THE LANDING APPROACH IN VARIABLE WINDS:
CURVED GLIDEPATH GEOMETRIES AND WORST-CASE WIND MODELING

by

Alexander B. Markov

TECHNISCHE HOGESCHOOL DELFT
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ABSTRACT

Both analytical and computational techniques that may be applied to predicting aircraft response to hazardous variable winds encountered on the landing approach and to modeling wind conditions have been investigated. An extensive literature review has identified a number of areas requiring further study. Of these areas, worst-case wind modeling techniques and modified glidepath geometries based on an estimate of the existing wind profile have been pursued extensively.

A nonlinear longitudinal and lateral dynamic model is posed for a twin engined STOL transport using look-up table, quasisteady, nonlinear aerodynamics. The aircraft controller is a feedback controller synthesized using linear state feedback optimal control theory for a suitable linearization of the general equations of motion. This dynamic model is employed in computer simulations predicting the aircraft dynamic behavior flying curved glidepath geometries that are based on an a priori estimate of the existing wind profile. These results are assessed to determine the effects of incorrect estimates of the wind profile and the suitability of the kinematic assumptions used in deriving the curved glidepath geometry.

Worst-case wind modeling techniques where the form of the wind model is not specified a priori are considered in detail, and extensions to an existing technique are proposed. New techniques are developed using more general functional maximization techniques, and in particular the worst-case wind modeling problem is viewed as a conflict of interest between the aircraft controller and a wind controller. Numerical examples of the application of the latter class of worst-case techniques are given for several formulations posed as linear quadratic problems, and two of these worst-case wind models are applied to the curved glidepath geometry problem. Some discussion is also included of the use of these methods in the certification process and in flight simulators; preliminary results from a flight simulator application are presented.

A number of recommendations for future work, based both on the results of the numerical simulations and on the literature review, are included.
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LIST OF SYMBOLS

The following is a list of major symbols. Some symbols, which are defined in the text and are used only once, or are secondary quantities related to a primary quantity in a way that is apparent from the text or from the notation conventions discussed in Appendix A, are not included. Aerodynamic derivative conventions are discussed in Appendix A, and thus aerodynamic derivatives are also not included. Numbers in parentheses refer to equations.

A Aerodynamic forces applied to the aircraft including thrust forces
AR Aspect ratio
b Wing span, m
\( C_D \) Drag coefficient including component of thrust in drag direction
\( C_{D1} \) Drag contribution due to \( \delta_F, \alpha_f \), see (3.5,20)
\( C_{D2} \) Drag contribution due to \( \dot{\alpha}_f, q_B \) and \( \delta_E \), see (3.5,19)
\( C_L \) Lift coefficient
\( C_{L1} \) Lift contribution due to \( \delta_F, \alpha_f \), see (3.5,19)
\( C_{L2} \) Lift contribution due to \( \dot{\alpha}_f, q_B \) and \( \delta_E \), see (3.5,19)
\( C_{B1} \) Drag contribution due to \( \delta_F, C_T \) and \( \alpha_f \), see (3.5,20)
\( C_{B2} \) Drag contribution due to \( \dot{\alpha}_f, q_B \) and \( \delta_E \), see (3.5,19)
\( C_{B3} \) Pitching moment contribution due \( \dot{\alpha}_f, q_B \) and \( \delta_E \), see (3.5,21)
\( C_{B4} \) Pitching moment contribution due \( \delta_F, C_T \) and \( \alpha_f \), see (3.5,21)

\( M \) thrust coefficient
\( C_w \) weight coefficient
\( C_{x1} \) pitching moment contribution due \( \dot{\alpha}_f, q_B \) and \( \delta_E \)
\( C_{x2} \) pitching moment contribution due \( \delta_F, C_T \) and \( \alpha_f \)
\( C_{y1} \) pitching moment contribution due \( \dot{\alpha}_f, q_B \) and \( \delta_E \)
\( C_{y2} \) pitching moment contribution due \( \delta_F, C_T \) and \( \alpha_f \)
\( C_{z1} \) pitching moment contribution due \( \dot{\alpha}_f, q_B \) and \( \delta_E \)
\( C_{z2} \) pitching moment contribution due \( \delta_F, C_T \) and \( \alpha_f \)

\( D \) Drag
\( D^* \) Drag including component of thrust in drag direction
\( x \) Aircraft centre of mass position perpendicular to conventional glidepath plane, positive above, m

\( C_{n1} \) Pitching moment contribution due \( \dot{\alpha}_f, q_B \) and \( \delta_E \), see (3.5,21)
\( C_{n2} \) Pitching moment contribution due \( \delta_F, C_T \) and \( \alpha_f \), see (3.5,21)
\( C_{n3} \) Pitching moment contribution due \( \dot{\alpha}_f, q_B \) and \( \delta_E \), see (3.5,21)
\( C_{n4} \) Pitching moment contribution due \( \delta_F, C_T \) and \( \alpha_f \), see (3.5,21)
\( C_{n5} \) Pitching moment contribution due \( \dot{\alpha}_f, q_B \) and \( \delta_E \), see (3.5,21)
\( C_{n6} \) Pitching moment contribution due \( \delta_F, C_T \) and \( \alpha_f \), see (3.5,21)

\( \bar{c} \) Mean aerodynamic chord, m
Position of curved glidepath perpendicular to conventional glidepath plane, positive above, m
Aircraft centre of mass position perpendicular to local curved glidepath plane
Isoperimetric constraint, (5.2.9) or (5.4.64)
Isoperimetric constraint on k-th disturbance input, van der Vaart's method, (5.1.11)
External force vector acting at aircraft centre of mass
General linear system matrix
Body-fixed reference frame, see Fig. 3.2
General Earth-fixed reference frame
Earth-fixed runway reference frame (see Fig. 2.2)
Glide slope reference frame, see Fig. 3.1
Inertial reference frame, see Fig. 3.1
Stability axes, see Fig. 3.2
Wind axes, see Fig. 3.3
Modified wind axes, see Fig. 3.3
Right hand side of general, nonlinear dynamic system or forcing time function for a linear dynamic system, depending on the context
General linear system control distribution matrix
General linear system disturbance distribution matrix
Gravity vector
Acceleration due to gravity
Integrand of general payoff functional
Vehicle angular momentum about centre-of-mass
Impulsive response matrix of the state of a general, time-varying linear system, see (5.2.5)
Altitude above ground level, m
Average rate of descent on a given approach curved glidepath geometry altitude above ground level, (3.7.11), m
Initial curved glidepath altitude, m
Decision height, m
Impulsive response function of i-th state variable due to k-th input
Transition height, m
Aircraft inertia matrix written with respect to $F_B$, (3.2.35)
Aircraft inertia matrix written with respect to $F_S$, (3.2.36)
Aircraft inertia components from $I_S$
Aircraft inertia components from $I_B$
Orthonormal dextral vector triad for $F_A$
Performance criterion or payoff functional
Payoff functional (5.2.8a)
Payoff functional (5.2.8b)
Impulsive response matrix of the output of a general, time-varying linear system, see (5.2.6a)
Impulsive response function of the i-th output variable due to the k-th input
Aerodynamic rolling moment about centre of mass in $F_S$, or lift, depending on the context
Aerodynamic rolling moment about centre of mass in $F_B$
L^*  Lift plus component of thrust in the lift direction
L_{BI}  Rotation matrix relating vector components in F_I to those in F_B
L_{BS}  Rotation matrix relating vector components in F_S to those in F_B
L_{BW}  Rotation matrix relating vector components in F_W to those in F_B
L_{BW}' Rotation matrix relating vector components in F_W' to those in F_B
L_{SI}  Rotation matrix relating vector components in F_I to those in F_S
L_{SW}  Rotation matrix relating vector components in F_W to those in F_S
L_{GL}  Rotation matrix relating vector components in F_I to those in F_GL
L_{AB}  The (i,j)-th element of L_{AB}
L_H  Horizontal tail moment arm
L_V  Vertical tail moment arm
M  Aerodynamic moment about the aircraft centre of mass, including thrust effects
M_{a}  Aerodynamic pitching moment about the centre of mass in F_S
M_{b}  Aerodynamic pitching moment about the centre of mass in F_B
m  Aircraft mass, kg
N  Aerodynamic yawing moment about the centre of mass in F_S
N_{b}  Aerodynamic yawing moment about the centre of mass in F_B
n  System order
n_A  Order of system optimized
P  Aircraft linear momentum vector
P  Roll rate in F_S (right wing down positive), rad./sec or dimension of disturbance vector D
P_B  Roll rate in F_B (right wind down positive), rad./sec
q  Pitch rate in F_S (nose up positive), rad./sec
q_{B}  Pitch rate in F_B (nose up positive), rad./sec
q_{i}  Aircraft centre of mass position vector relative to F_I
r  Yaw rate in F_S (nose right positive), rad./sec
r_{B}  Yaw rate in F_B (nose right positive), rad./sec
S  Isoperimetric constraint (5.465) or wing reference area (m^2) depending on the context
S_{WS}  Isoperimetric constraint in terms of wind velocity rate of change
s  Laplace variable
s \{x(t_f), t_T\}  General payoff functional terminal state weighting
T  Thrust vector
T_{e}  Throttle lag time constant
T  Time, sec.
t  terminal time
T  Initial time
t_T  Transition height time
t_w  Time constant associated with linear quadratic weighting conditions, see Table 5.1
(u,v,w)  Components of \ V in F_S
(u_B,v_B,w_B)  Components of \ V in F_B
(u_{E_{E}},v_{E_{E}},w_{E_{E}})  Components of \ V_E in F_B
Component of $V_E$ in $F_S$

Airspeed vector

Airspeed vector on which curved approach is based

Velocity of aircraft centre of mass with respect to $F_E (F_I)$

Ground speed vector on which curved approach is based

Component of $V$ in aircraft plane of symmetry

Approach airspeed

Curved approach airspeed

Linearization reference equilibrium airspeed

Wind velocity vector with respect to $F_E (F_I)$

Wind velocity vector with respect to $F_E (F_I)$ on which curved glidepath geometry is based

Linear mean wind velocity vector with respect to $F_E (F_I)$

Mean wind velocity vector with respect to $F_E (F_I)$

Vertical wind velocity vector with respect to $F_E (F_I)$

Turbulence velocity vector with respect to $F_E (F_I)$

Horizontal wind speed, $ms^{-1}$

$k_{ER}$ component of $W_V$, $ms^{-1}$

Components of $W_E$ in $F_I$

Components of $W_C$ in $F_I$

Components of $W_L$ in $F_I$

Components of $W_M$ in $F_I$

Linearization reference equilibrium value of $W_1$

Linearization reference equilibrium value of $W_2$

Components of $\frac{d}{dt}$ in $F_I$

Components of $\frac{d}{dt}$ in $F_S$

Components of $\frac{d}{dt}$ in $F_B$

Spatial coordinates in $F_I$

$\mathbf{R}$ components in $F_E$

$\mathbf{R}$ components in $F_{ER}$

$\mathbf{R}$ components in $F_{GL}$

$\mathbf{R}$ components in $F_I$

Curved glidepath coordinates in $F_I$

System output

Angle of attack of the aircraft's zero lift line, see Fig. 3.3

Angle of attack of the aircraft's fuselage reference line, see Fig. 3.3

Angle between fuselage reference line and zero lift line, see Fig. 3.3

Angle of attack of $\frac{d}{dt}$

Sideslip angle

The angle $\gamma$ makes with respect to the horizontal, positive downwards

Conventional glide slope angle with respect to the horizontal

Curved glide slope angle with respect to the horizontal

Perturbation state vector

Perturbation state vector of system which is optimized
\[ \Delta \delta \]
Perturbation control column matrix

\[ \Delta W \]
Perturbation wind column matrix

\[ \delta_A \]
Mean aileron deflection, right aileron up, left down positive, rad.

\[ \delta_E \]
Elevator deflection, trailing edge down positive, rad.

\[ \delta_F \]
Flap setting, deg.

\[ \delta_R \]
Rudder deflection, left rudder positive, rad.

\[ \delta_T \]
Throttle setting measured as a fraction of full throttle

\[ \delta_C_T \]
Commanded throttle setting

\[ \delta_a \]
Angle between \( T_e \) and \( V_e \), see (D.1,21), rad.

\[ \epsilon_T \]
Angle thrust vector makes with the fuselage reference line, positive upwards, rad.

\[ \zeta \]
Wind direction relative to runway centre line, see Fig. 2.2

\[ D \]
General disturbance column matrix

\[ \Theta, \Theta_1, \Theta_2 \]
General linear system disturbance distribution matrices

\[ \theta \]
Euler elevation angle for \( F_S \), rad.

\[ \theta_B \]
Euler elevation angle for \( F_B \), rad.

\[ \theta_c \]
Euler elevation angle for curved approach

\[ \lambda(t) \]
Column matrix of influence functions (Lagrangian multipliers)

\[ \nu, \mu_1, \mu_2 \]
Parameters to be determined so that the isoperimetric constraints are satisfied

\[ \rho \]
Density of air, kg./m³

\[ \tau \]
Time, sec

\[ \Psi(t) \]
Angular velocity of \( F_B \) with respect to \( F_I \), rad./s
### Frequently Used Acronyms, Superscripts and Subscripts

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>A</td>
<td>Subscript - ailerons</td>
</tr>
<tr>
<td>ABL</td>
<td>Atmospheric boundary layer</td>
</tr>
<tr>
<td>AS</td>
<td>Airspeed</td>
</tr>
<tr>
<td>B</td>
<td>Subscript or superscript - body-fixed reference frame</td>
</tr>
<tr>
<td>CATA</td>
<td>Canadian Air Transportation Administration</td>
</tr>
<tr>
<td>CAT II</td>
<td>Category II</td>
</tr>
<tr>
<td>CGA</td>
<td>Curved glidepath approach(s)</td>
</tr>
<tr>
<td>CGG</td>
<td>Curved glidepath geometry (geometries)</td>
</tr>
<tr>
<td>CTOL</td>
<td>Conventional take-off and landing</td>
</tr>
<tr>
<td>C</td>
<td>Subscript - curved glidepath approach or crosswind, depending on the context</td>
</tr>
<tr>
<td></td>
<td>Superscript - commanded</td>
</tr>
<tr>
<td>D</td>
<td>Subscript - decision height or drag, depending on the context</td>
</tr>
<tr>
<td>DME</td>
<td>Distance measuring equipment</td>
</tr>
<tr>
<td>DG</td>
<td>Differential game(s)</td>
</tr>
<tr>
<td>DH</td>
<td>Decision height</td>
</tr>
<tr>
<td>d</td>
<td>Subscript - desired or downdraft, depending on the context</td>
</tr>
<tr>
<td>E</td>
<td>Subscript - Earth-fixed reference frame or elevator, depending on the context</td>
</tr>
<tr>
<td></td>
<td>Superscript - Earth-fixed reference frame</td>
</tr>
<tr>
<td>ER</td>
<td>Subscript or superscript - Earth-fixed runway reference frame</td>
</tr>
<tr>
<td>e</td>
<td>Subscript - linearization reference equilibrium</td>
</tr>
<tr>
<td>F</td>
<td>Subscript - flare or flap, depending on the context</td>
</tr>
<tr>
<td>FAA</td>
<td>Federal Aviation Administration (USA)</td>
</tr>
<tr>
<td>f</td>
<td>Subscript - final or fuselage reference line, depending on the context</td>
</tr>
<tr>
<td>frl</td>
<td>Fuselage reference line</td>
</tr>
<tr>
<td>G</td>
<td>Subscript - gradient or glidepath, depending on the context</td>
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<tr>
<td>GL</td>
<td>Subscript or superscript - glideslope - localizer</td>
</tr>
<tr>
<td>GP</td>
<td>Glidepath</td>
</tr>
<tr>
<td>GTSP</td>
<td>Game-theoretic saddle point</td>
</tr>
<tr>
<td>H-J-B</td>
<td>Hamilton - Jacobi - Bellman</td>
</tr>
<tr>
<td>I</td>
<td>Subscript or superscript - inertial</td>
</tr>
<tr>
<td>IAS</td>
<td>Indicated airspeed</td>
</tr>
<tr>
<td>ICAO</td>
<td>International Civil Aviation Organization</td>
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<tr>
<td>ILS</td>
<td>Instrument Landing System</td>
</tr>
<tr>
<td>IMSL</td>
<td>International Mathematical and Statistical Libraries</td>
</tr>
<tr>
<td>IRWC</td>
<td>Impulsive response worst-case</td>
</tr>
<tr>
<td>L</td>
<td>Subscript - linear or lift, depending on the context</td>
</tr>
<tr>
<td>LQTPZS</td>
<td>Linear quadratic two player zero-sum</td>
</tr>
<tr>
<td>M</td>
<td>Subscript - mean</td>
</tr>
<tr>
<td>MLS</td>
<td>Microwave Landing System</td>
</tr>
<tr>
<td>MM</td>
<td>Middle marker</td>
</tr>
<tr>
<td>Max</td>
<td>Maximum</td>
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<tr>
<td>Min</td>
<td>Minimum</td>
</tr>
<tr>
<td>NASA</td>
<td>National Aeronautics and Space Administration</td>
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<td>National Oceanic and Atmospheric Administration</td>
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<tr>
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<td>National Severe Storms Laboratory</td>
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<td>NTSB</td>
<td>National Transportation Safety Board</td>
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<td>OM</td>
<td>Outer marker</td>
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Subscript initial
Subscript perverse
Subscript phugoid
Subscript rudder
Runway visual range
Subscript or superscript stability axes reference frame
Stability augmentation system
State linearized
State-disturbance linearized
Subscript short-period
Short take-off and landing
Sign function, see Appendix A
Subscript transition height or throttle depending on the context
University of Toronto Institute for Aerospace Studies
Subscript vertical
Very high frequency omni range
Subscript or superscript wind axes
Subscript or superscript modified wind axes
World Meteorological Organization
Subscript descriptor for moment of inertia
Subscript aircraft plane of symmetry or descriptor for moments of inertia depending on the context
Subscript descriptor for moment of inertia
Subscript descriptor for moment of inertia
1. INTRODUCTION

1.1 Historical Perspective and Motivation

The aerodynamic forces and moments that affect atmospheric flight vehicles are functions of many factors, including the shape of the vehicle, certain nondimensional quantities (e.g. Reynolds number, Prandtl number and Mach number), flow effects which are induced due to control and propulsion elements (e.g. rotors and propellers), and the wind field that the aircraft encounters as it moves through the atmosphere. While the early aviation pioneers had to directly or indirectly deal with each of these factors, they were probably dominantly concerned with actions which could be taken to counteract variable wind effects. This awareness was primarily a consequence of the sometimes catastrophic results of flight in variable winds, many aviators having lost their lives because of a poor understanding of its effects, and of methods for designing 'wind-proof' vehicles. (For example, in 1896 Otto Lilienthal was killed when his hang glider suffered a turbulence upset [1.1].)

The history of the engineering study of the dynamic effects of variable winds on aircraft response is almost as long as the era of powered flight. In 1915 Wilson [1.2] considered the analysis of aircraft response to discrete gusts (see also the review by Etkin [1.3]). Much of the work in the years preceding World War II focused on the gust loads placed on aircraft structures, with Donley, as reported by Taylor [1.4], being a leading authority. Relatively little was being done on investigating low altitude wind effects; the flight test work of Thompson, Peck and Beard [1.5], carried out in the middle thirties, was an exception.

After World War II investigations into the low altitude effects of variable winds continued to be relatively uncommon, much of the focus having turned to the development of spectral and correlation techniques for studying response to continuous turbulence (see the excellent reviews of Taylor [1.4] and Etkin [1.3]). One pioneering exception, appearing in the late forties, is Etkin's analysis of low altitude linear spatial wind gradient effects on glide and climb [1.6]. In the early sixties, Grosso [1.7] and Zbrozek [1.8] also considered wind gradient effects, but at high altitude.

During this period research on the hazardous effects of strong variable winds was concerned with high altitude gust encounters, and in particular on what eventually became to be known as the "jet upset" problem (Bisgood and Burnham [1.9], Burnham [1.10], Perry and Burnham [1.11], Strong [1.12], and Newberry [1.13]). Until the middle sixties low altitude wind encounters were generally not considered to be hazardous. In 1961 Browne [1.14] discussed operational considerations, relevant to large aircraft, on maintaining airspeed and path control in abnormal wind shear encounters, but concluded that with immediate adjustments to thrust and flight control settings, "safe and precise" flight was assured. In 1964
1.3

Burnham [1.15] suggested that low level wind gradients were of concern only to light aircraft pilots during take-off.

Such views of the hazards of low altitude winds began to change in the later sixties and seventies for a number of reasons. The jet upset problem was recognized as a wind instigated control problem rather than a structural problem. The effects of these severe gust encounters, even on large aircraft, were quite disconcerting and indirectly turned attention towards low altitude wind problems where the presence of the ground tends to magnify the associated risks.

Perhaps a second cause for these changing views can be found in the growing number of jet aircraft. This brought an increased awareness of the differences between the response characteristics of jet aircraft and propeller aircraft. Some of these characteristics (for example the longer thrust lags in response to control inputs associated with pure jet and turbofan engines) would tend to increase the risks associated with low level variable wind encounters.

Finally, and probably most importantly, this growing awareness was spurred by a number of low altitude aircraft incidents and accidents where variable winds were suspected of being contributing factors. By the middle seventies it was conclusively shown that severe variable wind encounters had been major contributing factors in some of these accidents (Laynor and Roberts [1.16], and the NTSB accident reports [1.17], [1.18]). In retrospect, Shrager [1.19] has concluded that low altitude variable winds may have been contributing factors in a number of accidents where winds had previously not been considered to be a factor.

Table 1.1, compiled from Refs. 1.15 through to 1.21, lists some of these accidents. It is apparent that these variable wind effects can be hazardous both during the final approach and during the take-off phases of flight. They all have the common denominator of wind induced airspeed instability and sudden, sustained losses of lift.

Such accidents, and the realization that low altitude variable wind effects would have to be assessed before Category III landing control systems could be certified for commercial use (Gera [1.23]), has resulted in an explosion of papers dealing with the many facets of this problem. These include work on atmospheric boundary layer wind modeling, digital simulation, design of 'wind-proof' aircraft controllers, control techniques for flying through severe low level variable winds, training and research simulators, and methods of warning aircraft of hazardous variable wind conditions. A representative sampling of the literature is given in Refs. 1.24 to 1.51, and will be discussed in more detail in the sequel.

1.2 Objectives and Scope

From the outset the scope of this research was restricted largely to the study of analytical and computational techniques which may be applied to evaluating aircraft system response to hazardous variable wind conditions encountered on the landing approach. Even with this restriction the number of possible branches which the work could have taken were many.
The elements of the general problem of the landing approach flight dynamics in the presence of variable winds are summarized in Fig. 1.1, and include not only the traditional flight dynamics areas of airframe and controller dynamics, but also the problems of suitably modeling the external navigation and guidance inputs and the variable wind disturbances. While all of these elements had to be considered to some extent in this investigation, three are especially relevant to studying the gross rigid body dynamics of aircraft exposed to hazardous wind conditions. These are the airframe dynamics, with the attendant considerations on aerodynamic modeling, the controller dynamics, and the variable wind models. All three are considered in depth in the sequel in the context of the following specific objectives:

1. To review the results available in the literature on the hazardous effects of low level variable winds.
2. To formulate a flight dynamics model suitable for studying aircraft response to hazardous variable wind encounters, wherever possible drawing on existing models.
3. To apply this model to assessing a class of curved glidepath landing approaches in the presence of variable winds.
4. To review and develop techniques of generating worst-case wind models, and to apply some of these models to the curved glidepath assessment.

Objectives (3) and (4) are discussed in more detail in the following subsections.

1.2.1 Curved Glidepath Approach Trajectories

Aircraft precision instrument approaches are normally flown tracking a rectilinear ground fixed reference trajectory. This is a consequence of two practical considerations:

1. This type of approach is the simplest to fly in the presence of a uniform wind although this is not necessarily true in variable winds. (See the comments to follow on pilot work load for certain types of curved approaches.)
2. Until recently limitations in the type of approach aids available made precision curved approaches impossible.

Developments in microwave landing system technology (Brown, Burrows, Goka and Park [1.52], Reed [1.53], and Interscan Australia Ltd. [1.54]), however, have eliminated the second reason and have given rise to the possibility of accurately defining curved approach trajectories. These might be required for various reasons, including (1) limiting the size of terminal area approach corridors, (2) avoiding obstructions on the approach, (3) avoiding passing over populated areas in order to adhere to noise abatement procedures, and (4) permitting multipath approaches and thus reducing terminal congestion. Since the early seventies a considerable amount of work has been carried out on such horizontal curved approach trajectories, particularly in the United States (Benner, McLaughlin et al. [1.55], Porter [1.56]...
and Sherman [1.57]). Most of the emphasis has been on simulator
evaluation, pilot display requirements and acceptable
tolerances.

This new landing aid technology has also opened the doors
to a somewhat different type of a curved approach trajectory,
namely the curved approach in the vertical plane (see Fig. 1.2).
The literature has occasionally made reference to such trajec-
tories (e.g. the "ideal descending route" of Ling [1.58]; see
also the discussion by Brockhaus [1.59] with regard to steep
STOL approaches). These might be required for a variety of
reasons including constant rate of descent procedures for
portions of the approach (Ref. 1.59, p. 10) and a steep initial
approach gradually transitioning to a shallow approach for the
last segment before landing; in this sense, such a curved
approach is just an extension of a two segment conventional
approach (see Fig. 1.2). The latter combines the obstacle
clearance advantages of a steep approach with the safety of a
conventional shallow approach for the final phase before
landing, while the constant rate of descent technique may lead
to a curved trajectory in ground coordinates because of
variable wind conditions, although this is not specifically
noted by Brockhaus. Such a curved flight path may also be
the result of the slow transition from a constant rate of
descent approach segment to a conventional glide slope.

Of interest to this investigation is the usefulness
of such vertical plane modified approach geometries in the
presence of variable winds (for brevity these approach
trajectories will be referred to as curved glidepath geometries
(CGG) or curved glidepath approaches (CGA)). Preliminary
study of the problem from this viewpoint was undertaken by
Hindson and Gould [1.60] in 1974. Their results suggested
that CGG may be desirable in the presence of wind gradients
in order to alleviate pilot workload, particularly for V/STOL
aircraft. This was followed by flight test evaluations in
the NAE airborne simulator in 1975-76, with encouraging
results (Hindson and Smith [1.61]).

The source of Hindson's and Gould's proposal was the
observation that vertical wind gradients of the horizontal
wind forces the pilot to continuously adjust his rate of
descent on the landing approach. For a constant airspeed
approach this implies continuously changing the throttle
settings. If there is complete, or even partial, a priori
knowledge of the wind profile (as might be provided by remote
sensing systems currently under development (Jerome [1.33],
Beaulieu [1.42], Beran [1.62], Beran, Hall, et al. [1.63],
and McCarthy, Blick and Elmore [1.64])), then it may be
possible to determine a flight trajectory which will permit
the pilot to fly an almost constant rate of descent approach
in tracking this trajectory. For a given airspeed such
approaches are nearly constant attitude and constant throttle
and consequently the need for pilot control inputs is
minimized.* Depending on the wind profile, this trajectory

* Beser [1.50] considers approach guidance for a VTOL aircraft
executing spiral approaches in the presence of winds, and
compares autopilot performance for cases where an accurate
estimate of the wind field is available to the autopilot to
cases where such estimates are not available or are incorrect.
The wind compensation in this technique is incorporated in
the control inputs rather than in the definition of the
reference approach flight path.
may be markedly curved in ground referenced coordinates.

Such CGG are most practically defined on the basis of kinematic considerations to be discussed in detail in Chapter 3. These kinematic considerations involve certain assumptions whose validity must be tested in the dynamic context. The dynamic model posed in this study is used to determine the level of variable wind activity at which the assumptions break down, in a practical sense, as well as assessing the effects of incorrect or only partial knowledge of the wind profile. These aspects of wind-based CGG were not considered in the NAE work ([1.60], [1.61]).

1.2.2 Worst-Case Wind Models

In studying aircraft response to variable winds, frequent use is made of stochastic techniques. Occasional extensions and modifications have been tried to permit application of these techniques to the landing approach through the atmospheric boundary layer where the turbulence characteristics are not isotropic or stationary, e.g. as by Reid, Markov and Graf [1.39] and Markov and Reid [1.65]. These techniques yield expectation values for the state of the aircraft, and while this information is useful in dealing with some important areas (e.g. fatigue loading and ride quality), it says very little about wind realizations that lie in the tails of the probability distributions. These winds have a low probability of occurring, and thus contribute very little to the expectation values. Nevertheless, when they do occur the results may be quite catastrophic.

Stochastic techniques are thus of limited usefulness with regard to the particular problems of investigating aircraft response to hazardous variable winds, of developing control methods and approach trajectories for reducing the hazards and pilot workload of flight in variable winds, and of providing wind models for training simulators and certain aspects of the certification process. The alternative is to consider deterministic (discrete) wind models which may be viewed as one realization of the overall wind field. Methods of developing such models are discussed in detail in Chapter 2, and include models based on physical considerations of the atmospheric boundary layer flows, as well as models based on somewhat more arbitrary considerations. Of particular relevance to this study, in view of the focus on hazardous wind effects, are worst-case deterministic wind models. Such models appear to be relatively unexplored, particularly for the case where the form of the wind inputs is not specified a priori. The latter is a problem in functional maximization, and clearly requires that certain types of constraints are placed on the candidate wind inputs. A technique for solving this problem for linear, time-invariant aircraft dynamic models has recently been proposed by van der Vaart, Jonkers and Kappetijn [1.66] and van der Vaart [1.67]. This technique was apparently developed independently of similar techniques used by Drenick [1.68] in 1970, as reported by Jones [1.69], to study worst-case seismic response of structures. Jones' review also indicates
that there is a growing interest in these worst-case techniques.

In this study the focus will be on worst-case techniques where the form of the wind input is not specified a priori. Van der Vaart's technique is reviewed, a number of extensions to it are developed, and some alternative formulations in a conflict of interest scenario are proposed.

1.3 Overview

Chapter 2 defines a number of wind related concepts and reviews the state of the art in deterministic wind modeling for all but the worst-case methods. A number of deterministic wind models which will be used in the CGG study are defined in Section 2.2.

Chapter 3 considers the aircraft equations of motion. A number of fundamental assumptions are made (Section 3.1 and Appendix B), and under these assumptions general nonlinear equations are derived using a Newton-Euler development (Sections 3.2 and 3.3). These equations are simplified further to incorporate the uniform gust (point) approximation and nonlinear, quasisteady, aerodynamics after consideration of a number of aerodynamic model and actuator dynamics model alternatives (Sections 3.4, 3.5, 3.6, and 3.8). Consideration is also given to linearized equations of motion (Sections 3.4 and 3.10). Such linearized equations are required in parts of this investigation. In particular, this includes the controller synthesis and worst-case wind modeling chapters. Finally, wind based CGG trajectories are considered in Section 3.7 and Appendix E.

Chapter 4 considers aircraft controller synthesis techniques from both the linear and nonlinear (Section 4.2) and optimal and classical (Section 4.3) perspectives and justifies the decision to proceed with linear optimal techniques. Optimal control theory is briefly reviewed in Appendix F, with the emphasis being on linear quadratic state feedback formulations. Such linear quadratic methods are discussed in the context of a number of controller modes, including airspeed-attitude hold controllers, lateral stability augmentation and guidance systems, and precision approach controllers (Sections 4.1 and 4.4). Section 4.4.4 briefly discusses suboptimal applications of these controllers.

Chapter 5 considers worst-case variable wind models. Section 5.1, largely focusing on van der Vaart's method, reviews existing models and discusses their limitations. Section 5.2 proposes a number of extensions to van der Vaart's method. Section 5.3 introduces the concept of worst-case modeling in a conflict of interest scenario where the wind is treated as a controller whose objectives oppose those of the aircraft controller. These concepts are generalized in the differential games context in Section 5.4 and Appendix G. Numerical examples of all of these worst-case methods are given in Section 5.5, and comparisons are made with
van der Vaart's method. Finally, this chapter closes by giving consideration to the possibility of applying these conflict of interest models to flight simulators and to the certification process.

Chapter 6 presents the results of the digital simulation of a STOL transport flying steep CGA in the presence of variable winds. The results are used to assess the usefulness of CGG defined on the basis of estimates of the variable wind altitude profiles. Two conflict of interest wind models, developed in Chapter 5, are applied to the CGG assessment.

Chapter 7 presents the conclusions and recommendations of this study.

The aircraft in the numerical examples is a hypothetical two-engined STOL transport whose mass and geometric properties are summarized in Appendix C, and whose aerodynamic properties are summarized in Appendix D.

The reader is referred to Appendix A for a discussion of the notation conventions employed in this study. MKS units are employed throughout, although occasionally they are supplemented by aeronautical units.
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<table>
<thead>
<tr>
<th>Date</th>
<th>Location</th>
<th>Aircraft Type</th>
<th>Flight Phase</th>
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<tr>
<td>Dec. 17, 1973</td>
<td>Logan International Airport, Boston</td>
<td>DC-10-30</td>
<td>Landing</td>
</tr>
<tr>
<td>Jan. 30, 1974</td>
<td>Pago Pago International Airport, American Samoa</td>
<td>B707-321B</td>
<td>Landing</td>
</tr>
<tr>
<td>June 24, 1975</td>
<td>John F. Kennedy International Airport, New York</td>
<td>B727-225</td>
<td>Landing</td>
</tr>
<tr>
<td>Aug. 7, 1975</td>
<td>Stapleton International Airport, Denver</td>
<td>B727-224</td>
<td>Take-off</td>
</tr>
<tr>
<td>June 23, 1976</td>
<td>Philadelphia International Airport, Philadelphia</td>
<td>DC-9-31</td>
<td>Landing</td>
</tr>
<tr>
<td>June 3, 1977</td>
<td>Tucson International Airport, Tucson</td>
<td>B727-224</td>
<td>Take-off</td>
</tr>
</tbody>
</table>

This list is not all-inclusive and has been compiled from Refs. 1.16 through to 1.22.
PILOT DEFINED MODES AND TARGET VALUES

\(Y_{1e} = Y_1 - Y_{lm}\)
\(Y_1^* = \) Desired Values of \(Y_1\)

FLIGHT DIRECTOR

\(Y_{em} = \) Estimated Subset of State Used by Flight Director to Compute \(Y_1^*\)

CONTROLLER DYNAMICS

1- Manual
2- Automatic

ACTUATOR DYNAMICS

1- Servo Dynamics
2- Engine Dynamics

AIRFRAME DYNAMICS

\(\Delta C\)

EXTERNAL NAVIGATION & GUIDANCE INPUTS (VOR, DME, MLS)

MEASUREMENT DYNAMICS

1- Sensor Dynamics
2- Estimation
3- Receivers

\(\gamma_A\) = Angle of Attack

First Part of Segmented Glidepath
Second Part of Segmented Glidepath
Conventional Glidepath
Curved Glidepath

FIG. 1.1 LANDING APPROACH FLIGHT DYNAMICS IN VARIABLE WINDS

FIG. 1.2 CONVENTIONAL, SEGMENTED AND CURVED GLIDEPATHS
2. WIND MODELING IN AIRCRAFT RESPONSE STUDY: DEFINITIONS AND REVIEW

2.1 Wind Definitions

The terms variable wind, gust, wind gradient, wind shear and turbulence have been used somewhat loosely in the previous discussion. They will now be defined more precisely.

