THEORY OF VENTILATING OR CAVITATING FLOWS ABOUT
SYMMETRIC SURFACE-PIERCING STRUTS

by

B. Yim

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

SHIP PERFORMANCE DEPARTMENT
RESEARCH AND DEVELOPMENT REPORT

September 1975
MAJOR DTNSRDC ORGANIZATIONAL COMPONENTS

DTNSRDC
COMMANDER 00
TECHNICAL DIRECTOR 01

OFFICER IN CHARGE
CARDEROCK 05

SYSTEMS
DEVELOPMENT
DEPARTMENT 11

SHIP PERFORMANCE
DEPARTMENT 15

STRUCTURES
DEPARTMENT 17

SHIP ACOUSTICS
DEPARTMENT 19

MATERIALS
DEPARTMENT 28

OFFICER IN CHARGE
ANNAPOLIS 04

AVIATION AND
SURFACE EFFECTS
DEPARTMENT 16

COMPUTATION
AND MATHEMATICS
DEPARTMENT 18

PROPUSSION AND
AUXILIARY SYSTEMS
DEPARTMENT 27

CENTRAL
INSTRUMENTATION
DEPARTMENT 29

(This document contains information affecting the national defense of the United States within the meaning of the Espionage Laws, Title 18, U. S. C., Sections 793 and 794. The transmission or the revelation of its contents in any manner to an unauthorized person is prohibited by law.)
THEORY OF VENTILATING OR CAVITATING FLOWS ABOUT SYMMETRIC SURFACE-PIERCING STRUTS

B. Yim

Two stable states of high-speed strut flow are considered in three dimensions under linear boundary conditions. The strut is thin, symmetric, without angle of attack, and has base-ventilation or base cavitation without side cavitation. The strut and cavity are represented by polynomial source distributions with unknown coefficients on the center plane. Coefficients of the polynomial for the cavity are obtained as solutions for integral equations of cavity source.

(Continued on reverse side)
(Block 20 continued)

using a least-squares method. All the singular double integrals in the present problem are integrated in closed forms by using recurrence formulas. Thus, common errors due to numerical integration of the singular integrals are avoided. Approximate cavity shape and cavity drag are obtained. Downwash is calculated at the hydrofoil plane, which is assumed to be located at the tip of and perpendicular to the strut. Downwash and drag are compared with the existing experimental results, and good agreement is found.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>1</td>
</tr>
<tr>
<td>ADMINISTRATIVE INFORMATION</td>
<td>1</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>BOUNDARY VALUE PROBLEM</td>
<td>2</td>
</tr>
<tr>
<td>SOLUTION</td>
<td>6</td>
</tr>
<tr>
<td>DRAG AND DOWNWASH</td>
<td>9</td>
</tr>
<tr>
<td>SCHEME OF ITERATION</td>
<td>9</td>
</tr>
<tr>
<td>NUMERICAL RESULTS AND DISCUSSION</td>
<td>11</td>
</tr>
<tr>
<td>CONCLUSIONS</td>
<td>15</td>
</tr>
<tr>
<td>APPENDIX A — INTEGRATIONS OF SINGULAR INTEGRALS</td>
<td>25</td>
</tr>
<tr>
<td>APPENDIX B — INTEGRATION OF LOGARITHMIC SINGULARITY</td>
<td>35</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>37</td>
</tr>
</tbody>
</table>

## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1</td>
<td>Strut Shapes</td>
<td>16</td>
</tr>
<tr>
<td>Figure 2</td>
<td>Cavity Shapes at 2.136 Chords Behind the Trailing Edge, Model 2</td>
<td>17</td>
</tr>
<tr>
<td>Figure 3</td>
<td>Cavity Shapes at 2.136 Chords Behind the Trailing Edge</td>
<td>17</td>
</tr>
<tr>
<td>Figure 4</td>
<td>$u$ for Pressure Distribution of Ventilating-Wedge Strut, $k_0 = 0.01415$ or $F_n = 8.41$</td>
<td>18</td>
</tr>
<tr>
<td>Figure 5</td>
<td>$u$ for Cavitating Wedge</td>
<td>18</td>
</tr>
<tr>
<td>Figure 6</td>
<td>$u$ for Ventilating-Strut Model 2</td>
<td>19</td>
</tr>
<tr>
<td>Figure 7</td>
<td>$u$ for Cavitating-Strut Model 2</td>
<td>19</td>
</tr>
<tr>
<td>Figure 8</td>
<td>$u$ Distributions of a Ventilating Parabolic Strut</td>
<td>20</td>
</tr>
<tr>
<td>Figure 9</td>
<td>$u$ for Ventilating Parabolic Strut</td>
<td>20</td>
</tr>
<tr>
<td>Figure 10</td>
<td>$u$ for Cavitating Parabolic Strut</td>
<td>21</td>
</tr>
</tbody>
</table>
Figure 11 – Drag Coefficients of Base-Vented or Base-Cavitating Parabolic Strut, $2t/c = 0.1$ .................................................. 21
Figure 12 – Cavity Drag of Base-Vented Parabolic Strut ................................................................. 22
Figure 13 – Cavity Drag of Ventilating Parabolic Strut ............................................................... 22
Figure 14 – Cavity Drag of Ventilating Parabolic Strut, Theory and Experiment .................. 23
Figure 15 – Change in Streamline Angle near a 12-Percent Parabolic Strut, $d/c = 1$ ................ 24
Figure 16 – Change in Streamline Angle near a 12-Percent Wedge Strut, $d/c = 1$ .................. 24
NOTATION

\( a_i \) Coefficients of polynomials for a strut

\( b_{ij} \) Coefficients of a double polynomial

\( c \) Chord length

\( D \) Drag

\( D_1 \) Projection of strut surface on the x-z plane

\( D_f \) Projection of image strut surface on the x-z plane

\( d \) Submerged length of strut

\( g \) Acceleration of gravity

\( H \) The z-coordinate of streamline at infinity

\( h \) The z-coordinate of streamline at cavity

\( k_0 = \frac{cg}{U^2} \)

\( m \) Source strength

\( P_a \) Atmospheric pressure

\( P_c \) Cavity pressure

\( r(\mathbf{X}, \mathbf{X}_1) \) The distance between two position vectors \( \mathbf{X} \) and \( \mathbf{X}_1 \)

\( S \) Projection of cavity surface on the x-z plane

\( \bar{U} \) The velocity at \( x \to -\infty \)

\( u, v, w \) The x, y, z components of the perturbation velocity

\( x, y, z \) The right-handed cartesian coordinate system

\( \alpha \) Cavitation number

SUBSCRIPTS

\( a \) Quantities for atmospheric pressure

\( c \) For cavity

\( f \) For free-surface image

\( L \) For leading edge

\( T \) For trailing edge
ABSTRACT

Two stable states of high-speed strut flow are considered in three dimensions under linear boundary conditions. The strut is thin, symmetric, without angle of attack, and has base-ventilation or base cavitation without side cavitation. The strut and cavity are represented by polynomial source distributions with unknown coefficients on the center plane. Coefficients of the polynomial for the cavity are obtained as solutions for integral equations of cavity source, using a least-squares method. All the singular double integrals in the present problem are integrated in closed forms by using recurrence formulas. Thus, common errors due to numerical integration of the singular integrals are avoided. Approximate cavity shape and cavity drag are obtained. Downwash is calculated at the hydrofoil plane, which is assumed to be located at the tip of and perpendicular to the strut. Downwash and drag are compared with the existing experimental results, and good agreement is found.

ADMINISTRATIVE INFORMATION

The work reported herein was authorized and funded by the Naval Materials Command under Task Area Z-F424 21 001, Work Unit 1-1520-001.

INTRODUCTION

Since the advent of hydrofoil boats, more knowledge of the flow about a ventilating strut has been needed. Unlike a hydrofoil, a two-dimensional approximation for a strut cannot be used properly, although there have been some attempts\(^1\) to solve the problem this way. Because of the free surface that the strut pierces, the problem is essentially three-dimensional. In addition, the cavity shape behind the strut is a function of the strut speed. Although many investigators have effectively solved free-streamline problems in two dimensions using complex variables, there is no such convenient mathematical tool in three dimensions.

