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ON THE SUM OF ACCRETIVE OPERATORS

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ON THE SUM OF ACCRETIVE OPERATORS

PROEFSCHRIFT

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door

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Contents

Chapter 1

Introduction

In this thesis we will deal with the sum of a linear and a nonlinear m-accretive operator in a Hilbert space. In particular we are interested in the question: when is the sum m-accretive as well ? This question has no immediate answer unless some additional assumptions are imposed on the operators involved.

As general references on m-accretive operators, let us mention [BB1], [CR], [CL1] and [D]. For the richer theory in the setting of Hilbert spaces, we refer to [BR1].

Let $\mathcal{H}$ be a real Hilbert space with innerproduct $((\cdot,\cdot))$ and let

$$\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to 2^{\mathcal{H}}$$

be a multivalued, possibly nonlinear operator in $\mathcal{H}$. As usual we shall identify a graph $\mathcal{A} \subset \mathcal{H} \times \mathcal{H}$ with its corresponding multivalued operator. An operator $\mathcal{A}$ in $\mathcal{H}$ is called *m-accretive* if the resolvent $(I + \lambda\mathcal{A})^{-1}$ is an everywhere defined (not necessarily strict) contraction in $\mathcal{H}$, for all positive real numbers λ. By expressing the contractivity of the resolvent in terms of the innerproduct of $\mathcal{H}$ it can be shown that the operator $\mathcal{A}$ is m-accretive if and only if $\mathcal{A}$ is *monotone*, that is,

$$((u_1 - u_2, v_1 - v_2)) \geq 0, \quad \text{for all } v_i \in \mathcal{A}u_i,\ i = 1,2,$$

and the operator $\epsilon I + \mathcal{A}$ is surjective, for each positive real number ϵ (equivalently for some $\epsilon > 0$). A monotone operator is called *maximal monotone* if it has no proper monotone extension. It is well-known that the operator $\mathcal{A}$ is m-accretive if and only if $\mathcal{A}$ is maximal monotone [BR1, Proposition 2.2]. The class of maximal monotone operators in $\mathcal{H}$ plays an important role in nonlinear functional analysis since it coincides with the class of negative generators of strongly continuous (nonlinear) contraction semigroups in $\mathcal{H}$ [BR1, Théorème 4.1]. An interesting subclass of the maximal monotone operators are the so-called subdifferentials $\partial\varphi$ of convex lower-semicontinuous functionals $\varphi : \mathcal{H} \to (-\infty, \infty]$ (see [BR1]).

Let $\mathcal{L}$ and $\mathcal{A}$ be two m-accretive operators in $\mathcal{H}$. We shall assume throughout that $\mathcal{L}$ is linear and that $0 \in \mathcal{A}0$. It is immediate that the sum $\mathcal{L}+\mathcal{A}$ with domain $D(\mathcal{L}+\mathcal{A}) = D(\mathcal{L}) \cap D(\mathcal{A})$ is monotone. Thus in order to prove the m-accretivity of the operator $\mathcal{L}+\mathcal{A}$ we need to show that the equation

$$\epsilon u + \mathcal{L}u + \mathcal{A}u \ni f, \quad \epsilon > 0, \tag{1.1}$$

has a solution $u \in D(\mathcal{L}) \cap D(\mathcal{A})$ for all $f \in \mathcal{H}$ (by the monotonicity of $\mathcal{L}+\mathcal{A}$, a solution of equation (1.1) is necessarily unique).

If the operator $\mathcal{A}$ is moreover everywhere defined and Lipschitz continuous then the sum $\mathcal{L}+\mathcal{A}$ is m-accretive. Indeed, equation (1.1) is equivalent to

$$u = (I + \frac{1}{\epsilon}\mathcal{L})^{-1}\{\frac{1}{\epsilon}f - \frac{1}{\epsilon}\mathcal{A}u\}. \tag{1.2}$$

Now take $\epsilon > 0$ sufficiently large such that the operator $\frac{1}{\epsilon}\mathcal{A}$ has a Lipschitz constant less than one. By the contractivity of the resolvent $(I + \frac{1}{\epsilon}\mathcal{L})^{-1}$ it follows that the right-hand side of equation (1.2) defines a strict contraction in $\mathcal{H}$ and therefore it admits a fixed point (see [BR1, page 34]). (Observe that we did not use that $\mathcal{L}$ is linear or that $0 \in \mathcal{A}0$.)

In the case that $\mathcal{A}$ is merely m-accretive one could try to approximate $\mathcal{A}$ by a sequence $\{\mathcal{A}_k\}_{k=1}^{k=\infty}$ of m-accretive, everywhere defined, Lipschitz continuous operators. The approximate equation

$$\epsilon u + \mathcal{L}u + \mathcal{A}_k u = f, \quad k = 1, 2, \ldots, \quad \epsilon > 0,$$

is then uniquely solvable. The next step is to obtain a priori estimates so that one can pass to the limit. We illustrate this approach by means of an example. This example possesses already some features of the setting which will be considered later.

An example

Let $\Omega \subset \mathbb{R}^N$ be a bounded open subset with a smooth boundary $\partial\Omega$. If Δ denotes the Laplace operator: $\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_N^2}$, then the operator $-\Delta$, with domain

$$D(-\Delta) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega),$$

is m-accretive in $L^2(\Omega)$ (see for example [BR3, page 206]). Here $L^2(\Omega)$ is equipped with the standard innerproduct denoted by $(\cdot,\cdot)$ and norm $\|\cdot\|_2$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be an everywhere defined non-decreasing function of class C^1, satisfying $\beta(0) = 0$. The function β is m-accretive in $\mathbb{R}$ [BR1, Proposition 2.7]. For each $\epsilon > 0$, consider the following equation

$$(I) \quad \begin{cases} \epsilon u - \Delta u + \beta(u) = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

We define the m-accretive operator B in $L^2(\Omega)$ by

$$\begin{cases} D(B) = \{u \in L^2(\Omega) : \text{the function } \beta(u) \text{ belongs to } L^2(\Omega)\}; \\ Bu(x) = \beta(u(x)) \text{ a.e. for } u \in D(B). \end{cases} \tag{1.3}$$

We claim that the equation (I) has a unique solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ for all $f \in L^2(\Omega)$, which means that the operator $-\Delta + B$ with domain $D(-\Delta) \cap D(B)$ is m-accretive in $L^2(\Omega)$.

For $k \in \mathbb{N}$ we define the Lipschitz continuous functions

$$\beta_k(x) = \begin{cases} \beta(-k), & x \quad \le -k; \\ \beta(x), & |x| \quad < k; \\ \beta(k), & x \quad \ge k, \end{cases}$$

and the corresponding operators B_k defined by $B_k u(x) = \beta_k(u(x))$ a.e. for all $u \in L^2(\Omega)$. The operators B_k, $k \in \mathbb{N}$, are m-accretive, everywhere defined and Lipschitz continuous in $L^2(\Omega)$. Therefore the approximate equation

$$\begin{cases} \epsilon u_k - \Delta u_k + B_k u_k = f, & \text{a.e. in } \Omega; \\ u_k = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

has a unique solution $u_k \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, for all $f \in L^2(\Omega)$, $k \in \mathbb{N}$. Moreover the sequence $\{u_k\}_{k=1}^{k=\infty}$ is bounded in $L^2(\Omega)$ by $\frac{1}{\epsilon}\|f\|_2$. By partial integration, using the fact that β_k is non-increasing and that $\beta_k(0) = 0$ for all $k \in I\!N$, we have that

$$\int_\Omega -\Delta u(x) B_k u(x) dx = \int_\Omega |\nabla u(x)|^2 \beta_k'(u(x)) dx \geq 0, \tag{1.5}$$

for all $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. It can be shown, multiplying equation (1.4) by $-\Delta u_k$, that inequality (1.5) implies the boundedness in $L^2(\Omega)$ of the sequences $\{-\Delta u_k\}_{k=1}^{k=\infty}$ and $\{B_k u_k\}_{k=1}^{k=\infty}$. By the compact embedding of $W^{2,2}(\Omega)$ into $L^2(\Omega)$ and the boundedness of the sequences $\{u_k\}_{k=1}^{k=\infty}$, $\{-\Delta u_k\}_{k=1}^{k=\infty}$ and $\{B_k(u_k)\}_{k=1}^{k=\infty}$, we can extract a convergent subsequence

$$u_{k_n} \to u, \qquad (n \to \infty),$$

and weakly convergent subsequences

$$-\Delta u_{k_n} \rightharpoonup v, \quad B_{k_n} u_{k_n} \rightharpoonup w, \qquad (n \to \infty),$$

in $L^2(\Omega)$ with u, v, $w \in L^2(\Omega)$. Since the operator $-\Delta$ is linear and closed, hence weakly closed, we get $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $v = -\Delta u$. It remains to show that $u \in D(B)$ and that $w = Bu$. Here the maximal monotonicity of the operator B plays a role. For all $z \in D(B)$ we have that

$$(u - z, w - Bz) = \lim_{k \to \infty} (u_{n_k} - z, B_{k_n} u_{k_n} - B_{k_n} z) \geq 0.$$

It follows that the graph $B \cup \{[u, w]\} \subset L^2(\Omega) \times L^2(\Omega)$ is monotone. Since B has no proper monotone extension it follows that $u \in D(B)$ and $w = Bu$. Hence, equation (I) has a unique solution $u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ for all $f \in L^2(\Omega)$. In other words the operator $-\Delta + B$ is m-accretive in $L^2(\Omega)$.

More generally, let $\beta \subset I\!R \times I\!R$ be an m-accretive graph, with $0 \in \beta(0)$. In this case, one can take the Yosida-approximation β_λ, $\lambda > 0$ (see below) which is everywhere defined, Lipschitz continuous and non-decreasing. By proceeding as above, one shows that

$$\int_\Omega -\Delta u(x) \beta_\lambda(u(x)) dx \geq 0, \quad \text{for all } u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \lambda > 0. \tag{1.6}$$

See [BR-C-P, Theorem 3.1].

The approximate equation $\epsilon u + \mathcal{L}u + \mathcal{A}_\lambda u = f$

We come back to equation (1.1) with $\mathcal{L}$ and $\mathcal{A}$ two m-accretive operators in $\mathcal{H}$ and $f \in \mathcal{H}$. One can define for a general m-accretive operator $\mathcal{A}$ in $\mathcal{H}$ the Yosida-approximation

$$\mathcal{A}_\lambda = \frac{1}{\lambda}(I - (I + \lambda\mathcal{A})^{-1}),$$

for all $\lambda > 0$. The Yosida-approximation $\mathcal{A}_\lambda$, $\lambda > 0$, is an everywhere defined Lipschitz continuous m-accretive operator [BR1]. (We note that $\mathcal{A}_\lambda$ is single-valued and that $\mathcal{A}_\lambda \subset \mathcal{A}(I + \lambda\mathcal{A})^{-1}$ in the sense of the corresponding graphs.) Therefore the equation

$$\epsilon u_\lambda + \mathcal{L}u_\lambda + \mathcal{A}_\lambda u_\lambda = f \quad \epsilon > 0, \tag{1.7}$$

has a unique solution $u_\lambda \in D(\mathcal{L})$ for all $f \in \mathcal{H}$ and $\lambda > 0$. By a result of Brezis, Crandall and Pazy [BR-C-P] (see also [BR1, Théorème 2.4]), equation (1.1) has a unique solution $u \in D(\mathcal{L}) \cap D(\mathcal{A})$ if and only if the net $\{\mathcal{A}_\lambda u_\lambda\}_{\lambda>0}$ remains bounded in $\mathcal{H}$. It is has been proven that the boundedness in $\mathcal{H}$ of the net $\{\mathcal{A}_\lambda u_\lambda\}_{\lambda>0}$ implies that the net $\{u_\lambda\}_{\lambda>0}$ converges strongly to some $u \in \mathcal{H}$. Therefore the net $\{(I + \lambda\mathcal{A})^{-1}u_\lambda\}_{\lambda>0}$ converges strongly to u as well, since

$$u_\lambda - (I + \lambda\mathcal{A})^{-1}u_\lambda = \lambda\mathcal{A}_\lambda u_\lambda \to 0, \quad (\lambda \downarrow 0).$$

Furthermore, by the boundedness of $\{\mathcal{A}_\lambda u_\lambda\}_{\lambda>0}$ there exists a weakly convergent subsequence

$$\mathcal{A}_{\lambda_n} u_{\lambda_n} \rightharpoonup w, \quad (n \to \infty),$$

for some $w \in \mathcal{H}$. Since $\mathcal{A}_\lambda \subset \mathcal{A}(I + \lambda\mathcal{A})^{-1}$, the monotonicity of $\mathcal{A}$ implies that

$$((u - z, w - \mathcal{A}z)) = \lim_{n\to\infty}((((I + \lambda_n\mathcal{A})^{-1}u_{\lambda_n} - z, \mathcal{A}_{\lambda_n}u_{\lambda_n} - \mathcal{A}z)) \geq 0,$$

for all $z \in D(\mathcal{A})$. By the maximal monotonicity of $\mathcal{A}$ we obtain that $u \in D(\mathcal{A})$ and $w \in \mathcal{A}u$. By the same argument, it follows that $u \in D(\mathcal{L})$ and that u satisfies equation (1.1).

In order to prove the boundedness of the net $\{\mathcal{A}_\lambda u_\lambda\}_{\lambda>0}$ one needs *a priori* estimates. A sufficient condition for $\{\mathcal{A}_\lambda u_\lambda\}_{\lambda>0}$ to be bounded in $\mathcal{H}$ for each $f \in \mathcal{H}$ is that the following inequality holds:

$$((\mathcal{L}u, \mathcal{A}_\lambda u)) \geq 0, \quad \text{for all } u \in D(\mathcal{L}),\ \lambda > 0. \tag{1.8}$$

Thus if $\mathcal{L}$ and $\mathcal{A}$ are m-accretive operators in $\mathcal{H}$ satisfying inequality (1.8), then the sum $\mathcal{L}+\mathcal{A}$ is m-accretive in $\mathcal{H}$. If the operators $\mathcal{L}$ and $\mathcal{A}$ satisfy inequality (1.8) then we shall say that $\mathcal{L}$ and $\mathcal{A}$ form an *acute angle* in $\mathcal{H}$ (see [SO], [K1]).

A result of Brezis and Strauss

Inequality (1.6) shows that the m-accretive operator $-\Delta$ with domain $W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$ and the operator B defined by (1.3) form an acute angle. This inequality is a special case of a result of Brezis and Strauss [BR-S, Lemma 2], who proved that if L is a linear m-accretive operator in $L^2(\Omega)$ such that

$$\begin{array}{l}\text{the resolvent } (I+\lambda L)^{-1} \text{ is order} - \text{preserving and} \\ \text{contractive in } L^p(\Omega) \text{ for all } p\in[1,\infty],\ \lambda>0,\end{array} \tag{1.9}$$

then

$$\int_\Omega (Lu)(x)\beta_\lambda(u(x))dx \geq 0, \quad \text{for all } u\in D(L), \lambda>0, \tag{1.10}$$

where $\beta\subset I\!R\times I\!R$ is an m-accretive graph satisfying $0\in\beta(0)$. We recall that the operator $-\Delta$ satisfies the property (1.9) (see [BR-S]). In the terminology of Bénilan and Crandall [B-C], an m-accretive operator L in $L^2(\Omega)$ satisfying (1.9) is called m-completely accretive in $L^2(\Omega)$. We note here that the operator L satisfies the property (1.9) if and only if $-L$ generates an order-preserving semigroup which is contractive in $L^p(\Omega)$ for all $p\in[1,\infty]$.

The m-accretive graph $\beta\subset I\!R\times I\!R$ is automatically a subdifferential of a convex and lower-semicontinuous function $j: I\!R\to[0,\infty]$ with $j(0)=0$ (see [BR1, page 43]). It can be shown that the inequality (1.10) with $\beta=\partial j$ is equivalent to the inequality

$$\int_\Omega j((I+\lambda L)^{-1}u(x))dx \leq \int_\Omega j(u(x))dx, \tag{1.11}$$

for all $u\in L^2(\Omega)$, $\lambda>0$ (see [B], [B-P]). The assumption (1.9) on L is necessary for inequality (1.10) to be true for all m-accretive graphs $\beta\subset I\!R\times I\!R$, $(0\in\beta(0))$. This follows from inequality (1.11) by choosing the convex and continuous functions $j(x)=|x|^p, p\in[1,\infty)$ and $j(x)=x\vee 0$.

The setting

In Chapter 3 we shall prove a *vector-valued* version of inequality (1.10). The structure we consider throughout, is motivated by the following two abstract differential equations.

$$(II)\quad \begin{cases} \frac{d}{dt}u(t) + Au(t) \ni f(t), & t \in (0,T]; \\ u(0) = 0, \end{cases}$$

$$(III)\quad \begin{cases} -\frac{d^2}{dt^2}u(t) + Au(t) \ni f(t), & t \in (0,T); \\ u(0) = u(T),\ u'(0) = u'(T). \end{cases}$$

Here A denotes a nonlinear, possibly multivalued, m-accretive operator in a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$ with normalization $0 \in A0$. The function f belongs to $L^2(0,T;H)$, where $L^2(0,T;H)$ is the Hilbert space equipped with the innerproduct $((u,v)) = \int_0^T \langle u(t), v(t)\rangle dt$. It is convenient to write each of these equations as an operator equation of the form

$$\mathcal{L}u + \mathcal{A}u \ni f. \tag{1.12}$$

The operator $\mathcal{A}$ is defined in $L^2(0,T;H)$ by

$$u \in D(\mathcal{A}) \text{ and } v \in \mathcal{A}u \text{ iff } u,\ v \in L^2(0,T;H) \text{ and } v(\cdot) \in Au(\cdot) \text{ a.e.}$$

The operators $\mathcal{L}$ corresponding to equation (II) and (III) are defined by

$$\begin{cases} D(\mathcal{L}) = \{u \in W^{1,2}(0,T;H) : u(0) = 0\}; \\ \mathcal{L}u = \frac{d}{dt}u, \quad \text{for } u \in D(\mathcal{L}), \end{cases} \tag{1.13}$$

respectively,

$$\begin{cases} D(\mathcal{L}) = \{u \in W^{2,2}(0,T;H) : u(0) = u(T),\ u'(0) = u'(T)\}; \\ \mathcal{L}u = -\frac{d^2}{dt^2}u, \quad \text{for } u \in D(\mathcal{L}). \end{cases} \tag{1.14}$$

The operators $\mathcal{A}$ and $\mathcal{L}$ are m-accretive operators in $L^2(0,T;H)$ (see for example [HA]). Instead of equation (1.12), we will first consider equation (1.1).

This motivates the following abstract setting. Let $(\Omega, \mathcal{M}, \nu)$ be a σ-finite measure space and let $L^2(\Omega;H)$ be the Hilbert space with the innerproduct

$$((u,v)) = \int_\Omega \langle u(\omega), v(\omega)\rangle d\nu(\omega), \quad u, v \in L^2(\Omega;H).$$

Let L be a linear m-accretive operator in $L^2(\Omega)$. In Section 2.4 we show, by using a result of Marcinkiewicz and Zygmund [E-G, page 203], that L is uniquely "extensible" to a linear m-accretive operator $\mathcal{L}$ in the Hilbert space $L^2(\Omega; H)$, with the property that

$$\mathcal{L}_\lambda(f \cdot x) = (L_\lambda f) \cdot x, \quad \text{for all } f \in L^2(\Omega),\ x \in H,\ \lambda > 0,$$

($f \cdot x$ is the function in $L^2(\Omega; H)$ defined by $(f \cdot x)(\omega) = f(\omega)x$ a.e.). Here $\mathcal{L}_\lambda$, (L_λ) denotes the Yosida-approximation of $\mathcal{L}$, (L).

We will be concerned with the sum $\mathcal{L} + \mathcal{A}$, where $\mathcal{L}$ is the m-accretive extension in $L^2(\Omega; H)$ of a linear m-accretive operator L in $L^2(\Omega)$, and where the operator $\mathcal{A}$ is the m-accretive operator in $L^2(\Omega; H)$ induced by the operator A:

$$u \in D(\mathcal{A}), v \in \mathcal{A}u \text{ iff } u, v \in L^2(\Omega; H) \text{ and } v(\cdot) \in Au(\cdot) \text{ a.e.}$$

Observe that the m-accretive operators in $L^2(0,T; H)$ defined by (1.13) and (1.14) are the extensions of the m-accretive operators in $L^2(0,T)$ defined by

$$\begin{cases} D(L) = \{u \in W^{1,2}(0,T) : u(0) = 0\}; \\ Lu = \frac{d}{dt}u, \quad \text{for } u \in D(L), \end{cases} \tag{1.15}$$

respectively,

$$\begin{cases} D(L) = \{u \in W^{2,2}(0,T) : u(0) = u(T),\ u'(0) = u'(T)\}; \\ Lu = -\frac{d^2}{dt^2}u, \quad \text{for } u \in D(L). \end{cases} \tag{1.16}$$

Operators forming an acute angle

Let us first restrict our attention to example (II). Let $\mathcal{L}$ be defined by (1.13). In the case that A is a subdifferential, $A = \partial\varphi$, with $\varphi \in \mathcal{J}_0$ (see Section 2.3), then the operators $\mathcal{L}$ and $\mathcal{A}$ form an acute angle in $\mathcal{H} = L^2(0,T; H)$, that is, $\mathcal{L}$ and $\mathcal{A}$ satisfy inequality (1.8) (see [BR1,Page 73]). The operator L defined by (1.15) satisfies (1.9) with $\Omega = (0,T)$. It will appear that this property is necessary and sufficient for the validity of inequality (1.8) for all subdifferentials $A = \partial\varphi$.

The class of linear m-accretive operators in $L^2(\Omega)$ for which (1.9) holds will be denoted by $\mathrm{M}(\Omega)$.

In Section 3.2 we prove that the inequality (1.8) holds ($\mathcal{H} = L^2(\Omega; H)$), for all subdifferentials $A = \partial\varphi$ with $\varphi \in \mathcal{J}_0$ if and only if $L \in \mathrm{M}(\Omega)$ [C-E, Theorem 1.1]. We refer to this result as the vector-valued version of inequality (1.10), since an m-accretive graph in $\mathbb{R}$ is a subdifferential .

We consider now example (III), so let $\mathcal{L}$ be defined by (1.14). It can be shown that inequality (1.8) holds for all m-accretive A in H with $0 \in A0$ (see [HA]). (For further information concerning abstract second order differential equations, we refer to [BB2], [BR4] and [M1,2,3].) It turns out that here the crucial point is that the operator $\mathcal{L}$ is the extension of the operator L defined by (1.16) which is symmetric and belongs to the class $\mathbf{M}(0,T)$.

In Section 3.3 we show, assuming that $\dim(H) > 1$, that inequality (1.8) holds for all m-accretive A in H with $0 \in A0$ if and only if $L \in \mathbf{M}(\Omega)$ and L is symmetric [C-E, Theorem 1.1].

It is of interest to note that if the operator A is linear, then $\mathcal{L}$ and $\mathcal{A}$ form an acute angle provided that L or A is symmetric (see Section 3.1). Whereas in the nonlinear case the class $\mathbf{M}(\Omega)$ is needed.

Let us recall that inequality (1.8) implies that the operator $\mathcal{L} + \mathcal{A}$ is m-accretive, hence equation (1.1) is uniquely solvable for all $f \in L^2(\Omega; H)$. In particular equation (1.12) with $\mathcal{A}$ replaced by $\mathcal{A}+\epsilon I$, $\epsilon > 0$ is uniquely solvable for all $f \in L^2(\Omega; H)$.

More can be said for equation (1.12). It follows from a result of Brezis and Haraux [BR-H] that inequality (1.8) implies that

$$\operatorname{int} R(\mathcal{L} + \mathcal{A}) = \operatorname{int}(R(\mathcal{L}) + R(\mathcal{A})) \tag{1.17}$$

holds, where $R(\mathcal{A})$ denotes the range of the operator $\mathcal{A}$ and " int " stands for the interior. Hence, when either $\mathcal{L}$ or $\mathcal{A}$ is surjective, then $\mathcal{L} + \mathcal{A}$ is surjective as well.

In particular equation (II) with $A = \partial\varphi$, a subdifferential, as is well-known, is solvable for all $f \in L^2(0,T; H)$.

Another consequence of the acute angle (1.8) is that the inequality

$$\epsilon^2\|u\|^2 + \|\mathcal{L}u\|^2 + \|v\|^2 \le \|f\|^2, \quad \epsilon > 0 \tag{1.18}$$

holds for equation (1.1), where $v \in \mathcal{A}u$ and $\epsilon u + \mathcal{L}u + v = f$ (see Section 3.4).

In fact, (1.17) is implied by the estimate (1.18).

The sum $\mathcal{L}+\mathcal{A}$ when $\mathcal{L}$ and $\mathcal{A}$ not forming an acute angle

Let L be a linear m-accretive operator in $L^2(\Omega)$ and A an m-accretive operator in H with $0 \in A0$. Let $\mathcal{L}$ and $\mathcal{A}$ denote the corresponding m-accretive extensions in $L^2(\Omega; H)$. We know that if $L \in \mathrm{M}(\Omega)$ and if L is symmetric then $\mathcal{L}$ and $\mathcal{A}$ form an acute angle, which implies that $\mathcal{L}+\mathcal{A}$ is m-accretive. We are now interested in the m-accretivity of $\mathcal{L}+\mathcal{A}$ when $\mathcal{L}$ and $\mathcal{A}$ do not form an acute angle.

We recall that if $\mathcal{L}+\mathcal{A}$ is m-accretive for all (linear) m-accretive A then L must be the negative generator of an analytic semigroup (see Section 4.3).

One approach to the problem of the m-accretivity of $\mathcal{L}+\mathcal{A}$, is to introduce a third operator $\mathcal{L}_0$ which forms an acute angle with both of the operators $\mathcal{A}$ and $\mathcal{L}$. More precisely we search for a symmetric operator $L_0 \in \mathrm{M}(\Omega)$ such that

$$\begin{cases} D(L_0) \subset D(L); \\ (Lu, L_0 u) \geq a\|L_0 u\|_2^2 - b\|u\|_2^2, \quad \text{for all } u \in D(L_0), \end{cases} \tag{1.19}$$

for some $a > 0$ and $b \in I\!R^+$. If such an operator L_0 exists, it is shown in Section 4.3 that $\{\mathcal{A}_\lambda u_\lambda\}_{\lambda>0}$ in equation (1.7) remains bounded and hence the m-accretivity of $\mathcal{L}+\mathcal{A}$ will follow.

Using this idea we prove in Section 4.3 that if $L \in \mathrm{M}(\Omega)$ is the negative generator of an analytic semigroup and if in addition L is normal then $\mathcal{L}+\mathcal{A}$ is m-accretive [EG2]. These conditions on the operator L in the case $\Omega = I\!R^N$ are satisfied when L is the negative generator of an order-preserving, translation invariant, analytic, contraction semigroup in $L^2(I\!R^N)$ (see example 4.3.4). It is a consequence of this result that the operator $(\frac{d}{dt})^\alpha + \mathcal{A}$ is m-accretive in $L^2(I\!R; H)$, $0 < \alpha < 1$. Here the operator $(\frac{d}{dt})^\alpha$, $0 < \alpha < 1$, denotes the fractional power of the m-accretive operator $\frac{d}{dt}$ in $L^2(I\!R; H)$ with domain $W^{1,2}(I\!R; H)$. It follows that equation (II) where $\frac{d}{dt}$ is replaced by $(\frac{d}{dt})^\alpha$, $0 < \alpha < 1$ is solvable for all $f \in L^2(0,T; H)$. We note that the operator $(\frac{d}{dt})^\alpha$, $0 < \alpha < 1$ on the interval $[0,T]$ is not a normal operator.

The m-accretivity of $\overline{\mathcal{L}+\mathcal{A}}$

In general $\mathcal{L}+\mathcal{A}$ is not m-accretive even if $L \in \mathrm{M}(\Omega)$ and A a linear m-accretive operator in H. Nevertheless it follows from [DP-G] (see Theorem 2.5.1) that if A is linear then

$$R(\epsilon I + \mathcal{L} + \mathcal{A}) \supset D(\mathcal{L}), \quad \epsilon > 0. \tag{1.20}$$

The inclusion (1.20) implies, in particular, that the operator $\overline{\mathcal{L}+\mathcal{A}}$, the closure of $\mathcal{L}+\mathcal{A}$, is m-accretive.

In Chapter 5, we formulate conditions for the operator L which ensure that $\overline{\mathcal{L}+\mathcal{A}}$ is m-accretive. As in Chapter 4, we explore the idea of introducing a suitable symmetric operator $L_0 \in \mathrm{M}(\Omega)$. It appears that if condition (1.19) is weakened to

$$\begin{cases} D(L_0) \subset D(L); \\ (Lu, L_0u) \geq 0, \quad \text{for all } u \in D(L_0), \end{cases} \tag{1.21}$$

then the operator $\overline{\mathcal{L}+\mathcal{A}}$ is m-accretive [EG1]. The method we use is, to solve first the singularly perturbed equation

$$u + \epsilon\mathcal{L}_0u + \mathcal{L}u + \mathcal{A}u \ni f, \quad \epsilon > 0. \tag{1.22}$$

It is then shown that for all $f \in L^2(\Omega; H)$ the corresponding solution u_ϵ of (1.22) converges strongly to a function u, which will be the unique solution of the equation $u + \overline{\mathcal{L}+\mathcal{A}}\, u \ni f$.

In the case that an operator $L \in L^2(I\!R^N)$ is translation invariant we may take L_0 to be $-\Delta$ which implies that $\overline{\mathcal{L}+\mathcal{A}}$ is m-accretive.

It is possible in some cases to take for L_0 the symmetric m-accretive operator L^*L (L^* denotes the adjoint of L). It can then be shown that the inclusion (1.20) holds. Since the operator L^* is monotone, the operator $L_0 = L^*L$ will always satisfy inequality (1.21). We show in Section 5.2 that $L^*L \in \mathrm{M}(\Omega)$ whenever $L \in \mathrm{M}(\Omega)$ is skew-adjoint.

We summarize some of the results. Let $L \in \mathrm{M}(\Omega)$ be a normal operator, and let A be an m-accretive operator in H with $0 \in A0$. Let the m-accretive operators $\mathcal{L}$ and $\mathcal{A}$ in the Hilbert space $\mathcal{H} = L^2(\Omega; H)$ denote the extensions of L and A respectively. Then

(i) The operators $\mathcal{L}$ and $\mathcal{A}$ form an acute angle for all m-accretive operators A in H, ($\dim(H) > 1$), if and only if L is selfadjoint;

(ii) The operator $\mathcal{L}+\mathcal{A}$ in $\mathcal{H}$ is m-accretive for all m-accretive operators A in H if and only if L is the negative generator of an analytic semigroup;

(iii) The inclusion (1.20) holds for all m-accretive operators A in H whenever L is skew-adjoint.

The considerations above motivates the following conjectures.