All motion of the atmosphere relative to the ground is wind. In general the wind velocity is not constant with respect to time or position, a fact which gives rise to the more descriptive terminology the variable wind. A sudden, brief change in the wind speed is often referred to by the term gust.

The wind velocity vector relative to an Earth-fixed reference frame \( \mathbf{E} \) will be given by

\[
\mathbf{W}(x_E, y_E, z_E, t) = \mathbf{W}_M(x_E, y_E, z_E, t) + \mathbf{W}(x_E, y_E, z_E, t).
\]

(2.1,1)

where its dependence on both time and position is explicitly noted. It has been common practice to separate this vector into two components, a mean wind and a zero-mean random turbulence component (e.g. Reid, Markov and Graf [1.39] and Huber [2.1]). This separation may be written

\[
\mathbf{W}(x_E, y_E, z_E, t) = \mathbf{W}_M(x_E, y_E, z_E, t) + \mathbf{W}(x_E, y_E, z_E, t).
\]

(2.1,2)

Here \( \mathbf{W}_M \) is the mean wind and \( \mathbf{W} \) is the turbulence component of the wind. The mean wind is frequently assumed to be time-independent, i.e. (2.1,2) may be written

\[
\mathbf{W}(x_E, y_E, z_E, t) = \mathbf{W}_M(x_E, y_E, z_E) + \mathbf{W}(x_E, y_E, z_E, t).
\]

(2.1,3)

In order to properly define a mean wind in the statistical sense, a suitable averaging technique must be employed. In the aircraft dynamics context such a technique should provide the mean value of the wind at a given location and time without referring to values of the wind temporally distant from the time of interest. For example, an ensemble average based on the wind speed at a given airport taken at a specified time of day on the same day each year averaged over the previous 50 years is clearly not particularly relevant to an aircraft on the landing approach to that airport.

These considerations restrict the averaging techniques to time averaging over time records of duration appropriate to the nature of the problem and the spectral content of the wind field. In Fig. 5 of Teunissen's review [1.27] the spectrum of the total wind speed measured at Brookhaven (by Templin [2.2]) is given. There is a spectral gap between the high frequency turbulence fluctua-
ations centred at approximately $10^2$ cycles per hour and the half day sunrise-sunset peak centred at approximately $10^{-1}$ cycles per hour. This region has been given the name the *micrometeorological gap*, with the lowest energy portions of it having periods of 5 minutes to 5 hours. Since the time intervals of interest to studying aircraft landing approach dynamics are of less than 5 minute duration, useful mean winds may be obtained by averaging with record lengths of a few minutes. The turbulence components are then defined as the wind velocity fluctuations about this average.

We now turn to the definitions of the terms *wind gradient* and *wind shear*. Formally speaking any spatial variation in the wind velocity at a given time is a gradient in the wind. In the aviation sense wind gradients are only important if they are significant along the flight path of the aircraft. But what is ultimately important in determining the effect of variable winds on the aircraft dynamics is the total rate of change of the wind field that the aircraft encounters. This total rate of change includes the contribution of mean wind gradients, turbulence gradients and time variations of the wind field.

The term *wind shear* itself, in the meteorological sense, is synonymous with wind gradient (Fujita [2.3]) and appears to have come into use in aviation circles in the early sixties. Nevertheless the literature has generally not adopted a common definition for the term. Shrager [1.22] discusses a number of definitions including one which restricts wind shear to include only spatial changes in the horizontal wind. In this definition spatially dependent updrafts and downdrafts are not included. Others have defined wind shear to be the spatial gradient in the mean wind (Gera [1.23], Beaulieu [1.42], and Huber [2.1]) arguing that while the fluctuating turbulence components of the wind effect ride quality and induce structural loads, other than in most unusual circumstances they will not create large, sustained dispersions from the desired reference state. Such distinctions are somewhat arbitrary and are not necessarily true; there may be spectrally low frequency turbulence components which have the potential to produce the same, significant disruptions as large gradients in the mean wind (Van der Vaart [1.67]).

It is also possible to categorize wind shear according to the direction of the gradients and their orientation relative to the flight path. Thus, for example, terms such as *vertical wind shear* arise, although once again there is disagreement on what the definition of such a shear should be. In the meteorological literature the term denotes the variation of horizontal wind (Fujita [2.3]) or total wind (Shrager [1.22], Nancoo [2.4]) along the vertical position coordinate, but Fujita suggests that in the aviation context the term can refer to variation in the vertical wind activity along the flight path. The latter does not appear to be a common interpretation.

These and other related definitions may be placed in
more precise terms. Let \((\frac{i_E}{i_E}, \frac{j_E}{j_E}, \frac{k_E}{k_E})\) be an Earth-fixed, dextral, orthonormal vector triad. Then let

\[
\vec{W} = W_1 \frac{i_E}{i_E} + W_2 \frac{j_E}{j_E} + W_3 \frac{k_E}{k_E} \\
\vec{V}_E = U_{E_1} \frac{i_E}{i_E} + V_{E_2} \frac{j_E}{j_E} + W_{E_3} \frac{k_E}{k_E}.
\]

(2.1,4)

(2.1,5)

Here \(\vec{W}\) has been defined previously and \(\vec{V}_E\) is the ground velocity vector of the aircraft centre-of-mass.

The gradient operator \(\vec{V}\) is defined in the usual way, i.e.

\[
\vec{V} = \frac{\partial}{\partial x_E} \vec{i} \frac{\partial}{\partial y_E} \vec{j} + \frac{\partial}{\partial z_E} \vec{k}.
\]

(2.1,6)

Thus the vector giving the magnitude and direction of the maximum spatial change of the \(i\)-th wind component \((i=1,3)\) is given by

\[
\vec{V}W_i = \frac{\partial}{\partial x_E} W_{i_1} \frac{i_E}{i_E} + \frac{\partial}{\partial y_E} W_{i_2} \frac{j_E}{j_E} + \frac{\partial}{\partial z_E} W_{i_3} \frac{k_E}{k_E}.
\]

(2.1,7)

The shear components are given by \(\frac{\partial}{\partial x_E} W_i\), \(\frac{\partial}{\partial y_E} W_i\), and \(\frac{\partial}{\partial z_E} W_i\), \(i=1,3\).

The representation \((2.1,7)\) is not particularly useful in the aviation sense. It does not give the wind shear along the flight path, nor does it include the effects of time-dependent wind changes. Such time-dependent changes are frequently associated with the turbulence component of the wind field, but may also occur in the mean wind (e.g. during a gust front passage).

To take these effects into account the total vector rate of change of the wind field that the aircraft encounters as it flies through the atmosphere must be considered. This is given by

\[
\frac{d\vec{W}}{dt} = (\vec{V}_E \cdot \vec{V}) \vec{W} + \frac{\partial \vec{W}}{\partial t}.
\]

(2.1,8)

Here \(\frac{d}{dt}\) (frequently written \(\frac{D}{Dt}\) in fluid mechanics) is the total or Lagrangian derivative following the motion. As well as the gradient term \((\vec{V}_E \cdot \vec{V})\), \(\frac{\partial \vec{W}}{\partial t}\) also contributes to the total change of the wind field. This latter term may be thought of as a time wind shear term*.

---

* That time wind shear is an appropriate term can be seen by treating \(\vec{W}\) as a function in the four dimensional space \((x_E, y_E, z_E, t)\). The 4-D equivalents to \(\vec{V}, \vec{W}\) and \(\vec{V}_E\) are

\[
\vec{V}_t = \frac{\partial}{\partial x_E} \vec{i} \frac{\partial}{\partial y_E} \vec{j} + \frac{\partial}{\partial z_E} \vec{k} + \frac{\partial}{\partial t} \vec{t}.
\]

(a)

\[
\vec{W} = W_{i_1} \vec{i}_E + W_{j_1} \vec{j}_E + W_{k_1} \vec{k}_E.
\]

(b)

\[
\vec{V}_E = U_{E_1} \vec{i}_E + V_{E_2} \vec{j}_E + W_{E_3} \vec{k}_E.
\]

(c)

It follows that in this formulation \(2.1,8\) may be written

\[
\frac{d\vec{W}}{dt} = (\vec{V}_E \cdot \vec{V}) \vec{W}.
\]

(d)
Summarizing and specializing the previous discussion, the following definitions will be adopted in this investigation:

1. **Wind Shear** - Spatio temporal gradients of the total wind velocity field.

2. **Vertical Wind Shear** \( \frac{\partial W}{\partial z_E} \) - Height dependent changes of the wind velocity field.

3. **Horizontal Wind Shear** \( \frac{\partial W}{\partial x_E} \) and/or \( \frac{\partial W}{\partial y_E} \) - Change of the wind velocity vector with respect to horizontal changes of position.

4. **Turbulent Wind Shear** (Beaulieu [1.42]) - Spatial gradients of the turbulence field.

5. **Time Wind Shear** \( \frac{\partial W}{\partial t} \) - Temporal changes of the wind velocity field at a specific location.

6. **Headwind or Tailwind Shears** - Variations of headwinds or tailwinds along the aircraft flight path.

7. **Crosswind Shear** - Variation of crosswind along the aircraft flight path.

8. **Downdrafts and Updrafts** - Sustained vertical activity of the wind field.

9. **Downburst** - Fujita [2.5] has coined this phrase in reference to severe downdraft activity associated with some thunderstorms. He has proposed that downdrafts that have a vertical wind velocity greater than 3.6 ms\(^{-1}\) (12 fts\(^{-1}\)) at an altitude of 91 m (300 ft) be considered downbursts.

The absolute intensity of these wind phenomena may be related in an imprecise way to the hazard posed to aviation. One classification scheme for reporting wind shear in the atmospheric boundary layer (according to the ICAO, Fichtl [2.6]) is as follows:

<table>
<thead>
<tr>
<th>Category</th>
<th>Vertical Shear Magnitude (ms(^{-1})/30m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light</td>
<td>0-2.5</td>
</tr>
<tr>
<td>Moderate</td>
<td>2.5-4.5</td>
</tr>
<tr>
<td>Strong</td>
<td>4.5-6.0</td>
</tr>
<tr>
<td>Severe</td>
<td>&gt;6</td>
</tr>
</tbody>
</table>

It is stressed that the effect of such shears on an aircraft depend on many factors unrelated to the shear magnitude itself. These include airspeed, aircraft configuration, engine response characteristics, and the presence of other meteorological factors. The latter may, in fact, lead to somewhat misleading situations. For example, if an aircraft encounters a downdraft cell of 4 ms\(^{-1}\) strength at 100 m altitude, the mean vertical shear of the downdraft will be \( 4/100 = 0.04 \text{ms}^{-1} \). In terms of the vertical shear of the horizontal wind which results from the divergence of the downdraft, the equivalent average value is 1.2 m/s per 30 meters. Such a vertical shear on its
own would not be considered significant but in combination with the downdraft may still cause significant control difficulties. Downbursts have been implicated in a number of aircraft accidents (Fujita and Caracena [2.5]) and their consideration is as important as that of shears of the horizontal wind alone.

It is possible to specialize these definitions further by considering the physical processes from which the underlying shears arise (e.g. thunderstorms). Such definitions are available in a number of sources (Barr, Gangsaas and Schaeffer [1.31] and Frost and Camp [1.37], among others) but are not necessary for the purposes of this study.

2.2 Deterministic Wind Models

The landing approach involves flight through the atmospheric boundary layer (ABL). The depth of this layer is known to vary widely and is typically in the range 100 to 1000 meters (Plate [2.7]). The wind characteristics in the ABL are at least functions of height and are very terrain dependent.

While one may study many facets of aircraft response to variable winds using stochastic techniques (see the discussion in Chapter 1, Section 2.2 and Etkin [1.3]) the aspects of the problem of interest to this investigation are best treated using deterministic models which represent one realization of the wind field. This is true when considering flight through the ABL where the turbulence properties are functions of at least height and wind probability models are not well established (see, e.g., Houboult [1.30] and van der Vaart [1.67]) and is especially true when considering the effects of winds that lie in the tails of the probability distribution whose contribution to the expected values of the aircraft response may be small but which nevertheless may create significant safety problems when they do occur. Deterministic models are also required in training simulators.

One may generate such deterministic wind models using a number of techniques and data sources including the following:

1. Models based on meteorological data - Examples of such data sources abound (typically Sinclair, Anthes and Panofsky [2.8], Grossman and Beran [2.9], Counihan [2.10], WMO [2.11], CATA [2.12] and NSSL [2.13] data, Frost, Fitchl, Connoll, and Hutte [2.14] and Ackerman [2.15]), although it should be noted that much of the data collected is for vertical wind shears rather than for horizontal wind shears.

2. Models based on ABL wind tunnels - These efforts have concentrated on obtaining boundary layer simulations that compare well with observed mean wind and turbulence properties (see, for example, Teunissen [2.16], Reid [2.17], Melbourne [2.18] and Reid, Etkin, Teunissen and Hughes [2.19]).

3. Situation specific dynamic models based on the physics of the atmospheric flow - Models exist of varying degrees of sophistication for the mean wind fields under the
following situations:

(i) Winds over homogeneous terrain – These include models for the vertical profiles of horizontal mean wind for neutral, stable and unstable boundary layers, for both the magnitude and direction of the mean wind (see Teunissen [1.27] and Barr, Gangsaas, Schaeffer [1.31], among others).

(ii) Winds over inhomogeneous terrain – These include models for the growth of internal boundary layers in the horizontal plane. Such boundary layers within boundary layers occur when the terrain homogeneity has a discontinuity, such as would be the case in the transition from a city to the flatter terrain that surrounds an airport (see, for example, Fichtl, Camp and Frost [2.20]).

(iii) Gust front models – Gust fronts and associated thunderstorms can be a source of particularly hazardous variable wind conditions and have spurred interest in their modeling (e.g. Lewellen and Williamson [1.34]). Such models have become quite sophisticated and have been used to predict wind and turbulence profiles associated with aircraft accidents (Williamson, Lewellen and Teske [2.21, 2.22]) and to study aircraft response in flight through thunderstorms (Frost and Crosby [1.45]).

(iv) Wind shears due to ground obstructions – Such wind fields are certainly functions of both horizontal and vertical position coordinates and are difficult to model accurately, although progress has been made (see the comments by Fichtl, Camp and Frost [2.20]).

4. Arbitrary wind models intended to stimulate aircraft dynamic response – Such models may be generated by drawing on experience gained from studying wind fields and their influence on aircraft response, or any other source of relevant insight. Examples of such models include the discontinuity profiles employed by Reid, Markov and Graf [1.39] and the discrete gust models of Jones and others (see the discussion in Etkin [1.3]).

5. Analytically maximizing worst-case wind models – Van der Vaart's technique [1.67] (see also the discussion in Chapter 1, Section 2.2) is an example of such a modeling procedure. This class of techniques looks for wind inputs which maximize, in the mathematical sense, certain functionals of the aircraft state. Such methods will be considered in detail in Chapter 5.

From this brief survey of work done on wind modeling in the ABL it is apparent that myriad models are available for use in aircraft response study. A number of investigators have proposed specific aeronautical models (Luers [2.23], Gerlach and Schuring [2.24], Barr, Gangsaas, and Schaeffer [1.31], Campagna [2.25], and Frost and Camp [1.37], among others). In the application to aircraft response studies the models used have generally been relatively simple (see the comments by Fichtl [2.6] and Houbolt [1.30]) with a few notable exceptions (e.g. the flight through thunderstorm study of Frost and Crosby [1.45]). The major reason for this appears to be that for the purposes of many studies evaluating
the wind shear response of the aircraft to an arbitrary, well chosen gradient will frequently do just as well as evaluating the response to variable winds generated by a complicated, condition specific ABL numerical model. The more complicated ABL models are generally more expensive to run on a computer and require considerably greater amounts of computer memory to store, particularly if they are three dimensional. In studies such as this one, where only the gross rigid body motions of the aircraft in response to variable wind conditions of unspecified origin are of interest, the complexity of the more sophisticated wind models does not enhance the validity of the results.

In view of this discussion, further consideration to sophisticated ABL numerical wind models based on physical considerations will not be given. Rather, the wind models that will be used are the worst-case wind models to be developed in Chapter 5, a sampling of wind models available in the sources cited previously (to be discussed as required) and a set of arbitrary profiles. This last group of wind models is discussed in the following.

2.2.1 A Family of Wind Profiles

For a neutrally stable ABL (this is usually the case with higher wind velocities) the mean horizontal wind speed altitude profile is approximated by (Teunissen [1.27])

\[
\frac{W_N}{W_G} = \left(\frac{h}{h_G}\right)^n.
\]  

(2.2.1)

Here

- \(W_N\) = wind speed for neutral ABL
- \(W_G\) = wind speed at the top of the ABL
- \(h\) = height above the ground
- \(h_G\) = height of the top of the ABL.

Typical values of \(h_G\) and \(n\) for several terrain types are (Teunissen [1.27]):

<table>
<thead>
<tr>
<th>terrain</th>
<th>(h_G) (meters)</th>
<th>(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>smooth (water, prairie)</td>
<td>300</td>
<td>0.16</td>
</tr>
<tr>
<td>woodland</td>
<td>400</td>
<td>0.28</td>
</tr>
<tr>
<td>city</td>
<td>500</td>
<td>0.40</td>
</tr>
</tbody>
</table>

These neutral profiles result in mild wind shears and will generally not lead to great control difficulties. In Ref. 1.39 severe wind shears are modeled by headwind and tailwind discontinuities superimposed onto this profile. This technique will also be adopted here, but a somewhat more elegant representation of the discontinuities is proposed. This may be done by adding hypertangent profiles similar to those discussed by Davies [2.26] in his atmospheric boundary layer stability studies. In particular, two half jets of the form
\[ W_{J_1} = W_{i_1} \left[ 1 - \tanh \left( \frac{(h - h_{i_1})}{\Delta s_{H_1}} \right) \right] \quad (2.2,2) \]

\[ W_{J_2} = W_{i_2} \left[ \tanh \left( \frac{(h - h_{i_2})}{\Delta s_{H_2}} \right) - 1 \right] \quad (2.2,3) \]

will be added. Here \( W_{J_1}, W_{J_2} \) = wind speed contributions of the first and second jets respectively,

\[ W_{i_1}, W_{i_2} = \text{inflection point values of the wind speed magnitudes for the first and second jets respectively,} \]

\[ h_{i_1}, h_{i_2} = \text{inflection point heights for first and second jets respectively,} \]

\[ \Delta s_{H_1}, \Delta s_{H_2} = \text{depth of shear zones for first and second jets respectively.} \]

In this model the total horizontal mean wind speed is given by

\[ W_H = W_N + W_{J_1} + W_{J_2}. \quad (2.2,4) \]

Equation (2.2,4) is qualitatively plotted in Fig. 2.1. If the overall mean wind is a headwind, then if \( h_{i_1} > h_{i_2} \) the model approximates an increasing headwind discontinuity and if \( h_{i_1} < h_{i_2} \) the model approximates a decreasing headwind discontinuity. As \( \Delta s_{H_1}, \Delta s_{H_2} \) approach zero these approximations approach true headwind and tailwind discontinuities.

* Note that in general \( W_{J_1}(0) = -W_{J_2}(0) \) and thus \( W_H(0) = 0 \).

The mean wind direction may be modeled similarly. An Earth-fixed runway reference frame \( F_{ER} \) and the wind direction angle \( \zeta(h) \) is defined in Fig. 2.2. \( F_{ER} \) is the angle between the mean wind at level \( h \) and the runway centreline.

In the northern hemisphere the wind direction tends to turn in a clockwise direction with height if we are looking down onto the Earth (in other words \( \zeta \) increases with height). The total (Teunissen [1.27] and Fichtl, Camp and Frost [2.20]) change in direction of this turn from the bottom to the top of the ABL is usually between 10 and 45 degrees, but has been measured to be as much as 180 degrees. Theoretical models attempting to predict the turning profile are available including the classical Ekman spiral result (Teunissen [1.27], Plate [2.7] and Fichtl, Camp and Frost [2.20]).

A family of turning profiles whose parameters may be adjusted to be representative of modest to abrupt direction changes of the wind that an aircraft might encounter can be defined as follows:

\[ \zeta = \zeta_0 + \kappa_\zeta h + \zeta_{J_1} + \zeta_{J_2}, \quad h \leq h_G \quad (2.2,5a) \]

\[ \zeta = \zeta_0 + \kappa_\zeta h_G + \zeta_{J_1} + \zeta_{J_2}, \quad h > h_G \quad (2.2,5b) \]

\( \zeta_{J_1}, \zeta_{J_2} \) are analogous to \( W_{J_1}, W_{J_2} \) and are given by

\[ \zeta_{J_1} = \zeta_{i_1} \left[ 1 - \tanh \left( \frac{(h - h_{i_1})}{\Delta s_{\zeta_1}} \right) \right] \quad (2.2,6) \]
Here $\zeta_1$, $\zeta_2$ = inflection point values of the direction perturbation magnitudes of the first and second direction shift jets respectively,

$$
\zeta_1, \zeta_2 = \text{inflection point heights for first and second direction shift jets respectively},
$$

$$
\Delta s_1, \Delta s_2 = \text{depth of shear zones for first and second direction shift jets respectively}.
$$

Fig. 2.3 presents the qualitative characteristics of the horizontal mean wind direction profiles. If $\zeta_1 > \zeta_2$ an increasing $\zeta$ with height direction shift occurs and vice versa for $\zeta_1 < \zeta_2$.

In terms of the reference frame defined in Fig. 2.2 the wind vector is therefore given by

$$
\vec{W} = -W_H \cos \zeta_{\text{ER}} - W_H \sin \zeta_{\text{ER}}.
$$

Here $W_H$ and $\zeta$ are given by (2.2.4) and (2.2.5) respectively.

In the models described above, updrafts and downdrafts have not been considered. These may also be modeled with hypertangent profiles.

Consider Fig. 2.4 where the geometry of the updraft/downdraft region is defined. This may be thought of as a model downburst cell. The vertical mean wind velocity model will then be given by

$$
W_V = W_{V0} k_{\text{ER}}
$$

where

$$
W_V = W_{V0} \left[ \tanh\left( \frac{x_{\text{ER}1} - x_{\text{ER}2}}{\Delta s_{V1}} \right) - \tanh\left( \frac{x_{\text{ER}1} - x_{\text{ER}2}}{\Delta s_{V2}} \right) \right]
$$

$$
\cdot \left[ \tanh\left( \frac{y_{\text{ER1}} - y_{\text{ER2}}}{\Delta s_{V1}} \right) - \tanh\left( \frac{y_{\text{ER1}} - y_{\text{ER2}}}{\Delta s_{V2}} \right) \right].
$$

The terms of this profile are defined analogously to those of the previous hypertangent profiles. The profile is plotted qualitatively in Fig. 2.5.

Physically the updraft or downdraft velocities must be zero at the ground. Terms which cause $W_V$ to go to zero as $h$ approaches zero may readily be added to (2.2.10) (e.g. by multiplying (2.2.10) by $\tanh(ch)$ for suitably chosen $c > 0$). Other physical characteristics, such as outflow near the ground level in a downdraft, may be included by appropriate combinations of (2.2.9) and the horizontal wind profiles.

Anticipating the analysis to follow, occasionally wind models which are linear with respect to position will also be required, typically in order to preserve the linearity of the equations of motion. The most general linear model of this kind is defined by

$$
W_L = W_{L0} + KR.
$$
\[ \mathbf{x} \text{ and } \mathbf{y} \text{ are, respectively, column matrices containing} \]
the components, expressed in \( F_E \), of the vectors \( \mathbf{x} \) and \( \mathbf{y} \).
\[ \mathbf{x}_L \text{ and } \mathbf{y}_L \text{ is the wind velocity vector at the aircraft} \]

centre of mass, \( \mathbf{x}_{L0} \text{ is the wind velocity vector at the origin} \]
of \( F_E \), and \( \mathbf{z} \text{ is the position vector of the aircraft} \)
centre of mass with respect to \( F_E \). \( \mathbf{z} \) is given by

\[ \mathbf{z} = x_{Ez} + y_{Ey} + z_{Ez} . \tag{2.2,12} \]

\( \mathbf{K} \) is the constant matrix of spatial wind gradients, given by

\[
\mathbf{K} = \begin{bmatrix}
\frac{\partial W_1^L}{\partial x_E} & \frac{\partial W_2^L}{\partial y_E} & \frac{\partial W_3^L}{\partial z_E} \\
\frac{\partial W_2^L}{\partial x_E} & \frac{\partial W_2^L}{\partial y_E} & \frac{\partial W_2^L}{\partial z_E} \\
\frac{\partial W_3^L}{\partial x_E} & \frac{\partial W_3^L}{\partial y_E} & \frac{\partial W_3^L}{\partial z_E}
\end{bmatrix} . \tag{2.2,13} \]
REFERENCES — CHAPTER 2


FIG. 2.1 HORIZONTAL MEAN WIND VELOCITY MODEL

FIG. 2.2 ER AND WIND DIRECTION & DEFINITIONS
FIG. 2.3 HORIZONTAL MEAN WIND DIRECTION MODEL

FIG. 2.4 UPDRAFT-DOWNDRAFT CELL GEOMETRY
Downdraft for $W_v > 0$

$Y_{ER} = 0$

$Y_{ER_{11}} = Y_{ER_{12}}$

FIG. 2.5 UPDRAFT-DOWNDRAFT MODEL
(See (2.2, 14))
3. AIRCRAFT EQUATIONS OF MOTION

The differential equations describing flight through the atmosphere are developed in numerous references (e.g. Etkin [3.1], Ashley [3.2], and McRuer, Ashkenas and Graham [3.3]). These equations may be posed to take into account some or all of the following:

1. Structural flexibility.
2. The curvature and rotation of the Earth.
3. Variable wind effects including models which take into account the finite size of the airplane.
4. Internal rotors, articulation of the controls and fuel sloshing.
5. Variation of the state of the atmosphere and gravitational forces with linear position.

The equations are traditionally derived using a Newton-Euler development, although there is no a priori reason why Lagrangian or Hamiltonian methods should not be used. The former lends to a physically intuitive and elegant development for the rigid body dynamics equations, and will be employed in the following derivations.

3.1 Fundamental Assumptions

Several overall simplifying assumptions will be made before proceeding further. These assumptions are justified in some detail in Appendix B. They are valid for fixed wing STOL and CTOL aircraft on the landing approach.

The assumptions are as follows:

1. The Earth is flat and any Earth-fixed reference frame is inertial. Thus the Earth-fixed reference frame $F_E$ and the runway reference frame $F_{ER}$, both defined in Chapter 2, and a general inertial reference frame $F_I$ can be considered to be equivalent.

2. The aircraft is a rigid body.

3. Air density, temperature and pressure, and the gravitational field are constant.

4. The mass properties of the aircraft are constant.

5. The aircraft has a vertical plane of symmetry.

6. The effects of internal and external rotors, articulation of the controls, and fuel sloshing are negligible to the gross rigid body motion of the aircraft.

As discussed in Appendix B, assumptions 3, 4, 5 and 6 can be relaxed with a modest increase in complexity. This is not the case with assumptions 1 and 2.
3.2 Reference Frames, Rotation Matrices and Angular Velocities

In the previous section $F_I$ was defined to be a general inertial reference frame. In this section and in the sequel it will be more precisely defined as that inertial reference frame whose axes and origin coincide with $F_{ER}$ (see Fig. 2.2 and Fig. 3.1).

Another useful Earth-fixed reference frame is $F_{GL}$, the glidepath-localizer reference frame. This reference frame will be defined to have an origin at the runway intercept point of the glidepath-localizer, and axes orientated as in Fig. 3.1.

It is convenient to define an aircraft-fixed reference frame $F_B$. In this study $F_B$ will be a body-fixed reference frame whose origin is at the centre of mass of the vehicle and whose x-axis is parallel to the fuselage reference line (frl). The z-axis is nominally downwards in the plane of symmetry and the y-axis orientation follows from the right hand rule. This modified wind axes reference frame will be referred to as $F_{W'}$ (see Fig. 3.3).

A number of transformations are required for the rotational relationships between these reference frames. There are a variety of methods that may be used to represent these rotations, including Euler angles, direction cosines, Euler variables (quaternions) and so forth (Hughes [3.4]). In this study the traditional Euler angle method will be employed. Since aircraft attitudes where the singularities in the Euler variable representations occur are outside the
sensible limits of the landing approach problem, this choice is not particularly restrictive, while at the same time maintaining the advantage of working with a rotational representation which is in common use in aircraft dynamics problems.*

In the following use is made of the notation conventions of Appendix A. In particular if $L_{BA}$ denotes a rotation matrix relating the components of a vector $\vec{v}$ expressed in $F_A$ ($\vec{v}^A$) to the components of the same vector in $F_B$ ($\vec{v}^B$), then

$$\vec{v}^B = L_{BA}\vec{v}^A.$$

The following definitions, geometric relationships, and matrices will be employed in the development of the equations of motion.

1. The rotation matrix relating $F_I$ and $F_S$:

$$L_{SI} = \begin{bmatrix}
\cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\
\sin \phi \sin \theta \cos \psi & \sin \phi \sin \theta \sin \psi & \sin \phi \cos \theta \\
-cos \phi \sin \psi & +\cos \phi \cos \psi & -\sin \phi \\
\cos \phi \sin \theta \cos \psi & \cos \phi \sin \theta \sin \psi & \cos \phi \cos \theta \\
+\sin \phi \sin \psi & -\sin \phi \cos \psi & \\
\end{bmatrix}$$

(3.2,2a)

An interesting application of quaternions to an aircraft dynamics problem is given by Goodman [3.5].

2. The rotation matrix relating $F_I$ and $F_B$:

$$L_{BI} = [l_{BI_{ij}}]$$

(3.2,3)

where the elements $l_{BI_{ij}}$ are identical to the elements $l_{ij}$ with $\psi, \theta, \phi$ replaced by $\psi_B, \theta_B, \phi_B$. $\psi_B, \theta_B,$ and $\phi_B$ are defined analogously to $\psi, \theta, \phi$.

3. The angular velocity of the aircraft with respect to $F_I$ written as components in $F_S$ and $F_B$:

$$\vec{\omega}^S = (p, q, r)$$

(3.2,4)

$$\vec{\omega}^B = (p_B, q_B, r_B).$$

(3.2,5)

From the definitions of $F_S$ and $F_B$, it follows that $q$ and $q_B$ are identical.

4. The angular rate cross-product matrices [3.1] in $F_S$ and $F_B$:

$$\vec{\omega}^S = \begin{bmatrix}
0 & -r & q \\
-r & 0 & -p \\
q & p & 0
\end{bmatrix}$$

(3.2,6)
5. The airspeed of the aircraft written as components in \( F_S, F_B, F_W \), and \( F_{\bar{W}} \):

\[
\begin{bmatrix}
0 & -r_B & q_B \\
r_B & 0 & -p_B \\
-q_B & p_B & 0
\end{bmatrix}
\]

(3.2.7)

\[\mathbf{w} = \mathbf{e}_B \cdot \mathbf{q}_B \]

\[\mathbf{v} = \mathbf{e}_B \cdot \mathbf{r}_B \]

\[\mathbf{w}_{\bar{W}} = \left( v_{xz}, v, 0 \right) \]

(3.2.10)

\[\mathbf{w}_{\bar{W}} = \left( v, 0, 0 \right) \]

(3.2.11)

From the definitions of \( F_S \) and \( F_B \), it follows that \( v \) and \( v_B \) are identical.

6. The groundspeed of the aircraft with respect to \( F_I \) written as components in \( F_S, F_B \), and \( F_I \):

\[\mathbf{v}^T_E = \left( u_E, v_E, w_E \right) \]

(3.2.12)

\[\mathbf{v}^T_E = \left( u_B, v_B, w_B \right) \]

(3.2.13)

\[\mathbf{v}^T_{\bar{I}} = \left( x_I, y_I, z_I \right) \]

(3.2.14)

\[\mathbf{v}^T_{\bar{I}} = \left( x_B, y_B, z_B \right) \]

(3.2.15)

(3.2.16)

(3.2.17)

\[\alpha_x = \arctan \left( \frac{w_B}{u_B} \right) \]

(3.2.18)

\[\beta = \arctan \left( \frac{v_{xz}}{v} \right) = \arcsin \left( \frac{v}{V} \right) \]

(3.2.19a)

or

\[\beta = \arctan \left( \frac{v_B}{v_{xz}} \right) = \arcsin \left( \frac{V_B}{V} \right) \]

(3.2.19b)

\[V_{xz} = \sqrt{u^2 + v^2} = \sqrt{u_B^2 + w_B^2} \]

\[V = \sqrt{u^2 + v^2 + w^2} = \sqrt{u_B^2 + v_B^2 + w_B^2} \]

(3.2.20)

(3.2.21)

Here \( \alpha \) is the angle of attack with respect to the zero lift line (zll), \( \alpha_0 \) is the angle between the fuselage reference line and the zero lift line, \( \alpha_f \) is the angle of attack with respect to the \( x_B \) axis, \( \alpha_{fe} \) is the equilibrium \( \alpha_f \), \( \alpha_x \) is the angle of attack with respect to the \( x_S \) axis, \( \beta \) is the sideslip angle, \( V \) is the total airspeed magnitude and \( V_{xz} \) is the magnitude of the airspeed component in the plane of symmetry.
8. The rotation matrix relating $F_S$ and $F_B$:

$$L_{BS} = \begin{bmatrix}
\cos \alpha_f & 0 & -\sin \alpha_f \\
0 & 1 & 0 \\
\sin \alpha_f & 0 & \cos \alpha_f
\end{bmatrix} \quad (3.2,22)$$

9. The rotation matrix relating $F_{W'}$ and $F_B$:

$$L_{BW'} = \begin{bmatrix}
\cos \alpha_f & 0 & -\sin \alpha_f \\
0 & 1 & 0 \\
\sin \alpha_f & 0 & \cos \alpha_f
\end{bmatrix} \quad (3.2,23)$$

10. The rotation matrix relating $F_W$ and $F_S$ (Etkin [3.1]):

$$L_{SW} = \begin{bmatrix}
\cos \beta \cos \alpha_x & -\sin \beta \cos \alpha_x & -\sin \alpha_x \\
\sin \beta & \cos \beta & 0 \\
\cos \beta \sin \alpha_x & -\sin \beta \sin \alpha_x & \cos \alpha_x
\end{bmatrix} \quad (3.2,24a)$$

$$= [l_{SW_{ij}}] \quad (3.2,24b)$$

The ordered rotations are $(-\beta, \alpha_x, 0)$.