There have been several interesting experimental results for ventilating or cavitating struts\(^2,3,4\) especially when a foil is attached to the strut. The change of cavity shape with

---


speed is most interesting to observe. When speed is low, there is a small depression behind the blunt-based strut on the free surface. This depression grows both in depth and length with added speed. Momentarily a ventilated cavity spreads along the entire strut, entrains behind it, and stays there for a considerable range of speeds. However, at a certain higher speed, the cavity length becomes longer near the deepest end of the strut than near the free surface, where the cavity length becomes zero. This has been called the choked stage. At this stage, cavity pressure is the vapor pressure, not the atmospheric pressure; that is, the ventilation ceases, and the strut is fully cavitating.

The physical explanation of the phenomenon can be given as follows. A large quantity of air is sucked in at high speed by ventilating cavity entrainment through the cavity behind the strut. The high air speed reduces the pressure of the cavity wall, and the mouth of the venting cavity becomes smaller at the free surfaces. With increasing air speed, the cavity wall lapses into an unstable state until it is completely closed, cutting off ventilation and forming a steady cavitating flow. Thus we may assume that there are two stable states of the high-speed strut flow; one is ventilating, and the other is cavitating.

In the present study, these two stable states of high-speed strut flow have been considered in three dimensions under linear boundary conditions. The strut is thin, symmetric, without angle of attack, and has base ventilation or base cavitation without side cavitation. The strut and cavity have been represented by polynomial source distributions with unknown coefficients on the center plane. Coefficients of the polynomial for the cavity have been obtained as solutions of integral equations for cavity source distribution, using a least-squares method. All of the singular double integrals in the present problem have been integrated in closed forms, using recurrence formulas. Thus, common errors due to numerical integration of the singular integrals have been avoided. The approximate cavity shape and cavity drag are obtained, and the downwash is obtained at a hypothetical hydrofoil plane, assumed to be located at the tip of and perpendicular to the strut. Downwash and drag have been compared with the existing experimental results, and good agreement has been obtained.

BOUNDARY VALUE PROBLEM

A vertical strut is located in a uniform flow, having velocity \( \bar{U} \). The mean free surface is considered to be the \( z=0 \) plane, where the origin 0 of the right-handed rectangular coordinates, \( 0-xyz \), is located at the point of intersection of the mean free surface with the trailing edge of the strut. The \( x \)-direction is the same as \( \bar{U} \). For convenience, we assume a uniform section shape for a symmetric strut

---

\[ y = 2\pi \sum_{i=0}^{N} \frac{a_i}{i+1} x^{i+1} \] (1)

in \( x_L < x < 0 \) and \( 0 > z > -d \), where coefficients \( a_i \) are given, and \( x_L \) is the x-coordinate of the straight leading edge. Then the linear source strength for the strut will be

\[ \frac{m_1}{U} = \frac{1}{2\pi} \frac{dy}{dx} = \sum_{i=0}^{N} a_i x^i \] (2)

in

\[ D_1 (x_L < x < 0, \ 0 > z > -d, \ y = 0) \]

where \( U = |\bar{U}| \). The cavity behind the strut may be represented by a source distribution

\[ \frac{m_2}{U} = a_0 + \sum_{i=1}^{N} \sum_{j=0}^{M} b_{ij} x^i |z|^j \] (3)

in

\[ S(0 < x < x_c, \ 0 > z > -d, \ y = 0) \]

where we first assume that \( S \) is rectangular. In Equations (2) and (3), a condition \( m_1 = m_2 = Ua_0 \) on the trailing edge \( x = 0 \) is automatically satisfied. Although the cavity shape is not known a priori, we have to assume the shape first and to formulate an iteration scheme. In the case of a supercavitating hydrofoil with cavity length longer than three chord lengths, the cavity drag is known to be quite insensitive to the cavity planform behind the foil.\(^6\) Thus a long cavity analysis will be tried first. The coefficients \( b_{ij} \) will be determined from boundary conditions. At the free surface, the linearized boundary condition for large \( u \), i.e., large Froude number, will be used. Considering perturbation-velocity components \( (u, u, w) \), we have on the free surface

\[ u = 0 \ \text{on} \ \ z = 0 \] (4)

This indicates that we have only to consider image-source distributions

\[
\frac{m_{1f}}{U} = - \sum_{i=0}^{N} a_i x^i 
\]

on

\[
D_f(x_L < x < 0, \ 0 < z < d, \ y = 0)
\]

and

\[
\frac{m_{2f}}{U} = - a_0 - \sum_{i=1}^{N} \sum_{j=0}^{M} b_{ij} x^i z^j 
\]

on

\[
S_f(0 < x < x_c, \ 0 < z < d, \ y = 0)
\]

The boundary condition on the cavity is from the Bernoulli equation along the cavity streamline

\[
\frac{P_c}{\rho} + \frac{1}{2} q_c^2 + gh = \frac{P_\infty}{\rho} + \frac{1}{2} U^2 + gH
\]

where \( P_c \) is cavity pressure

\( P_\infty \) is pressure at infinity

\( h, H \) are the z-coordinates of the streamline at the cavity and at infinity, respectively

\( g \) is the acceleration of gravity

\( q_c \) is the total speed at the cavity.

If we linearize the Bernoulli equation

\[
Uu = \frac{P_\infty}{\rho} + gH - \frac{P_c}{\rho} - gh
\]

At infinity

\[
\frac{P_\infty}{\rho} = \frac{P_a}{\rho} - gH
\]

where \( P_a \) is the atmospheric pressure.
Thus, on the vapor-cavity boundary, we have a nondimensional expression

\[ \frac{u}{U} = \frac{\sigma_a}{2} - k_0 \frac{h}{c} \]  

(8)

where \( \sigma_a = (P_a - P_c)/(1/2 \rho U^2) \) and \( k_0 = gc/U^2 \), with a standard chord length \( c \), where \( c \) is taken to be unity.

On a ventilating cavity

\[ P_c = P_a \]

hence

\[ \frac{u}{U} = -k_0 \frac{h}{c} \]  

(9)

These boundary conditions will be applied on the \( y = 0 \) plane. Since \( g \) and \( P_a \) are constants, and \( P_c \) is also approximately constant, \( \sigma_a \) is a function of only \( U \), and \( k_0 \) is a function of \( U \) and \( c \). For a choked cavity of a strut foil on a high-speed hydrofoil boat, \( U = 0 \) (100 fps), and both \( \sigma_a \) and \( k_0 \frac{h}{c} \) are small; however, \( k_0 \frac{h}{c} \) is much smaller than \( \sigma_a \). For example, if we take \( P_c \) as a vapor pressure of water\(^7\) at 70° F or 0.36 psia and the air pressure at approximately 14.7 psia, then

\[ \frac{u}{U} = h g \frac{P_a - P_c}{\rho gh} - 1 \quad U^2 = \frac{32.1g}{U^2} - gh/U^2 \]

Thus when \( U \) is sufficiently large, and \( h \) is small, the gravity effect can be negligible on the cavity, as well as on the free surface, compared with the cavitation number. However, if the scale of a hydrofoil is very large, and the submergence is almost 32 chords, say, the gravity effect on the cavity should be important even for reasonably high speed.

For a ventilating cavity, only the gravity effect should be considered, as stated by Equation (9). Since the steady gravity wavelength is

\[ \frac{\lambda}{c} = \frac{2\pi}{k_0} \]

the cavity will be affected by the free-surface waves, when \( \lambda \) is small or \( k_0 \) is large. In this case, the wave-number dependent, free-surface boundary condition \( \partial u / \partial x + k_0 \omega = 0 \) has to be used. However, the cavitation or ventilation does not take place unless the speed is great. For example, for \( c = 1 \) foot and \( U = 10 \) fps

\[
k_0 = gc/U^2 = 0.32172
\]
or

\[
\lambda/c = 19.53
\]

In such cases, when the wavelength is large, the cavity condition

\[
u/U = - gz/U^2 = - k_0 z/c
\]

and the free-surface condition on \( z = 0 \)

\[
u = 0
\]
can be effectively utilized to solve the problem approximately.

For two states of flow phenomena of the strut flow, when one is ventilating and the other is cavitating, only the approximate boundary conditions for both the free and the wetted surfaces have been considered as shown in Equations (8) and (9). We do not consider the cause of the phenomena, nor the airflow inside the cavity in our problem. Furthermore, as the linearized model, neither the effect of spray nor tip has been considered here.

