - (\frac{z_b}{bf/2})^2} \sum_{n=1}^{\infty} A_n P_n^1(\frac{z_b}{bf/2}).
\]

With the final boundary condition, equation (8-27), the coefficients \(A_n\) in the lift coefficient distribution \(C_f\), can be determined.

From the \(z\)-component of the linearised Euler equation it follows for the final boundary
condition in the situation of the uncambered and untwisted wing:

\[
-\frac{1}{1/2\rho(\Omega R)^2} \int_{-\infty}^{0} \frac{\partial p^{(1)}(x,y,z,t)}{\partial z} \Omega dt = 2\theta(z_0)
\]  \hspace{1cm} (8.30)

for a particle of air arriving at the midchordline at \( t=0 \).

The evaluation of this integral is carried out numerically. The particle of air arriving at \( t=0 \) at the location \( (0, y_0, 0) \) at the wing is assumed to travel along the linearised path given by:

\[
x(t) = 0 \ , \quad y(t) = y_0 \ , \quad z(t) = Wt \ ,
\]  \hspace{1cm} (8.31)

which is in accordance with the linearised Euler equation.

Although not immediately clear there is also an interference effect present in this situation. The reason for it is found in the fact that the load on the wing acts in the direction of the undisturbed flow \( W \). This load "causes" an induction velocity \( W' \) in the same direction. When the load is sufficiently large this induction velocity cannot be ignored. In such cases the semi-linearised Euler equation should be used. Its \( z \)-component yields:

\[
\frac{\partial w}{\partial t} + (W + W') \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} .
\]  \hspace{1cm} (8.32)

Here \( W' \) is the disturbance velocity in the direction of the undisturbed flow. Then the semi-linearised path of the particle is given by:

\[
x(t) = 0 \ , \quad y(t) = y_0 \ , \quad z(t) = (W + W')t .
\]  \hspace{1cm} (8.33)

This cannot directly be implemented in a numerical procedure since the disturbance velocity \( W' \) is not known. With an iterative scheme it is however no problem to use such a path along which the transport velocity is distorted.

\section*{8.5 The treatment of a partial span pitch step.}

A second interference effect is present when the pitch angle distribution along the span is discontinuous. Such a situation is found for example on wind turbine blades with a pitchable tip. The latter is often applied for power control of the turbine. In the schematic situation considered in this section, this is comparable with a step in pitch angle of the wing somewhere in the region of the outboard wing tip at radius \( R \).

The approximate solution \( p^{(1)} \) expressed in (8.28) is then extended to the solution
where $p^{(1)} + p^{(1)}_{tip}$, where $p^{(1)}_{tip}$ is a solution with a similar character as $p^{(1)}$, representing a pressure distribution on a blade with a step in pitch angle. Say the step in pitch is located at the spanwise location $z_{b,tip}$. The tip region has a span equal to $b_{tip}$ and extends radially outward from spanwise location $z_{b,tip}$.

With the local coordinates of the tip region indicated by the subscript $tip$, and the local coordinates of the ("fixed") inboard part of the blade indicated with the subscript $main$ the expression for $p_{i,tip}$ yields:

$$\frac{p^{(1)}_{tip}}{\frac{1}{2} \rho (\Omega R)^2} = \frac{C_{l,main}(\zeta_{main})}{\pi} \sin \phi_{main} \sin \frac{\zeta_{main}}{r_b} + \frac{C_{l,main}(\zeta_{main})}{\pi} \frac{cf}{r_b} \sin \chi_{main}$$

$$= \frac{\sin \chi_{main}}{\pi} \sum_{n=1}^{\infty} B_n P_n^1(\cos \phi_{main}) Q_n^1(\cos \chi_{main})$$

$$+ \frac{C_{l,tip}(\zeta_{tip})}{\pi} \sin \phi_{tip} \sin \frac{\zeta_{tip}}{r_b} + \frac{C_{l,tip}(\zeta_{tip})}{\pi} \frac{cf}{r_b} \sin \chi_{tip}$$

$$= \frac{\sin \chi_{tip}}{\pi} \sum_{n=1}^{\infty} D_n P_n^1(\cos \phi_{tip}) Q_n^1(\cos \chi_{tip})$$

$$\frac{\zeta_{main}}{b_{tip}/2} = \frac{z_{b,tip} - b_{tip}/2 - b/2}{b_{tip}/2}, \quad \zeta_{tip} = \frac{z_{b,tip} + b_{tip}/2}{b_{tip}/2}$$

$$C_{l,main}(\zeta_{main}) = \frac{b-b_{tip}}{c} \sqrt{1-\zeta_{main}^2} \sum_{n=1}^{\infty} B_n P_n^1(\zeta_{main})$$

$$C_{l,tip}(\zeta_{tip}) = \frac{b_{tip}}{c} \sqrt{1-\zeta_{tip}^2} \sum_{n=1}^{\infty} D_n P_n^1(\zeta_{tip})$$

As can be seen in equation (8-34) the approximate solution for the situation of a partial span pitch step consists of a standard first order solution on the main "fixed" part of the wing together with a first order solution on the pitchable tip of the wing (the outer part of the wing where the extra pitch angle is present). When the coefficients $B_n$ of the fixed part solution and the coefficients $D_n$ of the pitchable part solution of the wing are chosen such that a differentiable lift coefficient function is obtained over the complete wing having a large gradient at $z_{b,tip}$ then the induced velocity distribution at $z_{b,tip}$ will be (almost) discontinuous. The relation between the coefficients $B_n$ and $D_n$ of the load distributions on the main part and on the tip part should thus at least satisfy:

$$\sum_{n=1}^{\infty} n(n+1)B_n = \frac{b-b_{tip}}{b_{tip}} \sum_{n=1}^{\infty} (-1)^n n(n+1)D_n$$

in order to obtain a differentiable lift coefficient distribution over the wing.
When the numerical value of the expression in (8-36) is large it is assured that the velocity distribution along the mid chord line also exhibits a steep variation at the location of the step in pitch angle $z_{h, u_p}$. In the numerical evaluation of (8-36) a number of possibilities are present. In the code use is made of those combinations of $B_n$ and $D_n$ that yield conformal anti-symmetric distributions with respect to the position $z_{h, u_p}$.

The type of $C_l$ distributions generated in the above described way show a large similarity with the spanwise load distributions over wings with flaps or ailerons derived by Multhopp [31]. This is not so surprising since in both cases it is aimed to obtain continuous differentiable spanwise load distributions, together with discontinuous induced velocity distributions.

Figure 19 shows some examples of $C_l$ distributions determined with the present method, and therefore satisfying (8-36).

![Diagram](image.png)

**Figure 19:** Lift coefficient distributions related to a step in pitch angle at spanwise position $z_l/(b/2) = 0.5$
9 The acceleration potential problem of the wind turbine rotor

Introduction

In the present chapter the boundary value problem for a wind turbine blade is discussed. The difference with the rotating wing treated in section 8-4 is that the blade may have an arbitrary pitch angle distribution, and an arbitrary chord distribution. Furthermore the number of rotor blades rotating in the rotor plane can be larger than one. It will thus be possible to calculate the configurations found in practice.
For reasons of clearness the camber terms, representing cambered aerofoils, and additional terms due to partial span pitching blades (having a discontinuity in blade pitch angle along the span) will not be presented in this chapter. They have been derived to such an extend in the previous chapters 7 and 8 that implementation into numerical codes of the analytical results presented there is straightforward.

The structure of the solution is of course very similar to the result for the rotating wing presented in section 8-4 and will therefore need not much more attention. Emphasis is paid in this chapter to the fulfilment of the final (kinematic) boundary condition. This yields the integration of pressure gradients along the trajectories of the particles travelling to the rotor blade. For wind turbine rotors the velocity perturbation in axial direction is usually significant. This makes it necessary to perform the integrations along trajectories, which are at least non-linearized in axial direction. This is the direction in which the main forces on the flow are acting, and thus the direction in which considerable induction velocities can be expected.
Euler integration of the accelerations is also an option in the numerical codes presented in the chapters 10 and 11. Then the trajectories of the particles of air travelling to the rotor are calculated including the velocity disturbances in all three directions. Therefore the partial derivatives of the pressure field in all three directions are discussed here.
The integration of the near field term is given special attention. It will identify an important modification of two dimensional properties of applied aerofoils (in the present case a flat plate) in a three dimensional environment. Finally some modifications of the method will be discussed that have to be applied when more than one blade is taken into account.

9-1 The boundary value problem of the windturbine rotor and its solution

As was shown in the chapters 7 and 8 the first order problem for a rotating wing satisfies the Laplace equation for the pressure perturbation:
\[
\n\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 .
\]

(9-1)

This equation was derived under the small velocity perturbation assumption. The linearised Euler equation under those assumptions is given by:

\[
\frac{\partial V'}{\partial t} + (W \cdot \nabla) V' = -\frac{1}{\rho} \nabla p .
\]

(9-2)

The situation for a one bladed rotor will first be considered in order to keep the expressions as simple as possible. In a later stage it will be shown that the expressions can be easily adapted for multibladed configurations.

The first (obvious) boundary condition is the fact that the pressure perturbations should vanish far away from the aerodynamical active surface:

\[
p \to 0 \text{ when } x^2 + y^2 + z^2 \to \infty .
\]

(9-3)

---

**Figure 20:** Definition of local blade pitch angle $\theta_p$ and geometrical angle of attack $\alpha_{geo}$.
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When the projection of the rotor blade is assumed to be situated in the $x_z$ plane (the rotor blade plane) and the blade surface of the rotor blade with flat plate aerofoils is described as a surface function with function values along the $y_b$ axis:

$$y_b(x_b, z_b) = -\theta_p(z_b, t)x_b$$

where $|x_b| \leq \frac{c(z_b)}{2}$; $|z_b| \leq \frac{b}{2}$.

(9-4)

The expression $\theta_p(z_b, t)$ represents the (time dependent) setting angle of the blade, i.e. the local angle with respect to the plane of rotation, see figure 20.

In general the chord of a rotor blade is not constant along the span $b$, hence the expression $c(z_b)$ in (9-4). The first and second order derivatives of the chord distribution with respect to the non dimensionalised span are assumed to be small with respect to 1, in order to stay consistent with the linearisations yielding equation (8-22) of section 8-4.

Then the second boundary condition is, according to equation (8-24), described by the following linearized equation:

$$\frac{1}{\rho(\Omega R)^2} \frac{\partial p}{\partial y_b} = \frac{1}{\Omega^2 R} \frac{dw}{dt} = \left(1 - \frac{b/2 + z_b}{R} \right) \frac{x_b}{R} \frac{\partial \theta}{\partial z_b} - \frac{\Omega}{\Omega^2} \right)$$

(9-5)

on the rotor blade.

Finally the Kutta-Joukowski condition has to be satisfied:

$$p \rightarrow \infty \quad \text{on the leading edge of the rotor blade}$$

(9-6)

in such a way that

$$\frac{w}{\Omega r + u_\infty \Omega t + \nu \Omega t} = \theta_p(z_b, t) \quad \text{on the rotor blade}.$$ 

(9-7)

In the equations (9-3) and (9-5) the index $b$ indicates a coordinate system related to the blade, see figure 2.

Now assume that the velocities in the rotor plane are small with respect to the rotational velocity $\Omega r$. Furthermore assume that the pitch angle is small so that the conditions on the rotor blade can be transformed to the rotor plane. For a slender blade with a relatively small radial variation in pitch angle $\theta_p$ on a rotor with small variations of its rotational speed the equations (9-4), (9-5) and (9-7) may be further linearised to:

$$\frac{\partial p}{\partial y_b} = 0 \quad \text{on the rotor blade}$$

(9-8)

and
\[
\frac{w}{\Omega r} = \theta_p(z_b,t) \text{ on the rotor blade}. \tag{9-9}
\]

Completely analogous to equation (8-30) the general first order solution for the problem (9-1), (9-3), (9-6), (9-8) and (9-9) can then be derived:

\[
\frac{p^{(1)}}{2 \rho W^2} = -\frac{1}{\pi} \frac{1}{2} \rho W^2 c(z_b) \left( \frac{\sin \eta}{\cosh \eta + \cos \phi} \right) + \\
+ \frac{1}{2 \pi} \frac{1}{2} \rho W^2 c(z_b) \frac{c(z_b)}{r_b} \sin \chi + \\
+ \frac{1}{\pi} \sum_{\eta=1}^{\infty} A_n(t) P_n^1(\cos \theta) Q_n^1(\cosh \nu) \sin \chi.
\] \tag{9-10}

The first expression on the right hand side of equation (9-10) is the near field term written in local elliptical coordinates \( \phi \) and \( \eta \) (figure 3). The third expression is the far field term, written in prolate spheroidal coordinates \( \theta, \nu \) and \( \chi \) (figure 5); and the middle expression in the right hand side is the common field expression written in circular cylinder coordinates \( z_b, r_b \) and \( \chi \) (figure 4). The \( P_n^1 \) and \( Q_n^1 \) functions represent associate Legendre functions of the first and second kind. Legendre functions are the natural solutions for problems written in prolate spheroidal coordinates. In equation (9-10) it can be seen that close to the blade the common field expression exhibits a singular behaviour (caused by \( r_b^{-1} \)). This behaviour is also found in the far field term (when the \( Q_n^1 \) term is evaluated), but with the opposite sign. Thus the total expression does not have this essential singularity.

In expression (9-10) the function \( l(z_b,t) \) is used. This is the lift distribution over the blade. It can also be expressed in terms of associate Legendre functions of the first kind:

\[
l\left( \frac{z_b}{b/2},t \right) = \frac{1}{2} \rho W^2 b \sqrt{1 - \left( \frac{z_b}{b/2} \right)^2} \sum_{\eta=1}^{\infty} A_n(t) P_n^1(\frac{z_b}{b/2}). \tag{9-11}
\]

The coefficients \( A_n(t) \) are not determined at this stage. This is done by application of the final boundary condition:
for a particle of air arriving at the midchordline of the turbine blade.

The equations (9-10) and (9-11) have been written down for the situation that the pitch angle distribution \( \theta_p(z_p, t) \) is continuous with respect to the spanwise coordinate \( z_p \).
Expression (9-10) can however be extended with an expression equivalent to equation (8-34) for the situation of a blade with a partial span pitch control mechanism. In such situations there is a discontinuity in \( \theta_p(z_p, t) \) for a given value of \( z_p \), with a magnitude which may vary in time.

The introduction of the additional terms due to partial span pitch control does not essentially change the derivations performed in this and in the coming chapters. Therefore they have been left out, just as was done with the camber terms. If there is a need to implement such terms into a numerical code it is rather easy to add the extra terms based upon the expressions derived in chapter 7 and section 8-5.

The situation of a multibladed rotor can now be tackled with a straightforward approach. The solution (9-8) for a one bladed rotor can simply be expanded with similar equations for the other blades, all in their own local blade coordinates. It will be shown in section 9-5 that the character of the first order solution is unaffected by the introduction of a second or a third blade. The determination of the coefficients \( A_p(t) \) implies of course more labour, but is in principle unchanged as will also be explained in section 9-5.

The integration of equation (9-12) should take place along the trajectories of the particles of air. From the linearised Euler equation it can be seen that it is justified to take a straight path which is travelled with a constant speed \( W \). Thus:

\[
x(t) = 0 \quad , \quad y(t) = y_0 \quad , \quad z(t) = Wt \quad .
\]

(9-13)

The axial induction velocities near the rotor plane are however not so small for the situation of a wind turbine rotor. In that case the semi linearised Euler equation yields the following integration path:

\[
x(t) = 0 \quad , \quad y(t) = y_0 \quad , \quad z(t) = (W + W')t \quad .
\]

(9-14)

Finally it is possible to perform an Euler type of integration. In such case the particle of air is followed along its track. Then the velocities in the field must be known or in other words the pressure field must be determined. In practice such Euler integration procedure is performed iteratively starting with a path determined by (9-13). Once a first approximation of the field is known, it is possible to determine a perturbed path from the pressure gradients along the trajectory. This determines a second approximation of the pressure field and so on. The procedure will be enlightened in the next sections.
9-2 Partial derivatives of the basic distributions

The kinematic boundary condition (9-7) states that the velocity found at the rotor blade should be such that the particles of air move tangential to the blade surface. Suppose the pressure distribution is known. Then the accelerations in the field can be determined, and thus the path of the particles of air, and their velocities in the rotor plane.

The accelerations in the field are obtained by the partial derivatives of the pressure field given in equation (9-10). It thus requires the evaluation of the partial derivatives of the near, the common and the far field terms of equation (9-12).

Although the expression can be differentiated numerically, it is preferred to use the analytic expressions, which can be most easily obtained when the common and far field terms are differentiated in the local \((x_y, y_y, z_y)\) system.

The partial derivatives have to be taken in all three directions in the local coordinate system since this is rotating with respect to the inertial \((x, y, z)\) system in which the velocities are evaluated.

In a first linearised approach it might be sufficient to evaluate the derivatives in \(y_b\) direction only, since the relation \(z = -y_b\) is valid. But for more advanced applications the derivatives in the other directions are necessary to determine the velocities in the \(x\) and \(y\) directions as well. In appendix D the partial derivatives of the basic distributions are derived. Basic pressure distributions form the "building blocks" of the general first order solution given in the equations (9-10) and (9-11).

Thus the far field term is characterised by the basic expressions:

\[
\frac{P_{\text{far,basic}}}{1/2\rho W^2} = \frac{1}{\pi} P_n^1(\cos \theta)Q_n^1(\cosh v) \sin \chi
\]

(9-15)

and by substitution of (9-11) into (9-10) it can be seen that the basic common field term is characterised by:

\[
\frac{P_{\text{common,basic}}}{1/2\rho W^2} = -\frac{1}{\pi} \sqrt{1 - \left(\frac{z_b}{b/2}\right)^2} P_n^1(z_b) \frac{\sin \chi}{\sinh \nu \sin \theta}.
\]

(9-16)

In a similar way the basic near field expression is found to be:

\[
\frac{P_{\text{near,basic}}}{1/2\rho W^2} = -\frac{1}{\pi} \frac{b}{c(z_b)} \sqrt{1 - \left(\frac{z_b}{b/2}\right)^2} P_n^1(z_b) \frac{\sin \varphi}{\cosh v + \cos \varphi}.
\]

(9-17)

It should be taken into account that in the actual application of the above expressions in a
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...numerical procedure it is necessary to evaluate the common field term plus the far field term simultaneously.

The reason for it can be found in section 7-5, where the common field solution is introduced:

Both $p_{far}$ and $p_{common}$ exhibit a singularity of the order $O(r_b^{-1})$, where $r_b^2 = x_b^2 + y_b^2 = (b/2)^2 \sin^2 \theta$. When the far and common field expressions are taken together this singularity is annihilated. With the singularity of the order $O(r_b^{-1})$ in both basic pressure field expressions it can be expected that the partial derivatives (in $x_b$ and $y_b$ directions) will exhibit a singularity of the order $r_b^{-2}$. The combined evaluation of the far and common pressure field gradient close to the rotor blade may however have an essential singularity. In appendix D the partial derivatives of the combination of the far and common field with respect to the local coordinates are therefore also determined.

It is shown there that the partial derivatives of $p_{far} - p_{common}$ are all of the order $O(\ln \sinh v) = O(\ln (r_b))$.

Therefore the integration of the combination of the far and common pressure field gradient is possible. The combined pressure gradients are presented in the expressions (D-12) through (D-14) of appendix D. The integration of the pressure gradients can be performed without introducing large errors in the numerical evaluation when a suitable procedure is chosen. This will be explained in the next two sections.

9-3 The integration of the pressure gradients.

In its simplest application using the linearised Euler equation, the integration of the gradients of the basic pressure distributions $p_{basic}$, necessary for the fulfilment of the final boundary condition, takes place along the straight path given by (9-13). This straight path is given in inertial coordinates ($x$, $y$, $z$), the basic pressure fields are however given in blade coordinates according to the equations (4-3), (4-4) and (4-5).

The reason for writing the pressure fields in local blade coordinates is found in the fact that they then become steady; i.e. independent of time. The pressure field rotates together with the rotor blade!

The time dependency is "felt" in the inertial coordinate system, in which a constant rotational velocity of the blade and its pressure field is felt as a harmonic pressure variation (for a given fixed location). The transformation formulae from the inertial system to the local coordinate system ($x_b$, $y_b$, $z_b$) are given in (4-2), where $\psi = \Omega t = \pi/2$.

Travelling along the path given by (9-13), with a constant velocity $W$ (and thus with a constantly varying position) the experienced pressure variations are quite complicated. Such behaviour represents of course the interference effects of the rotor blade with itself. The pressure distribution on the blade will thus cause velocity gradients along the path of a particle of air in revolutions prior to the final revolution at which the particle arrives at the blade. This is the cause of the fact that the perturbation velocities at the more heavily loaded rotor blade are no longer small with respect to the undisturbed wind velocity $W$, and can therefore not be ignored, even in a first order approximation.
Not too much can a priori be said of the behaviour of the pressure gradients along the path. One thing however is quite clear. For azimuthal angles $\psi$ about equal to $\psi = \pi/2 + 2k\pi, \; k \in \mathbb{N}$, it is evident that the singularities in the pressure field (located at the rotor blade) are closer to the integration path than for other values of $\psi$. Therefore extrema in the values of the gradients can be expected at the instants where $t$ is about equal to $t = 2k\pi/\Omega, \; |k| \in \mathbb{N}$. So the behaviour of the pressure gradients is expected to be more or less periodical along the path (9-13), with a continuously increasing amplitude.

For small values of $k$ the effect of the singularities at the instantaneous position along the path will be a large at $t = 2k\pi/\Omega$, but will be small at $t = 2k\pi/\Omega + \pi/2$, so that a "peaky" behaviour can be expected.

The integration in the interval $(t_f, 0)$ where $t_f = -2\pi/\Omega$, will need special attention, since it implies the integration of an actual singularity at $t = 0$ (the integration path then coincides with the rotor blade).