11. The rotation matrix relating $F_W$ and $F_B$:

$$L_{BW} = [l_{BW_{ij}}] \quad (3.2,25)$$

where the elements $l_{BW_{ij}}$ are identical to the elements $l_{SW_{ij}}$ with $\alpha_x$ replaced by $\alpha_f$.

12. The geometric relationships (see Fig. 3.3)

$$\theta = \theta_B - \alpha_f \quad (3.2,26)$$

$$u = V \cos \beta \cos \alpha_x \quad (3.2,27a)$$

$$v = V \sin \beta \quad (3.2,27b)$$

$$w = V \cos \beta \sin \alpha_x \quad (3.2,27c)$$

$$u_B = V \cos \beta \cos \alpha_f \quad (3.2,28a)$$

$$v_B = V \quad (3.2,28b)$$

and

$$w_B = V \cos \beta \sin \alpha_f \quad (3.2,28c)$$

13. The windspeed with respect to $F_I$ written as components in $F_I$:

$$W_I^T = (W_1, W_2, W_3) \quad (3.2,29)$$

14. The acceleration due to gravity written as components in $F_I$:

$$g_I^T = (0, 0, g) \quad (3.2,30)$$

15. The aerodynamic forces (including thrust forces)
3.11 written as components in $F_S$ and $F_B$:

$$A^{ST} = (X, Y, Z)$$

(3.2,31)

$$A^{BT} = (X_B, Y_B, Z_B).$$

(3.2,32)

From the definitions of $F_S$ and $F_B$, it follows that $Y$ and $Y_B$ are identical.

16. The aerodynamic moments (including thrust moments) written as components in $F_S$ and $F_B$:

$$M^{ST} = (L, M, N)$$

(3.2,33)

$$M^{BT} = (L_B, M_B, N_B).$$

(3.2,34)

From the definitions of $F_S$ and $F_B$, it follows that $M$ and $M_B$ are identical.

17. The inertia matrix of the aircraft, including power plants, with respect to its centre of mass and expressed in $F_B$ (see Etkin [3.1]):

$$I^B = 
\begin{bmatrix}
I_{xx} & -I_{xy} & -I_{xz} \\
-I_{yx} & I_{yy} & -I_{yz} \\
-I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix}$$

(3.2,35)

From the definition of $F_B$ and the assumption of an aircraft plane of symmetry it can be shown that $I_{xy} = I_{yx} = I_{zy} = I_{yz} = 0$, and thus these terms will be dropped in the sequel. The plane of symmetry assumption and the definition of $I^B$ also imply that $I_{xz} = I_{zx}$.

18. The inertia matrix of the aircraft, including power plants, with respect to its centre of mass and expressed in $F_S$:

$$I^S = 
\begin{bmatrix}
I_{xx} & 0 & -I_{xz} \\
0 & I_{yy} & 0 \\
-I_{xz} & 0 & I_{zz}
\end{bmatrix}$$

(3.2,36)

The plane of symmetry assumption has been included in (3.2,36).

The inertia matrices $I^B$ and $I^S$ are related through a rotational transformation. From the definitions of $F_B$ and $F_S$, and following Etkin [3.1], Chapter 5, this transformation is

$$I^S = L_{SB}^T I^B L_{BS}$$

(3.2,37)

or in scalar form (noting that $L_{SB} = L_{BS}^T$)

$$I_{xx} = I_{xx} \cos^2 \alpha_e - 2I_{xz} \cos \alpha_e \sin \alpha_e + I_{zz} \sin^2 \alpha_e$$

(3.2,38a)
\[ I_{yy} = I^B_{yy} \] (3.2,38b)

\[ I_{zz} = I^B_{xx} \sin^2 \alpha_e + 2I^B_{xz} \cos \alpha_e \sin \alpha_e + I^B_{zz} \cos^2 \alpha_e \] (3.2,38c)

\[ I_{xz} = (I^B_{xx} - I^B_{zz}) \cos \alpha_e \sin \alpha_e + I^B_{xz} \cos^2 \alpha_e - I^B_{zz} \sin^2 \alpha_e \] (3.2,38d)

19. The rotation matrix relating \( F_I \) and \( F_{GL} \) (Fig. 3.1):

\[
L_{GL} = \begin{bmatrix}
\cos \gamma_G & 0 & \sin \gamma_G \\
0 & 1 & 0 \\
-\sin \gamma_G & 0 & \cos \gamma_G
\end{bmatrix}
\] (3.2,39a)

\[ = [L_{GL}]_{ij}. \] (3.2,39b)

3.3 Newton-Euler Development of the General Equations of Motion*

Newton-Euler techniques begin with the fundamental equations (Etkin [3.1])

\[ \dot{\mathbf{p}} = \mathbf{F} \] (3.3,1)

\[ \dot{\mathbf{h}} = \mathbf{M} \] (3.3,2)

\[ \mathbf{p} \] is the momentum vector of the body, \( \mathbf{h} \) is the angular momentum of the body about its centre of mass, \( \mathbf{F} \) is the external force vector acting at the centre of mass and \( \mathbf{M} \) is the external moment vector about the centre of mass. \( \mathbf{p} \) may be written

\[ \mathbf{p} = m \mathbf{V}_E \] (3.3,3)

where \( \mathbf{V}_E \) is the velocity vector of the aircraft with respect to \( F_E \) (\( F_I \)) and \( m \) is the mass of the aircraft. An expression for \( \mathbf{h} \) follows from the fundamental relationship

\[ \dot{\mathbf{h}} = \int \sum_{\text{mass}} [\mathbf{r} \times \mathbf{F} \times \mathbf{r}] \, dm \] (3.3,4a)

or

\[ \dot{\mathbf{h}} = \int \sum_{\text{mass}} [\mathbf{r} \times \mathbf{F} + \mathbf{r} \times (\omega \times \mathbf{r})] \, dm \] (3.3,4b)

where \( \mathbf{r} \) is the position vector of an element of mass \( dm \) of the body with respect to its centre of mass (see Fig. 3.4), \( \omega \) is the angular velocity vector of \( F_B \) with respect to \( F_I \), ' when applied to a vector represents rate of change with respect to \( F_I \) and ' when applied to a vector represents rate of change with respect to \( F_B \). (See McRuer, Ashkenas and Graham [3.3] for a more thorough discussion of vector differentiation.) Equation (3.3,4b) may be written in matrix notation...
as (replacing \( \mathbf{r} \times \mathbf{r} \) by \( -\mathbf{r} \times \mathbf{r}_B \) and dropping the subscript \( 'B' \) on \( \mathbf{r}_B \) for the sake of brevity)*

\[
\mathbf{h}^B = \frac{1}{\text{mass}} \int \mathbf{r}^B \times \mathbf{r}_B \times \mathbf{r}_B \, dm.
\]  

(3.3.5)

But

\[
\mathbf{r}^B = 0
\]

(3.3.6)

for a rigid body. Since \( \mathbf{r}^B \) is a constant with respect to the integration in (3.3.5), it follows that

\[
\mathbf{h}^B = \mathbf{r}^B \times \mathbf{r}_B
\]

(3.3.7)

where

\[
\mathbf{I}^B = -\frac{1}{\text{mass}} \int \mathbf{r}^B \times \mathbf{r}_B \times \mathbf{r}_B \, dm.
\]

(3.3.8)

\( \mathbf{I}^B \) is, by convention, given by (3.2.35)

The externally applied force \( \mathbf{F} \) is made up of an aerodynamic component \( \mathbf{A} \) (\( \mathbf{A} \) includes thrust forces) and a gravitational component \( mg \) such that

\[
\mathbf{F} = \mathbf{A} + mg.
\]

(3.3.9)

The externally applied moment \( \mathbf{M} \) arises entirely from aerodynamic effects (\( \mathbf{M} \) includes thrust moments).

The position of the aircraft in \( \mathbf{F}_I \) can be found by integrating the components of the aircraft groundspeed vector \( \mathbf{v}_E \) in \( \mathbf{F}_I \) (i.e. \( \mathbf{v}_E^I \)) where

\[
\mathbf{v}_E = \mathbf{v} + \mathbf{w}.
\]

(3.3.10)

\( \mathbf{v} \) is the aircraft airspeed vector and \( \mathbf{w} \) is the wind velocity vector with respect to \( \mathbf{F}_I \).

Substituting (3.3.9) and (3.3.10) into (3.3.1), the vector dynamic equations of motion become

\[
\mathbf{m} (\dot{\mathbf{v}} + \dot{\mathbf{w}}) = \mathbf{A} + mg
\]

(3.3.11)

and

\[
\mathbf{I} \dot{\mathbf{w}} = \mathbf{M}.
\]

(3.3.12)

Other than the gravitational force, the dominant forces and moments acting on the aircraft are due to aerodynamic causes and are largely determined by its orientation and configuration. It is accordingly advantageous to write the matrix equations of motion with respect to a body fixed reference frame. At this point this reference frame is chosen to be \( \mathbf{F}_B \), although the development is formally identical for the stability axes \( \mathbf{F}_S \) (since this, too, is a body fixed reference frame) with an appropriate reconciliation of notation.

* Superscripts on matrix quantities refer to the reference frame in which the components of the matrix are expressed. Overscore '\( \cdot \)\' refers to the matrix equivalent of the vector cross product. For more details, see Appendix A.
The reasons for this choice of reference frame for the nonlinear equations of motion will become clear when considering the aerodynamic data available for the example aircraft (see the discussion in the section to follow). The linearized equations of motion, however, will be written in \( F_S \) because of the resulting simplifications (a more detailed discussion will follow in Section 3.10).

Thus the matrix force and moment equations become (see Etkin [3.1] and Reid et al. [1.39])

\[
m(\ddot{y}^B + \dot{w}^B \dot{y}^B + L_{BI} \dot{w}^I) = \dot{A}^B + mL_{BI} \dot{e}^I
\]

(3.3.13)

and

\[
\dot{u}^B + \dot{w}^B = \dot{M}^B.
\]

(3.3.14)

Substituting for \( \dot{u}^B \) from (3.3.7), the moment equation becomes

\[
\dot{u}^B + \dot{w}^B + \dot{w}^B = \dot{M}^B.
\]

(3.3.15)

Writing out the equations (3.3.13) and (3.3.15) in scalar form yields

\[
m(\ddot{w}^B + \dot{q}^B \dot{w}^B - \dot{r}^B \dot{B}^B + \dot{\delta}_{BI} \dot{\delta}_{11} + \dot{\delta}_{BI} \dot{\delta}_{12} + \dot{\delta}_{BI} \dot{\delta}_{13})
\]

\[
= X^B + mg \dot{\delta}_{BI} \dot{e}_{13}
\]

(3.3.16a)

\[
m(\ddot{w}^B + \dot{r}^B \dot{w}^B - \dot{p}^B \dot{w}^B + \dot{\delta}_{BI} \dot{\delta}_{21} + \dot{\delta}_{BI} \dot{\delta}_{22} + \dot{\delta}_{BI} \dot{\delta}_{23})
\]

\[
= Y^B + mg \dot{\delta}_{BI} \dot{e}_{23}
\]

(3.3.16b)

for the force equations, and

\[
I_{xx} \dot{r}^B - I_{xz} \dot{r}^B + (I_{xx} - I_{yy}) \dot{p}^B
\]

\[
= L_B
\]

(3.3.17a)

\[
I_{yy} \dot{q}^B - I_{xz} \dot{r}^B + (I_{yy} - I_{xx}) \dot{q}^B
\]

\[
= M_B
\]

(3.3.17b)

\[
I_{zz} \dot{r}^B - I_{xz} \dot{r}^B + (I_{zz} - I_{yy}) \dot{p}^B
\]

\[
= N_B
\]

(3.3.17c)

for the moment equations.

Kinematic equations are also required for the linear and rotational position of the aircraft. The linear position equations follow from (3.3.10), i.e. in matrix form

\[
\dot{v}^E = \dot{v}^B + \dot{w}^B
\]

(3.3.18a)

or

\[
\dot{v}^I = \dot{v}^B + \dot{w}^I.
\]

(3.3.18b)
The rotational position equations are the Euler angle rate equations and are derived in Etkin [3.1]. The resulting scalar kinematic equations of motion are thus seen to be

\[
\begin{align*}
\dot{x}_I &= x_{B_{11}} u_B + x_{B_{12}} v_B + x_{B_{13}} w_B + W_1 \\
\dot{y}_I &= x_{B_{21}} u_B + x_{B_{22}} v_B + x_{B_{23}} w_B + W_2 \\
\dot{z}_I &= x_{B_{31}} u_B + x_{B_{32}} v_B + x_{B_{33}} w_B + W_3
\end{align*}
\]

(3.3,19a)

for linear position, and

\[
\begin{align*}
\dot{\varphi}_B &= p_B + q_B \sin \phi_B \tan \theta_B + r_B \cos \phi_B \tan \theta_B \\
\dot{\theta}_B &= q_B \cos \phi_B - r_B \sin \phi_B \\
\dot{\psi}_B &= q_B \sin \phi_B \sec \theta_B + r_B \cos \phi_B \sec \theta_B
\end{align*}
\]

(3.3,20a)

(3.3,20b)

(3.3,20c)

for angular position.

Equations (3.3,16), (3.3,17), (3.3,19) and (3.3,20) are identical to equations (2.31), (2.32), (2.33) and (2.34) of Reid et al. [1.39] with appropriate reconciliation of the notation.

An equation which is more useful in landing approach studies than the \( x_I \) and \( z_I \) equations is the equation of motion for \( d \), the normal glidepath deviation (positive for deviation above the glidepath). From Fig. 3.1 it follows that

\[
\begin{pmatrix}
\dot{x}_{GL} \\
\dot{y}_{GL} \\
\dot{z}_{GL}
\end{pmatrix} = L_{GLI} \begin{pmatrix}
\dot{x}_I \\
\dot{y}_I \\
\dot{z}_I
\end{pmatrix}
\]

(3.3,21)

where \( L_{GLI} \) is given by (3.2,39). Noting that

\[
d = -z_{GL}
\]

(3.3,22)

and substituting (3.3,19) into (3.3,21) yields the differential equation for \( d \), i.e.

\[
\ddot{d} = \sin \gamma_G \dot{x}_I + \cos \gamma_G \dot{h}
\]

(3.3,23a)

or

\[
\ddot{d} = (x_{B_{11}} \sin \gamma_G - x_{B_{13}} \cos \gamma_G) u_B + (x_{B_{21}} \sin \gamma_G - x_{B_{23}} \cos \gamma_G) v_B + (x_{B_{31}} \sin \gamma_G - x_{B_{33}} \cos \gamma_G) w_B + \sin \gamma_G W_1 - \cos \gamma_G W_3.
\]

(3.3,23b)

Here \( h \) is the altitude of the aircraft and is given by

\[
h = -z_I.
\]

(3.3,24)

3.4 Linearization Reference Equilibrium

Linearization of the general equations of motion is common practice, and is motivated by a number of practical
3.21

considerations:

1. Analytical techniques for studying linear systems of differential equations are more comprehensive and easier to apply.

2. Control system synthesis for linear systems is more systematic and less laborious (see Chapter 4).

3. The most commonly used aerodynamic modeling technique, stability derivatives (see the section to follow), requires that an aerodynamic linearization reference equilibrium be specified.

The linearization of the differential equations \((3.3,16)\), \((3.3,17)\), \((3.3,19)\) and \((3.3,20)\) is more conveniently considered if these equations are written in the vector form

\[
\dot{x}(t) = f(x, \dot{x}, \ddot{x}, t) \tag{3.4,1}
\]

where the dependence of \(f\) on control inputs \(\dot{x}\) (incorporated within the aerodynamic forces and moments), wind inputs \(\dot{W}\) and wind rate of change inputs \(\ddot{W}\) is explicitly noted. In writing \((3.4,1)\) in this form it has been tacitly assumed that the state derivatives \(\dot{x}\) are separable from \(f\) (i.e. \(X_B, Y_B, Z_B, L_B, M_B, N_B\) usually have some dependence on \(\dot{x}\), and thus this assumption assures that the \(\dot{x}\) terms may be moved to the left hand side of the equation). This is not particularly restrictive for aircraft dynamic models (see the following section on aerodynamic models).

3.22

A trajectory of the system \((3.4,1)\) is any vector function of time \(x_e(t)\) which satisfies \((3.4,1)\) for given \(\dot{x}_e(t)\) and \(W_e(t)\), i.e.

\[
\dot{x}_e(t) = f(x_e, \dot{x}_e, W_e, \ddot{W}_e, t). \tag{3.4,2}
\]

Mathematically speaking any of these trajectories are candidates for a reference equilibrium about which linearization can take place. Thus \((3.4,1)\) may be written

\[
\dot{x}_e + \Delta \dot{x} = f(x_e + \Delta x, \dot{x}_e + \Delta \dot{x}, W_e + \Delta W_e, \ddot{W}_e + \Delta \ddot{W}_e, t) \tag{3.4,3}
\]

where the '\(\Delta\)' symbol refers to a perturbation quantity and the subscript 'e' refers to a reference equilibrium quantity, i.e.

\[
a = a_e + \Delta a. \tag{3.4,4}
\]

Since in the linearization all terms of second order and greater in \(\Delta x\) will be dropped, equation \((3.4,3)\) may be written

\[
\dot{x}_e + \Delta \dot{x} = f(x_e, \dot{x}_e, W_e, \ddot{W}_e, t) + \Delta f_1(x_e, \dot{x}_e, W_e, \ddot{W}_e, \Delta x, \Delta \dot{x}, \Delta \ddot{W}, \Delta \dddot{W}, t) + \epsilon \tag{3.4,5}
\]

where \(\Delta f_1\) is the first order approximation to \(\Delta f\) and \(\epsilon\)
3.23

is a remainder term of second order in the state variable perturbations. Formally speaking (3.4,5) may be obtained using a Taylor series for vector functions which are functions of vectors, provided that \( f \) is continuous and infinitely differentiable in the elements of \( \mathbf{x} \) (for the required Taylor expansion see Deutsch [3.6]). In practice the substitution \( \mathbf{x}_e + \Delta \mathbf{x} \) is made (as in 3.4,3) and second order and higher terms in the perturbation quantities are dropped. The equilibrium portion will then drop out and only perturbation equations will remain.

Returning to (3.4,5), since \( \mathbf{x}_e \) is a trajectory of the system,

\[ \Delta \mathbf{x} = \Delta \mathbf{f}_1 + \mathbf{e} \]  

(3.4,6a)

must hold. If \( \Delta \mathbf{x} \) is small enough then \( \mathbf{e} \) need not be retained, i.e.

\[ \Delta \mathbf{x} = \Delta \mathbf{f}_1. \]  

(3.4,6b)

Having defined the mathematical meaning and indicated the significance of linearization reference equilibria, consideration should now be given to the question of what reference equilibria are most suitable for this study. To generate an \( \mathbf{x}_e(t) \) the nonlinear equations (3.4,1) could be solved for a particular set of initial conditions and some mean set of disturbance and control inputs. The equations could then be linearized about \( \mathbf{x}_e(t), \delta_e(t), \) and \( \mathbf{w}_e(t) \). This technique involves the generally difficult step of solving the nonlinear system of equations and defies the purpose of the operation for any but the simplest wind and control inputs. In general the resulting linearized equations will be time-varying.

Alternatively the reference equilibrium could be defined on the basis of a trimmed flight condition (e.g. straight and level flight). In such equilibria the state variables are assumed to have zero or constant rates of change, and are also particularly useful as reference equilibria about which to linearize the aerodynamics.

In light of this discussion, the reference equilibrium chosen is similar to that of Reid et al [1.39]. The differences arise due to the presence of equilibrium crosswinds in the development in this section. In this reference condition the aircraft is trimmed to fly on a constant airspeed \( (V_e) \) landing approach along a rectilinear glide slope of constant ground referenced glide slope angle \( \gamma_G \) in the presence of a constant headwind \( W_{1e} \) and constant crosswind \( W_{2e} \). Symmetric equilibrium flight conditions will be maintained by flying a crabbed approach. Perturbations in the aircraft state vector away from the reference equilibrium then arise from initial conditions on the equations, from variable winds, and from control inputs.

The linearized equations of motion that will be developed in Section 3.10 of this chapter will be written in stability axes \( F_S \) because of certain simplifications that result in the specification of the linearization reference equilibrium and in the introduction of linear quasisteady aerodynamics in this reference frame. This contrasts with the development of
the general nonlinear equations of motion in body axes \( F_B \). The latter was done to facilitate implementation of the nonlinear aerodynamic model, specified as it is in reference frame \( F_B \) (see Appendix D). It is again noted, however, that the general, nonlinear stability axes equations of motion can be obtained from the body axes equations of motion by a direct substitution of \( F_S \) variables for \( F_B \) variables.

From the definition of the linearization reference equilibrium of this section and with the anticipated development of the linearized equations of motion in \( F_S \), the linearized variables are specified by the following (see also Section 3.2 of this chapter):

\[
(u, v, w) = (V_e + \Delta u, \Delta v, \Delta w) \quad (3.4,7)
\]

\[
(p, q, r) = (\Delta p, \Delta q, \Delta r) \quad (3.4,8)
\]

\[
(W_1, W_2, W_3) = (W_{1e} + \Delta W_1, W_{2e} + \Delta W_2, \Delta W_3) \quad (3.4,9)
\]

\[
(x_I, y_I, z_I) = (x_{Ie} + \Delta x_I, \Delta y_I, z_{Ie} + \Delta z_I) \quad (3.4,10)
\]

\[
(X, Y, Z) = (X_e + \Delta X, \Delta Y, Z_e + \Delta Z) \quad (3.4,11)
\]

\[
(L, M, N) = (\Delta L, \Delta M, \Delta N) \quad (3.4,12)
\]

\[
(\psi, \theta, \phi) = (\psi_e + \Delta \psi, \theta_e + \Delta \theta, \Delta \phi) \quad (3.4,13)
\]

\[
\dot{x}_I = V_e \cos \theta_e \cos \psi_e + W_{1e} \quad (3.4,21a)
\]

\[
\delta^T = (\delta_e^T, \delta_{\psi}, \delta_{\theta}, \delta_{\phi}) = (\delta_x + \Delta \delta_x, \delta_{y_e} + \Delta \delta_{y_e}, \delta_{z_e} + \Delta \delta_{z_e}) \quad (3.4,14)
\]

\( \psi_e \) is chosen such that the lateral drift due to \( W_{2e} \) is zero (i.e. \( \dot{y}_{Ie} = 0 \)). In heavy aircraft such a crabbing technique is the normal procedure, steady sideslips usually being employed by lighter aircraft. The reference conditions for steady sideslip are discussed in Etkin [3.1] and will not be considered in this study.

Substituting (3.4,7) to (3.4,14) into the stability axes version of the general equations of motion (3.3,16), (3.3,17), (3.3,19) and (3.3,20) yields the linearization reference equilibrium relations:

\[
X_e = mg \sin \theta_e \quad (3.4,15)
\]

\[
Y_e = 0 \quad (3.4,16)
\]

\[
Z_e = -mg \cos \theta_e \quad (3.4,17)
\]

\[
L_e = 0 \quad (3.4,18)
\]

\[
M_e = 0 \quad (3.4,19)
\]

\[
N_e = 0 \quad (3.4,20)
\]
or

\[ x_{Ie} = [v_e \cos \theta_e \cos \psi_e + w_{Ie}]t + x_{I0} \]  

(3.4.21b)

\[ \dot{y}_{Ie} = y_{Ie} = 0 \]  

(3.4.22)

\[ \dot{z}_{Ie} = -v_e \sin \theta_e \]  

(3.4.23a)

or

\[ z_{Ie} = -v_e \sin \theta_e t + z_{I0} \]  

(3.4.23b)

\[ \dot{\psi}_e = 0 \]  

(3.4.24)

\[ \dot{\theta}_e = 0 \]  

(3.4.25)

\[ \dot{\psi}_e = 0 \]  

(3.4.26)

and

\[ \psi_e = \arcsin[-w_2/(v_e \cos \theta_e)] \]  

(3.4.27)

One more equation is required in order to be able to specify \( \theta_e \) and \( \psi_e \). This extra condition follows from the requirement that the aircraft track a ground referenced glide slope of angle \( \gamma_G \) in the reference equilibrium (see Fig. 3.5). Thus

\[ \frac{\dot{z}_{Ie}}{\dot{x}_{Ie}} = \tan \gamma_G \]  

(3.4.28)

If (3.4.21a) and (3.4.23a) are substituted into (3.4.28), and (3.4.27) into the resulting equation, a transcendental equation for \( \theta_e \) is obtained. However a more useful result is obtained by direct geometrical considerations. From Figs. 3.5 and 3.6 it may be shown that

\[ V_{pe} = w_{Ie} \cos \gamma_G + [w_{Ie}^2 (\cos^2 \gamma_G - 1) + v_{eV}^2]^{1/2} \]  

(3.4.29)

\[ \gamma_e = \arcsin[V_{pe} \sin \gamma_G/V_{eV}] \]  

(3.4.30)

\[ v_{eV} = V_e \cos \left[ \arcsin \left( \frac{W_e}{V_e} \right) \right] \]  

(3.4.31)

\[ \theta_e = \arcsin \left( \frac{\sin \gamma_e (v_e^2 - w_2^2)^{1/2}}{v_e} \right) \]  

(3.4.32)

From Fig. 3.5 we also have

\[ \sin \psi_e = -w_2/V_e \]  

(3.4.33)

\[ \sin \psi_e = -\frac{W_e}{v_{ep}} = -\frac{w_2}{v_e \cos \theta_e} \]  

(3.4.34)

and

\[ \frac{v_e^2}{v_e^2 - w_2^2} = \frac{\sin^2 \gamma_e}{\sin^2 (\theta_e)} \]  

(3.4.35)
It follows that in general

\[
\psi_e = \psi_e' \quad (3.4,36)
\]

\[
-\theta_e = \gamma_e. \quad (3.4,37)
\]

Relationships are also required for the equilibrium aero-
dynamic and thrust forces and angle of attack. These may be ob-
tained by considering a force balance along the \( \frac{1}{4}S \) and \( \frac{3}{4}S \) axes
(see Fig. 3.2). Dividing the dimensional force balances by
\( \frac{1}{2} \rho V_e^2 \), the nondimensional equilibrium conditions are

\[
C_T \cos(\alpha_f + \epsilon_T) - C_D e - C_W \sin \theta_e = 0 \quad (3.4,38)
\]

\[
-\frac{1}{2} \rho V_e^2 \cdot C_T \sin(\alpha_f + \epsilon_T) - C_L e + C_W \cos \theta_e = 0. \quad (3.4,39)
\]

Here \( \alpha_f \) is the angle of attack with respect to the flr, \( \epsilon_T \) is the angle the thrust line makes with respect to the
flr, \( C_T \) is the total thrust coefficient, \( C_D \) is the total
drag coefficient, \( C_W \) is the weight coefficient, and \( C_L \) is
the total lift coefficient.

In subsonic, high Reynolds number regimes, and for a
specified aircraft configuration (e.g. flap setting \( \delta_F = 40^\circ \),
undercarriage down), \( C_D e \) and \( C_L e \) will be functions of
only \( \alpha_f e \). Thus equations (3.4,38) and (3.4,39) form two
nonlinear algebraic equations for \( \alpha_f e \) and \( C_T e \) given
\( C_D(\alpha_f), \ C_L(\alpha_f), \ \epsilon_T, \ C_W \) and \( \theta_e \). \( C_D(\alpha_f) \) and \( C_L(\alpha_f) \)
are available from the aerodynamic data, \( \epsilon_T \) is a geometric
property of the aircraft, \( C_W \) is specified given \( m, g, \rho, \)
\( V_e \) and \( S \), and \( \theta_e \) can be computed from (3.4,32) given
\( \gamma_e, V_e, W_1 e \) and \( W_2 e \).

3.5 Aerodynamic Forces and Moments

3.5.1 Review of Modeling Techniques

One of the most difficult tasks of the aircraft dynami-
cist is to find suitable representations for \( \mathbf{A} \) and \( \mathbf{M} \). For
flight regimes where the fluid continuum hypothesis applies
the governing equations for these forces and moments are the
Navier-Stokes, the energy and the atmospheric state
equations. The Navier-Stokes and energy equations are partial
differential equations which are nonlinear and extremely
difficult to solve from both the analytical and computational
perspectives. The latter is a consequence of the enormous
computation times and computer storage requirements necessary
to solve problems of a realistic nature. Practically
speaking a direct approach such as this is not feasible.

Using various simplifications (e.g. inviscid flow),
wind tunnel and flight test results, and experience with
particular classes of aircraft, it is possible to formulate
several methods which are useful in specifying the aerodynamic
forces and moments. Broadly speaking these methods may be
summarized in three categories:
1. Analytical methods — These methods include inviscid flow theory to predict the pressure distribution and the induced drag, and boundary layer theory to estimate profile drag and separation conditions. These methods include vortex lattice (Goldhammer, Lopez and Shen [3.8]) and influence coefficient methods (Roskam [B.1]).

2. Wind tunnel and flight test data — These data may be summarized in a suitable form and used as a forcing input to the equations of motion.

3. Aerodynamic derivative methods — Aerodynamic derivatives are frequently referred to as stability derivatives (both terms will be used in this study), and will be discussed in more detail in the following (see also Etkin [3.1]). It is somewhat arbitrary to give these methods separate status inasmuch as both methods (1) and (2) above, or some combination thereof, may be used to estimate stability derivatives.

Of these the inviscid flow and boundary layer theory methods are the most difficult to apply computationally and do not necessarily yield accurate results (e.g. the vortex lattice method does not work well if significant separation regions exist). Such direct techniques will not be used in this investigation.

The second method is useful if a substantial body of data has been obtained for the aircraft aerodynamics (e.g. see Johnson [3.8], and the data of Appendix D). This information may be stored in look-up tables which are functions of the aircraft state, or in the form of aerodynamic derivatives which are functions of certain components of the aircraft state. The required forces and moments are then obtained by interpolation (extrapolation) between (beyond) the data points. A major disadvantage of this method is the practical consideration of obtaining the large body of data needed for each aircraft and aircraft configuration considered.

The third method, the aerodynamic derivative technique, has been traditionally employed to model the aerodynamic forces and moments in rigid body dynamic simulation. It was first proposed by G.H. Bryan in the early 1900's and essentially consists of a Taylor series expansion of the aerodynamic forces and moments about an aerodynamic reference equilibrium. In this expansion Bryan proposed that only linear terms be kept. Furthermore, the expansions are frequently assumed to include only terms in the Taylor series which involve the translational and/or rotational velocities (the exceptions to this are \( \dot{w} \) and \( \dot{v} \) derivatives as these can be shown to arise from strictly quasisteady considerations due to the downwash and sidewash of the aircraft [3.3]).

If the \( \delta_8 \) aerodynamic force \( X \) is considered as an example, then for the landing approach case it may be considered to be a function of \((u, v, w, p, q, r, \delta_E, \delta_T, \delta_A, \delta_R)\) where density, viscosity, Mach number and temperature effects have been assumed to be negligible. Here \( \delta_E, \delta_T, \delta_A, \) and \( \delta_R \) are, respectively, the elevator, throttle, aileron and rudder control settings. If such is
the case then

\[ X = X_e + \Delta X \]  
(3.5.1)

where

\[ \Delta X = X_u \Delta u + X_v \Delta v + X_w \Delta w + X_p \Delta p + X_q \Delta q \]

\[ + X_r \Delta r + \ldots + X_p \Delta p + X_q \Delta q + X_
u \Delta \nu \]

\[ + X_{\delta E} \Delta \delta E + X_{\delta T} \Delta \delta T + X_{\delta A} \Delta \delta A + X_{\delta R} \Delta \delta R \]

\[ + X_{\delta E} \Delta \delta E + \ldots + (\text{higher order terms}). \]  
(3.5.2)

Here

\[ X_u = \left( \frac{\partial X}{\partial u} \right)_e \]  
(3.5.3a)

\[ X_v = \left( \frac{\partial X}{\partial v} \right)_e \]  
(3.5.3b)

and so forth are referred to as aerodynamic derivatives. The higher order terms require a considerable amount of space to write out fully and involve terms such as \( X_{uu}, X_{uv}, X_{\nu \nu}, \ldots \) and so forth (see Thelander [3.9]). If it is assumed that only linear terms in \( \Delta u, \Delta v, \Delta w, \Delta p, \Delta q, \Delta r, \Delta \delta E, \Delta \delta T, \Delta \delta A, \Delta \delta R, \Delta \nu, \Delta \omega \) are significant, then (3.5.2) becomes

\[
\Delta X = X_u \Delta u + X_v \Delta v + X_w \Delta w + X_p \Delta p + X_q \Delta q + X_r \Delta r + \ldots + (\text{higher order terms}). \]  
(3.5.4)

This is the essence of Bryan's approximation. If it is further assumed that the aircraft is in a symmetric flight condition, then \( \Delta v, \Delta p, \Delta r, \Delta \nu, \Delta \delta A \) and \( \Delta \delta R \) are zero and (3.5.4) reduces to

\[ X = X_e + X_u \Delta u + X_w \Delta w + X_q \Delta q \]

(3.5.5)

Finally, for most conventional aircraft in subsonic flight, \( X_u, X_q \) and \( X_{\delta E} \) are negligible and thus equation (3.5.5) becomes

\[ X = X_e + X_w \Delta w + X_q \Delta q \]

(3.5.6)

Equation (3.5.6) is the form used by Reid et al [1.39].