**SOLUTION**

We represent \( u \) in terms of a source distribution

\[
\begin{align*}
u(x, y, z) &= \left( - \int_{-d}^{0} \int_{0}^{x_L} m_1 + \int_{0}^{d} \int_{0}^{x_L} m_{1f} \\
&\quad - \int_{0}^{x} \int_{0}^{d} m_2 + \int_{0}^{d} \int_{0}^{x} m_{2f} \right) \frac{\partial}{\partial x} \left( \frac{1}{r(X, X_1)} \right) dx_1 \, dz_1
\end{align*}
\]

(10)
where

\[ r(\bar{X}, \bar{X}_1) = \left\{ (x-x_1)^2 + y^2 + (z-z_1)^2 \right\}^{\frac{1}{2}} \]

If we use the boundary conditions of Equations (8) or (9) on \( y = 0 \), Equation (10) is a singular integral equation to solve for \( m_2 \) because the kernel is singular at \( \bar{X} = \bar{X}_1 \).

Inserting Equations (2), (3), (5), and (6) into Equation (10) we obtain

\[ u(x,y,z) = \sum_{i=0}^{N} a_i A_i + \sum_{i=1}^{N} \sum_{j=0}^{M} b_{ij} B_{ij} \]  

where

\[ A_0 = \left( - \int_{-d}^{d} \int_{x_L}^{x_C} \int_{x_L}^{x_C} - \int_{0}^{d} \int_{0}^{x_L} \int_{0}^{x_L} \right) \frac{\partial}{\partial x} \frac{1}{r} \, dx_1 \, dz_1 \]  

\[ A_i = \left( - \int_{-d}^{d} \int_{x_L}^{x_C} \int_{x_L}^{x_C} + \int_{0}^{d} \int_{0}^{x_L} \int_{0}^{x_L} \right) x_i \frac{\partial}{\partial x} \frac{1}{r} \, dx_1 \, dz_1 \]  

\[ B_{ij} = \left( -(-1)^j \int_{-d}^{d} \int_{0}^{x_C} \int_{0}^{x_C} + \int_{0}^{d} \int_{0}^{x_L} \int_{0}^{x_L} \right) x_i z_j \frac{\partial}{\partial x} \frac{1}{r} \, dx_1 \, dz_1 \]

These singular integrals will be analytically evaluated in Appendix A, using the concept of finite parts of singular integrals.\(^8\)

For a finite cavity we may include a closure condition

\[ \int_{x_1}^{x_C} (m_1 + m_2) \, dx = 0 \]  

\[ (15) \]

If Equations (11) through (14), (16), and either (8) or (9) are combined, then linear simultaneous equations for $b_{ij}$ and $\sigma_a$ and $k_0$ are obtained. Since the cavity planform shape will be assumed a priori, $\sigma_a$ (or $k_0$) cannot be given together. Thus when $\sigma_a$ (or $k_0$) is given first, the answer for $b_{ij}$ and the cavity planform will be obtained by iteration. If the simultaneous equations are solved in the normal way by a matrix inversion method, the number of simultaneous equations must be the same as the number of field points on the cavity plus the number $M$ of closure-condition Equation (16). Solution of the equation $b_{ij}$, $\sigma_a$, and $k_0$ depends on the location and number of field points and the number of terms in the cavity-source polynomial. Convergence of the solution can be numerically tested. If the number of terms is small, then the corresponding number of field points has to be small also. When the number of terms is given, the accuracy of the solution is only a function of the location of field points. To choose "better" field points, the collocation method\textsuperscript{9} was devised for airfoil lifting-surface theory and was widely used. However, in the collocation method the complicated behavior of convergence is not well understood.\textsuperscript{10}

In the present analysis, we use the least-squares method to solve the simultaneous equations.\textsuperscript{11} A similar technique has been used for airfoil lifting-surface theory.\textsuperscript{12} In this method, we can have a unique solution no matter how many equations we have for a given number of unknowns. Thus, we can choose as many field points as needed to increase accuracy for a given number of polynomial terms in Equation (3). The numerical programming is simple for the least-squares solution. Solutions for the cavity source strengths, i.e., $b_{ij}$, converge quite rapidly when the number of field points is increased.

\begin{equation}
a_0 (x_C - x_L) - \sum_{j=1}^{N} a_j \frac{x_L^{j+1}}{j+1} + \sum_{i=1}^{N} b_{ij} \frac{x_C^{i+1}}{i+1} = 0
\end{equation}

\begin{equation}
\sum_{i=1}^{N} b_{ij} \frac{x_C^{i+1}}{i+1} = 0
\end{equation}

for $j = 1, 2, \ldots, M$ (16)


DRAG AND DOWNWASH

The drag of the strut can be determined by integrating the pressure multiplied by the strut slope on the projection of the strut surface. The pressure at the leading edge is singular but it is integrable. The drag of the strut to be numerically evaluated can be written for the ventilating and cavitating cases in the form

\[
\frac{D}{\frac{1}{2} \rho U^2 D_1} = - \frac{1}{D_1} \int_{-a}^{a} \int_{0}^{c} \frac{4u}{U} \frac{dy}{dx} \, dx \, dz + \begin{cases} 
\frac{k_0}{c} \frac{d}{c} \frac{2t}{c} & \text{for ventilating case} \\
\frac{\sigma_a}{c} \frac{2t}{c} & \text{for cavitating case}
\end{cases}
\]  

(17)

where \( y(0) = t \).

When we consider a strut-foil system such as that on a hydrofoil boat, knowledge of the downwash on the hydrofoil due to the strut as well as the downwash due to the foil is important. The flow angle at the foil is significantly influenced by strut downwash, which may induce cavitation on the pressure side of the wing. Downwash due to the ventilating or cavitating strut is given by

\[
w(x,y,0) = - \int_{D_1 + D_f + S + S_f} \left( m_1 + m_{1f} + m_2 + m_{2f} \right) \frac{\partial}{\partial z} \left( \frac{1}{r(X, X_1)} \right) \, dx_1 \, dz_1
\]  

(18)

This equation can also be integrated analytically,\(^\text{13}\) although it is a little complicated as is shown in Appendix A.

SCHEME OF ITERATION

When \( U \) is very large, the cavity may be choked, so that the cavity length near the free surface becomes small. The small cavity length may have considerable influence on the pressure at the strut. However, since there is no simple way to predict the cavity length, we have to rely upon an iterative format. We will, then, construct a scheme similar to the Newton-Raphson scheme.

We assume an extra source distribution \( \sum_{j=1}^{M} b_{0j} z_l^j \) in \( D_1 \) for the strut in addition to that given in Equation (2)

\[
\frac{m_1}{U} = \sum_{i=1}^{N} a_i x_i + \sum_{j=1}^{M} b_{0j} z_l^j \quad \text{in} \quad D_1
\]

hoping that the extra source distribution will compensate for the effect of the assumed cavity length on the cavity boundary condition. The coefficients will be determined from the boundary condition. Because of the continuity of source strength at the trailing edge of the strut, we can write for the cavity source distribution

\[
\frac{m_2}{U} = a_0 + \sum_{i=0}^{N} \sum_{j=0}^{M} b_{ij} x_i |z_l^j| \quad \text{in} \quad S
\]

where \( i \) starts from 0 since \( m_1 (x = 0) = m_2 (x = 0) \).

If we assume a cavity shape \( y = x_0 \sqrt{1 - a z^2} (\Sigma C_i z_l^i) \) with unknown coefficients \( a \) and \( C_i \), integration of Equations (12) through (14) can be performed in closed form by approximating \( x_c \) with several broken lines

\[
x_c = a_1 + \beta_1 z
\]

For determination of the unknown coefficients \( a \) and \( C_i \), we use the Newton-Raphson method. That is, when \( \sigma \) is specified for a given strut (Equation (2)) the coefficients \( a \) and \( C_i \) are of such a nature that the calculated \( \sigma \) has to be equal to the specified \( \sigma \), and the resulting extra strut source or the coefficient \( b_{0j} \) has to be equal to zero. Thus, when a first approximation of \( a \) and \( C_i \), say, \( a^{(1)} \) and \( C_i^{(1)} \), are given, the next approximation can be obtained as follows from the Newton-Raphson scheme. When we solve simultaneous equations

\[
\begin{bmatrix}
\frac{\partial \sigma}{\partial a} & \frac{\partial \sigma}{\partial C_i} \\
\frac{\partial b_{ij}}{\partial a} & \frac{\partial b_{ij}}{\partial C_i}
\end{bmatrix}
\begin{bmatrix}
\Delta a \\
\Delta C_i
\end{bmatrix}
= \begin{bmatrix}
-\sigma + \sigma_0 \\
b_{0j}
\end{bmatrix}
\]

(22)
where $\sigma_0$ is the given cavitation number, and $\sigma$ is the calculated value,

$$a^{(2)} = a^{(1)} + \Delta a$$

$$C_{i}^{(2)} = C_{i}^{(1)} + \Delta C_{i}$$

will be the next approximation. By continuing this process until $\sigma$ and $b_{ij}$ are very close to the aiming values, $\sigma = \sigma_0$, and $b_{0j} = 0$.