The above considerations show that it will be convenient to cut the integration interval into pieces of a length $\Delta t = 2\pi/\Omega$. This makes it possible to arrange the peaks in the integrand at the begin and the end of each interval. A suitable numerical integration method capable of integrating singularities of the order $O(\varepsilon^{-1/2})$ at the begin- and at the end point of the interval, where $\varepsilon$ indicates the distance to begin- and/or end point, is the Gauss-Chebyshev integration method. In its general form this method is given by:

\[
\int_{-1}^{1} \frac{f(x)}{(1-x^2)^{1/2}} dx = \sum_{m=1}^{m_0} w_m f(x_m) + R_{m_0}
\]

where \( w_m = \frac{\pi}{m_0}, \; x_m = \cos\left(\frac{(2m-1)\pi}{2m_0}\right) \) \hspace{1cm} (9-18)

and \( R_{m_0} = \frac{\pi}{(2m_0)^{1/2}} f^{(2m_0)}(\xi) \) the remainder with $-1 < \xi < 1$.

With this method the integration can be performed up to the point $t_1 = -2\pi/\Omega$, without any problem and with a prescheduled order of accuracy. In practical situations using a numerical procedure, the integration interval (-\infty, t_1 ] will always be restricted to an interval $(t_{\xi_0}, t_1 ]$, where $t_{\xi_0} = -2k_0\pi/\Omega$ and $k_0$ is chosen such that the integration over (-\infty, t_{\xi_0}] has a negligible contribution to the complete integral.

The integration over the final interval before arrival at the midchordline $(t_1, 0]$ requires more consideration. At first consider an approximation of the integration path:

In local $(x_0, y_0, z_0)$ coordinates the linear integration path is given by:
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\[
\begin{align*}
    x_b &= -(R - \frac{b}{2} + z_b) \cos(\Omega t + \frac{\pi}{2}) \\
    y_b &= -Wt \\
    z_b &= (R - \frac{b}{2} + z_b) \sin(\Omega t + \frac{\pi}{2}) - R + \frac{b}{2}
\end{align*}
\]  

(9-19)

which for \( t \downarrow 0 \) this path can be approximated by:

\[
\begin{align*}
    x_b &= (R - \frac{b}{2} + z_b) \Omega t \\
    y_b &= -Wt \\
    z_b &= z_b_0 .
\end{align*}
\]  

(9-20)

The common field part and the far field part of \( p_{\text{basic}} \) both have a singularity at the location of the lifting line \((x_b = 0, y_b = 0, \mid z_b \mid \leq b/2 \) ).

It can be shown that the behaviour of the far field part is given by:

\[
\frac{P_{\text{far, basic}}}{\frac{1}{2} \rho W^2} = \frac{1}{\pi} \frac{b f}{r_b} \sin \chi \sqrt{1 - \left( \frac{z_b}{b/2} \right)^2} \bar{P}_n \left( \frac{z_b}{b/2} \right) + O \left( \frac{r_b}{b/2} \ln \left( \frac{r_b}{b/2} \right) \right) 
\]  

(9-21)

for \( \frac{r_b}{b/2} \rightarrow 0 \).

The common field part of \( p_{\text{basic}} \) is given by:

\[
\frac{P_{\text{common, basic}}}{\frac{1}{2} \rho W^2} = -\frac{1}{\pi} \frac{b f}{r_b} \sin \chi \sqrt{1 - \left( \frac{z_b}{b/2} \right)^2} \bar{P}_n \left( \frac{z_b}{b/2} \right) .
\]

Addition of both terms thus leads to a pressure field having a regular behaviour at the lifting line.

So when these two components are treated together, the integration of its gradient over the final interval \((t, 0] \) should not give any problem. It is indeed shown in appendix D that the pressure gradients of \( P_{\text{far}}, P_{\text{common}} \) are all of the order \( O(\ln \varepsilon) \) and can thus be integrated.

9-4 The integration of the near field term.

The basic near field term \( P_{\text{near, basic}} \) shows a singularity at the leading edge of the blade. This singularity is known as the pressure singularity at the leading edge of a flat plate aerofoil carrying a Birnbaum lift distribution. With the integration path given by equation (9-13), or with its approximation for small \( t \) according to (9-20), it can be seen that at the
time $t = -(c(z_b)/2)/(\Omega(R-b/2+z_b))$ the particle of air is very close to the leading edge, and therefore the numerical integration of the near field pressure term might yield relative large errors.

Equation (9-13) and its approximation (9-20) define a straight path, i.e. a path independent of character and magnitude of the pressure field. In the first part of this chapter it was argued that, although the deviations caused by the pressure field would indeed lead to a perturbed path, the accelerations felt along the real path would be about the same as those found along the path given in (9-13). Not much is known for the actual position of the real path, however at the immediate vicinity of the blade it is exactly determined by the tangential flow condition. A particle of air arriving at $t = 0$ at the midchordline of the blade must have travelled tangential to it in the instants between passage of the leading edge and arrival at the midchordline. Thus the real path of the particle of air in the time-interval $(-c(z_b)/2)/(\Omega(R-b/2+z_b))$, 0] is, in linearised theory, known to be:

$$
\begin{align*}
x_b &= (R - \frac{b}{2} + z_b)\Omega t \\
y_b &= -\frac{c(z_b)}{2} \\
z_b &= z_b
\end{align*}
$$

(9-23)

where $\varepsilon$ is the relative thickness of the aerofoil, and thus $|\varepsilon| \rightarrow 0$ under the present assumptions.

From (9-23) it can be seen that the previously determined integration path (9-13) for a particle of air (which is not its real path) does not give about the same accelerations generated by the near field term, as those found along the real path (9-23), because the distance $\varepsilon$ from the singularity becomes infinitely small.

Therefore it is concluded that the integration along (9-20) is not correct for the final part of the path, as far as the near field term is concerned.

In the present section two different ways of integrating the near field pressure term will be described:

- At first an approximate method using two interconnected straight paths is presented. This approach is used for the simpler approximations using the linearised or the semi linearised Euler equation.

- The second method uses the projection of the undisturbed path on the rotor plane for the integration of the near field pressure term. This approximation is used for the situations in which the particles path is determined by stepwise integration of the pressure field.

In the first approach it is thus assumed that the integration path consists of two "straight" paths instead of the path given by (9-20). In this approach the following assumption is made: One straight path is given by (9-20) and is valid for $t \leq -c(z_b)/(\Omega(R-b/2+z_b))$, and the other is given by (9-23), and is valid for $t \in (-c(z_b))/(\Omega(R-b/2+z_b)), 0]$. So the real path (9-20) of the particle of air in linearised theory is extended with a "straight" (in terms of local $x_b, y_b, z_b$ coordinates) path up to one chord length before the midchordline. Over this modified integration path the same assumption as before is taken,
stating that the accelerations found along the integration path are the same as those found along the real perturbed path.

The time-interval \((-c(z_b)/(\Omega(R-b/2+z_b)), 0)\) lies well within the final time interval \((t_1, 0)\), and so the initial assumptions still hold for all previously considered integration intervals.

The integration of the near field term of the basic pressure field over that last part of the final time-interval can take place analytically. Several steps are necessary and they include:
- First the substitution of the expression for \(I_{\text{basic}}\) in terms of the Legendre function \(P_n^1\).
- Second the sequence change of differentiation and integration including substitution of the relation between \(\Omega t\) and \(x_b\) according to the integration path given in (9-20).
- Third the substitution of the primitive function for the integrand.
- Fourth the evaluation of the primitive function.

The sequence of steps for the analytical integration is thus given by:

\[
0 \to \int_{c(z_b)}^{0} \frac{\partial}{\partial c(z_b)} \left[ -\frac{1}{\pi} \frac{I_{\text{basic}}(\frac{z_b}{b/2})}{\cos \eta + \cos \varphi} \right] \Omega dt =
\]

\[
= \frac{1}{\pi} \left[ 1 - \frac{c(z_b)^2}{b/2} \right] P_n^1(\frac{z_b}{b/2}) \left( \frac{b}{c(z_b)} \right) \int_{c(z_b)}^{0} \frac{\partial}{\partial y_b} \left[ -\frac{\sin \varphi}{\cosh \eta + \cos \varphi} \right] \Omega dt =
\]

\[
= \frac{1}{\pi} \left[ 1 - \frac{c(z_b)^2}{b/2} \right] P_n^1(\frac{z_b}{b/2}) \left( \frac{b}{c(z_b)} \right) \frac{c(z_b)^2}{2} \int_{c(z_b)}^{0} \frac{\partial}{\partial y_b} \left[ -\frac{\sin \varphi}{\cosh \eta + \cos \varphi} \right] \Omega dt =
\]

\[
= \frac{1}{\pi} \left[ 1 - \frac{c(z_b)^2}{b/2} \right] P_n^1(\frac{z_b}{b/2}) \left( \frac{b}{c(z_b)} \right) \frac{c(z_b)^2}{2} \sqrt{3} \int_{c(z_b)}^{0} \frac{\partial}{\partial y_b} \left[ -\frac{\sin \varphi}{\cosh \eta + \cos \varphi} \right] \Omega dt =
\]

\[
\text{for } |\epsilon| \to 0.
\]

(9-24)

With the assumption made of a slender rotor blade with a sufficiently large root cut-out, the final part of the projected integration path can indeed be approximated by the linear path given in (9-23) although the projection of the path (9-13) into the \(x_b-z_b\) plane (which is rotating with the rotor blade) is of course a circle.

It was considered to have a length equal to one chord length and covered the time-interval \((-c(z_b)/(\Omega(R-b/2+z_b)), 0)\).
But with a sufficiently slender rotor blade, not considering the locations close to the rotor centre, this "final path" may as well be extended to a distance of several times the chord, say over the time-interval \(-mc(z_b)/((\Omega(R-b/2+z_b)), 0)\).

Integration of the near field term of the basic pressure field then yields an integration analogous to (9-24). With the result now derived for the extended integration interval, it follows by substitution into the primitive function that the following approximation holds for sufficiently large \(m\):

\[
\int_{-\infty}^{0} -\frac{\partial}{\partial y_b} \left[ \frac{\sin\varphi}{\cosh\eta + \cos\varphi} \right] \Omega dt = \int_{-\infty}^{0} -\frac{\partial}{\partial y_b} \left[ \frac{\sin\varphi}{\cosh\eta + \cos\varphi} \right] \Omega dt = [-\frac{c(z_b)}{c(z_{bo})/2}] = -\frac{c(z_b)}{(R-b/2+z_{bo})}.
\]

With the interval length extended to \(t = -\infty\) the integration of the near field term, analogous to (9-24) thus yields:

\[
\int_{-\infty}^{0} \frac{\partial}{\partial z} \left[ -\frac{1}{\pi} \frac{l_{\text{basic}}(z_b)}{b/2} \frac{\sin\varphi}{\cosh\eta + \cos\varphi} \right] \Omega dt = \int_{-\infty}^{0} \frac{1}{\pi} \frac{l_{\text{basic}}(z_b)}{b/2} \frac{b}{1/2\rho W^2 c(z_{bo}) R - b/2 + z_{bo}} \Omega dt.
\]

For the interpretation of this result it is convenient to look first to the relative undisturbed flow at the rotor blade. This is the undisturbed velocity "felt" by an observer at the position \(z = z_{bo}\) on the midchordline and consists of a dominating component \(\Omega(R-b/2+z_b)\) in \(x_s\) - direction, and a minor component \(W\) in \(y_s\) - direction. The magnitude of the undisturbed velocity is therefore about equal to \(\Omega(R-b/2+z_b)\). With this in mind it follows for the lift coefficient \(c_{l, \text{basic}}\):

\[
c_{l,\text{basic}}(\frac{z_{bo}}{b/2}) = \frac{l_{\text{basic}}(\frac{z_{bo}}{b/2})}{1/2\rho \Omega^2(R-b/2+z_{bo})^2 c(z_{bo})}.
\]

The contribution to the inflow angle by the basic near field term is, according to the final
boundary condition (9-12) equal to:

\[
-(\frac{W}{\Omega R})^2 \frac{1}{Rb} \frac{1}{1/2\rho W^2} \frac{\partial \Omega}{\partial (\frac{z}{b/2})} = -\frac{1}{2\pi} c_{l, \text{basic}} (\frac{z_b}{b/2})
\]

(9-28)

This demonstrates that the near field pressure contribution over the time interval \((-mc(z_b)/(\Omega (R-b/2+z_b)), 0\) to the inflow angle is approximately equal to \(c_{l, \text{near}}/2\pi\).

The latter expression is a well known result for the angle of attack of a two dimensional flat plate aerofoil in uniform flow. It is also known as the effective angle of attack, or the angle of incidence, of a flat plate wing in uniform flow within the framework of Prandtl's lifting line theory, see section 6-2 and e.g. Schlichting [34]. It means that the integration of the near field pressure field along a straight path (with respect to the local \((x_p, y_p, z_p)\) coordinate system) results in the two dimensional relation between loads and velocities. Since the near pressure field is the solution of the two-dimensional problem close to the rotor blade, this seems an evident conclusion. It will however be shown in the remaining part of the present section that this conclusion is no longer valid for more realistic approximations of the integration path!

In the second approach for the integration of the near field pressure term it is assumed that the particles of air follow a circular path. This path is the projection of the undisturbed path in the rotor plane. With respect to the rotating \((x_p, y_p, z_p)\) coordinate system the projection of the undisturbed path is equivalent to a circle.

For this path the local coordinates satisfy:

\[
x_b = (R - \frac{b}{2} + z_{b_0}) \sin \Omega t
\]

\[
y_b = -\frac{c(z_b)}{2}
\]

\[
z_b = (R - \frac{b}{2} + z_{b_0}) \cos \Omega t - R + \frac{b}{2}
\]

(9-29)

where \(l \Omega t \to 0\) for the projection on the rotor plane.

It can be seen that the \(z_b\) coordinate of the path increases from \(-b/2\) at the time \(\Omega t = \arccos((R-b)/(R-b/2+z_{b_0}))\), to \(z_{b_0}\) at \(\Omega t = 0\). It can thus be seen that along the integration path (9-29) the near field term becomes 0 after a specific period, which is dependent upon the ultimate \(z_{b_0}\) coordinate on the rotor blade at \(\Omega t = 0\). Say that this happens after \(m_o\) intervals of a length \(\Delta \Omega t\). Then the integration of the near field term can be approximated by \(m_o\) integrals:
\[
\int_{\text{arcos} \left( \frac{-b}{R-b/z_{bo}} \right)}^{0} \frac{\partial}{\partial y_b} \left[ \frac{1}{1/2 \rho W^2 c(z_{bo})} \frac{\sin \varphi}{\cosh \eta + \cos \varphi} \right] d\Omega_t = 0
\]

\[
\frac{1}{n} \sum_{m=1}^{m_0} \int \frac{\partial}{\partial y_b} \left[ \frac{I_{\text{basic}}(x_{bo}/2)}{1/2 \rho W^2 c(z_{bo})} \frac{\sin \varphi}{\cosh \eta + \cos \varphi} \right] d\Omega_t = \]

Over the small time intervals \( \Delta\Omega_t \) the \( z_b \) coordinate is now assumed to be constant. Then the summation can be written as:

\[
\frac{1}{n} \sum_{m=1}^{m_0} \int \frac{\partial}{\partial y_b} \left[ \frac{I_{\text{basic}}(x_{bo}/2)}{1/2 \rho W^2 c(z_{bo})} \frac{\sin \varphi}{\cosh \eta + \cos \varphi} \right] d\Omega_t = \]

The nondimensional spanwise lift distribution is dependent on the \( z_b \) coordinate. This coordinate differs from integration interval to integration interval. Therefore the integration according to (9-31) will yield a different result compared with the integration according to (9-24).

According to (9-29) the integration takes place along a path with a very small \( y_b \) coordinate \( e c(z_{bo})/2 \). Therefore the last expression of equation (9-31) will always be regular. When \( \xi_1 \to 0 \) however the evaluation of (9-31) asks some more consideration.

At the blade \( \xi_1 \leq c(z_{bo})/2 \) the term between the brackets becomes 0.

Close to the leading edge of the blade, at \( \xi_1 = -c(z_{bo})/2 \) the term however shows a singular, though integrable behaviour. Therefore a special summation procedure should be used based upon integration intervals with decreasing length when approaching the leading edge.

The procedure is completely analogue to the Gauss-Chebychev numerical integration approach introduced in section 9-3 (equation (9-18)), and is used in the implementation of equation (9-31) in the actual numerical codes, see chapters 10 and 11.
In the sections 10-1 and 12-2 more emphasis will be paid to the differences resulting from the application of both approaches described above. However some qualitative remarks can be made here. To some extent the calculation of the near field contribution according the second approach of (9-31) can be compared with the calculation of loads on a skewed or swept wing. The integration of the near field term for such cases would also imply the inclusion of a spanwise location dependent amplitude just as in (9-31). The difference is of course the path of the particles. For swept (or skewed) wings the linearised path is straight (but not perpendicular to the wing), whereas in the above approximation the paths of the particles arriving at the rotor blade \( t = 0 \) are parts of concentric circles. But the local \( z_b \) coordinate of the particles of air travelling to the lifting surface (wing or rotor blade) in all these cases is dependent of the time, even for small \( t \). Since the load on the lifting surface in general depends upon the spanwise position it cannot be expected that the integration of the near field term yields the two dimensional relation found in equation (9-28).

9-5 The multibladed rotor.

Up to now the pressure field of one rotor blade has been derived. In most practical situations however the pressure-field and its related lift distribution are demanded for a rotor having \( B_o \) blades, evenly distributed over the rotor disc. But, by the fact that the Laplace equation (9-1) for the pressure-field is linear, this pressure-field can be constructed from the one-bladed configuration. In order to show this the following notation is introduced:

The pressure-field of one rotating rotor blade, at the time \( t=0 \) being at an azimuthal position \( \psi = \psi_o \) is written as \( p^{(0)}(x, t, \psi_o) \). Thus the pressure-field derived in section 9-1 is thus written as \( p^{(0)}(x, t, \pi/2) \).

Now suppose that the pressure-field must also be derived for one rotor blade, at the time \( t = 0 \) situated at azimuthal position \( \psi = -\pi/2 \). Then this derivation is completely analogous to the one performed in the previous chapters, thus leading to the pressure field \( p^{(1)}(x, t, -\pi/2) \). The pressure field

\[
p^{(1)}(x, t) = p^{(0)}(x, t, \frac{\pi}{2}) + p^{(0)}(x, t, -\frac{\pi}{2})
\]

(9-32)

then yields a competent candidate for the pressure-field of a two bladed rotor. By the linearity of the Laplace equation (9-1) it is evident that it is also satisfied by the pressure field (9-32). The first boundary condition (9-3) does not either give problems, because it is also linear. The second and third boundary condition (equations (9-8) and (9-9)) ask for more consideration.

The second boundary condition states that the pressure gradient the wind direction (z-direction) must be zero at the rotor blades. Of course it is true that both the components
satisfy this condition on their 'own' rotor blade. The problem is thus reduced to the question:

Is the gradient of \( p^{(1)}(x, t, -\pi/2) \) in the wind direction zero at the rotor blade at position \( \psi = \pi/2 \) at \( t = 0 \), and vice versa?

Since rotor blade 1 is completely situated in the far field of the pressure distribution of rotor blade 2, and vice versa, the question is equivalent to:

Is the pressure gradient in z-direction of the far field part of \( p_{\text{far}}(x, t, -\pi/2) \) zero at the rotor blade with azimuthal position \( \psi = \pi/2 \) at \( t = 0 \)?

From the presence of the \( \sin \chi \) function it can be seen that the expression for \( p^{(1)}_{\text{far}} \) is antisymmetric with respect to the rotor plane. Its derivative in z-direction (the direction normal to the rotor plane) is therefore symmetric, and thus in general not equal to zero in the rotor plane. Although it may be expected that at the location of rotor blade 1 the pressure gradient due to the pressure field of rotor blade 2 is small it needs some more consideration.

From chapter 7, equation (7-23), it is known that the far pressure field part can, in its general form, be written as

\[
\frac{1}{2} \frac{\partial p^{(1)}_{\text{far}}(x, t, -\pi/2)}{\rho W^2} = \frac{1}{\pi} \sum_{n=1}^{\infty} A_n(t) P_n^1(\cos \theta_2) \sin \chi_2 \sinh v_2 \sin \theta_2 Q_n^1(\cosh v_2) \sin \chi_2
\]  

\[ (9-33) \]

where the index 2 denotes the prolate spheroidal coordinate system related to the rotor blade present at position \( \psi = \pi/2 \) at \( t = 0 \).

In appendix D the partial derivative with respect to \( z \) is derived in terms of prolate coordinates. Since it is assumed that the rotor blade is represented in the present theory by its projection on the rotor plane \( (\chi_2 = 0 \mod \pi) \) it follows from appendix D by substitution of (9-33) into equation (D-6):

\[
\frac{1}{2} \frac{\partial p^{(1)}_{\text{far}}(x, t, -\pi/2)}{\rho W^2} = \frac{1}{\pi \sinh v_2 \sin \theta_2} \sum_{n=1}^{\infty} A_n(t) P_n^1(\cos \theta_2) \sin \chi_2 \sinh v_2 \sin \theta_2 Q_n^1(\cosh v_2)
\]  

\[ (9-34) \]

and this is indeed in general not equal to zero.

This would imply that it is not possible to simply add the pressure fields of both rotor blades and so obtain the correct expression for the complete configuration. But at this moment it must be realised that \( p \) is an approximate solution of the problem. The axial pressure variations at rotor blade 1, caused by the introduction of rotor blade 2 are not zero, as can be seen from (9-34). But still they are small compared with the variations in the pressure field of rotor blade 1 in the direct environment of the blade itself.
At the blade the composite pressure field is equivalent to its near field. The axial variation in the pressure field of rotor blade 1 at the rotor blade itself can thus be written like:

\[
\frac{1}{2} \frac{\partial p^{(1)}_{\text{near}}(x,t,-\pi/2)}{\partial \left( \frac{z}{b/2} \right) } = \begin{cases} 
- \frac{1}{\pi} \frac{b}{c} \frac{l(z_b,t)}{1 - \rho W^2 c(z_b)} & x_b = -\frac{c}{2} \\
0 & |x_b| \leq \frac{c}{2} \\
- \frac{1}{\pi} \frac{b}{c} \frac{l(z_b,t)}{1 - \rho W^2 c(z_b)} & x_b > \frac{c}{2}
\end{cases} \tag{9.35}
\]

which shows a singularity at \( x_b = -\frac{c}{2} \).