The first order separation of the longitudinal or symmetric variables from the lateral or asymmetric variables is commonly done and may result in uncoupling between the longitudinal and lateral equations of motion. This is only true if there are no rotor terms (as is assumed in this study), and if either or both of the following conditions are satisfied:
1. The aircraft remains in its symmetric flight mode (i.e. in its lateral equilibrium, $\Delta v = 0$, $\Delta p = 0$, etc.) when studying longitudinal response and in its longitudinal equilibrium (i.e. $\Delta u = 0$, $\Delta w = 0$, etc.) when studying lateral response.

2. The longitudinal and lateral coupling derivatives (e.g. $X_v$, $Y_u$ and so forth) are negligible; also indirect longitudinal and lateral coupling is negligible, i.e. longitudinal (lateral) derivatives are not functions of lateral (longitudinal) variables.

That (1) is not the case can be seen from the fact that any realistic disturbances will not be strictly longitudinal or lateral in nature and will thus excite the modes which lead to coupling.

That (2) is not always the case follows from the fact that propeller aircraft develop a yawing moment due to the rotating propeller (i.e. asymmetric disc loading, slipstream effects and so forth). This moment may be trimmed out at equilibrium but longitudinal changes in the airspeed ($\Delta u$) or in the throttle ($\Delta \delta_m$) will take the aircraft out of this trim condition and produce a yawing moment. Similarly, some lateral stability derivatives are strongly dependent on the lift coefficient $C_L$, and thus significant changes in $C_L$ (such as occur in wind shear) may result in significant changes in the lateral stability derivatives and thus the lateral response, even if the wind was strictly longitudinal in nature. Nevertheless for many purposes these coupling effects can be considered to be negligible for a first order linear approximation to the aircraft aerodynamics.

There are two extensions possible to the linear stability derivative expansion. The first is suggested by the $C_L$-coupling discussed above and states that if certain derivatives are made functions of the state of the aircraft, or at least functions of certain elements of or related to the state of the aircraft, then a more realistic representation of the forces and moments acting on the aircraft is obtained. This technique is mentioned in Etkin [3.1] and an example of its application is given in Case [3.10] (Orlik-Ruckemann [3.11] measures cross-coupling derivatives in a wind tunnel for a nonspecific aircraft model and discusses the conditions under which they are significant). The second suggests that the linear stability derivative expansion be extended to include nonlinear terms, i.e. $X_{uu}$, $X_{uw}$ and so forth, (see Etkin [3.1], Thelander [3.9], and Gersten [3.12]). While potentially providing a means of extending the validity of the stability derivative aerodynamic representation to a greater variety of problems, these two techniques suffer from the same theoretical objection. This is simply that the aerodynamic forces and moments are in reality functionals of the state of the aircraft and its past history. Physically this is due to the shed vortices altering the flow field about the aircraft itself. In effect this means that the concept of aerodynamic derivatives is only valid...
about an aerodynamic steady state. In an aerodynamic steady state the flow field about and behind the aircraft may be expressed strictly as a function of the position relative to the aircraft rather than as a function of time and position. Goldhammer et al [3.7] state that stability derivatives exist and are constant only in the following conditions:

1. If the velocities of deviation are small in relation to the displacements of deviation, and the acceleration is small in relation to the velocities, then the derivatives are constant and can be evaluated on the basis of purely quasi-steady aerodynamics.

2. If the aerodynamic perturbations are all proportional to the exponential function of time $e^{\lambda t}$, where $\lambda$ is a real, pure imaginary or complex constant, and the motion of deviation has existed for an infinite time, then the derivatives are constants dependent on $\lambda$.

The latter implies that the stability derivatives to be used for the phugoid and short-period mode calculations are, strictly speaking, different. These frequency effects are usually not taken into account, however, except in flutter computations (Duncan [3.13]).

An alternative to aerodynamic derivatives is the concept of indicial functions [Tobak [3.14] and Etkin [3.15]) or their nonlinear functional counterparts (Tobak and Pearson [3.16]). Indicial functions are measures of the step response of a particular aerodynamic force or moment to a particular step in an aerodynamic state variable. If the step response of lift coefficient to a unit change in angle of attack is found (call it $A_{L\alpha}(t)$), then if the system is linear the response to an arbitrary angle of attack change $\alpha(t)$ is given by the convolution integral as (Etkin [3.15])

$$C_L(t) = \int_0^t A_{L\alpha}(t-\tau) \frac{d\alpha(\tau)}{d\tau} d\tau + C_L(0).$$

The first order stability derivative approximation to this is given by

$$C_L(t) = C_{L\alpha} \alpha(t) + C_L(0)$$

where $C_{L\alpha}$ is the stability derivative (a constant at a particular equilibrium) relating lift and angle of attack about $C_L(0)$. It is evident from the expressions that the two do not yield the same result. However, as $t \to \infty$, and assuming $A_{L\alpha}(t) \to 1$, then

$$\int_0^t A_{L\alpha}(t-\tau) \frac{d\alpha(\tau)}{d\tau} d\tau \to C_{L\alpha} \alpha(t).$$

In particular, if $\alpha(t)$ is a step function, then

$$\Delta C_L(t) = A_{L\alpha} \alpha(t).$$

This involves a time dependent transient not taken into account by the stability derivative approximation.

Three further comments may be made here with regard to indicial functions. The first is that while they do yield unsteady flow effects, they are still derived with the
assumption of linear aerodynamics and are thus not valid for large disturbance cases.

The second comment is that their nonlinear counterparts, as discussed by Tobak and Pearson [3.16], while eliminating the above objection are very difficult to estimate analytically.

The third and final comment is that both indicial functions and their nonlinear counterparts have little data available on which to base empirical estimates.

3.5.2 Selection of Aerodynamic Models

From the discussion in the previous section it is apparent that aerodynamic derivatives and tabulated aerodynamic data are the most convenient methods of modeling the aerodynamics for simulation purposes. The major doubts about the validity of such models focuses on their ability to predict aerodynamic loads in large, rapid disturbance maneuvers where unsteady flow and separation effects may be significant. To some extent these difficulties may be alleviated by employing look-up table aerodynamic data which is a function of certain elements in the aircraft state. There is also some evidence that even for large disturbance maneuvers unsteady flow effects are negligible provided that the aircraft remains in an approximately linear aerodynamic regime, i.e. separation effects are not significant (Goldhammer, Lopez and Shen [3.7], Rhoads and Schuler [3.17] and Hoak, Carlson and Malthan [3.18]).

Indicial aerodynamics, while occasionally applied to simplified aircraft dynamic response studies (Etkin [3.15] and Wells [3.19]), was not considered further because of the lack of complete indicial function aerodynamic models for any of the classes of aircraft suitable for this study. More direct analytical methods (e.g. vortex lattice methods), while providing mathematically rigorous procedures for predicting the aerodynamic loads, are unlikely to be more accurate than aerodynamic data based on wind tunnel and flight test measurements.

For these reasons the decision was made to proceed with two quasisteady aerodynamic models. The first of these, which will be referred to as model AERO1, is a constant, linear, decoupled aerodynamic derivative model. It preserves the linearity and time-invariance of the linearized equations of motion to be developed later in this chapter, and thus provides a convenient aerodynamic model on which the controller synthesis and worst-case variable wind modeling work may be carried out (see Chapter 4, Chapter 5 and Appendix F).

The second model, which will be referred to as model AERO2, is a nonlinear, quasisteady look-up table aerodynamic model to be used with the general body-fixed nonlinear equations of motion. The look-up parameters are thrust coefficient $C_T$, angle of attack $\alpha$, and flap setting* $\delta_F$.

* Flap setting is included for generality. In the numerical examples only one flap setting is considered.
The lateral aerodynamics are coupled with the longitudinal dynamics, but not vice versa.

The aerodynamic representations are summarized below and are discussed in more detail in Appendix D. The form that they take is compatible with the structure of the available data and the form of the equations of motion in which they are to be used.

Model AERO1
(In stability axes $F_S$)

Longitudinal Aerodynamics

\[
\Delta X = X_u \Delta u + X_w \Delta w + X_q \Delta q + X_{\delta E} \Delta \delta E + X_{\delta T} \Delta \delta T \quad (3.5,10)
\]

\[
\Delta Z = Z_u \Delta u + Z_w \Delta w + Z_q \Delta q + Z_{\delta E} \Delta \delta E + Z_{\delta T} \Delta \delta T \quad (3.5,11)
\]

\[
\Delta M = M_u \Delta u + M_w \Delta w + M_q \Delta q + M_{\delta E} \Delta \delta E + M_{\delta T} \Delta \delta T \quad (3.5,12)
\]

Lateral Aerodynamics*

\[
\Delta Y = Y_v \Delta v + Y_p \Delta p + Y_r \Delta r + Y_{\delta A} \Delta \delta A + Y_{\delta R} \Delta \delta R \quad (3.5,13)
\]

\[
\Delta L = L_v \Delta v + L_p \Delta p + L_r \Delta r + L_{\delta A} \Delta \delta A + L_{\delta R} \Delta \delta R \quad (3.5,14)
\]

Model AERO2
(In body axes $F_B$)

Longitudinal Aerodynamics

\[
X_B = \frac{1}{2} \rho V^2 S \left[ -C_D \cos \alpha_f + C_L \sin \alpha_f + C_T \cos \epsilon_T \right] \quad (3.5,16)
\]

\[
Z_B = \frac{1}{2} \rho V^2 S \left[ -C_D \sin \alpha_f - C_L \cos \alpha_f - C_T \sin \epsilon_T \right] \quad (3.5,17)
\]

\[
M_B = \frac{1}{2} \rho V^2 S \bar{c} \quad (3.5,18)
\]

where

\[
C_L = C_{L_1} (\delta_F, \delta_T, \alpha_f) + C_{L_2} (\delta_F, \delta_T, \alpha_f, \delta_f, q_B, \delta_E) \quad (3.5,19)
\]

\[
C_D = C_{D_1} (\delta_F, \delta_T, \alpha_f) + C_{D_2} (\delta_F, \delta_T, \alpha_f, \delta_f, q_B, \delta_E) \quad (3.5,20)
\]

\[
c_B = c_m = C_{m_1} (\delta_F, \delta_T, \alpha_f) + C_{m_2} (\delta_F, \delta_T, \alpha_f, \delta_f, q_B, \delta_E) \quad (3.5,21)
\]

and

\[
C_{L_2} = \frac{-c}{2V} C_{L_1} (\delta_F, \delta_T, \alpha_f) \delta_f + \frac{c}{2V} C_{L_3} (\delta_F, \delta_T, \alpha_f) q_B
\]

\[+ C_{L_4} (\delta_F, \delta_T, \alpha_f) \delta_E \quad (3.5,22)\]

* $v$ derivatives are assumed to be negligible.
Finally, 
\[ C_D = \frac{c}{2V} C_{D_1} (\delta_F, C_T, \alpha_f)^2 + \frac{c}{2V} C_{D_2} (\delta_F, C_T, \alpha_f)^2 q + C_D (\delta_F, C_T, \alpha_f)^2 \delta_E \]  
\[ \frac{c}{2V} C_{D_1} (\delta_F, C_T, \alpha_f)^2 + \frac{c}{2V} C_{D_2} (\delta_F, C_T, \alpha_f)^2 q + C_D (\delta_F, C_T, \alpha_f)^2 \delta_E \]  
(3.5,23)

\[ C_m = \frac{c}{2V} C_{m_1} (\delta_F, C_T, \alpha_f)^2 + \frac{c}{2V} C_{m_2} (\delta_F, C_T, \alpha_f)^2 q_B + C_{m_1} (\delta_F, C_T, \alpha_f)^2 \delta_E \]  
\[ \frac{c}{2V} C_{m_1} (\delta_F, C_T, \alpha_f)^2 + \frac{c}{2V} C_{m_2} (\delta_F, C_T, \alpha_f)^2 q_B + C_{m_1} (\delta_F, C_T, \alpha_f)^2 \delta_E \]  
(3.5,24)

Finally, 
\[ C_T = C_T (V) \delta_T \]  
(3.5,25)

### Lateral Aerodynamics*

\[ Y_B = \frac{c}{2V} \delta_S C_{Y_B} (\delta_F, C_T, \alpha_f, \beta, P_B, R_B, \delta_A, \delta_R) \]  
\[ \frac{c}{2V} \delta_S C_{Y_B} (\delta_F, C_T, \alpha_f, \beta, P_B, R_B, \delta_A, \delta_R) \]  
(3.5,26)

\[ L_B = \frac{c}{2V} \delta_S C_{L_B} (\delta_F, C_T, \alpha_f, \beta, P_B, R_B, \delta_A, \delta_R) \]  
\[ \frac{c}{2V} \delta_S C_{L_B} (\delta_F, C_T, \alpha_f, \beta, P_B, R_B, \delta_A, \delta_R) \]  
(3.5,27)

\[ N_B = \frac{c}{2V} \delta_S C_{N_B} (\delta_F, C_T, \alpha_f, \beta, P_B, R_B, \delta_A, \delta_R) \]  
\[ \frac{c}{2V} \delta_S C_{N_B} (\delta_F, C_T, \alpha_f, \beta, P_B, R_B, \delta_A, \delta_R) \]  
(3.5,28)

\[ C_B = C_{Y_B} \beta + \frac{b}{2V} C_{Y_B} P_B + \frac{b}{2V} C_{Y_B} R_B + C_{Y_B} \delta_A + C_{Y_B} \delta_R \]  
\[ \frac{c}{2V} \delta_S C_{Y_B} (\delta_F, C_T, \alpha_f, \beta, P_B, R_B, \delta_A, \delta_R) \]  
(3.5,29)

\[ C_B = C_{L_B} \beta + \frac{b}{2V} C_{L_B} P_B + \frac{b}{2V} C_{L_B} R_B + C_{L_B} \delta_A + C_{L_B} \delta_R \]  
\[ \frac{c}{2V} \delta_S C_{L_B} (\delta_F, C_T, \alpha_f, \beta, P_B, R_B, \delta_A, \delta_R) \]  
(3.5,30)

where

\[ C_B = C_{Y_B} \beta + \frac{b}{2V} C_{Y_B} P_B + \frac{b}{2V} C_{Y_B} R_B + C_{Y_B} \delta_A + C_{Y_B} \delta_R \]  
(3.5,29)

\[ C_B = C_{L_B} \beta + \frac{b}{2V} C_{L_B} P_B + \frac{b}{2V} C_{L_B} R_B + C_{L_B} \delta_A + C_{L_B} \delta_R \]  
(3.5,30)

---

* The argument list \((\delta_F, C_T, \alpha_f)\) has been dropped from all the lateral stability derivatives for the sake of brevity.

### 3.6 The Uniform-Gust Approximation

The aerodynamic models presented in the previous section have tacitly assumed that there is no variation of the wind velocity from one point of the aircraft's surface to another. This is equivalent to assuming that the wind induced aerodynamic loads are determined by its velocity rate of change acting at the centre of mass. Thus the winds produce their aerodynamic effects via their interaction with the centre of mass dynamic variables (i.e. \(u, v, w, p, q, r\)), as governed by the equations of motion developed in Section 3 of this chapter. This has been referred to as the point approximation (Etkin [3.1]), a term which is unfortunately easily confused with the dynamic point approximation often used in performance analyses. Etkin [1.3] has recently suggested the more appropriate name uniform-gust approximation. This term will be used in the sequel.

A spectral component of a nonuniform wind field acting on a finite airplane is depicted in Fig. 3.7. The major effects of variable winds on the aerodynamic loads on the aircraft beyond those encompassed by the centre of mass contributions will be to produce extra moments about the three aircraft axes. These effects may be taken into account by a number of techniques (Etkin [1.3, 3.1]), at considerable increase in complexity. Most of these methods are intended
for application to stochastic studies of aircraft response to turbulence and require a probabilistic description of the turbulence field (e.g. via the appropriate correlation) as well as an extended aerodynamic model (e.g. in the quasi-steady aerodynamic derivative sense, the total derivatives may have to be broken down into the contributions of the tail, wing and fuselage).

The simplest and most frequently employed of these techniques define equivalent wind rolling, pitching and yawing rates \((p_g, q_g, r_g)\) which then interact with the aircraft dynamics by producing extra aerodynamic forces and moments. In Etkin's method [1.3] (see also McRuer, Ashkenas and Graham [3.3] and Roskam [8.1]) \(p_g, q_g\) and \(r_g\) are based on the wind gradients (in some suitable body fixed reference frame) at the centre of mass of the aircraft. Thus, for example, if only turbulence in the direction of the \(z_s\) axis is considered, then

\[
q_g = -\left(\frac{\partial \vec{w}_S}{\partial x_S}\right)_{x_S=y_S=z_S=0}
\]

(3.6.1)

where \(x_S\) is a linear position coordinate measured from the origin of \(F_S\) along \(z_S\), and \(\vec{w}_S^3\) is the third component of \(\vec{w}\) expressed in \(F_S\).

It is important to note that it is not generally possible to compute the extra aerodynamic inputs resulting from \(p_g, q_g\) and \(r_g\) by assuming that they act through the same aerodynamic derivatives as the centre of mass contributions, because of the detailed way in which these derivatives are obtained (Etkin [1.3], McRuer et al. [3.3] and Roskam [8.1]).

For example, \(\frac{\partial \vec{w}_S}{\partial x}\) and \(\frac{\partial \vec{w}_S}{\partial y}\) both produce equivalent yawing rates \(r_g\) (say \(r_{1g}, r_{2g}\) respectively), although their aerodynamic effects are clearly different. This suggests that even for this relatively simple technique of relaxing the uniform-gust approximation, the corresponding aerodynamic model increases considerably in complexity and requires a component by component breakdown of the aerodynamic derivatives.

In view of the previous discussion, and for the following reasons, the decision was made to adopt the uniform-gust approximation throughout this study:

1. In this investigation the primary concerns are (1) studying the airspeed stability, lift, and glidepath tracking characteristics of the aircraft on the landing approach in the presence of variable winds and (2) developing techniques for generating worst-case discrete wind models. This focus is a natural consequence of the observed characteristics of the low altitude incidents and accidents motivating this investigation (see the discussion in Chapter 1, Section 1 and Chapter 4, Section 2). In these incidents and accidents the ability to control roll, pitch and yaw did not seem to be significantly
compromised, as might be expected if \( p_g, q_g \) and \( r_g \) were significant from the safety point of view.

2. The development of discrete worst-case wind models, for practical reasons, favours the use of the uniform gust approximation, in which the number of wind inputs are kept to a minimum (see Chapter 5).

3. Comprehensive aerodynamic data for better than point approximation models was not available. This also prevented a direct validation of the uniform-gust approximation by comparison with simulations where it is relaxed.

The most general method of checking the resulting dynamic and aerodynamic model is to compare the predicted response of the aircraft to recorded real wind and control inputs with the observed response of the real aircraft. Such data is occasionally available (see, for example, the CATA report [3.20]), although the simultaneous availability of aerodynamic modeling data that is compatible with the real aircraft is uncommon. Such a general validation is beyond the scope of this study but it is recommended that it be pursued in future research.

3.7 Curved Approach Reference Trajectories

The concept of flying a curved spatial trajectory for reducing the undesirable effects of variable winds was introduced in Chapter 1, Section 1.2.1. Broadly speaking such a trajectory is encompassed by a dynamic state trajectory which falls within a specified range of a desired approach state envelope and a desired decision height state window (Fig. 3.8). Symbolically this may be written

\[
x_C = x(t; x_C(t_0), \delta_C, W_C)
\]

where the system dynamics require that

\[
\dot{x}_C = f(x_C, \delta_C, W_C, \dot{W}_C, t).
\]

The notation in (3.7,1) stresses the fact that a curved reference state trajectory \( x_C \) can only be determined if the initial conditions \( x_C(t_0) \), the control history (or control law) \( \delta_C \) and the wind inputs \( W_C \) are specified \( a \ priori \). A subscript lower case 'c' will be used to denote quantities related to the curved approach. To satisfy the envelope requirements, constraints of the form

\[
\varepsilon_{L_1}(t) \leq x_{C_1}(t) \leq \varepsilon_{U_1}(t)
\]

or some equivalent form, should also be implemented for \( t \in [t_0, t_d] \). Here \( \varepsilon_{L_1} \) and \( \varepsilon_{U_1} \) are, respectively, the lower and upper limits on the \( i \)-th component \( x_{C_1}(t) \) of the desired state trajectory \( x_C \), and \( t_d \) is the time of decision height passage.

The practical limitations for implementing such curved reference trajectories arise from a number of factors. These are as follows:
1. The accuracy with which the wind conditions at the time of aircraft passage are known.

2. The availability of landing aids with which general curved spatial trajectories may be defined.

3. The accuracy with which the aircraft dynamics and aerodynamics are known.

4. The onboard aircraft computing power with which to compute the general approach reference state trajectory.

Limitations (1) and (2) are no longer formidable technological barriers with the recent progress in the development of real time wind sensing systems and with the advent of microwave landing systems (see the discussion in Chapter 1, Section 1.2.1 and Chapter 4, Section 4.4.3). Limitations (3) and (4) are unfortunately more difficult to overcome.

In order to obtain a general \( x_c \), equation (3.7.2) will have to be solved iteratively either backwards in time until suitable initial conditions are found, or forwards in time until suitable terminal conditions are found, for given \( w_c \) and \( \delta_c \). This is not a practical computation for aircraft crews and onboard computers and thus this general technique, for the time being, is only of pedagogical interest.

The general approach can be simplified considerably, however, if the aircraft groundspeed vector is assumed to be well approximated by the vector sum of an airspeed vector \( v_c \) of specified magnitude and the estimated wind vector \( w_c \).

Thus

\[
\frac{V_E}{V_c} = \frac{V_c + w_c}{v_c}.
\]

In the following section it will be shown how (3.7.4) leads to a spatial trajectory which approximates the actual trajectory. The two will be the same if (1) \( V \) is identical to \( V_c \), (2) the dynamic effects of the variable wind is negligible and (3) the estimated wind velocity vector \( w_c \) corresponds to the actual wind velocity vector \( w \) encountered throughout the landing approach.

3.7.1 Kinematic Curved Glidepath Approach Reference Trajectories

The development in this section will ultimately lead to a kinematic formulation which defines a CGG that is based on an a priori estimate of the existing wind conditions and which compensates for some of the undesirable wind effects on the aircraft. This formulation will be used to generate a number of curved glidepaths that will be flown by the STOL aircraft in the simulations of Chapter 6.

The vector equation (3.7.4) may be expressed as a matrix equation in \( F_I \) components as

\[
V^I_E = L^I S^I v_c + w^I_c.
\]

Let us assume that \( v_c \) is constant (say \( \| V_c \| = V_c \)). Now
also assume that the CGA pitch angle \( \theta_c \) is also constant. \( V_c \) and \( \theta_c \) are not necessarily the airspeed and elevation Euler angles about which the linearization of the equations of motion occurs (see Section 3.4 of this chapter), although it is possible to define the linearization reference equilibrium in such a way as to make this true.

The scalar equations implied by (3.7,5) and the assumptions on \( \theta_c \) and \( V_c \) are a simplified version of the equations (3.3,19). They are, in stability axes variables

\[
(\theta_c, \, V_c \text{ constant}, \, u = V_c, \, v = w = \phi = 0, \, \psi = \psi_c(t))
\]

\[
\begin{align*}
\dot{x}_c & = \cos \theta_c \cos \psi_c(t) v_c + \dot{W}_c \cdot (x_{c0}, y_{c0}, z_{c0}, t) \tag{3.7,6a} \\
\dot{y}_c & = \cos \theta_c \sin \psi_c(t) v_c + \dot{W}_c \cdot (x_{c0}, y_{c0}, z_{c0}, t) \tag{3.7,6b} \\
\dot{z}_c & = -\sin \theta_c v_c + \dot{W}_c \cdot (x_{c0}, y_{c0}, z_{c0}, t) \tag{3.7,6c}
\end{align*}
\]

where the stability axes are chosen because of the simpler form that the equations (3.7,6) take in this reference frame. In writing (3.7,6) \( \psi \) is allowed to be a function of time (to be determined) to take into account variable crosswinds encountered during the approach.

The equations (3.7,6) are still not in a convenient form. They are a system of differential equations which must be solved repeatedly for different initial conditions \( x_{c0}, y_{c0}, \) and \( z_{c0} \), given \( \theta_c, V_c \) and \( W_c \), in order to determine a suitable approach trajectory. In the forward direction this is something of a hit and miss computation and is not of great practical use. These difficulties may be overcome to some extent by performing the integration in the reverse direction (i.e., starting with the desired final conditions and integrating backwards in time to determine the resulting initial conditions) although this suffers from problems of its own. These include no a priori knowledge of the final time, and the practical consideration that such a computation is still quite involved for onboard aircraft computers.

The situation may be considerably improved by making the following assumptions in addition to the two assumptions made above:

1. The pilot maintains lateral track by crabbing the aircraft into the wind and/or by sideslipping. Thus a curved lateral trajectory is not necessary and

\[
\dot{y}_c = y_{c0} = 0. \tag{3.7,7}
\]

2. The variations in \( \psi \) may be considered approximated by the solution of (3.7,6b) with the condition (3.7,7).

Thus
Equation (3.7,8) will hold exactly if the pilot maintains lateral track by crab alone, and if the rate of change of $W_{2c}$ that the aircraft is assumed to encounter accurately models the real wind changes and is "slow enough" so that dynamic effects are not significant.

3. Downdrafts and updrafts are considered negligible. Thus

$$W_{3c} = 0. \quad (3.7,9)$$

From equation (3.7,6c) and the constant $V_c$ and $\theta_c$ assumptions, this implies that

$$\dot{h}_c = \sin(\theta_c)V_c \quad \text{(constant)} \quad (3.7,10)$$

or, upon integrating,

$$h_c = \sin(\theta_c)V_c t + h_{c0}. \quad (3.7,11)$$

4. The horizontal wind velocity is assumed to be a function of height only, i.e. the wind that is measured along the glidepath – localizer region is related to height. This may always be done formally by creating a one-to-one correspondence between the wind vector and height along the glidepath.

5. The wind velocity above the ABL is constant. If it is not one can continue to define a CGA to an arbitrary altitude, although in practical terms most approaches begin between 300 and 600 meters above the airport elevation.

With these assumptions only equation (3.7,6a) remains to be solved. This may be integrated directly to yield

$$x_{Ic} = \int_0^t \cos \theta_c \cos \psi_c(t)V_c dt + \int_0^t W_{c}(h_c(t))dt + x_{IC0}. \quad (3.7,12)$$

In equations (3.7,11) and (3.7,12) it has been tacitly assumed that $t_0 = 0$, with no loss of generality. These equations form the basis for the CGG computation. If $\psi_c$ is constant then the first integral in (3.7,12) may be evaluated immediately, and thus $x_{Ic}(t)$ becomes

$$x_{Ic} = \cos \theta_c \cos \psi_c V_c t + \int_0^t W_{c}(h_c(t))dt + x_{IC0}. \quad (3.7,13)$$

Equations (3.7,11) and (3.7,12) contain the unknowns $x_{Ic0}$, $h_{IC0}$, and $\theta_c$ ($\psi_c(t)$ is given by (3.7,8)). The initial height at which the CGA begins is arbitrary and is specified a priori, but $x_{Ic0}$ and $\theta_c$ must be chosen so that the CGA passes through a transition point located at $(x_{IT}, 0, -h_T)$ with ground referenced flight path angle $\gamma_G$ (Fig. 3.1). This condition is included so that the aircraft is properly trimmed for landing. The transition height $h_T$ may or may not be the same as the decision height $h_D$. The value of $x_{IT}$ depends...
on the definition of $F_I$. With the origin and orientation of $F_I$ defined as in Section 3.2 of this chapter (see also Fig. 3.1), it follows that

$$x_{IT} = -h_T \cot \gamma_G. \tag{3.7,14}$$

The transition height flight path angle condition may be written

$$\left[ \frac{h_C}{x_{IT}} \right] = -\tan \gamma_G \tag{3.7,15}$$

where $t_T$ is the time of transition height passage. This condition may be used to derive a relationship for $\theta_C$ by making the observation that the problem is now formally identical to that posed by the condition (3.4,28) as used in determining a relationship for $\theta_e$ in the definition of the linearization reference equilibrium. Thus by considering the geometry of Figs. 3.5 and 3.6 with 'e' subscripted variables replaced by the corresponding curved approach variables, the solution is found to be completely analogous to the equations (3.4,29), (3.4,30), (3.4,31) and (3.4,32), i.e.

$$V_{E_C T} = \frac{W_1}{c_T} \cos \gamma_G + \left[ \frac{W_2}{c_T} \cos^2 \gamma_G - 1 \right] + \frac{V_{C V}}{c_T} \tag{3.7,16}$$

$$\gamma_C = \arcsin \left[ \frac{V_{E_C T}}{c_T} \sin \gamma_G / c_{CV} \right] \tag{3.7,17}$$

$$V_{C V} = V_c \cos \left[ \arcsin \left( \frac{W_2}{c_T} \right) \right] \tag{3.7,18}$$

$$\theta_C = \arcsin \left[ \sin \gamma_C \left( \frac{V_{C V}^2 - W_2^2}{c_T V_c} \right) ^{\frac{1}{2}} \right] \tag{3.7,19}$$

Knowing $\theta_C$ one may use (3.7,11) to determine $t_T$. This yields

$$t_T = \frac{(h_T - h_{C_0})}{V_c \sin \theta_C} \tag{3.7,20}$$

Finally, $x_{I_C 0}$ follows from (3.7,12) as

$$x_{I_C 0} = x_{IT} - \int_0^{t_T} \cos \theta_c \cos \psi_c(t) V_c \, dt \tag{3.7,21}$$

where $\psi_c(t)$ approximated by (3.7,8)*.

In Fig. 3.9 several curved glidepath approach paths are shown for a sampling of the family of wind profiles defined in Chapter 2, Section 2.1.

In Appendix E analytical solutions are developed for curved glidepath geometries that result from two simple forms

* For some purposes it may be advantageous to use a constant $\psi_c$, obtained from (3.7,8) by fixing $W_2$ at a suitably chosen constant value.
of the estimated wind profile. This is done in order to
demonstrate the procedure and to provide test solutions with
which the numerical procedures may be checked.

\( h_c \) and \( X_{tc} \), as they are now specified, are time
histories. As far as the glidepath tracking task is concerned,
such a specification has the undesirable added requirement
of the aircraft being at a given point at a certain time.
This problem may be avoided by defining purely spatial tra­
jectories. These may always be generated by treating \( h_c \) as
a function of \( x_I \), i.e. establishing a one-to-one correspon-
dence between (3.7,11) and (3.7,12) or (3.7,13), using time
as a parameter.

To summarize, the kinematic curved glidepath technique
has been proposed to provide a practical method of defining
approach trajectories which are functions of an estimate \( W_c \)
of the existing wind field. The method is referred to as a
kinematic method because it assumes that the dynamic effects
of the variable winds are negligible by treating \( V_c \) and \( \theta_c \)
as being well approximated by constants. Other important
assumptions that are made are that the aircraft controller
provides good lateral tracking (see (3.7,7)), that variations
in \( \psi \) may be approximated by (3.7,8), that downdrafts and
updrafts are negligible, that the estimated horizontal wind
velocity is a function of height only, and that \( h_{c0} \), \( h_T \)
and \( \gamma_G \) are specified a priori. Under these assumptions
the CGG is specified by (3.7,11), (3.7,12), and (3.7,16)
through to (3.7,21).

3.8 Actuator Dynamics

In the aerodynamic models of Section 3.5.2 of this chapter,
the aircraft controls were assumed to consist of elevator
\( (\delta_E) \), throttle \( (\delta_T) \), aileron \( (\delta_A) \) and rudder \( (\delta_R) \), as
is appropriate for the STOL transport which will be used in
the numerical examples (see Appendices C and D). Other
controls which are not implemented in this model, but could
readily be introduced, include direct lift controls, speed
brakes, spoilers and canards.

To this point in the development it has been assumed
that the aircraft controller (human pilot or automatic pilot)
obtains control response instantaneously and exactly as
commanded. This can never be true in reality. For example,
the aerodynamic control actuators contain nonidealities (e.g.
control line friction) and behave as a dynamic system between
the controller and the control surfaces. For the aerodynamic
controls these effects are fortunately largely of interest
in designing safe, reliable, certifiable control, feel and
actuator systems, and to studying control free stability.
They are generally not considered to be significant in pre­
dicting the closed-loop gross rigid body motion of the air­
craft in variable winds provided that the assumption is made
that the underlying control system is well designed. This
is particularly true in low speed flight, such as on the
landing approach, where aeroelastic effects on the control
surfaces of conventional aircraft are usually negligible.
Thus in this study actuator dynamics models for the elevator, aileron, and rudder controls will not be considered.* This is compatible with the characteristics of the aircraft variable wind accidents reviewed in Section 1 of Chapter 1, where aerodynamic control surface actuator dynamics do not appear to be a significant factor.

Engine response to throttle cannot be dealt with as easily, even though the actual throttle command to the engine can be assumed to reach the engine instantly. This is a consequence of engine acceleration characteristics, and may lead to significant lags between commanded thrust and actual thrust, particularly for pure jet and turbofan engines. For large turbofan engines the spool-up time from an idle setting to maximum thrust may be greater than 10 seconds. Such lags are too large to be ignored, and contribute to the hazards of low level variable wind encounters (e.g. see Long and Dale [1.41] and Huber [2.1]).

In this study Huber's [2.1] first order lag model for the engine characteristics will be adopted. By using a first order model the engine lag is introduced without creating extra engine dynamic modes not significant to the variable wind response characteristics being considered. It is given by

\[ \delta_T = -\frac{1}{T_{\delta_T}} \delta_T + \frac{1}{T_{\delta_T}} \delta_c \]

(3.8,1)

where \( T_{\delta_T} \) is the engine time constant, \( \delta_c \) is the commanded throttle setting, and \( \delta_T \) is the "throttle setting" used to compute the actual engine thrust (see section 3.5.2 of this chapter).