In general, partial derivatives should be computed numerically. Accuracy is not crucial for the iteration. When $i$ and $j$ of $C_i$ and $b_{0j}$ are large, the computing time will be the main problem. However, even for a choice of small values of $i$ and $j$, the calculation should be useful for understanding the cavity shape and for estimating cavity drag.

**NUMERICAL RESULTS AND DISCUSSION**

We cannot, obviously, obtain a real solution of cavity flow by using the inviscid model. Then what kind of model can best approximate the real solution? There are many kinds of inviscid models for two-dimensional cavity flow such as the Riabuchinsky symmetric model, the Roshko wake model, the Tulin double or single spiral vortex model, etc. As long as the cavity length is fairly large, these models differ little in their effect on flow phenomena near the foil. In general, to obtain the cavity length, the cavity-closure condition is applied for closed-cavity models. For the three-dimensional cavity model, which we are dealing with, the first approximation of the cavity planform influences the first approximation of the solution. In this respect, we have tried several methods numerically to have a better first approximation without considering the extra source on the strut shown in Equation (19).

First we tried a rectangular planform with the closure condition of Equation (16) and found that the cavity shape did not agree with experimental results. Naturally the cavity length can be controlled both by the singularity planform of the cavity and the closure condition. For a strut flow in which the free-surface effect makes the flow significantly deviate from the two-dimensional flow, it has been found that the closure condition, in addition to the given singularity planform, unreasonably restricts the solution. Therefore, when we neglected the closure condition for the simultaneous equations for the cavity source, results of the first approximation solution were quite good in both ventilating and cavitating cases. The cross sections of the cavity shapes in both the ventilating and cavitating cases, approximately two chords behind the parabolic and wedge struts (Figure 1), are shown in Figures 2 and 3 for drafts of one chord and two chords. Although Models 3 and 4 of struts differ in shape from the real parabolic strut, the physical quantities vary little. A closer approximation to the parabolic strut near the leading edge by Equation (1) is difficult. Therefore, we substitute Model 3 for the parabolic strut.
To relax the closure condition at the cavity end, we also considered a vertical line sink at \( x = x_c \) whose strength was the same as the total strut source per unit vertical length. This influenced the value of \( k_0 \) only. When we compared the flows with the same \( k_0 \) with and without the cavity-end sink, little difference was observed. However, either \( k_0 \) or \( \sigma \) were obtained as a solution, thus, to obtain higher values of \( k_0 \) or \( \sigma \), we had to reduce the length of the cavity planform and/or increase the strength of the cavity-end sink.

In the polynomial representation of cavity-source strength in Equation (6), sets of \( x \) and \( z \) terms \( (N,M) = (4,4), (4,5), (5,5), \) and \( (4,6) \) were tested. The result was that cavity shapes were changed a small amount; however, drag was changed very little.

The distribution of \( u \), the \( x \)-component of velocity on the struts shown in Figure 1 has been obtained and is shown in Figures 4 through 10. From Figures 4 through 10, the pressure distributions can be obtained immediately from the Bernoulli equation

\[
\frac{P - P_a}{\frac{1}{2} \rho U^2} = - \frac{2u}{U} - 2k_0 \frac{h}{c}
\]

together with Equation (9). The pressure on the wedge-shaped strut was normally greater than atmospheric pressure. The pressure was sensitive to the curvature of the strut surface, and the minimum pressure became less than atmospheric pressure at a comparatively small curvature of the strut surface as shown in Figures 4 through 10. For the approximate parabolic strut (Model 3), the pressure was less than atmospheric over most of the strut surface. However, it is believed that it would not cavitate unless the pressure should become lower than the vapor pressure. For a base-cavitating parabolic strut, the pressure on the after part of the strut falls slightly below the cavity pressure (Figure 10). These findings may be a manifestation of free-surface and three-dimensional effects because although in the two-dimensional cavity flow, we assume that the pressure in the flow is always larger than the cavity pressure, this is strictly an assumption and not a fact.\(^{14}\) Nondimensional velocity distributions for ventilating struts at moderate speeds (small \( k_0 \)) were almost the same as for the corresponding cavitating struts at much higher speeds with \( \sigma = 0(k_0) \), except the trailing edge (Figures 4 through 10), although cavity shapes were considerably different.

The strut drag coefficients, obtained by integrating the pressure distribution multiplied by the slope of the strut surface, are shown in Figures 11 and 12. For a parabolic two-dimensional foil with zero cavitation number,\(^{15}\) the drag coefficient is


\[ C_D = \frac{\pi}{8} \left( \frac{L}{c} \right)^2 \]

Of course this cannot be directly used for a parabolic strut because of three-dimensional and free-surface effects. If a parabolic strut of finite-aspect ratio is located in a high-speed flow of an infinite medium, we have only to consider the three-dimensional effect. The drag can be approximated by the well-known expression\textsuperscript{16}

\[ C_{D3} \left( \frac{L}{c} \right)^2 = \frac{C_{D2}}{\left( \frac{L}{c} \right)^2} \frac{\lambda}{\lambda + 1 + C_{D2}/(\pi t^2/c^2)} \]

where \( C_{D3} \) and \( C_{D2} \) are three- and two-dimensional drag coefficients, respectively, and \( \lambda \) is the aspect ratio. The free-surface effect tends to reduce the drag; this can be seen from the computed pressure distribution near the free surface in Figures 4 through 10. The free-surface effect of a two-dimensional finite wedge entering the water surface was known to reduce the drag almost 45 percent, compared to the base-vented wedge in an infinite medium.\textsuperscript{17} Thus we can expect a sizable free-surface effect which tends to reduce the cavity drag. However, we may not be able to neglect spray drag which can be quite large according to many experiments with nonventilating struts.

Although numerous experimental results (Reference 18) exist for high-speed strut flow, few data exist for symmetric base-vented struts. Therefore, we took a result for a base-vented parabolic strut from Reference 19 where the friction-drag coefficient was taken to be 0.003. Comparisons between the computed cavity drag of a parabolic strut, i.e., Model 3, and the residual drag determined from experiment\textsuperscript{19} are shown in Figures 11 and 12. The difference between the computed and measured drags may be mainly due to the spray drag.

---


Although there are several empirical formulas for the spray drag of nonventilating struts, they differ a great deal. Furthermore, spray drag measurements for a base-vented strut do not seem to exist. In general, the spray drag for a high Froude-number flow is a function of the thickness-to-chord ratio, the nondimensional distance from the leading edge to the point of maximum thickness, and the Reynolds number. When we consider existing spray drag measurements and their dependence on these parameters, the present result seems to be reasonable. The variation of drag coefficient with draft-to-chord ratio, shown in Figures 11 and 12, indicates a large three-dimensional free-surface effect for \( d/c < 2 \) and for small \( k_0 \).