This is the well known leading edge singularity at the flat plate aerofoil, and this singularity dominates the local pressure field completely. Therefore the far field pressure disturbances of one blade are, at the location of the other blade(s), small with respect to the variations in pressure field around the blade, and may be neglected in a first order approximation. Since its actual value exactly at the rotor blade is not zero it must however be realised that this effect needs further consideration and may no longer be neglected as soon as higher order asymptotic approximations are made!

The third boundary condition states that there must be a tangential flow along the rotor blade:

\[
\frac{w}{W} = \frac{\theta_r(z_b,t)}{\Omega R} \frac{\Omega R}{r} \quad \text{on the rotor blade} \quad . \tag{9.36}
\]

As was shown in section 9-1 this final condition can be written like:

\[
- \frac{(W/r)^2}{\Omega R} \frac{R}{r} \frac{1}{\partial \left( \frac{z}{b/2} \right) } \int_0^\infty \frac{\partial p^{(1)}}{\partial \left( \frac{z}{b/2} \right) } \Omega \, dt + \left( \frac{W}{\Omega R} \right) \left( \frac{r}{R} \right) = \frac{\theta_r(z_b,t)}{\Omega R} \tag{9.37}
\]

for a particle of air arriving at the midchordline of the turbineblade at \( t = 0 \).

Here the integration should take place along the path of the considered particle of air. As mentioned in the beginning of this chapter there are several possibilities for the choice of the integration path. According to linearised theory the paths can be assumed straight and travelled with a constant velocity. Then these paths do not depend upon the form of the composite field (the real path's do!) and so the addition of two (or more) pressure-fields for generating the complete expression does not give problems on this point. But also in the situation of a semi linearised path, or following the particle along its real perturbed path (a real Euler integration), the evaluation of the integral (9.37) for the
combined pressure field (9.32), necessary for the fulfilment of the final boundary condition, can be established without any problem.

For the numerical integration of the gradient of the pressure distribution \( p^{(1)} \) it is convenient to cut the integration path into pieces of a length \( \Delta t = 2 \pi/(\Omega W) \). In this way the beginning and end of each interval can be synchronised with the extrema in the value of \( \partial p / \partial \tau \), caused by the passage of the pressure singularity on the rotor blade. It will be evident that in the multi bladed situation the length of the (numerical) integration intervals reduces reciprocal with increasing number of blades. For a \( B_\theta \)-bladed rotor, the length of the intervals becomes \( \Delta t = 2 \pi/(\Omega WB_\theta) \). In the semi linearised situation the undisturbed wind velocity \( W \) is replaced by \( W + W' \) where \( W' \) is the induced velocity at the rotor blade determined in the previous iteration step.

This leads to the conclusion that indeed the pressure field defined in (9.1) is the right expression for the (first order approximate) pressure field of a two bladed rotor. The pressure field for multi bladed rotors follows in a similar way, and the modification of the numerical integration method is straightforward.
10 The calculation of loads, velocities and angles

Introduction

In the previous chapter the pressure field for the wind turbine rotor was derived. With such a pressure field it is possible to determine the accelerations of the particles of air everywhere around the considered wind turbine rotor. Integration of the accelerations along the paths of the particles then determines their velocities. Thus the velocities of the particles of air can also be calculated everywhere in the field. The present method used to derive acceleration potential solutions makes it possible to give a thorough definition of the induced velocity distribution at the rotor blade. Then detailed comparisons can be made with more classical methods using blade-element momentum theory.

With a given pressure distribution over the rotor blade it is furthermore easy to calculate the loads and moments on the blade. The loads determined in such a way are however obtained from basic equations valid for inviscid flow. These inviscid results have to be corrected to obtain loads which can be compared to experimental results. In the present chapter more details are given concerning the way in which loads on the rotor blades velocities and angles are determined. Besides that a method is presented to estimate the magnitude of the higher order correction on the blade loads once the first order result is known. Finally the definitions of a number of global wind turbine characteristics will be introduced in this chapter.

10-1 Inviscid loads and inflow angles

Once the coefficients $A_n (t)$ are determined according to the above described procedure the spanwise load over the rotor blade is given by equation (9-11).

Figure 21 shows an example of spanwise lift coefficient distributions generated with a version of the numerical code described later on in section 12-1.

The first order chordwise load distribution is equal to the theoretical result for the flat plate. Within the near field representation used in the present method (see the first term on
Figure 21: Example of spanwise lift coefficient distributions.

Figure 22: Example of chordwise load distributions at several radial stations.
the r.h.s. of (9-10)) the chordwise pressure distribution can be obtained from:

\[
\frac{P^{(1)}}{\frac{1}{2} \rho W^2} = -\frac{1}{\pi} \frac{l(z_{b,t})}{\frac{1}{2} \rho W^2 c(z_b)} \frac{\sin \varphi}{(\cosh \eta + \cos \varphi)}. \tag{10-1}
\]

The chordwise load distribution can now be obtained as the difference between the pressure on both sides of the blade. A typical example of the chordwise load distributions used in the acceleration potential method is given in figure 22.

With the definition of the local elliptical coordinate system (4-3) and substituting \( |x_b| < \frac{1}{2} c(z_b); \ y_b = +0 \) (the suction side, \( 0 < \phi < \pi \)), into (10-1) yields:

\[
\frac{P^{(1)}}{\frac{1}{2} \rho W^2} = -\frac{1}{\pi} \frac{l(z_{b,t})}{\frac{1}{2} \rho W^2 c(z_b)} \sqrt{1 - \left(\frac{x_b}{c(z_b)}\right)^2} \tag{10-2}
\]

which is the pressure distribution in rectangular coordinates.

The pressure distribution on the pressure side (\( \pi < \phi < 2\pi \)) is given by equation (10-2) as well, but then with an opposite sign.

The velocity potential \( \Phi \) corresponding to the near pressure field (10-1) is given by:

\[
\frac{\Phi}{Wc(z_b)} = -\frac{1}{\pi} \frac{l(z_{b,t})}{\frac{1}{2} \rho W^2 c(z_b)} (e^{-\eta \sin \varphi} - \varphi) \tag{10-3}
\]

which is the well known (two dimensional) potential for the Birnbaum velocity distribution over a flat plate under an angle of attack \( \alpha \), see e.g. Schlichting [34]. For the angle of attack \( \alpha \) associated to (10-3) the following relation holds:

\[
\alpha(z_{b,t}) = -\frac{1}{2\pi} \frac{l(z_{b,t})}{\frac{1}{2} \rho W^2 c(z_b)}. \tag{10-4}
\]

In the three dimensional situation the definition of an angle of attack is not straightforward.

For the situation of a wing in uniform perpendicular flow the Prandtl hypothesis is usually adopted, see e.g. Schlichting et al [34]. Suppose the wing is situated in the uniform flow under a geometrical angle of attack \( \alpha_{geo} \), determined by the undisturbed relative inflow velocity and the pitch of the wing.

Under this hypothesis it is then assumed that the characteristics of a specific wing section
are identical to the 2-d characteristics of the considered aerofoil under an effective angle of attack which equals $\alpha_{geo} - \alpha_{ind}$. The induced angle $\alpha_{ind}$ is attributed to the effect of the outer flow (related to the 3-D geometry) on the local 2-d flow, which induces velocity perturbations at the wing. For the wing in uniform flow the induced velocities and thus the induced angle $\alpha_{ind}$ can be calculated from the induced velocities in the Trefftz plane, see chapter 6. The local load on the wing is taken perpendicular to the direction of the effective flow. This reduction of the geometric flow angle to the effective angle, and the corresponding tilting of the load vector are the only effects of the 3-D flow taken into account within the Prandtl hypothesis.

The geometrical angle of attack $\alpha_{geo}$ for the situation of a wind turbine rotor blade is determined by the undisturbed relative inflow velocity and the local pitch angle $\theta_p$, see figure 20. For the application of the Prandtl hypothesis for rotor blades it is necessary to determine the induced velocities and thus the induced angle $\alpha_{ind}$ "at the blade".

These induced velocities can be obtained in different ways, yielding different results as will be shown later. The most frequently used method is the application of global momentum theory. For annular stream tubes passing through the rotor plane an assumed (local) rotor load is, by momentum considerations, related to induction velocities in the rotor plane. This can only be achieved when the local rotor load is assumed to be equal to the global load on the annulus. With the induced velocities obtained from momentum theory the induced angle, and thus the effective angle $\alpha_{geo} - \alpha_{ind}$ is determined. Then the left hand side of equation (10-4) is replaced by this effective angle which yields an improved estimate for the local rotor load. This combination of momentum and "blade element theory" is known as Blade Element Momentum theory (BEM-theory), see Glaubert [18] or Wilson et al [40]. Some methods have furthermore been developed to modify the averaged induced velocities from momentum theory for the presence of a finite number of blades. They are usually presented as a "tip-correction factor" such as the most commonly used Prandtl-tip correction factor, see e.g. Wilson et al [40]. The naming of these factors is unfortunately very misleading, since they intend to correct for the finiteness of the number of blades and not primarily for the finiteness of the blade itself.

The theoretical model developed so far in the present study makes it is however possible to calculate the effective angle and the induced angle in a more scrutinised way by directly using the components of expression (9-10):

- Integration of the near field term of this expression (the first term) yields the effective angle $\alpha(z_{as}, t)$.
- The induced angle is determined by integration of the common field and the far field terms (2nd and 3rd term of expression (9-10)).

The addition of the effective angle and the induced angle should then be equal to the geometrical angle of attack $\alpha_{geo}$ see figure 20, in order to fulfil the final boundary condition (9-12).
In Van Holten [24] it was shown that in the case of an uncambered rectangular wing in uniform perpendicular flow the integration of the first order near field expression results into an effective angle which satisfies equation (10-4). This shows that for that situation the present method and the Prandtl hypothesis are equivalent. Furthermore it was pointed out in section 9-4 that the integration of the near field term along a straight path with respect to the rotor blade according to equation (9-24) yields an equivalent result.

Application of the above described integration method is straightforward in the case of a wind turbine rotor blade. It does not need any further hypothesis and is easy to implement. The physical interpretation of the method using the integration of the far and common field expressions for the determination of induced angles and velocities is in agreement with Prandtl's ideas. With these terms in the pressure field representation the direction and the magnitude of the "outer" flow velocity at the rotor blade is determined. This velocity plays an important role in the determination of the local loads on the blade. Especially its direction is important since, in potential theory, the local load is directed perpendicular to this velocity.

In section 12-2 it will be shown that for rotor blades the calculation of effective angles from the near field term integration, and from the method using equation (10-4) give different results, especially in the blade root area. This demonstrates that in the rotor blade situation a local 2-d flow does not exist! So the two dimensional relation (10-4) is not valid and the Prandtl hypothesis cannot be applied! The difference can be attributed to a kind of "swept flow" effect caused by the more or less circular inflow path w.r. to the blade. Since the load of the root section of the blade is in general less than the load on the outboard part, the particles travelling along such paths do in general exhibit less induction from the blade load than found along a straight path perpendicular to the blade. This leads to larger effective angles on the majority of the blade, and has therefore an important consequence for especially the in plane components of the loads.

10-2 An estimate for the higher order effects upon the loads

Having determined the coefficients $A_n(t)$ for the first order solution it is possible to estimate the effects of a second order approximation of the full solution to the loads on the blade.

For this estimate the expressions are used which are derived in section 7-5 where the higher order solutions were treated. It was concluded that in the higher order expansion a virtual camber terms enters the near field solution. Furthermore, in equation (7-41), a relation was found between the coefficients $A_n$ and the coefficient $b_0$ of the virtual camber term. So once the coefficients $A_n$ are known it is possible to calculate the modification of the blade load corresponding to the higher order term of the pressure field representing the virtual camber:
\[ \frac{P_{\text{camber}}^{(2)}}{\frac{1}{2} \rho W^2} = \frac{1}{\pi} b_1(\frac{z_b}{b/2}, \phi) e^{-n} \sin \phi. \]  
(10-5)

Such virtual camber results at the blade in a pressure difference. The expression for the 
pressure difference in rectangular coordinates is found with the use of the local elliptical 
coordinate system (4-3) by substituting \(|x_b| < \frac{1}{2} c(z_b); y_b = +0 \) (the suction side, 
\(0 < \phi < \pi\)), into (10-5):

\[ \frac{P_{\text{camber}}^{(2)}}{\frac{1}{2} \rho W^2} = \frac{1}{\pi} b_1(\frac{z_b}{b/2}, \phi) \sqrt{1 - \left(\frac{x_b}{c(z_b)/2}\right)^2}. \]  
(10-6)

Equation (10-6) shows that the virtual camber function indeed adds a parabolic load 
distribution in chordwise direction over the rotor blade.

Equation (10-6) does not represent the total effect of the complete second order solution of 
the problem \( p^{(2)} \) on the load: The coefficients have been determined by application of the 
final boundary condition through an integration of the gradient of the first order pressure 
distribution \( p^{(1)} \), and this is numerically not the same as integration of the gradient of the 
second order solution \( p^{(2)} \). In an asymptotical sense this expression is however correct.

Equation (10-6) does however give a good impression of the magnitude of the extra 
parabolic load distribution and thus of the modification of the loads through second order 
effects. It can therefore be used in a simple way to decide whether or not it is necessary to 
develop a second order solution of the considered problem.

10-3 Modification of the loads due to viscosity

Load distributions, such as the lift distribution determined in the previous chapter are 
obtained from basic equations valid for inviscid flow. When comparisons with 
experimental results are made it is necessary to introduce a viscous correction procedure 
for the load on the rotor blade. From wing theory a method can be adopted to modify the 
inviscid results. Such method is applicable for situations of moderate load, where the 
aerofoil sections are operating in an unstalled mode. For wind turbines however it is 
necessary to develop a method for stalled conditions as well, since those occur very 
frequently during operation. It should be emphasised here that the modifications due to 
viscosity used in the present method are a posteriori corrections. Or in other words, first a 
complete inviscid solution is obtained and after that the inviscid loads are corrected for 
viscosity. There is no viscous-inviscid interaction procedure within the method which would 
treat the viscous corrections in a more thorough way.
In Van Bussel [13] a method was selected which was found to be most suitable for the wind turbine application. The method uses the local inviscid lift coefficient value \( C_l(z_b) \) to determine an angle according to:

\[
\alpha_{2d}(z_{b,t}) = \frac{C_l(z_{b,t})}{2\pi} = \frac{1}{2\pi} \frac{l(z_{b,t})}{\mu(\dot{w}^2 + (\Omega r)^2)c(z_b)}. \tag{10-7}
\]

It must be emphasised here that this "two-dimensional angle" \( \alpha_{2d} \) has lost its physical meaning for wind turbine rotor blade applications, in view of the explanation given in section 10-1.

With this two-dimensional angle \( \alpha_{2d} \) the corresponding (viscous) lift and drag coefficients are determined using aerofoil data. Usually the data are taken from wind tunnel measurements, such as presented in Abbott et al. [1], and Timmer [38], but it is also possible to use the results of numerical codes for the determination of aerofoil characteristics, e.g. Drela [16].

When the calculated angle \( \alpha_{2d} \) falls outside the range of angles at which the viscous lift and drag coefficients are known, an approach is followed, which makes use of the theoretical behaviour of a flat plate aerofoil at large angles. Either the flat plate behaviour itself is used, or an interpolation is made between the known aerofoil values and this flat plate behaviour.

The theoretical flat plate behaviour for large angles can be described by the following formulas:

\[
C_l\left(\frac{z_b}{b/2}\right) = C_N \sin^2 \alpha, \tag{10-8}
\]
\[
C_d\left(\frac{z_b}{b/2}\right) = C_N \sin \alpha \cos \alpha,
\]

where \( C_N \) denotes the force coefficient of a flat plate perpendicular to the flow.

The above deep stall model is a simple representation in which the normal force on the flat plate is taken into account only. For an infinite long flat plate (the two dimensional situation) at the relevant Reynolds numbers for wind turbine rotors the value of that normal force coefficient equals \( C_N = 2.2 \), see e.g. Hoerner [23].

The interpolation procedure between the given values for \( C_l \) and \( C_d \) (either from measurements or from numerical codes for the determination of aerofoil characteristics) to the values given by (10-8) is active for values of the calculated angle of attack \( \alpha_{2d} \) which satisfy \( |\alpha_{2d}| < 0.15\pi \). The value 0.15\pi is chosen in a rather arbitrary way. Usually characteristics of aerofoils are provided up to angles in the order of 12 to 15 degrees \( (\alpha_{2d} = 0.07 \text{ to } 0.08) \). Twice this value is normally large enough for the two dimensional aerofoil to operate in deep stall, see e.g. Abbott et al [1]. The most simple interpolation, using a linear relation is then adopted between the last measured (or calculated) value for \( C_l \) and \( C_d \) and the values \( C_N \sin^2(0.15\pi) \) and \( C_N \sin(0.15\pi)\cos(0.15\pi) \) respectively.

Figure 23 shows a situation in which the \( C_l - \alpha \) and \( C_d - \alpha \) curves for a given aerofoil
section are known from measurements over the complete range of angles of attack. It shows that the differences in the measured curves, and the curves obtained by application of the above described procedure are limited.

Figure 23: $C_t - \alpha$ and $C_d - \alpha$ curves for a given aerofoil compared with result of approximation procedure.
10-4 Wind turbine characteristics

When the loads on a wind turbine rotor are determined they are usually presented in a way which is characteristic for wind turbine aerodynamics, see e.g. Betz [4], Glauert [18] or Wilson [40].

An important parameter in these presentations is the tipspeed ratio $\lambda$ defined as:

$$\lambda = \frac{\Omega R}{W}.$$  \hspace{1cm} (10-9)

With $\lambda$ as parameter the global characteristics are usually displayed in terms of $C_p - \lambda$
$C_{D_{ax}} - \lambda$ and $C_T - \lambda$ plots, where the following definitions are used:

For the power coefficient:

$$C_p = \frac{P}{{\frac{1}{2}\rho W^3\pi R^2}},$$  \hspace{1cm} (10-10)

for the axial force coefficient:

$$C_{D_{ax}} = \frac{D_{ax}}{{\frac{1}{2}\rho W^2\pi R^2}},$$  \hspace{1cm} (10-11)

and for the torque coefficient:

$$C_T = \frac{T}{{\frac{1}{2}\rho W^2\pi R^3}}.$$  \hspace{1cm} (10-12)

The power $P$ is obtained by multiplication of the rotor shaft torque $T$ with the angular speed $\Omega$, so that the following relation is valid: $C_p = C_T \lambda$.

This means that the global characteristics of a rotor with given geometry are determined with two of the three above mentioned relations, usually the $C_p - \lambda$ and the $C_{D_{ax}} - \lambda$ curves.

From global momentum theory, see Betz [4], Glauert[18], Wilson [40], it is known that the maximum power coefficient $C_p$ for wind turbines exerting forces only in the direction opposite to the wind does not exceed the value $C_{p_{max}} = 16/27$. The axial force coefficient $C_{D_{ax}}$ equals 8/9 for maximum power and satisfies $C_{D_{ax}} \leq 1$ according to global momentum theory.

With the present method the spanwise load distribution is given by equation (9-11). Integration of the far and common field pressure gradients determines the induced velocities at the rotor blade, as explained in section 10-2. Then the effective angle is
known and with it the direction of the local load, which is perpendicular to the effective inflow direction.

The local load can then be decomposed into an axial and a tangential component. Integration of the axial component and addition over all the blades determines the axial force $D_{ax}$. Integration of the moment of the tangential component and addition over all the blades determines the torque $T$. Examples of $C_{p}-\lambda$ and $C_{Dar}-\lambda$ curves calculated with the present method will be presented in section 12-1 (figures 26, 27, 31 and 32).

For comparisons with other methods or with experimental results it is sometimes necessary to determine other quantities than the global characteristics introduced above, such as spanwise load and moment distributions. Since both spanwise and chordwise load distributions are known with the present method and furthermore a consistent method for determining the local flow angles has been defined it is easy to derive those quantities. In chapter 12 some examples will be given.
11 Implementation of the theoretical wind turbine model into numerical codes.

Introduction

Once the general expression for the pressure field is known, a suitable integration procedure is established for the pressure gradients and a method for viscous correction of the inviscid solution is determined the implementation into computer codes can be established. In this implementation process some more assumptions and choices have to be made.

These choices are treated in more detail in the present chapter. They comprise the implementation of aerofoil characteristics and the choices made for the iterative integration of the pressure gradients. Furthermore some special pre-processing considerations have to be taken into account for dynamic conditions. In such conditions the loads, the operational conditions and/or the geometry are time dependent.

In three consecutive sections the implementation of the final boundary condition for different situations will be discussed. In section 11-3, the stationary situations will be considered. The implementation of the formulas for axisymmetric dynamic situations is discussed in section 11-4. It will then be possible to determine loads on rotor blades during pitching excursions, during start up procedures and safety stops.

In section 11-5 the application of the theoretical model for yawed flow conditions is treated. This is the situation where the rotor axis is no longer parallel to the direction of the oncoming undisturbed wind. This will of course result in periodic load variations on the rotor blades.

11-1 Implementation of aerofoil characteristics

In chapter 10 a method was presented for the implementation of viscous aerofoil characteristics in the inviscid acceleration potential model for the wind turbine rotor.

In the model implemented in the present codes a flat plate aerofoil is assumed, although camber could have been introduced as well.

Specific aerofoil series used for wind turbine rotor blades are however in general rather close to (viscous) flat plate behaviour. In cases where the zero lift angle $\alpha_0$ is not equal to 0 the geometry used in the calculations is modified by adding the zero lift angle $\alpha_0$ to the actual geometry.