3.9 Nonlinear Equations of Motion with Quasisteady Aerodynamics

The nonlinear quasisteady aerodynamic model AER02 defined in Section 3.5 of this chapter is now introduced into the general force and moment equations of motion (3.3,16) and (3.3,17). These are augmented with the linear and angular position equations (3.3,19) and (3.3,20), with an exact equation for angle of attack \( \alpha_f \), obtained by differentiating (3.2,18), with an exact equation for total airspeed \( V \), obtained by differentiating (3.2,21), with the normal glidepath deviation (d) equation (3.3,23), with the engine dynamics equation (3.8,1), and with an exact equation for the sideslip angle \( \beta \), obtained by differentiating (3.2,19b). Finally, all explicit time derivatives of the response variables are moved to the left hand side of the differential equations. The equations of motion that result are as follows:

* Huber [2.1] points out that modern elevator actuators do not have lags that are significant from the gross rigid body motion point of view. For further general discussion of actuator dynamics modeling, see Etkin [3.1] and McRuer, Ashkenas and Graham [3.3].
\[ \dot{u}_B = \frac{\rho V SC}{4m} \left[ -\cos \alpha_f C_{D_a} + \sin \alpha_f C_{L_a} \right] \dot{\alpha}_f \]

\[ = -q_B^2 u_B + r_B v_B - \beta_{BI11} \dot{\hat{w}}_1 - \beta_{BI12} \dot{\hat{w}}_2 - \beta_{BI13} \dot{\hat{w}}_3 \]

\[ + \frac{1}{\rho \rho^2} \left\{ -\cos \alpha_f \left[ C_{D_1} + \frac{C_{D}}{2V} C_{D} q_B \right] \right. \]

\[ + C_{D_E} \delta_E \right] \sin \alpha_f \left[ C_{L_1} + \frac{C_{L}}{2V} C_{L} q_B + C_{L_E} \delta_E \right] \]

\[ + \cos \epsilon_T C_T \right\} / m + g \xi_{BI13} \]

(3.9,1)

\[ \dot{w}_B = \frac{\rho V SC}{4m} \left[ -\sin \alpha_f C_{D_a} - \cos \alpha_f C_{L_a} \right] \dot{\alpha}_f \]

\[ = -p_B u_B + q_B v_B - \beta_{BI31} \dot{\hat{w}}_1 - \beta_{BI32} \dot{\hat{w}}_2 - \beta_{BI33} \dot{\hat{w}}_3 \]

\[ + \frac{1}{\rho \rho^2} \left\{ -\sin \alpha_f \left[ C_{D_1} + \frac{C_{D}}{2V} C_{D} q_B \right] \right. \]

\[ + C_{D_E} \delta_E \right] \cos \alpha_f \left[ C_{L_1} + \frac{C_{L}}{2V} C_{L} q_B + C_{L_E} \delta_E \right] \]

\[ - \sin \epsilon_T C_T \right\} / m + g \xi_{BI33} \]

(3.9,2)

\[ \dot{q}_B = \frac{\rho V SC}{4I_{YY}} C_{m_D} \dot{\alpha}_f \]

\[ = \frac{I_{xz}}{I_{yy}} (r_B - p_B) - \frac{(I_{xx} - I_{zz})}{I_{yy}} r_B p_B \]

\[ + \frac{1}{\rho \rho^2} \left\{ C_{m_D} + \frac{C_m}{2V} C_m q_B + C_{m_E} \delta_E \right\} / I_{yy} \]

(3.9,3)

\[ \dot{\alpha}_f = \frac{u^2}{V^2} \left[ \dot{w}_B \dot{u}_B - \dot{w}_B ^2 \right] = 0 \]

(3.9,4)

\[ \dot{v} = \frac{1}{V} \left[ u_B \dot{u}_B + \dot{v}_B \dot{v}_B + \dot{w}_B \dot{w}_B \right] = 0 \]

(3.9,5)

\[ \dot{q}_B = q_B \cos \phi_B - r_B \sin \phi_B \]

(3.9,6)

\[ \dot{x_I} = \beta_{BI11} u_B + \beta_{BI21} v_B + \beta_{BI31} w_B + w_1 \]

(3.9,7)

\[ \dot{h} = -\beta_{BI31} u_B - \beta_{BI32} v_B - \beta_{BI33} w_B - w_3 \]

(3.9,8)

\[ \dot{d} = (\beta_{BI11} \sin \gamma_G - \beta_{BI13} \cos \gamma_G) u_B + (\beta_{BI21} \sin \gamma_G - \beta_{BI23} \cos \gamma_G) v_B \]

\[ + (\beta_{BI31} \sin \gamma_G - \beta_{BI33} \cos \gamma_G) w_B + \sin \gamma_W - \cos \gamma_W \]

(3.9,9a)

or

\[ \dot{d} = \sin \gamma_G \dot{x_I} + \cos \gamma_G \dot{h} \]

(3.9,9b)

\[ \delta_T = -\frac{1}{T_{\delta_T}} \delta_T + \frac{1}{T_{\delta_T}} \delta_C \]

(3.9,10)

\[ \dot{\beta}_B = -r_B u_B + p_B w_B - \beta_{BI21} \dot{w}_1 - \beta_{BI22} \dot{w}_2 - \beta_{BI23} \dot{w}_3 + g \xi_{BI23} + \frac{Y_R}{m} \]

(3.9,11)
\[
\dot{\beta} = \frac{1}{V_{xz}} \left[ \dot{\beta} - \dot{V} V^{-1} \dot{V}_B \right]
\]

where*

* The dependence of \( C_{T_{0_T}} \) only on airspeed \( V \) is compatible with the aircraft used in the numerical examples (see Appendix D).
motion. Similarly, if the trim flight condition is symmetric, if the initial conditions on the longitudinal response variables are all such that the aircraft is in its longitudinal equilibrium, and if there are no longitudinal wind and control inputs, then the lateral response may be determined independently of the longitudinal equations of motion.

3.10 Linearized Equations of Motion

Techniques for analyzing linear time-invariant models of dynamic systems and synthesizing the corresponding controllers are well developed and relatively easy to apply. It is thus of great utility to linearize the general, nonlinear equations of motion.

In this section several sets of linearized equations of motion, each dependent on slightly different assumptions, will be developed. The linearization reference equilibrium has been defined in Section 3.4 of this chapter, i.e. equation (3.4,7) and following. The linearized equations of motion will be developed in stability axes $F_S$, for reasons already discussed in Section 3.4.

If terms which are second order or greater in the perturbation Euler angles are ignored, then the rotation matrix $L_{SI}$ is approximated by

$$
L_{SI} = \begin{bmatrix}
\cos \theta_e \cos \phi_e & \cos \theta_e \sin \phi_e & -\sin \theta_e \\
-sin \phi_e & \cos \phi_e & 0 \\
\sin \theta_e \cos \phi_e & \sin \theta_e \sin \phi_e & \cos \theta_e 
\end{bmatrix}
$$

Substituting (3.10,1) and (3.4,7) and following into the stability axes form of the scalar equations (3.3,19), (3.3,20) and (3.3,23), and dropping second and higher order terms, the perturbation kinematic equations become

$$
\Delta \dot{x}_1 = \cos \theta_e \cos \phi_e \Delta u - \sin \phi_e \Delta v + \sin \theta_e \cos \phi_e \Delta w
-V \sin \theta_e \cos \phi_e \Delta \theta - V \sin \phi_e \cos \phi_e \Delta \psi + \Delta W_1
$$

$$
\Delta \dot{y}_1 = \cos \theta_e \sin \phi_e \Delta u + \cos \phi_e \Delta v + \sin \theta_e \sin \phi_e \Delta w
+V \cos \theta_e \cos \phi_e \Delta \psi - V \sin \theta_e \sin \phi_e \Delta \theta + \Delta W_2
$$

$$
\Delta \dot{\phi} = +\sin \theta_e \Delta u - \cos \theta_e \Delta w + V \cos \theta_e \Delta \theta - \Delta W_3
$$
\[ \Delta \ddot{d} = \sin \gamma \dot{x} \Delta \dot{x} + \cos \gamma \Delta \dot{h} \quad (3.10,3) \]

\[ \Delta \ddot{\phi} = \Delta p + \Delta r \tan \theta_e \quad (3.10,4a) \]

\[ \Delta \ddot{\psi} = \Delta q \quad (3.10,4b) \]

\[ \Delta \dot{\psi} = \Delta r \sec \theta_e \quad (3.10,4c) \]

where use of (3.3,24) has been made. Implicit in these are the linearization reference equilibrium conditions (3.4,21) through to (3.4,26).

The stability axes form of the dynamic moment equations (3.3,17) may also be readily linearized using a similar procedure. They are found to be

\[ I_{xx} \Delta \dot{p} - I_{xz} \Delta \dot{r} = \Delta L \quad (3.10,5a) \]

\[ I_{yy} \Delta \dot{q} = \Delta M \quad (3.10,5b) \]

\[ I_{zz} \Delta \dot{r} - I_{xz} \Delta \dot{p} = \Delta N. \quad (3.10,5c) \]

Implicit in these equations are the linearization reference equilibrium conditions (3.4,18), (3.4,19) and (3.4,20).

The linearization is not quite as straightforward for the force equations. Two procedures are possible. In the first it is assumed that only second order or greater terms in the state may be dropped, but not terms which are second order in a state/disturbance quantity (e.g. \( \Delta \theta \Delta \dot{W}_1 \)). This will be referred to as state linearization (SL). In the second approach all second order terms including such terms will be dropped. This will be referred to as state-disturbance linearization (SDL).

The rationale for keeping terms of the form \( \Delta \theta \Delta \dot{W}_1 \) is somewhat nebulous, inasmuch as large disturbances will tend to make the state deviations large and thus probably invalidate the linearization. It may nevertheless be argued, especially for closed-loop systems, that a "large" disturbance may still produce relatively small deviations, in the linear sense, in the state variables. Thus to drop terms such as \( \Delta \theta \Delta \dot{W}_1 \) would lead to greater errors than those resulting from dropping the higher order state terms (e.g. \( \Delta u^2 \)).

The SL force equations corresponding to the stability axes form of (3.3,16) are found to be

\[ m[\Delta \ddot{u} + (\cos \theta_e \cos \psi_e - \sin \theta_e \cos \psi_e \Delta \theta - \sin \psi_e \cos \theta_e \Delta \psi) \Delta \dot{W}_1 + \Delta X - mg \cos \theta_e \Delta \theta \quad (3.10,6a) \]

\[ + (\cos \theta_e \sin \psi_e + \cos \theta_e \cos \psi_e \Delta \psi - \sin \theta_e \sin \psi_e \Delta \theta) \Delta \dot{W}_2 \]

\[ - (\sin \theta_e + \cos \theta_e \Delta \theta) \Delta \dot{w}_3] = \Delta \ddot{v} + \Delta r \Delta \dot{V}_e \]

in which

\[ m[\Delta \ddot{\nu} + (\sin \theta_e \cos \psi_e - \sin \psi_e - \cos \psi_e \Delta \psi) \Delta \dot{W}_1 + (\sin \theta_e \sin \psi_e \Delta \psi + \cos \psi_e \Delta \psi) \Delta \dot{W}_2 + \cos \theta_e \Delta \psi \Delta \dot{W}_3] \]

\[ = \Delta Y + mg \cos \theta_e \Delta \phi \quad (3.10,6b) \]
\[ m [\dot{\mathbf{w}} - \mathbf{q}_e \mathbf{v}_e + (\sin \theta_e \cos \psi_e - \sin \theta_e \sin \psi_e \Delta \theta + \cos \theta_e \cos \psi_e \Delta \theta) + \Delta \sin \psi_e \dot{\mathbf{w}}_1 + (\sin \theta_e \sin \psi_e + \sin \theta_e \cos \psi_e \Delta \theta + \cos \theta_e \sin \psi_e \Delta \theta - \cos \psi_e \Delta \theta) \dot{\mathbf{w}}_2 + (\cos \theta_e - \sin \theta_e \Delta \theta) \dot{\mathbf{w}}_3] = \Delta z - mg \sin \theta_e \Delta \theta \]

\( (3.10,6c) \)

Similarly the SDL force equations are found to be

\[ m (\dot{\mathbf{u}} + \cos \theta_e \cos \psi_e \Delta \mathbf{w}_1 + \cos \theta_e \sin \psi_e \Delta \mathbf{w}_2 - \sin \theta_e \Delta \mathbf{w}_3) = \Delta x - mg \cos \theta_e \Delta \theta \quad (3.10,7a) \]

\[ m (\dot{\mathbf{v}} + \mathbf{v}_e \Delta r - \sin \psi_e \Delta \mathbf{w}_1 + \cos \psi_e \Delta \mathbf{w}_2) = \Delta y + mg \cos \theta_e \Delta \psi \quad (3.10,7b) \]

\[ m (\dot{\mathbf{w}} - \mathbf{v}_e \mathbf{q}_e + \sin \theta_e \cos \psi_e \Delta \mathbf{w}_1 + \sin \theta_e \sin \psi_e \Delta \mathbf{w}_2 + \cos \theta_e \Delta \mathbf{w}_3) = \Delta z - mg \sin \theta_e \Delta \theta. \quad (3.10,7c) \]

Implicit in equations (3.10,6) and (3.10,7) are the reference equilibrium conditions (3.4,15), (3.4,16) and (3.4,17).

The following properties of the SL and SDL equations of motion are noted:

1. Including wind models which are functions of the aircraft's position will generally implicitly make the SL and SDL equations of motion nonlinear, with one important exception. This is that the SDL equations will maintain their linearity if the wind model is a linear function of position, such as that given by (2.2,11). Maintaining linearity is generally advantageous in analysis and will be seen to be particularly advantageous in some of the worst-case modeling techniques to be discussed in Chapter 5.

2. Even with longitudinally and laterally uncoupled aerodynamics (such as the AERO1 model of Section 5.2 of this chapter), the SL longitudinal and lateral equations of motion are strongly coupled through terms such as \( \cos \theta_e \cos \psi_e \Delta \mathbf{w}_2 \) in (3.10,6a). In the SDL equations these terms are dropped and only a weak coupling remains because of nonzero \( \psi_e \) (equivalently nonzero \( \mathbf{w}_2 \)). This coupling may be avoided by linearizing about \( \mathbf{w}_2 = 0 \) or by rotating reference frame \( F_I \) about \( \mathbf{k}_I \) to a new reference frame \( F'_I \) through an angle \( \psi_e \). The latter procedure requires redefining the linearization reference equilibrium (i.e. \( y_{\Psi_e} = 0 \)) and the wind inputs (e.g. \( \mathbf{w}_2 \) is no longer a direct crosswind relative to the runway) in a cumbersome way. Thus the former procedure will be employed on the SDL equations when decoupling is desired (e.g. to reduce the order of the dynamic system for more convenient and cost effective application of the worst-case wind modeling methods of Chapter 5).

3. The SL equations of motion, with the aerodynamic model AERO1 of Section 5.2 of this chapter, may be written in a matrix form convenient for machine computation, that is

\[ \mathbf{A}_1 \dot{\mathbf{x}} = \mathbf{A}_2 \mathbf{q} + \mathbf{B}_1 \mathbf{A}_2 \Delta \mathbf{w} + \mathbf{B}_2 \Delta \mathbf{w} + \mathbf{B}_3 \Delta \mathbf{w} + \mathbf{b}_1 \quad (3.10,8) \]
where $\Delta x$ is the perturbation state vector, $\Delta \hat{\delta}$ is the perturbation control vector, $\Delta \hat{W}$ is the perturbation wind velocity vector, in $F_I$ components, $A_1$, $A_2$, $B_1$, $B_2$ and $B_3$ are associated time-invariant system matrices and

$\mathbf{b}_1$ is a column vector containing the state-disturbance terms (e.g. terms such as $-\sin \psi_{\epsilon} \cos \delta \Delta \hat{W}_1$ in (3.10,6a)). This form remains essentially unchanged for the SDL and the longitudinal and lateral decoupled equations of motion*, except that

$$b_1 = \mathbf{0}.$$ (3.10,9)

The associated matrices will, of course, be different. For example, for the SL and SDL equations of motion, $\Delta x$ is given by

$$\Delta x^T = [\Delta u \Delta v \Delta \dot{\varphi} \Delta \dot{\theta} \Delta \psi \Delta x_I \Delta y_I \Delta h \Delta d].$$ (3.10,10a)

whereas for the laterally decoupled longitudinal equations

$$\Delta x^T = [\Delta u \Delta w \Delta q \Delta \dot{\varphi} \Delta \dot{\theta} \Delta x_I \Delta y_I \Delta h \Delta d].$$ (3.10,10b)

The details of the matrices of (3.10,8) are left to the numerical examples, where specific cases will be considered

---

* The matrix form of the laterally decoupled longitudinal linear equations of motion is given by Reid et al [1.39].

Based on the SDL equations of motion. In scalar form, with the linear quasisteady aerodynamic model AERO1 (see Section 3.5.2 of this chapter), and with time derivatives of response variables moved to one side, the force and moment SDL equations are found to be

$$m\Delta \dot{u} - X_u \Delta \dot{w} = X_u \Delta u + X_w \Delta w + X_q \Delta q$$

$$-mg \cos \theta_{\epsilon} \Delta \theta + X_{\theta} \Delta \theta_E + X_{\theta} \Delta \theta_T$$

$$-m \cos \theta_{\epsilon} \cos \psi_{\epsilon} \Delta \hat{W}_1 - m \cos \theta_{\epsilon} \sin \psi_{\epsilon} \Delta \hat{W}_2$$

$$+ m \sin \theta_{\epsilon} \Delta \hat{W}_3.$$ (3.10,11a)

$$m\Delta \dot{v} = Y_v \Delta v + Y_p \Delta p + (Y_{\varphi} - mV_{\epsilon}) \Delta r + mg \cos \theta_{\epsilon} \Delta \phi$$

$$+ Y_{\delta} \Delta \delta_A + Y_{\delta} \Delta \delta_R + m \sin \psi_{\epsilon} \Delta \hat{W}_1$$

$$- m \cos \psi_{\epsilon} \Delta \hat{W}_2.$$ (3.10,11b)

$$(m - Z_3) \Delta \dot{w} = Z_u \Delta u + Z_w \Delta w + (Z_q + mV_{\epsilon}) \Delta q$$

$$-mg \sin \theta_{\epsilon} \Delta \theta + Z_{\theta} \Delta \theta_E + Z_{\theta} \Delta \theta_T$$

$$- m \sin \theta_{\epsilon} \cos \psi_{\epsilon} \Delta \hat{W}_1 - m \sin \theta_{\epsilon} \sin \psi_{\epsilon} \Delta \hat{W}_2$$

$$- m \cos \theta_{\epsilon} \Delta \hat{W}_3.$$ (3.10,11c)

$$I_{xx} \Delta \dot{\varphi} - I_{xx} \Delta \dot{r} = L_{v} \Delta v + L_{p} \Delta p + L_{r} \Delta r + L_{\theta} \Delta \theta_A + L_{\theta} \Delta \theta_R$$

(3.10,12a)
The linear and angular position equations are given by
(3.10,2), (3.10,3) and (3.10,4).
REFERENCES — CHAPTER 3


FIG. 3.1 CURVED AND CONVENTIONAL GLIDEPATH GEOMETRIES, AND DEFINITIONS OF $F_{ER}$, $F_I$, AND $F_{GL}$

FIG. 3.2 EQUILIBRIUM FLIGHT CONDITION FORCES AND GEOMETRY, AND DEFINITION OF $F_B$ AND $F_S$

Note: The definition of $F_B$ has assumed that the flight condition is symmetric.
FIG. 3.3 DEFINITION OF F_w AND F_w', AND AERODYNAMIC ANGLE GEOMETRY

Note: k_w and k_w' are always in the plane of symmetry.

FIG. 3.4 ANGULAR MOMENTUM CONTRIBUTION OF THE MASS ELEMENT dm
$V_{ev} = V_e \cos \psi_e$

**FIG. 3.5** REFERENCE EQUILIBRIUM GEOMETRY

**FIG. 3.6** VERTICAL PLANE PROJECTION OF REFERENCE EQUILIBRIUM GEOMETRY
FIG. 3.7 NONUNIFORM WIND FIELDS
(Figure abridged from McRuer et al. [3.3])

FIG. 3.8 GENERAL CURVED GLIDEPATH APPROACH TRAJECTORY
$V_e = 40 \text{ ms}^{-1}$
$
\gamma_G = 7^\circ
$
$h_{co} = 300 \text{ m}$
$h_T = 30 \text{ m}$
$W_{2c} = 0$

--- CONVENTIONAL GLIDEPATH

FIG. 3.9 CURVED GLIDEPATH GEOMETRIES FOR DIFFERENT $W_{1c}$
4. AIRCRAFT CONTROLLER

In the previous chapter several systems of differential equations of different levels of sophistication were developed. These equations describe the gross rigid body motions of the aircraft in the presence of wind and control inputs (Fig. 4.1). While this forms a significant portion of the analytical effort required, consideration has not been given to modeling the control inputs themselves. Since the winds acting on the aircraft are unpredictable and can be strong enough to cause it to deviate far from the desired state trajectory, a controller which determines the control inputs as a function of state, or certain measurable quantities which are functions of the state, is required. This is known as closed-loop control and is schematically depicted in Fig. 4.2.

The union of control theory and aircraft flight mechanics did not occur until well after the end of World War II. Prior to this aircraft closed-loop control systems were relatively crude and were designed largely by trial and error. Since that time the advent of the digital computer and the application of modern optimal and adaptive control theory have brought about dramatic changes, to the extent that many current closed-loop control systems can now be designed entirely on paper and with the aid of digital computers, and only "fine-tuned" by flight testing. For thorough and

interesting historical perspectives the reader is referred to McRuer, Ashkenas and Graham [3.3].

The logic behind the control inputs may be provided by one of two sources, namely (1) an automatic pilot and (2) a human pilot. Both have their inherent advantages and limitations. It is the objective in this section to outline controller models which are adequate approximations to the former, noting that much of the underlying theory for the synthesis of automatic pilots can also be applied to human pilot modeling. In doing this recall that we wish to formulate an analytic model which realistically describes the closed-loop rigid body response of an aircraft to variable winds. This is not necessarily the same as accurately modeling the individual elements that characterize automatic and human pilots. The former is a performance black box type of approach and is quite adequate for the purposes of this study.

4.1
4.1 Controller Modes and Requirements

The following automatic pilot modes will be considered in more detail in the sequel:

1. Airspeed hold (autothrottle)
2. Attitude hold
3. Lateral stability augmentation and guidance
4. Glidepath-localizer tracking
5. Go-around

Although go-around controllers will not be incorporated in the numerical examples, they are included in the discussion in order to highlight a current controversy regarding what control techniques are appropriate for this maneuver in certain types of variable wind conditions. Other modes which might ultimately be incorporated in a general model but are not considered further in this study include glideslope-localizer capture, flare and touchdown, automatic rollout and braking, and automatic taxiing.

In order to properly consider the synthesis of modes (1) to (4) a set of requirements must be defined which acceptable controllers should satisfy. These are best stated in qualitative terms at this point, followed by some numerical examples to indicate the magnitude of the acceptable deviations. The requirements are stated for the glidepath-localizer tracking mode, this mode encompassing the first three modes, and are taken from McRuer et al. [3.3] (p. 625):

4.4 Guidance and Control Requirements

1. To establish and maintain the aircraft on a specified spatial pathway or beam.
2. To reduce flight path errors to zero in a stable, well-damped, and rapidly responding manner.
3. To establish an equilibrium flight condition.
4. To limit the speed or angle of attack excursions from this established equilibrium flight condition.

4.5 Regulation Requirements

1. To maintain the established flight path in the presence of variable wind disturbances.
2. To provide a degree of short term attitude stability in the presence of disturbances.

The path keeping and airspeed deviation requirements are specified by performance objectives and certification regulations. A sampling of the numerical values that these requirements may take is given in Fig. 4.3 and Table 4.1, as taken from Hofmann, Clement et al. [4.1], Graham, Clement and Hofmann [4.2] and Johnson and McRuer [4.3].

A kinematic CGA (see Chapter 1, Section 2.1, and Chapter 3, Section 3.4.1) would ideally be carried out at constant attitude and airspeed. For this limiting case the controller that would be required is one that maintains airspeed and
attitude only, rather than one which attempts to track a particular spatial trajectory. Simulations carried out using such a controller would then provide an indicator of the dynamic errors that might be anticipated due to the kinematical description of the curved approach. Such errors would occur even if the wind profile were known exactly. This is a very informative performance measure, and it will be considered further in the sequel.

A controller which tracks a reference airspeed and attitude is a considerably simpler proposition than one which tracks a given spatial trajectory (as well as providing airspeed and attitude stability). The general requirements would be similar to the requirements for the present approach control system with the outer spatial pathway tracking loop removed.

4.2 Linear Versus Nonlinear Controllers

Most controllers designed on paper for aircraft application are designed with the use of linear, time-invariant theory, for which analysis techniques are well understood and algorithmic procedures are frequently available. These controllers are modified as found necessary by analog simulation and flight testing to handle situations which are not suitably taken into account by the linear techniques. Finally, if situations are encountered where the controllers designed by all the above methods are inadequate (e.g. severe variable wind conditions), then the human pilot is expected to intervene and take command from the automatic pilot.

With regard to a simulation study of aircraft response to variable winds, this suggests that it may be necessary to design controllers which are based on linear theory. Nonlinearities which do arise may be handled by using a number of techniques, including (1) piecewise linear controllers, (2) limiters, and (3) time-varying linear controllers. In general nonlinearities occur when the aircraft state variables and their derivatives experience large deviations from the linearization reference equilibrium. This not only invalidates linearized equations of motion but may also lead to nonlinear aerodynamic effects. These include the following:

1. Inertial roll coupling - rapid sustained rolling maneuvers which lead to instabilities.
2. Large angles of attack near the stall region - separation effects lead to nonlinearities.
3. Large angles of sideslip - also results in separation effects.
4. Spins (incipient or fully developed) - may lead to high angles of attack and sideslip as well as rapid rotation about the z-axis (in an extreme scenario this is exemplified by the flat spin).

Not all of these possibilities need be considered in this
study. The first can be eliminated because winds are not likely to place the aircraft in a sustained rapid roll (e.g. through 360 degrees), although momentary rapid rolls may occur in severe turbulence (see also the comments in Chapter 3, Section 3.6). The fourth can be eliminated because its consequences at the low landing approach altitudes are catastrophic. For any but the smallest aircraft recovery will not be possible in the altitude available. Thus only two and three remain to which further consideration should be given.

For CTOL and most STOL aircraft the nominal angle of attack at which the approach is flown will not be in the non-linear region (see Fig. 4.4). Of course, the nominal sideslip angle will be zero. Thus, provided that the aircraft does not significantly depart from these nominal values, the linearity assumption will hold and a satisfactory aircraft controller may be designed using linear theory.

If the disturbances are substantial, however, as will be the case for flight in hazardous variable winds, this may not be a valid approximation. This is particularly true for the angle of attack because the nominal value of the absolute angle of attack \( \alpha \) is not zero.

Table 4.2 summarizes, in qualitative terms, the effects of step winds along the \( \mathbf{i}_S \), \( \mathbf{j}_S \) and \( \mathbf{k}_S \) axes on an aircraft initially flying in its reference equilibrium \( \{u = V_e, (v, w, p, q, r) = 0, \gamma_x = 0\} \). These are initial effects that would result assuming unsteady flow phenomena are negligible; the response after a certain amount of time has passed from the moment of the step wind change encounter can only be determined via a dynamic analysis. Also assumed is that the aircraft is initially in straight and level flight so that a step headwind increase will only be a change of the wind along \( \mathbf{i}_S \), a step crosswind will only be a change of the wind along \( \mathbf{j}_S \) and so forth.

The important features of Table 4.2 are as follows:

1. Step headwinds (tailwinds) and step downdrafts (updrafts) produce step changes in lift.

2. Step headwinds (tailwinds) do not change the angle of attack within the conditions specified above and in the table.

3. Step updrafts (downdrafts) do not change \( u \) and thus the measured airspeed.

4. Step changes in headwinds (tailwinds) and downdrafts (updrafts) do not produce sideslip. This is due to the initial condition \( v = 0 \) and the definition of the sideslip angle as given in equation (3.2,19a).

5. Step changes in crosswind do not produce a step change in lift although they do produce step changes in side force.

As an example, a step decreasing (increasing) downdraft (updraft) may produce an instantaneous increase in angle of attack. In practical terms the immediate danger in such a wind change is that the aircraft will be taken to an angle of
attack regime at or beyond the stall leading to a significant loss of lift and/or to severe control difficulties due to separated flow. Assuming that the controls and stability augmentation systems have been designed in such a way that the aircraft remains controllable in pitch and roll in the high angle of attack regime, the primary concerns become its airspeed stability, its lift characteristics, and its deviation from the glidepath.

The previous example provides a scenario for inadvertent flight into a nonlinear regime. But there is also a landing approach scenario where correct pilot (automatic and/or human) procedure may call for command flight into nonlinear, high angle of attack regimes.

In practice piloting procedure calls for glidepath tracking with elevator alone* for path deviations which are not large. For small airspeed deviations no use of throttle is required, but in any case airspeed stability may be enhanced by the use of an autothrottle. Thus the simplest control strategy is seen to be (1) control glidepath deviations with elevator and (2) control airspeed deviations with throttle.

The aircraft is now subjected to a variable wind which is of such a type as to produce a sustained loss of airspeed rather than a fluctuating airspeed about the nominal value. If the airspeed loss is significant enough the initial pilot reaction may be to apply power. This loss of airspeed results in a loss of lift and will lead to a deviation below the glidepath if it is sustained, as has been postulated. This deviation from the glidepath may be corrected by appropriate elevator action, but this is fundamentally opposed to the pilot's training of maintaining airspeed safely above the stall speed. If this airspeed has dropped near enough to the stall speed the pilot may attempt to regain the lost airspeed by pitching the nose down as well as adding power. At high altitude this is the correct control action, but in this landing approach scenario such control action increases the glidepath deviation even further. There may, in fact, not be sufficient altitude and the wind conditions may be such as to prevent normal airspeed recovery before striking an obstruction. What may be required is (1) go-around power and (2) an attitude which will result in a high angle

* This statement is made with considerable trepidation in that this strategy, sometimes referred to as the "bastard method of flight control", has been criticized as the cause of the misconception that one can, at a given airspeed, change the flight path angle with elevator alone (Melvin [4.4]). This of course, is not what is happening in glidepath tracking where the aircraft is trimmed and the throttles set to maintain a desired flight path angle. If the aircraft is then slightly above or below this glidepath (but essentially paralleling it) and flying on the front side of the drag versus speed curve, then a small, temporary change in elevator setting will return it to the desired glidepath and cause no sustained airspeed instability. For an interesting discussion of a STOL aircraft flying the landing approach on the backside of the drag versus speed curve, see Franklin and Innis [4.5].
of attack (large lift coefficient). Only when the aircraft has checked its sink rate and has established a positive rate of climb (in inertial coordinates) can the attitude be reduced to help restore airspeed.*

The most notable feature of the above discussion is that the suggested procedures ask the pilot to temporarily let his airspeed drop below the target value in an attempt to prevent ground contact. While this is certainly in conflict with normal pilot training and will rarely be necessary, it has been suggested that it may be the difference between an effective recovery and an accident. At least three incidents/accidents have been documented where such a procedure could have been employed. In the first, an L-1011 preceding the Boeing 727 which crashed at John F. Kennedy International Airport on June 24, 1975 (see Table 1.1 and Ref. 1.17) recovered using such a procedure [4.6, 4.7]. In the second, flight simulator tests indicated that such a procedure may have prevented the crash of a DC-9-31 approaching to land in a rainstorm at Philadelphia International Airport on June 23, 1976 [4.7, 4.8]. Collins [4.8] points out, however, that to expect pilots to follow such procedures before they have been proven and with no a priori training is unacceptable.

Finally, in the third case the NTSB concluded that a Boeing 727-224, whose take-off climb performance was degraded because of hazardous variable winds, could have cleared obstacles past the departure end of the runway if it had optimized the use of its available aerodynamic potential [1.20].

This discussion suggests that consideration should be given to controllers which provide the appropriate response in this extreme variable wind scenario. Such an effort may be rewarded by a controller which effectively applies appropriate control action in order to maximize the aerodynamic performance as required for a low altitude variable wind recovery. This would be an additional flight director and automatic pilot mode to the conventional go-around mode.

Such a special mode does not appear to have been considered in the literature and is recommended for further research.

To return to the problem of the analysis of nonlinear controllers, early work in this area consisted of investigating the effects of certain parasitic nonlinearities (e.g. of the saturation type (Alex and Heermann [4.9]), and of investigating the effects of making certain autopilot gains functions of the state of the aircraft (e.g. of the dynamic pressure [4.9]). This work inevitably had to be complimented with digital and analog computer work since nonlinear analytical techniques are not well established and very difficult to

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* The success of this combination depends on engine thrust available and the induced drag characteristics of the aircraft.
apply. Recent work has focused on the use of optimal control theory to obtain suboptimal feedback controllers for nonlinear systems (see the discussion in the introduction of the paper by Garrard and Jordan [4.10] and Ling [1.58]). For systems where the nonlinearities can be expressed as a power series in the state vector, a systematic procedure has been developed [4.10]. Garrard and Jordan [4.10] indicate that this procedure does not appear to be a clear-cut improvement over controllers that are designed using linear quadratic optimal control theory, and set out to apply optimal nonlinear feedback control theory to the design of a flight control system which can provide acceptable dynamic response over the entire range of the angle of attack which a modern high performance interceptor jet aircraft may encounter. The procedure outlined may be quite useful in analyzing a high angle of attack controller in the scenario discussed previously, and involves the use of an approximate perturbational technique solution to the Hamilton-Jacobi partial differential equation (see Appendix F, Section 1). The procedure is not easily amenable to computer calculation and is very laborious. It does, however, appear to provide a considerable improvement over linear quadratic optimal controllers.

In view of these unavoidable practical and theoretical complications in synthesizing nonlinear controllers, in the following past practice will be adopted and linear theory will be employed. Only when such controllers prove to be inadequate will they be modified (via the use of limiters and so forth).