Open problems

Conjecture I: If $L \in \mathbf{M}(\Omega)$ and if $-L$ generates an analytic semigroup then the operator $\mathcal{L}+\mathcal{A}$ in $\mathcal{H}$ is m-accretive for all m-accretive A in H with $0 \in A0$.

Conjecture II: If $L \in \mathbf{M}(\Omega)$ then the operator $\overline{\mathcal{L}+\mathcal{A}}$ in $\mathcal{H}$ is m-accretive for all m-accretive A in H with $0 \in A0$.

An application to a semilinear elliptic system

Part two is devoted to some applications of the theory developed in Part one. In Chapter 6 we prove an existence result for a semilinear second order elliptic system. We mention first a special case of a result due to Brezis and Nirenberg [BR-N, Theorem III. 6′, Remark III.5] which is related to the elliptic system considered in Chapter 6. Let g be a continuous everywhere defined m-accretive mapping in $I\!R^N$, with respect to some innerproduct $\langle\cdot,\cdot\rangle$, and assume that $g(0) = 0$. Let $\Omega \subset I\!R^n$ be a bounded open subset with a smooth boundary and let M be a linear m-accretive operator in $L^2(\Omega; I\!R^N)$, with respect to the innerproduct $((\cdot,\cdot))$, such that $M - cI$ is accretive for some positive constant $c > 0$. Consider the following system

$$Mu + g(u) = f, \tag{1.23}$$

where $f \in L^2(\Omega; I\!R^N)$. It has been proven that if for all $R > 0$ there exists $C_R > 0$ such that

$$\langle x, g(x)\rangle \geq R|g(x)| - C_R \text{ for all } x \in I\!R^N, \tag{1.24}$$

and the set

$$\{u \in D(M) : \|u\|_1 \leq 1, \|Mu\|_1, ((Mu, u)) \leq 1\}, \tag{1.25}$$

is relatively compact in $L^1(\Omega; I\!R^N)$, then there exists $u \in L^2(\Omega; I\!R^N)$ such that $g(u) \in L^1(\Omega; I\!R^N)$ satisfying the system

$$\overline{M}u + g(u) = f,$$

where $\overline{M}$ denotes the closure of M in $L^2(\Omega; I\!R^N) \times L^1(\Omega; I\!R^N)$. The outline of the proof of this result is as follows : By a contraction argument, there exists a unique $u_\lambda \in D(M)$ satisfying the approximate equation

$$Mu_\lambda + g_\lambda(u_\lambda) = f, \quad \lambda > 0, \tag{1.26}$$

where g_λ denotes the Yosida-approximation of g. Multiplying (1.26) by u_λ we have

$$((Mu_\lambda, u_\lambda)) + ((g_\lambda(u_\lambda), u_\lambda)) \leq \|f\|_2\|u_\lambda\|_2.$$

Since $c\|u_\lambda\|_2^2 \leq ((Mu_\lambda, u_\lambda))$ and $((g_\lambda(u_\lambda), u_\lambda)) \geq 0$, $\lambda > 0$ it follows that the net $\{\|u_\lambda\|_2\}_{\lambda>0}$ is bounded and hence the nets

$$\{((g_\lambda(u_\lambda), u_\lambda))\}_{\lambda>0} \text{ and } \{((Mu_\lambda, u_\lambda))\}_{\lambda>0}$$

are bounded as well. Inequality (1.24) implies now that $\{g_\lambda(u_\lambda)\}_{\lambda>0}$, and therefore also $\{Mu_\lambda\}_{\lambda>0}$, remains bounded in $L^1(\Omega; I\!R^N)$. Thus we can extract a weakly convergent subsequence $u_{\lambda_n} \rightharpoonup u$ in $L^2(\Omega; I\!R^N)$ and, by the compactness of the set (1.25) in $L^1(\Omega; I\!R^N)$, a convergent subsequence $u_{\lambda_n} \to u$ in $L^1(\Omega; I\!R^N)$. We take the subsequence $\{u_{\lambda_n}\}_{n=1}^\infty$ such that $u_{\lambda_n} \to u$ almost everywhere. Therefore $g_{\lambda_n}(u_{\lambda_n}) \to g(u)$ almost everywhere. Using inequality (1.24) and the boundedness of the net $\{((g_\lambda(u_\lambda), u_\lambda))\}_{\lambda>0}$ it can be shown that the sequence $\{g_{\lambda_n}(u_{\lambda_n})\}$ is uniformly integrable. It follows by the convergence theorem of Vitali that $g_{\lambda_n}(u_{\lambda_n}) \to g(u)$ in $L^1(\Omega; I\!R^N)$ which proves the result.

It can be shown that if $g = \partial\varphi$ with $\varphi : I\!R^N \to I\!R^+$ convex and of class C^1 then g satisfies (1.24) (see [BR-N, Remark III.4]).

We obtain stronger results if the operator M in $L^2(\Omega; I\!R^N)$ is of the form

$$Mu = (L_1u_1, L_2u_2, \ldots, L_Nu_N), \quad u = (u_1, u_2, \ldots, u_N),$$

where $L_1, L_2, \ldots, L_N$ are N strictly elliptic m-accretive second order differential operators in $L^2(\Omega)$ with smooth coefficients and

$$D(L_i) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad i = 1, \ldots, N.$$

If $L_1 = L_2 = \ldots = L_N$ and g a subdifferential as above it follows from Theorem 3.2.1 and Lemma 2.2.3 that equation (1.23) has a unique solution $u \in \{W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\}^N$ for all $f \in L^2(\Omega; I\!R^N)$. More generally we treat the following system

$$\begin{cases} L_iu_i + A_i(u_1, \ldots, u_N) \ni f_i, & \text{in } \Omega, \quad i = 1, \ldots, N; \\ u_i = 0 & \text{on } \Omega, \quad i = 1, \ldots, N, \end{cases} \tag{1.27}$$

where $A = (A_1, \ldots A_N)$ is an m-accretive graph in $I\!R^N$ with respect to some innerproduct. It will be proven that there exist unique $u_i \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, $i = 1, \ldots, N$,for all $f_1, \ldots, f_N \in L^2(\Omega)$ satisfying (1.27).

Let $a_{ij} \in C^1(\overline{\Omega})$, $i, j = 1, \ldots, N$ be such that $a_{ij} = a_{ji}$ and, for some constant $\alpha > 0$, $\sum_{i,j} a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2$ on Ω for all $\xi = (\xi_1, \ldots, \xi_N) \in I\!R^N$. If

$L_1 = L_2 = \ldots = L_N = -\sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$, the existence and uniqueness result for system (1.27) follows from Theorem 3.3.1. In the case that the operators L_i, $i = 1, \ldots, N$ are different we use an inequality due to Sobolevskii [SO] (see also [BR-E]). This inequality, which will be stated precisely in Section 6.3, implies that L_i satisfies (1.19) with $L_0 = -\Delta$. More generally we shall consider (1.27) with right-hand side depending on $u_1, \ldots, u_N$, using a fixed point theorem.

Nonlinear Volterra equations

The last chapter deals with nonlinear Volterra equations in Banach spaces. We consider the equation

$$u(t) + \int_0^t b(t-s)Au(s)ds \ni f(t), \quad t \in [0,T], \tag{1.28}$$

where A is an m-accretive operator in a Banach space X, b a real, integrable kernel and f a given function taking values in X. As a general reference on Volterra equations we mention [GR-L-S]. Equations of the type (1.28) have been studied first by Barbu [BB] and Londen [LO] in the case where X is a Hilbert space, and A is the subdifferential of a convex function. The case where A is m-accretive in a Hilbert space has been considered by Gripenberg [GR1].

Existence results for equation (1.28) are obtained by approximating the operator A by its Yosida-approximation A_λ, $\lambda > 0$, and subsequently by passing to the limit as $\lambda \downarrow 0$. It follows from a contraction argument that the approximate equation

$$u_\lambda(t) + \int_0^t b(t-s)A_\lambda u_\lambda(s)ds = f(t), \quad t \in [0,T], \tag{1.29}$$

has a unique solution $u_\lambda \in L^1(0,T;X)$ whenever $b \in L^1(0,T)$ and $f \in L^1(0,T;X)$. The convergence of the net $\{u_\lambda\}_{\lambda>0}$ has then to be established. This has been done in [LO] and [GR1] for kernels satisfying

$$b \in AC[0,T], \quad b(0) > 0, \quad b' \in BV[0,T]. \tag{1.30}$$

It can be shown that a kernel b satisfies (1.30) if and only if $b \in L^1(0,T)$ and there exist a real number $\gamma > 0$ and a function $k \in BV[0,T]$ such that

$$\gamma b(t) + \int_0^t b(t-s)k(s)ds = 1, \quad t \in [0,T] \tag{1.31}$$

holds. This equivalence is used in [CR-N] to "invert the kernel b" and to write equation (1.28) as an abstract differential equation of the form

$$\begin{cases} \gamma \frac{d}{dt} u(t) + Au(t) \ni G(u)(t), & t \in [0,T]; \\ u(0) = x, \end{cases} \tag{1.32}$$

where G is a mapping satisfying certain Lipschitz conditions. Existence of solutions of the equation (1.28) is obtained via existence results for equations of the type (1.32), even when X is a general Banach space. It has been proven by Crandall and Liggett [CR-L] that equation (1.32) (with initial value $u(0) \in \overline{D(A)}$), where $G(u) = f \in L^1(0,T;X)$, has a unique generalized solution $u \in C([0,T];X)$ which, roughly speaking, may be obtained as a uniform limit of solution of an implicit difference scheme.

Motivated by positivity preserving and invariance properties for equation (1.28) in an ordered Banach space, Clément and Nohel [CL-N1] introduced a class of kernels b, called *completely positive* in [CL-N2]. By [CL-N2, Theorem 2.2], a kernel $b \in L^1(0,T)$ is completely positive if and only if there exist a real number $\gamma \geq 0$ and a function $0 \leq k \in L^1(0,T)$ non-increasing such that (1.30) holds (see also [CL-P] and [CL-M]).

In [CL-N1] existence of solutions of (1.28), where b is completely positive, is proven when A is a linear m-accretive operator in a Banach space.

The nonlinear case has been treated by Gripenberg [GR3]. If the kernel is completely positive and the function f is of the form $u_0 + \int_0^t b(t-s)g(s)ds$ with $g \in L^1(0,T;X)$, then equation (1.28) reduces to the equation

$$\begin{cases} \frac{d}{dt}(\gamma u(t) + \int_0^t k(t-s)u(s)ds) + Au(t) \ni k(t)u_0 + g(t), \ t \in [0,T]; \\ u(0) = u_0, \end{cases} \tag{1.33}$$

where $\gamma \geq 0$ and $0 \leq k \in L^1(0,T)$ non-increasing.

In [GR3, Theorem 1], Gripenberg showed that the existence of generalized solutions of equation (1.33) can be obtained, by approximating the operator

$$\mathcal{L}u(t) := \frac{d}{dt}(\gamma u(t) + \int_0^t k(t-s)u(s)ds) \tag{1.34}$$

by its Yosida-approximation, instead of approximating the nonlinearity. In Section 7.3 we give a modified proof of this result, in the spirit of this thesis, based on personal communication from dr. G. Gripenberg. We note that, by

[GR3, Theorem 3], generalized solutions of equation (1.33) can also be obtained for kernels $k = k_1 + k_2$, where $0 \leq k_1 \in L^1(0,T)$ is non-increasing and $k_2 \in BV[0,T]$.

In Section 7.2 we consider equation (1.33) with $u(0) = 0$, where $\gamma \geq 0$, $0 \leq k \in L^1(0,T)$ non-increasing, in the Hilbert space setting. Equation (1.33) fits into the theory developed in part one. The operator $\mathcal{L}$ in $L^2(0,T;H)$ expressed by (1.34) is the extension of an operator $L \in \mathrm{M}(0,T)$. If A is a subdifferential it follows immediately from Theorem 3.2.1 and Lemma 2.2.4 that the equation (1.33) is solvable for all $g \in L^2(0,T;H)$.

For m-accretive operators A in H which are not necessarily subdifferentials we obtain some new results in the case $\gamma = 0$. If $\gamma = 0$ and the kernel k is in addition convex we have that equation (1.33) is solvable for right-hand side $g \in D(\mathcal{L})$. Furthermore, if the operator $Lu(t) = \frac{d}{dt}\int\limits_0^t k(t-s)u(s)ds$ is the negative generator of an analytic semigroup in $L^2(0,T)$ then equation (1.33), with $\gamma = 0$, is solvable for all $f \in L^2(0,T;H)$. This is for example the case if $k(t) = t^{-\alpha}$, $t \in (0,T)$, $0 < \alpha < 1$. In particular the equation

$$(\tfrac{d}{dt})^\alpha u(t) + Au(t) \ni f(t), \quad t \in [0,T],$$

$0 < \alpha < 1$, has a solution $u \in D((\frac{d}{dt})^\alpha)$ for all $f \in L^2(0,T;H)$. Here $(\frac{d}{dt})^\alpha$ denotes the fractional power of the operator defined by (1.13).

Part I

Abstract theory

Chapter 2

Preliminaries

2.1 Introduction

In the first section we give the definition of a nonlinear, possibly multivalued, m-accretive operator in a Banach space. However, our attention will be restricted to m-accretive operators in a Hilbert space. We state some known results which provide a way to show the m-accretivity of the sum of two m-accretive operators in a Hilbert space.

In Section 2.3 and 2.4 we set up the structure, motivated in Chapter 1. In Section 2.3 we define the m-accretive extension $\mathcal{A}$ in $L^2(\Omega; H)$ of an m-accretive operator A in a Hilbert space H. Also the notion of a subdifferential will be introduced. In Section 2.4 we show that a linear m-accretive operator L in $L^2(\Omega)$ has a unique linear m-accretive "extension" $\mathcal{L}$ in $L^2(\Omega; H)$ such that $\mathcal{L}_\lambda(f \cdot x) = (L_\lambda f) \cdot x$ for all $f \in L^2(\Omega)$, $x \in H$, $\lambda > 0$.

In general it is not true that the operator $\mathcal{L}+\mathcal{A}$ is closed even if the operator A is linear. If A is linear then $\mathcal{L}$ and $\mathcal{A}$ commute in the sense of resolvents. In the last section we shall state some general theorems concerning the sum of two linear and resolvent commuting operators. A result of Da Prato and Grisvard [DP-G] tells us that, if A is linear, then $\overline{\mathcal{L}+\mathcal{A}}$, the closure of $\mathcal{L}+\mathcal{A}$, is m-accretive. It follows from another result in [DP-G] that $\mathcal{L}+\mathcal{A}$ is m-accretive if L is the negative generator of an analytic semigroup. This will also be a consequence of a result of Dore and Venni [D-V] where imaginary powers of operators plays a role. A basic assumption on the operators in [DP-G] and [D-V], besides that they are resolvent commuting, is that the sum of the angles of the sectors in which the spectra lie is less then π. Baillon and Clément [BA-C] proved that an additional condition is necessary in order to show the closedness of the sum. They constructed an example of two resolvent commuting closed operators in a Hilbert space with spectra on the non-negative real axis for which the sum is not closed. By [DP-G] and [D-V], both of the operators in this example can not be m-accretive.

2.2 m-Accretive operators

Definition of m-accretivity and some notations

Let X be a Banach space with norm $\|\cdot\|$. As usual we identify a graph $A \subset X \times X$ with its corresponding nonlinear, possibly multivalued operator $A : X \to 2^X$. The effective domain $D(A)$ of A is defined by

$$D(A) = \{x \in X : Ax \neq \emptyset\}.$$

We denote by $R(A)$ the range of A. The operator $A : D(A) \subset X \to 2^X$ is called *accretive* if

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\| \quad \text{for all } y_i \in Ax_i,\ i = 1,2,\ \lambda > 0.$$

The operator A is called m-accretive if A is accretive and $R(I + \lambda A) = X$, for all $\lambda > 0$. We will denote by J_λ^A the resolvent of A, that is, $J_\lambda^A = (I + \lambda A)^{-1}$. Thus A is m-accretive if J_λ^A is an everywhere defined (nonlinear) contraction for all $\lambda > 0$. The Yosida-approximation of A is defined, for $\lambda > 0$, by $A_\lambda = \frac{1}{\lambda}(I - J_\lambda^A)$ which is m-accretive as well.

m-Accretive operators in a Hilbert space

Let H be a real Hilbert space with innerproduct $\langle \cdot, \cdot \rangle$ and norm $|\cdot| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$. An operator $A : D(A) \subset H \to 2^H$ is then accretive if and only if

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0 \quad \text{for all } y_i \in Ax_i,\ i = 1,2.$$

An operator in H satisfying this property is also called *monotone*. A *maximal monotone* operator is defined as a monotone operator which has no proper monotone extension. Recall that the notions of m-accretivity and maximal monotonicity in a Hilbert space coincide. Furthermore the m-accretive operators in a Hilbert space H are precisely the negative generators of nonlinear strongly continuous semigroups on closed and convex subsets $C \subset H$.

In this thesis we are mainly concerned with the sum of two m-accretive operators in a Hilbert space. In particular to find conditions on the operators which ensure us that the sum is m-accretive. It should be noted that the sum of accretive operators in a Hilbert space is obviously accretive. Here we state some basic tools. First a simple case.

Lemma 2.1. [$BR1$, *Lemma* 2.4]. *Let $A \subset H \times H$ be m-accretive and let B be an accretive, Lipschitz continuous and everywhere defined operator in H. Then $A + B \subset H \times H$ is m-accretive.*

Throughout this section, let A, $B \subset H \times H$ denote two m-accretive operators. Lemma 2.1 shows that for all $f \in H$ there exists a unique $u_\lambda \in D(B)$ satisfying

$$\epsilon u_\lambda + A_\lambda u_\lambda + B u_\lambda \ni f, \quad \epsilon > 0. \tag{2.1}$$

A method to show that $f \in R(\epsilon I + A + B)$, $\epsilon > 0$, is to consider the approximate equation (2.1) and the following theorem due to Brezis, Crandall and Pazy [BR-C-P]

Theorem 2.2. *Let $\epsilon > 0$. Then $f \in R(\epsilon I + A + B)$ if and only if $A_\lambda u_\lambda$ remains bounded in H as $\lambda \downarrow 0$. If $f \in R(\epsilon I + A + B)$ then $u_\lambda \to u$ as $\lambda \downarrow 0$, where $u \in D(A) \cap D(B)$ is the solution of*

$$\epsilon u + Au + Bu \ni f. \tag{2.2}$$

Moreover $\|u_\lambda - u\| = o(\sqrt{\lambda})$ and $A_\lambda u_\lambda \to v$ as $\lambda \downarrow 0$, where v is the element with minimal norm of the closed and convex set $Au \cap (f - u - Bu)$.

The following lemma provides a useful sufficient condition for $A_\lambda u_\lambda$ to remain bounded.

Lemma 2.3. [$BR2$]. *Assume that $D(A) \cap D(B) \neq \emptyset$. If*

$$\langle A_\lambda u, Bu \rangle \geq 0 \quad \text{for all } u \in D(B),\ \lambda > 0, \tag{2.3}$$

then $A + B \subset H \times H$ is m-accretive.

It has been shown by Brezis and Haraux [BR-H] that inequality (2.3) has the following interesting consequence (see also [BR2]).

Theorem 2.4. *If inequality (2.3) holds under the same assumption of Lemma 2.3 then*

$$\overline{R(A+B)} = \overline{R(A) + R(B)} \text{ and } \operatorname{int} R(A+B) = \operatorname{int}(R(A) + R(B)).$$

In several cases it is not true that the sum $A+B$ is m-accretive but the closure $\overline{A+B}$ of (the graph) $A+B$ is m-accretive. If equation (2.2) has a solution in $D(A) \cap D(B)$ for f in a dense subset in H then $\overline{A+B}$ is m-accretive. We also note that if $\overline{A+B}$ is m-accretive then the solution u_λ of the approximate equation (2.1) converges to the solution u of

$$\epsilon u + \overline{A+B}\, u \ni f, \quad f \in H.$$

We end this section by stating a nonlinear Trotter product formula proved by Brezis and Pazy [BR-P], see also [BR1, Prop. 4.3 and 4.4].

Proposition 2.5. *(i) Assume that the operator $\overline{A+B}$ is m-accretive and denote by $\{S(t)\}_{t\geq 0}$, $\{S_A(t)\}_{t\geq 0}$ and $\{S_B(t)\}_{t\geq 0}$ the semigroup generated by $-\overline{A+B}$, $-A$ and $-B$ respectively. Then for all $u \in \overline{D(A) \cap D(B)}$,*

$$\{J^A_{\frac{t}{n}} J^B_{\frac{t}{n}}\}^n u \to S(t)u, \quad \text{if } n \to \infty, \tag{2.4}$$

uniformly on compact subsets of $[0,\infty)$.
(ii) Let $C \subset \overline{D(A)} \cap \overline{D(B)}$ be closed and convex, such that $J^A_\lambda C \subset C$ and $J^B_\lambda C \subset C$ for all $\lambda > 0$. If in addition A, B are single valued and $A+B$ is closed then for all $u \in C \cap \overline{D(A) \cap D(B)}$,

$$\{S_A(\frac{t}{n})S_B(\frac{t}{n})\}^n u \to S(t)u, \quad \text{if } n \to \infty, \tag{2.5}$$

uniformly on compact subsets of $[0,\infty)$.

2.3 The operator $\mathcal{A}$

Let $(\Omega, \mathcal{M}, \nu)$ be a σ-finite measure space and let $(H, \langle\cdot,\cdot\rangle, |\cdot|)$ be a real Hilbert space. Set $\mathcal{H} = L^2(\Omega; H)$, the Hilbert space of (equivalence classes) H-valued Bochner measurable functions $u : \Omega \to H$ which are square integrable, and with the innerproduct $((\cdot,\cdot))$, defined by

$$((u,v)) = \int_\Omega \langle u(\omega), v(\omega)\rangle d\nu(\omega), \quad \text{for } u,\ v \in \mathcal{H}.$$

Let $A \subset H \times H$ be m-accretive. If $\nu(\Omega) < \infty$ or if $0 \in A0$, then the operator $\mathcal{A} \subset \mathcal{H} \times \mathcal{H}$ defined by

$$\begin{cases} D(\mathcal{A}) = \{u \in \mathcal{H} : \exists\ v \in \mathcal{H} \text{ such that } v(\cdot) \in A(u(\cdot)) \text{ on } \Omega \text{ a.e.}\}; \\ \mathcal{A}u = \{v \in \mathcal{H} : v(\cdot) \in A(u(\cdot)) \text{ on } \Omega \text{ a.e.}\}, \end{cases} \tag{3.1}$$

is m-accretive.

Next, we introduce an important class of m-accretive operators. Set

$$\mathcal{J}_0 := \{\psi : H \to [0,\infty] : \psi \text{ l.s.c., convex, with } \psi(0) = 0\}. \tag{3.2}$$

Recall that the subdifferential $\partial\varphi$, $\varphi \in \mathcal{J}_0$, is defined by

$$(\partial\varphi)u = \{w \in H : \varphi(v) - \varphi(u) \geq \langle w, v - u\rangle \text{ for all } v \in D(\varphi)\}, \tag{3.3}$$

where $D(\varphi) = \{u \in H : \varphi(u) < \infty\}$. The set $D(\varphi)$ is called the effective domain of $\partial\varphi$. It is well-known that $\partial\varphi$ is m-accretive [BR1]. Furthermore, the Yosida-approximation is again a subdifferential. In fact, $(\partial\varphi)_\lambda = \partial\varphi_\lambda$, where $\varphi_\lambda : H \to [0,\infty)$ is the convex and Fréchet differentiable function given by

$$\varphi_\lambda(u) = \min_{v\in H}\{\frac{1}{2\lambda}|u - v|^2 + \varphi(v)\}, \quad u \in H.$$

Note that $0 \in (\partial\varphi)0$ since $\varphi(0) = 0$. It can be shown that if $A = \partial\varphi$, then its extension $\mathcal{A}$ defined by (3.1) is again a subdifferential. More precisely $\mathcal{A} = \partial\Phi$ where $\Phi : \mathcal{H} \to [0,\infty)$ is given by

$$\Phi(u) = \int_\Omega \varphi(u(\omega))d\nu(\omega), \quad u \in \mathcal{H}.$$

We refer again to [BR1] for the facts that Φ is lower semicontinuous, convex and that, indeed, $\mathcal{A} = \partial\Phi$. Moreover,

$$\Phi_\lambda(u) = \int_\Omega \varphi_\lambda(u(\omega))d\nu(\omega), \quad u \in \mathcal{H}.$$

Finally, we introduce a special class of lower semicontinuous convex functions which will be useful in the sequel.

Let $K \subset H$ be a closed and convex subset and define for $x \in H$

$$I_K(x) = \begin{cases} 0 & \text{if } x \in K; \\ +\infty & \text{otherwise.} \end{cases}$$

I_K is called the indicator function of K. Observe that if $\varphi = I_K$ then $\Phi = I_{\mathcal{H}_K}$, where

$$\mathcal{H}_K = \{u \in \mathcal{H} : u(\omega) \in K,\ \omega \in \Omega \text{ a.e. }\}.$$

Note that $\mathcal{H}_K \subset \mathcal{H}$ is also closed and convex.

2.4 The operator $\mathcal{L}$

Let $(X, \|\cdot\|)$ be a Banach space and let L be a linear densely defined operator in X. We use the notation $\rho(L)$ and $\sigma(L)$ for the resolvent set and spectrum of L. Consider the assumption

$(H1)$ $(-\infty, 0) \subset \rho(L)$ and for some constant $M \geq 1$, $\|J_\lambda^L\| \leq M$ for all $\lambda > 0$.

In the literature, an operator L in X satisfying (H1) is sometimes called a positive operator (see for example [TR]). Note that an operator L satisfying (H1), with constant $M = 1$, is m-accretive.

Let $L : D(L) \subset L^p(\Omega) \to L^p(\Omega)$, $p \in [1, \infty)$ be a linear operator satisfying (H1). Then there exists a unique linear operator $\mathcal{L}$ in $L^p(\Omega; H)$ satisfying (H1) with the same constant M as for L, such that

$$\langle x, (J_\lambda^{\mathcal{L}} u)(\omega)\rangle = (J_\lambda^L \langle x, u\rangle)(\omega) \tag{4.1}$$

for any $u \in L^p(\Omega; H)$, $x \in H$, $\lambda > 0$ and $\omega \in \Omega$ almost everywhere.

This extension is based on the following result of Marcinkiewicz and Zygmund [E-G, Page 203], see also [RF]:

A bounded linear operator $T : L^p(\Omega) \to L^p(\Omega)$ is uniquely extensible to a bounded linear operator $\tilde{T} : L^p(\Omega; H) \to L^p(\Omega; H)$ such that

$$\tilde{T}(f \cdot x)(\omega) = ((Tf) \cdot x)(\omega)$$

for all $f \in L^p(\Omega)$, $x \in H$ and $\omega \in \Omega$ almost everywhere ($(f \cdot x)(\omega) := f(\omega)x$, $\omega \in \Omega$ a.e.). Moreover $\|\tilde{T}\| = \|T\|$.

Applying this result to the bounded linear operator J_λ^L in $L^p(\Omega)$, $\lambda > 0$, we obtain a bounded linear operator $\tilde{J}_\lambda^L$ in $L^p(\Omega; H)$ such that

$$\tilde{J}_\lambda^L \langle f \cdot y\rangle(\omega) = \langle (J_\lambda^L f) \cdot y\rangle(\omega)$$

for all $f \in L^p(\Omega)$, $y \in H$ and $\omega \in \Omega$ almost everywhere. Using the linearity of the operator J_λ^L one verifies that

$$\langle x, \tilde{J}_\lambda^L (f \cdot y)(\omega)\rangle = (J_\lambda^L \langle x, f \cdot y\rangle)(\omega).$$

Next, let $u \in L^p(\Omega; H)$. Then there exists a sequence $\{u_n\}_{n=1}^\infty$ of simple functions in $L^p(\Omega; H)$ such that $u_n \to u$ in $L^p(\Omega; H)$ if $n \to \infty$. Since J_λ^L and

$\tilde{J}^L_\lambda$ are bounded operators we can choose the sequence $\{u_n\}_{n=1}^\infty$ such that

$$\begin{cases} J^L_\lambda\langle x, u_n\rangle \to J^L_\lambda\langle x, u\rangle; \\ \tilde{J}^L_\lambda u_n \to \tilde{J}^L_\lambda u, \qquad \text{pointwise a.e. as } n \to \infty. \end{cases}$$

Hence,

$$\langle x, (\tilde{J}^L_\lambda u)(\omega)\rangle = \lim_{n\to\infty}\langle x, (\tilde{J}^L_\lambda u_n)(\omega)\rangle = \lim_{n\to\infty}(J^L_\lambda\langle x, u_n\rangle)(\omega) = (J^L_\lambda\langle x, u\rangle)(\omega),$$

$\omega \in \Omega$ almost everywhere. It remains to show the existence of a unique linear operator $\mathcal{L}$ in $L^p(\Omega; H)$ such that $J^{\mathcal{L}}_\lambda = \tilde{J}^L_\lambda$ for all $\lambda > 0$. For that, consider the resolvent equation

$$\lambda J^L_\lambda - \mu J^L_\mu = (\lambda - \mu)J^L_\lambda J^L_\mu, \quad \lambda, \mu > 0.$$

Then,

$$\lambda \tilde{J}^L_\lambda - \mu \tilde{J}^L_\mu = (\lambda - \mu)\tilde{J}^L_\lambda \tilde{J}^L_\mu, \quad \lambda, \mu > 0.$$

Thus the family $\{\tilde{J}^L_\lambda,\ \lambda > 0\}$ is a so-called pseudoresolvent on $I\!R^+$ [P]. Moreover, $\tilde{J}^L_\lambda$ is injective. For, if $\tilde{J}^L_\lambda u = 0$, then $J^L_\lambda\langle x, u\rangle = \langle x, \tilde{J}^L_\lambda u\rangle = 0$ for all $x \in H$, hence $u = 0$. Therefore, there exists a linear operator $\mathcal{L}$ in $L^p(\Omega; H)$ such that $J^{\mathcal{L}}_\lambda = \tilde{J}^L_\lambda$ for all $\lambda > 0$ [P].