From the inviscid lift coefficient determined with the code a two dimensional angle $\alpha_{zd}$ is calculated according to equation (10-7). As explained in chapter 10 this angle does not have any physical meaning. It is just used as a parameter for the inclusion of the
corresponding two dimensional viscous aerofoil characteristics into the calculations. The
angle $\alpha_2 + \alpha_1$ is used to determine the viscous lift and drag coefficient according to the
two dimensional aerofoil data.

Figure 24: Spanwise $C_t$ - distribution, Tjæreborg geometry (table 1). Top: inviscid result, bottom: after application viscous correction procedure CDALCX.
In case the angle falls outside the range of 2-d data available a large angle flat plate viscous behaviour, or an interpolation procedure between both is used. This was already described in section 10-3.

The algorithm is implemented into a numerical subroutine called CDALCX and more details can be found in Van Bussel [13].

It must be emphasised that the subroutine CDALCX uses measured values whenever they are provided in the input table. In figure 23 however the measured data above stall were deliberately discarded, in order to make the comparison possible between the synthesised curves and the measured curves.

Finally it should be mentioned that in figure 23 a maximum $C_d$ value of 2.2 is adopted. This is the maximum $C_d$ value for an infinite strip at the relevant Reynolds number, see Hoerner [23] and should thus be used for comparison with experimental aerofoil data obtained in a windtunnel.

In the actual application for numerical calculations for wind turbines the maximum value for $C_d$ is reduced to a value in the order of 1.8, depending upon the slenderness of the rotor blade. The values are obtained from flat plate drag values with corresponding aspect ratio which can also be found in Hoerner [23].

Figure 24 shows the effect upon the $C_l$ distribution for the Tjæreborg configuration given in table 1.
The top part of the figure shows the acceleration potential result for the spanwise $C_l$ distribution at a tipspeed ratio $\lambda = 8$ for different blade pitch settings. The bottom part of the figure shows the result after application of the subroutine CDALCX.

A slight decrease in the $C_l$ value can be observed due to the fact that the lift curve slope of the actual (viscous) aerofoil is slightly less then $2\pi$, the inviscid potential value for a flat plate aerofoil.

Near the root region the reduction of the $C_l$ value is more severe. This is caused by the fact that the inviscid $C_l$ values reach the value for which the aerofoil stalls.

**11-2 The implementation of the kinematic boundary condition.**

The implementation of the equations from chapter 9 into numerical codes deals with the determination of the coefficients $A_n(t)$ in the equations (9-10) and (9-11) as well as the calculation of induced velocities at the rotor blade described in the previous chapter.

The coefficients $A_n(t)$ are determined by solving the equation (9-12), representing the boundary condition for tangential flow along the rotor blades. This equation (9-12) however cannot be solved analytically in the three dimensional rotor blade situation which is presently dealt with. Therefore numerical procedures have to be applied.

The integration of the pressure gradients was already discussed in the sections 9-2 and 9-3. The pressure gradients must be integrated along the paths of the particles of air travelling
to the rotor blades. In order to determine the induced velocities at the blade the contributions of the far and common field expressions are determined parallel with the near field contribution.

Since a priori nothing is known with respect to the paths of these particles (the coefficients $A_n(t)$ are not known) a start up process has to be defined. In a first numerical approximation the paths of the particles of air are assumed to be straight and unperturbed. Furthermore it is initially assumed that the track of these particles of air is travelled with an imposed constant speed.

For the numerical procedures a finite number of points must be chosen on the rotor blade where the boundary condition will be fulfilled. The choice of the number of these collocation points $N_0$ then also determines the number of coefficients $A_n(t)$ taken into account. This can be seen when it is realised that each basic pressure distribution (a pressure distribution with $A_n = 1$ for a given $n = n_1$ and $A_n = 0$ for $n \neq n_1$) induces velocities at all $N_0$ collocation points. Taking $N_0$ different basic pressure distributions thus leads to $N_0$ induced velocities at the $N_0$ collocation points. Satisfying the kinematic boundary condition in the collocation points then is equivalent to solving a set of $N_0$ equations with $N_0$ unknown coefficients $A_n(t)$. For the dynamic cases the procedure has to be applied at each time step.

A systematic survey of the number of collocation points and their spanwise positions is performed in Van Bussel [9]. It turned out that in general 10 spanwise collocation points was sufficient for accurate calculations. Since the first order calculations do not allow for chordwise modification of the load distribution no further chordwise collocation points are added. With the 10 collocation points evenly distributed over the span (at spanwise locations $z/(b/2) = -0.9, -0.7, -0.5, -0.3, -0.1, 0.1, 0.3, 0.5, 0.7$ and $0.9$) it turns out that the coefficients in the matrix are all of about equal order of magnitude. More information can be found in Van Bussel [9] and [13].

The iterative process is in general semi linear. This means that once a set of coefficients $A_n(t)$ is obtained these coefficients are used in the pressure field to determine the paths of the particles arriving at the collocation points on the rotor blade at the time $t$. This yields a new set of coefficients $A_n(t)$ and with this new set completely new paths of the particles of air are determined. More details of the procedure determining the paths of the particles of air in dynamic conditions are given in section 11-4.

In the following sections the implementation of the final boundary condition for different situations will be discussed. The situations where the coefficients $A_n$ are independent of time will be considered first. The load distribution over the blade is thus a solely a function of the position on the blade.

Next the implementation of the formulas for axisymmetric unsteady situations is discussed. With the numerical code elaborated in this section it is possible to determine loads on rotor blades during pitching excursions, during start up procedures and safety stops. In general these kind of conditions will be referred to as dynamic inflow situations. The unsteadiness in these kind of situations is assumed to be such that it does not directly interfere with unsteady aerofoil behaviour. By definition this is the case when the pressure distribution over the rotor blade can still be written according to (9-10).
Finally the application of the theoretical model for yawed flow conditions is treated. Although the formulas of chapter 9 have been derived for axial flow conditions only, it is relatively easy to implement conditions where the direction of the undisturbed oncoming wind is not parallel to the rotor axis. As long as the yaw angle, which is defined as the angle between rotor axis and the direction of the oncoming wind is small the small perturbation assumptions remain valid. Thus the equations (9-10) and (9-11) remain valid as well. The final boundary condition (9-12) is however modified somewhat, since the geometrical angle of attack \( \alpha_{\text{geo}} \) has become azimuth dependent.

11-3 Numerical codes for steady calculations.

In its simplest implementation for the calculation of loads on the rotor blades (the code PREDICHAT1, see Van Bussel [13], an iterative procedure is developed for the calculation of the stationary coefficients \( A_n \). It starts from an assumed straight, unperturbed path, which is travelled by the particles of air with an imposed constant speed. For the speed the value \( 0.6667 \times W \) is used, the optimum value in the rotor plane determined by axial momentum theory, see e.g. Betz [4] or Wilson et al [40]. The accelerations experienced during the travel are integrated in order to obtain the velocities at the collocation points. This yields a first guess of the stationary coefficients \( A_n \).

In fact this procedure is a special application of the semi-linearised expression (5-30). The perturbation velocity \( W' \) is in such case constant, and equal to \(-0.33333 \times W\). Then the expression can be reduced to:

\[
\mathbf{w}(r_0) = W + W' + w' = 0.6667 \times W + w'(r_0) = -\frac{1}{\rho W_{\infty}} \int_{-\infty}^{r_0} \frac{\partial p}{\partial z} dz
\]  

(11-1)

for the velocity in the \( z \) direction.

It is clear that the assumption of an imposed constant speed is the least conflicting when this constant speed is approximately equal to the real actual speed. By choosing the speed as close as possible to the actual speed at the rotor plane, where the pressure gradients are the largest, the calculation according to (11-1) along the \( z \)-axis with that speed will yield a reasonable approximation.

With this first guess for the coefficients \( A_n \) the pressure field is now well determined. From the pressure field the perturbed (axial) velocities of the particles travelling to the collocation points can be calculated through integration of the accelerations. Within the code PREDICHAT1 the convective velocity of the particles of air is kept constant along the straight unperturbed path, although its value is collocation point dependent. In repeating this iterative procedure the ultimate stationary coefficients are determined. Details of the implementation of the present method into the code PREDICHAT1 can be found in Van Bussel [13].
In the velocity potential method the rotor blade is represented by a system of bound and trailing vorticity, see e.g. Afjeh et al [2], Simoes et al [35], Bareiss et al [3]. In terms of such vortex wake models this approach is equivalent to a deformed vortex wake, where the axial velocity of the wake is constant, but radially dependent. The wake radius is kept equal to the rotor radius, thus ignoring any wake divergence.

In a more elaborated version (PREDICCHAT2) the option of a perturbed path with a time depended velocity can be chosen. Essentially it does not differ too much from the PREDICCHAT1 approach. The first iteration is in fact identical. But from the second iteration step onwards it takes more "bookkeeping". Furthermore the integration of the pressure field together with the determination of the perturbed path takes place "backwards", i.e. travelling back in time and away from the considered collocation point because the starting position of a particle of air at \( t = - \infty \) is a priory not known.

The path and the velocities of the particles of air determined in the \( i \)th iteration step are used to calculate the accelerations in the \((i+1)\)th iteration experienced from the \(i\)th pressure distribution. This leads to modifications in the path and the velocities and thus to the next iteration step. Iteration is continued until convergence is established based upon the velocities calculated at the collocation points (i.e. the calculated velocities must match the pressure distribution). In fact the convergence criterium is applied with respect to the velocity at \( t = - \infty \), which is in the numerical code of course replaced by \( t_{so} = - \frac{2 \pi K_o}{\Omega} \) for a large \( K_o \). The criterium requires the components of the velocity at the time \( t = t_{so} \) of the particle of air travelling to the rotor blade to deviate less then \( 10^{-5} \) of the undisturbed windspeed \( W \).

In the velocity potential representation there is an analogous procedure to the iterative process described above which is often referred to as wake relaxation, see e.g. Simoes et al [35] or Bareiss et al [3]. Such wake relaxation processes are however much more time consuming since they require a two dimensional integration over the complete vortex wake in each iteration step. Compared to the one dimensional integration along the particles path required in the present method this means a considerable difference in calculational effort.

Both codes PREDICCHAT1 and PREDICCHAT2 are operational at present. In fact both are nowadays modes of the same code. The PREDICCHAT1 settings are chosen when questions arise in the pre-design phase. Optimisation with respect to blade geometry and blade pitch setting, rotational speed etcetera can be assessed conveniently with a fast computational code. With the PREDICCHAT1 version answers are obtained within seconds on an ordinary PC (386 or 486 microprocessor). The calculation time with the PREDICCHAT2 settings increases up to the order of one to several minutes. This is well suited for the aerodynamic analysis of existing geometries, comparisons with measured data etc.

Apart from the determination of loads on the rotor blade it is sometimes interesting to calculate the flow about the rotor of the wind turbine. Of special interest, e.g. for comparison with other more elaborate models such as based upon a vortex lattice or a vortex particle approach are the velocities in the wake of the rotor.
For the calculation of velocities in the field governed by the type of pressure distributions given in chapter 9 two numerical codes have been developed. The first code, designated VIAIX, is based upon the same assumptions as taken for the code PREDICCHAT1. In fact first a PREDICCHAT1 calculation is performed to determine the coefficients \( A_n \).

The herewith determined pressure field is then used to calculate the velocities in the field assuming that the paths of the particles travelling to the specific positions in the field are straight and are travelled with a constant speed.

The second code is designated VITRAX, and is the equivalent of VIAIX, but now under the PREDICCHAT2 assumptions.

### 11-4 Numerical codes for dynamic inflow calculations.

The time dependent (dynamic inflow) solutions are obtained according to the following procedure:

First a stationary calculation is carried out using PREDICCHAT2. This calculation uses the initial values of the relevant parameters (such as pitch angle and wind speed at \( "t = 0" \)). With the now well determined stationary pressure field the unsteady paths of the particles (and their time and position dependent velocities) are calculated using a step by step variation procedure. Every next time step the whole process of determination of the accelerations (now time dependent), the velocities and the paths is repeated, thus calculating the dynamic inflow velocities in the rotor plane.

This yields a new set of coefficients \( A_\phi(t) \) at every new time step and with this new set completely new paths of the particles of air will be found as will be clarified.

The set of coefficients \( A_\phi(t) \) obtained at previous time steps is conserved. They determine the instantaneous pressure field necessary for the calculation of the accelerations in the time prior to arrival at the rotor blade. The paths of the particles arriving at the collocation points on the rotor blade at the time \( t + \Delta t \) are different from the paths of the particles arriving a the time \( t \). Through the different acceleration “felt” in the final time step the position of the latter particles at the time \( t - \Delta t \) with respect to the rotor blades will differ from the position of the particles at the time \( t \) prior to arrival at \( t + \Delta t \). In such way a completely different path is found with an integration of the accelerations backward in time (just as in the code PREDICCHAT2). For the time steps prior to the introduction of the dynamics (before \( "t = 0" \)) the steady coefficients \( A_\phi \) determined by PREDICCHAT2 are used. The code in which this calculation is implemented is designated PREDICDYN.

The latter process is equivalent to a vortex wake calculation with a dynamically varying wake (a process called dynamic wake adaptation). Such a dynamic vortex wake calculation can be carried out numerically, but in general the computation time (in the order of 12 hours on a work station) is prohibitive for more practical applications. Therefore the code is usually simplified to a “hybrid wake” calculation, where only a rather small fraction of the wake is dynamically adapted to the unsteady conditions, see Bareiss et al [3].

The dynamic inflow calculations performed with PREDICDYN can be distinguished into
two types: axisymmetric dynamic inflow calculations, asymmetric dynamic inflow calculations:

The axisymmetric dynamic inflow cases assume identical loads on all blades, which of course will vary, but simultaneously. Examples of such situations are start up procedures, control actions and safety stops, where the pitch angles of the blades are fixed or changed simultaneously. Also the treatment the effect of coherent axial wind gusts on the blade loads falls into this category.

The asymmetric dynamic inflow cases concern calculations for situations where blade pitching is not performed simultaneously for all blades. Examples of such situations are usually failure conditions. But also a (schematic) wind gust acting on a part of the rotor disc can be considered as asymmetric dynamic inflow case.

A number of axisymmetric dynamic inflow cases will be considered in section 12-5. Results of such calculations with PREDICDYN have also been published in Hasegawa et al [22] as well as in Snel et al [36]. Asymmetric dynamic inflow cases will not be treated separately here. The numerical treatment of such cases is very similar to the yawed case described in the next section.

11-5 Numerical codes for yawed flow calculations.

The implementation of the model for yawed flow is not essentially different from the implementation for the dynamic inflow cases as described in the previous section. Some modifications however have to be made for calculations of yawed flow cases.

Yawed flow calculations are dynamic calculations all the way up from the beginning. Thus preprocessing with a steady PREDICHAT calculation, as was always done in the dynamic inflow situations is thus strictly spoken, not possible.

An engineering approach is adopted however for yawed cases, by assuming that the wind turbine is originally operating in steady uniform perpendicular flow. For such the pressure field can again be calculated with PREDICHAT2. At some instant the rotor is assumed to yaw instantaneously into its desired position. Then the calculations after this sudden yaw are continued with the dynamic code PREDICDYN until periodicity in the results is obtained.

This again makes it possible to combine steady preprocessing with dynamic calculations, which is important for achieving results within reasonable calculation time.

It turned out that after 4 consecutive revolutions following sudden yaw movement a periodic solution was obtained. The method however fails to converge for yaw angles larger than about 45 degrees. It should however be remembered that application of the present method presumes small perturbations with respect to the windspeed \( W \), certainly in directions perpendicular the undisturbed wind. For yaw angles of 45 degrees the induced velocities perpendicular to the wind will also be considerable and therefore application of the code for such cases is doubtful and should not be performed.

The cases discussed in section 12-5 will therefore be restricted to a maximum yaw angle of about 30 degrees.
12 Predictions with the numerical codes and comparison with experimental results

Introduction

Based upon the expressions derived in the chapters 9 and 10 numerical codes have been established and are applied for a number of wind turbine configurations in several steady and dynamic conditions. In the present chapter the results of the codes will be presented, and when possible comparisons with other numerical predictions and with experimental results are shown.

First the results for steady cases are presented, followed by a discussion about the differences in the calculation of the local inflow at the rotor blade with the present method, and from more classical approaches using the 2-d relation between inflow angle and local load. For some cases the expected higher order corrections are calculated as well as some velocity distributions in the vicinity of the rotor blades.

Figure 25: Calculated axial (induced) velocities of particles of air travelling to collocation points on the blade for "constant" and "variable" axial velocity.
Then some dynamic inflow cases are treated. At first the effects of an instantaneous change in blade pitch angle are shown, followed by the results found from a calculation during a safety stop. The latter case implies a combination of a continuous blade pitch excursion in combination with a decreasing rotational speed, until the situation that blade feather in combination with an idling rotational speed is established.

Finally the situation of a wind turbine rotor in yawed condition is considered. In such case the direction of the (uniform) incoming flow is not parallel to the direction of the rotor axis. Such situation evidently leads to periodic changes in the loads and velocities as will be shown.

12-1 Predictions for steady cases

The steady code PREDICCHAT1 was validated with other predictions and with a large number of experimental results, see Van Bussel [13]. Within PREDICCHAT1 the paths of the particles of air are unperturbed (straight), and their imposed velocities are constant, but different for each radial station. In the PREDICCHAT2 code the particles of air moving to the collocation points have travelled along a curved path with a varying velocity.
Figure 27: PREDIC4AT1 and PREDIC4AT2 results for the WEG MS-1 geometry (table 2) compared with measurements.

Figure 28: Benchmark results for the WEG MS-1 geometry compared with measurements, Van Grol et al [20].
In figure 25 a comparison is shown between the calculated axial (induced) velocities of particles of air travelling to collocation points on the blade for "constant" and "variable" axial velocity. The "constant velocity" results are obtained for the situation where the accelerations are integrated along a straight unperturbed path, and the "variable velocity" is obtained for a real perturbed path. The general behaviour is quite similar. The influence of the blade passages can be distinguished clearly on the right hand side (close to the rotor plane). Note that the accelerations are sometimes positive along the trajectory!! This shows that the global idea of a monotonous decrease in axial velocity when approaching the rotor plane obtained from global momentum theory is not valid when details of the actual flow are considered.

The influence of the improved modelling of the trajectories of the particles on the overall results is present, but rather limited. Figures 26 and 27 show comparisons between PREDICHA1 and PREDICHA2 results (for stationary situations) together with the measured result. In figure 26 the calculations have been performed for the Tjæreborg geometry (see table 1), whereas in fig. 27 the WEG MS-1 geometry is used, see table 2. In fig. 28 the comparison is shown with the results of other codes examined in the CEC project "Wind Turbine Benchmark Exercise", Van Grol et al [20]. From this comparisons it can be seen that the predictability of the curves seems to be slightly better when using the PREDICHA1 code. The reason for it is not clear. The trajectory modelling used in PREDICHA2 is expected to be closer to reality than the rather forced straight path constant velocity method used in PREDICHA1 for the integration of the accelerations.

Figure 29 shows a picture of the Lagerwey 10/35 wind turbine during some tests. One blade was equipped with "tufts", small pieces of wool fitted to the suction side. With the tufts it is possible to examine the boundary layer behaviour to some extent. It is for example possible to determine the radial position at which flow separation due to aerofoil stall is seen.

Figure 29 is in fact a composite picture, where the blade with the tufts was photographed three times on three different azimuthal positions (within 2 revolutions). The inboard blade area at which stall occurs can be distinguished clearly by the more or less radial position of the woollen tufts. It extends from the root section towards the most inboard chord marked with a black line.
Figure 29: Lagerwey 10/35 wind turbine with blade equipped with "tufts", small pieces of wool fitted to the suction side (composite picture).
Figure 30 shows the prediction with PREDICCHAT1 for the radial position at which flow separation due to aerofoil stall occurs. The data for the calculations have been taken from table 3. Separation was considered to take place whenever the calculated $\alpha_{st}$ (see section 10.7) was above the angle at which two-dimensional stall occurs (in this case for $\alpha_{st} > 12$ degrees). As can be seen the predictions are in reasonable accordance with the experimental values.

In fact there is a small overprediction of the radial station at which the rotor blade stalls. In other words the experiments show a smaller stalled blade area than calculated. This is in accordance with findings from other experiments, and from recently developed semi-empirical models which, at least qualitatively, predict such stall delay, see Snel et al [36]. These semi-empirical models take into account a delay in boundary layer separation and thus a stall delay due to centrifugal effects. Such kind of viscous modelling is not taken into account in the present method.

Figure 31 and 32 show the calculated $C_p - \lambda$ and $C_{pax} - \lambda$ curves for the Tjæreborg rotor geometry. Details of the geometry of the rotor are given in table 1. The figures show the result from inviscid calculations, so without implementation of a viscous correction, and the results with the subroutine CDALCX, according to the code PREDICCHAT2. With increasing blade pitch angle the loads on the rotor blade decrease, as can be observed by the reduction of $C_{pax}$ with increasing pitch angle for a given value of $\lambda$.

![Figure 30](image-url)

**Figure 30:** Prediction with PREDICCHAT1 for the radial position at which flow separation due to aerofoil stall occurs, Lagerwey 13/35 rotor blade.
Figure 31: Calculated inviscid and viscous $C_p - \lambda$ curves for the Tjaereborg geometry (PREDICHAT2).

Figure 32: Calculated inviscid and viscous $C_{Dax} - \lambda$ curves for the Tjaereborg geometry (PREDICHAT2).
In general it can be concluded that both PREDICHAT1 and PREDICHAT2 predict overall blade loads with an accuracy equal to other methods currently used. Its advantage with respect to BEM methods is that much more detail can be obtained with regard to load and velocity distributions. Looking into more details of the velocity distributions about the rotor blades with the present gives more insight in the local flow. In the next section the local inflow of the aerofoils is treated in more detail.