4.3 Classical Versus Optimal Controller Synthesis Techniques

Classical control theory focuses on the question of determining the feedback gains of a given form of the controller so that the design objectives are satisfied. Thus, for example, we may consider the problem of finding values for the constant gains $K_0$, $K_E$, $K_0$ of the proportional plus rate plus integral attitude controller

$$\delta_E^c = K_0 \delta_e + K_0 \delta_e^r + K_0 \int \delta_e \, dt \quad (4.3.1)$$

so that satisfactory attitude holding properties are obtained. Here

$$\delta_e = \theta_d - \theta \quad (4.3.2)$$

is an error signal which depends on the desired pitch attitude $\theta_d$ and the actual attitude $\theta$, and $\delta_E^c$ is the commanded elevator deflection. A large part of the synthesis effort that is required comes into the selection of a suitable form of the controller.

This classical approach to controller synthesis originally encompassed only single-input, single-output feedback systems whose dynamics were modeled by a constant coefficient linear differential equation (Elgerd [4.11]).
The analysis methods were Laplace plane and frequency domain techniques and included root-locus, Nyquist plots and Bode plots. Modern classical control theory (which will be referred to as classical control theory in the sequel for brevity) still employs these techniques but has been extended to include multiloop systems (McRuer et al. [3.3, 4.12]) and certain types of nonlinearities (Lefschetz [4.13]), at great increase in complexity. For the most part, however, practical applications have been restricted to systems whose dynamics may be approximated by linear constant coefficient differential equations. This is not as restrictive an approximation as might be first thought (also see the discussion in the previous section) because it can be shown that feedback has the characteristic of linearizing the performance of the dynamic system to which it is applied (Ref. 3.3, p. 52). For cases where the dynamic range of the controller exceeds the linear limits, it is often sufficient to design several controllers for a number of time-invariant trim conditions and switch from one to the other as the aircraft enters its different dynamic regimes in order to obtain satisfactory performance. The controller for each dynamic regime may then be designed with a linear, time-invariant theory. Alternatively one can use limiters and controller parameters which are state dependent in order to extend the useful range of the controller. Finally, it is interesting to note that recent developments in classical control theory have embraced certain aspects of functional minimization theory in attempting to obtain optimum values for controller parameters (e.g. MacNamara [4.14]).

Optimal control theory is an outgrowth of variational calculus and is coming into greater use primarily because of the widespread use of the digital computer. In contrast to classical control theory, optimal control theory looks for control inputs which minimize a functional whose minimization reflects good controller performance. In the closed-loop sense the control inputs must be generated via an optimal controller whose form is generally not known a priori. Thus in parallel to the classical example given previously, one might look for a control law

\[ u^*(t) = f(x, t) \]  

(4.3.3)

which minimizes the performance criterion

\[ J = \int_0^\infty \left[ k_1 \delta^2(t) + k_2 \delta^C_\xi(t) \right] dt. \]  

(4.3.4)

The \( \delta_\xi \) and \( \delta^C_\xi \) terms of \( J \) are weighted by positive constants \( k_1 \) and \( k_2 \), and \( x \) is the state of the aircraft and is constrained to satisfy the equations of motion

\[ \dot{x} = f(x, \delta^C_\xi, t). \]  

(4.3.5)
Optimal control theory is discussed in a number of excellent texts, including those by McCauseland [4.15], Bryson and Ho [4.16], Kirk [4.17], Anderson and Moore [4.18], and Kwakernaak and Sivan [4.19]. It appears to offer the most hope for the development of algorithmic techniques for designing aircraft controllers, particularly for time-invariant, linear systems with quadratic performance criteria (Power [4.20] and Nakagawa, Muotsru and Kanesada [4.21]).

One may also consider disturbances in carrying out the optimization process. Of particular interest are controllers which are optimal given the worst-possible disturbance inputs, i.e. wind inputs which maximize what the controller is trying to minimize. Such controller design falls under the category of two-sided optimization, and consists of posing the mathematical problem as a minimaximization (see, e.g. Bryson and Ho [4.16], also Chapter 5 to follow).

In summary the advantages and disadvantages of the classical and optimal controller design methods are outlined in the following and in Fig. 4.5.

### Classical Controller Design

#### Advantages

1. Provides the designer with the opportunity to directly select the feedback structure and loop closures as desired.

2. Allows the designer to directly apply his intuition and experience in the design process.

3. Uses well tried analysis methods in selecting the controller parameters.

4. Provides for direct insights into the system-controller interactions and possible problem areas.

5. Provides a wealth of previous applications from which the designer can gain insights into his/her own system (typically Refs. 4.22 to 4.33). These include turbulence penetration closed-loop studies (e.g. McClean [4.22] and Gilbert [4.23]), autothrottle synthesis (e.g. Sullivan [4.26]), longitudinal and lateral glideslope-localizer tracking (e.g. Graham, Clement and Hofmann [4.2]), some with flare and touchdown modes (e.g. Hofmann, Clement and Graham [4.30]), some with rollout modes (e.g. Paires, Schmidt and Mann [4.32]), some with decrab capability (e.g. Arness [4.31]) and some with describing function human pilot models (e.g. Johnson and McRuer [4.3]).
Disadvantages

1. Will generally not provide a controller which is optimal in the sense of minimizing a desired performance criterion (acceptance criteria are somewhat qualitative).

2. The analysis techniques are extremely complicated and laborious for systems with more than one feedback loop (i.e. multiloop systems).

3. The analysis techniques are not readily amenable to algorithmic computer techniques, although work is in progress in this regard (Gresham, Mitchell and McDaniel [4.34] and Meyer and Cicolani [4.35]). This is a particularly serious objection when viewed in the context of studies considering several aircraft and/or several flight conditions where the control systems will have to be tailored independently to each aircraft or condition.

4. Nonlinear and variable parameter control systems are extremely difficult to synthesize analytically and usually have to be designed and tested with the use of analog and/or digital computers.

5. The analysis is in the frequency domain.

Optimal Controller Design

Advantages

1. Provides an optimal control system in the sense of minimizing a given performance criterion.

2. The technique is easier to adapt to algorithmic computer design, particularly for linear controllers with quadratic performance criteria.

3. A problem of considerable difficulty in comparing closed-loop response of two or more different aircraft is that of designing controllers which are equivalent. Optimal control theory may provide a solution to this problem by allowing control systems to be optimized with regards to nondimensional performance criteria. If these criteria are chosen appropriately, it may be possible to directly compare the closed-loop response of the aircraft.

4. The analysis is in the time domain. This is intuitively more appealing.

Disadvantages

1. Does not provide for as many physical insights into the interactions between the controlled system and the controllers.

2. Applications to the present date are not as widespread as for classical control, and thus computational routines and experience are not as readily available. However this state of affairs is changing rapidly. Recent work includes that of Huber [2.1], Harrison [4.36], Crawford [4.37], Power [4.20], Govindaraj, Rynaski and Fann [4.38], Dunn [4.39] and Hofmann, Riedel and McRuer [4.40].
3. The desired handling qualities are frequently expressed in terms of design objectives which originate in the Laplace plane. This makes the a priori selection of optimal performance indices somewhat arbitrary, although some progress has been made on this problem for linear systems with quadratic performance criteria in characterizing these criteria in terms of desired complex plane characteristics (e.g. Govindaraj, Rynaski and Fam [4.38]).

4. The most conveniently applied optimal methods lead to controllers which require feedback of the full state vector of the aircraft dynamic system. In practice measurements of only certain elements of the state vector are available to the controller. This complicates the design process by requiring that estimators be designed which provide the missing elements.

5. Another practical consideration is that in certain cases optimal controllers are quite sensitive to off nominal characteristics of the real system to be controlled. This may significantly degrade the performance of the optimal controller.

After consideration of the merits and demerits of these controller design methods, the decision was made to proceed with state feedback optimal synthesis techniques. This decision was based on the following factors:

1. The disadvantages (4) and (5) of optimal methods are not relevant to this study where perfect state measurements and nominal system characteristics can be assumed at all times.

2. It is not required that the elements of the controllers be modeled accurately, only that the overall closed-loop response be realistic. Thus the advantage of the classical techniques where loop closures are chosen by the designer is not required here.

3. Controllers will be required for a number of flight conditions. The computationally more algorithmic linear quadratic optimal techniques are more conveniently applied to repeated controller design.

4. The optimal techniques naturally lead into the worst-case wind modeling techniques of Chapter 5, both from the theoretical and computational perspectives.

Appendix F briefly reviews optimal control theory and presents a number of linear quadratic formulations which will be required in the sequel.

4.4 Automatic Pilots Synthesized Using Optimal Control Theory

In this section the characteristics of the controller modes that will be required are discussed with the objective of relating these characteristics to the linear quadratic formulations which will be used to model them. The focus will be on deterministic formulations because of the discrete
nature of the wind models employed in this study. The details of the controller analysis are left to Appendix F, and the details of the models used in the computations to the sections on numerical examples.

4.4.1 Airspeed-Attitude Hold Automatic Pilot

An ideal kinematic curved approach, as defined in Chapter 3, Section 3.7.1, and as further discussed in Section 4.1 of this chapter, would be carried out maintaining constant attitude and airspeed throughout the approach. With the selection of an appropriate linearization reference equilibrium, this is just a regulator problem (see Appendix F). The aircraft equations of motion are the SL or SDL equations of motion or simpler equations derived from these (the SL equations are identical to the SDL equations only if the wind inputs are zero). Of particular interest are infinite terminal time solutions, since for time-invariant systems these yield constant control laws. Also, if integral and/or rate control are desired, one may implement the methods of Formulation 6 and Section F.4 of Appendix F.

4.4.2 Lateral Stability Augmentation and Guidance Systems

Of the three rigid body lateral modes that most CTOL and STOL aircraft display, two are very often poorly behaved. These are the Dutch roll mode (poor damping) and the spiral mode (weakly convergent, neutral, or weakly divergent, all of which are not acceptable for good lateral dynamic response and control). Thus before consideration can be given to lateral path-keeping or heading functions, the basic lateral stability of the aircraft must be improved.

Such systems are known as stability augmentation systems (SAS). With optimal methods SAS and the guidance system may be treated simultaneously, in a similar fashion to the airspeed-attitude hold controller of the previous section. The objective is to stabilize the lateral system, and thus the infinite terminal time regulator for time-invariant problems is important because of the asymptotic stability results of Formulation 2 of Appendix F. Integral control is also desirable because of its wind-proofing properties.
4.4.3 Glidepath-Localizer Tracking Controller

Before giving further consideration to the required controller, a description of the data available from ground approach aids and of the approach procedure will be given. Fig. 3.1 gives some of the numbers and reference points that are associated with conventional glidepaths. The signals normally available to the aircraft are proportional to the angular deviation of the aircraft from the glidepath and localizer (Van Sickle [4.41]), rather than directly to the lateral \( y' \) and glide slope vertical \( d' \) deviations. The latter may be related to the former if the range from the signal transmitters is known (e.g. with distance measuring equipment (DME)).

These signals are subject to nonidealities such as far field and near field effects, geographical distortion, noise, beam misalignment and so forth, but for the purposes of this investigation it will be assumed that the ground facilities transmit ideal signals. It will also be assumed that the aircraft has onboard equipment which will convert the angular deviations and DME measurements into linear position coordinates. Such a procedure is required in defining CGG, and is feasible with MLS systems. The procedure may also have advantages in improving the accuracy of the controller tracking and in removing the need for variable gains because of the range dependent sensitivity of the angular deviations (see the discussion by Schaeenzer [4.42]). It is also compatible with the equations of motion developed in Chapter 3, formulated as they are to express the linear position of the aircraft with respect to \( F_1' \) in terms of \( x_1' \), \( y_1' \), \( z_1' \).

The CGA procedure that will be assumed is as follows:

1. An estimate of the wind profile is transmitted to the aircraft from ground facilities. On board computers use this information to compute a CGA trajectory in \( F_1' \) coordinates.

2. The aircraft is equipped with equipment to receive ground guidance facility information in order to obtain an accurate estimate of the aircraft’s linear position in \( F_1' \). This information may be available in a reference frame different from \( F_1' \), in which case a suitable transformation would also be required.

3. The controller feedbacks are designed on the basis of \( y_1' \) and \( z_1' \) (or \( d' \)) feedbacks.

The CGA technique is inapplicable if range information is not available. This is an additional complexity in the system, but is not a serious drawback on its own because virtually all transport category and many business aircraft have DME equipment (the resolution and accuracy of such systems may have to be upgraded when used as an aid for precision instrument approaches). This limitation is a consequence of the fact that the CGG will generally not be along a line formed by the nulls of the transmitted signals, as is the case with conventional systems. Rather it is a
path which is computed internally with onboard computers. The particular trajectory that results exists only within the onboard computer, and the resulting deviation signals depend on the position of the aircraft (as estimated from the MLS and DME systems) relative to this computed trajectory.

The controller synthesis may proceed using a number of alternative formulations. In the application to tracking a rectilinear ground referenced approach trajectory, this is essentially a regulator problem, provided that the reference equilibrium is defined appropriately. In particular, the glidepath deviation is completely defined by $d$. In the kinematic curved approach this is not true, the trajectories to be tracked being those of Chapter 3, Section 3.7.1. These are given as time histories, and thus one may use the trajectory tracking optimal controllers of Formulations 5 and 6 of Appendix F.

Finally, infinite terminal time controllers with integral feedback may be considered. These controllers do not require a time-varying formulation, and also provide a certain amount of wind proofing via the integral feedback. Such integral feedback is also desirable for good tracking performance of CGG.

4.4.4 Suboptimal Application of Optimal Feedback Laws

The fact that the controllers to be used will be synthesized employing linear, quadratic theory in no way prevents their application to other systems of equations of motion which may be more faithful dynamic models (e.g. the linear optimal feedback laws with the nonlinear equations of motion). The controllers will, of course, not be optimal if applied to a system other than the one for which the computations were carried out, although the optimal techniques may still yield controllers which provide satisfactory characteristics.

Similarly, it may also be necessary to modify the control inputs generated by optimal feedback laws because they exceed physical limits. This problem can largely be eliminated by appropriate selection of the control weighting terms, although this may not be sufficient in some circumstances (e.g. the use of linear feedback laws with nonlinear equations of motion). One may also require limiters on the state or output that is fed back to the controller because of the large control and state deviations that may occur if the state variables deviate far enough from the reference equilibrium.

Finally, it is worth noting that it is not necessary to find an optimal feedback law for the total problem in one iteration. It may be advantageous to follow a multiloop closure sequence similar to that used in classical controller synthesis. For example, one may find an optimal controller to give lateral stability and then treat the resulting stabilized system in a subsequent analysis to determine the optimal controller that gives good localizer tracking.
REFERENCES — CHAPTER 4


TABLE 4.1a*

ICAO LOW-VISIBILITY-LANDING ILS CATEGORIES

<table>
<thead>
<tr>
<th>CATEGORY</th>
<th>RUNWAY VISUAL RANGE (RVR) ft</th>
<th>DECISION HEIGHT (DH) ft</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>2,400 (730m)</td>
<td>200 (60m)</td>
</tr>
<tr>
<td>II-A</td>
<td>1,600 (490m)</td>
<td>150 (45m)</td>
</tr>
<tr>
<td>II-B</td>
<td>1,200 (365m)</td>
<td>100† (30m)</td>
</tr>
<tr>
<td>III-A</td>
<td>700 (215m)</td>
<td></td>
</tr>
<tr>
<td>III-B</td>
<td>150 (45m)</td>
<td></td>
</tr>
<tr>
<td>III-C</td>
<td>zero</td>
<td></td>
</tr>
</tbody>
</table>

† Sometimes called CAT II.

* Taken from Johnson and McRuer [4.3].

TABLE 4.1b†

OVERALL ILS GLIDE SLOPE SYSTEM PERFORMANCE REQUIREMENTS

<table>
<thead>
<tr>
<th>ILS Performance Category</th>
<th>Overall Flight Tracking Performance Requirement on Glide Slope</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(No performance standard; only equipment complement, pilot training and proficiency, and operational standards).</td>
</tr>
<tr>
<td>II</td>
<td>From 700 ft altitude to the decision altitude (100 ft), ± 35 μA or ± 12 ft with respect to indicated &quot;on-path&quot; position, whichever is larger, without sustained oscillations.</td>
</tr>
<tr>
<td>III</td>
<td>From 700 ft altitude to the flare initiation height [circa 50 ft], ± 35 μA or ± 12 ft with respect to indicated &quot;on-path&quot; position, whichever is larger, without sustained oscillations and repeatable touchdown on the runway within the longitudinal limits 200 ft and 2500* ft from the runway threshold and with a 2σ dispersion of 1500 ft about the nominal touchdown point.</td>
</tr>
</tbody>
</table>

* For a medium large jet transport (e.g., DC-8 Series 60) this distance results from applying the requirement that the pilot be able to see at least four bars of the 3000-ft touchdown zone lights on 100-ft centers.

† Taken from Hofmann et al. [4.1].
### TABLE 4.2 INITIAL STEP WIND EFFECTS

<table>
<thead>
<tr>
<th></th>
<th>Headwind $W_h$</th>
<th>Crosswind $W_c$</th>
<th>Downdraft $W_d$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$W_h = \text{Headwind}$</td>
<td>$W_c = \text{Crosswind}$</td>
<td>$W_d = \text{Downdraft}$</td>
</tr>
<tr>
<td></td>
<td>$W_h = -W_1$</td>
<td>$W_c = W_2$</td>
<td>$W_d = W_3$</td>
</tr>
</tbody>
</table>

**Step Headwind**
- Increase
- Decrease

**Step Crosswind**
- Increase
- Decrease

**Step Downdraft**
- Increase
- Decrease

<table>
<thead>
<tr>
<th><strong>Measured</strong></th>
<th><strong>Step Headwind</strong></th>
<th><strong>Step Crosswind</strong></th>
<th><strong>Step Downdraft</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Measured Airspeed ($u$)</td>
<td>SI</td>
<td>NC</td>
<td>NC</td>
</tr>
<tr>
<td>Total Airspeed ($V$)</td>
<td>SI</td>
<td>SI</td>
<td>SI</td>
</tr>
<tr>
<td>Angle of Attack ($\alpha$)</td>
<td>NC</td>
<td>NC</td>
<td>SD</td>
</tr>
<tr>
<td>Lift ($L = \frac{1}{2} \rho V^2 \Sigma_{\alpha} C_{\alpha}$)</td>
<td>SI</td>
<td>NC</td>
<td>SI</td>
</tr>
<tr>
<td>Angle of Sideslip ($\beta$)</td>
<td>NC</td>
<td>SD</td>
<td>SI</td>
</tr>
<tr>
<td>Side Force (positive in the negative y direction)</td>
<td>NC</td>
<td>SD</td>
<td>SI</td>
</tr>
</tbody>
</table>

**Note:** Aircraft Initially At Equilibrium
- $u = V_e$
- $x = y = w = p = q = r = 0$
- $C_L = C_{Le} = C_L \alpha$

**Increasing Tailwind** = Decreasing Headwind
**Increasing Updraft** = Decreasing Downdraft

- SI = Step Increase
- SD = Step Decrease
- NC = Negligible

**Change**
**FIG. 4.1 OPEN-LOOP SYSTEM**

**FIG. 4.2 CLOSED-LOOP SYSTEM**

**FIG. 4.3 RECOMMENDED CATEGORY II WINDOW**

(See Graham, Clement and Hofmann [4.2])

At 100 feet altitude this corresponds to 1 "dot" of glideslope error and 1/3 "dot" of localizer error on a standard ILS.
FIG. 4.4 QUALITATIVE DESCRIPTION OF LIFT VERSUS ANGLE OF ATTACK CURVE

- CONSERVATIVE DESIGN
- TIME DOMAIN ANALYSIS
- J SELECTED A PRIORI
- GENERALLY MORE DIFFICULT ANALYSIS AND MORE COMPLICATED SOLUTION PROPERTIES
- LINEAR QUADRATIC FORMULATIONS SUITED TO ALGORITHMIC SOLUTION
- DISTURBANCES TAKEN INTO ACCOUNT DIRECTLY IN J
- SENSITIVITY
- DIFFICULTIES WITH REALIZABILITY

FIG. 4.5: FEEDBACK CONTROLLER SYNTHESIS
5. WORST-CASE WIND MODELING TECHNIQUES

In analyzing aircraft response to variable winds, suitable mean wind and turbulence models must be defined. We have already noted (Chapter 1, Section 1.2.2 and Chapter 2) the limitations of using stochastic methods for evaluating the aircraft response to wind shear and spectrally low frequency winds on the landing approach for certain types of studies, such as the investigation of methods for handling low altitude variable wind encounters, the development of ground based simulator wind models, and certain aspects of aircraft certification. Thus there is a continuing need for a deterministic analysis.

There are several methods and data sources available for defining such deterministic wind inputs, including the following important ones (for a more detailed discussion see Chapter 2, Section 2.2):

1. Models based on meteorological data.
2. Models based on ABL wind tunnel results.
3. Situation specific dynamic models of atmospheric flows affecting aircraft flight (e.g. thunderstorm outflow models).
4. Arbitrary profiles intended to stimulate aircraft dynamic response.
5. Analytically maximizing worst-case wind models.

The first four categories contain wind models which are either situation dependent or are chosen in a somewhat arbitrary manner to produce some type of unusual aircraft behavior (e.g. step discontinuities in the wind profile), or to provide a convenient form for analysis (e.g. constant wind gradients). The fourth category is conceptually distinct from the first three in that its methods look for wind models which maximize a particular functional of the state of the aircraft. Thus the concept of a "worst-case" wind input or model is put in a formal mathematical framework while at the same time the difficult task of modeling complex atmospheric flows is avoided. This final category appears to be relatively unexplored and will be the subject of this chapter.

It is obvious from the outset that in the absolute sense no worst-case definition can be made, i.e. an arbitrarily large wind gradient will always cause the aircraft controller to lose control. This suggests that any meaningful worst-case formulation must constrain the wind inputs in some way. An obvious constraint is to limit the magnitude of the wind acting on the aircraft, but this does not yield realistic wind fields and is not particularly amenable to analysis for many problems. A more useful candidate is to constrain the inner product of the wind vector, i.e.

\[ \int_{t_0}^{t_f} \mathbf{w}(t) \mathbf{w}(t) \, dt = E \]  

(5.1)

for some positive constant \( E \). This integral can be thought of as the time integral of the input wind energy to the aircraft system, or by dividing by the total time, as the time averaged
mean square wind intensity. Constraints of the type (5.1) may, of course, be readily related to typical values for real wind profiles.

One may also consider constraints that arise because of the characteristics of real atmospheric flows (e.g., zero horizontal velocity at the surface). Finally, one may constrain the form of the wind inputs and search for worst-case parameters which can be varied within these forms.

5.1 Review of Analytical Techniques of Generating Worst-Case Wind Models

The only attempt of which the author is aware at analytically generating worst-case winds without specifying the form of the wind function are the efforts of Van der Vaart et al. at the Delft University of Technology [1.66, 1.67] (see also the comments in Chapter 1, Section 2.2). Attempts at determining the values of the parameters of a given form of the wind inputs so that certain undesirable characteristics exceed safe values are only slightly more common. Corbin [5.1] uses a search procedure in combining sines and cosines to determine worst-case turbulence histories given certain constraints on the histories for the flare and touchdown case, and appears to have been the inspiration for van der Vaart's work. Rosenberg [5.2] assumes discrete gust models which are compatible with the energy distribution of the von Kármán form of the turbulence power spectrum and uses a simplex expanding and contracting hill-climber algorithm to determine certain parameters of these gust models so that touchdown vertical velocity and touchdown zone deviation exceed acceptable values. This latter class of techniques will not be pursued further here; Jones [1.69] reviews it in greater depth.

Van der Vaart's approach considers linear, time-invariant systems of the form

\[ \dot{x}(t) = Fx(t) + \eta(t) \]  

(5.1.1)

where \( \eta(t) \) is a disturbance time history, and there are no restrictions on the stability of the system. The system may represent an open-loop system or a closed-loop system where the feedback control law has been incorporated into \( F \). The initial conditions are given by

\[ x(0) = 0 . \]  

(5.1.2)

The problem is to find constrained inputs \( \eta \) maximizing \( x(t_f) \).
Under these conditions the solution of (5.1.1) is given by [1.67]

\[ x(t) = \int_0^t \Phi(t-\tau)\eta(t) d\tau. \quad (5.1.3) \]

\( \Phi(t) \) is the system transition matrix*, given by

\[ \Phi(t) = \exp^{\int_0^t (sI - \mathbf{F})^{-1} } \quad (5.1.4) \]

where \( s \) is the complex Laplace variable.

Van der Vaart further restricts the problem to consideration of the worst-case response of a particular component \( x_i \) of the state \( x \) to a particular disturbance component \( \eta_k \) at a specific terminal time \( t_f \). In this case it may readily be shown that

\[ x_i(t_f) = \int_0^{t_f} h_{ik}(t_f-\tau) \eta_k(\tau) d\tau \quad (5.1.5) \]

where \( h_{ik}(t) \) is the response of the \( i \)-th state variable to an impulse applied at the \( k \)-th disturbance input, given by the \( (i,k) \)-th component of

\[ \mathbf{H}(t) = \Phi(t) \mathbf{G} \quad (5.1.6) \]

Under the above conditions it is intuitively obvious that if the input magnitude of \( \eta_k \) is inequality constrained, i.e.

\[ |\eta_k(t)| \leq k_k \quad (5.1.7) \]

then (5.1.5) will be maximized (minimized) if \( \eta_k \) has amplitude \( k_k \) and the same (opposite) sign as \( h_{ik}(t_f-\tau) \) for all \( \tau \) in the interval \([0,t_f] \). Thus

\[ \eta_k^*(t) = \pm \text{sgn}(h_{ik}(t_f-\tau))k_k \quad (5.1.8) \]

where \text{sgn} refers to the sign function and is defined in Appendix A. This \( \eta_k \) is depicted in Fig. 5.1.

If there is no magnitude constraint on \( \eta_k \) then

\[ \eta_k = \pm \infty \quad (5.1.9) \]

will do just as well as

\[ \eta_k = \pm \text{sgn}(h_{ik}(t-\tau)) \lim_{k_k \to \infty} k_k \quad (5.1.10) \]

Ref. 1.67 does not explicitly consider magnitude constraints. The magnitude constraints that are referred to in this reference are those that would result if an integral constraint of the form

\[ \int_0^{t_f} \eta_k^2(\tau) d\tau = E_k \quad (5.1.11) \]

were applied to the problem rather than just a constraint on the magnitude of \( \eta_k \). Here \( E_k \) can be thought of as a constraint on the input wind energy and leads to a constrained minimization problem whose solution is

* This is also sometimes referred to as the fundamental matrix or matrizant.
\[ n_k(t) = k' k_{ik} (t_f - t) \]  \hspace{1cm} (5.1,12)

with

\[ k'_k = \pm \sqrt{\frac{E_k}{\int_0^{t_f} h_{ik}^2(t) dt}} \]  \hspace{1cm} (5.1,13)

The proof of (5.1,12) and (5.1,13) is given in the Appendix of Ref. 1.67, and is a special case of the more general results that will be developed in the following section. Equation (5.1,12) is not a bang-bang type of input as would be the case if \( n_k \) were just magnitude constrained. It is a very elegant result which is related to the integral of the kinetic energy content of the wind seen by the aircraft and may be generated with relative ease on the computer. It does not suffer from the point discontinuities that a solely magnitude constrained problem has, and as such avoids certain numerical difficulties in solving the system differential equations. Ref. 1.67 also uses some stochastic results relating certain deterministic outputs to the system expectation values to show that worst-case winds defined in this way are related to the expectation values of the i-th response variable to white noise in the k-th disturbance input. This is of some importance to designing wind shear proof aircraft. Since aircraft tend to behave as low pass filters, they can be expected to respond similarly regardless of whether they are disturbed by white noise or atmospheric turbulence. Thus if an aircraft has been designed to have low expectation values to atmospheric turbulence, it would also exhibit good response characteristics to the worst-case inputs defined in (5.1,12). How valid this argument is depends on the break frequency of the input turbulence spectra characteristics in relation to the filtering properties of the aircraft.

The method of Ref. 1.67 is not without disadvantages, some of which are quite important. These are:

1. The initial conditions are assumed to be zero (this condition can be relaxed without great difficulty, although this is not explicitly done in Ref. 1.67).
2. The method has been derived only for time-invariant systems (the same comment applies here as for the previous one).
3. The resulting inputs are worst-case in the sense defined for a particular state variable and disturbance input pair rather than for the overall system. It is not immediately apparent how to extend the results to such an overall system because of the large number of ways in which one might define the cost function. This will be given further consideration in the following section.
4. Although not stated specifically, the conditions that are given in Ref. 1.67 are only necessary conditions. For the elegant result that is obtained it is straightforward to verify that it is indeed a maximum or a minimum, but this may not always be the case in more general problems.
5. It is not possible to use the method to reconstruct a set of wind inputs that yield particular aircraft state
trajectories. Such a procedure, if available, may be very useful in determining wind histories for input into aircraft simulators for studying a particular accident or incident.

6. While the method can be extended to nonlinear systems, globally maximizing results may lead to discontinuities in the disturbance inputs (see the comments by Jones [1.69]). The analysis and numerical computations are also correspondingly more complex. Finally, in the local sense, more than one maximum may exist.

7. The method is not useful in a feedback system where the feedback structure is not known a priori. Thus, for example, this technique could not be used to generate worst-case inputs for a simulator being flown by a human pilot, although the inputs that are defined in the context of the open-loop system or a describing function feedback model of the pilot may still prove to be very useful.

8. The method maximizes the final value only rather than an integral measure of all values along the path.

9. The method has no way of conveniently weighting (i.e. indirectly constraining) the value of the wind so that certain realistic physical wind constraints are satisfied (e.g. wind speed goes to zero near the ground).

In the following section some extensions to the work of Van der Vaart and his co-workers are presented. These eliminate some of the objections discussed above.

5.2 Extensions to the Impulsive Response Worst-Case (IRWC) Method

We consider systems of the form

\[ \dot{x}(t) = f(t)x(t) + g(t)u(t) + f(t) \]  \hspace{1cm} (5.2.1)

with output

\[ y(t) = M(t)x(t) \]  \hspace{1cm} (5.2.2)

and initial conditions \( x(t_0) \). \( f(t) \) is a time-varying forcing function that is specified a priori. We assume that the outputs \( y_i(t) \), \( i = 1, q \) are linearly independent, i.e. the \( q \times n \) matrix \( M(t) \) is of rank \( q \). The rank property implies that the associated system of linear algebraic equations may not be reduced to an order \( < q \) and the latter implies the linear independence of the \( y_i(t) \). The interval of interest is \([t_0, t_f]\).

If we further assume that \( f(t) \) and \( M(t) \) are continuous and \( g(t) \) and \( u(t) \) are piecewise continuous for all \( t \in [t_0, t_f] \), then the solution of (5.2.1) is given by (Kwakernaak and Sivan [4.19])

\[ x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \left[ H(t, \tau)g(\tau) + \dot{\Phi}(t, \tau)\tilde{f}(\tau) \right] d\tau \]  \hspace{1cm} (5.2.3)

and

\[ y(t) = \Psi(t, t_0)y(t_0) + \int_{t_0}^{t} \left[ K(t, \tau)g(\tau) + \dot{\Psi}(t, \tau)\tilde{f}(\tau) \right] d\tau \]  \hspace{1cm} (5.2.4)

where \( \Phi(t) \) and \( \Psi(t) \) are continuous.

Here

\[ H(t, \tau) = \Phi(t, \tau) \tilde{g}(\tau) \]  \hspace{1cm} (5.2.5a)
where $a_{x_1}, a_1(t) \geq 0, \quad i = 1, q, \quad t \in [t_0, t_f]$ and

$$J_2 = \hat{E}_f^T [\hat{y}(t_f) - \gamma_d(t_f)] + \int_{t_0}^{t_f} \hat{E}_f^T (t) [\hat{y}(t) - \gamma_d(t)] \, dt$$

(maximize or minimize) (5.2,8b)

where $\hat{E}_f, \hat{E}(t), \quad t \in [t_0, t_f]$ are column weighting matrices whose elements are of arbitrary sign*. The term $\gamma_d(t)$ is a desired output trajectory which is assumed to be continuous.

We shall also consider quadratic payoff functionals of the form

$$J_3 = \gamma^T (t_f) \gamma(t_f) + \int_{t_0}^{t_f} \gamma^T (t) Q(t) \gamma(t) \, dt \quad \text{(maximize)}$$

$$Q(t), \quad q \geq 0, \quad t \in [t_0, t_f].$$

Such payoff functionals are a special case of a more general linear quadratic problem and are more naturally discussed in a later section.

We shall attempt to find $\gamma(t)$ time histories which maximize (minimize) these payoff functionals subject to the isoperimetric constraint

$$\text{subject to isoperimetric constraint}$$

* The restrictions on $a_{x_1}, a_1(t)$ ensure that the problem is meaningful. It is not clear at this point what restrictions are required on $\hat{E}_f, \hat{E}(t)$. We shall return to this question in the subsequent proofs.
The results will be presented in the form of two theorems and a number of corollaries.

Theorem 5.1

Given the equations of motion (5.2,1) with initial conditions \( g(t_0) \) and output equations (5.2,2), \( \Phi(t) \) and \( \Theta(t) \) continuous, \( \theta(t) \) piecewise continuous for all \( t \in [t_0, t_f] \) so that the solutions to (5.2,1) and (5.2,2) are given by (5.2,3) and (5.2,4) respectively, then all continuous \( \theta(t) \) satisfying the necessary extremum conditions for maximizing (minimizing) the payoff functional \( J_2 \) given by (5.2,8b) subject to the integral constraint (5.2,9) are given by

\[
\theta(t) = -\frac{1}{2\mu} \left[ \int_{t_0}^{t_f} \Phi^T(t_f,t) \dot{\Phi} - \int_{t_0}^{t_f} \Phi^T(t_f,t) \dot{\Phi}(t) \, dt \right] \quad (5.2,10)
\]

where \( \mu \) is a constant Lagrange multiplier given by

\[
\mu = \frac{1}{E} \left[ \int_{t_0}^{t_f} \left( \int_{t}^{t_f} \Phi^T(t_f,t) \dot{\Phi} - \int_{t}^{t_f} \Phi^T(t_f,t) \dot{\Phi}(t) \, dt \right) \, dt \right] \quad (5.2,11)
\]

and the positive sign is necessary for a minimizing result, the negative sign for a maximizing result.