We note here that the chord length \( c \) is only a small portion of the total strut-cavity length. For a parabolic strut, i.e., Model 3, of one-chord draft, the three-dimensional drag coefficients as a function of \( k_0 \) and \( \sigma \) are shown in Figure 13 together with results from two-dimensional theory. This figure shows that \( C_D \) is very nearly a linear function of \( k_0 \) or \( \sigma \). In fact, we have seen that the nondimensional velocity distributions on a strut do not change too much for different values of \( k_0 \) or \( \sigma \) in Figures 4 through 10. The calculated differences in drag coefficients come mainly from the direct contribution of \( k_0 \) or \( \sigma \), represented in the second term of the right-hand side of Equation (17). This can be understood when we compare curves of drag coefficients with the curve of

\[
\frac{C_D}{(\frac{2t}{c})^2} = \frac{C_{D0}}{(\frac{2t}{c})^2} + \frac{\sigma}{(\frac{2t}{c})^2} \quad \text{or} \quad \frac{C_{D0}}{(\frac{2t}{c})^2} + \frac{k_0 \, d}{2t}
\]

represented by dashed curves in Figure 13, where \( C_{D0} \) is the value of \( C_D \) for \( k_0 = 0 \). In Figure 13, another interesting fact is that the drag coefficient for a ventilating strut is very close to that for a cavitating strut at the same values of \( k_0 \), \( \sigma \) and \( d/c \). However, we note that \( \sigma \) for the ventilating strut is approximately equal to 16 \( k_0 \) for the cavitating strut at the same speed. Therefore, when a strut suddenly switches to cavitation from ventilation, the drag jumps to quite a large value. Another comparison between calculated and experimental results is shown in Figure 14, where good agreement can be observed.


The effect of the strut on downwash at the foil is easy to calculate and is shown in Figures 15 and 16. The calculated downwashes agree with the experimental results fairly well, although the strut cavity shape with a foil in place may be different from that without a foil. In fact, downwash does not seem to be too sensitive to the cavity shape.

The previously mentioned numerical results have all been first approximations, without successive iterations to determine different cavity planforms. In addition to this problem, we may incorporate angle of attack, using a lifting-surface theory that is being considered for a supercavitating hydrofoil of finite span. This problem is physically and mathematically interesting in itself and is related to the engineering-design problem of a high-speed, strut-foil system of a hydrofoil boat. A computer program has been written as a future unit of the strut-foil system so that all the subroutines can be utilized for both the cavitating and lifting foil as well as the interference between the strut and the foil.

CONCLUSIONS

Flows of ventilating or cavitating struts have been analyzed numerically, using a three-dimensional mathematical model. With a double-polynomial representation of the cavity source, integral equations for the cavity source are solved, and various interesting physical phenomena are found. Cavity drag and downwash seem to agree well with existing experiments. However, since the measured data are widely scattered and a large portion of the measured total drag is the friction drag which is difficult to estimate correctly or separate from other components, we need to be cautious in drawing quick conclusions.

The present analysis can be easily extended to a symmetric strut of any shape. For better prediction of cavity shape, an iterative scheme may be needed. However, for the prediction of strut pressure distribution or drag, the first approximation seems to be enough, and no iteration is necessary as long as the cavitation number or Froude number is obtained as a solution because the pressure on the strut is almost insensitive to both cavity shape and cavitation number. Comparison with the two-dimensional solution shows that the latter considerably overestimates the drag. Numerical experiments with various cavity models have been very helpful in determining a suitable cavity model and the source polynomial for this problem. Thus, the present method can be confidently applied to other three-dimensional cavity problems.


Figure 1a - Ventilating Strut

Figure 1b - Cavitating Strut

Figure 1c - Various Models

Figure 1 - Strut Shapes
Figure 2 – Cavity Shapes at 2.136 Chords Behind the Trailing Edge, Model 2

Figure 3 – Cavity Shapes at 2.136 Chords Behind the Trailing Edge
Figure 4 – u for Pressure Distribution of Ventilating-Wedge Strut, 
\( k_0 = 0.01415 \) or \( F_n = 8.41 \)

Figure 5 – u for Cavitating Wedge
Figure 6 – u for Ventilating-Strut Model 2

Figure 7 – u for Cavitating-Strut Model 2
Figure 8 - $u$ Distributions of a Ventilating Parabolic Strut

Figure 9 - $u$ for Ventilating Parabolic Strut
Figure 10 – $u$ for Cavitating Parabolic Strut

Figure 11 – Drag Coefficients of Base-Vented or Base-Cavitating Parabolic Strut, $2t/c = 0.1$
Figure 12 – Cavity Drag of Base-Vented Parabolic Strut

Figure 13 – Cavity Drag of Ventilating Parabolic Strut
Figure 14 — Cavity Drag of Ventilating Parabolic Strut, Theory and Experiment
Figure 15 – Change in Streamline Angle near a 12-Percent Parabolic Strut, d/c = 1

Figure 16 – Change in Streamline Angle near a 12-Percent Wedge Strut, d/c = 1
APPENDIX A
INTEGRALS OF SINGULAR INTEGRALS

INTEGRALS IN INTEGRAL EQUATIONS

Equations (13) and (14) are composed of integrals of the type:

\[ I = \int_0^d \int_{x_L}^{x_T} x_i^j z_1 \frac{\partial}{\partial x} \frac{1}{r} \, dx \, dz_1 = - \int_0^d \int_{x_L}^{x_T} x_i^j z_1 \frac{\partial}{\partial x} \frac{1}{r} \, dx \, dz_1 \]

\[ = - \int_0^d x_i^j z_1 \frac{1}{r} \left[ \int_{x_i=x_L}^{x_T} dz_1 + \int_0^d \int_{x_L}^{x_T} i x_i^{j-1} z_1 \frac{1}{r} \, dx \, dz_1 \right] \]

where

\[ r(\bar{X}, \bar{X}_1) = \left\{ (x-x_1)^2 + y^2 + (z-z_1)^2 \right\}^{\frac{1}{2}} \]

For the single integral of Equation (24), when we write

\[ r(\bar{X}, \bar{X}_1) = (Az_1^2 + 2Bz_1 + C)^{\frac{1}{2}} \]

we have

\[ Y(\bar{X}, \bar{A}, d, j+1) = \int_0^d \frac{z_1^j}{r(\bar{X}, \bar{X}_1)} \, dz_1 = \left[ \frac{1}{A_j} z_1^{j-1} \frac{1}{r} \right] \]

\[ - \frac{(2j-1)}{j} \frac{B}{A} \int_0^d \frac{z_1^{j-1}}{r} \, dz_1 - \frac{(j-1)}{j} \frac{C}{A} \int_0^d \frac{z_1^{j-2}}{r} \, dz_1 \]

\[ = \frac{1}{A_j} z_1^{j-1} \frac{1}{r} \]

where

\[ \bar{A} = (A, B, C) \]
\[ Y(\vec{X}, \vec{A}, d, 1) \equiv \int_{0}^{d} \frac{dz_1}{r} = \frac{1}{\sqrt{A}} \log \left( \frac{Az_1 + B}{\sqrt{A}} + r \right) \bigg|_{0}^{d} \] (26)

All of the integrals appearing in \( Y(\vec{X}, \vec{A}, d, j+1) \) may be evaluated by using available integral tables.\textsuperscript{13} If we insert Equation (26) into Equation (25), we can obtain \( Y(\vec{X}, \vec{A}, d, 2) \); then if we insert \( Y(\vec{X}, \vec{A}, d, 1) \) and \( Y(\vec{X}, \vec{A}, d, 2) \) into Equation (25), we can obtain \( Y(\vec{X}, \vec{A}, d, 3) \). If we continue this process we can obtain \( Y(\vec{X}, \vec{A}, d, j+1) \) for any integer \( j \).

For the double integral appearing in Equation (24), we write

\[ F(\vec{X}, d, i+1, j+1) \equiv \int_{0}^{d} \int_{x_L}^{x_T} x_1^{i} z_1^{j} \frac{1}{r} \, dx_1 \, dz_1 \] (27)

where \( r^2 = A_1 x_1^2 + 2 B_1 x_1 + C_1 \)

\[ A_1 = 1 \]
\[ B_1 = -x \]
\[ C_1 = x^2 + y^2 + (z-z_1)^2 \]

Using Equations (25) and (26), Equation (27) can be written

\[
F(\vec{X}, d, i+1, j+1) = \int_{0}^{d} \frac{1}{i} \left[ x_1^{i-1} \, r - (2i-1) B_1 \int_{x_L}^{x_T} \frac{x_1^{i-1}}{r} \, dx_1 \right] \bigg|_{x_1=x_L}^{x_T} - (i-1) C_1 \int_{x_L}^{x_T} \frac{x_1^{i-2}}{r} \, dx_1 \bigg|_{x_1=x_L}^{x_T} \] (28)

and, for \( i = 0 \),

\[
F(\vec{X}, d, 1, j+1) = \int_{0}^{d} \log (x_1 - x + r) \bigg|_{x_L}^{x_T} z_1^{j} \, dz_1 \] (29)
If we integrate Equation (29) by parts, then

\[
F(\bar{X}, d, l, j+1) = \left[ \frac{z_i^{j+1}}{j+1} \log (x_1 - x + r) \right]_{x_1 = x_L}^{x_T} - \int_0^d \frac{z_i^{j+1}}{j+1} \left\{ \frac{z_1 - z}{(z_1 - z) + y^2} + \frac{\gamma(z_1 - z) + \delta}{(z_1 - z)^2 + y^2} \right\} dz_1
\]

where \( \gamma = A(x_1 - x) \)
\( \delta = B(x_1 - x) \)

with \( A \) and \( B \) the same as in Equation (25).