Remark 4.1. Note that the operator $\mathcal{L}$ is given by

$$\begin{cases} D(\mathcal{L}) &= \{u \in L^p(\Omega; H) : \langle u, x\rangle \in D(L) \text{ for all } x \in H \text{ and there} \\ & \quad\ \text{exists a } v \in L^p(\Omega; H) \text{ such that } L\langle u, x\rangle = \langle v, x\rangle\}; \\ \mathcal{L}u &= v \text{ for } u \in D(\mathcal{L}), \end{cases}$$

In the case $p = 2$ the extension result above can be proven directly, using completely orthonormal sets in $L^2(\Omega)$ and H, as we will point out here. Let $\{f_\tau\}_{\tau\in I}$ respectively $\{x_\sigma\}_{\sigma\in J}$ be a complete orthonormal set of $L^2(\Omega)$ respectively of H. Then $\{f_\tau \cdot x_\sigma\}_{\tau\in I, \sigma\in J}$ is a complete orthonormal set of $L^2(\Omega; H)$. For $T : L^2(\Omega) \to L^2(\Omega)$ linear and bounded we define

$$\tilde{T}(\sum_{i=1}^n f_{\tau_i} \cdot x_{\sigma_i}) := \sum_{i=1}^n (Tf_{\tau_i}) \cdot x_{\sigma_i}.$$

Set $u = \sum_{i=1}^n f_{\tau_i} \cdot x_{\sigma_i}$. Then,

$$\|\tilde{T}u\| = \{\sum_{i=1}^n \|Tf_{\tau_i}\|_2^2\}^{\frac{1}{2}} \le \|T\|\{\sum_{i=1}^n \|f_{\tau_i}\|_2^2\}^{\frac{1}{2}} = \|T\|\|u\|.$$

It follows that $\tilde{T}$ give rise to a bounded operator in $L^2(\Omega; H)$, also denoted by $\tilde{T}$, with $\|\tilde{T}\| \leq \|T\|$ and such that $\tilde{T}(f \cdot x) = (Tf) \cdot x$ for all $f \in L^2(\Omega), x \in H$. To show that $\|\tilde{T}\| = \|T\|$ take $f \in L^2(\Omega)$ with $\|f\|_2 = 1$ and $x \in H$ with $|x| = 1$. Then,

$$\|Tf\|_2 = \|(Tf) \cdot x\| = \|\tilde{T}(f \cdot x)\| \leq \|\tilde{T}\| \|f \cdot x\| = \|\tilde{T}\|.$$

Thus $\|T\| \leq \|\tilde{T}\|$ and equality follows.

Remark 4.2. Let L be a linear operator in $L^p(\Omega)$ satisfying (H1). If the resolvent family consists of positive (with respect to the order) operators, that is, $J^L_\lambda f \geq 0$ for all $f \geq 0$, $\lambda > 0$, then there exists a unique linear operator $\mathcal{L}$ in $L^p(\Omega; X)$ such that $\mathcal{L}$ satisfies (H1), with constant M, and

$$\langle x^*, (J^{\mathcal{L}}_\lambda u)(\omega)\rangle = (J^L_\lambda \langle x^*, u\rangle)(\omega)$$

for all $u \in L^p(\Omega; X), x^* \in X^*, \lambda > 0$ and $\omega \in \Omega$ almost everywhere. Here X^* denotes the dual space of X and $\langle \cdot, \cdot \rangle$ the pairing between X^* and X.

This follows from the known fact that a positive bounded linear operator T in $L^p(\Omega)$ is uniquely extensible to a bounded linear operator $\tilde{T}$ in $L^p(\Omega; X)$ such that $\tilde{T}(f \cdot x) = (Tf) \cdot x$ for all $f \in L^p(\Omega), x \in X$ and $\|\tilde{T}\| = \|T\|$. For the sake of completeness we prove it here, see also [RF].

For any function $S = \sum_{i=1}^{n} \chi_{E_i} x_i$ in $L^p(\Omega; X)$ with $E_i \in \mathcal{M}$ pairwise disjoint, $\nu(E_i) < \infty$ and $x_i \in, i = 1, ..., n$, define $\tilde{T}(S) := \sum_{i=1}^{n} (T\chi_{E_i}) x_i$. Then,

$$\begin{aligned}
\|\tilde{T}(S)\|_p &= \|\sum_{i=1}^{n} (T\chi_{E_i}) x_i\|_p = \{\int_\Omega \|\sum_{i=1}^{n} (T\chi_{E_i}) x_i\|^p d\nu\}^{\frac{1}{p}} \\
&\leq \{\int_\Omega (\sum_{i=1}^{n} |T(\chi_{E_i})| \|x_i\|)^p d\nu\}^{\frac{1}{p}} = \{\int_\Omega (\sum_{i=1}^{n} T(\chi_{E_i}) \|x_i\|)^p d\nu\}^{\frac{1}{p}} \\
&\leq \|T\| \{\int_\Omega (\sum_{i=1}^{n} \chi_{E_i} \|x_i\|)^p d\nu\}^{\frac{1}{p}} = \|T\| \{\int_\Omega \|(\sum_{i=1}^{n} \chi_{E_i} x_i\|)^p d\nu\}^{\frac{1}{p}} \\
&= \|T\| \|S\|_p.
\end{aligned}$$

Hence, $\tilde{T}$ is bounded on the dense subset of simple functions of $L^p(\Omega; X)$ with $\|\tilde{T}\| \leq \|T\|$. One verifies that $\tilde{T}$ is extensible to a bounded operator $\tilde{T}$ in $L^p(\Omega; X)$ with $\|\tilde{T}\| = \|T\|$.

2.5 Sum of commuting operators, the linear case

As pointed out in the introduction we will consider the sum "$\mathcal{L}+\mathcal{A}$" in the Hilbert space $\mathcal{H} := L^2(\Omega; H)$, where $\mathcal{L}$ is the extension of a linear m-accretive in $L^2(\Omega)$ as described in Section 2.4, and $\mathcal{A}$ the extension of a nonlinear possibly multivalued m-accretive operator $A \subset H \times H$ defined by (3.1). Observe that, in the case A is linear, the operators $\mathcal{L}$ and $\mathcal{A}$ are *resolvent commuting*, that is,

$$J_\lambda^{\mathcal{L}} J_\mu^{\mathcal{A}} = J_\mu^{\mathcal{A}} J_\lambda^{\mathcal{L}}, \quad \text{for all } \lambda, \mu > 0,$$

(equivalently for some $\lambda, \mu > 0$). Then $\mathcal{L}+\mathcal{A}$ is closable, and by a result of Da Prato and Grisvard [DP-G], the operator $\overline{\mathcal{L}+\mathcal{A}}$ is m-accretive. In fact they proved the following

Theorem 5.1. *Let X be a Banach space. Let A and B be two linear m-accretive resolvent commuting operators in X and let one of them be densely defined. Then,*

$$R(\epsilon I + A + B) \supset D(A) + D(B), \quad \text{for all } \epsilon > 0.$$

Recall that a linear m-accretive operator in a reflexive space is automatically densely defined.

We need the following definition. Set

$$S(\theta) = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \le \theta\},$$

for $0 \le \theta < \pi$.

A densely defined linear operator L in X is said to be of *type* (ω, M) if there exist $0 \le \omega < \pi$ and $M \ge 1$ such that

(i) $\sigma(L) \subset S(\omega) \cup \{0\}$

(ii) for all $0 \le \theta < \pi - \omega$ there exists $M(\theta) \ge 1$, with $M(0) = 1$, such that $\|J_z^L\| \le M(\theta)$ for all $z \in S(\theta)$.

It is well-known that an operator L in X satisfying (H1) is of type (ω, M) for some $\omega \in (0, \pi)$, $M \ge 1$. In the case L is m-accretive, L is of type $(\frac{\pi}{2}, 1)$ and if L is of type (ω, M) with $0 < \omega < \frac{\pi}{2}$ then $-L$ generates an analytic semigroup. Another result of Da Prato and Grisvard tells us that if A and B are two, resolvent commuting operators in a Banach space X of type (ω_A, M_A)

and (ω_B, M_B) respectively, such that $\omega_A + \omega_B < \pi$ then the equation

$$x + Ax + Bx = f, \tag{5.1}$$

is uniquely solvable if f belongs to certain interpolation spaces. For the precise statement we need the following definition. Let L satisfy (H1) in a Banach space X, $1 \le r < \infty$ and $0 < \theta < 1$. Define

$$D_L(\theta, r) := \{x \in X : \|x\|_{\theta,r} := \{\int_0^\infty \|t^{-\theta+1} L_t x\|^r \frac{dt}{t}\}^{\frac{1}{r}} < \infty\};$$

$$D_L(\theta, \infty) := \{x \in X : \|x\|_{\theta,r} := sup_{t>0} \|t^{-\theta+1} L_t x\| < \infty\}.$$

Theorem 5.2. $[DP - G]$ *Let A and B be two, resolvent commuting operators of type (ω_A, M_A) and (ω_B, M_B) respectively, such that $\omega_A + \omega_B < \pi$. Let $1 \le r \le \infty$ and $0 < \theta < 1$. Then $A + B$ is closable, $\overline{A+B}$ is of type (ω, M) with $\omega \le \max(\omega_A, \omega_B)$ and for every $f \in D_A(\theta, r)$ there exists a unique $x \in D(A) \cap D(B)$ satisfying (5.1). Furthermore Ax and Bx belong to $D_A(\theta, r)$ and there exists a constant M such that $\|Ax\|_{\theta,r} + \|Bx\|_{\theta,r} \le M\|f\|_{\theta,r}$*

In the case that the underlying space is a Hilbert space :

Theorem 5.3. *Let A and B be two, resolvent commuting operators in a Hilbert space H of type (ω_A, M_A) and (ω_B, M_B) respectively, such that $\omega_A + \omega_B < \pi$ and $D_A(\theta, 2) = D_{A^*}(\theta, 2)$ for some $0 < \theta < 1$. Then (5.1) is uniquely solvable for all $f \in H$ with solution $x \in D(A) \cap D(B)$ and $A + B$ is of type (ω, M) with $\omega \le \max(\omega_A, \omega_B)$.*

If A is an m-accretive operator in H with $N(A) = \{0\}$ then $D_A(\theta, 2) = D_{A^*}(\theta, 2)$ for all $0 < \theta < \frac{1}{2}$ as can be seen as follows. It is well-known that for such an operator one can define fractional powers A^z, $0 \le \mathrm{Re}z \le 1$ [KO]. In [K2] Kato proved that the purely imaginary powers of A are bounded : $\|A^{is}\| \le e^{\frac{\pi}{2}|s|}$, $s \in \mathbb{R}$. From this it follows that the complex interpolation space $[H, D(A)]_\theta$ coincides with $D(A^\theta)$ [A2]. Furthermore it is known that $[H, D(A)]_\theta = D_A(\theta, 2)$ [PE]. Finally it has been shown in [K1] that $D(A^\theta) = D(A^{*^\theta})$ for all $0 < \theta < \frac{1}{2}$.

It follows from Theorem 5.3 that if A and B are two m-accretive, resolvent commuting operators in a Hilbert space such that $-A$ generates an analytic semigroup then $A+B$ is m-accretive. This is also a consequence of the following theorem due to Dore and Venni [DO-V] where the boundedness of the purely imaginary powers is assumed:

$(H2)$ there exist constants $M \geq 1$ and $\omega \geq 0$ such that
$\|A^{is}\| \leq Me^{\omega|s|}$, for all $s \in \mathbb{R}$

Theorem 5.4. *Let A and B be two, resolvent commuting operators in a Hilbert space H of type (ω_A, M_A), (ω_B, M_B) such that $\omega_A + \omega_B < \pi$ and for some $0 < \epsilon < 1$, $\epsilon + A$ satisfies (H2). Then for all $f \in H$ there exists a unique $x \in D(A) \cap D(B)$ satisfying (5.1) and $A + B$ is of type (ω, M) with $\omega \leq \max(\omega_A, \omega_B)$.*

In [DO-V] and [PR-S] it has been proved that (H2) implies that A is of type (ω, M) if $\omega < \pi$. The converse question is negatively answered in [BA-C]. They showed the existence of an operator A, in a Hilbert space, of type $(0, M)$, for some $M > 1$, such that the imaginary powers A^{is} are unbounded for all $s \in \mathbb{R}\backslash\{0\}$.

Another example in [BA-C] is the construction of two resolvent commuting operators A and B of type $(0, M_A)$ and $(0, M_B)$ such that $A + B$ is not closed. Note that by Theorem 5.3 and Theorem 5.4 the constants M_A and M_B in this example are necessarily greater than one.

Chapter 3

The operators $\mathcal{L}$ and $\mathcal{A}$ forming an acute angle

3.1 Introduction

Let $(\mathcal{H}, ((\cdot,\cdot)))$ be a real Hilbert space and let $\mathcal{L}$ and $\mathcal{A}$ denote two, for the moment linear, m-accretive and resolvent commuting operators in $\mathcal{H}$. As we have seen in Section 2.5, the operator $\mathcal{L} + \mathcal{A}$ is m-accretive whenever $-\mathcal{L}$ generates an analytic semigroup. In the particular case that the operator $\mathcal{L}$ is symmetric even more is true. In this case the operators $\mathcal{L}$ and $\mathcal{A}$ form an acute angle:

$$((\mathcal{L}_\lambda u, \mathcal{A}_\mu u)) \geq 0, \quad \text{for all } \lambda,\ \mu > 0,\ u \in \mathcal{H} \tag{1.1}$$

Indeed, by using the commutativity of $(\mathcal{L}_\lambda)^{\frac{1}{2}}$ and $\mathcal{A}_\mu$ and the accretivity of $\mathcal{A}_\mu$ we have that

$$\begin{aligned} ((\mathcal{L}_\lambda u, \mathcal{A}_\mu u)) &= (((\mathcal{L}_\lambda)^{\frac{1}{2}} u, (\mathcal{L}_\lambda)^{\frac{1}{2}} \mathcal{A}_\mu u)) \\ &= (((\mathcal{L}_\lambda)^{\frac{1}{2}} u, \mathcal{A}_\mu (\mathcal{L}_\lambda)^{\frac{1}{2}} u)) \geq 0 \end{aligned}$$

By Lemma 2.2.3 inequality (1.1) implies, even if $\mathcal{L}$ and $\mathcal{A}$ are nonlinear, that $\mathcal{L} + \mathcal{A}$ is m-accretive. We will prove a nonlinear version of inequality (1.1).

Throughout this chapter we consider the following situation. Let $(\Omega, \mathcal{M}, \nu)$ be a σ-finite measure space and let $(H, \langle\cdot,\cdot\rangle)$ a real Hilbert space with norm $|\cdot| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. We denote by $\mathcal{H}$ the Hilbert space $L^2(\Omega; H)$ with innerproduct $((u,v)) = \int_\Omega \langle u(\omega), v(\omega)\rangle d\nu(\omega)$, $u,\ v \in \mathcal{H}$.

We assume that $\mathcal{L}$ is the linear m-accretive extension of a linear m-accretive operator L as discussed in Section 2.4 and $\mathcal{A}$ is assumed to be the m-accretive operator in $\mathcal{H}$ defined by (2.3.1), where $A \subset H \times H$ is a nonlinear, possibly multivalued m-accretive operator with normalization $0 \in A0$.

Motivated by the paper of Brezis and Strauss [BR-S], where $H = I\!R$, we consider the following assumption on L

$$\begin{cases} J_\lambda^L f \geq 0 & \text{for every } 0 \leq f \in L^2(\Omega), \lambda > 0; \\ \|J_\lambda^L f\|_p \leq \|f\|_p & \text{for every } f \in L^1(\Omega) \cap L^\infty(\Omega), p \in [1,\infty], \lambda > 0. \end{cases} \tag{1.2}$$

We shall denote by $\mathbf{M}(\Omega)$ the class of linear m-accretive operators L in $L^2(\Omega)$ satisfying (1.2).

Under the assumption $L \in \mathbf{M}(\Omega)$ we prove that if L is symmetric or if $A = \partial\varphi$, a subdifferential, with $\varphi \in \mathcal{J}_0$, then inequality (1.1) holds. This is a vector-valued extension of the result of Brezis and Strauss mentioned in the main introduction. Recall that if $H = I\!R$ then A is automatically a subdifferential. Furthermore it will be shown that the assumption $L \in \mathbf{M}(\Omega)$ is necessary for inequality (1.1) to be true for all subdifferentials and if (1.1) holds for m-accretive A then in addition L must be symmetric.

In the last section we show some consequences of inequality (1.1).

3.2 The case where $\mathcal{A}$ is a subdifferential

The main result in this section is the following

Theorem 2.1. *Let L be a linear m-accretive operator in $L^2(\Omega)$. Then inequality (1.1) holds for all $A = \partial\varphi$ with $\varphi \in \mathcal{J}_0$ if and only if $L \in \mathbf{M}(\Omega)$.*

Proof. Assume that $L \in \mathbf{M}(\Omega)$ and recall that if $A = \partial\varphi$ then $\mathcal{A} = \partial\Phi$, where $\Phi : \mathcal{H} \to [0,\infty]$ is given by $\Phi(u) = \int_\Omega \varphi(u(\omega))d\nu(\omega)$ for every $u \in \mathcal{H}$ and $\mathcal{A}_\mu = (\partial\Phi)_\mu = \partial\Phi_\mu$, $\mu > 0$, see Section 2.4. By the definition of a subdifferential,

$$((\mathcal{L}_\lambda u, \partial(\Phi_\mu)u)) = \frac{1}{\lambda}((u - J_\lambda^{\mathcal{L}} u, \partial(\Phi_\mu)u)) \geq \frac{1}{\lambda}(\Phi_\mu(u) - \Phi_\mu(J_\lambda^{\mathcal{L}} u)).$$

Thus in order to show inequality (1.1) it is sufficient (and necessary) to show that

$$\Phi_\mu(J_\lambda^{\mathcal{L}} u) \leq \Phi_\mu(u), \quad \text{for all } \lambda, \mu > 0. \tag{2.1}$$

Since a lower semicontinuous function is the pointwise supremum of continuous affine functions we have

$$\varphi_\mu(x) = sup_{\alpha \in I}(c_{\alpha,\mu} + \langle x_{\alpha,\mu}, x\rangle), \quad \text{for all } x \in H, \mu > 0,$$

for certain $c_{\alpha,\mu} \in I\!R$ and $x_{\alpha,\mu} \in H$, where $\alpha \in I$ and I an index set [BR3]. Note that $c_{\alpha,\mu} \leq 0$ for all $\alpha \in I$ since $\varphi_\mu(0) = 0$.

We prove that (2.1) holds for simple functions. Let $u \in \mathcal{H}$ be such a function, that is, $u = \sum_{i=1}^n \chi_{E_i} x_i$, where $x_i \in H$, $E_i \in \mathcal{M}$ pairwise disjoint such that $\nu(E_i) < \infty$ and χ_{E_i} the characteristic function of E_i, $i = 1, ..., n$. Set $E = \cup_{i=1}^n E_i$ and let from now on $\lambda, \mu > 0$, $\alpha \in I$. It follows from the assumption $L \in \mathbf{M}(\Omega)$ that $0 \leq J_\lambda^L \chi_E \leq 1$. Hence,

$$c_{\alpha,\mu} 1 \leq J_\lambda^L c_{\alpha,\mu} \chi_E \quad \text{a.e.}$$

By using that $\langle x_{\alpha,\mu}, (J_\lambda^{\mathcal{L}} u)(\omega)\rangle = J_\lambda^L \langle x_{\alpha,\mu}, u\rangle(\omega)$ a.e. (see Section 2.3) we obtain

$$c_{\alpha,\mu} 1 + \langle x_{\alpha,\mu}, (J_\lambda^{\mathcal{L}} u)(\omega)\rangle \leq (J_\lambda^L (c_{\alpha,\mu} \chi_E + \langle x_{\alpha,\mu}, u\rangle))(\omega) \quad \text{a.e.}$$

From the positivity of the resolvent J_λ^L and the inequality

$$c_{\alpha,\mu} \chi_E(\omega) + \langle x_{\alpha,\mu}, u(\omega)\rangle \leq \varphi_\mu(u(\omega)) \quad \text{a.e.,}$$

it follows that

$$c_{\alpha,\mu} 1 + \langle x_{\alpha,\mu}, (J_\lambda^{\mathcal{L}} u)(\omega)\rangle \leq (J_\lambda^L \varphi_\mu(u))(\omega)) \quad \text{a.e.} \tag{2.2}$$

Note that $\varphi_\mu(u) \in L^1(\Omega) \cap L^\infty(\Omega)$, since $\varphi_\mu(u) = \sum_{i=1}^n \varphi(x_i)\chi_{E_i}$. By taking the supremum, with respect to $\alpha \in I$, in inequality (2.2) we obtain

$$\varphi_\mu((J_\lambda^{\mathcal{L}} u)(\omega)) \leq (J_\lambda^L \varphi_\mu(u))(\omega) \quad \text{a.e.} \tag{2.3}$$

Inequality (2.3), together with the assumption that the resolvent

$$J_\lambda^L : L^1(\Omega) \cap L^\infty(\Omega) \to L^1(\Omega)$$

is a contraction with respect to the L^1-norm, implies that

$$\begin{aligned}
\Phi_\mu(J_\lambda^{\mathcal{L}} u) &= \int_\Omega \varphi_\mu((J_\lambda^{\mathcal{L}} u)(\omega)) d\nu(\omega) \\
&\leq \int_\Omega (J_\lambda^L \varphi_\mu(u))(\omega) d\nu(\omega) \\
&\leq \int_\Omega (\varphi_\mu(u))(\omega) d\nu(\omega) = \Phi_\mu(u).
\end{aligned}$$

We have proven that inequality (2.1) holds for all simple functions. Since the set of simple functions in $\mathcal{H}$ is a dense subset of $\mathcal{H}$ and $\Phi_\mu : \mathcal{H} \to [0, \infty)$ is a continuous function the inequality holds for all $u \in \mathcal{H}$.

Now we show that the assumption $L \in \mathbf{M}(\Omega)$ is necessary for the validity of inequality (1.1), or equivalently of (2.1), for all subdifferentials $A = \partial\varphi$, $\varphi \in \mathcal{J}_0$. Indeed, take $y \in H$ with $|y| = 1$.
(i) Let $p \in [1, \infty)$ and define $\varphi : H \to [0, \infty)$ by

$$\varphi(x) = |x|^p, \quad \text{for all } x \in H.$$

Then φ is a continuous function and $\Phi : \mathcal{H} \to [0, \infty)$ is given by

$$\Phi(u) = \int_\Omega |u|^p d\nu, \quad \text{for all } u \in \mathcal{H}.$$

One can verify that Φ is lower semicontinuous function by using Fatou's lemma. Let $u = f \cdot y$ where $f \in L^1(\Omega) \cap L^\infty(\Omega)$. Inequality (2.1) implies that

$$\int_\Omega |J_\lambda^L f|^p d\nu = \int_\Omega |J_\lambda^{\mathcal{L}}(f \cdot y)|^p d\nu \leq \int_\Omega |f \cdot y|^p d\nu = \int_\Omega |f|^p d\nu,$$

which shows the necessity of the contraction property of J_λ^L in L^p.
(ii) Set $K = \{z \in H : z = ty,\ t \in [0,1]\}$ and let $\Phi = I_{\mathcal{H}_K}$. Then (2.1) with $u = f \cdot y$, $0 \leq f \leq 1$ shows that $0 \leq J_\lambda^L f \leq 1$. Hence

$$\|J_\lambda^L h\|_\infty \leq \|h\|_\infty, \quad \text{for all } h \in L^1(\Omega) \cap L^\infty(\Omega).$$

(iii) If $K = \{y \in H : z = ty,\ t \in [0, \infty)\}$ and $\Phi = I_{\mathcal{H}_K}$ then (2.1) implies that the resolvent J_λ^L is order-preserving.

Remark 2.2. We note that inequality (1.1) with $\mathcal{A} = \partial\Phi$ is equivalent to each of the following assertions.
(i) $\Phi(J_\lambda^{\mathcal{L}} u) \leq \Phi(u)$ for all $u \in \mathcal{H}$, $\lambda > 0$.

(ii) $((\mathcal{L}_\lambda u, v)) \geq 0$ for all $u \in D(\partial\Phi)$, $v \in \partial\Phi u$.

(iii) $((\mathcal{L}u, (\partial\Phi)_\mu u)) \geq 0$ for all $u \in D(\mathcal{L})$, $\mu > 0$.

See [BR1, Théorème 4.4].

3.3 The case where $\mathcal{L}$ is symmetric

Besides $L \in \mathrm{M}(\Omega)$ we need an additional condition on L for inequality (1.1) to be true for general m-accretive $A \subset H \times H$. This condition is that L should be a subdifferential. Recall that a linear m-accretive operator in a Hilbert space is a subdifferential if and only if L is symmetric. From now on we consider $A \subset H \times H$ to be a general m-accretive operator.

Theorem 3.1. *Let L be a linear m-accretive operator in $L^2(\Omega)$ and assume that $\dim(H) > 1$. Then inequality (1.1) holds for all m-accretive $A \subset H \times H$ with $0 \in A0$ if and only if $L \in \mathrm{M}(\Omega)$ and L is symmetric.*

Remark 3.2. In the case $\dim(H) = 1$ we are in the situation of Theorem 2.1 since A is in this case a subdifferential.

Proof of Theorem 3.1. Assume that $L \in \mathrm{M}(\Omega)$ is symmetric. Let $u = \sum_{i=1}^n \chi_{E_i} x_i \in \mathcal{H}$ be a simple function as described in the proof of Theorem 2.1. Then

$$
\begin{aligned}
((\mathcal{L}_\lambda u, \mathcal{A}_\mu u)) &= \int_\Omega \langle \mathcal{L}_\lambda u, \mathcal{A}_\mu u \rangle d\nu \\
&= \frac{1}{\lambda} \int_\Omega \langle \sum_{i=1}^n \chi_{E_i} x_i - \sum_{i=1}^n J_\lambda^L(\chi_{E_i}) x_i, \sum_{j=1}^n \chi_{E_j} A_\mu x_j \rangle d\nu \\
&= \frac{1}{\lambda} \sum_{i=1}^n \langle x_i, A_\mu x_i \rangle \int_\Omega (\chi_{E_i} - J_\lambda^L(\chi_{E_i}) \chi_{E_i}) d\nu \\
&\quad - \frac{1}{\lambda} \sum_{i=1}^n \sum_{\substack{j=1 \\ j \neq i}}^n \langle x_i, A_\mu x_j \rangle \int_\Omega J_\lambda^L(\chi_{E_i}) \chi_{E_j} d\nu \\
&= \frac{1}{\lambda} \sum_{i=1}^n \langle x_i, A_\mu x_i \rangle \int_\Omega (\chi_{E_i} - J_\lambda^L(\chi_{E_i}) \chi_E) d\nu \\
&\quad + \frac{1}{\lambda} \sum_{i=1}^n \langle x_i, A_\mu x_i \rangle \int_\Omega J_\lambda^L(\chi_{E_i}) (\sum_{\substack{j=1 \\ j \neq i}}^n \chi_{E_j}) d\nu
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{\lambda}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\langle x_i, A_\mu x_j\rangle \int_\Omega J_\lambda^L(\chi_{E_i})\chi_{E_j}d\nu \\
= \; & \frac{1}{\lambda}\sum_{i=1}^{n}\langle x_i, A_\mu x_i\rangle \int_\Omega(\chi_{E_i} - J_\lambda^L(\chi_{E_i})\chi_E)d\nu \\
& +\frac{1}{\lambda}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\langle x_i, A_\mu x_i - A_\mu x_j\rangle \int_\Omega J_\lambda^L(\chi_{E_i})\chi_{E_j}d\nu.
\end{aligned}
$$

We claim that both terms are non-negative. Since $0 = A_\mu 0$ and A_μ is accretive, we have that $\langle x_i, A_\mu x_i\rangle \geq 0$ for $i = 1, \ldots, n$. Using the assumption that

$$J_\lambda^L : L^1(\Omega) \cap L^\infty(\Omega) \to L^1(\Omega)$$

is a positive contraction with respect to the L^1-norm we obtain

$$\int_\Omega J_\lambda^L(\chi_{E_i})\chi_E d\nu \leq \int_\Omega J_\lambda^L(\chi_{E_i})d\nu \leq \int_\Omega \chi_{E_i}d\nu \quad \text{for } i = 1, \ldots, n.$$

Hence, the first term is non-negative. In order to show that the second term is also non-negative we introduce for convenience the following notation :

$$
\begin{aligned}
c_{ij} &:= \int_\Omega J_\lambda^L(\chi_{E_i})\chi_{E_j}d\nu; \\
x_{ij} &:= \langle x_i, A_\mu x_i - A_\mu x_j\rangle, \quad \text{for } i, j = 1, \ldots, n.
\end{aligned}
$$

With these notations the second term becomes

$$\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} c_{ij}x_{ij}.$$

Observe that for $i, j = 1, \ldots, n$, $0 \leq c_{ij} = c_{ji}$, since J_λ^L is a positive symmetric operator in $L^2(\Omega)$ and that $x_{ij} + x_{ji} \geq 0$ since A_μ is accretive. Therefore,

$$
\begin{aligned}
\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} c_{ij}x_{ij} &= \sum_{i=1}^{n}\Big(\sum_{j<i} c_{ij}x_{ij} + \sum_{j>i} c_{ij}x_{ij}\Big) \\
&= \sum_{i=1}^{n}\Big(\sum_{j<i} c_{ij}x_{ij} + \sum_{j<i} c_{ji}x_{ji}\Big) \\
&= \sum_{i=1}^{n}\sum_{j<i} c_{ij}(x_{ij} + x_{ji}) \geq 0.
\end{aligned}
$$

Using the continuity of the operators $\mathcal{L}_\lambda$, $\mathcal{A}_\mu$ and the fact that the set of simple functions is dense in $\mathcal{H}$ it follows that inequality (1.1) holds for every $u \in \mathcal{H}$.

We have observed in the previous section that the assumption $L \in \mathbf{M}(\Omega)$ is necessary for inequality (1.1) to be true for all subdifferentials $A = \partial\varphi$ with $\varphi \in \mathcal{J}_0$. If inequality (1.1) holds for general $A \subset H \times H$ m-accretive (with $0 \in A0$) and if $\dim(H) > 1$ then in addition L must be symmetric. Indeed, let $x_1,\ x_2 \in H$ be two orthonormal vectors and let A be the bounded operator H defined by

$$\begin{cases} Ax_1 &= -x_2; \\ Ax_2 &= x_1; \\ Ax &= 0, \quad \text{for every } x \in \{x_1, x_2\}^\perp. \end{cases}$$

Then A and $-A$ are accretive since $\langle Ax, x\rangle = 0$ for all $x \in H$. Therefore, since $\pm A$ are bounded, $\pm A$ are m-accretive operators in H. By assuming that inequality (1.1) holds we have

$$\pm((\mathcal{L}_\lambda u, \mathcal{A}_\mu u)) \geq 0 \text{ for all } \lambda,\ \mu > 0 \text{ and } u \in \mathcal{H}.$$

Since A is single-valued and everywhere defined it follows, by letting $\mu \downarrow 0$, that

$$((\mathcal{L}_\lambda u, \mathcal{A}u)) = 0 \quad \text{for all } \lambda > 0 \text{ and } u \in \mathcal{H}. \tag{3.1}$$

Let $f_1,\ f_2 \in L^2(\Omega)$ and take $u := f_1 \cdot x_1 + f_2 \cdot x_2$ in (3.1). Then

$$((\mathcal{L}_\lambda(f_1 \cdot x_1 + f_2 \cdot x_2), f_2 \cdot x_1)) = ((\mathcal{L}_\lambda(f_1 \cdot x_1 + f_2 \cdot x_2), f_1 \cdot x_2)).$$

Hence,

$$\int_\Omega (L_\lambda f_1) f_2 d\nu = \int_\Omega (L_\lambda f_2) f_1 d\nu \quad \text{for every } f_1,\ f_2 \in L^2(\Omega) \text{ and } \lambda > 0.$$

Finally, since $L_\lambda = L_\lambda^*$ for all $\lambda > 0$ we get $L = L^*$.