Proof

The variational necessary conditions for a maximum (minimum) are given by Theorem F.5. We may apply these conditions to an equivalent augmented problem which does not have an isoperimetric constraint. The augmenting equation is given by

\[
\gamma(t) = \pi^T(t) \gamma(t) \quad (5.2,12a)
\]

where

\[
\gamma(0) = -E, \quad \gamma(t_f) = 0 \quad (5.2,12b)
\]

\[
\gamma(t) = \int_{t_0}^{t} \pi^T(t) \gamma(t) \, dt + \gamma(0) \quad (5.2,13)
\]

In preparation for the application of the variational necessary conditions the augmented Hamiltonian is defined as

\[
H_a = \pi^T \mathbf{M} \dot{x} - \mathbf{K} \pi + \frac{1}{2} \pi^T \mathbf{F} \pi + \mathbf{G} \pi(t) + \mathbf{H} \pi \quad (5.2,14)
\]
where $\lambda$ is a column vector of Lagrange multipliers corresponding to $\xi$ and $\mu$ is a Lagrange multiplier corresponding to $\gamma$. Differentiating $H_a$ once with respect to $\xi$ and $\gamma$ and twice with respect to $\mu$ we obtain

\[
\frac{\partial H_a}{\partial \xi} = \mathbf{M}^T \lambda + \mathbf{F}^T \lambda; \quad \frac{\partial H_a}{\partial \gamma} = 0 \tag{5.2,15a}
\]

\[
\frac{\partial^2 H_a}{\partial \mu^2} = \mathbf{M}^T \lambda + 2\mu \eta \tag{5.2,15b}
\]

\[
\frac{\partial H_a}{\partial \mu} = 2\mu \xi \tag{5.2,15c}
\]

* Because $J_2$ is a functional involving $\gamma$ one may be tempted to deal directly with a differential equation involving $\gamma$. Thus differentiating (5.2,2) we have

\[
\dot{\gamma}(t) = \mu(t) \dot{\lambda}(t) + \mathbf{M}(t) \ddot{\lambda}(t). \tag{a}
\]

Substituting (5.2,1) for $\dot{\lambda}(t)$ we obtain

\[
\dot{\gamma}(t) = [\dot{\mu}(t) + \mathbf{M}(t) \dot{\mathbf{F}}(t)] \mathbf{g}(t) + \mathbf{M}(t) \dot{\gamma}(t) + \mathbf{M}(t) \ddot{\lambda}(t). \tag{b}
\]

Since the $y_i(t), i = 1, q$ are postulated to be linearly independent, $\mathbf{M}(t)$ will have rank $q$ for $t \in [t_0, t_f]$ and for the case where $q < n$ an infinity of solutions for $\lambda$ will exist for (5.2,2). One may mechanically attempt to get around this difficulty by multiplying (5.2,2) by $\mathbf{M}^T$ and taking the inverse of the $n \times n$ matrix $\mathbf{M}^T \mathbf{M}$ to obtain

\[
\lambda = [\mathbf{M}^T \mathbf{M}]^{-1} \mathbf{M}^T \gamma. \tag{continued}
\]

First applying the condition (F.1,22c) we set (5.2,15b) to zero and solve for $\eta$ to obtain

\[
\eta = -\frac{1}{2\mu} \mathbf{M}^T \lambda. \tag{5.2,16}
\]

Thus we have been able to take the important step of uniquely solving for $\eta$ in terms of $\lambda$.

Proceeding further with the application of the necessary conditions, applying equation (F.1,22b) with (5.2,15a) we obtain the Euler-Lagrange equations

\[
\dot{\lambda} = -\mathbf{F}^T \lambda - \mathbf{M}^T \beta \tag{5.2,17a}
\]

\[
\ddot{\mu} = 0 \quad (\mu = \text{constant}). \tag{5.2,17b}
\]

The terminal conditions for $\lambda$ depend on the term preceding the integral portion of the payoff functional $J_2$ (see F.1,23b).

Footnote continued

Formally substituting (c) into (b) we obtain

\[
\dot{\gamma} = [\dot{\gamma} + \mathbf{M}(t) \dot{\mathbf{F}}(t)] \mathbf{g}(t) + \mathbf{M}(t) \dot{\gamma}(t) + \mathbf{M}(t) \ddot{\lambda}(t). \tag{d}
\]

Superficially this would seem to be the desired result, but the inverse of $\mathbf{M}^T \mathbf{M}$ will generally not exist. This follows from theorems in linear algebra (Ref. F.3, pp. 138 and 139) which show that $\mathbf{M}^T \mathbf{M}$ is also of rank $q < n$ and the $n \times n$ matrix $\mathbf{M}^T \mathbf{M}$ will thus be singular.
We have \( \delta t_f = 0 \) since \( t_f \) fixed

\[
\lambda(t_f) = \frac{3 \Phi^T(t_f) [x(t_f) - x_d(t_f)]}{\delta x(t_f)}.
\] \( (5.2,18a) \)

or

\[
\lambda(t_f) = M^T(t_f) \beta_f.
\] \( (5.2,18b) \)

The value of the constant \( \mu \) cannot be determined from such a boundary condition because \( y(t) \) is just a variable representation of the isoperimetric constraint \( (5.2,9) \) and this implies a terminal constraint on \( y(t_f) \) which in turn implies that \( \delta y(t_f) \) is zero, and a boundary condition of the form \( (5.2,18a) \) need not be satisfied. Thus \( \mu \) can be viewed as a parameter which must be adjusted so that the solution to the problem satisfies the isoperimetric constraint. The exact form of \( \mu \) will be determined at the end of the proof.

From linear system theory (Ref. 4.19, p. 12) it is possible to show that the transition matrix of the linear system \( (5.2,17a) \) is given by \( \Phi^T(t_f, t) \), where \( \Phi(t, t_0) \) is the transition matrix for the system \( (5.2,1) \) and we have substituted \( t_f \) for \( t_0 \) since the costate system has final value conditions given at \( t_f \). It follows from this fact and from the final conditions \( (5.2,18b) \) that \( \lambda(t) \) is given by

\[
\lambda(t) = \Phi^T(t_f, t) M^T(t_f) \beta_f - \int_{t_f}^t \Phi^T(\tau, t_0) M^T(\tau) \beta(\tau) d\tau. \] \( (5.2,19) \)

Having thus obtained the solution for \( \lambda(t) \), \( (5.2,19) \) may be substituted into \( (5.2,16) \) to obtain the necessary form of \( \eta(t) \). Using the definitions of \( \Phi(t, \tau) \) and \( \eta(t, \tau) \) from \( (5.2,5a) \) and \( (5.2,6a) \), the desired result \( (5.2,10) \) is obtained.

What remains is to determine \( \mu \) so that the isoperimetric constraint \( (5.2,9) \) is satisfied. This may be done by substituting \( (5.2,10) \) into \( (5.2,9) \) and solving for \( \mu \) in terms of \( E \). This determines \( \mu \) to its sign, and its sign follows from the convexity condition (also known as the Legendre-Clebsch condition) \( (F.1,22d) \) and \( (5.2,15c) \), i.e.

\[
\mu \leq 0 \quad (\mu \geq 0)
\] \( (5.2,20) \)

for a maximum (minimum). Thus \( \mu \) is given by \( (5.2,11) \) and the proof is complete.

The main result of this theorem is obtained using only necessary conditions for a weak extremum. Rather than applying sufficient conditions to complete the proof (e.g. such as those discussed in Bryson and Ho[4.16], Chapter 6), we may...
argue heuristically as follows. The way the problem was formulated guarantees that a maximum (minimum) exists for $J_2$. This is a consequence of the well-behaved nature of linear systems with the continuity properties given at the beginning of this section, the integral constraint on the inner product of the disturbance input vector and the form of the payoff functional $J_2$. Since the necessary conditions provide an explicit, unique expression for a continuous $\beta(t)$ in (5.2,10), this solution must be the maximum (minimum) solution.

If the case where

$$\beta(t) = 0, \quad t \in [t_0, t_f]$$  \hspace{1cm} (5.2,21)

is considered, then an extended version of van der Vaart's result is obtained, i.e. the resulting $\beta(t)$ maximizes (minimizes) the weighted sum of the final values of the output vector. More specifically we have the following corollary.

**Corollary 5.1.1**

Given the same assumptions as in Theorem 5.1, then all the vector functions $\beta(t)$ satisfying the necessary extremum conditions for maximizing (minimizing) the payoff functional $J'_2$, given by

$$J'_2 = \beta(t_f) \gamma(t_f) - \gamma_d(t_f)$$  \hspace{1cm} (5.2,22)

with the integral constraint (5.2,9) are given by

$$\gamma(t) = -\frac{1}{2\mu} K^T(t_f, t) \beta_f$$  \hspace{1cm} (5.2,23)

where $\mu$ is a constant Lagrange multiplier given by

$$\mu = \pm \frac{1}{2} \sqrt{\frac{1}{\mu} \int_{t_0}^{t_f} \beta_f^2(t_f, t) K^T(t_f, t) \beta_f dt}.$$  \hspace{1cm} (5.2,24)

The positive sign is necessary for a minimizing result and the negative sign for a maximizing result.

We may now consider one more corollary which shall take us a step closer to van der Vaart's result.

**Corollary 5.1.2**

Given the same conditions as in Theorem 5.1, then all the extremum functions $\beta(t)$ satisfying the necessary conditions for maximizing (minimizing)

$$J'_2 = \beta \gamma(t_f)$$  \hspace{1cm} (5.2,25)

subject to the integral constraint (5.2,9) are given by

$$\eta_k(t) = -\frac{1}{2\mu} k(t_f, t) \beta_k$$  \hspace{1cm} (5.2,26)

where

$$\mu = \pm \frac{1}{2} \sqrt{\frac{1}{\mu} \int_{t_0}^{t_f} \beta_k^2(t_f, t) k^2(t_f, t) dt}.$$  \hspace{1cm} (5.2,27)

The positive sign is necessary for a minimizing result and the negative sign for a maximizing result.
Proof

This corollary follows immediately from Corollary 5.1.1 by setting

\[ \beta_i = 0, \quad i \neq v, \quad i = 1, q. \]  
(5.2.28a)
\[ Y_d(t) = 0 \]  
(5.2.28b)

The result can then be obtained by considering explicitly the matrix multiplications in (5.2.23) and (5.2.24).

We may reduce this result to van der Vaart's result if we consider maximization (minimization) of the v-th output due to the k-th input alone for zero initial conditions and time-invariant systems, and set the system outputs to the system state (i.e. \( \Psi(t) = \mathbb{1} \)), and \( \beta_v \) to 1. In this case (5.2.26) and (5.2.27) become identical to (5.1.12) and (5.1.13) respectively, where

\[ k_k' = -\frac{1}{2\mu}. \]  
(5.2.29)

At this point it is appropriate to pause and consider the worst-case properties of the solutions given by Theorem 5.1 and its corollaries.

The first observation that can be made is that the \( \beta(t) \) are independent of \( Y_d(t) \), the desired output time trajectories. This result is a consequence of the structure of \( J_2 \), i.e. there is no coupling between the \( \dot{Y} \) and \( Y_d \) terms. In fact we may write \( J_2 \) as

\[ J_2 = c_1(\beta(t), \beta_f, \dot{Y}(t), \dot{Y}(t_f)) + c_2(\beta(t), \beta(t_f), Y_d(t), Y_d(t_f)) \]  
(5.2.30)

where the second term may be evaluated a priori once \( Y_d(t) \) and \( t_f \) are specified, i.e. it is independent of \( Y(t) \), and the first term is independent of \( Y_d(t) \). Thus from the point of view of maximizing (minimizing) \( J_2 \), the second term can be thought of as an additive constant and maximizing (minimizing) the payoff functional given by the first term will yield the same result as maximizing (minimizing) \( J_2 \).

The second observation that we can make is that a maximum solution, or for that matter a minimum solution, does not imply a worst-case solution, i.e. a solution which maximizes the absolute value of the deviation from \( Y_d(t) \). How well such an objective is achieved will depend on the sign properties of \( \beta(t) \) and \( \beta_f \) with respect to the signs of \( (\dot{Y}(t) - Y_d(t)) \), \( t \in [t_0, t_f] \). In fact, for \( \beta_f \), \( \beta(t) \) as given in (5.2.8a), if*

\[ \beta_f = \text{diag}[\text{sgn}(\dot{Y}(t_f) - Y_d(t_f))] \beta_f \]  
(5.2.31a)
and
\[ \beta(t) = \text{diag}[\text{sgn}(\dot{Y}(t) - Y_d(t))] \beta(t) \]  
(5.2.31b)

* See Appendix A for an explanation of the \( \text{diag}[\text{sgn}(z)] \) notation.
then a maximization of $J_2$ will produce the worst possible solution, in the sense described above, obtainable by maximization for given $\mathbf{g}_f$ and $\mathbf{g}(t)$. Similarly, a minimization of $J_2$ for the case where $\mathbf{g}_f$ and $\mathbf{g}(t)$ are given by the negative of (5.2,31a) and (5.2,31b) will produce the worst possible solution obtainable by minimization. These conclusions are a consequence of the fact that defining $\mathbf{g}_f$, $\mathbf{g}(t)$ in this way leads to a sign definite payoff functional for which it is clear whether one must proceed with minimization or maximization to produce a worst-case solution.

But from the properties of the sgn function, it is evident that $J_2$ defined with $\mathbf{g}_f$, $\mathbf{g}(t)$ given by (5.2,31a) and (5.2,31b) respectively is identical to $J_1$ given by (5.2,8a). Thus we may view $J_1$ as in general providing a more stringent worst-case criterion than $J_2$, and as such, if useful worst-case solutions can be found to it, may be of greater utility.

The main result for the maximization of $J_1$ is presented in the following theorem.

**Theorem 5.2**

Given the same assumptions as in Theorem 5.1, then all the vector functions $\mathbf{g}(t)$ satisfying the necessary extremum conditions for maximizing the payoff functional $J_1$ given by (5.2,8a) with the integral constraint (5.2,9) are given by

$$\mathbf{g}(t) = \frac{1}{2\mu} \left[ \mathbf{K}^T(t_f, t) \text{diag}[\text{sgn}(\mathbf{y}(t_f) - \mathbf{y}_d(t_f))] \mathbf{a}_f \right.$$  

$$- \int_{t_f}^{t} \mathbf{K}^T(r, t) \text{diag}[\text{sgn}(\mathbf{y}(r) - \mathbf{y}_d(r))] \mathbf{a}(r) \, dr \right] \tag{5.2,32}$$

where $\mu$ is a constant Lagrange multiplier given by (5.2,11) with (5.2,31a) and (5.2,31b).

**Proof**

The form of the worst-case solution for $J_1$ is identical to that for $J_2$, and one is tempted to formally make the substitutions (5.2,31a) and (5.2,31b) throughout the proof for Theorem 5.1 and thus obtain the result of Theorem 5.2. This procedure is unfortunately not rigorous in that the sgn functions introduce singularities in the derivatives of the Hamiltonian for the problem, given by

$$H_a = \mathbf{q}^T \text{diag}[\text{sgn}(\mathbf{M}(\mathbf{x} - \mathbf{x}_d))] \mathbf{M}(\mathbf{x} - \mathbf{x}_d) + \lambda^T [\mathbf{F}(\mathbf{x}) + \mathbf{g}(t)] + \mu \mathbf{u}^T \mu \tag{5.2,33}$$

with respect to those components of $\mathbf{x}$ which have sign changes (possibly more than one) in the interval $[t_0, t_f]$.

We may resolve these difficulties by considering an equivalent problem which has interior discontinuities in some

* We consider the augmented Hamiltonian because we have reduced the original problem with the isoperimetric constraint (5.2,9) to an unconstrained augmented problem (cf. the method employed in the proof of Theorem 5.1).
of the state variables, for which necessary conditions were discussed in Appendix F, Section 1. Let the times where $H_a$ has a singularity in at least one of its derivatives with respect to the components of $\mathbf{x}$ be denoted by $t_i$, $i = 1, \ldots, t_f$, $t_0 < t_1 < t_f$, $t_i < t_{i+1}$. We do not know these times or the number of such times a priori. Then the equivalent problem is defined by a maximization of the payoff functional

$$J_{1a} = g^T \text{diag}(x_2(t_f) - x_1(t_f)) + \mu_1^T \mu_1$$

where

$$\dot{x}_1 = f_2 + \mu_0 + \mathbf{f}; \quad t \in [t_0, t_f]$$

$$\dot{x}_2 = 0; \quad t_{i-1}^- < t < t_i; \quad i = 1, \ldots, t_f, t_{i+1}$$

Here $t_{i+1} = t_f$ and

$$x_1 \overset{\Delta}{=} \mathbf{x},$$

$$x_2 \overset{\Delta}{=} \text{sgn}(x_2 - x_d) = \text{sgn}(y - y_d).$$

By $t_i^+$ we mean the time just after $t_i$ and by $t_i^-$ we mean the time just before (more precisely $t_i + \epsilon$ and $t_i - \epsilon$ respectively for arbitrarily small positive $\epsilon$).

The veracity of the claim that this formulation is equivalent to the original formulation hinges on showing that $J_{1a}$ is equivalent to $J_{1a}$. We shall return to this point at the end of the proof, for the moment assuming that it is true and looking for $\mathbf{u}(t)$ which maximizes $J_{1a}$.

We have thus been able to pose the problem of maximizing $J_1$, in terms of an equivalent maximization problem which has discontinuities in some of the state variables at interior points, for which necessary conditions for an extremum are discussed in Theorem F.6. We define $i+1$ Hamiltonians

$$H_{a_i} = g^T \text{diag}(\text{sgn} x_2) M(x_1 - x_d) + \mu_1^T [f x_1 + \mu_0 + \mathbf{f}]$$

$$+ \mu_0^T, \quad t_{i-1}^- < t < t_i, \quad i = 1, \ldots, t_f.$$ (5.2.37)

Here $\lambda_1$ is the vector of Lagrange multipliers corresponding to the $x_1$ system and are thus equivalent to $\lambda$ in Theorem 5.1. The vector of Lagrange multipliers $\lambda_2$ corresponding to the $x_2$ system do not appear in (5.2.37) because of the fact that $x_2$ is constant in each of the $i+1$ time intervals. Because of this $\lambda_2$ is of no significance to this problem and the necessary conditions are identical in form to those in Theorem 5.1, with

$$\lambda = \lambda_1$$

(5.2.38)

and with the added interior conditions

$$H_{a_i}(t_i^+) - H_{a_{i+1}}(t_i^-) = 0; \quad i = 1, \ldots.$$ (5.2.39)
This last condition implies that $t_i$ must be such that

$$\alpha X^T \text{diag} \{ \text{sgn} x_2 \} M(x_2 - x_d) \bigg|_{t_i^-} - \alpha X^T \text{diag} \{ \text{sgn} x_2 \} M(x_2 - x_d) \bigg|_{t_i^+} = 0 .$$

(5.2.40a)

This may be written as

$$\sum_{k=1}^{q} \left[ \alpha_k \text{sgn}(y_k - y_{dk}) (y_k - y_{dk}) \right]_{t_i^-} - \alpha_k \text{sgn}(y_k - y_{dk}) (y_k - y_{dk}) \bigg|_{t_i^+} = 0$$

(5.2.40b)

where we have used the definition of $X_2$ and $Y$ given by (5.2.36b) and (5.2.2) respectively.

For any of the singularity times $t_i$, the elements of the summation (5.2.40b) will obviously be zero except for those elements $k \in \xi_i$, where $\xi_i$ is the set of indices of the components of $Y - Y_d$ which change sign at this time. But since $Y(t)$ and $Y_d(t)$ are continuous ($\dot{Y}(t)$ is continuous because of the restrictions placed on $F(t)$, $\Theta(t)$ and $M(t)$, as discussed at the beginning of this section, and $Y_d(t)$ was restricted to be continuous), for these elements

$$y_k(t_i) - y_{dk}(t_i) = 0 .$$

(5.2.41)

Thus the condition (5.2.39) is automatically satisfied at any point where $X_2$ has a discontinuity.

In effect we have shown that the application of the necessary conditions for an extremum without regard for the point singularities in the derivatives of the Hamiltonian with respect to the elements of $X$ is a valid procedure for this problem. As mentioned at the beginning of the proof, the results of the theorem are then readily obtained with the formal substitution of (5.2.31a) and (5.2.31b) throughout the proof of Theorem 5.1, where the negative sign for $\mu$ is chosen because we are looking for a maximum of $J$. In going through this procedure, we must formally exclude the possibility that the time $t_i$ is a time when a sign change in one or more of the components of $[Y(t) - Y_d(t)]$ occurs.

We also note that we could have considered an equivalent problem identical to the one defined above with the additional constraints

$$x_{2k}(t_i^-) + x_{2k}(t_i^+) = 0$$

(5.2.42)

for $k \in \xi_i$. This is unnecessary, however, inasmuch as the formulation of the problem guarantees that these constraints will be satisfied, i.e. in combination and from the definition of $X_2$, they merely express the fact that a sign change in $[Y_k(t) - Y_{dk}(t)]$ must occur at $t_i$. This is also implied by the definition of $X_2$ and by the fact the times $t_i$ are times where $X_2$ has discontinuities of the first kind.

We may now return to the question of whether $J_{1a}$ is equivalent to $J_{1a}$. To show this we prove the following lemma.
Lemma 5.1

Given the payoff functional

$$J^e = s\{y(t_f), t_f\} + \sum_{i=1}^{i+1} \int_{t_{i-1}}^{t_i} g(y, n, t) dt$$ (5.2,43)

subject to the differential equation constraints

$$\dot{x}(t) = f(x, n, t), \quad x(t_0) \quad \text{given}$$ (5.2,44a)

$$\dot{y}(t) = h(x, n, t)$$ (5.2,44b)

for which a vector function $n^*(t)$, taken from an admissible function space $n$, exists maximizing $J^e$ (possibly only locally), then $n^*(t)$ also maximizes (possibly only locally) on $n$ the payoff functional $J$ given by

$$J = s\{y(t_f), t_f\} + \int_{t_0}^{t_f} g(y, n, t) dt$$ (5.2,45)

subject to (5.2,44a) and (5.2,44b).

Proof of Lemma 5.1

The $s$ terms of $J^e$ and $J$ are identical and Riemann integrals such as those in (5.2,43) and (5.2,45) are unaffected by a finite number of point singularities and by the removal of a finite number of points in the integrand, i.e.

$$\sum_{i=1}^{i+1} \int_{t_{i-1}}^{t_i} g(y, n, t) dt = \int_{t_0}^{t_f} g(y, n, t) dt$$ (5.2,46)

where $t_{i+1} = t_f$. This implies that

$$J(n) = J^e(n), \quad n \in \Omega.$$ (5.2,47)

But we may now argue by contradiction. Suppose $n^*(t)$ does not maximize (5.2,45) on $\Omega$, i.e. there is a vector $n_1(t) \in \Omega$ given by

$$n_1(t) = n^*(t) + h(t)$$ (5.2,48)

such that

$$J(n_1) > J(n^*)$$ (5.2,49)

Here $\Omega_1$ is a neighbourhood of $n^*(t)$ whose members are also members of $\Omega$ and which does not contain any other maxima of (5.2,43), i.e.

$$\|h(t)\| < \delta$$ (5.2,50)

for some suitably chosen positive $\delta$ and a norm compatible with $\Omega$. Further suppose that $n_1(t)$ is different from $n^*(t)$ in some finite* time interval $(t_a, t_b)$, $t_a \neq t_b$ (there may

* Two function vectors $n_1(t), n_2(t) \in \Omega$ which differ from each other only at a finite number of times, rather than in finite time intervals, will be considered to be identical because of the Riemann integral properties of $J^e$ and $J$. We may also define $\Omega_1$ and thus $\Omega_1$ in such a way as to exclude these cases.
be more than one such interval). Then from (5.2,47) and
(5.2,49) this means that
\[ J^e(n_1) = J(n_1) > J(n^*) = J^e(n^*) \]  \hspace{1cm} (5.2,51a)
or that
\[ J^e(n_1) > J^e(n^*). \]  \hspace{1cm} (5.2,51b)
This contradicts the fact that \( n^* \) is a global maximum in \( N_1 \)
for \( J^e \) and thus no such \( n_1(t) \) can exist. It follows that
\( n^*(t) \) must also be a maximum of (5.2,45). This completes the
proof of Lemma 5.1.

Returning to the proof of Theorem 5.2, Lemma 5.1 may now
be used as a justification to conclude that \( J_{1a} \) and \( J_{1a} \)
are indeed equivalent. This justifies the procedure used in
proving this theorem, and completes the proof.

If we substitute (5.2,32) into (5.2,4) we obtain an
expression for the worst-case response which is actually an
implicit equation for \( q(t) \) because of the presence of the
\( sgn \{ y(t_f) - y_d(t_f) \} \) terms in the right hand side. This makes the
problem of finding a suitable solution difficult in the most
general case, both from the point of view of determining the
solution and finding out if it is unique.

We may illustrate these problems by considering the case
where
\[ q(t) = 0, \quad t \in [t_0, t_f] \]  \hspace{1cm} (5.2,52)
i.e. the corollary paralleling corollary 5.1.1.

**Corollary 5.2.1**

Given the same assumptions as in Theorem 5.2, then all
the vector functions \( n(t) \) satisfying the necessary extremum
conditions for maximizing the payoff functional \( J'_1 \) given by
\[ J'_1 = \mathbf{g}_f^T [y(t_f) - y_d(t_f)] \]  \hspace{1cm} (5.2,53)
with the integral constraint (5.2,9) are given by
\[ n(t) = -\frac{1}{2\mu} \mathbf{K}^T(t_f, t) \text{diag}[sgn[y(t_f) - y_d(t_f)]] \mathbf{g}_f \]  \hspace{1cm} (5.2,54)
where \( \mu \) is a constant given by the negative root of (5.2,24)
with the substitution (5.2,31a) for \( \mathbf{g}'_f \).

The uniqueness of (5.2,54), if this is indeed the case,
is not obvious because of the presence of the \( sgn[y(t_f) - y_d(t_f)] \)
term. This depends on the signs of the final values of the \( q \)
outputs, which are not known a priori. Thus (5.2,54) represents
\( 2^q \) potential solutions of which possibly more than one leads to a
\( \text{diag}[sgn[y(t_f) - y_d(t_f)]] \) matrix which agrees with that assumed in \( n(t) \). Of these successful solutions the one which is the
worst-case solution may be determined by evaluating \( J'_1 \) for
each case and selecting the one that is maximizing.

The situation is better clarified by considering a particu-
lar case for which
Substituting (5.2,54) into (5.2,4) and examining the result for \( t = t_f \), we have, in scalar form

\[
Y_i(t_f) = n \sum_{j=1}^{n} \psi_{ij}(t_f, t_0)x_j(t_0)
- \frac{1}{2} \int_{t_0}^{t_f} \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} k_{ij}^2(t_f, \tau) \sum_{j=1}^{n} k_{kj}(t_f, \tau) \operatorname{sgn} Y_k(t_f) a_{f_i} \right] d\tau.
\]

(5.2,56a)

The equations (5.2,56a) may be separated into three parts as follows:

\[
Y_i(t_f) = \sum_{j=1}^{n} \psi_{ij}(t_f, t_0)x_j(t_0)
- \frac{1}{2} \int_{t_0}^{t_f} \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} k_{ij}^2(t_f, \tau) \sum_{j=1}^{n} k_{kj}(t_f, \tau) \operatorname{sgn} Y_k(t_f) a_{f_i} \right] d\tau
- \frac{1}{2} \int_{t_0}^{t_f} \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} k_{ij}^2(t_f, \tau) \sum_{l=1}^{n} k_{lj}(t_f, \tau) \operatorname{sgn} Y_l(t_f) a_{f_i} \right] d\tau.
\]

(5.2,56b)

We see that (5.2,56a) (equivalently (5.2,56b)) form implicit equations for \( Y_k(t_f) \) with no obvious method of resolving the ambiguous \( \operatorname{sgn} Y_i \). Some insight is gained by noting that for finite \( E \) we have \( u < 0 \), and thus, since \( a_{f_i} > 0 \), term two will have the same sign as \( Y_i(t_f) \). Thus if this term dominates over terms one and three, we are able to resolve the sign ambiguity. This result may be stated in the form of a corollary.

**Corollary 5.2.2**

Necessary and sufficient conditions for finding a consistent \( \text{diag}[\operatorname{sgn} Y(t_f)] \) matrix for (5.2,54) for \( Y(t_f) \neq 0 \) and for \( Y_d(t_f) = 0, f(t) = 0 \) are that the following inequalities be satisfied. We have, for \( Y_i(t_f) < 0 \) \( Y_i(t_f) > 0 \), \( i = 1, q \)

\[
\frac{1}{2} \int_{t_0}^{t_f} \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} k_{ij}^2(t_f, \tau) \sum_{j=1}^{n} k_{kj}(t_f, \tau) \operatorname{sgn} Y_k(t_f) a_{f_i} \right] d\tau
- \frac{1}{2} \int_{t_0}^{t_f} \left[ \sum_{j=1}^{n} \sum_{k=1}^{n} k_{ij}^2(t_f, \tau) \sum_{l=1}^{n} k_{lj}(t_f, \tau) \operatorname{sgn} Y_l(t_f) a_{f_i} \right] d\tau.
\]

(5.2,57)

If \( Y_i(t_f) = 0 \), then one must consider \( Y_i(t_f - \varepsilon) \) for arbitrarily small \( \varepsilon > 0 \) in order to determine the appropriate sign for \( \operatorname{sgn} Y_i(t_f) \) in accordance with the definition of this function for \( Y_i(t_f) = 0 \) (see Appendix A). Thus the conditions (5.2,57) continue to hold with \( t_f \) everywhere replaced by \( t_f - \varepsilon \), and we consider the limit on both sides of the inequality as \( \varepsilon \) becomes arbitrarily small.

The situation becomes more obvious if we consider even
more restricted versions of this problem. In the case where we wish to maximize the absolute value of the final value of the i-th output,

\[ a_{fi} = 0, \quad \lambda \neq i \]  

(5.2,58)

and (5.2,57) becomes for \( \mu < 0, \; y_i(t_f) < 0 \) (\( y_i(t_f) > 0 \)), \( i=1,q \),

\[ \frac{1}{2\mu} \int_{t_0}^{t_f} \sum_{j=1}^{n} k_{ij}^2(t_f, \tau) \text{sgn} \; y_i(t_f) a_{fi} \; \text{d} \tau \leq \frac{\sum_{j=1}^{n} \psi_{ij}(t_f, t_0) x_j(t_0)}{\mu}. \]

(5.2,59)

Consideration of (5.2,59) for the positive and negative \( y_i(t_f) \) cases readily shows that one of the inequalities must always be satisfied for at least one sign of \( y_i(t_f) \). It is also important to note that it may be satisfied for both signs of \( y_i(t_f) \), in which case the worst-case solution is determined by trial and error. In effect, the following corollary has been proven.

Corollary 5.2.3

There may be more than one consistent choice for the \( \text{diag} \{ \text{sgn} \; [y^*(t_f) - y_d(t_f)] \} \) matrix of the worst-case solution (5.2,54) (and thus (5.2,32)). This multiplicity of solutions implies that local maxima exist.

This corollary follows intuitively from the presence of absolute values in the payoff functionals \( J_1 \) and \( J_1' \). Thus one would expect solutions which make the \( y_i(t_f) \) maximally negative, and others which tend to make them maximally positive. For the case where (5.2,52), (5.2,55) and (5.2,58) hold, it is apparent from (5.2,59) that there are at most only two such cases, i.e. the solutions leading to a maximally negative and maximally positive \( y_i(t_f) \).*

We may continue with these considerations at some length, but without significant gains in being able to determine, a priori, the worst-case solution for \( J_1 \) in its most general form. For the case of maximizing (5.2,53), it has been shown how the problem reduces to at most \( 2^{q'} \) iterations, where \( q' \) denotes the number of nonzero \( a_{fi} \), \( i=1,q \), but for the general payoff functional (5.2,8a) the problem is, in effect, a two point boundary value problem, i.e. it must be solved for \( y(t) \), given \( x(t_0) \), simultaneously with \( z(t) \) (equivalently \( y(t) \)), for which terminal conditions are given at \( t_f \).

From the form of the worst-case solution (5.2,32), this amounts to knowing \( \text{sgn} \; [y(t) - y_d(t)] \) backwards in time, i.e. from \( t_f \) to \( t_0 \), simultaneously with solving for \( y(t) \) forwards in time. This is clearly a very difficult condition with which to work.

It has thus been possible to eliminate some of the disadvantages present in van der Vaart's worst-case solution.

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* In particular we note that for zero initial conditions \( (x_j = 0, \; j=1,n) \), (5.2,59) holds for both cases and the two local maxima yield the same value for \( J_1' \).
(see the discussion at the end of Section 5.1) by considering direct extensions of his method. For the most part, however, much of the elegance of his initial result is lost in going to these more general cases (and even more is lost going to non-linear cases), the extended results being difficult to apply in practice. This leads to the consideration of other techniques which might be better applied to this problem so that practical, general worst-case solutions may be obtained. These will be the subject of the following two sections.

5.3 Perverse Wind Controllers

In aircraft response study there exists a conceptual separation between control inputs and wind inputs. This separation is a natural one, but from the analytical point of view there is no reason why one should not choose to view wind inputs as control inputs. From this altered perspective one can immediately conceive of wind controllers (strategies) that strive to achieve their own objectives. Because wind inputs tend to produce undesirable response characteristics, these wind objectives, for realistic wind controllers, would oppose desired closed-loop aircraft response characteristics. In a sense this gives the wind a mind of its own: it becomes an intelligent adversary.

From the point of view of defining desired handling qualities that the wind controller must give the aircraft system (including any human or automatic pilots), then once the wind controller forms are defined the analysis may proceed using classical controller techniques, with the objective of destabilizing the aircraft system in some way* given certain constraints on the wind inputs. But in this section we are concerned with analytically maximizing worst-case wind inputs and controllers, and thus the somewhat qualitative statements of controller objectives in classical controller design are not suitable. What we are looking for are optimal wind controllers, i.e. a feedback law for the wind inputs that maximizes a given functional. This functional, of course, must be defined in such a way that its maximization expresses the disturbance-like nature of the wind as well as any other constraints that might be required to make the problem meaningful and the wind field physically realistic.