The integrals in \( F(\bar{X}, d, l, j+1) \) can be integrated term by term. We will consider here a linear function of \( z \) for \( x_L \) and \( x_T \) for later use. That is, we will use \( x_L = a_1 + b_1 z \) and \( x_T = a_2 + b_2 z \), so that the leading and the trailing edges or the cavity end are not necessarily perpendicular to the mean free surface but are represented by any straight line on the \( y = 0 \) plane. Then, making use of Equations (25) through (27) and (30), Equation (24) can be written

\[
I = -\int_0^d \left\{ (a_2 + b_2 z_1) z_i^j \frac{1}{r(\bar{X}, \bar{X}_T)} - (a_1 + b_1 z_1) z_i^j \frac{1}{r(\bar{X}, \bar{X}_L)} \right\} dz_1
\]

\[
+ i \int_0^d \int_{x_L}^{x_T} x_1^{i-1} z_i^j \frac{1}{r} dx_1 dz_1
\]

\[
= -a_2^i \sum_k \left[ \frac{b_2}{a_2} \right]^k Y(\bar{X}, \bar{A}_2^1, d, j+k+1)
\]

\[
+ a_1^i \sum_k \left[ \frac{b_1}{a_1} \right]^k Y(\bar{X}, \bar{A}_1^1, d, j+k+1)
\]

\[
+ i F(\bar{X}, d, i, j+1)
\]

(31)
where

\[ iC_k = \frac{i!}{k!(i-k)!} \]

\[
Y(\bar{X}, \bar{A}_i^1, d, j+k+1) = \int_0^d z_1^{j+k} \left\{ (x_1 - a_i - b_1 z_1)^2 + y^2 + (z - z_1)^2 \right\}^{-\frac{1}{2}} dz_1
\]

\[
\equiv \int_0^d z_1^{j+k} \left( A_i^1 z_1^2 + 2B_i^1 z_1 + C_i^1 \right)^{-\frac{1}{2}} dz_1 \tag{32}
\]

and

\[ \bar{A}_i^1 = (A_i^1, B_i^1, C_i^1), \quad i = 1, 2 \tag{33} \]

Integral (32) by the same method as used is evaluated for Equation (25). However, Integral (27) is considerably different when \( x_L \) and \( x_T \) are functions of \( z \). As in Equation (28), we write

\[
F(\bar{X}, d, i+1, j+1) = \int_0^d \frac{1}{i} \left[ \frac{x_1^{i-1}}{r} - (2i-1) B_1 \int_0^{x_1^{i-1}} \frac{r}{a_1 + b_1 z_1} \, dx_1 \right] \tag{34}
\]

\[
-(i-1)C_1 \int_0^{x_1^{i-2}} \frac{r}{a_1 + b_1 z_1} \, dx_1 \]

Then

\[
F(\bar{X}, d, 1, j+1) = \int_0^d - \sum_{i=1}^2 (-1)^i \log (a_i + b_1 z_1 - x + r_i) z_i^i dz_1 \tag{35}
\]

where

\[
r_i^2 = (a_i + b_1 z_1 - x)^2 + y^2 + (z - z_1)^2 \tag{36}
\]
Integrating Equation (35) by parts

\[ F(\bar{X}, d, 1, j+1) = -\sum (-1)^j \left[ \frac{z_i^{j+1}}{j+1} \log (a_i + b_i z_i - x + r_i) \right] \]

\[ = \int_0^d \frac{z_i^{j+1}}{(z_i - z)^2 + y^2} + \frac{z_i - z}{(z_i - z)^2 + y^2} + \frac{\beta (z_i - z)^2 + \gamma (z_i - z) + \delta}{(z_i - z)^2 + y^2} \] \tag{37}

where

\[ \beta = b_i A_i \]
\[ \gamma = b_i B_i + A_i (b_i z_i + a_i - x) \]
\[ \delta = B_i (b_i z_i + a_i - x) \]

The integrals in the right-hand side of Equation (37) can be integrated term by term, using the following integrations

\[ \int \frac{Z^m r}{Z^2 + y^2} \ dZ \]

if \( m = 2i + 1 \)

\[ = \int \frac{Z^{2i} \pm y^{2i}}{Z^2 + y^2} Zr dZ \pm \int \frac{y^{2i} Ze}{Z^2 + y^2} \ dZ \]

\[ = (Z^{2(i-1)} - Z^{2(i-2)} y^2 + \ldots \pm y^{2(i-1)}) Zr dZ \]

\[ + \int \frac{y^{2i} Zr}{Z^2 + y^2} \ dZ \] \tag{39}
with upper sign \{ \text{when } i \text{ is odd} \}
lower sign \{ \text{even,} \}

if \ m = 2i

\[ = \int (Z^2(0-1) - Z^2(0-2) y^2 + \ldots \mp y^2(0-1)) \, r \, dZ \]

\[ \mp \int \frac{y^{2i}r}{Z^2 + y^2} \, dZ \]

\[ \int \frac{Z^{2i+1}}{Z^2 + y^2} \, dZ \]

\[ = \frac{Z^{2(i-1)+2}}{2(i-1)+2} - \frac{Z^{2(i-2)+2}}{2(i-2)+2} y^2 + \ldots \pm \frac{Z^2}{2} y^{2(i-1)} \]

\[ \mp \frac{y^{2i}}{2} \log (Z^2 + y^2) , \]

\[ \int \frac{Z^m}{(Z^2 + y^2)} \, dZ = \frac{Z^{2(i-1)+1}}{2(i-1)+1} - \frac{Z^{2(i-2)+1}}{2(i-2)+1} y^2 + \ldots \pm Zy^{2(i-1)} \]

\[ \mp y^{2i-1} \tan^{-1} \frac{Z}{y} , \]

\[ \int \frac{Z^m}{(Z^2 + y^2)_{r}} \, dZ = \int \frac{(Z^{2i} \pm y^{2i})Z^j}{(Z^2 + y^2)_{r}} \, dZ \mp \int \frac{y^{2i}Z^j}{(Z^2 + y^2)_{r}} \, dZ \]

\[ = \int (Z^2(i-1) - Z^2(i-2) y^2 + \ldots \pm y^2(i-1)) \frac{Z^j}{r} \, dZ \]
\[ \pm \int \frac{y^{2i} Z^j}{(Z^2 - y^2)^r} \, dZ \quad \text{with upper sign} \}
\text{lower sign} \}
\begin{cases} \text{when } i \text{ is} \{ \begin{array}{ll} \text{odd} & \text{even} \\
\text{and } j = 0, 1. & \end{array} \end{cases} \]

where we use

\[ \int \frac{dZ}{(Z^2 + y^2)^r} = \frac{\beta_2}{2y \rho^2} \log \frac{z^2 + y^2}{(\gamma_1 Z + \delta_1 - r)^2 + (\gamma_2 Z + \delta_2)^2} \]
\[ + \frac{\beta_1}{y \rho^2} \left( \tan^{-1} \frac{y}{Z} + \tan^{-1} \frac{\gamma_2 Z + \delta_2}{\gamma_1 Z + \delta_1 - r} \right) \]  \hspace{1cm} (41)

and

\[ \int \frac{Z}{(Z^2 + y^2)^r} \, dZ = \frac{\beta_1}{2 \rho^2} \log \frac{(\gamma_1 Z + \delta_1 - r)^2 + (\gamma_2 Z + \delta_2)^2}{Z^2 + y^2} \]
\[ + \frac{\beta_2}{\rho^2} \left( \tan^{-1} \frac{y}{Z} + \tan^{-1} \frac{\gamma_2 Z + \delta_2}{\gamma_1 Z + \delta_1 - r} \right) \]  \hspace{1cm} (42)

where

\[ r = A Z^2 + 2 B Z + C, \quad \theta = C - A y^2 \]
\[ \beta_1 = \pm \sqrt{\frac{1}{2} \rho^2 + \frac{1}{2} \theta}, \quad \beta_2 = \pm \sqrt{\frac{1}{2} \rho^2 - \frac{1}{2} \theta} \]

with the sign chosen by \( \beta_1 \beta_2 = By \)

\[ \rho^2 = (\theta^2 + 4 B^2 y^2)^2 \]  \hspace{1cm} (43)
\[ \gamma_1 = \frac{1}{\rho^2} (A y \beta_2 + B \beta_1) \]
\[ \gamma_2 = \frac{1}{\rho^2} (Ay\beta_1 - B\beta_2) \]

\[ \delta_1 = \frac{1}{\rho^2} (B\beta_2 + C\beta_1) \]

\[ \delta_2 = \frac{1}{\rho^2} (B\beta_1 - C\beta_2) \]