3.4 Consequences of the acute angle

The main implication of the acute angle (1.1) is that the sum $\mathcal{L} + \mathcal{A}$ is m-accretive. Here we give some other consequences.

Theorem 4.1. *Let L and A be as in Theorem 2.1 and 3.1 such that inequality (1.1) holds. Let $f \in \mathcal{H}$ and $\epsilon > 0$. Then* *(i) For all* $\lambda > 0$ *there exists a unique* $u_\lambda \in D(\mathcal{A})$ *satisfying*

$$\epsilon u_\lambda + \mathcal{L}_\lambda u_\lambda + \mathcal{A}u_\lambda \ni f.$$

For all $\mu > 0$ *there exists a unique* $u_\mu \in D(\mathcal{L})$ *satisfying*

$$\epsilon u_\mu + \mathcal{L}u_\mu + \mathcal{A}_\mu u_\mu = f.$$

Moreover $\lim_{\lambda\downarrow 0} u_\lambda = \lim_{\mu\downarrow 0} u_\mu = u$, *where* $u \in D(\mathcal{L}) \cap D(\mathcal{A})$ *is the unique solution of* $\epsilon u + \mathcal{L}u + \mathcal{A}u \ni f$.

(ii) The estimate

$$\epsilon^2\|u\|^2 + \|\mathcal{L}u\|^2 + \|v\|^2 \leq \|\epsilon u + \mathcal{L}u + v\|^2 \tag{4.1}$$

holds for all $u \in \overline{D(\mathcal{L}) \cap D(\mathcal{A})}$, $v \in \mathcal{A}u$.
(iii) $\overline{R(\mathcal{L}+\mathcal{A})} = \overline{R(\mathcal{L}) + R(\mathcal{A})}$ *and* $\operatorname{int} R(\mathcal{L}+\mathcal{A}) = \operatorname{int}(R(\mathcal{L}) + R(\mathcal{A}))$.

(iv) Let $\{S(t)\}_{t>0}$ *denote the semigroup generated by* $-(\mathcal{L}+\mathcal{A})$. *Then for all* $u \in \overline{D(\mathcal{L}) \cap D(\mathcal{A})}$,

$$\lim_{n\to\infty} \{J^{\mathcal{L}}_{\frac{t}{n}} J^{\mathcal{A}}_{\frac{t}{n}}\}^n u = S(t)u, \tag{4.2}$$

uniformly on compact subsets of $[0,\infty)$.

Proof. (i) See Theorem 2.2.2.
(ii) Let $\epsilon > 0$, $u \in D(\mathcal{L}) \cap D(\mathcal{A})$ and $v \in \mathcal{A}u$ and consider the approximation equation

$$\epsilon u_\lambda + \mathcal{L}u_\lambda + \mathcal{A}_\lambda u_\lambda = \epsilon u + \mathcal{L}u + v, \quad \lambda > 0.$$

Then, using inequality (1.1) and the accretivity of the operators $\mathcal{L}$ and $\mathcal{A}$ it follows that

$$\|\epsilon u_\lambda\|^2 + \|\mathcal{L}u_\lambda\|^2 + \|\mathcal{A}_\lambda u_\lambda\|^2 \leq \|\epsilon u + \mathcal{L}u + v\|^2, \quad \lambda, \epsilon > 0.$$

By Theorem 2.2.2,

$$u_\lambda \to u, \quad \mathcal{L}u_\lambda \to \mathcal{L}u, \quad \mathcal{A}_\lambda u_\lambda \to v, \quad \text{if } \lambda \downarrow 0,$$

and (4.1) follows.
(iii) See Theorem 2.2.4
(iv) See Proposition 2.2.5

Next, we prove an invariance result.

Theorem 4.2. *Let $K \subset H$ be a closed and convex set such that $0 \in K$ and let L be a linear m-accretive operator in $L^2(\Omega)$ which satisfies*

$$f \in L^2(\Omega),\ 0 \le f \le 1 \text{ a.e. implies } \ 0 \le J_\lambda^L f \le 1 \text{ a.e. for all } \lambda > 0.$$

Let $A \subset H \times H$ be m-accretive and assume that $\overline{\mathcal{L}+\mathcal{A}} \subset \mathcal{H}\times\mathcal{H}$ is m-accretive. If $J_\lambda^A K \subset K$ for all $\lambda > 0$ then the solution of

$$u + \overline{\mathcal{L}+\mathcal{A}}u \ni f, \quad f \in \mathcal{H}_K \tag{4.3}$$

belongs to $\mathcal{H}_K$

Proof. Let $f \in \mathcal{H}_K$. It follows from the proof of Theorem 2.1 that inequality (2.3) holds for all $\varphi \in \mathcal{J}_0$. Since $(I_K)_\mu(x) = 0$ if and only if $x \in K$ it follows from (2.3) with $\varphi = I_K$ that

$$0 \le (I_K)_\mu((J_\lambda^{\mathcal{L}} f)(\omega)) \le (J_\lambda^L (I_K)_\mu(f))(\omega) = 0,$$

almost everywhere, and therefore $J_\lambda^L f \in \mathcal{H}_K$, for all $\lambda > 0$. Consider the equation

$$u + \mathcal{L}_\lambda u + \mathcal{A}u \ni f, \quad f \in \mathcal{H}_K. \tag{4.4}$$

This equation can be written as

$$u = J^{\mathcal{A}}_{\frac{\lambda}{\lambda+1}}\left(\frac{1}{\lambda+1} J_\lambda^{\mathcal{L}} u + \frac{\lambda}{\lambda+1} f\right) \tag{4.5}$$

Observe that the right-hand side defines a strict (nonlinear) contraction in $\mathcal{H}$. Thus, there exists a unique solution, which we denote by u_λ, of equation (4.4). We claim that $u_\lambda \in \mathcal{H}_K$. Namely, if $u \in \mathcal{H}_K$ then $J_\lambda^{\mathcal{L}} u \in \mathcal{H}_K$ and therefore

$$\frac{1}{\lambda+1} J_\lambda^{\mathcal{L}} u + \frac{\lambda}{\lambda+1} f \in \mathcal{H}_K,$$

since it is a convex combination of elements in $\mathcal{H}_K$. Hence the right-hand side of (4.5) defines a strict contraction in $\mathcal{H}_K$. Finally, one shows that $u_\lambda \to u$, if $\lambda \downarrow 0$ where u is the solution of equation (4.3). By the closedness of $\mathcal{H}_K$ it follows that $u \in \mathcal{H}_K$.

Remark 4.3. Provided that $\overline{\mathcal{L}+\mathcal{A}} \subset \mathcal{H} \times \mathcal{H}$ is m-accretive, the Trotter product formula gives that, denoting the semigroup generated by $-\overline{\mathcal{L}+\mathcal{A}}$ by $\{S(t)\}_{t\ge 0}$, for $x \in \overline{D(\mathcal{L}) \cap D(\mathcal{A})}$,

$$\{J^{\mathcal{L}}_{\frac{t}{n}} J^{\mathcal{A}}_{\frac{t}{n}}\}^n x \ \to \ S(t)x, \quad \text{if } n \to \infty,$$

uniformly on bounded subsets of $[0,\infty)$, (see Proposition 2.2.5). In order to prove Theorem 4.2 one could also use this formula. It follows, under the assumptions of Theorem 4.2, that $\{J^{\mathcal{L}}_{\frac{t}{n}} J^{\mathcal{A}}_{\frac{t}{n}}\}^n$ leaves $\mathcal{H}_K$ invariant, so that

$$S(t)(\overline{D(\mathcal{L}) \cap D(\mathcal{A})} \cap \mathcal{H}_K) \cap \mathcal{H}_K, \quad \text{for all } t \geq 0.$$

This implies that $J_\lambda^{\overline{\mathcal{L}+\mathcal{A}}} \mathcal{H}_K \cap \mathcal{H}_K$, for all $\lambda > 0$ [BR1, Prop.4.5].

Denote by $\{T(t)\}_{t\geq 0}$, $\{\mathcal{T}(t)\}_{t\geq 0}$ the C_0-contraction semigroup generated by $-L$ and $-\mathcal{L}$ respectively. It is clear that if $L \in \mathrm{M}(\Omega)$ then the semigroup $\{T(t)\}_{t\geq 0}$ satisfies (1.2) with J^L_λ replaced by $T(\lambda)$. Note also that if L is symmetric then the semigroup is symmetric as well. Then, setting

$$T_{(\lambda)} := \frac{1}{\lambda}(I - T(\lambda)),$$

it follows from the proof of Theorem 2.1 and Theorem 3.1 that, under the same assumptions, the inequality

$$((\mathcal{T}_{(\lambda)} u, \mathcal{A}_\mu u)) \geq 0 \quad \text{for all } u \in \mathcal{H},\ \lambda,\ \mu > 0 \tag{4.6}$$

holds. Inequality (4.6) provides another way to approximate the equation $u + \mathcal{L}u + \mathcal{A}u \ni f$.

Proposition 4.4. *Let $\mathcal{L}$ be a linear m-accretive operator in a Hilbert space $\mathcal{H}$ and $\mathcal{A} \subset \mathcal{H} \times \mathcal{H}$ an m-accretive operator such that $D(\mathcal{L}) \cap D(\mathcal{A}) \supset \{0\}$ and (4.6) holds. Then $\mathcal{L} + \mathcal{A} \subset \mathcal{H} \times \mathcal{H}$ is m-accretive and the solution $u_\lambda \in D(\mathcal{A})$ of*

$$u_\lambda + \mathcal{T}_{(\lambda)} u_\lambda + \mathcal{A} u_\lambda \ni f, \quad f \in \mathcal{H} \tag{4.7}$$

converges strongly to the solution $u \in D(\mathcal{L}) \cap D(\mathcal{A})$ of $u + \mathcal{L}u + \mathcal{A}u \ni f$.

Proof. By taking the inner product of (4.7) with $\mathcal{T}_{(\lambda)} u_\lambda$ it follows that $\|\mathcal{T}_{(\lambda)} u_\lambda\|$ remains bounded if $\lambda \downarrow 0$. Thus we can find subsequences

$$u_{\lambda_n} \rightharpoonup u, \quad \mathcal{T}_{(\lambda_n)} u_{\lambda_n} \rightharpoonup w, \quad v_n \rightharpoonup f - u - w, \quad \text{as } n \to \infty,$$

for certain u, $w \in \mathcal{H}$ and $v_n \in \mathcal{A} u_{\lambda_n}$, $n = 1, 2, \ldots$. Then for all $z \in \mathcal{H}$

$$\begin{aligned} ((\mathcal{T}_{(\lambda_n)} u_{\lambda_n}, z)) &= ((\frac{1}{\lambda_n} \int_0^{\lambda_n} \mathcal{T}(s) u_{\lambda_n} ds, z)) \\ &= ((u_{\lambda_n}, \frac{1}{\lambda_n} \int_0^{\lambda_n} \mathcal{T}^*(s) z ds)) \to ((u, z)) \end{aligned}$$

as $n \to \infty$, since the dual semigroup $\{\mathcal{T}^*(t)\}_{t \geq 0}$ is also strongly continuous. Since $\mathcal{T}_{(\lambda)} u_\lambda = \mathcal{L}(\frac{1}{\lambda}\int_0^\lambda \mathcal{T}(s) u_\lambda ds)$ and $\mathcal{L}$ is a linear and closed operator we obtain $u \in D(\mathcal{L})$ and $w = \mathcal{L}u$. One verifies that

$$\limsup_{n \to \infty}((u_{\lambda_n}, v_n)) \leq ((u, f - u - \mathcal{L}u)),$$

and by the m-accretivity of $\mathcal{A}$ this implies that $u \in D(\mathcal{A})$ and $f - u - \mathcal{L}u \in \mathcal{A}u$. By a standard argument we have that $u_\lambda \rightharpoonup u$ if $\lambda \downarrow 0$. Finally, taking the innerproduct of

$$u_\lambda - u + \mathcal{T}_{(\lambda)} u_\lambda - \mathcal{T}_{(\lambda)} u + \mathcal{T}_{(\lambda)} u - \mathcal{L}u + \mathcal{A}u_\lambda - \mathcal{A}u \ni 0,$$

with $u_\lambda - u$ we obtain

$$\|u_\lambda - u\| \leq \|\mathcal{T}_{(\lambda)} u - \mathcal{L}u\| \to 0 \quad \text{if } \lambda \downarrow 0.$$

Chapter 4

Angleboundedness and the m-accretivity of $\mathcal{L} + \mathcal{A}$

4.1 Introduction

Let L be a linear m-accretive operator in $L^2(\Omega)$ and A an m-accretive operator in H. In Section 3.3 it is proven that the operator $\mathcal{L} + \mathcal{A}$ is m-accretive if L is symmetric and in the class $\mathbf{M}(\Omega)$. We extend this result to a larger class of operators.

Even if the operator A is linear, $\mathcal{L} + \mathcal{A}$ need not to be m-accretive. If $\mathcal{L} + \mathcal{A}$ is m-accretive for all m-accretive operators A in H then L must be the negative generator of an analytic semigroup (Remark 3.5). As we know already this condition on L is sufficient if the operator A is linear (see Section 2.4). In the nonlinear case, if L is symmetric then $\mathcal{L} + \mathcal{A}$ need not to be m-accretive. In the last section we give such an example where the operator A is a subdifferential.

Motivated by Chapter 3 we assume that L belongs to $\mathbf{M}(\Omega)$. We conjecture that $\mathcal{L} + \mathcal{A}$ is m-accretive if $L \in \mathbf{M}(\Omega)$ is the negative generator of an analytic semigroup. In the case L is normal we answer this conjecture affirmativily in Section 4.3.

We show in Section 4.2 that angleboundedness is equivalent to the notion of regular accretivity. It is well-known that if L is regularly accretive then $-L$ generates an analytic semigroup, see [K1]. On the other hand we show that if L is normal and the negative generator of an analytic semigroup then $L + cI$ is anglebounded (hence regularly accretive) for some $c \geq 0$.

An example of an operator L which is normal and the negative generator of an analytic semigroup is the operator $L = (\frac{d}{dt})^\alpha$, in $L^2(I\!R)$, $0 < \alpha < 1$, the fractional power of the operator $\frac{d}{dt}$ with domain $W^{1,2}(I\!R)$. In Chapter 7 we consider more generally, operators of the form $(Lu)(t) = \frac{d}{dt}\int_0^t k(t-s)u(s)ds$, where k is a real kernel.

4.2 Angleboundedness and regular accretivity

Let H be a real Hilbert space with innerproduct $(\cdot,\cdot)$. An accretive operator L in H is called *anglebounded* if for an $a \geq 0$,

$$|(Lf,g) - (f,Lg)| \leq 2a(Lf,f)^{\frac{1}{2}}(Lg,g)^{\frac{1}{2}}, \quad \text{for all } f,\ g \in D(L).$$

This notion is introduced by Amann, see e.g. [A1]. For equivalent formulations of angleboundedness we refer to [BA-H].

Let $V \subset H$ be a continuously and densily embedded Hilbert space and denote the innerproduct of V by $(\cdot,\cdot)_V$ and $\|f\|_V := (f,f)_V^{\frac{1}{2}}$, $f \in V$. Let

$$\varphi(\cdot,\cdot) : V \times V \to I\!R$$

be bilinear, continuous and coercive, that is, there exists an $\alpha > 0$ such that

$$\alpha\|f\|_V^2 \leq \varphi(f,f), \quad \text{for all } f \in V.$$

It is well-known that there exists a unique m-accretive operator L such that

$$\begin{cases} D(L) \subset V \\ \varphi(f,g) = (Lf,g) \quad \text{for all } f \in D(L),\ g \in V \end{cases}$$

Such an operator L is said to be *regularly accretive*, see [K1].

We recall that an m-accretive operator L is called *strictly* m-accretive if $L - cI$ is accretive for some constant $c > 0$.

Theorem 2.1. *The following assertions are equivalent*
(i) L is regularly accretive.
(ii) L is strictly m-accretive and anglebounded.

Proof. (i) $\Rightarrow$ (ii). By assumption, there exists constants $c_1, c_2 > 0$ such that $\|\cdot\| \leq c_1\|\cdot\|_V$ and $|\varphi(f,g)| \leq c_2\|f\|_V\|g\|_V$ for all $f,\ g \in V$. Therefore

$$(Lf,f) = \varphi(f,f) \geq \alpha\|f\|_V^2 \geq \frac{\alpha}{c_1^2}\|f\|^2,$$

and

$$\begin{aligned}|(Lf,g)-(f,Lg)| &= |\varphi(f,g)-\varphi(g,f)| \\ &\leq 2c_2\|f\|_V\|g\|_V \\ &\leq \frac{2c_2}{\alpha}\varphi(f,f)^{\frac{1}{2}}\varphi(g,g)^{\frac{1}{2}} \\ &= \frac{2c_2}{\alpha}(Lf,f)^{\frac{1}{2}}(Lg,g)^{\frac{1}{2}},\end{aligned}$$

for all $f,\ g \in D(L)$.
(ii) $\Rightarrow$ (i). Set

$$(f,g)_V := \frac{1}{2}\{(Lf,g)+(f,Lg)\},\ f,\ g \in D(L).$$

Then $(\cdot,\cdot)_V$ defines an innerproduct on $D(L)$ since L is strictly accretive. Denote by $(V,(\cdot,\cdot)_V)$ the completion of the pre-Hilbert space $(D(L),(\cdot,\cdot)_V)$. Note that V can be identified with a subspace, also denoted by V, of H which is continuously and densely embedded since L is strictly accretive and $D(L)$ is dense in H. Define

$$\begin{cases} \varphi_1(f,g) := (f,g)_V, & \text{for } f,\ g \in V; \\ \varphi_2(f,g) := \frac{1}{2}\{(Lf,g)-(f,Lg)\}, & \text{for } f,\ g \in D(L). \end{cases}$$

Since L is anglebounded we have

$$|\varphi_2(f,g)| \leq a(Lf,f)^{\frac{1}{2}}(Lg,g)^{\frac{1}{2}} = a\|f\|_V\|g\|_V,$$

for all $f,\ g \in D(L)$. Using that $D(L)$ is a dense subset of V we can extend $\varphi_2(\cdot,\cdot)$ to a bilinear continuous form, again denoted by φ_2, on V.
Now we define the continuous bilinear form φ on V by

$$\varphi(f,g) := \varphi_1(f,g)+\varphi_2(f,g), \quad f,\ g \in V.$$

Observe that $\varphi(f,f) = \varphi_1(f,f) = \|f\|_V^2$, so that φ is coercive. Furthermore, if $f,\ g \in D(L)$ $(\subset V)$ then $\varphi(f,g) = (Lf,g)$. Hence $\varphi(f,g) = (Lf,g)$ for all $f \in D(L),\ g \in V$ and we have proved that L is regularly accretive.

If H is a complex Hilbert space the notion of regular accretiveness is defined as follows [K1] : Consider a sesquilinear form

$$\varphi(\cdot,\cdot) : D(\varphi)\times D(\varphi) \subset H\times H \to \mathbb{C}\,.$$

The form φ can be written as

$$\varphi = Re\varphi + iIm\varphi,$$

where

$$\begin{aligned}(Re\varphi)(f,g) &:= \frac{1}{2}\{\varphi(f,g) + \overline{\varphi(g,f)}\};\\ (Im\varphi)(f,g) &:= \frac{1}{2i}\{\varphi(f,g) - \overline{\varphi(g,f)}\},\end{aligned}$$

Then φ is called a *regular* form (or a *closed sectorial* form [K3]) if

(i) $Re\varphi$ is non-negative, that is $(Re\varphi)(f,f) \geq 0$ for all $f \in D(\varphi)$.

(ii) $Re\varphi$ is closed.

(iii) for an $a \geq 0$, $|(Im\varphi)(f,f)| \leq a(Re\varphi)(f,f)$ for all $f \in D(\varphi)$.

(iv) $D(\varphi)$ is dense in H.

If φ is a regular form then there exists a unique m-accretive operator L such that $D(L) \subset D(\varphi)$ and $\varphi(f,g) = (Lf,g)$ for all $f \in D(L)$, $g \in D(\varphi)$ [K2]. The operator L will be called *regularly accretive.*

If L is a linear operator in a real Hilbert space we denote by $L_{\mathbb{C}}$ its complexification in the complexification $H_{\mathbb{C}}$ of H. Now we can state

Theorem 2.2. *Let L be a linear operator in a real Hilbert space and* $a \geq 0$. *Then the following assertions are equivalent.*

(i) L is m-accretive and anglebounded (with constant a).

(ii) $L_{\mathbb{C}}$ *is m-accretive and* $|Im(L_{\mathbb{C}} f,f)| \leq aRe(L_{\mathbb{C}} f,f)$ *for all* $f \in D(L_{\mathbb{C}})$.

(iii) $L_{\mathbb{C}}$ *is regularly accretive (with* $|(Im\varphi)(f,f)| \leq a(Re\varphi)(f,f)$ *).*

(iv) $L_{\mathbb{C}}$ *is of type* $(arctg(a),1)$ *and* $M(\theta) = 1$ *for all* $\theta \in [0, \frac{\pi}{2} - arctg(a))$, *(see Section 2.4).*

(v) $e^{i\theta}L_{\mathbb{C}}$ *is m-accretive for all* $\theta \in [0, \frac{\pi}{2} - arctg(a))$.

Proof. (i) $\Rightarrow$ (ii). Writing $f \in H_{\mathbb{C}}$ as $f = f_1 + if_2$, f_1, $f_2 \in H$ we have

$$(L_{\mathbb{C}} f,f) = (Lf_1,f_1) + (Lf_2,f_2) + i\{(Lf_2,f_1) - (f_2,Lf_1)\}.$$

Thus

$$\begin{aligned} |Im(L_{\mathbf{C}}\, f, f)| &= |(Lf_2, f_1) - (f_2, Lf_1)| \\ &\leq 2a(Lf_1, f_1)^{\frac{1}{2}}(Lf_2, f_2)^{\frac{1}{2}} \\ &\leq a\{(Lf_1, f_1) + (Lf_2, f_2)\} = aRe(L_{\mathbf{C}}\, f, f), \end{aligned}$$

for all $f \in D(L_{\mathbf{C}})$.

(ii) $\Rightarrow$ (i). By assumption

$$|(Lf_2, f_1) - (f_2, Lf_1)| \leq a\{(Lf_1, f_1) + (Lf_2, f_2)\},$$

for all $f_1,\ f_2 \in D(L)$. Then

$$\begin{aligned} |(Lf_2, f_1) - (f_2, Lf_1)| &= |(L(\frac{f_2}{\alpha}), \alpha f_1) - (\frac{f_2}{\alpha}, L(\alpha f_1))| \\ &\leq a\{\alpha^2(Lf_1, f_1) + \frac{1}{\alpha^2}(Lf_2, f_2)\}. \end{aligned}$$

Choosing $\alpha = \left\{\frac{(Lf_2,f_2)}{(Lf_1,f_1)}\right\}^{\frac{1}{4}}$ we obtain

$$|(Lf_2, f_1) - (f_2, Lf_1)| \leq 2a(Lf_1, f_1)^{\frac{1}{2}}(Lf_2, f_2)^{\frac{1}{2}}.$$

Here we assumed that $(Lf_1, f_1) \neq 0 \neq (Lf_2, f_2)$ otherwise consider $L + \epsilon I$, $\epsilon > 0$, instead of L.

(ii) $\Rightarrow$ (iii). The expression

$$(f, g)_V := \frac{1}{2}\{(L_{\mathbf{C}}\, f, g) + (f, L_{\mathbf{C}}\, g)\} + (f, g)$$

defines an innerproduct on $D(L_{\mathbf{C}})$. We denote by $(V, (\cdot,\cdot))$ the completion of $(D(L_{\mathbf{C}}), (\cdot,\cdot))$ and $\|f\|_V := (f, f)_V^{\frac{1}{2}}$ for $f \in V$. Note that V is dense in H. Define

$$\varphi_1(f, g) := (f, g)_V - (f, g),$$

for $f,\ g \in V$. Then φ_1 is a closed bilinear and non-negative form. Next, we define for $f,\ g \in D(L_{\mathbf{C}})$

$$\varphi_2(f, g) := \frac{1}{2i}\{(L_{\mathbf{C}}\, f, g) - (f, L_{\mathbf{C}}\, g)\}.$$

Observe that φ_2 is symmetric on $D(L_{\mathbf{C}})$ and

$$|\varphi_2(f, g)| = |Im(L_{\mathbf{C}}\, f, f)| \leq aRe(L_{\mathbf{C}}\, f, f) = a\varphi_1(f, f),$$

for $f \in D(L_{\mathbb{C}})$. This implies that

$$|\varphi_2(f,g)| \le 2a\varphi_1(f,f)^{\frac{1}{2}}\varphi_1(g,g)^{\frac{1}{2}}, \tag{2.1}$$

for all $f,\ g \in D(L_{\mathbb{C}})$. Indeed, by replacing f by $e^{-i\arg\varphi_2(f,g)}f$ we may assume that $\varphi_2(f,g)$ is real. Then, since φ_2 is symmetric the polarization principle reads

$$\varphi_2(f,g) = \frac{1}{4}\{\varphi_2(f+g,f+g) - \varphi_2(f-g,f-g)\}.$$

Therefore

$$\begin{aligned} \varphi_2(f,g) &= \frac{a}{4}\{\varphi_1(f+g,f+g) - \varphi_1(f-g,f-g)\} \\ &= \frac{a}{2}\{\varphi_1(f,f) + \varphi_1(g,g)\}. \end{aligned}$$

Set $\alpha := \{\frac{\varphi_1(g,g)}{\varphi_1(f,f)}\}^{\frac{1}{4}}$. Then

$$\begin{aligned} |\varphi_2(f,g)| &= |\varphi_2(\alpha f, \frac{g}{\alpha})| \\ &\le \frac{a}{2}\{\alpha^2\varphi_1(f,f) + \frac{1}{\alpha^2}\varphi_1(g,g)\} \\ &= 2a\varphi_1(f,f)^{\frac{1}{2}}\varphi_1(g,g)^{\frac{1}{2}}. \end{aligned}$$

Due to inequality (2.1) we can extend φ_2 to a continuous sesquilinear form φ_2 on $V \times V$. Define

$$\varphi(f,g) := \varphi_1(f,g) + i\varphi_2(f,g), \quad \text{for } f,\ g \in V.$$

As already observed $Re\varphi = \varphi_1$ is closed and non-negative. Furthermore,

$$\begin{aligned} |(Im\varphi)(f,f)| &= |\varphi_2(f,f)| = |Im(L_{\mathbb{C}}f,f)| \\ &\le aRe(L_{\mathbb{C}}f,f) = a(Re\varphi)(f,f), \end{aligned}$$

for all $f \in D(L_{\mathbb{C}})$ and hence

$$|(Im\varphi)(f,f)| \le a(Re\varphi)(f,f), \quad \text{for all } f \in V$$

by continuity. We have proved now that φ with $D(\varphi) = V$ is a regular form. Clearly, $D(L_{\mathbb{C}}) \subset D(\varphi)$ and $\varphi(f,g) = (L_{\mathbb{C}}f,g)$ for all $f \in D(L_{\mathbb{C}}),\ g \in D(\varphi)$. Hence $L_{\mathbb{C}}$ is the regularly accretive operator associated with φ.

(iii) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (iv). This implication is well-known see e.g. [K3].

(iv) $\Rightarrow$ (ii). If $\|(I+zL_{\mathbb{C}})^{-1}\| \leq 1$ for $z \in \Sigma_{\frac{\pi}{2}-\omega}$ with $\omega = \arctan(a)$ then for $z = x+iy$

$$\begin{aligned}
\|u\|^2 &\leq \|(I+zL_{\mathbb{C}})u\|^2 = \|u\|^2 + (zL_{\mathbb{C}}u,u) + (u,zL_{\mathbb{C}}u) + |z|^2\|L_{\mathbb{C}}u\|^2 \\
&= \|u\|^2 + z(L_{\mathbb{C}}u,u) + \overline{z(L_{\mathbb{C}}u,u)} + |z|^2\|L_{\mathbb{C}}u\|^2 \\
&= \|u\|^2 + 2(xRe(L_{\mathbb{C}}u,u) - yIm(L_{\mathbb{C}}u,u))|z|^2\|L_{\mathbb{C}}u\|^2.
\end{aligned}$$

In particular $xRe(L_{\mathbb{C}}u,u) - yIm(L_{\mathbb{C}}u,u) \geq 0$. Then

$$x = \cos(\frac{\pi}{2}-\omega),\ y = \sin(\frac{\pi}{2}-\omega) \Rightarrow Im(L_{\mathbb{C}}u,u) \leq \tan(\omega)Re(L_{\mathbb{C}}u,u)$$

and

$$x = \cos(\frac{\pi}{2}-\omega),\ y = -\sin(\frac{\pi}{2}-\omega) \Rightarrow -Im(L_{\mathbb{C}}u,u) \leq \tan(\omega)Re(L_{\mathbb{C}}u,u).$$

Hence $|Im(L_{\mathbb{C}}u,u)| \leq Re(L_{\mathbb{C}}u,u)$ for all $u \in D(L_{\mathbb{C}})$.

(iv)$\Rightarrow$ (v). Note that (v) is a reformulation of (iv).

Some remarks concerning Theorem 2.2 are in order.

Remark 2.3. (i) An operator satisfying property (ii) of Theorem 2.2 is called m-sectorial in [K3].
(ii) The implication "(ii) $\Rightarrow$ (iii)" corresponds to [K3, Ch.6, Thm 1.27]. However, the proof presented here is different.
(iii) Recall that property (iv) implies that the operator $-L$ generates an analytic semigroup. In [BA-H] a different proof is given of the fact that an m-accretive and anglebounded operator is the negative generator of an analytic semigroup.

4.3 The m-accretivity of $\mathcal{L}+\mathcal{A}$

In this section we will weaken the assumption of selfadjointness of the operator L in Theorem 3.3.1.

Theorem 3.1. *Let $L \in \mathbf{M}(\Omega)$ be normal and the negative generator of an analytic semigroup. Then $\mathcal{L}+\mathcal{A} \subset \mathcal{H}\times\mathcal{H}$ is m-accretive. Moreover there exists a constant $c > 0$ such that*

$$\|\mathcal{L}u\| \leq c\|u+\mathcal{L}u+v\|, \quad \text{for all } u \in D(\mathcal{L})\cap D(\mathcal{A}),\ v \in \mathcal{A}u. \tag{3.1}$$

The main step in proving this theorem is to introduce a third operator $\mathcal{L}_0$ in $\mathcal{H}$ which, roughly speaking, forms simultaneously an acute angle with $\mathcal{L}$ and $\mathcal{A}$. This idea is expressed in the following

Lemma 3.2. *Let L be a linear m-accretive operator in $L^2(\Omega)$ such that there exists a symmetric operator $L_0 \in \mathrm{M}(\Omega)$ satisfying*
(i) $D(L_0) \subset D(L)$
(ii) there exists constants $0 < a$ and $b \in I\!R^+$ such that

$$(Lu, L_0 u) \geq a\|L_0 u\|_2^2 - b\|u\|_2^2, \quad \text{for all } u \in D(L_0).$$

Then $\mathcal{L} + \mathcal{A} \subset \mathcal{H} \times \mathcal{H}$ is m-accretive. Furthermore inequality (3.1) holds.