12-2 Calculation of local inflow at the blade

Local inflow conditions calculated with the present method show rather large differences with predictions from BEM-methods. An example is presented in figure 33, where the effective inflow angle along the span calculated with PREDICHAT for the Tjæreborg geometry (table 1) is compared with calculations from the BEM-code PROPSI.

The latter code is an SI-systems version of the PROP-code described by Wilson et al [40]. This difference between the two predictions is not so surprising since BEM methods use average induced values over annuli, sometimes modified with "tip-correction factors", whereas the acceleration potential method directly calculates local induced velocities. A similar difference can be seen when the spanwise load distributions are compared. Figure 34 shows the corresponding lift coefficient distributions. Note that the lift coefficient values calculated with PREDICHAT are larger than those resulting from PROPSI for the larger values of the tipspeed ratio (at low wind speeds). Furthermore the effective angles calculated with PREDICHAT are larger. Therefore the predicted power using PREDICHAT is larger than the prediction from PROPSI. For the lower \( \lambda \) cases (at high wind speeds) the situation is more or less opposite. There the \( C_l \) values are slightly smaller, whereas the effective angle is significantly smaller in the PREDICHAT predictions, when compared to the PROPSI results.

Apart from the fact that BEM methods use averages over annuli there is another cause for the differences: the hypothesis of a local 2-d flow is not valid for wind turbine rotor blades as will be clarified below. Results presented in Hasegawa et al [21] show that with the present method the calculation of effective angles from the near field term integration, and from a modified code using the Prandtl hypothesis with equation (10-4) give different results. The discrepancy is quite large in the blade root area and decreases with increasing blade radius. Figure 35 shows this effect for the Tjæreborg geometry (table 1), and for the T.U. Delft field aerodynamics rotor (table 4) at various operational conditions. It depicts the ratio of the angle determined by equation (10-4) and the angle calculated from the near field integration. The difference is attributed to a kind of "swept flow effect" caused by the more or less circular inflow path with respect to the blade, which is not taken into account when using equation (10-4). The effect does not seem to be too sensitive for inflow variations, but is definitely geometry dependent.
Figure 33: Effective inflow angle along the span calculated with PREDICHCAT2 and with the BEM-method PROPSI for the Tjæreborg geometry.

Figure 34: Spanwise lift coefficient distribution calculated with PREDICHCAT2 and with the BEM-method PROPSI for the Tjæreborg geometry.
The induced velocity distributions and the load distributions obtained with the present method are assumed to be more plausible than those from BEM methods. An argument for this statement is the fact that equation (10-4), the crucial relation in BEM-methods to relate effective angles to local blade loads, is based on a physically incorrect argument!! The acceleration potential method does not need such a local 2-d flow hypothesis since effective angles are obtained directly from the near field term integration.

A firm proof of such a statement is very difficult to give, since it would imply a comparison of predictions with experimental values. Measuring induced velocity distributions is impossible. Experimental determination of load distributions can be done, but such experiments require very specialised equipment and difficult to perform to a reasonable accuracy. What can be done relatively easy and is therefore quite often carried out on experimental wind turbines is the measurement of e.g. blade root flapping moments.

Comparisons with such measurements will be made in section 12-5, where the dynamic situation is often preceded with a steady phase.

Despite the fundamental impossibility a number of researchers have (unsuccessfully) tried to apply the two dimensional relation between the loads measured locally on rotating blades and the measured effective inflow angle. In these situations the latter is usually "measured" indirectly by means of directional pitot tubes, Madsen [30] or wind vanes, Butterfield et al [14].

![Graph showing spanwise variation of the calculated lift curve slope](Figure 35: Spanwise variation of the calculated lift curve slope calculated with PREDICHAT2.)
In view of the results presented above such efforts are doomed to fail, since the 2-d relation is not valid in the 3-d flow about rotating wind turbine blades.

In Bruining et al [6] some preliminary efforts were presented to relate a measured pressure load with an effective inflow angle through a theoretical model. Within the framework of the present research it is clear that such an approach is inevitable.

12-3 An estimate for higher order effects

As was shown in chapter 7 it is possible to derive second order expressions for the pressure field. In section 10-2 it was shown that it is possible to make an estimate of the effects of such second order approximation of the exact solution on the blade loads. The formulas for the virtual camber at the rotor blade, given in section 10-2 were implemented in an extension of the code to explore the magnitude of this second order effect. In the figures 36 through 39 the result of calculations with this extension are depicted. Figures 36 and 37 show the calculated pressure distributions for two configurations: the Tjæreborg geometry and the T.U.Delft windtunnel model geometry. The latter geometry is given in table 5. A comparison is made between the first order calculation and the calculation including the virtual camber estimated from the coefficients found in the first order calculation.

Note that in these figures 36 and 37 the chord of the rotor blade is exaggerated in order to enhance the visibility of the isobars.

It can be seen that the influence of virtual camber is rather limited. The differences are mainly found at the low loaded areas near the trailing edge at the root and the tip. Figures 38 and 39 show the dependency of the virtual camber term for both geometries at several operational conditions. The figures show the maximum correction of the pressure (present at the mid chord line) caused by virtual camber.

From the figures 36 through 39 it is clear that no major effects can be expected by actual implementation of the complete second order approximation. This conclusion is supported by some specialised measurements of chordwise pressure distributions on rotating wind turbine blades reported in literature. The measurements performed by Ronsten [33] with a small production rotor instrumented with chordwise pressure holes at several spanwise locations in a large windtunnel, and the measurements by Bruining et al [7] with a similar experimental set-up under atmospheric conditions do not yield much difference between the observed chordwise pressure distributions and the known 2-d pressure distributions, at least for small effective inflow angles.

Thus the magnitude of the actual virtual camber is assumed to be rather limited and therefore the higher order effects have not been included in the codes developed so far.
**Figure 36**: Pressure distribution on the Tjæreborg blade at a normal operational condition.

**Figure 37**: Pressure distribution on the T.U.Delft model rotor blade at a normal operational condition.
**Figure 38:** Estimation of mid chord line pressure correction due to virtual camber effect at different operational conditions, Tjæreborg geometry.

**Figure 39:** Estimation of mid chord line pressure correction due to virtual camber effect at different operational conditions, T.U. Delft windtunnel geometry.
12-4 The calculation of velocity distributions

As described in section 11-3 the code VIAAX calculates the velocities by integrating the pressure gradients along straight paths travelled with a constant velocity, equivalent to the way the velocities at the rotor blade are calculated with PREDICHAT1. The PREDICHAT2 equivalent of the code is VITRAX. There integration of velocities along perturbed paths with varying velocities.

In figures 40 and 41 some predictions with VITRAX are depicted. They show the distribution of the axial, tangential (circumferential) and radial induced dimensionless velocities in a plane parallel and very close to the rotor plane (at a distance $z/R = 0.025$ in front of the rotor plane). The rotor blade is positioned along the horizontal line in the centre of the picture. The thick lines indicate velocities above nominal which means for example for the axial component higher than the undisturbed windspeed. The increment between two successive contours is 0.05. The presence of the blade and the high induced (axial and radial) velocities close to the tip and the root of the rotor blade can be clearly examined. The latter are the "footprints" of the root and tip vortex originating from the (large) pressure gradients at the root and the tip region on the blade. Note that the velocity pattern rotates with the blades. The patterns shown in the figures 40 and 41 give some indication of the actual induced velocity field around the rotor blades, and emphasise the crude approximation used in BEM theory concerning the induced velocities in the rotor plane.

With the codes VIAAX and VITRAX it is possible to compare calculated velocities in the field about the rotor with measured velocities. Of course such measured velocities can only be obtained from an experimental set up in the windtunnel. In Vermeer et al [39] some results of such comparison are presented.

12-5 Predictions for dynamic inflow cases

The PREDICHAT2 code is used as starting point for dynamic inflow calculations. This means that at the starting point of the dynamic inflow calculations ("at $t = t_0$") the particles of air moving to the collocation points have travelled along a curved path with a varying velocity. This steady situation is then used as the input for PREDICDYN. In this dynamic code the complete flow is modelled according to a time stepping scheme. For particles of air travelling along a (now time dependent) path towards the collocation points use is made of the accelerations experienced by the initial steady rotor calculation of the paths when $t < t_0$. The procedure is described in section 11-3. In this way the unsteady environment for $t \geq t_0$ is extended towards $t = -\infty$ with a steady situation.
Figure 40: Velocity distribution in a plane parallel to the rotor plane, just in front of the rotor, Tjæreborg geometry.

Figure 41: Velocity distribution in a plane parallel to the rotor plane, just in front of the rotor, T.U. Delft field aerodynamics rotor.
In figure 42 a typical example is shown of the variation of the spanwise load distribution under such dynamic inflow conditions. The geometry of the Tjæreborg wind turbine used in the example is given in table 1. It shows the variation in the spanwise load caused by a stepwise collective variation in the pitch angle setting of the blades at the time $t_0 = 6$ sec. Fig. 43 shows the corresponding contour plot. From these figures a number of effects can be noticed. Apart from the sudden drop in load and the gradual increase towards a new equilibrium after the sudden stepwise variation in geometry, a kind of periodic changes in the lift coefficient distribution can be seen which damp out after some seconds. The period is equal to about one second, which is the blade passage frequency (the rotational speed of the three bladed Tjæreborg machine is about 22 RPM). The sudden change in blade angle causes an immediate change in blade load at $t_0 = 6$ sec. Furthermore this sudden change in blade load is "felt" by particles of air travelling to the "next" blades. The effect will of course be most pronounced when these particles of air are in vicinity of the pitching blade. Those particles will arrive at "their" blade with a delay equal to a multiple of the blade passing period, which is about 1 second. Therefore it is also a typical example of effects of fluctuations in the instantaneous inflow caused by a sudden change of the blade load. Translated to a vorticity representation this is effect of the fluctuation of the vorticity strength in the wake (shed vorticity) generating variations in the instantaneous blade loads. Finally note that 4 seconds after the stepwise collective pitch variation a new equilibrium has not yet been established. This is not so surprising since the distance travelled with the undisturbed windspeed (8 m/s) in this 4 seconds equals 32 m, which is about equal to the rotor radius (30.56 m). From stationary calculations it was already found that the particles have to travel a distance equal to several times the rotor diameter before convergence is obtained in the loads and velocities.

Comparison with experimental results took place with respect to the blade root bending moments and the shaft torque. The experimental results were taken from Øye [32] and concern the Tjæreborg wind turbine, see table 1. Figure 44 shows the measured and calculated blade root flapping moment for a situation with a collective pitch step of 2 degrees up and down at $t = 6$ and $t = 34$ seconds respectively. The blade root moment is measured (and calculated) at a local radius equal to $r = 2.75$ m.

Figure 45 shows the corresponding measured and calculated shaft torque. In both situations it concerns the response on a (almost) stepwise variation. It can be seen that the qualitative agreement between the measured and the calculated results is good. The calculated time constants are also in reasonable good agreement with the measured ones, where the calculated values tend to be somewhat smaller. This difference between the calculated and the measured response of the blade flapping moments and the rotor shaft torque just after the changes in pitch angle may be partially attributed to structural dynamic response, which is not modelled in PREDICDYN.

Furthermore it can be seen that at approximately $t = 20$ seconds the calculations have reached their new equilibrium value.
**Figure 42:** Spanwise load distribution at a step in pitch angle at $t = 6$ seconds, Tjæreborg wind turbine

**Figure 43:** Contour plot corresponding to figure 41 showing the periodicity of the blade load variations synchronous with the blade passage frequency.
The results of figures 44 and 45 show that, for pitching transients in conditions with substantial induced velocities, the dynamic inflow effects are important for both flap wise moments and rotor shaft torque. Compared to quasi-stationary calculations overshoots of a factor 1.3 are present in both calculations and measurements.

Figure 46 shows a comparison of calculated and measured blade root flap wise moments for the situation of a safety stop. In such case the wind turbine initially runs under normal conditions. After some time, in this example at $t = 6$ seconds, the blades pitch collectively to the feather position ($\theta_r = 90^\circ$). The calculations were carried out with the measured rotational speed (initially 22.3 RPM slowing down to approximately 0 RPM), and measured blade pitch angle and show a good agreement with the measured values.

As a result from the calculations with PREDICDYN an with other advanced vortex models, supported with measurements from the Tjaereborg machine and from the T.U. Delft windtunnel model (see table 5) it was possible to modify blade element momentum theory calculations in order to take into account the above described dynamic inflow effects upon the loads. It was found that BEM models could be significantly improved by implementation of first order time derivative equation with an "empirical" coefficient derived from both experiments and advanced models. In Snel et al [37] more details can be found.

![Figure 44: Measured and calculated blade root flapping moment for a collective pitch step of 2 degrees up and down, Tjaereborg geometry.](image)
**Numerical predictions**

**Figure 45:** Measured and calculated shaft torque for a collective pitch step of 2 degrees up and down, Tjæreborg geometry.

**Figure 46:** Calculated and measured blade root flap wise moments for the situation of a safety stop, Tjæreborg geometry.
12.6 Predictions for yawed flow cases

As was said in the previous chapter PREDICCHAT2 is also used as a preprocessing procedure for yawed flow calculations. Such a steady situation is used as input for a yawed flow calculation with PREDICDYN according to a time marching scheme. For particles of air travelling along dynamic paths towards the collocation points use is made of the accelerations experienced by the initial steady rotor calculation of the paths when \( t < 0 \). At \( t = 0 \) the rotor is assumed to yaw instantaneously. After some time the coefficients \( A_n(t) \) become periodic with the rotor azimuth angle and a stable yawed flow solution is obtained. In general it is found that a periodic solution was realised after taking four revolutions into account.

Figure 47 shows the conventions used for the yawed flow situations. The wind turbine is rotating in clockwise direction, seen in the direction with the wind. The azimuth angle \( \psi \) is taken 0 in the "bottom down" position. At \( \psi = 90^\circ \) the considered rotor blade is positioned horizontally pointing towards the direction of the wind. The position \( \psi = 180^\circ \) is the "top up" situation and \( \psi = 270^\circ \) is again horizontal, pointing away from the wind, and rotating in downward direction.

![Diagram of yawed flow](image)

**Figure 47:** Sign and phase conventions for yawed flow.
Numerical predictions

Figure 48 shows the computed spanwise load distribution for the Tjæreborg rotor at a yaw angle $\gamma = 32^\circ$. Figures 49 shows a corresponding situation for the T.U. Delft windtunnel model at a yaw angle $\gamma = 30^\circ$. The figures 50 and 51 show contour plots of these load distributions. In these figures the load distribution is presented in terms of a nondimensional axial load coefficient $C_{dax}$ as a function of the azimuthal angle $\psi$. The reason for using this way of representing the load distribution in stead of using $C_l$ comes from the fact that the magnitude of the relative inflow velocity is a function of the azimuthal position for yawed flow conditions. A constant load is in such cases not equivalent to a constant $C_l$. With the definition of $C_{dax}$ chosen here it is assured that the value is representative for the actual load at the considered position in the rotor plane.

The shape of the load distributions shows some differences for both cases since the geometry of the two rotors differs considerably.

It can be seen that in both situations the minimum load is found around the azimuthal angle $\psi = 0^\circ$. From the contour plots it can be seen that in fact the minimum is found just before arriving at the "bottom down" position. Rotation to the position pointing towards the wind at $\psi = 90^\circ$ involves a more or less constant growth of load along the whole span of the blade. The load at the tip region stays more or less constant over the second quadrant, where the load in the root region keeps growing. The maximum load is found around the "top up" position at $\psi = 180^\circ$. Over the third quadrant the load decreases, where the largest drop takes place at the tip.

![Graphical representation of load distribution](image.png)

**Figure 48:** Axial load distributions for the Tjæreborg geometry at a yaw angle $\gamma = 32^\circ$. 
This leads to a spanwise load distribution around $\psi = 270^\circ$ which has its maximum in the blade root region. The load at the blade root falls down remarkably over the fourth quadrant, whereas the tip load stays more or less constant. Comparison of the distributions found at the two horizontal positions $\psi = 90^\circ$ and $\psi = 270^\circ$ shows that shape of the load distribution has changed significantly. It should be observed that in both these horizontal positions of the blade the load has shifted towards the windward direction.

The change in load observed here is consistent with the ideas of Glauert, see e.g. Bramwell [5], where the concept is introduced of the rotor as a whole represented by a steady elliptically loaded wing in the situation of a helicopter rotor in forward flight. As on a real wing the centre of pressure on such a hypothetical wing is shifted towards the virtual leading edge which is the edge of the rotor disc pointing towards the oncoming wind. With the present definition of the azimuthal angle, this rotor disc edge is represented by the blade tip area in the first and second quadrant. So the shift of load towards the tip around $\psi = 90^\circ$ fits in this concept.

The shift of load towards the root around $\psi = 270^\circ$ can also be covered with an in analogous concept. There the root area takes the role of the virtual leading edge. Of course the latter was not part of the original global ideas about the disc loading since

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**Figure 49:** Axial load distributions for the T.U.Delft windtunnel model, yaw angle $\gamma = 30^\circ$. 
Figure 50: Contour plot of axial load distributions, Tjæreborg geometry, yaw angle $\gamma = 32^\circ$.

Figure 51: Contour plot of axial load distribution, T.U.Delft windtunnel model, yaw angle $\gamma = 30^\circ$. 
Figure 52: Induced axial velocity distributions, Tjæreborg wind turbine, $\gamma = 32^\circ$.

Figure 53: Induced axial velocity distributions, T.U.Delft windtunnel model, yaw angle $\gamma = 30^\circ$. 

the root cut out was not taken into account. When such root cut out would have been accounted for then the appearance of root loading effects around $\psi = 270^\circ$ are plausible. The results of the present method, where discrete blades including a root cut out are modeled, would support such an extension. Furthermore they are qualitatively consistent with the results of Mangle and Squire, see Bramwell [5], where a prescribed rotor load having a maximum close to the blade tip leads to a minimum in induced velocity in the blade root area around $\psi = 270^\circ$.

The figures 52 and 53 depict the corresponding induced axial velocity distributions for the Tjæreborg geometry and the T.U.Delft windtunnel model respectively. In both situations a considerable phase shift can be observed, which is not so surprising in view of the axial load distributions discussed above.

Figures 54 and 55 are taken from Snel et al [37]. They show some measurements for yawed cases together with a number of predictions, included the predictions from PREDICDYN. Figure 54 concerns the flap wise moment (in the direction of the rotor axis) for the Tjæreborg machine while figure 55 shows the flat wise moment (in the direction perpendicular to the tip chord) for the T.U. Delft windtunnel model. It can be seen that the predictions with PREDICDYN (denoted as TUD in the graphs), are quite reasonable. From the windtunnel model experiments it is known that the measured flat wise moment is lowered by "centrifugal stiffening" of the blades. Therefore the predictions with purely aerodynamic models will yield too high values.

Furthermore it can be noticed that the measured flapping moments suffer from a dynamic response on the periodic excitation which is most likely of a structural origin (a blade flapping eigenfrequency).

Figure 56 shows a comparison of measured total horizontal velocity in a rotor plane of the T.U.Delft windtunnel model compared with the calculated values from PREDICDYN. The measurements have been performed as close as possible behind the rotor plane, at an axial distance of 3.2 mm. behind the rotor ($z/R = 0.053$). The same trends can be observed such as the variation in phase angle with increasing radial position although the position of the minimum value and the predicted levels are not too accurate.

Figures 57 and 58 show comparisons of predictions and measurements of the yawing and tilting moments for the Tjæreborg machine, again at a yaw angle $\gamma = 32^\circ$. The yawing and tilting moments are the moments in horizontal and vertical direction respectively, observed at the rotor centre due to asymmetric loading of the rotor in yaw. The predictions with PREDICDYN in these figures, which are also taken from Snel et al [37], are marked with TUD.

The conclusion from these comparisons is again that the PREDICDYN predictions are in reasonable agreement with the measurements. The phase angle seems to be quite correct for the yawing moment, but the variation in the moment is less than measured. The measured tilting moment shows a phase lag with respect to the calculations. It is most likely that this also can be attributed to a dynamic structural response, see Snel et al [37].
Figure 54: Variation of flap wise blade root moment, Tjæreborg wind turbine, $\gamma = 32^\circ$, Snel et al [37]

Figure 55: Variation of flat wise blade root moment, T.U. Delft windtunnel model, $\gamma = 30^\circ$, Snel et al [37]
Figure 56: Comparison of measured and calculated horizontal velocities at $\gamma = 30^\circ$, T.U. Delft windtunnel model.

Figure 57: Yawing moments at the nacelle of the Tjæreborg wind turbine, $\gamma = 32^\circ$, Snel et al [37]
Figure 58: Tilting moments at the nacelle of the Tjæreborg wind turbine, $\gamma = 32^\circ$, Snel et al [37]
13 Conclusions

The use of first order Legendre functions offers the possibility of developing efficient calculation methods for the calculation of spanwise wing and rotor blade loads and induced velocities. The standard Prandtl lifting line method using harmonic functions in a velocity potential solution can be modified using such functions, with equivalent results.

The extension of the theory to the situation of interfering wings in a plane staggered formation is straightforward, as well as to the situation with a discontinuity in pitch angle, such as encountered at a partial span pitchable rotor blade.

The asymptotic acceleration potential method yields good possibilities for development of rather fast computer codes for the calculation of steady loads on wind turbine rotors. The codes can be elaborated in a rather straightforward way for the calculation of, axisymmetric dynamic inflow cases, as well as for yawed flow cases.

The overall results of the code PREDICCHAT1 for steady applications are comparable to results from blade element-momentum theory results. On one hand PREDICCHAT 1 is probably more time consuming, on the other hand it generates much more detail of the load and the surrounding flow.

Overall results from calculations with the code PREDICCHAT2, where an Euler integration is established for the calculation of the velocities, do not deviate much from the PREDICCHAT1 results. This suggests that the effect of the axial deformation of the flow (present in PREDICCHAT1) is probably the most important for proper calculation of loads on horizontal axis wind turbines.