Thus \( F(\bar{X}, b, i, j+1) \) can be evaluated for any \( j \), although it is a little complicated. The first term in the right-hand side of Equation (28) can be evaluated from the following recurrence relation

\[ \int x_1^m rdx_1 = \frac{x_1^{m-1}}{(m+2)A_1} r^3 - \frac{(2m+1)}{m+2} \frac{B_1}{A_1} \int x_1^{m-1} rdx_1 \frac{(m-1)C_1}{(m+2)A_1} \int x_1^{m-2} rdx_1 \]  

(44)

and

\[ \int rdx_1 = \frac{x_1 + \frac{B_1}{A_1}}{2} r + \frac{A_1 C_1 - B_1^2}{2A_1 \sqrt{A_1}} \log \left( \frac{A_1 x_1 + B_1}{\sqrt{A_1}} + r \right) \]

where \( A_1, B_1, \) and \( C_1 \) are the same as in Equation (28). Now we can evaluate \( F(\bar{X}, d, 2, j+1) \) in Equation (28) by using Equation (44) and \( F(\bar{X}, d, 1, j+1) \); then \( F(\bar{X}, d, 3, j+1) \) by using Equation (44), \( F(\bar{X}, d, 1, j+1) \), and \( F(\bar{X}, d, 2, j+1) \). By this process for \( F(\bar{X}, d, i+1, j+1) \) with any integer \( i \), we use Equation (44) and integrals \( F(\bar{X}, d, i-1, j+1) \) and \( F(\bar{X}, d, i, j+1) \). Although this process is quite complicated, it is far faster in computing time and more accurate than evaluating the singular integral by a numerical scheme.
EVALUATION OF DOWNWASH INTEGRAL

The downwash

\[
\frac{w}{U} = -\int \int m \frac{\partial}{\partial z} \frac{1}{r} \ dx_1 \ dz_1
\]

has a general term

\[
I_2 = \int \int x^i z^j \frac{\partial}{\partial z} \frac{1}{r} \ dx_1 \ dz_1
\]

When we exchange \((x, x_1)\) with \((z, z_1)\), they become exactly the same as the integral already evaluated for the integral equation. As was mentioned earlier, the evaluation of integrals on \(y = 0\) was a little simpler than on \(y \neq 0\) for the integral equation; however, for the downwash, we need integrals for \(y \neq 0\).
APPENDIX B

INTEGRATION OF LOGARITHMIC SINGULARITY

The numerical integration of a logarithmic singularity which arises in the pressure integration for drag, using the Simpson rule, is performed in the following way. We consider four values in an interval \((0, x)\) where the logarithmic singularity is located at, say, \(x = 0\)

\[ y(x_1), y(x_2), y(x_3), y(x_4) \]

We assume a function

\[ y(x) = a_1 + a_2 x + a_3 x^2 + a_4 \log x \tag{46} \]

Then we can obtain

\[ y(x) = \sum_{i=1}^{3} \left\{ y(x_i) - a_4 \log (x_i) \right\} \frac{1}{x(x_1)(x_2)(x_1-x_2)(x_2-x_3)(x_3-x_2)} + a_4 \log x \]

where

\[ \xi_1 = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} , \xi_2 = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} , \xi_3 = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \]

and

\[ a_4 = \frac{-\sum_{i=1}^{3} y_i \xi_i (x_4) + y(x_4)}{\log x_4 - \sum_{i=1}^{3} \log (x_i) \xi_i (x_4)} \tag{47} \]

Therefore, the integration from \(x = 0\) to \(x = x_1\)

\[ \int_{0}^{x_1} y(x)dx = \sum_{i=1}^{3} \left\{ y(x_i) - a_4 \log (x_i) \right\} \left[ L_1 (x) + a_4 x (\log x - 1) \right]_{0}^{x_1} \tag{48} \]
where

\[
L_1(x) = \frac{1}{(x_1 - x_2)(x_1 - x_3)} \left\{ \frac{1}{2} (x - x_2)^2 (x - x_3) - \frac{(x - x_2)^3}{6} \right\}
\]

\[
= \frac{(x - x_2)^2}{6(x_1 - x_2)(x_1 - x_3)} \left\{ 3(x - x_3) - (x - x_2) \right\} , \text{ etc.}
\]

This can be nicely incorporated with the integration from the Simpson rule, since the Simpson rule has also the same form for uneven intervals.
REFERENCES