Proof of Theorem 3.1. Without loss of generality we assume that L is strictly accretive, otherwise consider $L_\epsilon := L + \epsilon$, $\epsilon > 0$. We will apply Lemma 3.2 with $L_0 := L + L^*, D(L_0) = D(L)(= D(L^*))$. The operator L_0 is clearly symmetric. First we show that the operator L_0 belongs to the class $\mathrm{M}(\Omega)$. Since $-L$ is a generator of an analytic contraction semigroup we have that L is of type $(\omega_L, 1)$ with $\omega_L < \frac{\pi}{2}$ (see Section 2.5). Similarly the operator L^* is of type $(\omega_{L^*}, 1)$ with $\omega_{L^*} = \omega_L$. Therefore

$$\omega_L + \omega_{L^*} < \pi.$$

Since L is normal it is evident that

$$D_L(\theta, 2) = D_{L^*}(\theta, 2), \quad \text{for all } \theta \in (0,1)$$

and it follows from Theorem 2.5.3 that L_0 is m-accretive. The fact that the resolvent of L_0 is order-preserving and contractive in $L^p(\Omega)$ for all $p \in [1, \infty]$ can be seen, using the Trotter product formula

$$T_0(t)f = \lim_{n\to\infty} \{J^L_{\frac{t}{n}} J^{L^*}_{\frac{t}{n}}\}^n f, \quad \text{for every } f \in L^2(\Omega),$$

uniformly on bounded intervals of t, see proposition 2.2.5. Here $\{T_0(t)\}_{t\geq 0}$ denotes the semigroup generated by $-L_0$. It follows that the semigroup $\{T_0(t)\}_{t\geq 0}$ consists of order-preserving and contractive operators in $L^p(\Omega)$ for all $p \in [1, \infty]$. We have proved now that $L_0 \in \mathrm{M}(\Omega)$. Next we claim that there exists a constant $c > 0$ such that

$$(Lf, L_0 f) \geq c\|L_0 f\|^2, \quad \text{for all } f \in D(L_0). \tag{3.2}$$

Indeed, by the commutativity of L_λ and L_λ^* we have that

$$\begin{aligned}(L_\lambda f,(L_\lambda+L_\lambda^*)f) &= \frac{1}{2}((L_\lambda+L_\lambda^*)f,(L_\lambda+L_\lambda^*)f)\\ &= \frac{1}{2}\|(L_\lambda+L_\lambda^*)f\|^2.\end{aligned}$$

Let $f\in D(L_0)$. Then by passing to the limit, inequality (3.2) is proven and the proof is complete using Lemma 3.2.

Proof of Lemma 3.2. First we observe that (i) and (ii) implies that $D(L_0)=D(L)$. Namely,

$$a\|L_{0,\lambda}u\|_2^2-b\|J_\lambda^{L_0}u\|_2^2\leq(LJ_\lambda^{L_0}u,L_{0,\lambda}u)\leq(Lu,L_{0,\lambda}u)\leq\|Lu\|_2\|L_{0,\lambda}u\|_2,$$

for all $u\in D(L)$, $\lambda>0$. Thus $\|L_{0,\lambda}u\|_2$ remains bounded if $\lambda\downarrow 0$ for $u\in D(L)$ so that $u\in D(L_0)$.

Consider the approximation equation

$$u_\lambda+\mathcal{L}u_\lambda+\mathcal{A}_\lambda u_\lambda=f,\quad \lambda>0.$$

Taking the inner product with $\mathcal{L}_0u_\lambda$ and using Theorem 3.3.1 we obtain

$$a\|\mathcal{L}_0u_\lambda\|^2\leq\|f\|\|\mathcal{L}_0u_\lambda\|+b\|u_\lambda\|\leq\|v\|\{\|\mathcal{L}_0u_\lambda\|+b\}.$$

Thus $\|\mathcal{L}_0u_\lambda\|$ remains bounded if $\lambda\downarrow 0$ and it follows that $\|\mathcal{L}u_\lambda\|$ remains bounded as well if $\lambda\downarrow 0$. By Theorem 2.2.2 it follows that $v\in R(I+\mathcal{L}+\mathcal{A})$.

By Theorem 3.1 and Theorem 2.2 we have

Corollary 3.3. *Let $L\in\mathrm{M}(\Omega)$ be normal and satisfying one of the assertions of Theorem 2.2. Then $\mathcal{L}+\mathcal{A}\subset\mathcal{H}\times\mathcal{H}$ is m-accretive and (3.1) holds.*

In the next example all the conditions in Theorem 3.1 are fulfilled

Example 3.4. Let L be the negative generator of a positive, analytic and translation invariant contraction semigroup in $L^2(\mathbb{R}^N)$. Then $\mathcal{L}+\mathcal{A}\subset\mathcal{H}\times\mathcal{H}$ is m-accretive and (3.1) holds. This result follows from Theorem 3.1 and Remark 5.3.8 (iii) which tells us that such an operator L is normal and belongs to $\mathrm{M}(\mathbb{R}^N)$.

We point out now that if L is normal then the assumption "$-L$ generates an analytic semigroup" is equivalent to assertion (ii) (and therefore all assertions) of Theorem 2.2. Indeed, the spectral theorem in multiplication form for a normal operator L in a Hilbertspace H says that L is unitarily equivalent to a multiplication operator, that is, there exist a measure space (M, μ) with μ a finite positive measure, a unitary operator $U : H \to L^2(M, \mu)$ and a (complex-valued) measurable function f on M which is finite almost everywhere, so that

(i) $g \in D(L)$ if and only if $f(\cdot)(Ug)(\cdot) \in L^2(M, \mu)$;
(ii) If $h \in U(D(L))$, then $(ULU^{-1})(m) = f(m)h(m)$.

Then one shows that $\sigma(L)$ is precisely the essential range of f. Recall that the essential range of f is the set

$$\{\lambda \in \mathbb{C} : \mu\{m \in M : |f(m) - \lambda| < \epsilon\} > 0 \text{ for all } \epsilon > 0\}.$$

If L is m-accretive and such that it is the negative generator of an analytic semigroup we have that $Ref \geq 0$ almost everywhere and the essential range of f is contained in a sector

$$\{z \in \mathbb{C} : |arg(z + c)| \leq \frac{\pi}{2} - \epsilon\},$$

for certain $0 < \epsilon \leq \frac{\pi}{2}$ and c a non-negative constant. Hence, there exists a constant $a \geq 0$ such that

$$|Im(f + c)| \leq aRe(f + c).$$

Then

$$\begin{aligned} |Im((L + c)h, h)| &= |Im \int_M (f + c)|Uh|^2 d\mu| = |\int_M Im(f + c)|Uh|^2 d\mu| \\ &\leq \int_M |Im(f + c)||Uh|^2 d\mu \leq a \int_M Re(f + c)|Uh|^2 d\mu \\ &= aRe \int_M (f + c)|Uh|^2 d\mu = aRe((L + c)h, h). \end{aligned}$$

Hence $L + cI$ satisfies (ii) of Theorem 2.2.

Remark 3.5. About the necessity of the assumption "$-L$ generates an analytic semigroup". We define the m-accretive operator A in $L^2(0, T)$ by setting $A = \frac{d}{dt}$ with $D(A) = \{u \in W^{1,2}(0, T); u(0) = 0\}$. It is well-known that if $\mathcal{L} + \mathcal{A}$ is m-accretive then L must be the negative generator of an analytic semigroup. For the sake of completeness we prove it here.

Let u_λ be the solution of the equation $u+\mathcal{L}u+\mathcal{A}u = f_\lambda$ where $f_\lambda = e^{\lambda t}g$ with $\mathrm{Re}\lambda > 0$ and $g \in L^2(\Omega)$. Denote by $\{T(t)\}_{t\geq 0}$ the semigroup generated by $-L$. One verifies that u_λ is given by

$$u_\lambda = (e^{\lambda t} - e^{-t}T(t))(\lambda + I + L)^{-1}g$$

Then

$$\mathcal{A}u_\lambda = (\lambda e^{\lambda t} + e^{-t}T(t)(I+L))(\lambda+I+L)^{-1}g$$

By the closedness of $\mathcal{A}$ and $\mathcal{L}+\mathcal{A}$ we have that there exists a constant c independent of λ such that $\|\mathcal{A}u_\lambda\| \leq c\|f_\lambda\|$. Using this estimate and the expression of u_λ we obtain that $\|(\lambda+I+L)^{-1}g\|_2 \leq \frac{M}{|\lambda|}\|g\|_2$ for some constant M and for all $\mathrm{Re}\lambda > 0$. Hence $I+L$ and therefore L is the negative generator of an analytic semigroup, see e. g. [P, Theorem 2.5.2].

In the proof of Theorem 3.1 we used a result of Da Prato and Grisvard in order to prove that if L is m-accretive, normal and the negative generator of an analytic semigroup then $L+L^*$ is m-accretive. Here we give another proof of this fact.

Let A and B be m-accretive resolvent commuting operators in a Hilbert space $(H,(\cdot,\cdot))$ and assume that A is normal and the negative generator of an analytic semigroup. We show that $A+B$ is m-accretive. For that consider the approximation equation

$$u_\lambda + Au_\lambda + B_\lambda u_\lambda = f, \quad f\in H,\ \lambda>0. \tag{3.3}$$

If $\|Au_\lambda\|$ remains bounded if $\lambda \downarrow 0$ then the result follows by substracting the weakly convergent subsequences

$$u_{\lambda_n} \rightharpoonup u, \quad Au_{\lambda_n} \rightharpoonup v, \quad B_{\lambda_n}u_{\lambda_n} \rightharpoonup f-u-v, \quad \text{if } n\to\infty,$$

for certain $u,\ v\in H$. Observe that $u_{\lambda_n} - J^B_{\lambda_n}u_{\lambda_n} = \lambda_n B_{\lambda_n}u_{\lambda_n} \to 0$ if $n\to\infty$, thus $J^B_{\lambda_n}u_{\lambda_n} \rightharpoonup u$ if $n\to\infty$. Since A and B are closed and linear operators, hence weakly closed, and $B_{\lambda_n}u_{\lambda_n} = BJ^B_{\lambda_n}u_{\lambda_n}$, it follows that $u\in D(A)\cap D(B)$, $v = Au$ and $Bu = f-u-Au$.

Let us prove the boundedness of $\|Au_\lambda\|$, $\lambda>0$. To this end let $0<\epsilon<1$ be fixed and $A_\epsilon := A+\epsilon I$. Then by the observation following Example 3.4, A_ϵ is regularly accretive, say with the corresponding form φ and $D(\varphi) = V$. Since $A_\epsilon^{\frac{1}{2}}$ is normal as well we have that $D(A_\epsilon^{\frac{1}{2}}) = D(A_\epsilon^{*\frac{1}{2}})$. In [K2] it is proved that this implies that $V = D(A_\epsilon^{\frac{1}{2}}) = D(A_\epsilon^{*\frac{1}{2}})$, see also [L]. We repeat here the argument used in [K2]. From $D(A_\epsilon^{\frac{1}{2}}) = D(A_\epsilon^{*\frac{1}{2}})$ it follows that $A_\epsilon^{*\frac{1}{2}}A_\epsilon^{-\frac{1}{2}}$ is a bounded operator and therefore $A_\epsilon^{*-\frac{1}{2}}A_\epsilon^{\frac{1}{2}}$ is also bounded. Then for some constant $c > 0$

$$\|u\|_V^2 = (A_\epsilon u, u) = (A_\epsilon^{*-\frac{1}{2}}A_\epsilon^{\frac{1}{2}}A_\epsilon^{\frac{1}{2}}u, A_\epsilon^{\frac{1}{2}}u) \leq c\|A_\epsilon^{\frac{1}{2}}u\|^2, \tag{3.4}$$

for all $u \in D(A_\epsilon)$. Using the well-known fact that $D(A_\epsilon)$ is a core of $A_\epsilon^{\frac{1}{2}}$, inequality (3.4) implies that $D(A_\epsilon^{\frac{1}{2}}) \subset V$. On the other hand for some constant $c_1 > 0$

$$|(A_\epsilon^{\frac{1}{2}}u, A_\epsilon^{*\frac{1}{2}}v)| \leq c_1\|u\|_V\|v\|_V,$$

for all $u \in D(A)$, $v \in D(A_\epsilon^{*\frac{1}{2}}) = D(A_\epsilon^{\frac{1}{2}})$. By setting $w = A_\epsilon^{*\frac{1}{2}}v$ we have

$$|(A_\epsilon^{\frac{1}{2}}u, w)| \leq c_1\|u\|_V\|A_\epsilon^{*-\frac{1}{2}}w\|_V \leq c_2\|u\|_V\|w\|,$$

for all $w \in H$. Hence,

$$\|A_\epsilon^{\frac{1}{2}}u\| \leq c_2\|u\|_V, \quad \text{for all } u \in D(A). \tag{3.5}$$

Using that $D(A)$ is dense in V and that $A_\epsilon^{\frac{1}{2}}$ is closed, inequality (3.5) implies that $V \subset D(A_\epsilon^{\frac{1}{2}})$. We have shown that $V = D(A_\epsilon^{\frac{1}{2}}) = D(A_\epsilon^{*\frac{1}{2}})$. It follows now that $A_\epsilon^{\frac{1}{2}}$ and $A_\epsilon^{*\frac{1}{2}}$ form an acute angle and are comparable, that is, for certain positive constants c, m, M

$$(A_\epsilon^{\frac{1}{2}}u, A_\epsilon^{*\frac{1}{2}}u) \geq c\|A_\epsilon^{\frac{1}{2}}u\|\|A_\epsilon^{*\frac{1}{2}}u\|,$$

and

$$m\|A_\epsilon^{*\frac{1}{2}}u\| \leq \|A_\epsilon^{\frac{1}{2}}u\| \leq M\|A_\epsilon^{*\frac{1}{2}}u\|,$$

for all $u \in D(A_\epsilon^{\frac{1}{2}}) = D(A_\epsilon^{*\frac{1}{2}})$.

By taking the inner product of equation (3.3) with the selfadjoint operator $A_\epsilon^{*\frac{1}{2}}A_\epsilon^{\frac{1}{2}}$ we get

$$(1-\epsilon)\|A_\epsilon^{\frac{1}{2}}u_\lambda\|^2 + (A_\epsilon u_\lambda, A_\epsilon^{*\frac{1}{2}}A_\epsilon^{\frac{1}{2}}u_\lambda) + (B_\lambda u_\lambda, A_\epsilon^{*\frac{1}{2}}A_\epsilon^{\frac{1}{2}}u_\lambda) = (f, A_\epsilon^{*\frac{1}{2}}A_\epsilon^{\frac{1}{2}}u_\lambda).$$

Since

$$(A_\epsilon u_\lambda, A_\epsilon^{*\frac{1}{2}}A_\epsilon^{\frac{1}{2}}u_\lambda) = (A_\epsilon^{\frac{1}{2}}A_\epsilon^{\frac{1}{2}}u_\lambda, A_\epsilon^{*\frac{1}{2}}A_\epsilon^{\frac{1}{2}}u_\lambda) \geq c\|A_\epsilon u_\lambda\|\|A_\epsilon^{*\frac{1}{2}}A_\epsilon^{\frac{1}{2}}u_\lambda\|$$

and, due to the assumption that A and B are resolvent commuting,

$$(B_\lambda u_\lambda, A_\epsilon^{*\frac{1}{2}}A_\epsilon^{\frac{1}{2}}u_\lambda) = (B_\lambda A_\epsilon^{\frac{1}{2}}u_\lambda, A_\epsilon^{\frac{1}{2}}u_\lambda) \geq 0, \quad \lambda > 0,$$

it follows that $c\|A_\epsilon u_\lambda\| \leq \|f\|$, for all $\lambda > 0$. Hence $\|Au_\lambda\|$ remains bounded if $\lambda \downarrow 0$ which completes the proof.

4.4 A counterexample

For the sake of completeness we give an example of a symmetric m-accretive operator L in $L^2(0,1)$ and a subdifferential $A = \partial\varphi \subset H \times H$ with $\varphi \in \mathcal{J}_0$, such that $\mathcal{L}+\mathcal{A} \subset \mathcal{H}\times\mathcal{H}$ is closed but not m-accretive. Here $\mathcal{H} = L^2(0,1;H)$. Define the operator L by

$$\begin{cases} D(L) &= \{u \in W^{4,2}(0,1) : u^{(2)}(0) = u^{(2)}(1) = u^{(3)}(0) = u^{(3)}(1)\}; \\ Lu &= u^{(4)} \quad \text{for } u \in D(L). \end{cases}$$

The operator L is a selfadjoint and accretive operator in $L^2(0,1)$ and therefore m-accretive. Let P be a nonempty, closed and proper cone in H. By proper we mean that $(P \cap \{-P\} \doteq \{0\})$. Let $A = \partial I_P$, where I_P is the indicator function of the closed and convex set P. Then the extension $\mathcal{A}$ is given by

$$\begin{cases} D(\mathcal{A}) &= \mathcal{H}_P; \\ \mathcal{A} &= \partial I_{\mathcal{H}_P}, \end{cases}$$

where $\mathcal{H}_P = \{u \in \mathcal{H} : u(\omega) \in P,\ \omega \in (0,1) \text{ a.e.}\}$. In order to show that $\mathcal{L}+\mathcal{A}$ is not m-accretive we will use the following

Proposition 4.1. *Let L be a linear m-accretive operator in a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and let $C \subset H$ be a closed and convex subset. Then, the following conditions are equivalent.*

(i) 1. $\langle Lu, v\rangle \geq 0$ for all $v \in \partial I_C u$ and $u \in D(L) \cap C$.
2. $\overline{D(L) \cap C} = C$.
3. $L + \partial I_C$ is m-accretive.
(ii) $J_\lambda^L C \subset C$ for all $\lambda > 0$.

Remark 4.2. Proposition 4.1 is a slight modification of [BR1, Proposition 4.5]. Note that $(L+\partial I_C)^0$, the principal section of $(L+\partial I_C)$, is equal to L if and only if (i) 1 holds. Indeed, if $u \in D(L) \cap C$ then,

$$|Lu + v|^2 \geq |Lu|^2 \quad \text{for every } v \in \partial I_C u,$$

if and only if condition (i) 1 holds since,

$$v \in \partial I_C u \text{ if and only if } \lambda v \in \partial I_C u,\ \lambda > 0.$$

By showing that (i) 1, (i) 2 are satisfied and that (ii) does not hold for $C = \mathcal{H}_P$ and the operator $\mathcal{L}$, Proposition 4.1 implies that $\mathcal{L} + \mathcal{A}$ is not m-accretive.

First, (i) 1 holds. In fact,

$$((\mathcal{L}u, v)) = 0, \quad \text{for all } v \in \partial I_{\mathcal{H}_P} u, \quad u \in D(\mathcal{L}) \cap \mathcal{H}_P. \tag{4.1}$$

Namely, for $u \in D(L)$, $Lu = 0$ almost everywhere on the set where $u = 0$ [G-T, Lemma 7.7], so that $\operatorname{supp}(\mathcal{L}u) \subset \operatorname{supp}(u)$ for all $u \in D(\mathcal{L})$. Furthermore, for $u \in \mathcal{H}_P$, $v \in \partial I_{\mathcal{H}_P} u$ implies that $((u, v)) = 0$ and $((v, w)) \leq 0$ for all $w \in \mathcal{H}_P$. In particular $\operatorname{supp}(v) \cap \operatorname{supp}(u) = \emptyset$. Therefore we may conclude that (4.1) holds.

Next, (i) 2 holds since a function in $\mathcal{H}_P$ can be approximated by $C_0^\infty(0,1;P)$ functions.

Finally, by using a contradiction argument we show that (ii) is not satisfied. Suppose that $J_\lambda^{\mathcal{L}} \mathcal{H}_P \subset \mathcal{H}_P$ for all $\lambda > 0$. Take $0 \neq x \in P$ and $0 \leq f \in L^2(0,1)$ then, since P is a cone, $f \cdot x \in \mathcal{H}_P$. By assumption $J_\lambda^L f \cdot x = J_\lambda^{\mathcal{L}}(f \cdot x) \in \mathcal{H}_P$ and hence $J_\lambda^L f \geq 0$ since P is assumed to be proper. It follows that J_λ^L is a contraction in $L^\infty(0,1)$ for all $\lambda > 0$, since $J_\lambda^L 1 = 1$. In particular, the "part of L in $C[0,1]$," that is, the operator L_∞ defined by

$$\begin{cases} D(L_\infty) &= \{u \in C^4[0,1] : u^{(2)}(0) = u^{(2)}(1) = u^{(3)}(0) = u^{(3)}(1)\}; \\ L_\infty u &= u^{(4)} \quad \text{for } u \in D(L_\infty), \end{cases}$$

is accretive in $(C[0,1], \|\cdot\|_\infty)$. By a result due to K. Sato [SA], this is equivalent with

$$\max_{x \in S(u)} \{(L_\infty u)(x) \operatorname{sign} u(x)\} \geq 0 \quad \text{for all } u \in D(L_\infty), \tag{4.2}$$

where $S(u) = \{x \in [0,1] : |u(x)| = \|u\|_\infty\}$. If

$$u(x) = (1 - (x - \frac{1}{2})^4)\chi(x) \quad \text{for } x \in [0,1],$$

with $\chi \in C_0^\infty(0,1)$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $(\frac{1}{4}, \frac{3}{4})$, then $u \in D(L_\infty)$ and $S(u) = \{\frac{1}{2}\}$. For this $u \in D(L_\infty)$ we have

$$\max_{x \in S(u)} \{(L_\infty u)(x) \operatorname{sign} u(x)\} = (L_\infty u)(\frac{1}{2}) \operatorname{sign} u(\frac{1}{2}) = -24 < 0,$$

which contradicts (4.2), see also [SI]. Hence we may conclude that (ii) is not satisfied.

To show the closedness of $\mathcal{L}+\mathcal{A}$, consider a sequence $\{u_n\}_{n=1}^{\infty}$ in $D(\mathcal{L})\cap D(\mathcal{A})$ such that

$$\begin{cases} u_n \to u; \\ \mathcal{L}u_n + v_n \to w \quad \text{as } n\to\infty, \text{ for certain } u,\ w \in L^2(0,1), \end{cases}$$

where $v_n \in \mathcal{A}u_n$ for $n = 1,2,\ldots$. Set $w_n = \mathcal{L}u_n + v_n$. Using (4.1) we obtain

$$\|\mathcal{L}u_n\|^2 = ((w_n, \mathcal{L}u_n)) \leq \|w_n\|\|\mathcal{L}u_n\| \quad \text{for all } n = 1,2,\ldots$$

Hence, the sequence $\{\mathcal{L}u_n\}_{n=1}^{\infty}$ is bounded in $\mathcal{H}$ and consequently $\{v_n\}_{n=1}^{\infty}$ is bounded as well. Therefore, there exists a subsequence of $\{u_n\}_{n=1}^{\infty}$, which we denote again by $\{u_n\}_{n=1}^{\infty}$, such that $\{\mathcal{L}u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are weakly convergent. Since $\mathcal{L}$ and $\mathcal{A}$ are m-accretive it follows that $u \in D(\mathcal{L})\cap D(\mathcal{A})$ and $\{\mathcal{L}u_n\}_{n=1}^{\infty}$ respectively $\{v_n\}_{n=1}^{\infty}$ has the weak limit $\mathcal{L}u$ respectively $v \in \mathcal{A}u$, so that $w \in \mathcal{L}u + \mathcal{A}u$. Thus $\mathcal{L}+\mathcal{A}$ is closed.

Remark 4.3. Let L be a linear m-accretive operator in $L^2(\Omega)$ which is local, that is $\operatorname{supp}(Lf) \subset \operatorname{supp}(f)$ for all $f \in D(L)$, and such that $\overline{D(L)\cap L^2_+(\Omega)} = L^2_+(\Omega)$. Note that the latter is satisfied if $C_0^\infty(\Omega) \subset D(L)$. Then from the above we have that the statement $\mathcal{L}+\mathcal{A} \subset \mathcal{H}\times\mathcal{H}$ is m-accretive for all $A = \partial\varphi$, $\varphi \in \mathcal{J}_0$ implies that $J_\lambda^L \geq 0$ for all $\lambda > 0$.

Let us take in the example above $H = L^2(\Omega)$, $P = L^2_+(\Omega)$. Then, $A = \partial I_{L^2_+(\Omega)}$ and $J_\lambda^A f = f^+$. The operator A has the property that it satisfies the nonlinear analog of assumption (3.1.2), that is,

$$\begin{cases} J_\lambda^A u \geq J_\lambda^A v \quad \text{for all } \lambda > 0 \text{ if } u \geq v,\ u,v \in L^2(\Omega); \\ \|J_\lambda^A u - J_\lambda^A v\|_p \leq \|u-v\|_p \text{ for all } u,v \in L^1(\Omega)\cap L^\infty(\Omega), \lambda > 0, p \in [1,\infty]. \end{cases}$$

In the terminology of [B-C], A is called an m-completely accretive operator in $L^2(\Omega)$. This example shows that one can not interchange the role of the linear and nonlinear operator.

Chapter 5

The m-accretivity of $\overline{\mathcal{L}+\mathcal{A}}$

5.1 Introduction

In the previous chapters we have obtained results where $\mathcal{L}+\mathcal{A}$ is m-accretive. It is not true in general that $\mathcal{L}+\mathcal{A}$ m-accretive, even if $L \in \mathrm{M}(\Omega)$ and A is linear. However, by Theorem 2.5.1, if A is linear then

$$R(\epsilon I+\mathcal{L}+\mathcal{A}) \supset D(\mathcal{L})+D(\mathcal{A}), \text{ for all } \epsilon>0.$$

This is due to the fact that $\mathcal{L}$ and $\mathcal{A}$ are resolvent commuting. In particular it follows that $\overline{\mathcal{L}+\mathcal{A}}$ is m-accretive. In Section 5.2 we show that if the operator L, not necessarily in the class $\mathrm{M}(\Omega)$, forms an acute angle with a symmetric operator $L_0 \in \mathrm{M}(\Omega)$ such that $D(L_0) \subset D(L)$, then $\overline{\mathcal{L}\mid_{D(\mathcal{L}_0)}+\mathcal{A}}$ is m-accretive and for $0<\alpha \leq \frac{1}{2}$,

$$J_\lambda^{\overline{\mathcal{L}|_{D(\mathcal{L}_0)}+\mathcal{A}}} D(\mathcal{L}_0^\alpha) \subset D(\mathcal{L}_0^\alpha), \text{ for all } \lambda>0,$$

[EG1]. (By $\mathcal{L}\mid_{D(\mathcal{L}_0)}$ we mean the operator $\mathcal{L}$ with domain $D(\mathcal{L}_0)$.) If moreover $D(L_0^\alpha) \subset D(L)$, $0<\alpha \leq \frac{1}{2}$, then one can conclude that

$$R(\epsilon I+\mathcal{L}+\mathcal{A}) \supset D(\mathcal{L}_0^\alpha), \text{ for all } \epsilon>0.$$

In Proposition 3.1 we give sufficient (and necessary) conditions for L such that $L_0 := L^*L \in \mathrm{M}(\Omega)$. Here L^* denotes the adjoint of L. Then we have

$$R(\epsilon I+\mathcal{L}+\mathcal{A}) \supset D(\mathcal{L}), \text{ for all } \epsilon>0,$$

and for $0<\beta<1$,

$$J_\lambda^{\overline{\mathcal{L}+\mathcal{A}}} D(\mathcal{L}^\beta) \subset D(\mathcal{L}^\beta), \text{ for all } \lambda>0.$$

We show that if $L \in \mathrm{M}(\Omega)$ is skew-adjoint then $L^*L \in \mathrm{M}(\Omega)$.

If $\Omega = I\!R^N$ and the semigroup generated by $-L$ is translation invariant we can take $L_0 = -\Delta$ if $W^{2,2}(I\!R^N) \subset D(L)$. In the case that $W^{s,2}(I\!R^N) \subset D(L)$, $0 < s \leq 1$ we obtain

$$R(\epsilon I + \mathcal{L} + \mathcal{A}) \supset W^{s,2}(I\!R^N; H), \text{ for all } \epsilon > 0.$$

5.2 A regularity result

The main idea we use here in order to deal with situations where the sum $\mathcal{L} + \mathcal{A}$ is not m-accretive, is to find suitable conditions on L which ensure the existence of a symmetric operator $L_0 \in \mathrm{M}(\Omega)$ with $D(L_0) \subset D(L)$ forming an acute angle with L : $(Lu, L_0u) \geq 0$ for all $u \in D(L_0)$. If such an operator L_0 exists, one can solve, and pass to the limit, the perturbed equation

$$u_\epsilon + \epsilon \mathcal{L}_0 u_\epsilon + \mathcal{L}u_\epsilon + \mathcal{A}u_\epsilon \ni f, \quad \epsilon > 0.$$

We use this perturbation argument to prove the following

Lemma 2.1. *Let L be a linear m-accretive operator in $L^2(\Omega)$. Assume that there exists a symmetric operator $L_0 \in \mathrm{M}(\Omega)$ such that*

$$\begin{array}{ll} i)\ D(L_0) \subset D(L) & \\ ii)\ (Lu, L_0u) \geq 0, & \text{for all } u \in D(L_0) \end{array} \tag{2.1}$$

Let $\epsilon > 0$ and $0 < \alpha \leq \frac{1}{2}$.
Then $\overline{\mathcal{L} |_{D(\mathcal{L}_0)} + \mathcal{A}} \subset \mathcal{H} \times \mathcal{H}$ is m-accretive and the solution u of the equation

$$\epsilon u + \overline{\mathcal{L} |_{D(\mathcal{L}_0)} + \mathcal{A}}\ u \ni f, \tag{2.2}$$

belongs to $D(\mathcal{L}_0^\alpha)$, whenever $f \in D(\mathcal{L}_0^\alpha)$.
If moreover $D(L_0^\alpha) \subset D(L)$, then $R(\epsilon I + \mathcal{L} + \mathcal{A}) \supset D(\mathcal{L}_0^\alpha)$, and the solution u of the equation

$$\epsilon u + \mathcal{L}u + \mathcal{A}u \ni f, \tag{2.3}$$

belongs to $D(\mathcal{L}_0^\alpha)$ if $f \in D(\mathcal{L}_0^\alpha)$.