From calculations with PREDICCHAT2 it is concluded that the Prandtl hypothesis, relating a chordwise load distribution to an effective angle of attack through a two-dimensional relation is not valid for wind turbine rotor blades. With the effective angle directly calculated from the three dimensional integration of the (two dimensional) near field term a considerable smaller value is found for the lift curve slope $dc/d\alpha$, especially in the blade root region.

Experimental research in which it is tried to measure such two dimensional relationship at a rotating wind turbine rotor blade is thus bound to fail.

Compared to PREDICCHAT1 calculations the modified lift curve slope found with PREDICCHAT2 leads to slightly lower loads, but to similar values for the torque and edgewise blade moments.
The combined code PREDICCHAT2 / PREDICDYN for calculations of axisymmetric
dynamic inflow cases has shown its capability to predict unsteady loads. A corresponding
lifting line method, and certainly lifting surface methods require much more labour for
implementation of the algorithms in the code, and more computer power.
Furthermore these approaches are less equipped to perform physical interpretations of the
calculated effects.

The inclusion of a dynamic inflow model in the present design codes is imperative for the
calculation of loads during fast collective pitch control actions and aerodynamic brake
actions of the rotor.

The results from yawed flow calculations show that the load on the rotor shifts towards
the upstream side of the rotor disc, in agreement with the concept proposed by Glauert.
14 Acknowledgement

It is hard to sum up and acknowledge all the people that have contributed in some way or another to the work presented in this report, personally without forgetting someone. This is something I will therefore not try to do here. Still I would like to acknowledge some of them in particular.

At first I would like to mention the students that participated in the development of the method. It is my experience that working with students on the development of new theory and new codes can be a very inspiring job. Therefore I would like to thank especially Ferd Grosveld and Harmen Hogenhuis all in the beginning of wind energy research in Delft, through Andreas Heege, Kristian Duin, Jaap de Boer and Mathieu Zweerts in later stages of the research.

Furthermore I would like to acknowledge some of my colleagues during wind energy research at T.U. Delft over a period covering more than 18 years.
At first the colleagues of the tipvane research period: prof. dr. Theo van Holten, who initiated the use of the presently discussed methods for wind turbine applications, and furthermore dr. Gijs van Kuik as a sparring partner during the development of models for (tipvane) wind turbines.

Then dr. Yutaka Hasegawa is especially acknowledged for his part in the development of the PREDICDYN code during his post-doc research fellow stay at Delft University of Technology. His "asian" patience and persistence were a perfect match to my "wild" ideas and premature conclusions.

Ruud van Rooy was a skilled and hard (late) working assistant in the modification of the code for the calculation of partial span pitchable blades.

Nord-Jan Vermeer performed an elaborated set of windtunnel measurements, which were used frequently for comparisons with predictions of several codes, including the codes presented here.

TU Denmark (Stig Øye) is also kindly acknowledged for their permission to use the experimental data from the Tjæreborg wind turbine.
Furthermore the CEC is kindly acknowledged for their financial support of the development of PREDICDYN. This development took place under the CEC-JOULE R&D programme.

Finally I would like to thank the board of the faculty of Civil Engineering for providing me the opportunity to finish a period of research covering more than a decade with this publication. The flexible interpretation they gave of the status of a sabbatical year was beneficial for both the completion of this thesis and for the attention I could give to my family.
15 References


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### Aerofoil characteristics (synthesised)

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Blade geometry:

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Table 2 (continued): MS-1 configuration
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<tr>
<td>Rotational speed (variable RPM)</td>
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<td>Span ratio (for calculations) b/R</td>
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<td>Blade angle distribution (degrees)</td>
<td>θ = 10</td>
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<tr>
<td>Aerofoil</td>
<td>NACA 4412</td>
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</table>

**Aerofoil characteristics**

<table>
<thead>
<tr>
<th>Angle (degrees)</th>
<th>$c_1$</th>
<th>$c_d$</th>
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</thead>
<tbody>
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<td>-0.200</td>
<td>0.0210</td>
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<td>-4.00</td>
<td>0.000</td>
<td>0.0150</td>
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<td>-2.00</td>
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<td>0.400</td>
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<td>4.00</td>
<td>0.800</td>
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<td>10.00</td>
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<td>12.00</td>
<td>1.270</td>
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<tr>
<td>14.00</td>
<td>1.250</td>
<td>0.1050</td>
</tr>
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</table>

Table 3: Lagerwey configuration
### Number of blades
2

### Radius (meter)
5.00

### Rotational speed (variable RPM)
50-110

### Span ratio (for calculations) b/R
0.86

### Chord distribution
\(c/R = 0.100\)

### Blade angle distribution (degrees)
\(\theta = 0\)

### Aerofoil
NLF 416

### Aerofoil characteristics

<table>
<thead>
<tr>
<th>Angle (degrees)</th>
<th>(c_1)</th>
<th>(c_d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-12.705</td>
<td>-0.947</td>
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<td>-11.191</td>
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Table 4: T.U. Delft field aerodynamics rotor
Table 4 (continued):  T.U. Delft field aerodynamics rotor

Aerofoil characteristics

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<tr>
<th>Angle (degrees)</th>
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### Table 5: Windtunnel model configuration

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<td>Rotational speed (operational RPM)</td>
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<tr>
<td>Chord distribution</td>
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<td>Blade angle distribution (degrees)</td>
<td>( \theta = 6-6.67t/R )</td>
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<tr>
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<tr>
<td></td>
<td>for ( 0.3 \leq t/R \leq 0.9 )</td>
</tr>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>for ( 0.9 \leq t/R \leq 1.0 )</td>
</tr>
<tr>
<td>Aerofoil</td>
<td>NACA 0012</td>
</tr>
<tr>
<td>Angle (degrees)</td>
<td>$c_l$</td>
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Table 5 (continued): Windtunnel model configuration
Appendix A: Explicit expressions for the Legendre-functions, recurrence relations and asymptotic behaviour

Explicit expressions

The first few Legendre polynomials $P_n(x)$ are given by the expressions:

\[ P_0(x) = 1, \quad (A-1) \]

\[ P_1(x) = x, \quad (A-2) \]

\[ P_2(x) = \frac{1}{2}(3x^2-1), \quad (A-3) \]

\[ P_3(x) = \frac{1}{2}(5x^3-3x), \quad (A-4) \]

and

\[ P_4(x) = \frac{1}{8}(35x^4-30x^2+3). \quad (A-5) \]

The first few Legendre functions of the second kind (and of order zero) $Q_n(x)$ are given by:

\[ Q_0(x) = \begin{cases} \frac{1}{2} \ln \frac{1+x}{1-x}, & |x| < 1 \\ \frac{1}{2} \ln \frac{x+1}{x-1}, & |x| > 1 \end{cases}, \quad (A-6) \]
Expressions for Legendre functions

\[
Q_1(x) = P_1(x)Q_0(x) - 1, \quad \text{(A-7)}
\]

\[
Q_2(x) = P_2(x)Q_0(x) - \frac{3}{2}x, \quad \text{(A-8)}
\]

\[
Q_3(x) = P_3(x)Q_0(x) - \frac{5}{2}x^2 + \frac{2}{3}, \quad \text{(A-9)}
\]

and

\[
Q_4(x) = P_4(x)Q_0(x) - \frac{35}{8}x^3 + \frac{55}{24}x. \quad \text{(A-10)}
\]

Recurrence relations

The Legendre functions of first and second kind of order \( m \neq 0 \) are obtained from the zero order functions by:

\[
P_n^m(x) = \begin{cases} 
(1-x^2)^{\frac{m}{2}} \frac{d^m P_n(x)}{dx^m} & |x|<1 \\
(x^2-1)^{\frac{m}{2}} \frac{d^m P_n(x)}{dx^m} & |x|>1
\end{cases} \quad \text{(A-11)}
\]

and

\[
Q_n^m(x) = \begin{cases} 
(1-x^2)^{\frac{m}{2}} \frac{d^m Q_n(x)}{dx^m} & |x|<1 \\
(x^2-1)^{\frac{m}{2}} \frac{d^m Q_n(x)}{dx^m} & |x|>1
\end{cases} \quad \text{(A-12)}
\]

These Legendre functions of order \( m \neq 0 \) are often called the associate Legendre functions of the first and second kind. From (A-12) and (A-13) it is clear that \( P_n^0(x) = P_n(x) \) and
Expressions for Legendre functions

\( Q_n^0(x) = Q_n(x) \).

For the determination of the Legendre polynomials and functions of a higher degree than given in (A-1) through (A-10) and for numerical evaluations the recurrence relation:

\[
(n-m)P_n^m(x) = (2n-1)xP_{n-1}^m(x) - (n+m-1)P_{n-2}^m(x)
\]

\( m \geq 0, \ n \geq 0 \)  \hspace{1cm} \text{(A-13)}

also valid for the \( Q_n^m(x) \) functions, is very useful.

The following general expression for the \( Q_n(x) \) functions will be used in the asymptotic expansions:

\[
Q_n(x) = P_n(x)Q_0(x) + W_{n-1}(x) \quad \text{for } n \geq 1
\]

where \( W_{n-1}(x) = \sum_{m=1}^{n} \frac{1}{m} P_{m-1}(x)P_{n-m}(x) \).  \hspace{1cm} \text{(A-14)}

From eqs. (A-1) through (A-3), and using (A-13) it follows for the Legendre polynomials:

\[
P_n(1) = 1 \quad \quad n \geq 0 \hspace{1cm} \text{(A-15)}
\]

\[
\lim_{x \to \infty} P_n(x) = \infty \quad \quad n \geq 1 \hspace{1cm} \text{(A-16)}
\]

The associate Legendre functions of the first kind \( P_n^m(x) \) satisfy:

\[
P_n^m(1) = 0 \quad \quad n \geq 0, \ m \geq 1 \hspace{1cm} \text{(A-17)}
\]

\[
\lim_{x \to \infty} P_n^m(x) = \infty \quad \quad n \geq 1, \ m \geq 1 \hspace{1cm} \text{(A-18)}
\]
Using the recurrence relation (A-13) explicit expressions for all the \( P_n''(x) \) and \( Q_n''(x) \) functions can be obtained. A relation can then be derived between the Legendre functions of the second kind (and zero order) \( Q_n(x) \) and the Legendre polynomials \( P_n(x) \):

\[
Q_n(x) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(x)}{(x-x')} dx'
\]  
(A-19)

**Asymptotic behaviour**

The explicit expressions (A-6) through (A-10) for the Legendre functions of the second kind and zero order, together with the recurrence relation (A-13) for functions of a higher degree give the possibility to examine the asymptotic behaviour of the Legendre functions of the second kind \( Q_n(x) \) for \( x \uparrow 1 \). Keeping in mind that \( P_n(1) = 1, n \geq 0 \), it follows that the singularity of these Legendre functions is determined by the logarithmic behaviour of \( Q_n(x) \) at \( x = 1 \):

\[
Q_n(x) = \ln |x^2 + 1| - \ln(\sqrt{|x^2 - 1|}) = \\
-\ln(\sqrt{|x^2 - 1|}) + \ln 2 + (x^2 - 1) + O((x^2 - 1)^2)
\]  
(A-20)

for \( x \uparrow 1 \) and \( n \geq 0 \)

From the definition (A-6) it follows for the behaviour of \( Q_n(x) \) for large \( x \):

\[
\lim_{x \to \infty} Q_0(x) = 0
\]  
(A-21)

and with the series expansion:

\[
\lim_{x \to \infty} \frac{1}{2} x \ln \left( \frac{x+1}{x-1} \right) = 1 + \frac{1}{3x^2} + \frac{1}{5x^4} + \ldots
\]  
(A-22)

substituted in the explicit expressions it follows:

\[
\lim_{x \to \infty} Q_n(x) = 0 \quad n \geq 0
\]  
(A-23)
Furthermore it can be shown, using the recurrence relation:

\[ Q_n^{m-1}(x) = \frac{1}{\sqrt{(x^2-1)}}[(n-m)xQ_n^{m}(x)-(n+m)Q_{n-1}^{m}(x)] \quad (A-24) \]

that also the associated Legendre functions of the second kind \( Q_n^{m}(x) \) approach zero for \( x \to \infty \):

\[ \lim_{x \to \infty} Q_n^{m}(x) = 0 \quad n \geq 0, \ m \geq 0 \quad (A-25) \]

The behaviour of the \( Q_n^{m}(x) \) functions in the point \( x=1 \) is also of special importance.

By differentiation of the expression (A-11) it then follows for the \( Q_n^{1}(x) \) functions, with the use of (A-13):

\[ Q_n^{1}(x) = P_n^{1}(x)Q_0(x) + P_n(x)Q_0^{1}(x) + \sqrt{|x^2-1|} \frac{dW_{n-1}(x)}{dx} \quad \text{for } n \geq 1. \quad (A-26) \]

Making use of the definition of \( Q_n(x) \) and \( Q_n^{m}(x) \) (equations (A-6) and (A-13) respectively) it follows:

\[ Q_0^{1}(x) = \sqrt{|x^2-1|} \frac{-1}{(x^2-1)} = \frac{-1}{\sqrt{|x^2-1|}} \quad \text{for } |x| \neq 1 \quad (A-27) \]

The \( P_n^{1}(x) \) functions are regular around \( x=1 \):

\[ \lim_{x \to 1} P_n^{1}(x) = 0 \quad n \geq 0 \quad (A-28) \]

The value of the polynomial \( P_n(x) \) in the point \( x=1 \) is known from (A-15): \( P_n(1)=1 \), and the third term of (A-26) is also regular around \( x=1 \) since \( W_{n-1}(x) \) is a polynomial in \( x \) of degree \( n-1 \).

The Legendre functions \( Q_n(x) \) show only a logarithmic singularity at \( x=1 \) as can be concluded from the equations (A-6) and (A-11) in the previous section. This shows that the second term of (A-26) dominates the behaviour of \( Q_n^{1}(x) \).

From equation (A-26) with the use of (A-20), (A-27) and (A-14) the asymptotic behaviour of the \( Q_n^{1}(x) \) functions at \( x=1 \) can thus be written as:
\[ Q_n^1(x) = \frac{-1}{\sqrt{|x^2-1|}} - \frac{n(n+1)}{2} \sqrt{|x^2-1|} \ln(\sqrt{|x^2-1|}) + \]
\[ - \sqrt{|x^2-1|} \left[ \frac{n(n+1)}{2} \ln 2 - \sum_{m=1}^{n} \frac{m(m-1)}{2(n+1-m)} \right] + O((x^2-1)\ln(\sqrt{|x^2-1|})) \]
\text{for } x \uparrow 1 \quad (A-29)

Differentiating (A-26) it follows for the \( Q_n^2(x) \) functions:
\[ Q_n^2(x) = P_n^2(x)Q_0(x) + 2P_n^1(x)Q_0^1(x) + P_n(x)Q_0^1(x) + (x^2-1) \frac{d^2W_{n-1}(x)}{dx^2} \]
\text{for } n \geq 1 \quad (A-30)

With the definition of \( Q_n(x) \) and \( Q_n^m(x) \), (equations (A-6) and (A-13) respectively) it follows:
\[ Q_0^2(x) = (x^2-1) \frac{2x}{(x^2-1)^2} = \frac{2x}{x^2-1} \quad \text{for } |x| \neq 1 \]
\text{for } |x| \neq 1 \quad (A-31)

The asymptotic behaviour of the \( Q_n^2(x) \) functions at \( x=1 \) can then be written as:
\[ Q_n^2(x) = \frac{2x}{x^2-1} P_n(x) + \frac{d^2P_n(x)}{dx^2} (x^2-1) \ln(\sqrt{x^2-1}) - 2 \frac{dP_n(x)}{dx} + \]
\[ + O(\sqrt{x^2-1}) \quad \text{for } x \uparrow 1 \quad (A-32)

In (A-27) and (A-31) explicit expressions have been found for \( Q_0^1(x) \) and \( Q_0^2(x) \).
Both have the form:
\[ Q_0^m(x) = \frac{Y_m(x)}{|x^2-1|^{m/2}} \quad \text{for } m=1,2 \quad |x| \neq 1 \]
\text{for } m=1,2 \quad (A-33)

where \( Y_m(x) \) is a polynomial in \( x \).

In the following part of this appendix it will be shown that (A-33) is valid for all \( m \geq 1 \).
Suppose (A-33) is also satisfied for a given \( m = m_0 \), \( m_0 \neq 1,2 \).
Expressions for Legendre functions

For $Q_{0}^{m_{0}+1}(x)$ it follows:

$$Q_{0}^{m_{0}+1}(x) = |x^{2} - 1|^{-\frac{m_{0}+1}{2}} \frac{d^{m_{0}+1}}{dx^{m_{0}+1}} Q_{0}(x) =$$

$$= \frac{1}{m_{0}+1} \left[ \frac{d}{dx} \left[ |x^{2} - 1|^{\frac{m_{0}+1}{2}} \frac{d^{m_{0}}}{dx^{m_{0}}} Q_{0}(x) \right] + (m_{0}+1)2x(x^{2}-1) |x^{2} - 1|^{-\frac{m_{0}+1}{2}} \frac{d^{m_{0}}}{dx^{m_{0}}} Q_{0}(x) \right]$$

$$= \frac{1}{m_{0}+1} \left[ \frac{d}{dx} \left[ |x^{2} - 1|^{\frac{m_{0}+1}{2}} Q_{0}(x) \right] - (m_{0}+1)2x(x^{2}-1) |x^{2} - 1|^{\frac{m_{0}-1}{2}} Q_{0}(x) \right]$$

(A-34)

So:

$$Q_{0}^{m_{0}+1}(x) = \frac{1}{m_{0}+1} \left[ \frac{m_{0}+1}{2} 2x(x^{2}-1) |x^{2} - 1|^{\frac{m_{0}-1}{2}} Q_{0}(x) + \right.$$

$$+ \left. |x^{2} - 1|^{\frac{m_{0}+1}{2}} \frac{d}{dx} Q_{0}(x) \right]$$

$$= \frac{-m_{0}}{2} \frac{2x(x^{2}-1)}{|x^{2} - 1|^{3/2}} Q_{0}(x) + \frac{\sqrt{|x^{2} - 1|}}{2} \frac{d}{dx} Q_{0}(x)$$

(A-35)

Since it was assumed that $Q_{0}^{m_{0}}(x)$ satisfies (A-34) it follows for the first expression in the right hand side of (A-34):

$$-\frac{m_{0}}{2} \frac{2x(x^{2}-1)}{|x^{2} - 1|^{3/2}} Q_{0}^{m_{0}}(x) = \frac{Y_{m_{0}+1,1}(x)}{|x^{2} - 1|^{\frac{m_{0}+1}{2}}}$$

(A-36)

where $Y_{m_{0}+1,1}(x)$ is a polynomial in $x$. 


The second term in the right hand side of (A-35) contains the derivative of $Q^m_0(x)$. Carrying out this differentiation it follows from the assumption that $Q^m_0(x)$ satisfies (A-33) for $m_0$:

$$\frac{d}{dx} Q^m_0(x) = -\frac{m_0 2x(x^2-1)}{m_0^2 |x^2-1|^2} Y^m_0(x) + \frac{1}{m_0} \frac{d}{dx} Y^m_0(x)$$  \hspace{1cm} (A-37)

Since the derivative of $Y^m_0(x)$ is again a polynomial in $x$, equation (A-37) can be rewritten to:

$$\frac{d}{dx} Q^m_0(x) = \frac{Y^m_{m_0+1,2}(x)}{|x^2-1|^2} \frac{m_0+1}{m_0} \hspace{1cm} |x| \neq 1$$  \hspace{1cm} (A-38)

Where $Y^m_{m_0+1}(x)$ is again a polynomial in $x$.

Substitution of (A-36) and (A-38) into (A-34) yields:

$$Q^m_{m_0+1}(x) = \frac{Y^m_{m_0+1,1}(x)}{|x^2-1|^2} + \frac{Y^m_{m_0+1,2}(x)}{|x^2-1|^2} = \frac{Y^m_{m_0+1}(x)}{|x^2-1|^2} \hspace{1cm} |x| \neq 1$$  \hspace{1cm} (A-39)

Where $Y^m_{m_0+1}(x) = Y^m_{m_0+1,1}(x) + Y^m_{m_0+1,2}(x)$ is a polynomial in $x$.

It has thus been shown that if (A-33) is satisfied for a certain $m=m_0$, than (A-33) is also valid for $m=m_0+1$, and therefore by induction:

$$Q^m_0(x) = \frac{Y^m_m(x)}{|x^2-1|^m} \hspace{1cm} \text{for all } m \geq 1 \hspace{1cm} |x| \neq 1$$  \hspace{1cm} (A-40)

Having derived equation (A-40) for the $Q^m_0(x)$ functions, the asymptotic behaviour of these functions for $x \to 1$ is also known.
The asymptotic behaviour of the \( Q_n^m(x) \) functions with \( n \geq 1 \) follows from the equations (A-12) and (A-25):

\[
Q_n^m(x) = P_n^m(x)Q_0(x) + \binom{m}{1} P_n^{m-1}(x)Q_0^1(x) + \cdots + \binom{m}{m-1} P_n^1(x)Q_0^{m-1}(x) + P_n(x)Q_0^m(x) + \frac{d^m W_n(x)}{dx^m} \tag{A-41}
\]

for \( n \geq 1 \)

where \( \binom{m}{k} \) denotes the binomial coefficient.

All \( P_n^m(x) \) functions show a regular behaviour for the argument \( x \uparrow 1 \), and with (A-40) it follows that the asymptotic behaviour of \( Q_n^m(x) \) for \( x \uparrow 1 \) is dominated by the term:

\[
P_n(x)Q_0^m(x) = \frac{P_n(x)Y_m(x)}{|x^2 - 1|^{m/2}} \tag{A-42}
\]

So the asymptotic behaviour of \( Q_n^m(x) \) satisfies:

\[
Q_n^m(x) = O((|x^2 - 1|^{-m/2}) \quad \text{for all } m \geq 1 \quad x \uparrow 1 \tag{A-43}
\]
Appendix B: Velocities and accelerations in a rotating reference frame.