## INITIAL DISTRIBUTION

<table>
<thead>
<tr>
<th>Copies</th>
<th>Copies</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>WES/LIBRARY</td>
</tr>
<tr>
<td>1</td>
<td>CHONR 438</td>
</tr>
<tr>
<td>1</td>
<td>NAVOCEANO 1640</td>
</tr>
<tr>
<td>2</td>
<td>NRL</td>
</tr>
<tr>
<td>1</td>
<td>2627 Library</td>
</tr>
<tr>
<td>1</td>
<td>Orlando</td>
</tr>
<tr>
<td>3</td>
<td>ONR</td>
</tr>
<tr>
<td>1</td>
<td>Boston</td>
</tr>
<tr>
<td>1</td>
<td>Chicago</td>
</tr>
<tr>
<td>1</td>
<td>Pasadena</td>
</tr>
<tr>
<td>1</td>
<td>USNA LIBRARY</td>
</tr>
<tr>
<td>1</td>
<td>NAVPGSCOL LIBRARY</td>
</tr>
<tr>
<td>1</td>
<td>NROTC &amp; NAVADMINU, MIT</td>
</tr>
<tr>
<td>1</td>
<td>NAVWARCOL</td>
</tr>
<tr>
<td>1</td>
<td>NAVAIRDEVcen ADL</td>
</tr>
<tr>
<td>1</td>
<td>NELC LIBRARY</td>
</tr>
<tr>
<td>4</td>
<td>NAVUSEACEN</td>
</tr>
<tr>
<td>1</td>
<td>San Diego</td>
</tr>
<tr>
<td>1</td>
<td>6005/Fabula</td>
</tr>
<tr>
<td>1</td>
<td>Pasadena</td>
</tr>
<tr>
<td>1</td>
<td>2501/Hoyt</td>
</tr>
<tr>
<td>1</td>
<td>NAVWPNScen</td>
</tr>
<tr>
<td>1</td>
<td>NAVCIVENGRLAB L31/LIB</td>
</tr>
<tr>
<td>1</td>
<td>NSWC WHITE OAK</td>
</tr>
<tr>
<td>1</td>
<td>NSWC DAHLGREN LAB/LIB</td>
</tr>
<tr>
<td>1</td>
<td>NUSC NPTLAB</td>
</tr>
<tr>
<td>1</td>
<td>NUSC NLONLAB</td>
</tr>
<tr>
<td>6</td>
<td>NAVSEA</td>
</tr>
<tr>
<td>1</td>
<td>SEA 03221, Larry Benen</td>
</tr>
<tr>
<td>1</td>
<td>SEA 033</td>
</tr>
<tr>
<td>1</td>
<td>SEA 0331</td>
</tr>
<tr>
<td>6</td>
<td>NAVSEA (Continued)</td>
</tr>
<tr>
<td>1</td>
<td>SEA 0372</td>
</tr>
<tr>
<td>1</td>
<td>SEA 038</td>
</tr>
<tr>
<td>1</td>
<td>SEA 09G32</td>
</tr>
<tr>
<td>1</td>
<td>NAVFAc 032C</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPyD CHASN/LIB</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPyD LBEACH/LIB</td>
</tr>
<tr>
<td>2</td>
<td>NAVSHIPyD MARE</td>
</tr>
<tr>
<td>1</td>
<td>Library</td>
</tr>
<tr>
<td>1</td>
<td>250</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPyD PEARL/LIB</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPyD PHILA 240</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPyD PTSmH/LIB</td>
</tr>
<tr>
<td>1</td>
<td>NAVSHIPyD BREM/LIB</td>
</tr>
<tr>
<td>7</td>
<td>NAVSEC</td>
</tr>
<tr>
<td>1</td>
<td>SEC 6034B</td>
</tr>
<tr>
<td>1</td>
<td>SEC 6110</td>
</tr>
<tr>
<td>1</td>
<td>SEC 6114H</td>
</tr>
<tr>
<td>1</td>
<td>SEC 6120</td>
</tr>
<tr>
<td>1</td>
<td>SEC 6136</td>
</tr>
<tr>
<td>1</td>
<td>SEC 6140B/Foncannon</td>
</tr>
<tr>
<td>1</td>
<td>SEC 6148</td>
</tr>
<tr>
<td>1</td>
<td>SUBASE NLON/SUB LIB</td>
</tr>
<tr>
<td>1</td>
<td>AFOSR/NAM</td>
</tr>
<tr>
<td>1</td>
<td>AFFDL/FYS/Olsen</td>
</tr>
<tr>
<td>12</td>
<td>DDC</td>
</tr>
<tr>
<td>1</td>
<td>BUSTAND/P. Klebanoff</td>
</tr>
<tr>
<td>1</td>
<td>LC/SCI &amp; TECH DIV</td>
</tr>
<tr>
<td>1</td>
<td>MMA LIB</td>
</tr>
<tr>
<td>1</td>
<td>MMA/MARITIME RES CEN</td>
</tr>
<tr>
<td>1</td>
<td>NSF ENGR DIV LIB</td>
</tr>
<tr>
<td>Copies</td>
<td>Institution</td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
</tr>
<tr>
<td>1</td>
<td>DOT Lib</td>
</tr>
<tr>
<td>1</td>
<td>U Bridgeport/Uram</td>
</tr>
<tr>
<td>1</td>
<td>U Cal Berkeley/Dept Name</td>
</tr>
<tr>
<td>3</td>
<td>U Cal Name</td>
</tr>
<tr>
<td>1</td>
<td>U Cal San Diego/Ellis</td>
</tr>
<tr>
<td>1</td>
<td>U Cal Scripps Lib</td>
</tr>
<tr>
<td>3</td>
<td>CIT</td>
</tr>
<tr>
<td>1</td>
<td>Catholic U</td>
</tr>
<tr>
<td>1</td>
<td>Colorado State U Engr Res Cen</td>
</tr>
<tr>
<td>1</td>
<td>U Connecticut/Scottron</td>
</tr>
<tr>
<td>1</td>
<td>Cornell/Sears</td>
</tr>
<tr>
<td>1</td>
<td>Florida Atlantic U OE Lib</td>
</tr>
<tr>
<td>2</td>
<td>Harvard U</td>
</tr>
<tr>
<td>1</td>
<td>U Hawaii/Bretschneider</td>
</tr>
<tr>
<td>1</td>
<td>U Illinois/Robertson</td>
</tr>
<tr>
<td>1</td>
<td>U Iowa Inst Hydru Res Lib</td>
</tr>
<tr>
<td>2</td>
<td>U Iowa IHR</td>
</tr>
<tr>
<td>1</td>
<td>Johns Hopkins U/Phillips</td>
</tr>
<tr>
<td>1</td>
<td>U Kansas CIV Engr Lib</td>
</tr>
<tr>
<td>1</td>
<td>Kansas St U Engr Exp/Nesmith</td>
</tr>
<tr>
<td>1</td>
<td>Long Island U/Price</td>
</tr>
<tr>
<td>6</td>
<td>MIT Ocean Engr</td>
</tr>
<tr>
<td>1</td>
<td>MIT Parsons Lab/Ippen</td>
</tr>
<tr>
<td>4</td>
<td>U Michigan Name</td>
</tr>
<tr>
<td>1</td>
<td>U Michigan Willow Run Labs</td>
</tr>
<tr>
<td>4</td>
<td>U Minnesota SAFHL</td>
</tr>
<tr>
<td>2</td>
<td>Notre Dame</td>
</tr>
<tr>
<td>1</td>
<td>New York U/Pierson</td>
</tr>
<tr>
<td>2</td>
<td>NYU Courant Inst</td>
</tr>
<tr>
<td>1</td>
<td>Penn State U ARL Lib</td>
</tr>
<tr>
<td>1</td>
<td>Princeton U/Mellor</td>
</tr>
<tr>
<td>1</td>
<td>St Johns U/Lurye</td>
</tr>
<tr>
<td>1</td>
<td>SWRI Applied Mech Review</td>
</tr>
<tr>
<td>1</td>
<td>SWRI/Abramson</td>
</tr>
<tr>
<td>2</td>
<td>Stanford U</td>
</tr>
<tr>
<td>1</td>
<td>Stanford Res Inst Lib</td>
</tr>
</tbody>
</table>
Copies | Copies
---|---
3 | 1
SIT DAVIDSON LAB
1 Lib
1 Breslin
1 Tsakonas

1 TEXAS U ARL LIB
1 UTAH STATE U/Jeppson
1 U WASHINGTON APL LIB

2 WEBB INST
1 Lewis
1 Ward

1 WHOI OCEAN ENGR DEPT
1 WPI ALDEN HYDR LAB LIB

1 SNAME
1 BETHLEHEM STEEL NEW YORK/LIB
1 BETHLEHEM STEEL SPARROWS

2 BOLT BERANEK AND NEWMAN
1 Lib
1 Jackson

1 CAMBRIDGE ACOUS/Junger
1 CORNELL AERO LAB/Ritter
1 EASTERN RES GROUP

1 ESSO DES DIV
1 GEN DYN ELEC BOAT/Boatwright

1 GIBBS & COX
1 HYDRONAUTICS LIB
1 LOCKHEED M&S/Waid

2 DOUGLAS AIRCRAFT
1 Hess
1 Smith

1 NEWPORT NEWS SHIPBUILDING LIB

1 NIELSEN ENGR/Spangler
1 NAR SPACE/Ujihara
1 OCEANICS
1 SPERRY SYS MGMT LIB
1 ROBERT TAGGART
1 TRACOR

CENTER DISTRIBUTION

Copies Code
---|---
1 | 11
W. M. Ellsworth

1 | 115
R. J. Johnston

1 | 1151
W. C. O'Neill

1 | 1170
D. A. Jewel

1 | 1154
HYSTU

1 | 15
W.E. Cummins

1 | 1500
J.B. Hadler

1 | 1504
V. J. Monacella

1 | 1505
S. F. Crump

1 | 1506
M. K. Ochi

1 | 1507
D. Cieslowski

1 | 152
R. Wermter

1 | 1521
P. C. Pien

1 | 1532
G. F. Dobay

1 | 154
W. B. Morgan

1 | 1542
B. Yim

1 | 1544
R. A. Cumming

1 | 1552
J. H. McCarthy, Jr.

1 | 1556
G. L. Santore

1 | 156
G. R. Hagen

1 | 1562
M. Martin

1 | 1564
J. P. Feldman

1 | 1568
G. G. Cox

1 | 1572
C. M. Lee

1 | 17
W. W. Murray

43
<table>
<thead>
<tr>
<th>Copies</th>
<th>Code</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18</td>
<td>G. H. Gleissner</td>
</tr>
<tr>
<td>1</td>
<td>1843</td>
<td>J. W. Schot</td>
</tr>
<tr>
<td>1</td>
<td>19</td>
<td>M. M. Sevik</td>
</tr>
<tr>
<td>30</td>
<td>5211</td>
<td>Reports Distribution</td>
</tr>
<tr>
<td>1</td>
<td>5221</td>
<td>Library (C)</td>
</tr>
<tr>
<td>1</td>
<td>5222</td>
<td>Library (A)</td>
</tr>
</tbody>
</table>