Remark 2.2. If $\overline{\mathcal{L} |_{D(\mathcal{L}_0)} + \mathcal{A}} \subset \mathcal{H} \times \mathcal{H}$ is m-accretive then for all $f \in \mathcal{H}$ there exists a unique generalized solution u of equation (2.3) in the sense that there exist sequences $u_n \in D(\mathcal{L}_0) \cap D(\mathcal{A}), f_n \in \mathcal{H},\ n = 1, 2, \cdots$, such that

$$u_n \to u,\ f_n \to f, \text{ as } n \to \infty \text{ and } \epsilon u_n + \mathcal{L}u_n + \mathcal{A}u_n \ni f_n.$$

Recall that a linear m-accretive operator L admits fractional powers L^α, $0 < \alpha < 1$ which are m-accretive as well [Y]. $D(L)$ is a core of L^α and for $u \in D(L)$ one has the expression

$$L^\alpha u = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \frac{1}{\lambda^\alpha} L_\lambda u d\lambda,$$

where the integral is absolutely convergent. The semigroup generated by $-L$ we denote by $\{T(t)\}_{t\geq 0}$. Then the formula for the semigroup $\{T_\alpha(t)\}_{t\geq 0}$ generated by $-L^\alpha$ yields

$$T_\alpha(t) = \int_0^\infty f_{\alpha,t}(s)T(s)ds, \ t > 0, \ \ T_\alpha(0) = I,$$

where the family $\{f_{\alpha,t}\}_{t>0}$ consists of non-negative functions satisfying $\int_0^\infty f_{\alpha,t}(s)ds = 1$. It follows from the representation of T_α that $L \in \mathrm{M}(\Omega)$ implies that $L^\alpha \in \mathrm{M}(\Omega)$, $0 < \alpha < 1$. Furthermore it should be observed that L^α is symmetric if L is symmetric.

Proof of Lemma 2.1. Let $f \in \mathcal{H}$ and $\epsilon > 0$. Since for all $\delta > 0$, the operator $\delta\mathcal{L}_0 + \mathcal{L}$ is m-accretive, there exists a unique solution $u_{\delta,\mu} \in D(\mathcal{L}_0)$ of the equation

$$\epsilon u_{\delta,\mu} + \delta\mathcal{L}_0 u_{\delta,\mu} + \mathcal{L}u_{\delta,\mu} + \mathcal{A}_\mu u_{\delta,\mu} = f. \tag{2.4}$$

Note that $||u_{\delta,\mu}|| \leq \frac{1}{\epsilon}||f||$ due to the accretivity of the operators involved. Let $f \in D(\mathcal{L}_0^{\frac{1}{2}})$. By taking the innerproduct of (2.4) with $\mathcal{L}_0 u_{\delta,\mu}$ we obtain

$$\begin{aligned} \epsilon||\mathcal{L}_0^{\frac{1}{2}} u_{\delta,\mu}||^2 + \delta||\mathcal{L}_0 u_{\delta,\mu}||^2 \ &+ \ ((\mathcal{L}u_{\delta,\mu}, \mathcal{L}_0 u_{\delta,\mu})) + \\ &+ \ ((\mathcal{A}_\mu u_{\delta,\mu}, \mathcal{L}_0 u_{\delta,\mu})) = ((\mathcal{L}_0^{\frac{1}{2}} f, \mathcal{L}_0^{\frac{1}{2}} u_{\delta,\mu})) \end{aligned}$$

Using assumption (2.1) and the fact that $((\mathcal{A}_\mu u_{\delta,\mu}, \mathcal{L}_0 u_{\delta,\mu})) \geq 0$ it follows that

$$\epsilon||\mathcal{L}_0^{\frac{1}{2}} u_{\delta,\mu}||^2 \leq ||\mathcal{L}_0^{\frac{1}{2}} f|| ||\mathcal{L}_0^{\frac{1}{2}} u_{\delta,\mu}||.$$

Hence, $||\mathcal{L}_0^{\frac{1}{2}} u_{\delta,\mu}||$ and subsequently $\delta||\mathcal{L}_0 u_{\delta,\mu}||^2$ remains bounded. Then there exists a unique $u_\delta \in D(\mathcal{L}_0) \cap D(\mathcal{A})$ satisfying

$$\epsilon u_\delta + \delta\mathcal{L}_0 u_\delta + \mathcal{L}u_\delta + \mathcal{A}u_\delta \ni f, \tag{2.5}$$

(see Proposition 3.3.3) and $\mathcal{L}_0^{\frac{1}{2}} u_\delta$ respectively $\delta \mathcal{L}_0 u_\delta$ are the limits of weakly convergent subsequences $\mathcal{L}_0^{\frac{1}{2}} u_{\delta,\mu_n}$ respectively $\delta \mathcal{L}_0 u_{\delta,\mu_n}$. Therefore,

$$\|\mathcal{L}_0^{\frac{1}{2}} u_\delta\| \le \liminf_{n\to\infty} \|\mathcal{L}_0^{\frac{1}{2}} u_{\delta,\mu_n}\|,$$

and

$$\delta \|\mathcal{L}_0 u_\delta\|^2 \le \liminf_{n\to\infty} \delta \|\mathcal{L}_0 u_{\delta,\mu_n}\|^2.$$

Thus $\|\mathcal{L}_0^{\frac{1}{2}} u_\delta\|$ and $\delta\|\mathcal{L}_0 u_\delta\|^2$ remains bounded if $\delta \downarrow 0$. In particular,

$$\delta \mathcal{L}_0 u_\delta \to 0, \quad \text{as } \delta \downarrow 0,$$

which implies that

$$D(\mathcal{L}_0^{\frac{1}{2}}) \subset \overline{R(\epsilon I + \mathcal{L}\,|_{D(\mathcal{L}_0)} + \mathcal{A})}.$$

Since $D(\mathcal{L}_0^{\frac{1}{2}})$ is a dense subset of $\mathcal{H}$ and

$$\overline{R(\epsilon I + \mathcal{L}\,|_{D(\mathcal{L}_0)} + \mathcal{A})} \subset \overline{R(\epsilon I + \overline{\mathcal{L}\,|_{D(\mathcal{L}_0)} + \mathcal{A}})} = R(\epsilon I + \overline{\mathcal{L}\,|_{D(\mathcal{L}_0)} + \mathcal{A}}),$$

we conclude that

$$R(\epsilon I + \overline{\mathcal{L}\,|_{D(\mathcal{L}_0)} + \mathcal{A}}) = \mathcal{H},$$

and therefore the operator $\overline{\mathcal{L}\,|_{D(\mathcal{L}_0)} + \mathcal{A}} \subset \mathcal{H} \times \mathcal{H}$ is m-accretive which proves the first statement of the theorem.

In order to obtain the estimates for the other statements we show that

$$(Lu, L_0^\beta u) \ge 0, \quad \text{for all } u \in D(L_0),\ 0 < \beta < 1. \tag{2.6}$$

Let $u \in D(L_0)$, $0 < \beta < 1$. Then

$$\begin{aligned} (Lu, L_0^\beta u) &= (Lu, \frac{\sin\beta\pi}{\pi} \int_0^\infty \frac{1}{\lambda^\beta} L_{0,\lambda} u) \\ &= \frac{\sin\beta\pi}{\pi} \int_0^\infty \frac{1}{\lambda^\beta} (Lu, L_{0,\lambda} u) \\ &\ge \frac{\sin\beta\pi}{\pi} \int_0^\infty \frac{1}{\lambda^\beta} (L J_\lambda^{L_0} u, L_{0,\lambda} u) \ge 0. \end{aligned}$$

Take $f \in D(\mathcal{L}_0^\alpha)$ and let $0 < \alpha \leq \frac{1}{2}$. Then, by taking the innerproduct of (2.4) with $L_0^{2\alpha} u_{\delta,\mu}$, using that the operator $L_0^{2\alpha} \in \mathrm{M}(\Omega)$ is symmetric, one shows that $\epsilon \|\mathcal{L}_0^\alpha u_{\delta,\mu}\| \leq \|\mathcal{L}_0^\alpha f\|$. By observing that $u_{\delta,\mu}$ converges strongly to the solution u of (2.2) as $\delta, \mu \downarrow 0$ it follows by the closedness of $\mathcal{L}_0^\alpha$ that $u \in D(\mathcal{L}_0^\alpha)$.

Assume now that $D(L_0^\alpha) \subset D(L)$. By the closed graph theorem and the closedness of L and L_0^α we have

$$\|Lu\| \leq c(\|L_0^\alpha u\| + \|u\|), \quad \text{for all } u \in D(L_0^\alpha), \tag{2.7}$$

for some positive constant c. Making use of the fact that $D(L_0)$ is a core of $D(L_0^{2\alpha})$ one verifies that (2.6) holds, with $\beta = 2\alpha$, for all $u \in D(L_0^{2\alpha})$. Instead of equation (2.4) we consider the equation

$$\epsilon u_{\delta,\mu} + \delta \mathcal{L}_0^{2\alpha} u_{\delta,\mu} + \mathcal{L} u_{\delta,\mu} + \mathcal{A}_\mu u_{\delta,\mu} = f.$$

In a similar way as we have obtained the estimates so far, it follows that $\|\mathcal{L}_0^\alpha u_\delta\|$ and $\delta \|\mathcal{L}_0^{2\alpha} u_\delta\|^2$ remain bounded if $\delta \downarrow 0$, where

$$\epsilon u_\delta + \delta \mathcal{L}_0^{2\alpha} u_\delta + \mathcal{L} u_\delta + \mathcal{A} u_\delta \ni f.$$

In particular, $\delta \mathcal{L}_0^{2\alpha} u_\delta \to 0$ if $\delta \downarrow 0$, and by inequality (2.7), $\|\mathcal{L} u_\delta\|$ remains bounded as well. Therefore we can extract the following convergent subsequences in $\mathcal{H}$:

$$\begin{array}{lll} u_{\delta_n} & \to u = (\epsilon I + \overline{\mathcal{L}+\mathcal{A}})^{-1} f, & \\ \mathcal{L} u_{\delta_n} & \rightharpoonup w, & \\ v_{\delta_n} & \rightharpoonup f - \epsilon u - w, & \text{as } n \to \infty, \end{array}$$

for certain $w \in \mathcal{H}, v_{\delta_n} \in \mathcal{A} u_{\delta_n}, n = 1, 2, \cdots$. Since $\mathcal{L}_0^\alpha$ and $\mathcal{L}$ are linear and closed operators we have $u \in D(\mathcal{L}_0^\alpha)$ and $w = \mathcal{L} u$. Due to the m-accretivity of $\mathcal{A}$ it follows that $u \in D(\mathcal{A})$ and $f - \epsilon u - \mathcal{L} u \in \mathcal{A} u$.

5.3 Examples

In this section we describe some situations where we can apply Lemma 2.1. Let L be an m-accretive operator in $L^2(\Omega)$. We distinguish two situations, namely, where we can take $L_0 = L^* L$, respectively $L_0 = -\Delta$. The latter will be the case if $\Omega = \mathbb{R}^N$ and the operator L commutes with translations. As a first application of Lemma 2.1 we have

Proposition 3.1. *Let L be a linear m-accretive operator in $L^2(\Omega)$ with a core D such that*

i) If $u \in D$ then $| u | \in D(L)$.

ii) $(Lu^+, Lu^-) \leq 0$ for all $u \in D$, where $u^+ = u \vee 0$, $u^- = (-u)^+$.

iii) If $u \in D$ then $0 \vee (u \wedge 1) \in D(L)$ and $\|L(0 \vee (u \wedge 1)\|_2 \leq \|Lu\|_2$.
Let $\epsilon > 0$ and $0 < \beta < 1$.
Then $R(\epsilon I + \mathcal{L} + \mathcal{A}) \supset D(\mathcal{L})$ and $\overline{\mathcal{L} |_{D(\mathcal{L}^*\mathcal{L})} + \mathcal{A}} \subset \mathcal{H} \times \mathcal{H}$ is m-accretive. Moreover, if $f \in D(\mathcal{L}^\beta)$ then the solution u of

$$\epsilon u + \overline{\mathcal{L} |_{D(\mathcal{L}^*\mathcal{L})} + \mathcal{A}}\, u \ni f \tag{3.1}$$

belongs to $D(\mathcal{L}^\beta)$.

Proof. It is well-known that L^*L is selfadjoint and m-accretive. We show that L^*L can play the role of L_0 in Lemma 2.1. For that, observe that the non-negative closed and bilinear form Q corresponding to the positive selfadjoint operator L^*L is given by

$$\begin{cases} Q : D(L) \times D(L) \to I\!R \\ Q(u,v) = (Lu, Lv), \qquad u, v \in D(L) \end{cases}$$

By the Beurling-Deny conditions, formulated in terms of Q, the operator L^*L belongs to the class $\mathbf{M}(\Omega)$ if and only if i), ii) and iii) holds for a core D of L, see for example [DV,Thm 1.3.2, Thm 1.3.3 and Lemma 1.3.4]. Observe that $(Lf, L^*Lf) \geq 0$ for all $f \in D(L^*L)$ since L^* is accretive. Note also that $D((L^*L)^{\frac{1}{2}}) = D(Q) = D(L)$. The purely imaginary powers of $(L^*L)^{\frac{1}{2}}$ and L satisfies estimate (H2) (see Section 2.4) with $M = 1$ and $\omega = \frac{\pi}{2}$ [K2]. By [A2] it follows that

$$D((L^*L)^{\frac{\beta}{2}}) = [L^2(\Omega), D((L^*L)^{\frac{1}{2}})]_\beta = [L^2(\Omega), D(L)]_\beta = D(L^\beta).$$

Here $[\ ,\ ]_\beta$ denotes the complex interpolation. The result follows now from Lemma 2.1.

As a consequence of the following theorem we can apply Proposition 3.1 to skew-adjoint operators in the class $\mathbf{M}(\Omega)$.

Theorem 3.2. *Let $L \in \mathbf{M}(\Omega)$ be a skew-adjoint operator.*
Then $L^*L \in \mathbf{M}(\Omega)$ as well, or equivalently, L satisfies the assumptions of Proposition 3.1.

Proof. Denote the by $\{T(t)\}_{t\geq 0}$ the semigroup generated by $-L$. Recall that the semigroup $\{T(t)\}_{t\geq 0}$ consists of unitary operators. Set

$$T_{(\lambda)} := \frac{1}{\lambda}(I - T(\lambda)).$$

Then

$$T^*_{(\lambda)}T_{(\lambda)}f \to L^*Lf \quad \text{if } \lambda \downarrow 0 \ \text{ for all } f \in D(L^*L),$$

as can be seen by writing

$$T^*_{(\lambda)}T_{(\lambda)} = \frac{1}{\lambda}\int_0^\lambda T^*(t)\{\frac{1}{\lambda}\int_0^\lambda T(s)L^*Lf ds\}dt.$$

So that $T^*_{(\lambda)}T_{(\lambda)} \to L^*L$ in the sense of resolvents that is,

$$J_\mu^{T^*_{(\lambda)}T_{(\lambda)}} f \to J_\mu^{L^*L} f \quad \text{if } \lambda \downarrow 0 \ \text{ for all } f \in L^2(\Omega),\ \mu > 0.$$

Thus for the positivity of the resolvent $J_\mu^{L^*L}$ it suffices to show that $-T^*_{(\lambda)}T_{(\lambda)}$ generates a positive semigroup. This follows from the following inequality

$$\begin{aligned}(T_{(\lambda)}u^+, T_{(\lambda)}u^-) &= \frac{1}{\lambda^2}((I - T(\lambda))u^+, (I - T(\lambda))u^-)\\ &= -\frac{1}{\lambda^2}\{(T(\lambda)u^+, u^-) + (u^+, T(\lambda)u^-)\} \leq 0,\end{aligned}$$

for all $u = u^+ - u^- \in L^2(\Omega)$. Here we used that $T(\lambda)$ is a unitary and order preserving operator. In order to prove that the resolvent $J_\mu^{L^*L}$ is a contraction in $L^1(\Omega)$ observe that

$$\begin{aligned}T^*_{(\lambda)}T_{(\lambda)} &= \frac{1}{\lambda^2}\{(I - T^*(\lambda))(I - T(\lambda))\}\\ &= \frac{1}{\lambda^2}\{I - T^*(\lambda) + I - T(\lambda)\}.\end{aligned}$$

Since $T(\lambda)$ as well as $T^*(\lambda)$ defines an contraction in $L^1(\Omega)$ it follows that $I - T(\lambda)$ and $I - T^*(\lambda)$ are m-accretive operators in $L^1(\Omega)$. Hence, $T^*_{(\lambda)}T_{(\lambda)}$ is an m-accretive operator in $L^1(\Omega)$. Next, consider the equation

$$u_\lambda + \mu T^*_{(\lambda)}T_{(\lambda)}u_\lambda = f,$$

with $0 \leq f \in L^1(\Omega) \cap L^2(\Omega)$ and $\mu > 0$ fixed. As shown above the solution u_λ is non-negative and $u_\lambda \to u$ in $L^2(\Omega)$ if $\lambda \downarrow 0$ where $u = J_\mu^{L*L}f$. By subtracting a pointwise convergent subsequence $\{u_{\lambda_n}\}_{n=1}^{\infty}$ and applying Fatou's lemma we obtain

$$\|u\|_1 \le \liminf_{n\to\infty} \|u_{\lambda_n}\|_1 \le \|f\|_1.$$

We have proved now that $J_\mu^{L^*L}$ is a positive contraction in $L^1(\Omega)$ and by duality it is also a contraction in $L^\infty(\Omega)$.

Example 3.3. One verifies that the operator $Lu = \frac{du}{dt}$ in $L^2(0,T)$ with domain

$$D(L) = \{u \in W^{1,2}(0,T) : u(0) = 0\} \text{ or}$$
$$D(L) = \{u \in W^{1,2}(0,T) : u(0) = u(T)\}$$

and the operator $Lu = \frac{du}{dt}$ in $L^2(I\!R)$ with domain $D(L) = W^{1,2}(I\!R)$ satisfies the assumptions of Proposition 3.1. However, for these operators one can show that equation (2.3) has a unique strong solution $u \in D(\mathcal{L}) \cap D(\mathcal{A})$ if f is of bounded variation on $[0,T]$ [BR1].

Note that the operator with the periodic boundery values and the operator on the real line satisfy the assumptions of Theorem 3.2.

Example 3.4. A first order differential operator on $I\!R^N$ with smooth coefficients. We define

$$\begin{cases} D(L) = \{u \in L^2(I\!R^N) : \sum_i a_i \frac{\partial u}{\partial x_i} \in L^2(I\!R^N) \text{ in distributional sense }\} \\ Lu = \sum_i a_i \frac{\partial u}{\partial x_i} \text{ for } u \in D(L), \end{cases}$$

where $a_i, i = 1, \cdots, N$, are bounded continuous differentiable functions with bounded derivative, and $\sum_i \frac{\partial a_i}{\partial x_i} \le 0$ on $I\!R^N$. The operator L is m-accretive and $W^{1,2}(I\!R^N) \subset D(L)$ is a core of L, see [M, page 321-322]. We show that the assumptions of Proposition 3.1 are satisfied with $D = W^{1,2}(I\!R^N)$.

First of all i) holds since $W^{1,2}(I\!R^N)$ itself is a sublattice. Clearly ii) is true. It remains to show iii) : $u \in D$ implies $0 \vee (u \wedge 1) \in D$ and

$$L(0 \vee (u \wedge 1)) = \sum_i a_i \frac{\partial u}{\partial x_i} \mathcal{X}_{[0<u<1]},$$

where $\mathcal{X}_{[0<u<1]}$ denotes the characteristic function of the set $[0 < u < 1] = \{x \in I\!R^N : 0 < u(x) < 1\}$. Hence $\|L(0 \vee (u \wedge 1))\|_2 \le \|Lu\|_2$ and iii) is proved.

It can be shown that $L \in \mathrm{M}(I\!R^N)$ so that, by Theorem 3.2.1, $\mathcal{L} + \mathcal{A}$ is m-accretive if $A = \partial\varphi$, $\varphi \in \mathcal{J}_0$.

Now we consider the situation where we can take $L_0 = -\Delta := -\sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$. We introduce the fractional Sobolev spaces $W^{s,2}(I\!R^N;H)$, $s>0$, by defining

$$W^{s,2}(I\!R^N;H) := \{u \in L^2(I\!R^N;H) : x \to |x|^s \tilde{f}(x) \text{ belongs to } L^2(I\!R^N;H)\}.$$

Here $\tilde{u}$ denotes the Fourier transform of u in $L^2(I\!R^N;H)$.

Proposition 3.5. *Let L be a linear m-accretive operator in $L^2(I\!R^N)$ which commutes with the group of translation.*
Let $\epsilon > 0$ and $0 < s \leq 1$.

i) If $W^{2,2}(I\!R^N) \subset D(L)$ then $\overline{\mathcal{L}\mid_{W^{2,2}(I\!R^N;H)} + \mathcal{A}} \subset \mathcal{H}\times\mathcal{H}$ is m-accretive.

ii) If $W^{s,2}(I\!R^N) \subset D(L)$ then $R(\epsilon I + \mathcal{L} + \mathcal{A}) \supset W^{s,2}(I\!R^N;H)$ and if $f \in W^{s,2}(I\!R^N;H)$ then the solution u of (2.3) belongs to $W^{s,2}(I\!R^N;H)$.

Proof. Apply Lemma 2.1, where $L_0 = -\Delta$ and observe that $D(L_0^{\frac{s}{2}}) = W^{s,2}(I\!R^N)$.

Remark 3.6. The fact that under the assumptions of Proposition 3.5 inequality (2.1) holds is a special case of the following situation. If a symmetric m-accretive operator L_0 with $D(L_0) \subset D(L)$ is resolvent commuting with L i.e. $J_\mu^{L_0} J_\lambda^L = J_\lambda^L J_\mu^{L_0}$ for all $\lambda, \mu > 0$ (equivalently for some $\lambda, \mu > 0$) then (2.1) holds. Indeed,

$$(L_\lambda f, L_{0,\mu} f) = ((L_{0,\mu})^{\frac{1}{2}} L_\lambda f, (L_{0,\mu})^{\frac{1}{2}} f) = (L_\lambda (L_{0,\mu})^{\frac{1}{2}} f, (L_{0,\mu})^{\frac{1}{2}} f) \geq 0.$$

If $f \in D(L_0)$ and $\lambda, \mu \downarrow 0$ it follows that $(Lf, L_0 f) \geq 0$.

Example 3.7. A second order differential operator on $I\!R^N$ with constant coefficients.
Define the m-accretive operator L in $L^2(I\!R^N)$ by setting

$$\begin{cases} D(L) = \{u \in L^2(I\!R^N) : -\sum\limits_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum\limits_i b_i \frac{\partial u}{\partial x_i} + cu \in L^2(I\!R^N)\} \\ Lu = -\sum\limits_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum\limits_i b_i \frac{\partial u}{\partial x_i} + cu, \text{ for } u \in D(L) \end{cases} \tag{3.2}$$

where $(a_{ij})_{i,j=1}^N$ is a real positive semidefinite symmetric matrix, $b_i \in I\!R$ and $c \geq 0$. By Proposition 3.1 we have that $\overline{\mathcal{L}\mid_{W^{2,2}(I\!R^N;H)} + \mathcal{A}}$ is m-accretive. In the case that $(a_{ij})_{i,j=1}^N$ is positive definite it follows from Proposition 4.3.2 that $\mathcal{L} + \mathcal{A}$ is m-accretive.

We end this section with some remarks concerning the structure of translation invariant m-accretive operators in $L^2(\mathbb{R}^N)$.

Remark 3.8. (i) An m-accretive operator L in $L^2(\mathbb{R}^N)$ is given by (3.2) with $(a_{ij})_{i,j=1}^N$ a real positive semidefinite symmetric matrix, $b_i \in \mathbb{R}$ and $c \geq 0$ if and only if the operator L is local, commutes with translations, and the resolvent J_λ^L is order-preserving for all $\lambda > 0$ [BG-F, page 190].

(ii) There is a one to one correspondence between translation invariant m-accretive operators $L \in L^2(\mathbb{R}^N)$, having an order-preserving resolvent family $\{J_\lambda^L,\ \lambda > 0\}$, and continuous negative definite functions $\varphi : \mathbb{R}^N \to \mathbb{C}$, given by

$$\begin{cases} D(L) = \{f \in L^2(\mathbb{R}^N) : \varphi \tilde{f} \in L^2(\mathbb{R}^N)\}; \\ \tilde{Lf} = \varphi \tilde{f} \text{ for } f \in D(L), \end{cases} \tag{3.3}$$

see [BG-F]. In connection with Proposition 3.5 i), we note that, by [BG-F, Corollary 7.16], a locally bounded negative definite function is at most of order $| x |^2$ if $| x | \to \infty$. Hence $W^{2,2}(\mathbb{R}^N) \subset D(L)$.

For an expression of a continuous negative definite function, the so-called Lévy-Khinchine formula, we refer to [BG-F] or [CO].

(iii) Let the operator L be as in (ii). It is known that the semigroup $\{S(t)\}_{t\geq 0}$, generated by $-L$, can be expressed as

$$(S(t)f)(s) = (f * \mu_t)(s) := \int_{\mathbb{R}^N} f(s-r)d\mu_t(r),\ t > 0, s \in \mathbb{R}^N, \tag{3.4}$$

$f \in L^2(\mathbb{R}^N)$, where $\{\mu_t\}_{t>0}$ is a family of bounded positive measures (see [BG-F]). It follows that $L \in \mathrm{M}(\mathbb{R}^N)$ due to the fact that the positive measures $\{\mu_t\}_{t\geq 0}$ in expression (3.4) are bounded by 1 ($\mu_t(\mathbb{R}^N) = \|S(t)\| \leq 1$). Hence $\|S(t)f\|_p \leq \|f\|_p$ for all $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $p \in [1, \infty]$. We note also that L is a normal operator.

Part II

Applications

Chapter 6

Elliptic systems

6.1 Introduction

Let $\Omega \subset I\!R^n$ be an open and bounded subset with a C^2 boundary $\partial\Omega$. In the first section we consider the elliptic system

$$\begin{array}{rclcll} -\Delta u_1 & = & \sum_{j=1}^{N} a_{1j}u_j & + & g_1 & \\ -\Delta u_2 & = & \sum_{j=1}^{N} a_{2j}u_j & + & g_2 & \text{in } \Omega \\ \vdots & & \vdots & & \vdots & \\ -\Delta u_N & = & \sum_{j=1}^{N} a_{Nj}u_j & + & g_N & \\ \multicolumn{5}{l}{u_1 = u_2 = \ldots = u_N = 0} & \text{on } \partial\Omega, \end{array} \qquad (1.1)$$

where $g_i \in L^2(\Omega)$, $i = 1, \ldots, N$, $A = (a_{ij})_{i,j=1}^{N}$ a real $N \times N$ matrix. The system is assumed to be cooperative, that is, $a_{ij} \geq 0$ for $i \neq j$. de Figueiredo and Mitidieri proved among other things in [F-M1] that a necessary and sufficient condition for the unique solvability and the positivity of the solution for positive data of system (1.1) is that the principal minors of $\lambda_0 I - A$ are positive (see also [F-M2]). Here λ_0 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions. We shall give another proof of this result [CL-E].

In the second section we prove an existence result for the following semilinear elliptic system

$$\begin{array}{rclll} L_1u_1 & + & A_1(u_1,u_2,...,u_N) & \ni & g_1(\omega,u_1,u_2,...,u_N) \\ L_2u_2 & + & A_2(u_1,u_2,...,u_N) & \ni & g_2(\omega,u_1,u_2,...,u_N) \\ \vdots & & \vdots & & \vdots \\ L_Nu_N & + & A_N(u_1,u_2,...,u_N) & \ni & g_N(\omega,u_1,u_2,...,u_N) \\ u_1 = u_2 = \ldots = u_N = 0, & & & & \end{array} \quad \begin{array}{l} \\ \text{in } \Omega \\ \\ \text{on } \partial\Omega \end{array} \tag{1.2}$$

Here $L_1, L_2, \ldots, L_N$ are N m-accretive strictly elliptic second order differential operators with smooth coefficients and $A = (A_1, A_2, ...A_N)$ an m-accretive graph in $I\!R^N$ with respect to some innerproduct. The function

$$g = (g_1, g_2, ..., g_N) : \Omega \times I\!R^N \to I\!R^N$$

satisfies the Caratheodory condition and a growth condition.

6.2 A maximum principle for a linear cooperative system

Let $A = (a_{ij})_{i,j=1}^N$ be a real $N \times N$ matrix.

Theorem 2.1. *Assume that A is cooperative: $a_{ij} \geq 0$ for $i \neq j$. Then the system (1.1) is uniquely solvable and the solution is non-negative for non-negative data g_i, $i = 1, \ldots, N$, if and only if*

$$\text{the principal minors of } \lambda_0 I - A \text{ are positive} \tag{2.1}$$

Proof. If A is a cooperative matrix then, by [BM-P, Ch.6, Thm 2.3], condition (2.1) holds if and only if the matrix $\lambda_0 I - A$ is invertible and the matrix $(\lambda_0 I - A)^{-1}$ is non-negative. Taking this equivalence into account we prove first the necessity of (2.1).
If $x \in I\!R^N$ satisfies $(\lambda_0 I - A)x = 0$ and if u_0 is the principal eigenvector of $-\Delta$ normalized by $\max_{x \in I\!R^N} u_0(x) = 1$ then $u = u_0 \cdot x$ satisfies system (1.1) with $g = 0$. From this we have $x = 0$. Hence $(\lambda_0 I - A)$ is invertible. For $v \in I\!R^N_+$, let $x := (\lambda_0 I - A)^{-1} v$. Then, the function $u = u_0 \cdot x$ satisfies system (1.2) with $g = u_0 \cdot v$ which is non-negative. Since $u_0 > 0$ in Ω, we have $x \in I\!R^N_+$.
Next, we prove the sufficiency. By [BM-P, Ch. 6, Thm 2.3], we have that the

real parts of the eigenvalues of $\lambda_0 I - A$ are positive. By using Lyapunov's theorem, there exists a symmetric positive definite matrix B such that

$$B(\lambda_0 I - A) + (\lambda_0 I - A)^T B$$

is positive definite, see [BM-P, Ch. 6, Thm 2.3]. Define the following inner-product on $I\!R^N$:

$$(x,y)_B := x^T B y \quad \text{for all } x, y \in I\!R^N.$$

Then there exists a constant $c > 0$ such that

$$((\lambda_0 I - A)x, x)_B \geq c(x,x)_B.$$

Define L by setting

$$\begin{cases} D(L) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \\ Lu = -\frac{1}{c}(\Delta + \lambda_0 I)u \quad \text{for } u \in D(L) \end{cases}$$

and define $A_0 = \frac{1}{c}(\lambda_0 - c - A)$. Note that L is m-accretive in $L^2(\Omega)$ and that A_0 is m-accretive in $(I\!R^N, (\cdot,\cdot)_B)$. Denote by $\mathcal{L}$ and $\mathcal{A}_0$ the m-accretive extensions in $L^2(\Omega; I\!R^N)$ of L and A_0 respectively. Then $\mathcal{L}$ and $\mathcal{A}_0$ form an acute angle. This is a consequence of the fact that $\mathcal{L}_\lambda$ and $\mathcal{A}_{0,\mu}$ commute and that $\mathcal{L}$ is symmetric, see Section 3.1. Set $\mathcal{K} = L^2(\Omega; I\!R^N_+)$. It is well-known that $J_\lambda^{\mathcal{L}}\mathcal{K} \subset \mathcal{K}$ and since A is cooperative we have that $e^{-tA_0} I\!R^N_+ \subset I\!R^N_+$ and therefore $J_\lambda^{\mathcal{A}_0}\mathcal{K} \subset \mathcal{K}$. By the Trotter product formula (2.2.4) it follows that the semigroup $\{S(t)\}_{t\geq 0}$ generated by $-(\mathcal{L} + \mathcal{A}_0)$ leaves $\mathcal{K}$ invariant. This implies, by the well-known formula $J_\lambda^{\mathcal{L}+\mathcal{A}_0} = \frac{1}{\lambda}\int_0^\infty e^{-\frac{t}{\lambda}} S(t)dt$, that $J_1^{\mathcal{L}+\mathcal{A}_0}\mathcal{K} \subset \mathcal{K}$ and the proof is complete.