General expressions

A particle of air, having a velocity $\mathbf{V_b}$ in blade reference system $(x_b, y_b, z_b)$, rotating with respect to the inertial $(x, y, z)$ reference system will have a velocity $u$ in $x$-direction of this basic reference system determined by:

$$ u = \frac{dx}{dt} = \frac{\partial x}{\partial \psi} \frac{d\psi}{dt} + \frac{\partial x}{\partial x_b} u_b + \frac{\partial x}{\partial y_b} v_b + \frac{\partial x}{\partial z_b} w_b \tag{B-1} $$

where $\Omega$ denotes the rotational velocity of the rotor blade, and $\mathbf{V_b} = u_b \mathbf{i} + v_b \mathbf{j} + w_b \mathbf{k}$. Here $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ denote the unit vectors in $x, y$ and $z$ direction respectively.

Similar expressions can be written down for the $v$ and $w$ component of the velocity in the basic reference system.

In an analogous way one can write for the accelerations of the particle of air expressed in the basic reference system:

$$ \frac{du}{dt} = \frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial \psi^2} \frac{d^2 \psi}{dt^2} + \frac{\partial^2 x}{\partial x_b^2} u_b^2 + \frac{\partial^2 x}{\partial y_b^2} v_b^2 + \frac{\partial^2 x}{\partial z_b^2} w_b^2 + 
\begin{aligned}
+2 \frac{\partial^2 x}{\partial \psi \partial x_b} \Omega u_b + 2 \frac{\partial^2 x}{\partial \psi \partial y_b} \Omega v_b + 2 \frac{\partial^2 x}{\partial \psi \partial z_b} \Omega w_b + \\
+2 \frac{\partial^2 x}{\partial x_b \partial y_b} u_b v_b + 2 \frac{\partial^2 x}{\partial x_b \partial z_b} u_b w_b + 2 \frac{\partial^2 x}{\partial y_b \partial z_b} v_b w_b + \\
+ \frac{\partial x}{\partial \psi} \dot{\Omega} + \frac{\partial x}{\partial x_b} \dot{u}_b + \frac{\partial x}{\partial y_b} \dot{v}_b + \frac{\partial x}{\partial z_b} \dot{w}_b.
\end{aligned} \tag{B-2} $$

And again similar expressions can be written down for the $v$ and $w$ component of the acceleration in the basic reference system.
Expressions for particles of air moving over the wing surface

When the particle of air moves over the wing surface, which is expressed as a surface function:

$$y_b = y_b(x_b, z_b) \quad \text{(B-3)}$$

the velocity of the particle in the wing coordinate system satisfies:

$$v_b = \frac{\partial y_b}{\partial t} = \frac{\partial y_b}{\partial x_b} u_b + \frac{\partial y_b}{\partial z_b} w_b \quad \text{(B-4)}$$

and the acceleration of the particle in the wing coordinate system:

$$\frac{\partial y_b}{\partial t^2} = \frac{\partial^2 y_b}{\partial x_b^2} u_b^2 + \frac{\partial^2 y_b}{\partial z_b^2} w_b^2 + 2 \frac{\partial^2 y_b}{\partial x_b \partial z_b} u_b w_b + \frac{\partial y_b}{\partial x_b} \dot{u}_b + \frac{\partial y_b}{\partial z_b} \dot{w}_b \quad \text{(B-5)}$$

Then the expression for the velocity in the inertial reference system can be written as:

$$u = \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \psi} \Omega + \left( \frac{\partial x}{\partial x_b} + \frac{\partial x}{\partial y_b} \frac{\partial y_b}{\partial x_b} \right) u_b + \left( \frac{\partial x}{\partial z_b} + \frac{\partial x}{\partial y_b} \frac{\partial y_b}{\partial z_b} \right) w_b \quad \text{(B-6)}$$

and the expression for the acceleration in the inertial reference system yields:

$$\frac{du}{dt} = \frac{\partial^2 x}{\partial \psi^2} \Omega^2 + \left( \frac{\partial^2 x}{\partial x_b^2} + \frac{\partial^2 x}{\partial y_b^2} \frac{\partial y_b}{\partial x_b} \right) u_b^2 + \left( \frac{\partial^2 x}{\partial y_b^2} \frac{\partial y_b}{\partial x_b} + \frac{\partial^2 x}{\partial z_b^2} \frac{\partial y_b}{\partial z_b} \right) u_b w_b + \left( \frac{\partial^2 x}{\partial x_b \partial y_b} \frac{\partial y_b}{\partial x_b} + \frac{\partial^2 x}{\partial x_b \partial z_b} + \frac{\partial^2 x}{\partial y_b \partial z_b} \right) \Omega u_b + \left( \frac{\partial^2 x}{\partial x_b \partial y_b} \frac{\partial y_b}{\partial x_b} + \frac{\partial^2 x}{\partial x_b \partial z_b} + \frac{\partial^2 x}{\partial y_b \partial z_b} \right) \Omega \dot{u} + \left( \frac{\partial^2 x}{\partial x_b \partial y_b} \frac{\partial y_b}{\partial x_b} + \frac{\partial^2 x}{\partial x_b \partial z_b} + \frac{\partial^2 x}{\partial y_b \partial z_b} \right) u_b \omega_b + \left( \frac{\partial^2 x}{\partial x_b \partial y_b} \frac{\partial y_b}{\partial x_b} + \frac{\partial^2 x}{\partial x_b \partial z_b} + \frac{\partial^2 x}{\partial y_b \partial z_b} \right) \dot{u} \omega_b$$

$$+ \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \psi} \Omega + \left( \frac{\partial x}{\partial x_b} + \frac{\partial x}{\partial y_b} \frac{\partial y_b}{\partial x_b} \right) \dot{u} + \left( \frac{\partial x}{\partial z_b} + \frac{\partial x}{\partial y_b} \frac{\partial y_b}{\partial z_b} \right) \dot{w} \quad \text{(B-7)}$$
Application to a wing rotating at a cylindrical surface

The coordinate transformation for the wing rotating on a cylindrical surface with a radius $R$ yields:

$$
x = R \cos \psi + x_b \sin \psi - y_b \cos \psi
$$
$$
y = R \sin \psi - x_b \cos \psi - y_b \sin \psi
$$
$$
z = -z_b .
$$

(B-8)

For the partial derivatives of the inertial coordinates one can derive:

$$
\frac{\partial x}{\partial \psi} = -R \sin \psi + x_b \cos \psi + y_b \sin \psi
$$
$$
\frac{\partial y}{\partial \psi} = R \cos \psi + x_b \sin \psi - y_b \cos \psi
$$
$$
\frac{\partial z}{\partial \psi} = 0 ,
$$

(B-9)

$$
\frac{\partial x}{\partial x_b} = \sin \psi
$$
$$
\frac{\partial y}{\partial x_b} = -\cos \psi
$$
$$
\frac{\partial z}{\partial x_b} = 0
$$

$$
\frac{\partial x}{\partial y_b} = -\cos \psi
$$
$$
\frac{\partial y}{\partial y_b} = -\sin \psi
$$
$$
\frac{\partial z}{\partial y_b} = 0
$$

(B-10)

And for the second derivatives:

$$
\frac{\partial^2 x}{\partial \psi^2} = -R \cos \psi - x_b \sin \psi + y_b \cos \psi
$$
$$
\frac{\partial^2 y}{\partial \psi^2} = -R \sin \psi + x_b \cos \psi + y_b \sin \psi
$$
$$
\frac{\partial^2 z}{\partial \psi^2} = 0
$$

$$
\frac{\partial^2 x}{\partial \psi \partial x_b} = \cos \psi
$$
$$
\frac{\partial^2 y}{\partial \psi \partial x_b} = \sin \psi
$$
$$
\frac{\partial^2 z}{\partial \psi \partial x_b} = 0
$$

$$
\frac{\partial^2 x}{\partial \psi \partial y_b} = \sin \psi
$$
$$
\frac{\partial^2 y}{\partial \psi \partial y_b} = -\cos \psi
$$
$$
\frac{\partial^2 z}{\partial \psi \partial y_b} = 0
$$

(B-11)

$$
\frac{\partial^2 x}{\partial \psi \partial z} = 0
$$
$$
\frac{\partial^2 y}{\partial \psi \partial z} = 0
$$
$$
\frac{\partial^2 z}{\partial \psi \partial z} = 0 .
$$

All the other second derivatives with respect to the local coordinates $x_b$, $y_b$ and $z_b$ are
equal to 0.
Now a specific position of the wing on the cylindrical surface can be chosen. With $\psi = \pi/2$ it follows:

$$x = x_b, \quad y = R - y_b, \quad z = -z_b. \quad \text{(B-12)}$$

The non zero first order derivatives are:

$$\frac{\partial x}{\partial \psi} = -R + y_b; \quad \frac{\partial y}{\partial \psi} = x_b; \quad \frac{\partial x}{\partial x_b} = 1; \quad \frac{\partial y}{\partial y_b} = -1; \quad \frac{\partial z}{\partial z_b} = -1. \quad \text{(B-13)}$$

The non zero second order derivatives are:

$$\frac{\partial^2 x}{\partial \psi^2} = -x_b; \quad \frac{\partial^2 x}{\partial \psi \partial y_b} = 1; \quad \frac{\partial^2 y}{\partial \psi^2} = -R + y_b; \quad \frac{\partial^2 y}{\partial \psi \partial x_b} = 1. \quad \text{(B-14)}$$

Substitution of (B-13) through (B-14) into (B-6) and (B-7) results into the following expressions for the velocities:

$$u = (-1 + \frac{y_b}{R})\Omega R + u_b \quad \text{(B-15)}$$

$$v = \frac{x_b}{R}\Omega R - \frac{\partial y_b}{\partial x_b}u_b - \frac{\partial y_b}{\partial z_b}w_b$$

$$w = -w_b$$

and for the accelerations:

$$\frac{du}{dt} = -\frac{x_b}{R}\Omega^2 R + 2\frac{\partial y_b}{\partial x_b}\dot{u}_b + 2\frac{\partial y_b}{\partial x_b}\Omega w_b + (-1 + \frac{y_b}{R})\dot{\Omega} R + \ddot{u}_b$$

$$\frac{dv}{dt} = (-1 + \frac{y_b}{R})\Omega^2 R + 2\Omega u_b - \frac{\partial^2 y_b}{\partial x_b^2}u_b^2 - 2\frac{\partial y_b}{\partial x_b}\dot{u}_b w_b - \frac{\partial y_b}{\partial z_b}w_b^2 +$$

$$+ \frac{x_b}{R}\dot{\Omega} R - \frac{\partial y_b}{\partial x_b}\ddot{u}_b - \frac{\partial y_b}{\partial z_b}\dot{w}_b$$

$$\frac{dw}{dt} = -\dot{w}_b. \quad \text{(B-16)}$$

Using the first and the third expression of (B-15) it is possible to express $u_b$ and $w_b$ in terms of the inertial velocities $u$ and $w$. 
\[ u_b = (1 - \frac{y_b}{R}) \Omega R + u_b \]  \hspace{1cm} (B-17)

\[ w_b = -w \]

and substitution into the second expression of (B-15) then yields an expression for the inertial velocity \( v \) in terms of the local wing geometry and the inertial velocities:

\[ v = \left( \frac{x_b}{R} - \frac{\partial y_b}{\partial x_b}(1 - \frac{y_b}{R}) \right) \Omega R - \frac{\partial y_b}{\partial x_b} u + \frac{\partial y_b}{\partial z_b} w . \]  \hspace{1cm} (B-18)

Rewriting the first and the third expression of (B-15), and substituting into the second expression of (B-16), making use of (B-17) then yields for the total acceleration in the \( y \) direction:

\[
\frac{dv}{dt} = \left[ (1 - \frac{y_b}{R}) - \frac{x_b}{R} \frac{\partial y_b}{\partial x_b} + 2(1 - \frac{y_b}{R})(\frac{\partial y_b}{\partial x_b})^2 - (1 - \frac{y_b}{R}) \frac{\partial^2 y_b}{\partial x_b^2} R \right] \Omega^2 R + \\
+ 2 \left[ 1 + \left( \frac{\partial y_b}{\partial x_b} \right)^2 - (1 - \frac{y_b}{R}) \frac{\partial^2 y_b}{\partial x_b^2} R \right] \Omega u + \\
+ 2 \left[ - \frac{\partial y_b}{\partial x_b} \frac{\partial y_b}{\partial z_b} + (1 - \frac{y_b}{R}) \frac{\partial^2 y_b}{\partial x_b \partial z_b} R \right] \Omega w + \\
- \frac{\partial^2 y_b}{\partial x_b^2} u^2 + 2 \frac{\partial^2 y_b}{\partial x_b \partial z_b} u w - \frac{\partial^2 y_b}{\partial z_b^2} w^2 + \\
+ \left[ \frac{x_b}{R} - (1 - \frac{y_b}{R}) \frac{\partial y_b}{\partial x_b} \right] \Omega \left( \frac{\partial y_b}{\partial x_b} \frac{du}{dt} + \frac{\partial y_b}{\partial z_b} \frac{dw}{dt} \right) .
\]  \hspace{1cm} (B-19)
Application to a wing rotating in a plane (the windturbine geometry)

The coordinate transformation for the wing as a rotorblade rotating in a plane having a (maximum) radius \( R \) yields:

\[
x = (R - \frac{b}{2} + z_b)\cos\psi + x_b\sin\psi
\]
\[
y = (R - \frac{b}{2} + z_b)\sin\psi - x_b\cos\psi
\]
\[
z = -y_b.
\]

From (B-20) one can easily derive the first and second order partial derivatives with respect to \( \psi, z_b, y_b \) and \( z_b \). Substituted into the equations (B-6) and (B-7) and applied for \( \psi = \pi/2 \) results into the following expressions for the velocities:

\[
u = -(1 - \frac{bf/2}{R} + \frac{z_b}{R})\Omega R + u_b
\]
\[
\nu = \frac{x_b}{R}\Omega R + w_b
\]
\[
w = -\frac{\partial y_b}{\partial x_b}u_b - \frac{\partial y_b}{\partial z_b}w_b
\]

and for the accelerations:

\[
\frac{du}{dt} = -\frac{x_b}{R}\Omega^2 R - 2\Omega w_b - (1 - \frac{bf/2}{R} + \frac{z_b}{R})\dot{\Omega} R + \dot{u}_b
\]
\[
\frac{dv}{dt} = -(1 - \frac{bf/2}{R} + \frac{z_b}{R})\Omega^2 R + 2\Omega u_b + \frac{x_b}{R}\dot{\Omega} R + \dot{w}_b
\]
\[
\frac{dw}{dt} = -\frac{\partial y_b}{\partial x_b}u_b^2 - 2\frac{\partial^2 y_b}{\partial x_b \partial z_b}u_by_b - \frac{\partial^2 y_b}{\partial z_b^2}w_b - \frac{\partial y_b}{\partial x_b}\dot{u}_b - \frac{\partial y_b}{\partial z_b}\dot{w}_b.
\]

Substitution of the expressions for \( u_b \) and \( w_b \), derived from the first and the second expression of (B-21) into the third expression yields:
\[ \dot{w} = \left[ \frac{x_b}{R} \frac{\partial y_b}{\partial x_b} - (1 - \frac{b/2}{R} + \frac{z_b}{R}) \frac{\partial y_b}{\partial z_b} \right] \Omega R - \frac{\partial y_b}{\partial x_b} u - \frac{\partial y_b}{\partial z_b} v. \tag{B-23} \]

In a similar way it follows from the third expression of (B-22), using the first two expressions and the expressions for \( u_b \) and \( w_b \) from (B-21):

\[ \frac{d\dot{w}}{dt} = \left[ \frac{x_b}{R} \frac{\partial y_b}{\partial x_b} + (1 - \frac{b/2}{R} + \frac{z_b}{R}) \frac{\partial y_b}{\partial z_b} - (1 - \frac{b/2}{R} + \frac{z_b}{R})^2 \frac{\partial^2 y_b}{\partial x_b^2} R + \right. \\
+ 2 \frac{x_b}{R} \left( 1 - \frac{b/2}{R} + \frac{z_b}{R} \right) \frac{\partial^2 y_b}{\partial x_b \partial z_b} R - \left( \frac{x_b}{R} \right)^2 \frac{\partial^2 y_b}{\partial z_b^2} R \right] \Omega^2 R + \\
+ 2 \left[ \frac{\partial y_b}{\partial x_b} - (1 - \frac{b/2}{R} + \frac{z_b}{R}) \frac{\partial^2 y_b}{\partial x_b^2} R + \frac{x_b}{R} \frac{\partial^2 y_b}{\partial x_b \partial z_b} R \right] \Omega u + \\
+ 2 \left[ - \frac{\partial y_b}{\partial z_b} - (1 - \frac{b/2}{R} + \frac{z_b}{R}) \frac{\partial^2 y_b}{\partial z_b^2} R + \frac{x_b}{R} \frac{\partial^2 y_b}{\partial z_b \partial x_b} R \right] \Omega v + \\
+ \frac{\partial^2 y_b}{\partial x_b^2} u^2 - 2 \frac{\partial^2 y_b}{\partial x_b \partial z_b} u v - \frac{\partial^2 y_b}{\partial z_b^2} v^2 + \\
\left[ \frac{x_b}{R} \frac{\partial y_b}{\partial x_b} - (1 - \frac{b/2}{R} + \frac{z_b}{R}) \frac{\partial y_b}{\partial z_b} \right] \Omega R - \frac{\partial y_b}{\partial x_b} \frac{du}{dt} - \frac{\partial y_b}{\partial z_b} \frac{dv}{dt}. \tag{B-24} \]
Velocities and accelerations in a rotating frame
Appendix C: Partial derivatives in prolate spheroidal coordinates.

In this appendix the relation between the partial derivatives in the rectangular \((x_b, y_b, z_b)\) coordinate system and in the prolate spheroidal coordinate system \((v, \theta, \chi)\) is derived.

The transformation formulae from rectangular \((x_b, y_b, z_b)\) coordinate system to the prolate spheroidal coordinate system \((v, \theta, \chi)\), depicted in figure 5, yields:

\[
\begin{align*}
\frac{x_b}{b/2} &= \sinh v \sin \theta \cos \chi \\
\frac{y_b}{b/2} &= \sinh v \sin \theta \sin \chi \\
\frac{z_b}{b/2} &= \cosh v \cos \theta .
\end{align*}
\] (C-1)

For the performance of the required differentiation with respect to the non-dimensionalised coordinates \(x/(b/2), y/(b/2), z/(b/2)\) it is necessary to determine the Jacobi-matrix \((J)\).

\[
(J) = \begin{pmatrix}
\frac{\partial(x_b)}{\partial v} & \frac{\partial(y_b)}{\partial v} & \frac{\partial(z_b)}{\partial v} \\
\frac{\partial(x_b)}{\partial \theta} & \frac{\partial(y_b)}{\partial \theta} & \frac{\partial(z_b)}{\partial \theta} \\
\frac{\partial(x_b)}{\partial \chi} & \frac{\partial(y_b)}{\partial \chi} & \frac{\partial(z_b)}{\partial \chi}
\end{pmatrix} .
\] (C-2)

By substitution of (C-1) into (C-2), and performing all the partial derivatives it follows:

\[
(J) = \begin{pmatrix}
\cosh v \sin \theta \cos \chi & \cosh v \sin \theta \sin \chi & \sinh v \cos \theta \\
\sinh v \cos \theta \cos \chi & \sinh v \cos \theta \sin \chi & -\cosh v \sin \theta \\
-\sinh v \sin \theta \sin \chi & \sinh v \sin \theta \cos \chi & 0
\end{pmatrix} .
\] (C-3)
Partial derivatives in ellipsoidal coordinates

The relation between the partial derivatives with respect to the prolate speriodal coordinates \((v, \theta, \chi)\) and the rectangular coordinates \((x_b, y_b, z_b)\) is then given by:

\[
\begin{pmatrix}
\frac{\partial F}{\partial v} \\
\frac{\partial F}{\partial \theta} \\
\frac{\partial F}{\partial \chi}
\end{pmatrix} = (J)
\begin{pmatrix}
\frac{\partial x_b}{\partial F} \\
\frac{\partial y_b}{\partial F} \\
\frac{\partial z_b}{\partial F}
\end{pmatrix}
\]

\[(C-4)\]

where \(F\) is a function of \(x_b, y_b\) and \(z_b\).

According to (C-1) the function \(F\) can also be written as a function of the coordinates \((v, \theta, \chi)\). Strictly spoken this function \(F\) of the prolate spheroidal coordinates is not the same as the original function, depending upon the rectangular coordinates \(x_b, y_b, z_b\); it is the convolution of \(F\) with the function determined by the coordinate transformation (C-1).

But since it is common use to name this convolution also \(F\) this has been done here also.

When the function \(F\) is originally written in \((v, \theta, \chi)\) coordinates, the relation with partial derivatives in the rectangular \((x_b, y_b, z_b)\) coordinates follows from (C-4), written in the inverse way:

\[
\begin{pmatrix}
\frac{\partial F}{\partial v} \\
\frac{\partial F}{\partial \theta} \\
\frac{\partial F}{\partial \chi}
\end{pmatrix} = (J)^{-1}
\begin{pmatrix}
\frac{\partial F}{\partial x_b} \\
\frac{\partial F}{\partial y_b} \\
\frac{\partial F}{\partial z_b}
\end{pmatrix}
\]

\[(C-5)\]

In order to evaluate equation (C-5) it is necessary to invert matrix \((J)\). First the determinant of \((J)\) is calculated:
\[
\text{Det } (J) = |J| = \cosh v \sin \theta \cos \chi \begin{pmatrix}
\sinh v \cos \theta \sin \chi & -\cosh v \sin \theta \\
\sinh v \sin \theta \cos \chi & 0
\end{pmatrix} + \\
- \cosh v \sin \theta \sin \chi \begin{pmatrix}
\sinh v \cos \theta \cos \chi & -\cosh v \sin \theta \\
-\sinh v \sin \theta \sin \chi & 0
\end{pmatrix} + \\
+ \sinh v \cos \theta \begin{pmatrix}
\sinh v \cos \theta \cos \chi & \sinh v \cos \theta \sin \chi \\
-\sinh v \sin \theta \sin \chi & \sinh v \sin \theta \cos \chi
\end{pmatrix}.
\]

(C-6)

Evaluation of the sub-determinants then yields:

\[
|J| = \sinh v \sin \theta (\sinh^2 v + \sin^2 \theta) = \sinh v \sin \theta (\cosh^2 v - \cos^2 \theta) .
\]

(C-7)

A well known procedure to calculate the elements of the inverse matrix is to make use of the sub-determinants of the original matrix.

When \( \alpha_{ij} \) is the element in the i'th row and the j'th column of the matrix \((J)^{-1}\) its value is given by:

\[
\alpha_{ij} = \frac{(-1)^{i+j} \Delta_{ji}}{|J|} .
\]

(C-8)

Here \( \Delta_{ji} \) is a sub-determinant of \((J): it is the determinant of the matrix which remains after erasing the j'th row and the i'th column in the matrix \((J)\).

Using (E-8), and evaluating the \( \Delta_{ji} \)'s from (C-3) it follows:

\[
(J)^{-1} = \frac{1}{|J|} \begin{pmatrix}
\sinh v \cosh v \sin^2 \theta \cos \chi & \sinh v \cosh v \sin^2 \theta \sin \chi & \sinh^2 v \sin \theta \cos \theta \\
\sinh^2 v \sin \theta \cos \theta \cos \chi & \sinh^2 v \sin \theta \cos \theta \sin \chi & -\sinh v \cosh v \sin^2 \theta \\
-(\sinh^2 v + \sin^2 \theta) \sin \chi & (\sinh^2 v + \sin^2 \theta) \cos \chi & 0
\end{pmatrix} .
\]

(C-9)
Substitution of (C-9) into (C-5) now yields for the partial derivatives of $F$ w.r. to $x_{b}$, $y_{b}$ and $z_{b}$ respectively, in terms of prolate spheroidal coordinates:

$$
\frac{\partial F}{\partial \left(\frac{x_{b}}{b/2}\right)} = \frac{\cosh v \sin \theta \cos \chi}{(\sinh^2 v + \sin^2 \theta)} \frac{\partial F}{\partial v} + \frac{\sinh v \cos \theta \cos \chi}{(\sinh^2 v + \sin^2 \theta)} \frac{\partial F}{\partial \theta} - \frac{\sin \chi}{\sinh v \sin \theta} \frac{\partial F}{\partial \chi}.
$$  \hspace{1cm} (C-10)

$$
\frac{\partial F}{\partial \left(\frac{y_{b}}{b/2}\right)} = \frac{\cosh v \sin \theta \sin \chi}{(\sinh^2 v + \sin^2 \theta)} \frac{\partial F}{\partial v} + \frac{\sinh v \cos \theta \sin \chi}{(\sinh^2 v + \sin^2 \theta)} \frac{\partial F}{\partial \theta} + \frac{\cos \chi}{\sinh v \sin \theta} \frac{\partial F}{\partial \chi},
$$  \hspace{1cm} (C-11)

$$
\frac{\partial F}{\partial \left(\frac{z_{b}}{b/2}\right)} = \frac{\sinh v \cos \theta}{(\sinh^2 v + \sin^2 \theta)} \frac{\partial F}{\partial v} - \frac{\cosh v \sin \theta}{(\sinh^2 v + \sin^2 \theta)} \frac{\partial F}{\partial \theta}.
$$  \hspace{1cm} (C-12)
Appendix D: Partial derivatives of the basic distributions

The accelerations in the field of a windturbine rotor are obtained by the partial derivatives of the pressure field.

It thus requires the evaluation of the partial derivatives of the near, the common and the far field terms in of the expression given in chapter 9. Although the expressions can be differentiated numerically, it is preferred to use the analytic expressions, which can be obtained when the common and far field terms are differentiated in the local \((x_b, y_b, z_b)\) coordinate system.

The far field term is characterised by the basic expressions:

\[
\frac{p_{\text{far, basic}}}{\frac{1}{2} \rho W^2} = \frac{1}{\pi} P_n^1(\cos \theta) Q_n^1(\cosh \nu) \sin \chi.
\]  

\[(D-1)\]

Differentiation of the expression \((D-1)\) with respect to the prolate spheroidal coordinates yields:

\[
\frac{\partial}{\partial \nu} \left( \frac{p_{\text{far, basic}}}{\frac{1}{2} \rho W^2} \right) = \frac{1}{\pi} P_n^1(\cos \theta) \sin \chi \left( \frac{\cosh \nu}{\sinh \nu} n Q_n^1(\cosh \nu) - \frac{n+1}{\sinh \nu} Q_{n-1}^1(\cosh \nu) \right),
\]

\[(D-2)\]

\[
\frac{\partial}{\partial \theta} \left( \frac{p_{\text{far, basic}}}{\frac{1}{2} \rho W^2} \right) = \frac{1}{\pi} Q_n^1(\cosh \nu) \sin \chi \left( \frac{\cos \theta}{\sin \theta} n P_n^1(\cos \theta) - \frac{n+1}{\sin \theta} P_{n-1}^1(\cos \theta) \right),
\]

\[(D-3)\]

and:

\[
\frac{\partial}{\partial \chi} \left( \frac{p_{\text{far, basic}}}{\frac{1}{2} \rho W^2} \right) = \frac{1}{\pi} P_n^1(\cos \theta) Q_n^1(\cosh \nu) \cos \chi.
\]

\[(D-4)\]

Making use of the expressions \((C-10)\) through \((C-12)\) from appendix C it follows for the partial derivatives in the local (rectangular) \((x_b, y_b, z_b)\) coordinate system:
\[\frac{\partial}{\partial \left(\frac{x_b}{b/2}\right)} \left( \frac{P_{\text{far,basic}}}{1/2 \rho W^2} \right) = \frac{1}{\pi \sinh v \sin \theta} \left[ (n-1)P_n^1(\cos \theta)Q_n^1(\cosh v) + \right.\]
\[-\frac{(n+1)}{\sinh^2 v + \sin^2 \theta} \left( \cosh v \sin^2 \theta P_n^1(\cos \theta)Q_{n-1}^1(\cosh v) + \right.\]
\[\left. \left. + \sinh^2 v \cos \theta P_{n-1}^1(\cos \theta)Q_n^1(\cosh v) \right) \right] , \]

\[\frac{\partial}{\partial \left(\frac{y_b}{b/2}\right)} \left( \frac{P_{\text{far,basic}}}{1/2 \rho W^2} \right) = \frac{1}{\pi \sinh v \sin \theta} \left[ (n-1)\sin^2 \chi + 1\right] P_n^1(\cos \theta)Q_n^1(\cosh v) + \]
\[-\frac{(n+1)}{\sinh^2 v + \sin^2 \theta} \left( \cosh v \sin^2 \theta P_n^1(\cos \theta)Q_{n-1}^1(\cosh v) + \right.\]
\[\left. \left. + \sinh^2 v \cos \theta P_{n-1}^1(\cos \theta)Q_n^1(\cosh v) \right) \right] , \]

and

\[\frac{\partial}{\partial \left(\frac{z_b}{b/2}\right)} \left( \frac{P_{\text{far,basic}}}{1/2 \rho W^2} \right) = -\frac{(n+1)}{\pi} \frac{\sin \chi}{(\sinh^2 v + \sin^2 \theta)} \left( \cos \theta P_n^1(\cos \theta)Q_{n-1}^1(\cosh v) + \right.\]
\[\left. - \cosh v P_{n-1}^1(\cos \theta)Q_n^1(\cosh v) \right) . \]

The basic common field term is characterised by:

\[\frac{P_{\text{common,basic}}}{1/2 \rho W^2} = -\frac{1}{\pi} \sqrt{1 - \left(\frac{z_b}{b/2}\right)^2} \int P_n^1\left(\frac{z_b}{b/2}\right) \frac{\sin \chi}{\sinh v \sin \theta} . \]  

Making use again of the expressions from appendix C it can be derived for the partial derivatives with respect to the \(x_b, y_b\) and \(z_b\) coordinates:
Partial derivatives of basic distributions

\[ \frac{\partial}{\partial \left( \frac{x_b}{b/2} \right)} \left( \frac{P_{\text{common,basic}}}{1/2 \rho W^2} \right) = \frac{1}{\pi} \frac{\sin 2\chi}{\sinh^2 \nu \sin^2 \theta} \sqrt{1 - \left( \frac{z_b}{b/2} \right)^2} \frac{P_n^1 \left( \frac{z_b}{b/2} \right)}{P_{n-1}^1 \left( \frac{z_b}{b/2} \right)} , \]  

(D-9)

\[ \frac{\partial}{\partial \left( \frac{y_b}{b/2} \right)} \left( \frac{P_{\text{common,basic}}}{1/2 \rho W^2} \right) = -\frac{1}{\pi} \frac{\cos 2\chi}{\sinh^2 \nu \sin^2 \theta} \sqrt{1 - \left( \frac{z_b}{b/2} \right)^2} \frac{P_n^1 \left( \frac{z_b}{b/2} \right)}{P_{n-1}^1 \left( \frac{z_b}{b/2} \right)} . \]  

(D-10)

and:

\[ \frac{\partial}{\partial \left( \frac{z_b}{b/2} \right)} \left( \frac{P_{\text{common,basic}}}{1/2 \rho W^2} \right) = -\frac{1}{\pi} \frac{\sin \chi}{\sinh \nu \sin \theta} \left( \frac{n+1}{n} \right) \sqrt{1 - \left( \frac{z_b}{b/2} \right)^2} \frac{(-)^{n+1} P_n^1 \left( \frac{z_b}{b/2} \right)}{P_{n-1}^1 \left( \frac{z_b}{b/2} \right)} + \frac{n+1}{n} \frac{P_{n-1}^1 \left( \frac{z_b}{b/2} \right)}{P_{n-1}^1 \left( \frac{z_b}{b/2} \right)} . \]  

(D-11)

It should be taken into account that in the actual application of the above expressions in a numerical procedure it is necessary to evaluate the common field term plus the far field term simultaneously.

The reason for it can be found in paragraph 7-4, where the common field solution is introduced. Both \( P_{\text{far}} \) and \( P_{\text{common}} \) exhibit a singularity of the order \( r_b^{-1} \), where

\[ r_b^2 = \chi_x^2 + \chi_y^2 = (b/2)^2 \sinh^2 \nu \sin^2 \theta . \]

Taken together this singularity is annihilated. The partial derivatives (in \( x_b \) and \( y_b \) directions) thus exhibit a singularity of the order \( r_b^{-2} \), which makes combined evaluation even more necessary.

With the expressions:
\[ \partial \left( \frac{P_{\text{far, basic}}}{1/2 \rho W^2} - \frac{P_{\text{common, basic}}}{1/2 \rho W^2} \right) = \]
\[ \frac{1}{\pi \sinh^2 \theta} \left[ (n-1) \sin \theta \sin \chi P_n^1(\cos \theta)Q_n^1(\cosh \psi) + \right. \]
\[ - \frac{(n+1)}{(\sinh^2 \nu + \sin^2 \theta)} \left( \sinh \cosh \nu \sin \theta P_n^1(\cos \theta)Q_{n-1}^1(\cosh \nu) + \right. \]
\[ + \sinh^3 \sin \theta \cos \theta P_{n-1}^1(\cos \theta)Q_n^1(\cosh \nu) \right] - 2 \sqrt{1 - \left( \frac{z_b}{b/2} \right)^2} P_n^1(\frac{z_b}{b/2}) \]

(D-12)

\[ \partial \left( \frac{P_{\text{far, basic}}}{1/2 \rho W^2} - \frac{P_{\text{common, basic}}}{1/2 \rho W^2} \right) = \]
\[ \frac{1}{\pi \sinh^2 \theta} \left[ (n-1) \sin^2 \chi + 1 \right] \sin \sin \theta \sin \chi P_n^1(\cos \theta)Q_n^1(\cosh \nu) + \]
\[ - \frac{(n+1)}{(\sinh^2 \nu + \sin^2 \theta)} \left( \sinh \cosh \nu \sin \theta P_n^1(\cos \theta)Q_{n-1}^1(\cosh \nu) + \right. \]
\[ + \sinh^3 \sin \theta \cos \theta P_{n-1}^1(\cos \theta)Q_n^1(\cosh \nu) \right] + \cos 2 \chi \sqrt{1 - \left( \frac{z_b}{b/2} \right)^2} P_n^1(\frac{z_b}{b/2}) \]

and
\[ \partial \left( \frac{P_{\text{far, basic}}}{1/2 \rho W^2} - \frac{P_{\text{common, basic}}}{1/2 \rho W^2} \right) = \]
\[ \frac{1}{\pi \sinh \sin \theta} \left[ (n+1) \right] \sin \cosh \sin \theta \sin \theta P_n^1(\cos \theta)Q_{n-1}^1(\cosh \nu) + \]
\[ - \sinh \cosh \sin \theta P_{n-1}^1(\cos \theta)Q_n^1(\cosh \nu) \right] \]
\[ + \frac{(n+1)}{(1 - \left( \frac{z_b}{b/2} \right)^2) P_n^1(\frac{z_b}{b/2}) + P_{n-1}^1(\frac{z_b}{b/2})} \]

(D-13)
for $\nu \downarrow 0$ the expression is valid: $\sinh \nu Q_n^1(\cosh \nu) = -1 + O(\sinh^2 \nu \ln \nu)$. The terms in the partial derivatives of $P_{for} - P_{common}$ between the square brackets always occur in the combination $\sinh \nu Q_n^1(\cosh \nu)$. When it is further realised that for $r_b \downarrow 0$ the relation $\cos \theta = z_b/(b/2) + O((r_b/(b/2)^3)$ holds it can then be seen quite easily that the leading term in the partial derivatives of $P_{for} - P_{common}$ of order $O(1)$ disappears. Therefore remaining terms and with it the partial derivatives are thus of the order $O(\ln \nu)$.

In the rotor(blade)plane ($y_b = 0$, or $\sin \chi = 0$) it can be seen that the expressions (D-12) through (D-15) reduce to:

$$\frac{\partial}{\partial (\frac{z_b}{b/2})} \left( \frac{P_{for, basic}}{1/2 \rho W^2} - \frac{P_{common, basic}}{1/2 \rho W^2} \right) = \frac{\partial}{\partial (\frac{z_b}{b/2})} \left( \frac{P_{for, basic}}{1/2 \rho W^2} - \frac{P_{common, basic}}{1/2 \rho W^2} \right) = 0 \quad (D-15)$$

and

$$\frac{\partial}{\partial (\frac{y_b}{b/2})} \left( \frac{P_{for, basic}}{1/2 \rho W^2} - \frac{P_{common, basic}}{1/2 \rho W^2} \right) =
\frac{1}{\pi \sinh^2 \sin \theta^2} \left( \sinh \nu \sin \theta P_n^1(\cos \theta)Q_n^1(\cosh \nu) + \right.
+ \sqrt{1 - \left(\frac{z_b}{b/2}\right)^2} P_n^1\left(\frac{z_b}{b/2}\right) \right). \quad (D-16)$$

Since the following relations are valid: $\sinh \nu Q_n^1(\cosh \nu) = -1 + O(\sinh^2 \nu \ln \nu)$ and $\cos \theta = z_b/(b/2) + O((r_b/(b/2)^3)$ for $r_b \downarrow 0 (\nu \downarrow 0)$, it is clear that the partial derivative in the $y_b$ direction in the rotor plane can be integrated with an adequate integration procedure.
Partial derivatives of basic distributions
Samenvatting

Gedurende de laatste twintig jaar heeft het gebruik van windenergie opnieuw de aandacht gekregen. Windenergie werd in het verleden gebruikt voor directe mechanische aandrijving. Tegenwoordig ligt de aandacht evenwel sterk bij de opwekking van electriciteit. Dit heeft de ontwikkeling van een geheel nieuw type windmolens, de windturbine vereist. De wielen van deze windturbines, rotorbladen genoemd, dienen op een aerodynamisch verantwoorde manier te worden ontworpen. Kennis en ervaring uit de luchtvaart-aerodynamica kan hierbij veelal goed worden gebruikt. Toch zijn er belangrijke verschillen aan te wijzen, die er toe geleid hebben dat zich aan de stam van de aerodynamica een nieuwe tak aan het ontwikkelen is, de windturbine-aerodynamica.

Een belangrijk verschil met de "klassieke" luchtvaart is het feit dat windturbines bij voortdurend onderhevig zijn aan sterk variërende belastingen. Dit als direct gevolg van de variaties in windsnelheid en -richting zoals deze worden aangetroffen in de eerste 150 meter van aardse grenslaag, de plaats waar deze windturbines zich bevinden. Met name de windturbinerotor ondervindt daadwerkelijk belastingen met een specifiek dynamisch karakter.

Bij het ontwerpen van windturbinerotoren dient daarom met deze wisselende aerodynamische belastingen rekening te worden houden. Dat maakt het belangrijk om methoden te ontwikkelen waarmee dergelijke belastingen kunnen worden voorspeld. In de praktijk van het ontwerpen wordt evenwel nog steeds gebruik gemaakt van zgn. bladeelement-impulsmethoden waarmee het in principe niet mogelijk is om een gedetailleerde beeld te krijgen van de belastingenverdeling over een rotorblad, zeker niet onder wisselende aerodynamische omstandigheden. Wel is het relatief eenvoudig om dergelijke methoden te modificeren voor dergelijke situaties. Omdat metingen veelal moeizaam en duur zijn wordt hiervoor in belangrijke mate gebruik gemaakt van voorspellingen gemaakt met zeer uitgebreide aerodynamische rekenmethodes. Deze zijn echter complex van opzet en tevens zeer rekenintensief.

In dit proefschrift is een rekenmodel ontwikkeld gebaseerd op de asymptotische versnellingspotentieel methode. Met de numerieke codes die op basis daarvan zijn geschreven zijn het mogelijk om een grote mate van detail van zowel de rotorbelastingen als ook van de daarbij behorende stromingspatronen te bepalen.

In de eerste hoofdstukken wordt de basis gelegd voor de gebruikte methode. vervolgens wordt de klassieke aanpak van Prandtl voor het vleugel probleem, waarbij een oplossing in harmonische functies wordt gevonden, vergeleken met een oplossingsmethode die gebruik maakt van Legendre functies. Tevens wordt de oplossingsmethode voor het vleugelprobleem uitgebreid voor een formatievlucht situatie waarbij de onderlinge interferentie van de vleugels op een juiste wijze in rekening wordt gebracht.
Samenvatting

Vervolgens wordt de asymptotische versnellingspotentiaal oplossing voor het vleugelprobleem gepresenteerd. De resulterende aerodynamische belastingen in spanwijde richting worden ook hierbij beschreven met behulp van Legendre functies.

Daarna wordt het probleem behandeld van een roterende rechthoekige vleugel. Voor de situatie waarbij de vleugel roept op een denkbeeldig cilinderoppervlak wordt een vergelijk gemaakt met de resultaten die eerder zijn afgeleid voor de formatievleugel. Ook voor de situatie waarbij de vleugel roept om een punt dat in het verlengde ligt van een der tippen wordt een bijbehorend drukveld afgeleid. Eenzelfde oplossingsmethode als voor deze rudimentaire rotorgeometrie, ook op basis van asymptotische versnellingspotentiaal theorie, wordt vervolgens gebruikt om de belastingen en de omstroming van realistische rotorconfiguraties van windturbine te kunnen berekenen.

Net als bij de vleugel in uniforme aanstroming blijkt het nodig te zijn om een integratie van het resulterende drukveld uit te voeren om de coëfficiënten voor de Legendrefuncties in de oplossing te bepalen. Deze integratie dient ervoor om de oplossing te laten voldoen aan de kinematische randvoorwaarde, en behelst de integratie van de drukgradiënt (de versnellingen) van een luchtdeeltje langs zijn baan op weg naar het rotorblad.

Het ontwikkelde theoretische model is geïmplementeerd in een aantal sterk verwante numerieke codes. De code PREDICCHAT, voor de berekening van rotorbladbelastingen in stationaire situaties is in een tweetal uitvoeringen gebruikt. Met de meest eenvoudige uitvoering is een vergelijking gemaakt met de resultaten verkregen via andere methoden en met metingen. Uit deze globale vergelijking kan worden geconcludeerd dat de kwaliteit van voorspellingen overeenkomt met die van elders ontwikkelde methoden. Berekeningen met de meer uitgebreide uitvoering (PREDICCHAT2) tonen aan dat er effecten zijn die voorheen niet zijn opgemerkt. Zo blijkt de gekromde aanstroming van het binnendeel van een rotorblad een aanzienlijk effect te hebben op de liftgradiënt van de profielen die daar worden gebruikt.

De PREDICCHAT codes worden ook gebruikt als "pre-processor" voor dynamische berekeningen. Deze dynamische berekeningen, vervat in de code PREDICDYN, behelzen vooral situaties waarbij de variaties in bijvoorbeeld windsnelheid een schaal hebben in de orde van een rotorstraal. Ook bladverstel-acties bij een vast toerental en situaties waarbij de rotor door bladverstelling aerodynamisch wordt geremd kunnen met PREDICDYN worden berekend. Tenslotte wordt een aantal situatie behandeld waarbij de windturbine-rotor scheef op de wind wordt gezet. De voorspellingen gedaan met PREDICDYN zijn in voorkomende gevallen vergeleken met metingen. Een centrale rol daarbij wordt gespeeld door meetresultaten van een grote Deense onderzoeksturbine, de Tjæreborg windturbine, met een diameter van 60 m. en een geïnstalleerd vermogen van 3 MW, en door meetresultaten verkregen van een modelrotor-opstelling in de Delftse openstraal windtunnel.

Geconcludeerd wordt dat er een bevredigende overeenkomst is tussen de voorspellingen en de meetresultaten. Bovendien is er uit berekeningen van stromingsdetails een beter inzicht verkregen in de oorzaak van een aantal dynamische belastingen die kunnen optreden bij windturbines.
Curriculum Vitae

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