6.3 A semilinear elliptic system

Let $\Omega \subset I\!R^n$ be an open and bounded subset with C^2 boundary $\partial\Omega$. We define for $k = 1, \ldots, N$ the following elliptic differential operators

$$L_k u = -\sum_{i,j}^{n} \frac{\partial}{\partial x_i}(a_{ij}^{(k)} \frac{\partial u}{\partial x_j}) + \sum_{i}^{n} \frac{\partial}{\partial x_i}(a_i^{(k)} u) + a^{(k)} u,$$

where,

$$(H1)\begin{cases} a_{ij}^{(k)},\ a_i^{(k)} \in C^1(\overline{\Omega}),\ a^{(k)} \in L^\infty(\Omega), \quad i,j = 1,\ldots,n; \\ \sum_{i,j}^{n} a_{ij}^{(k)}\xi_i\xi_j \geq \alpha \sum_i^n \xi_i^2 \text{ on } \Omega,\ \xi = \{\xi_1,\ldots,\xi_N\} \in \mathbb{R}^N \text{ for some } \alpha > 0; \\ a^{(k)} \geq 0,\ a^{(k)} + \frac{1}{2}\sum_i \frac{\partial a_i^{(k)}}{\partial x_i} + \alpha\lambda_0 \geq \delta \text{ a.e. for some } \delta > 0. \end{cases}$$

Here λ_0 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. Let $\langle\cdot,\cdot\rangle$ denote an innerproduct in $\mathbb{R}^N$ and let

$$A = (A_1, ..., A_N) : D(A) \subset \mathbb{R}^N \to 2^{\mathbb{R}^N}$$

be an m-accretive map in $(\mathbb{R}^N, \langle\cdot,\cdot\rangle)$. If $B = (b_{ij})_{i,j=1}^N$ is the positive definite $N \times N$ matrix such that $\langle x, y\rangle = x^T By$, $x, y \in \mathbb{R}^N$ then the accretivity of A is expressed by

$$\sum_{i,j=1}^{N} b_{ij}(x^i - y^i)(x_j - y_j) \geq 0 \quad \text{for all } x^i \in A_i x,\ y^i \in A_i x, \tag{3.1}$$

where $x = (x_1, ..., x_N)$, $y = (y_1, ..., y_N)$.

The function $g = (g_1, g_2, ..., g_N) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to satisfy the following two conditions

$$(H2)\begin{cases} (i)\ g(\cdot,x) \text{ is measurable for all } x \in \mathbb{R}^N \text{ and } g(\omega,\cdot) \text{ is continuous,} \\ \quad \omega \in \Omega \text{ almost everywhere (the Caratheodory condition).} \\ (ii)\ |g(\omega,x)| \leq h(\omega) + c|x|^\beta \text{ for some } h \in L^2(\Omega),\ c \in \mathbb{R}^N \text{ and} \\ \quad 0 \leq \beta < 1 \text{ for all } x \in \mathbb{R}^N \end{cases}$$

Define for $u \in L^2(\Omega; \mathbb{R}^N)$, $G(u)(\omega) = g(\omega, u(\omega))$. It is well-known that under the assumptions on g that G defines a bounded and continuous map in $L^2(\Omega; \mathbb{R}^N)$ (see for example [BD]).

The innerproduct in $L^2(\Omega; \mathbb{R}^N)$ will be denoted by $((\cdot,\cdot))$ and $\|\cdot\| := ((\cdot,\cdot))^{\frac{1}{2}}$.

Theorem 3.1. *Let $L_1, \ldots, L_N$ and $g = (g_1, g_2, \ldots, g_N) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy $(H1)$ and $(H2)$. Assume that $A = (A_1, A_2, \ldots, A_N)$ is m-accretive in $(\mathbb{R}^N, \langle\cdot,\cdot\rangle)$ with $0 \in D(A)$.*
Then there exists $u = (u_1, u_2, \ldots, u_N) \in \{W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\}^N$ satisfying (1.2). Moreover if $f = (f_1, f_2, \ldots, f_N) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy $(H2)$ and if

u, v are solutions of (1.2) with right-hand side g respectively f then $\|u-v\| \le \frac{1}{\delta}\|Gu - Fv\|$, where F is defined by $F(u)(\omega) = f(\omega, u(\omega))$, $\omega \in \Omega$.

The basic tool in the proof of this theorem is the following inequality due to Sobolevskii [SO]:

Let $\Omega \subset \mathbb{R}^N$ be as above and let L and M be two second order strictly elliptic operators with bounded coefficients and leading coefficients belonging to $C^1(\overline{\Omega})$. Then there exist constants $a > 0$, $b \in \mathbb{R}^+$ such that

$$(Lu, Mu) \ge a\|u\|^2_{W^{2,2}(\Omega)} - b\|u\|_2^2 \quad \textit{for all } u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega). \tag{3.2}$$

See also [L-U, page 182], [BR-E][1] and [SK][1] .

Proof of Theorem 3.1. Since Ω is bounded we can assume that $0 \in A0$, otherwise consider the m-accretive operator $A_0 = A - x_0$, where $x_0 \in A0$. It is well-known that the operators L_k, $k = 1, \dots, N$, with domain $D(L_k) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ are m-accretive in $L^2(\Omega)$ and that $\delta\|u\|_2^2 \le (L_k u, u)$ for $u \in D(L_k)$ [BR-S]. Set

$$\begin{cases} D(\mathcal{L}) = \{W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\}^N \subset L^2(\Omega; \mathbb{R}^N) \\ \mathcal{L}u = (L_1 u_1, L_2 u_2, \dots, L_N u_N) \quad \text{for } u = (u_1, u_2, \dots, u_N) \in D(\mathcal{L}) \end{cases}$$

Then $\mathcal{L}$ is an m-accretive operator in $L^2(\Omega; \mathbb{R}^N)$. Let $L_0 = -\Delta$, the Laplacian with domain $D(L_0) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. The m-accretive extension in $L^2(\Omega; \mathbb{R}^N)$ is denoted by $\mathcal{L}_0$ and the extension of A in $L^2(\Omega; \mathbb{R}^N)$ by $\mathcal{A}$.

We will show that the map $(\mathcal{L} + \mathcal{A})^{-1}G : L^2(\Omega; \mathbb{R}^N) \to L^2(\Omega; \mathbb{R}^N)$ is compact and then using a fixed point theorem the result will follow. First, $R(\mathcal{L} + \mathcal{A}) = L^2(\Omega; \mathbb{R}^N)$. For that, consider the equation

$$\epsilon u_\lambda + \mathcal{L}u_\lambda + \mathcal{A}_\lambda u_\lambda = f, \quad f \in \mathcal{H}, \ \epsilon, \lambda > 0.$$

Then

$$\epsilon((u_\lambda, \mathcal{L}_0 u_\lambda)) + ((\mathcal{L}u_\lambda, \mathcal{L}_0 u_\lambda)) + ((\mathcal{A}_\lambda u_\lambda, \mathcal{L}_0 u_\lambda)) = ((f, \mathcal{L}_0 u_\lambda)).$$

Since $L_0 \in \mathrm{M}(\Omega)$ is symmetric we have by Theorem 3.3.1 that

$$((\mathcal{A}_\lambda u_\lambda, \mathcal{L}_0 u_\lambda)) \ge 0.$$

Hence $((\mathcal{L}u_\lambda, \mathcal{L}_0 u_\lambda)) \le \|f\|\|\mathcal{L}_0 u_\lambda\|$. Using Sobolevskii's inequality we obtain that there exist constants $a > 0$, $b \in \mathbb{R}^+$ such that

$$((\mathcal{L}u_\lambda, \mathcal{L}_0 u_\lambda)) \ge a\|\mathcal{L}_0 u_\lambda\|^2 - b\|u_\lambda\|^2, \quad \lambda > 0. \tag{3.3}$$

[1] The author thanks Patrick Fitzpatrick for pointing out these references.

It follows that $\|\mathcal{L}_0 u_\lambda\|$ remains bounded if $\lambda \downarrow 0$. Similarly as in the proof of Lemma 4.3.2 it follows that $\mathcal{L}+\mathcal{A}$ is m-accretive. Hence there exists a unique $u_\epsilon \in D(\mathcal{L}) \cap D(\mathcal{A})$ satisfying

$$\epsilon u_\epsilon + \mathcal{L}u_\epsilon + \mathcal{A}u_\epsilon \ni f$$

for all $\epsilon > 0$ and $f \in \mathcal{H}$. Since $\delta\|u\|_2^2 \le (L_k u, u)$ for all $u \in D(L_k)$ we have that

$$\delta\|u\|^2 \le ((\mathcal{L}u, u)) \quad \text{for all } u \in D(\mathcal{L}). \tag{3.4}$$

Using this estimate we get $\|u_\epsilon\| \le \frac{1}{\delta}\|f\|$ one shows, by passing to the limit, that there exists a unique $u \in D(\mathcal{L}) \cap D(\mathcal{A})$ such that $\mathcal{L}u + \mathcal{A}u \ni f$. Observe, using (3.3) and (3.4), that $\|\mathcal{L}u\| \le c\|f\|$ for some constant $c > 0$. It follows that the map $(\mathcal{L}+\mathcal{A})^{-1} : L^2(\Omega; I\!R^N) \to D(\mathcal{L})$ is bounded ($D(\mathcal{L})$ equipped with the graph norm of $\mathcal{L}$). By a well-known embedding theorem $D(\mathcal{L})$ is compactly embedded in $L^2(\Omega; I\!R^N)$.

Therefore we may conclude that the mapping

$$(\mathcal{L}+\mathcal{A})^{-1}G : L^2(\Omega; I\!R^N) \to L^2(\Omega; I\!R^N)$$

is compact. If we can show that the set

$$S = \{v \in L^2(\Omega; I\!R^N) : v = \sigma(\mathcal{L}+\mathcal{A})^{-1}Gv, \sigma \in [0,1]\}$$

is bounded it follows that $(\mathcal{L}+\mathcal{A})^{-1}G$ has a fixed point (see e.g. [G-T]). Taking the innerproduct of $\mathcal{L}v + \mathcal{A}v \ni \sigma G(v)$ with v and using (3.4), (H2) and that $((\mathcal{A}u, u)) \ge 0$ for all $u \in D(\mathcal{A})$ we obtain that for all $\epsilon > 0$ there exists C_ϵ such that

$$\delta\|v\|^2 \le \sigma((G(v), v)) \le \sigma\{\epsilon\|v\|^2 + C_\epsilon\}$$

and the boundedness of S is proved. Since $(\mathcal{L}+\mathcal{A})^{-1}$ is Lipschitz continuous with constant $\frac{1}{\delta}$ the last statement of the theorem follows.

Remark 3.2. If $A : I\!R^N \to I\!R^N$ is an everywhere defined, continuous (hemicontinuous) and accretive map with respect to $\langle\cdot,\cdot\rangle$ then A is m-accretive [BR1,Prop.2.4]. Note that if $A = (A_1, ..., A_N)$ is continuously differentiable then (3.1) is equivalent to

$$\sum_{i,j=1}^{N} b_{ij} \sum_{k=1}^{N} \frac{\partial A_i}{\partial x_k}(x)\xi_k\xi_j \ge 0, \quad \text{for all } x, \xi = (\xi_1, ..., \xi_N) \in I\!R^N$$

Remark 3.3. Let $\mathcal{L}_0$ and $A = (A_1, ..., A_N)$ be as in the proof of Theorem 2.1. We have used the fact that, by Theorem 3.3.1, $((\mathcal{L}_0 u, \mathcal{A}_\lambda u)) \geq 0$ for all $u \in D(\mathcal{L}_0)$. For this particular case the inequality can be proven by partial integration as we will indicate. Note that $0 = A_\lambda 0$ since $0 \in A0$ and recall that $A_\lambda = (A_{\lambda,1}, ..., A_{\lambda,N})$ is Lipschitz continuous. Then for $u = (u_1, ..., u_N) \in D(\mathcal{L}_0)$ we have by partial integration

$$\begin{aligned}
((\mathcal{L}_0 u, \mathcal{A}_\lambda u)) &= \int_\Omega \sum_{i,j}^{N} b_{ij}(-\Delta) u_j(x) A_{\lambda,i}(u(x))dx \\
&= -\sum_{k=1}^{n} \sum_{i,j=1}^{N} b_{ij} \int_\Omega \frac{\partial^2 u_j}{\partial x_k^2}(x) A_{\lambda,i}(u(x))dx \\
&= \sum_{k=1}^{n} \sum_{i,j=1}^{N} b_{ij} \int_\Omega \frac{\partial u_j}{\partial x_k}(x) \frac{\partial}{\partial x_k} A_{\lambda,i}(u(x))dx \\
&= \sum_{k=1}^{n} \sum_{i,j=1}^{N} b_{ij} \int_\Omega \frac{\partial u_j}{\partial x_k}(x) \sum_{l=1}^{N} \frac{\partial A_{\lambda,i}}{\partial x_l}(u(x)) \frac{\partial u_l}{\partial x_k}(x)dx \\
&= \sum_{k=1}^{n} \sum_{i,j=1}^{N} b_{ij} \int_\Omega \sum_{l=1}^{N} \frac{\partial A_{\lambda,i}}{\partial x_l}(u(x)) \frac{\partial u_l}{\partial x_k}(x) \frac{\partial u_j}{\partial x_k}(x)dx \geq 0.
\end{aligned}$$

Chapter 7

Nonlinear Volterra equations

7.1 Introduction

In the first section of this chapter we will deal with the following equation in a real Hilbert space H on the interval $J = \mathbb{R},\ \mathbb{R}^+$ or $[0,T]$

$$\begin{cases} \epsilon u(t) + \frac{d}{dt}(\gamma u(t) + \int\limits_0^t k(t-s)u(s)ds) + Au(t) \ni f(t), & t \in J; \\ u(0) = 0, \quad \text{if } J = \mathbb{R}^+ \text{ or } [0,T]. \end{cases} \tag{1.1}$$

Here A is an m-accretive operator in H, $f \in L^2(J;H)$, $\epsilon, \gamma \geq 0$ and the kernel k satisfies

$$0 \leq k \in L^1(0,T) \text{ non-increasing and } T = \infty \text{ if } J = \mathbb{R}, \mathbb{R}^+. \tag{1.2}$$

We prove results concerning existence of strong solutions of (1.1) based on the theory developed in the previous chapters. Also the case where A is a subdifferential is considered. Some of the results obtained here are not new and should be regarded as an illustration of the theory. However the results where $\gamma = 0$ is assumed appear to be new.

In the second section, equation (1.1) is considered in a real Banach space X. The operator A is then an m-accretive operator in X. We take here $J = [0,T]$, $\epsilon = 0$, $f \in L^1(0,T;X)$. Gripenberg proved in [GR3] that for all $f \in L^1(0,T;X)$ there exists a "generalized" solution of (1.1). Based on private communication from dr. G. Gripenberg[1] a proof of this result is given, which is along the lines of the setting in this thesis.

[1]The author likes to thank Gustaf Gripenberg for useful discussions.

Let us summarize some known facts as a preparation of the next two sections.

We denote by "$*$" the convolution operation

$$(a * u)(t) := \int_0^t a(t-s)u(s)ds,$$

for $a \in L^1(0,T)$ $(L^1(I\!R^+))$, $u \in L^p(0,T;X)$ $(L^p(I\!R^+;X),\ L^p(I\!R;X))$ and $p \in [1,\infty]$, where the integral should be interpreted as a Bochner integral. Recall that by Young's inequality,

$$\|a * u\|_p \le \|a\|_1 \|u\|_p, \qquad p \in [1,\infty].$$

For $J = I\!R^+$ or $[0,T]$ and $p \in [1,\infty)$ we set

$$W_0^{1,p}(J) := \{u \in W^{1,p}(J) : u(0) = 0\}$$

and $W_0^{1,p}(I\!R) = W^{1,p}(I\!R)$. Define the operator L in $L^2(J)$ by

$$\begin{cases} D(L) := \{u \in L^2(J) : \gamma u + k * u \in W_0^{1,2}(J)\}; \\ Lu := \frac{d}{dt}(\gamma u + k * u), \end{cases} \tag{1.3}$$

where $\gamma \ge 0$ and the kernel k satisfying (1.2). Then $L \in \mathbf{M}(J)$.

In the case $J = I\!R$ the m-accretivity of L can be shown using the Fourier transform and is a consequence the fact that (1.2) implies that

$$\int_0^\infty k(y) \sin \omega y dy \ge 0 \quad \text{for all } \omega \in I\!R^+.$$

Using (1.2) one shows that $(Lu, u^-) \le 0$ for $u = u^+ - u^- \in D(L)$ which implies the positivity of the resolvent J_λ^L. Since the m-accretive operator L is the negative generator of a positive translation semigroup it follows that $L \in \mathbf{M}(J)$, see Remark 5.3.8 (iii).

Let $J = I\!R^+$ or $[0,T]$ and let $\gamma > 0$. Then there exists a function $0 \le b \in L^\infty(J)$ satisfying

$$\gamma b + k * b = 1,$$

see [LE]. Note that $b \le \frac{1}{\lambda}$. In [CL-N2] it has been proven that the solution r_λ of

$$u + \frac{1}{\lambda} b * u = \frac{1}{\lambda} b, \quad \lambda > 0$$

is non-negative, belongs to $L^1(J)$ and $\|r_\lambda\|_1 \leq 1$. One shows that $J_\lambda^L u = r_\lambda * u$ (see [CL-M]) and it follows that $L \in \mathrm{M}(J)$. For $\gamma = 0$ it has been shown in [GR1] that there exists a positive locally finite measure β satisfying $k * \beta = 1$, see also [GR-L-S,Theorem 5.5.5]. Then $J_\lambda^L u = \rho_\lambda * u$, where ρ_λ is a positive measure with $\int_J d\rho_\lambda(s) \leq 1$ which is the solution of

$$\mu + \frac{1}{\lambda}\beta * \mu = \frac{1}{\lambda}\beta.$$

(Here we are dealing with convolutions of measures.) Again we get $L \in \mathrm{M}(J)$. Furthermore, $k(0^+) = \infty$ if and only if the measure β has no discrete part. Finally, observe that if $\gamma > 0$ then $L^{-1}u = b * u$ and similarly if $\gamma = 0$ then $L^{-1}u = \beta * u$.

7.2 Volterra equations in a Hilbert space

We define a function $u : J \to H$ to be a *strong solution* of equation (1.1) if $u \in L^2(J; H)$, $\gamma u + k * u \in W_0^{1,2}(J; H)$, $u(t) \in D(A)$ almost everywhere and u satisfies (1.1) almost everywhere.

The case A is a subdifferential

Proposition 2.1. *Let $\epsilon > 0$, $\gamma \geq 0$ and let k satisfy (1.2). If $A = \partial\varphi$ with $\varphi \in \mathcal{J}_0$ then for all $f \in L^2(J; H)$ there exists a unique strong solution u of equation (1.1). In the case $J = [0, T]$ and $\gamma > 0$ the existence and the unicity follows also if $\epsilon = 0$. Furthermore $\|\frac{d}{dt}(\gamma u + k * u)\| \leq \|f\|$.*

Proof. Set $\mathcal{H} = L^2(0, T; H)$ and define the m-accretive operator L in $L^2(J)$ by (1.3). Let $\mathcal{L}$ denote the m-accretive extension in $\mathcal{H}$ of L and $\partial\Phi$ the m-accretive extension in $\mathcal{H}$ of $\partial\varphi$. Since $L \in \mathrm{M}(J)$, inequality (3.1.1) holds with $\mathcal{A} = \partial\Phi$ which, by Lemma 2.2.3, implies that the operator $\mathcal{L} + \partial\Phi$ is m-accretive in $\mathcal{H}$. This proves the proposition for the case $\epsilon > 0$. In the case $J = [0, T]$ and $\epsilon = 0$ the existence of a strong solution follows from the equality

$$\mathrm{int} R(\mathcal{L} + \partial\Phi) = \mathrm{int}(R(\mathcal{L}) + R(\partial\Phi)), \tag{2.1}$$

which is, by Theorem 2.2.4, implied by inequality (3.1.1), and the surjectivity of $\mathcal{L}$. In order to obtain the unicity of the strong solution if $\gamma > 0$, let u, v be two solutions of the equation

$$\mathcal{L}u + \partial\Phi u \ni f, \quad f \in L^2(0, T; H).$$

Then subtracting the two corresponding equations and subsequently taking the innerproduct in H with the difference $w(s) := u(s) - v(s)$ we obtain, using that $\partial\varphi$ is accretive,

$$\int_0^t \frac{d}{ds}(\gamma w + k * w)(s)w(s)ds = 0, \quad t \in [0,T].$$

Hence,

$$\gamma w^2(t) + \int_0^t \frac{d}{ds}(k * w)(s)w(s)ds = 0, \tag{2.2}$$

and the unicity follows since the integral term in (2.2) is non-negative for all $t \in [0,T]$ so that $w^2(t) = 0$ for $t \in [0,T]$.

As another example, but in the same spirit, we take $J = [0,T]$ and consider the following initial value problem

$$\begin{cases} \frac{d}{dt}(u(t) + \int_0^t k(t-s)u(s)ds) + \partial\varphi(u(t)) \ni f(t), & t \in (0,T]; \\ u(0) = x, \end{cases} \tag{2.3}$$

Proposition 2.2. *Let $\varphi \in \mathcal{J}_0$. Assume that $k \in L^2(0,T)$ satisfies (1.2). Then for all $x \in D(\varphi)$ and $f \in L^2(0,T;H)$ there exists a unique strong solution of (2.3). Furthermore*

$$\begin{aligned} \|\frac{d}{dt}(u + k * u)\| \quad &\le \quad \|f\| + \{(1 + \|k\|_1)\varphi(x) + \\ &\quad + \|x\|\|k\|_2(\|f\| + \|x\|\|k\|_2 + (1 + \|k\|_1)^{\frac{1}{2}}\varphi^{\frac{1}{2}}(x))\}^{\frac{1}{2}}. \end{aligned}$$

Proof. Define the (nonlinear) operator $\mathcal{L}^x$ by

$$\begin{cases} D(\mathcal{L}^x) = \{u \in W^{1,2}(0,T;H) : u(0) = x\}; \\ \mathcal{L}^x u = \frac{d}{dt}(u + k * u), \text{ for } u \in D(\mathcal{L}^x). \end{cases}$$

Then one verifies that $\mathcal{L}^x$ is m-accretive with resolvent given by

$$J_\lambda^{\mathcal{L}^x} u = J_\lambda^{\mathcal{L}}(u - \lambda kx) + (1 - J_\lambda^L 1)x, \tag{2.4}$$

where L denotes the operator defined by (1.3) with $\gamma = 1$ and $\mathcal{L}$ its extension. We show that the following inequality holds

$$(((\mathcal{L}^x)u, (\partial\Phi)_\mu u)) \ge -(1 + \|k\|_1)\varphi_\mu(x) - \|x\|\|k\|_2\|(\partial\Phi)_\mu u\|_2. \tag{2.5}$$

This inequality implies (2.1) where $\mathcal{L}$ is replaced by $\mathcal{L}^x$ (see the proof of Lemma 2.2.3), which gives the existence part of the proposition. The unicity of the strong solution can be shown as done in Proposition 2.1. We show now inequality (2.5). For the following inequality we refer to (the proof of) Theorem 3.2.1

$$\begin{aligned}\varphi_\mu(J_\lambda^{\mathcal{L}^x}u) &= \varphi_\mu(J_\lambda^{\mathcal{L}}(u-\lambda kx)+(1-J_\lambda^L 1)x)\\ &\le J_\lambda^L\varphi_\mu(u-\lambda kx)+(1-J_\lambda^L 1)\varphi_\mu(x).\end{aligned}$$

Integrating this inequality over $(0,T)$ and using that $L\in \mathbf{M}(0,T)$ we get

$$\begin{aligned}\Phi_\mu(J_\lambda^{\mathcal{L}^x}u) &\le \int_0^T J_\lambda^L\varphi_\mu((u-\lambda kx)(t))dt+\lambda\int_0^T \frac{d}{dt}(J_\lambda^L 1+k*J_\lambda^L 1)(t)dt\ \varphi_\mu(x)\\ &\le \Phi_\mu(u-\lambda kx)+\lambda(1+\|k\|_1)\varphi_\mu(x)\\ &= \Phi_\mu(u)+\Phi_\mu(u-\lambda kx)-\Phi_\mu(u)+\lambda(1+\|k\|_1)\varphi_\mu(x).\end{aligned}$$

Hence,

$$((\mathcal{L}_\lambda^x u,(\partial\Phi)_\mu u))\ge -\frac{\Phi_\mu(u-\lambda kx)-\Phi_\mu(u)}{\lambda}-(1+\|k\|_1)\varphi_\mu(x).$$

If $u\in D(\mathcal{L}^x)$ we obtain, by passing to the limit with respect to λ, and using that Φ_μ is Fréchet differentiable, that

$$((\mathcal{L}^x u,(\partial\Phi)_\mu u))\ge ((kx,(\partial\Phi)_\mu u))-(1+\|k\|_1)\varphi_\mu(x), \tag{2.6}$$

and inequality (2.5) follows. The last assertion of the proposition follows from (2.5).

The case A is general

We write equation (1.1) as

$$\epsilon u+\mathcal{L}u+\mathcal{A}u\ni f, \tag{2.7}$$

where, as before $\mathcal{L}$ is the extension of the operator L defined by (1.3) and $\mathcal{A}$ the extension of A.

First we observe that, if $J=\mathbb{R}$, it is a direct consequence of Proposition 5.3.5 that $R(\epsilon I+\mathcal{L}+\mathcal{A})\supset W^{1,2}(\mathbb{R};H)$, $\epsilon>0$. Furthermore it follows that

$\overline{\mathcal{L}+\mathcal{A}}$ is m-accretive and that if $f \in W^{s,2}(\mathbb{R}; H)$, $0 < s < 1$, then the solution of

$$\epsilon u + \overline{\mathcal{L}+\mathcal{A}}\, u \ni f, \quad \epsilon > 0$$

belongs to $W^{s,2}(\mathbb{R}; H)$.

If $J = \mathbb{R}^+$ or $[0,T]$ one can apply Lemma 5.2.1 with $L_0 = D^*D$ where $D = \frac{d}{dt}$ with domain $W_0^{1,2}(J)$. Indeed

$$(Lu, L_0u) = (D(u + k * u), D^*Du) = (Du, D^*Du) + (D(k * Du), Du) \geq 0$$

for all $u \in D(L_0)$. Then it follows that equation (2.7) has a unique strong solution for all $f \in W_0^{1,2}(J)$.

Remark 2.3. The results obtained here are not optimal in the sense that one can prove the existence of strong solutions of (2.7) for f of bounded variation [GR2].

Next, we prove some results which, to our knowledge, are new. We show that much stronger regularity results can be obtained if $\gamma = 0$.

Proposition 2.4. *Let $J = \mathbb{R}$ and let L be defined by (1.3) with $\gamma = 0$ and k satisfying (1,2). Assume in addition that k is convex. Then the inclusion $R(\epsilon I + \mathcal{L} + \mathcal{A}) \supset D(\mathcal{L})$ holds for all $\epsilon > 0$*

Proof. We show that Proposition 5.3.1 is applicable. Observe that the sublattice $W^{1,2}(\mathbb{R})$ is a core of L and hence i) of Proposition 5.3.1 is satisfied for $D = W^{1,2}(\mathbb{R})$. In order to prove that ii) holds we approximate the function k by k_λ, where

$$k_\lambda(x) := k(x + \lambda), \quad \lambda, x > 0.$$

Set

$$L_{(\lambda)}u := \frac{d}{dt}(k_\lambda * u) = k_\lambda(0)(u - h_\lambda * u),$$

where

$$h_\lambda := -\frac{1}{k_\lambda(0)}k'_\lambda.$$

Then for $u = u^+ - u^- \in W^{1,2}(\mathbb{R})$ we have

$$\begin{aligned}(L_{(\lambda)}u^+, L_{(\lambda)}u^-) &= k_\lambda^2(0)(u^+ - h_\lambda * u^+, u^- - h_\lambda * u^-) \\ &= k_\lambda^2(0)((\check{h} * h_\lambda * - \check{h}_\lambda * - h_\lambda *)u^+, u^-). \end{aligned} \tag{2.8}$$

Here $\check{h}_\lambda(x) := h_\lambda(-x)$. Since $h_\lambda \in L^1(I\!R^+)$ satisfies (1.2) and $\|h_\lambda\|_1 = 1$ one can verify that

$$\check{h}_\lambda * h_\lambda \leq h_\lambda + \check{h}_\lambda.$$

Therefore (2.8) implies that

$$(L_{(\lambda)}u^+, L_{(\lambda)}u^-) \leq 0.$$

Furthermore,

$$(L_{(\lambda)}u^+, L_{(\lambda)}u^-) = (k_\lambda * \frac{du^+}{dt}, k_\lambda * \frac{du^-}{dt}) \to (k * \frac{du^+}{dt}, k * \frac{du^-}{dt}) = (Lu^+, Lu^-)$$

if $\lambda \downarrow 0$. Thus $(Lu^+, Lu^-) \leq 0$ for all $u = u^+ - u^- \in W^{1,2}(I\!R)$. Since i) and ii) of Proposition 5.3.1 implies the positivity of the translation invariant C_0-semigroup generated by $-L^*L$ it follows that $L^*L \in \mathrm{M}(I\!R)$ (see Remark 5.3.8 (iii)), which is equivalent to the assumptions i), ii) and iii) of Proposition 5.3.1.

The symbol of the operator L defined by (1.3) with $\gamma = 0$ is given by $ix\hat{k}(ix)$ where $\hat{k}$ denotes the Laplace transform of k. Then by the definition of $W^{s,2}(I\!R)$ it is clear that $W^{s,2}(I\!R) \subset D(L)$ whenever

$$\limsup_{x\to\infty} \frac{|\hat{k}(ix)|}{|x|^{s-1}} < \infty. \tag{2.9}$$

By Proposition 5.3.5 we have

Proposition 2.5. *Let $J = I\!R$, $0 < s < 1$, and let L be defined by (1.3) with $\gamma = 0$ and k satisfying (1.2). Assume in addition that k satisfies (2.9). Then $R(\epsilon I + \mathcal{L} + \mathcal{A}) \supset W^{s,2}(I\!R; H)$, $0 < s \leq 1$, $\epsilon > 0$. Furthermore the solution u of equation (2.7) with $f \in W^{s,2}(I\!R; H)$ belongs to $W^{s,2}(I\!R; H)$*

Observe that if there exists a non-negative constant a such that

$$|Im\hat{k}(ix)| \leq aRe\hat{k}(ix), \tag{2.10}$$

then the operator L with $\gamma = 0$ satisfies assertion (ii) of Theorem 4.2.2 (and therefore all the assertions of Theorem 4.2.2). We conclude from Corollary 4.3.3 the following

Proposition 2.6. *Let $J = I\!R$ and let L be defined by (1.3) with $\gamma = 0$ and k satisfying (1.2). Assume in addition that k satisfies (2.10). Then the operator $\mathcal{L} + \mathcal{A} \subset \mathcal{H} \times \mathcal{H}$ is m-accretive.*

Remark 2.7. Let L be as in Proposition 2.6. By Remark 3.3.5 and the observation following example 3.3.4 it follows that condition (2.10) is necessary for $\mathcal{L}+\mathcal{A}$ to be m-accretive for all m-accretive $A \subset H \times H$.

Remark 2.8. In [PR-S] it has been proven that if $k \in C^\infty(0,\infty)$ is completely monotonic such that $\lim_{t\downarrow 0} -\frac{tk'(t)}{k(t)} > 0$ then (2.10) holds for some positive constant a (see also [PR]). Recall that k is completely monotonic if $(-1)^j k^{(j)} \geq 0, j = 0,1,2,\ldots$.

An example of a kernel k satisfying the assumptions of Proposition 2.6 is the kernel $k(t) = t^{\alpha-1}e^{-\epsilon t}$, $0 < \alpha < 1, \epsilon > 0$.

Remark 2.9. Note that Proposition 2.6 is a special case of example 4.3.4. It should also be observed that this proposition can also be applied if $J = I\!R^+$ or $[0,T]$ in the case $k \in L^2(0,T)$ is a restriction of a kernel $k \in L^2(I\!R^+)$ satisfying (2.10). In the case $J = I\!R^+$ or $[0,T]$, the operator L defined by (1.3) is not a normal operator.

7.3 Volterra equations in a Banach space

Let $(X, \|\cdot\|)$ be a real Banach space and let $A \subset X \times X$ be an m-accretive operator satisfying $0 \in A0$. We denote by $\mathcal{A}$ its m-accretive extension in $L^1(0,T;X)$, $T \in I\!R^+$. Define the operator L by setting

$$\begin{cases} D(L) = \{u \in L^1(0,T) : \gamma u + k * u \in W_0^{1,1}(0,T)\}; \\ Lu = \frac{d}{dt}(\gamma u + k * u), \end{cases}$$

where $\gamma \geq 0$ and k satisfying (1.2). If $\gamma = 0$ then we assume that $k(0^+) = \infty$. Note that if $\gamma = 0$ and $k(0^+) < \infty$ we are dealing with a bounded operator. The operator L is m-accretive in $L^1(0,T)$ with order-preserving resolvent, see Section 7.1. Let the operator $\mathcal{L}$ be the m-accretive extension in $L^1(0,T;X)$ of L (see Remark 2.4.2).

It will be shown that for all $f \in L^1(0,T;X)$ there exists a so-called generalized solution $u \in L^1(0,T;X)$ of the equation

$$\mathcal{L}u + \mathcal{A}u \ni f, \tag{3.1}$$

which is obtained as the L^1-limit of solutions of the approximate equation

$$\mathcal{L}_\lambda u_\lambda + \mathcal{A} u_\lambda \ni f, \quad \lambda > 0. \tag{3.2}$$

By a contraction argument one shows that equation (3.2) has a unique solution $u_\lambda \in D(\mathcal{A})$ for all $f \in L^1(0,T;X)$. We present here a somewhat simplified proof of the following theorem due to G. Gripenberg [GR3].

Theorem 3.1. *The net $\{u_\lambda\}_{\lambda>0}$ is convergent in $L^1(0,T;X)$. Furthermore, if $\gamma > 0$ then the limit function u belongs to $C([0,T];X)$ and if in addition $f \in L^\infty(0,T;X)$ then $u_\lambda \to u$ in $L^\infty(0,T;X)$ as $\lambda \downarrow 0$. Finally, if v is a strong solution of (3.1), then $u = v$.*

We use some facts and notations already introduced in the introduction. Such as, there exist non-negative bounded measures ρ_λ, β such that

$$J_\lambda^L f(t) = \int_0^t f(t-s)d\rho_\lambda(s), \quad L^{-1}f(t) = \int_0^t f(t-s)d\beta(s), \quad f \in L^1(0,T)$$

and $\int_0^T d\rho_\lambda(s) \le 1$. If $\gamma > 0$ then $d\rho_\lambda(s) = r_\lambda(s)ds$ and $d\beta(s) = b(s)ds$, where $0 \le r_\lambda \in L^1(0,T)$ satisfies $\|r_\lambda\|_1 \le 1$, $\lambda > 0$, and $b \in L^\infty(0,T)$ with $0 \le b \le \frac{1}{\gamma}$.

For the proof of Theorem (3.1) we need the following two lemmas

Lemma 3.2. *If $f \in BV(0,T;X)$ then $u_\lambda \in BV(0,T;X)$ and*

$$\begin{aligned}\mathrm{Var}(u_\lambda,[t_1,t_2]) \le\ & \lambda \mathrm{Var}(f,[t_1,t_2]) + \int_0^{t_1} \mathrm{Var}(f,[t_1-s,t_2-s])d\beta(s) \\ & + \|f(0^+)\| \int_{t_1}^{t_2} d\beta(s)\end{aligned} \tag{3.3}$$

for $0 < t_1 < t_2 < T$. Furthermore

$$\|u_\lambda(t)\| \le \lambda \|f(t)\| + L^{-1}\|f(\cdot)\|(t), \ t \in (0,T), \tag{3.4}$$

and

$$\|u_\lambda(0^+)\| \le \lambda \|f(0^+)\|. \tag{3.5}$$

If $\gamma > 0$ then

$$\|u_\lambda(t+h) - u_\lambda(t)\| \le \lambda \mathrm{Var}(f,[t,t+h]) + \frac{h}{\gamma}(\|f(0^+)\| + \mathrm{Var}(f,[0,t+h])), \tag{3.6}$$

for $0 < t < t+h < T$.

In particular it follows that

$$\|u_\lambda(0^+)\| + \mathrm{Var}(u_\lambda, [0,T]) \le c(\|f(0^+)\| + \mathrm{Var}(f, [0,T]))$$

for some constant $c > 0$. This inequality will be frequently used.

Lemma 3.3. *Let* $b \in L^1(0,T)$, $v \in BV(0,T;X)$. *Then the function*

$$(b * v)(t) = \int_0^t b(t-s)v(s)ds$$

is absolutely continuous and differentiable almost everywhere on $(0,T)$. *Moreover*

$$\int_0^T \|\frac{d}{dt}(b*v)(t)\|dt \le \|b\|_1(\|v(0^+)\| + \mathrm{Var}(v, [0,T])).$$

Assuming these lemmas for the moment we proceed with

Proof of Theorem 3.1. First we consider the case where $\gamma = 0$. Note that equation (3.2) is equivalent to

$$u_\lambda = J_\lambda^{\mathcal{A}}(\lambda f + J_\lambda^{\mathcal{L}} u_\lambda). \tag{3.7}$$

Rewriting equation (3.2) with λ replaced by μ as

$$\mathcal{L}_\lambda u_\mu + \mathcal{A}u_\mu \ni f + \mathcal{L}_\lambda u_\mu - \mathcal{L}_\mu u_\mu,$$

we obtain in the same way

$$u_\mu = J_\lambda^{\mathcal{A}}(\lambda f + \lambda(\mathcal{L}_\lambda u_\mu - \mathcal{L}_\mu u_\mu) + J_\lambda^{\mathcal{L}} u_\mu). \tag{3.8}$$

Due tot the m-accretivity of the operator A, the equations (3.7) and (3.8) imply

$$\|u_\lambda(t) - u_\mu(t)\| \le \lambda\|(\mathcal{L}_\lambda - \mathcal{L}_\mu)u_\mu\|(t) + J_\lambda^L\|u_\lambda - u_\mu\|(t). \tag{3.9}$$

Since $I - J_\lambda^L = \lambda L J_\lambda^L$ it follows from (3.9) that

$$J_\lambda^L\|u_\lambda - u_\mu\|(t) \le L^{-1}\|(\mathcal{L}_\lambda - \mathcal{L}_\mu)u_\mu\|(t).$$

So that, again using (3.9), we have

$$\|u_\lambda(t) - u_\mu(t)\| \le \lambda\|(\mathcal{L}_\lambda - \mathcal{L}_\mu)u_\mu\|(t) + L^{-1}\|(\mathcal{L}_\lambda - \mathcal{L}_\mu)u_\mu\|(t). \tag{3.10}$$

Note that $\mathcal{L}_\lambda u = \mathcal{L}J_\lambda^{\mathcal{L}}u = \frac{d}{dt}(k * J_\lambda^{\mathcal{L}}u) = \frac{d}{dt}(J_\lambda^L k * u)$. Hence, by Lemma 3.2 and Lemma 3.3

$$\begin{aligned}
\int_0^T \|\mathcal{L}_\lambda - \mathcal{L}_\mu)u_\mu(t)\|dt &= \int_0^T \|\frac{d}{dt}((J_\lambda^L k - J_\mu^L k) * u_\mu(t))\|dt \\
&\le \|J_\lambda^L k - J_\mu^L k)\|_1\{\|u_\mu(0^+)\| + \mathrm{Var}(u_\mu, [0,T])\} \\
&\le c\|J_\lambda^L k - J_\mu^L k)\|_1\{\|f(0^+)\| + \mathrm{Var}(f, [0,T])\}
\end{aligned}$$

for some constant $c > 0$. Therefore $\|(\mathcal{L}_\lambda - \mathcal{L}_\mu)u_\mu\| \to 0$ in $L^1(0,T)$ if $\lambda,\ \mu \downarrow 0$. It follows now from (3.10) that $\{u_\lambda\}_{\lambda>0}$ is a Cauchy-net in $L^1(0,T;X)$.

We continue with the case $\gamma > 0$. Without loss of generality we may assume that $\gamma = 1$. Denote by D the m-accretive operator $D = \frac{d}{dt}$ with domain $W_0^{1,1}(0,T)$. The extension in $L^1(0,T;X)$ of D will be denoted by $\mathcal{D}$. Now we rewrite equation (3.2) as

$$\mathcal{D}_\lambda u_\lambda + \mathcal{A}u_\lambda \ni f + \mathcal{H}_\lambda u_\lambda - \mathcal{D}(k * u_\lambda),$$

where

$$\mathcal{H}_\lambda u_\lambda = \mathcal{D}_\lambda u_\lambda - \mathcal{L}_\lambda u_\lambda + \mathcal{D}(k * u_\lambda).$$

From [CR-E, Lemma 1.7] it follows that

$$\|u_\lambda(t) - u_\mu(s)\| \le \frac{\mu}{\lambda+\mu}\|J_\lambda^{\mathcal{D}} u_\lambda(t) - u_\mu(s)\| + \frac{\lambda}{\lambda+\mu}\|u_\lambda(t) - J_\mu^{\mathcal{D}} u_\mu(s)\|$$
$$+\frac{\lambda\mu}{\lambda+\mu}[u_\lambda(t) - u_\mu(s), \mathcal{H}_\lambda u_\lambda(t) - \mathcal{H}_\mu u_\mu(s) - \mathcal{D}(k * u_\lambda)(t) + \mathcal{D}(k * u_\mu)(s)]_+$$

where $[x,y]_+ = \inf_{\lambda>0} \frac{1}{\lambda}(\|x + \lambda y\| - \|x\|)$. Therefore

$$\begin{aligned}
\|u_\lambda(t) - u_\mu(s)\| \le{}& \frac{\mu}{\lambda+\mu}\|u_\mu(s)(1 - J_\lambda^D 1)(t)\| + \frac{\lambda}{\lambda+\mu}\|u_\lambda(t)(1 - J_\mu^D 1)(s)\| \\
&+\frac{\mu}{\lambda+\mu}\|J_\lambda^{\mathcal{D}}(u_\lambda(\cdot) - u_\mu(s))(t)\| + \frac{\lambda}{\lambda+\mu}\|J_\lambda^{\mathcal{D}}(u_\lambda(t) - u_\mu(\cdot))(s)\| \\
&+\frac{\lambda\mu}{\lambda+\mu}r(\lambda,\mu,t,s),
\end{aligned}$$

where

$$r(\lambda,\mu,t,s)=[u_\lambda(t)-u_\mu(s),\mathcal{H}_\lambda u_\lambda(t)-\mathcal{H}_\mu u_\mu(s)-\mathcal{D}(k*u_\lambda)(t)+\mathcal{D}(k*u_\mu)(s)]_+$$

Then we rewrite this inequality as

$$D_\lambda\|u_\lambda(\cdot)-u_\mu(s)\|(t)+D_\mu\|u_\lambda(t)-u_\mu(\cdot)\|(s)\le q(\lambda,\mu,t,s). \tag{3.11}$$

Here

$$q(\lambda,\mu,t,s)=r(\lambda,\mu,t,s)+\|u_\lambda(t)\|D_\mu 1(s)+\|u_\mu(s)\|D_\lambda 1(t).$$

We introduce the following m-accretive operators in $L^1[(0,T)\times(0,T)]$. Set $Y=L^1[0,T]$ and recall that $L^1(0,T;Y)\cong L^1((0,T)\times(0,T))$. Let the operator D_1 denote the extension of D in $L^1(0,T;Y)$. Considering D as an operator in Y we denote by D_2 its extension in $L^1(0,T;Y)$ with domain $L^1(0,T;W_0^{1,1}(0,T))$ Note that the operator $D_{1,\lambda}+D_{2,\mu}$ is surjective with an order-preserving inverse and that the closure $\overline{D_1+D_2}$ exists since D_1 and D_2 commute in the sense of resolvents. Furthermore,

$$\overline{D_1+D_2}^{-1}f(t,s)=\int_0^{\min(t,s)} f(t-\tau,s-\tau)d\tau$$

for $f\in L^1[(0,T)\times(0,T)]$. Since $\overline{D_1+D_2}$ and $D_{1,\lambda}+D_{2,\mu}$ are resolvent commuting we obtain from (3.11) that

$$\int_0^{\min(t,s)}\|u_\lambda(t-\tau)-u_\mu(s-\tau)\|d\tau\le(D_{1,\lambda}+D_{2,\mu})^{-1}\int_0^{\min(t,s)} q(\lambda,\mu,t-\tau,s-\tau)d\tau \tag{3.12}$$

We show now that there exists a continuous function ψ on $(-T,T)$ satisfying $\psi(0)=0$ such that

$$\limsup_{\lambda,\mu\downarrow 0}\int_0^{\min(t,s)} q(\lambda,\mu,t-\tau,s-\tau)d\tau\le\psi(t-s)\ \text{ uniformly on } (-T,T). \tag{3.13}$$

For that we need to estimate the following three integrals

$$I_1(\lambda,\mu,t,s)\ :=\ \int_0^{\min(t,s)} r(\lambda,\mu,t-\tau,s-\tau)d\tau; \tag{3.14}$$

$$I_2(\lambda,\mu,t,s) \;:=\; \int_0^{\min(t,s)} \|u_\lambda(t-\tau)\| D_\mu 1(s-\tau)d\tau; \tag{3.15}$$

$$I_3(\lambda,\mu,t,s) \;:=\; \int_0^{\min(t,s)} \|u_\mu(s-\tau)\| D_\lambda 1(t-\tau)d\tau. \tag{3.16}$$

First we estimate I_2. For $t \geq s$,

$$\begin{aligned} I_2(\lambda,\mu,t,s) &= \int_0^{s} \|u_\lambda(t-\tau)\| \frac{1}{\mu} e^{-\frac{1}{\mu}(s-\tau)} d\tau \\ &= \int_0^{\frac{s}{\mu}} \|u_\lambda(t-s+\mu\sigma)\| e^{-\sigma} d\sigma. \end{aligned}$$

Using (3.4) it follows that

$$\limsup_{\lambda,\mu\downarrow 0} I_2(\lambda,\mu,t,s) \leq \|f\|_\infty L^{-1}1(t-s) \;\text{ uniformly on } (-T,T).$$

Similarly one treats I_3. We continue with I_1.

$$\begin{aligned} |r(\lambda,\mu,t,s)| \;\leq\; & \|f(t)-f(s)\| + \|\mathcal{H}_\lambda u_\lambda(t)\| + \|\mathcal{H}_\mu u_\mu(s)\| + \\ & +[u_\lambda(t)-u_\mu(s), -\mathcal{D}(k*u_\lambda)(t)+\mathcal{D}(k*u_\mu)(s)]_+ \end{aligned}$$

The corresponding four integrals of the right-hand side of this inequality we denote by I_1^1, I_1^2, I_1^3 and I_1^4 respectively, so that $I_1 \leq I_1^1 + I_1^2 + I_1^3 + I_1^4$. First of all

$$I_1^1(\lambda,\mu,t,s) \leq |t-s| \mathrm{Var}(f,[0,T]).$$

Next, we estimate I_1^2.

$$\begin{aligned} I_1^2(\lambda,\mu,t,s) &\leq \int_0^T \|\mathcal{H}_\lambda u_\lambda(\tau)\| d\tau \\ &= \int_0^T \|\frac{d}{dt}(J_\lambda^{\mathcal{D}} - J_\lambda^{\mathcal{L}} - k*J_\lambda^{\mathcal{L}} + k*)u_\lambda(\tau)\| d\tau. \end{aligned}$$

Since

$$(I+\lambda D)J_\lambda^L + \lambda D(k*J_\lambda^L) = I$$

we have

$$J_\lambda^L - J_\lambda^D = (I - J_\lambda^D)(k * J_\lambda^L).$$

Thus we obtain by Lemma 3.3 that

$$I_1^2(\lambda,\mu,t,s) \le c(\|(I - J_\lambda^D)J_\lambda^L k\|_1 + \|(I - J_\lambda^L)k\|_1)(\|f(0^+)\| + \mathrm{Var}(f,[0,T]))$$

for some constant $c > 0$. The estimation of I_1^3 can be done the same way. We proceed with the estimation of I_1^4. Recall that $|[x,y]_+| \le \|y\|$, $[x,y+z] \le [x,y]+[x,z]$, $|[x,y]-[x,z]| \le \|y-z\|$ and $[x,cx+y] = c\|x\|+[x,y]$, $c \in I\!R$, see [CR]. Set $k_m(t) := \min\{m,k(t)\}$, $m \in I\!R^+, t \in I\!R^+$ almost everywhere and denote by $I_{1,m}^4$ the integral I_1^4 where k is replaced by k_m. Note that $k_m \to k$ in $L^1(0,T)$ as $m \to \infty$. By Lemma 3.3 it follows that $\lim\limits_{m\to\infty} I_{1,m}^4 = I_1^4$. Again let $t \ge s$. Then

$$I_{1,m}^4(\lambda,\mu,t,s) =$$

$$\begin{aligned}
&= \int_0^s [u_\lambda(t-\tau) - u_\mu(s-\tau), -\frac{d}{dt}k_m * u_\lambda(t-\tau) + \frac{d}{dt}k_m * u_\mu(s-\tau)]_+ d\tau \\
&= \int_0^s [u_\lambda(t-\tau) - u_\mu(s-\tau), -k_m(0^+)(u_\lambda(t-\tau) - u_\mu(s-\tau)) - \\
&\quad (\int_0^{t-\tau} u_\lambda(t-\tau-\xi)dk_m(\xi) - \int_0^{s-\tau} u_\mu(s-\tau-\xi)dk_m(\xi))]_+ d\tau \\
&\le \int_0^s \{-k_m(0^+)\|u_\lambda(t-\tau) - u_\mu(s-\tau)\| - \\
&\quad (\int_0^{s-\tau} \|u_\lambda(t-\tau-\xi) - u_\mu(s-\tau-\xi)\|dk_m(\xi) - \int_{s-\tau}^{t-\tau} \|u_\lambda(t-\tau-\xi)\|dk_m(\xi))\}d\tau \\
&= \int_0^s \frac{d}{dt}\int_0^{s-\tau} k_m(\xi)\|u_\lambda(t-\tau) - u_\mu(s-\tau)\|d\xi d\tau - \int_0^s \int_{s-\tau}^{t-\tau} \|u_\lambda(t-\tau-\xi)\|dk_m(\xi)d\tau \\
&= \int_0^s k_m(\xi)\|u_\lambda(t-\xi) - u_\mu(s-\xi)\|d\xi - \int_0^s \int_{s-\tau}^{t-\tau} \|u_\lambda(t-\tau-\xi)\|dk_m(\xi)d\tau \\
&\le -\int_0^s \int_{s-\tau}^{t-\tau} \|u_\lambda(t-\tau-\xi)\|dk_m(\xi)d\tau
\end{aligned}$$

$$\leq \ c\|f\|_\infty \int_0^s (k_m(s-\tau) - k_m(t-\tau))d\tau$$
$$\leq \ c\|f\|_\infty \int_s^t k_m(\tau)d\tau$$

for some constant $c > 0$. Hence $I_1^4(\lambda,\mu,t,s) \leq c\|f\|_\infty \int_s^t k(\tau)d\tau$, for some constant $c > 0$. Combining the estimates we have obtained for I_1, I_2, I_3 we get (3.13). It follows from (3.12) and (3.13) that for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_0^T \|u_\lambda(\tau)-u_\mu(\tau)\|d\tau \leq (D_{1,\lambda}+D_{2,\mu})^{-1}\psi(t-s)\mid_{(T,T)} +\epsilon(D_{1,\lambda}+D_{2,\mu})^{-1}1\mid_{(T,T)}$$

for all $0 < \lambda,\ \mu < \delta$. This implies

$$\begin{aligned}\limsup_{\lambda,\mu\downarrow 0} \int_0^T \|u_\lambda(\tau) - u_\mu(\tau)\|d\tau \ &\leq \ \overline{D_1 + D_2}^{-1}\psi(t-s)\mid_{(T,T)} \\ &\quad +\epsilon\overline{D_1 + D_2}^{-1}1\mid_{(T,T)} \\ &= \ 0 + \epsilon T = \epsilon T.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary it follows that the net $\{u_\lambda\}_{\lambda>0}$ is convergent in $L^1(0,T;X)$.

So far we assumed that $f \in BV(0,T;X)$. The convergence of $\{u_\lambda\}_{\lambda>0}$ for all $f \in L^1(0,T;X)$ follows from the estimate

$$\|u_\lambda - v_\lambda\| \leq \lambda\|f-g\| + L^{-1}\|f-g\| \tag{3.17}$$

where u_λ, v_λ denotes the solution of (3.2) with right-hand side f respectively g. Inequality (3.17) can be obtained in a similar way as inequality (3.4). Next we prove the last three statements of the theorem. If $\gamma > 0$ it follows from (3.6) that the family $\{u_\lambda\}_{\lambda>0}$ is equicontinuous if f is Lipschitz continuous on $[0,T]$. Since, by (3.4), $\|u_\lambda\|_\infty \leq c\|f\|_\infty$ for some constant $c > 0$ we may conclude that the limit function u belongs to $C([0,T];X)$ and that $u_\lambda \to u$ in $L^\infty(0,T;X)$ if $\lambda \downarrow 0$. Using (3.17) one shows that $u \in C([0,T];X)$ for all $f \in L^1(0,T;X)$ and that if $f \in L^\infty(0,T;X)$ then the convergence of u_λ to u is uniform.

Let v be a strong solution of (3.1). Let u_λ, $\lambda > 0$, be the solution of (3.2) and u its L^1-limit. Writing $\mathcal{L}_\lambda v + \mathcal{A}v = f + \mathcal{L}_\lambda v - \mathcal{L}v$ and using (3.17) we obtain

$$\|u_\lambda - v\| \le \lambda\|\mathcal{L}_\lambda v - \mathcal{L}v\| + L^{-1}\|\mathcal{L}_\lambda v - \mathcal{L}v\|.$$

Since $\mathcal{L}_\lambda v \to \mathcal{L}v$ if $\lambda \downarrow 0$ we get that $u = v$.

It remains to prove Lemma (3.2) and Lemma (3.3). For the proof of Lemma (3.3) we refer to [GR3]. We show now

Proof of Lemma (3.2). From (3.7), the contractivity of J_λ^A and the positivity of J_λ^L it follows that

$$\|u_\lambda(t+h) - u_\lambda(t)\| \le \lambda\|f(t+h) - f(t)\| +$$

$$+ J_\lambda^L\|u_\lambda(\cdot+h) - u_\lambda(\cdot)\|(t) + J_\lambda^L\|u_\lambda(\cdot)\|(t+h) - J_\lambda^L\|u_\lambda(\cdot+h)\|(t). \tag{3.18}$$

This inequality can also be written as

$$\lambda L J_\lambda^L\|u_\lambda(\cdot+h) - u_\lambda(\cdot)\|(t) \le$$

$$\le \lambda\|f(t+h) - f(t)\| + J_\lambda^L\|u_\lambda(\cdot)\|(t+h) - J_\lambda^L\|u_\lambda(\cdot+h)\|(t). \tag{3.19}$$

Using that L^{-1} is positive we get

$$J_\lambda^L\|u_\lambda(\cdot+h) - u_\lambda(\cdot)\|(t) \le L^{-1}\|f(\cdot+h) - f(\cdot)\|(t) +$$

$$+ \frac{1}{\lambda}L^{-1}\{J_\lambda^L\|u_\lambda(\cdot)\|(t+h) - J_\lambda^L\|u_\lambda(\cdot+h)\|(t)\}. \tag{3.20}$$

Inequality (3.20) combined with (3.18) gives

$$\begin{aligned}
\|u_\lambda(t+h) - u_\lambda(t)\| &\le \lambda\|f(t+h) - f(t)\| + L^{-1}\|f(\cdot+h) - f(\cdot)\|(t) \\
&\quad + (I + \frac{1}{\lambda}L^{-1})\{J_\lambda^L\|u_\lambda(\cdot)\|(t+h) - J_\lambda^L\|u_\lambda(\cdot+h)\|(t)\} \\
&= \lambda\|f(t+h) - f(t)\| + L^{-1}\|f(\cdot+h) - f(\cdot)\|(t) \\
&\quad + \frac{1}{\lambda}L^{-1}\|u_\lambda(\cdot)\|(t+h) - \frac{1}{\lambda}L^{-1}\|u_\lambda(\cdot+h)\|(t) \\
&= \lambda\|f(t+h) - f(t)\| + L^{-1}\|f(\cdot+h) - f(\cdot)\|(t) \\
&\quad + \frac{1}{\lambda}\int_t^{t+h}\|u_\lambda(t+h-s)\|d\beta(s).
\end{aligned} \tag{3.21}$$

In the same way as we have derived (3.21) one shows that (3.4) holds. Then for all $\epsilon > 0$ there exists a $\delta > 0$ such that $0 \le t < \delta$ implies

$$\|u_\lambda(t)\| \le \lambda \|f(0^+)\| + \epsilon.$$

Here we used that β has no discrete part (see Section 7.1). In particular (3.5) follows. Choose, for a given $\epsilon > 0$, a number $N \in \mathbb{N}^+$ large enough such that $\frac{h}{N} < \delta$. Then it follows from (3.21) that

$$\begin{aligned}
\|u_\lambda(t+h) - u_\lambda(t)\| &\le \sum_{i=1}^{N} \|u_\lambda(t + \frac{i}{N}h) - u_\lambda(t + \frac{i-1}{N}h)\| \\
&\le \lambda \mathrm{Var}(f, [t, t+h]) + \\
&\quad + \int_0^{t+h} \mathrm{Var}(f, [\min(0, t-s), t+h-s]) d\beta(s) \\
&\quad + (\|f(0^+)\| + \frac{\epsilon}{\lambda}) \int_t^{t+h} d\beta(s).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, one verifies that the inequalities (3.3) and (3.6) follow.

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Samenvatting

In dit proefschrift beschouwen we de som van een lineaire ($\mathcal{L}$) en een niet-lineaire ($\mathcal{A}$) m-accretieve operator in een Hilbertruimte.

We bewijzen dat, onder bepaalde voorwaarden, de som $\mathcal{L} + \mathcal{A}$ ook m-accretief is. In hoofdstuk 1 worden deze voorwaarden toegelicht aan de hand van enkele voorbeelden.

In hoofdstuk 3 geven we nodige en voldoende voorwaarden zodat de operatoren $\mathcal{L}$ en $\mathcal{A}$ een "scherpe hoek" (acute angle) vormen. Dit resultaat is een vectorwaardige uitbreiding van een ongelijkheid van Brezis and Strauss [BR-S]. Voor m-accretieve operatoren $\mathcal{L}$ en $\mathcal{A}$ die een "scherpe hoek" vormen geldt dat de som $\mathcal{L} + \mathcal{A}$ m-accretief is. In hoofdstuk 4 wordt de m-accretiviteit van de operator $\mathcal{L} + \mathcal{A}$ bewezen onder algemenere voorwaarden. In veel gevallen is niet de operator $\mathcal{L} + \mathcal{A}$ maar de afsluiting, $\overline{\mathcal{L} + \mathcal{A}}$, m-accretief. Deze situatie wordt bestudeerd in hoofdstuk 5. Tot zover het eerste deel van het proefschrift.

Het tweede deel, bestaande uit hoofdstuk 6 en 7, behandelt enkele toepassingen van de in het eerste deel ontwikkelde theorie. In hoofdstuk 6 bewijzen we een existentie resultaat voor een semi-lineair elliptisch systeem en in hoofdstuk 7, beschouwen we niet-lineaire Volterra vergelijkingen.

[BR-S] H. Brezis and W.A. Strauss, *Semilinear second order elliptic equations in L^1*, J. Math. Soc. Japan, 25 (1973) 565-590.

Curriculum Vitae

De schrijver van dit proefschrift werd geboren op 18 maart 1964 te Alphen aan den Rijn. Na het behalen van het Atheneum diploma aan het Ashram College te Alphen aan den Rijn, begon hij in 1983 met de studie wiskunde aan de Rijksuniversiteit te Leiden, waar hij in 1984 het propedeutisch examen aflegde. In december 1987 slaagde hij voor het doctoraal examen wiskunde met bijvak informatica. Van september 1986 tot december 1987 was hij werkzaam als student-assistent bij de Subfaculteit der Wiskunde en Informatica.

Als onderzoeker in opleiding trad hij in februari 1988 in dienst van de Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO). Het onderzoek werd uitgevoerd aan de Faculteit der Technische Wiskunde en Informatica van de Technische Universiteit Delft onder begeleiding van Prof. dr. Ph. Clément. Gedurende het onderzoek verbleef de schrijver drie maanden op de Universiteit te Besançon (Frankrijk) bij Prof. dr. Ph. Bénilan.

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