The IRWC method (Van der Vaart's method and its extensions) is not an example of the latter group of techniques. While the IRWC method does produce a worst-case wind history for a particular functional, it is a programming method, i.e. it produces a time history rather than a control law, and as such is conceptually different, and of less generality than methods that do yield maximizing wind control laws.

Because of our desire to find optimal wind control laws it is natural to consider linear systems with quadratic payoff

* For example, the problem of finding a proportional feedback of airspeed wind controller that causes neutral airspeed stability in the combined system.
functionals for which the analysis is more straightforward. Two different classes of linear quadratic methods will be considered.

5.3.1 Indirect Methods

The basic idea behind these methods is that the wind inputs are treated as control inputs which attempt to make the aircraft track a specified state or output trajectory which is known to be particularly perverse. If wind inputs can be found which nearly achieve such trajectories then we may think of these inputs as worst-case in this sense.

While the selection of these perverse state trajectories is quite arbitrary, one may be aided considerably in this process by using state trajectories which have been documented to have occurred in aircraft incidents and accidents. In this sense the technique may also be thought of as a way of estimating wind inputs that might have been present in a given situation*, although how accurately this reflects the actual winds will depend on the payoff functional and the values given to the weighting matrices. It may be possible to determine weighting matrices which lead to a reconstruction of the wind inputs quite accurately. This is especially true if the

* This, in turn, may be quite useful in aircraft accident/incident investigations where wind conditions are suspected to have been a contributing factor.

perverse output trajectories \( y_p(t) \) are achievable output trajectories.

The theory required to handle the indirect or perversity function method, for linear equations of motion with quadratic payoff (perversity) functionals, has been summarized in Appendix F, Section 1. The changes are strictly notational. The equations of motion for which an optimal controller is to be found are now

\[
\begin{align*}
\dot{x}(t) &= F(t)x(t) + G(t)\eta(t) \\
x(t_0) &= x_0
\end{align*}
\]

(5.3,1a)

(5.3,1b)

with output equations

\[
y(t) = M(t)x(t)
\]

(5.3,1c)

where \( \eta(t) \) is the disturbance vector.

The payoff criterion is

\[
J = \|x(t_f) - x_p(t_f)\|_F^2 + \int_{t_0}^{t_f} [\|x(t) - x_p(t)\|_S^2 + \|\eta(t)\|_R(t)]^2 dt
\]

(5.3,2)

with the same restrictions on \( S, G(t) \) and \( R(t) \) as in Appendix F, Section 1.1, Formulation 4. The objective is still to minimize \( J \). The perversity enters the problem by appropriate selection of \( x_p(t) \) or, equivalently, \( y_p(t) \).
If constant disturbance set-points are desired, the theory of Formulation 4 still applies, with

\[ x_p(t) = x_{ps} \quad \text{(constant).} \quad (5.3,3) \]

If, however, \( x_{ps} \) is restricted to be a set point which the system can achieve as \( t \to \infty \), then the restrictions of Formulation 3 of Appendix F, Section 1.1 are applicable.

In some situations where there are time referenced control inputs, it may be advantageous to enhance the ability of the wind to track a perverse trajectory by using integral feedback to the wind controller. In this situation the theory of Formulation 6 of Appendix F, Section 1.1, with appropriate reconciliation of the notation, is applicable.

It is important to note that although the SDL equations of motion derived in Section 3 are of the form

\[ \ddot{x}(t) = F(t)x(t) + G(t)y(t) + \theta_1(t)\eta(t) + \theta_2(t)\dot{\eta}(t) \]

\[ (5.3,4) \]

the presence of the \( \dot{\eta}(t) \) and \( y(t) \) terms cause no great difficulty in applying this method. Also the feedback controller for \( y(t) \) may be determined separately by a number of methods, including the optimal control methods discussed in Appendix F. This feedback controller may then be incorporated into \( F(t) \), and the wind controller may be designed on the basis of the closed-loop system.

Alternatively, and this is a very important advantage, the wind controller may be designed on the basis of the open-loop system. This controller may then be incorporated into \( F(t) \), and the control inputs may then attempt to counteract the new system that results with wind controller augmentation. Such a wind control law may, in fact, be programmed on aircraft simulators, because of its feedback form, and this may yield considerable insight into the ability of pilots to achieve desired trajectories in winds that attempt to move the aircraft state to a perverse trajectory.

To recapitulate, the advantages of the indirect wind controller method are as follows:

1. Provides a wind control law.
2. Uses existing linear optimal control theory with quadratic payoff functionals for which analytical and numerical procedures are well understood.
3. Provides necessary and sufficient conditions.
4. May be applied to determine wind inputs giving state trajectories which approach a specified state trajectory.
5. Because of the nature of the payoff functionals this approach considers not only final values, as in van der Vaart's version of the IRWC method, but also deviations from a perverse trajectory for all \( t \in [t_0, t_f] \).
6. This method provides an overall worst-case wind input rather than one for a particular input-output pair as is the case for van der Vaart's version of the IRWC method.

The disadvantages of the method are as follows:
1. The perverse state trajectory must be specified a priori.
2. The weighting matrices of the payoff functional are somewhat arbitrary. Their numerical values must be provided a priori.
3. The numerical methods that are required are generally more involved than for the IRWC method.
4. The wind inputs are not integral constrained, although by appropriate selection of weighting matrices one may provide indirect constraints on wind magnitude and energy.

5.3.2 Direct Methods

One may also consider the maximization of quadratic payoff functionals of the form

\[ J = \int_{t_0}^{t_f} \left( \| x(t) - x_d(t) \|^2_Q + \| \dot{x}(t) - \dot{x}_d(t) \|^2_R + \| \eta(t) \|^2_S \right) dt \]

(5.3,5)

where \( Q(t) \) and \( S \) are positive semidefinite and \( R(t) \) is negative definite, or the minimization of cost criteria such as (5.3,5) with \( Q(t) \) and \( S \) negative semidefinite and \( R(t) \) positive definite. These conditions on \( Q, S, \) and \( R \) will tend to result in histories which make the \( x(t) \) move farther away from \( x_d(t) \), the desired state trajectory within the constraints placed on \( \eta(t) \) by the weighting matrix \( R \).

The analytical tools required to solve this problem are formally identical to the linear quadratic tracking problem with known input disturbances (for this application it would be known input control histories). This is discussed in detail in Appendix F, Section F.2. Section G.1 of Appendix G considers this worst-case wind approach in more detail.

An advantage of this method over the indirect method is that it requires no specification of a perverse state trajectory. (This may also be thought of as a disadvantage depending on the nature of the worst-case investigation.) Otherwise the direct and indirect methods have similar advantages and disadvantages (see the discussion at the end of the previous section).

5.4 Conflict of Interest Wind and Aircraft Controller Models

In Section 5.3 we introduced the concept of the wind being controlled by an intelligent entity whose objectives, as appropriate for the disturbing nature of the wind, are generally opposed to those of the aircraft controller. The resulting mathematical model (for the direct method) then consisted of looking for a wind control law which maximized a quadratic payoff functional whose weighting matrices are

* In particular we note that a solution may not exist in \([t_0, t_f]\) for all \( J \). See Remark 1 following Theorem 5.5 in Section 5.4.2.
defined so that they express the wind objectives while at the same time constraining the wind inputs in a suitable way. The aircraft controller was defined \textit{a priori} and incorporated in the equations of motion. The essence of the intelligence of the wind is captured by the control law nature of the solution, i.e. the wind input is a function of the state of the aircraft system rather than an open-loop maximizing time history*.

It is the purpose of this section to indicate that these concepts may be generalized by viewing the aircraft controller and the wind controller as intelligent adversaries (rather than as independent intelligences) playing a conflict of interest game on the aircraft dynamics equations (see Fig. 5.2). In this scenario the wind and aircraft controllers determine their strategies simultaneously given the following information:

1. They both have full knowledge of each other's objectives.
2. They both have full knowledge of the system dynamics (i.e. the game board), system states, and control inputs.
3. Each controller may or may not have information regarding their opponent's strategy itself.

The statement of this scenario is broad enough so that its analysis may be approached in a number of ways, including ad hoc classical controller synthesis methods. As argued in Section 5.3, however, such methods are incompatible with the objective of finding worst-case wind models because of their imprecise definition of the term \textit{worst-case}. Thus we once more turn our attention to functional optimization methods. Given that we are simultaneously looking for optimal control strategies for two distinct controllers with separate objectives acting on one dynamic system, we can immediately conclude that optimal control methods are inapplicable. What is required is a theoretical framework for two-sided optimization*.

Such a framework, to the extent that it is developed, does exist within the differential game theory. It is the intention of the following subsections to review the relevant elements of this theory, which is generally not well known in aeronautical circles, and to indicate how it may be applied to this conflict of interest scenario and to the problem of obtaining worst-case wind models.

* This is something of a conceptual artifice: the open-loop and closed-loop maximizing wind time histories are identical for given $\mathbf{x}(t_0)$ because of the one-sided nature of the problem.

* To recapitulate, in the previous analysis the objective was to find a worst-case wind controller, but now we are simultaneously looking for a best aircraft controller given the worst wind controller, and vice versa.
5.4.1 Differential Game Theory* — An Overview

Although game theory** and differential game theory differ substantially in the analysis and types of application appropriate for the two fields, it is still appropriate to briefly discuss games before considering differential games (DG). This is loosely analogous to considering parameter optimization problems before considering optimal control problems.

Game theory, in its most elementary forms, deals with discrete optimization problems involving two players with conflicting interests. It has been extended to consideration of more than two players and interests which are not directly conflicting, as well as to continuous payoff functions. In its most general forms, the possibility of coalitions and games of imperfect information must be considered.

An example of a $2 \times 2$ discrete game is given in Fig. 5.3a. Here player $u$ is minimizing and player $n$ is maximizing, with the payoff for each move given by $g_{ij}$, $i,j = 1,2$. It is evident that if $u$ plays first he should pick the row with the smallest maximum (since $n$ will subsequently pick the column with the maximum for that row), and similarly if $n$ plays first he would pick that column with the largest minimum.

For this game the optimal choices for $u$ and $n$ give a payoff of 7 regardless of the order of play, i.e.

$$\max \min g_{ij} = 7 = \min \max g_{ij}$$

(5.4.1a)

$$g(u_1, n_2) \leq g(u_1, n_1) \leq g(u_2, n_2).$$

(5.4.1b)

The choice $u_1, n_2$ is the minimax solution of the game.

This happy situation does not always occur, i.e. it is possible to have games where the order of play matters, e.g. the $2 \times 2$ discrete game given in Fig. 5.3b. Without going into further details*, the resolution to this difficulty lies in asking each side to make a random selection of strategies according to some fixed probability. In this way probability mixes can be found such that the expectation of the minimax and maximin solutions are the same.

From discrete games it is natural to consider games where $g(u, n)$ is continuous in $u, n$. In such games we look for $u^*, n^*$ such that

* For the most part the introductory remarks in this subsection are based on those of Bryson and Ho [4.16, Chapter 9].

** For a lucid and interesting discussion of game theory see the text by Jones [5.3].

* See Bryson and Ho [4.16], Chapter 9. This leads into the important (to game theory) minimax principle of Von Neumann and Morganstern "...by randomization the difference between minimax and maximin can be equalized on an expected value basis."
\[ g(u^*, n) \leq g(u^*, n^*) \leq g(u, n^*) \quad (5.4,2) \]

for all \( u, n \). The necessary conditions for \( u^*, n^* \) are

\[ \frac{\partial g}{\partial u} = 0 \quad ; \quad \frac{\partial g}{\partial n} = 0 \quad (5.4,3a) \]
\[ \frac{\partial^2 g}{\partial u^2} > 0 \quad ; \quad \frac{\partial^2 g}{\partial n^2} < 0 \quad \text{for all} \quad u, n \quad (5.4,3b) \]

Sufficient conditions are \((5.4,3a)\) and \((5.4,3b)\) with the inequalities in \((5.4,3b)\) changed to strict inequalities.

If \( u^*, n^* \) satisfy the sufficient conditions, then they are called a game-theoretic saddle point (GTSP). A GTSP is not equivalent to the calculus saddle point conditions, i.e. \((5.4,3a)\) with

\[ \frac{\partial^2 g}{\partial u^2} \frac{\partial^2 g}{\partial n^2} - \left( \frac{\partial^2 g}{\partial u \partial n} \right)^2 \leq 0 \quad (5.4,4) \]

In fact a solution may be a calculus saddle point but not a GTSP (see Figs. 5.4a and 5.4b, respectively, for examples of (1) a GTSP and (2) a calculus saddle point which is not a GTSP). For problems where \( g(u, n) \) contains no cross-terms in \( u \) and \( n \), the minimax and maximin solutions, if they exist, are equal (for more details see Bryson and Ho [4.16]).

We are now ready to begin our consideration of DG as a natural extension to continuous games. Thus we look for continuous game theory solutions to payoff functionals constrained by systems of differential equations. In the most direct extension to the two player continuous games discussed above, we may consider the problem of finding \( y^*(t) \), \( n^*(t) \)

such that

\[ J(y^*, n) \leq J(y^*, n^*) \leq J(y, n^*) \quad (5.4,5) \]

subject to

\[ \dot{x} = f(x, y, n, t), \quad x(t_0) = x_0 \quad (5.4,6) \]
\[ u \in U, \quad n \in H, \quad x \in X, \quad t \in [t_0, t_f]. \]

Here the payoff functional \( J \) is given by

\[ J = s(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), y(t), n(t), t) dt. \quad (5.4,7) \]

The control \( y \) is minimizing and the control \( n \) is maximizing (this follows from \((5.4,5)\)). This is very much a conflict of interest problem whose form is suggestive of the type of formulation required for worst-case controller and aircraft controller conflict of interest models. In fact this description is a heuristic definition of two player direct conflict of interest DG.

Such DG problems and their generalizations were first considered by Isaacs [5.4] (as indicated by Baron [5.5]) in the middle fifties. His approach to the theory was very similar to dynamic programming in optimal control problems, although at this time he had not specifically cast the theory in an extended optimal control framework. In the following decade some work was carried out in this regard, and the
natural extension to the application of variational methods to differential games was also investigated (Berkovitz [5.6], Ho, Bryson and Baron [5.7] and Starr and Ho [5.8]). Nevertheless it was not until 1965, with the publication of Isaacs' book [5.9] on the subject that the theory became reasonably well known in the control field. This resulted in a number of papers treating a variety of control related DG formulations. These include many player differential games (Starr and Ho [5.8, 5.10], Case [5.11], and Lukes and Russell [5.12]) (i.e. more than two players) and differential games with imperfect information (Cilletti [5.13] and Roxin, Liu and Sternberg [5.14]) (imperfect information games usually lead to stochastic games). These results, while frequently quite abstract in nature, have occasionally been applied to aerospace problems, usually in pursuit-evasion game models. This application is important from the military aerospace point of view in that it forms the framework in which one may analyse certain combat situations (e.g. air-to-air missile guidance and avoidance). Typical applications in this regard are Baron [5.5], Baron and Kleinman [5.15], Anderson [5.16] and Shinar and Gutman [5.17]. The application of this theory to other aerospace problems is a relatively unexplored area. In particular the application to wind/aircraft controller conflict of interest models does not appear to have been attempted.

While DG theory was initially developed independently of, but concurrently with optimal control theory, the somewhat satisfying conclusion that can be drawn is that it encompasses optimal control theory in that optimal controls may be thought of as solutions to one player DG. This explicit theoretical connection is quite important in describing the DG theory inasmuch as one may frequently take advantage of the parallels between the two theories; also the notation and nomenclature are similar. Nevertheless DG theory is both analytically and computationally more complex than optimal control theory and thus great care must be used when making analogies between the two fields (see the comments in Isaacs [5.18]).

In the sequel we are primarily interested in two player (the aircraft controller and the wind) conflict of interest DG. Except for a number of general N-player concepts, only these aspects of the theory will be stressed. The review will be quite brief, with the objective being to outline a number of useful results for linear quadratic DG. These will be applied to the decoupled linearized equations to provide numerical examples of conflict of interest wind and aircraft controller models.

We begin by defining a number of DG concepts.

1. N-Player Differential Games - An abstract, rigorous definition of a differential game is quite involved and is not germane to the purposes of this investigation*. Rather,

* The interested reader is referred to Friedman [5.19] for more details.
we will define N-player DG nonrigorously in the following way.

We consider the ordinary system of differential equations *

\[ \dot{x} = f(x, u_1, \ldots, u_N, t) \quad (5.4,8a) \]

\[ x(t_0) = x_0 \quad (5.4,8b) \]

where \( u_1, \ldots, u_N \) are the inputs of the N players, taken from admissible player input history sets \( U_i \). If \( N = 1 \) we may think of the player input as a control or a disturbance input depending on the interpretation of the source. The game is played such that each player attempts to minimize a payoff functional \( J_i, i = 1, N \) given by†

\[ J_i(x(t_0), u_1(t), \ldots, u_N(t), t_0) = s_i(x(t_f), t_f) \]

\[ + \int_{t_0}^{t_f} g_i(x(t), u_1(t), \ldots, u_N(t), t) dt. \]

\[ (5.4,9) \]

This is a very general way of describing the differential game situation. Because of this there is no guarantee that the solution exists or that the player inputs will be finite throughout the interval \([t_0, t_f]\). To assure this we must

* There is no reason why we cannot deal with DG played on partial differential equations. However one might anticipate the increase in complexity to be substantial.

† The arguments of \( J_i \) imply that it depends on the initial conditions \( x(t_0) \) and \( t_0 \), as well as the control inputs \( u_i(t), t \in [t_0, t_f] \).

place constraints on the system of equations (5.4,8) and particularly on the payoff functional (5.4,9). These may take the form of player and state inequality constraints, as well as restricting the \( J_i \) to forms which have particular properties (e.g. sign definiteness).

We observe that the parallels between the above definition and optimal control are quite evident and that if \( N = 1 \) we have an optimal control problem.

The nature of the DG will be determined largely by the number of players and the details of the payoff functionals \( J_i \). Some different types of DG and their respective solutions are now considered.

2. Conflict of Interest DG – This is the type of game of interest to this investigation. These are two player games where the payoff that one player attempts to minimize the other attempts to maximize. In this case it is known as a direct conflict of interest DG and the solution, if it exists, is known as a minimax solution or game theoretic saddle point. Equations (5.4,5), (5.4,6), and (5.4,7) form a direct conflict of interest differential game.

It is also possible to consider indirect conflict of interest DG, i.e. two player games where one player attempts to minimize a payoff functional while the other attempts to maximize a different functional which expresses objectives somewhat conflicting to the first player's interests. An interesting discussion of such indirect conflict situations
for linear systems with quadratic payoff functionals is given by Krikelis and Rekasius [5.20].

Direct and indirect conflict of interest DG are special cases of many player games. The fact that in the general definition of many player DG we have defined the \( J_i \) as payoff functionals to be *minimized* is not restrictive. In two player games, for example, we may wish to minimize \( J_1 \) and maximize \( -J_2 = J_2' \) where \( J_1 \) and \( J_2 \) are defined in the minimization context of the general definition. It follows that for direct conflict of interest in a two player game

\[
J_2' = J_1 \tag{5.4,10}
\]

must hold. It also follows that the game is zero-sum, i.e.

\[
J_1 + J_2 = 0 \tag{5.4,11a}
\]

for all admissible plays of the game. It is evident that many player DG will usually not be zero-sum, i.e.

\[
\sum_{i=1}^{N} J_i = 0 \tag{5.4,11b}
\]

for all admissible plays, and that because of the presence of more than two players a zero-sum many player DG will generally not result in great theoretical simplification.

Finally, we note that in optimal control it is possible to show that for a given performance criterion and initial conditions one may define equivalent open-loop and closed-loop strategies. This is not always possible in DG. Even in zero-sum, two player games where the open-loop and closed-loop solutions are equivalent (if they exist) it is possible to show that the time interval of existence need not be the same (Schmitendorf [5.21]).

3. Types of Solutions for General N-Player DG — In the general N-player situation it is not clear what properties the solution should have. There are a number of possibilities.

Let \( \mathbf{k} \) be a strategy vector (control law) for the N-player DG defined by (5.4,8) and (5.4,9) where

\[
\mathbf{k}^T = [k_1^T(x,t), \ldots, k_N^T(x,t)] \tag{5.4,12}
\]

We will denote the *Nash equilibrium* strategy vector by \( \mathbf{k}^* \) and we will define it as the strategy vector \( \mathbf{k}^* \) such that (Starr and Ho [5.8])

\[
J_i(k_1, \ldots, k_i-1, k_i^*, k_{i+1}, \ldots, k_N^*) \geq J_i(k_1^*, \ldots, k_N^*) \tag{5.4,13}
\]

for all admissible \( k_i \). It is secure against any attempt by one player to unilaterally alter his strategy, and thus provides the minimum \( J_i \) provided that all players use their Nash strategies. There may be more than one Nash solution for a given problem and thus we must state the equilibrium controls for all the players in order to define a particular equilibrium solution.
If it is not clear that all the players use their Nash controls then the i-th player may wish to minimize his payoff functional against the worst possible set of strategies that his opponents could select. We will denote this strategy by \( \hat{\xi}_i(g,t) \) and we will call it the minimax strategy for the i-th player. It is defined by (Starr and Ho [5.8])

\[
\max_{k_1, \ldots, k_i, \ldots, k_N} J_i(k_1, \ldots, k_i, \ldots, k_N) \\
\text{s.t.} \\
\max_{k_1, \ldots, k_i, \ldots, k_N} J_i(k_1, \ldots, k_i, \ldots, k_N)
\]

for all admissible \( k_i \). It is equivalent to finding the solution of a two player, zero-sum problem where the opponent of player \( i \) chooses all the controls except the i-th and tries to maximize \( J_i \) (or equivalently minimize \(-J_i\)). It follows that Nash and minimax solutions are identical in two player zero-sum DG. This is not true for \( N > 2 \) or for nonzero-sum games.

One may also consider the possibility of obtaining a further minimization from the Nash equilibrium by searching for a negotiated solution between the players, or blocks of players. Such a solution, if enforced, could produce an improvement in the payoff for at least some of the players but is susceptible to cheating. It is referred to as the noninferior strategy vector. The case of a single coalition among all the players is referred to as the pareto-optimal solution. Such solutions are of no use for the purposes of this investigation, although they are of some use in modeling negotiating situations.

As indicated earlier, the analytical procedures used in solving DG are similar to those of optimal control theory. They may be split into two categories, one a generalized dynamic programming method (equivalently a generalized H-J-B equation (F.1,15)) and the other a collection of methods employing variational calculus. These techniques are illustrated by two important theorems.

Consider the minimax problem defined by (5.4,5), (5.4,6), and (5.4,7). The following results hold for this problem.

**Theorem 5.3**

Given (1) \( y(t), \eta(t) \) are smooth, i.e. admissible \( y(t), \eta(t) \) are assumed to be continuous and to have continuous first derivatives, (2) \( g(x, y, \eta, t) \) and \( s(x(t_f), t_f) \) have continuous first and second partial derivatives with respect to all their arguments, (3) admissible \( y(t), \eta(t) \) and \( x(t) \) are not bounded, (4) initial conditions \( x_0 \) and initial time \( t_0 \) are specified, (5) the terminal time \( t_f \) is fixed, and (6) there are no terminal conditions on the \( x(t_f) \), then necessary first order conditions that the control vectors \( y(t), \eta(t) \) produce a minimax value of the payoff functional (5.4,7) subject to the differential equation constraints (5.4,6) are, along the minimax trajectory, as follows:
\[ \lambda = \frac{\partial H}{\partial \dot{y}} \]  
(5.4,15a)

\[ H_y = 0 \]  
(5.4,15b)

\[ H_{\dot{y}} = 0 \]  
(5.4,15c)

with terminal boundary condition

\[ \lambda(t_f) = \frac{\partial S}{\partial \dot{y}}(y(t_f), t_f). \]  
(5.4,16)

\( H \) is the Hamiltonian for this problem and is given by

\[ H(x, y, \eta, \lambda, t) = g(x, y, \eta, t) + \lambda^T f(x, y, \eta, t). \]  
(5.4,17)

**Proof**

One may compare this theorem with Theorem F.5 of Appendix F, Section F.1. By restricting the assumptions of Theorem F.5 to match those of this theorem we see that the first order necessary conditions for the optimal control problem and the minimax problem are completely analogous. This result is not surprising in that first order necessary conditions are conditions for a stationary point, rather than for a maximum, minimum or saddle point. Thus the proof of this result parallels the proof of the analogous optimal control problem (e.g. Bryson and Ho [4.16], Section 2.3).

We treat the differential equation constraints (5.4,6) in the usual way, i.e. by defining an augmented payoff functional

\[ J_a = S(x(t_f), t_f) + \int_{t_0}^{t_f} \left[ g(x(t), y(t), \eta(t), t) + \lambda^T(t) \{ f(x(t), y(t), \eta(t), t) - \dot{x}(t) \} \right] dt \]  
(5.4,18)

where we have adjoined the equations (5.4,6) with \( J \) by multiplier functions (Lagrange multipliers) \( \lambda(t) \). Using the definition of \( H \) in (5.4,17) and integrating the last term on the right hand side in (5.4,18) by parts we obtain

\[ J_a = S(x(t_f), t_f) - \lambda^T(t_f) x(t_f) + \lambda^T(t_0) x(t_0) + \int_{t_0}^{t_f} \lambda^T(t) \{ f(x(t), y(t), \eta(t), t) - \dot{x}(t) \} dt. \]  
(5.4,19)

Now, taking the first variation* of \( J_a \) with respect to variations in \( y, \eta, \) and using the fact that \( t_0, t_f \) are fixed, we obtain

* For more details of this procedure, see Gelfand and Fomin [4.43].
\[ \delta J_a = \left[ \frac{\partial H}{\partial x} \right]_{x=x(t_f)} \delta x_{t=t_f} + \left[ \frac{\partial H}{\partial \eta} \right]_{x=x(t_0)} \delta \eta_{t=t_0} + \int_{t_0}^{t_f} \left[ \frac{\partial H}{\partial x} \delta x + \frac{\partial H}{\partial \eta} \delta \eta \right] dt. \tag{5.4,20} \]

The variation \( \delta x \) could be determined in terms of \( \delta y \) and \( \delta \eta \), but such an operation would be difficult. It is more convenient to choose the \( \lambda(t) \) so that the coefficients of \( \delta x \) vanish, i.e., choose \( \lambda(t) \) such that

\[ \lambda(t) = -\frac{\partial H}{\partial x} \tag{5.4,21a} \]

and

\[ \lambda(t_f) = \frac{\partial \eta}{\partial x}(t_f). \tag{5.4,21b} \]

Thus (5.4,20) becomes

\[ \delta J_a = \lambda(t_0) \delta x_{t=t_0} + \int_{t_0}^{t_f} \left[ \frac{\partial H}{\partial x} \delta x + \frac{\partial H}{\partial \eta} \delta \eta \right] dt. \tag{5.4,22} \]

But from assumption (4) \( x(t_0) \) is fixed and thus

\[ \delta x(t_0) = 0. \tag{5.4,23} \]

Furthermore, for a stationary value of \( J_a \) we must have

\[ \delta J_a = 0 \tag{5.4,24} \]

for arbitrary variations \( \delta y(t) \), \( \delta \eta(t) \). This can only happen if

\[ \frac{\partial H}{\partial y} = 0 \quad t \in [t_0, t_f] \tag{5.4,25a} \]

and

\[ \frac{\partial H}{\partial \eta} = 0 \quad t \in [t_0, t_f]. \tag{5.4,25b} \]

Equation (5.4,21a), (5.4,25a) and (5.4,25b) with boundary condition (5.4,21b) are the required results. This completes the proof.

Remark 1 – The equations (5.4,15) and (5.4,16) are the Euler-Lagrange equations for this problem. If we remove the equations and arguments associated with the maximizing input \( \eta \), they are the same as the Euler-Lagrange equations for the optimal control problem (cf. Theorem F.5).

Remark 2 (see also Bryson and Ho [4.16]) – The interpretation of the equation (5.4,22) is as follows. \( \lambda(t_0) \) is the gradient of \( J \) with respect to variations in the initial conditions \( x(t_0) \) while holding \( y(t) \) and \( \eta(t) \) constant and satisfying the system differential equations (5.4,6). Since \( t_0 \) is arbitrary, \( \lambda(t) \) are the influence functions on \( J \) of variations in \( x(t) \).

The functions \( \partial H/\partial y \), \( \partial H/\partial \eta \) may be interpreted as impulsive response functions of \( J \) since each component of \( \partial H/\partial y \), \( \partial H/\partial \eta \) represents the variation in \( J \) due to a unit impulse in the corresponding component of \( \delta y \), \( \delta \eta \) at time \( t \) while holding \( x(t_0) \) constant and satisfying the system differential equations (5.4,6).
Remark 3 - The conditions in Theorem 5.3 are only necessary conditions for a stationary point. Thus results that are obtained by their application are not always minimax, i.e. they do not always satisfy the saddle point property (5.4,5). Bryson and Ho [4.16] point out that such GTSP do not exist unless we assume $J$ (and thus $H$) are separable in $u$ and $v$ (a heuristic argument for this can be made via the continuous game examples discussed earlier). This situation is not an argument against the use of variational methods for such problems. One may look for solutions such that the maximizing player announces his strategy, with the task of the minimizing player being to find in optimal strategy given that information. Alternatively, the candidate minimax solutions may have to be tested with two one-sided problems to determine if they do satisfy (5.4,5). In both cases the variational methods may prove to be quite useful.

Remark 4 - In optimal control problems the interpretation of the optimal results as open-loop time histories or as closed-loop strategies makes no difference to the resulting control (assuming that initial conditions for both cases are the same). This is not the case for many DG problems (see the discussion in Bryson and Ho [4.16], Chapter 9). The reason for this is that if one player plays a nonminimax control strategy or control history, his opponent can only take advantage of this situation if he plays an optimal closed-loop strategy, i.e. if his control inputs change in such a way that they take advantage of any nonoptimal play that may occur.

Following Bryson and Ho, we may then say that a general procedure for solving the minimax problem defined by (5.4,5), (5.4,6) and (5.4,7) is to (1) solve for $u^*$ and $v^*$ by applying Theorem 5.3 and (2) verify the saddle point condition (5.4,5) by solving two one-sided problems and checking to see that the two one-sided problem solutions are identical to $u^*$, $v^*$ obtained by the application of Theorem 5.3. Necessary second order conditions for this verification step follow from the usual optimal control variational calculus analysis, and include [4.16]

\[ h_{uu}^* \geq 0, \ h_{vv}^* \leq 0, \]  
(5.4,26)

no conjugate point\(^\dagger\) for the problem

\[ J(u^*,v^*) = \min_u J(u,v^*) \]  
(5.4,27)

where

\[ v^* = \begin{cases} \eta^*(t; x_0, t_0) \\ k_\eta(x,t) \end{cases} \]  
(5.4,28)

and no conjugate point for the problem

\[ J(u^*,v) = \max_v J(u^*,v) \]  
(5.4,29)

where

\(^\dagger\)Conjugate points are discussed in more detail in Bryson and Ho [4.16] and in Remark 1 following Theorem 5.5 of this chapter.
We now consider a generalized H-J-B equation approach to this problem. The key result is summarized in the following theorem.

**Theorem 5.4**

Consider the minimax problem defined by (5.4,5), (5.4,6) and (5.4,7). If we assume (1) \( t_f \) is fixed, (2) \( \bar{x}(t_f) \) is unconstrained and (3) the value of \( J \) along a minimax trajectory, denoted \( J^*(\bar{x}(t_f), t_f) \), exists, is continuous, and possesses continuous first and second partial derivatives at all points of interest in \((x,t)\) space, then \( J^* \) satisfies the generalized Hamilton-Jacobi-Bellman equation

\[
\frac{\partial J^*}{\partial t} + H^*(\bar{x}, J^*, t) = 0 \quad (5.4,31a)
\]

\[
J^*(\bar{x}(t_f), t_f) = s(\bar{x}(t_f), t_f) \quad (5.4,31b)
\]

where

\[
H^*(\bar{x}, J^*, t) = \max_{\bar{u}} \min_{\bar{y}} H(\bar{x}, \bar{y}, \bar{u}, \bar{z}, J^*, t) \quad (5.4,32)
\]

See (5.4,33) to follow.

**Proof**

For abstract, rigorous proofs the reader is referred to Case [5.1] and Friedman [5.19]. A heuristic proof is given by Isaacs [5.9] for a simpler version of the equation. One may also generalize the one-sided optimization proof given by Bryson and Ho (Ref. 4.16, Section 4.2) or Kirk (Ref. 4.17, Section 3.11). The latter procedure is adopted here.

The minimax return function is

\[
J^*(\bar{x}(t), t) = s(\bar{x}(t_f), t_f) + \max_{\bar{y}(\tau), \bar{u}(\tau), \bar{z}(\tau), \bar{r}(\tau), \tau} \min_{\bar{y}(\tau), \bar{u}(\tau), \bar{z}(\tau), \bar{r}(\tau), \tau} \int_t^{t_f} g(\bar{x}(\tau), \bar{y}(\tau), \bar{z}(\tau), \bar{r}(\tau), \tau) d\tau \quad (5.4,33)
\]

The boundary condition (5.4,31b) follows immediately from this definition.

Suppose that \( \bar{y}' \) and \( \bar{z}' \) are candidate minimax solutions. Now consider two separate optimization problems. In the first the objective is to find a \( \bar{y}(\tau), t_0 \leq \tau \leq t_f \) minimizing \( J \) given a priori knowledge of \( \bar{y}' \). In the second the objective is to find an \( \bar{y}(\tau), t_0 \leq \tau \leq t_f \) maximizing \( J \) given a priori knowledge of \( \bar{y}' \). The optimal return functions for these problems are, respectively,

\[
J^*_1(\bar{x}, t) = s(\bar{x}(t_f), t_f) + \min_{\bar{y}(\tau), t \leq \tau \leq t_f} \int_t^{t_f} g(\bar{x}(\tau), \bar{y}(\tau), \bar{z}(\tau), \bar{r}(\tau), \tau) d\tau \quad (5.4,34a)
\]

and

\[
J^*_2(\bar{x}, t) = s(\bar{x}(t_f), t_f) + \max_{\bar{y}(\tau), t \leq \tau \leq t_f} \int_t^{t_f} g(\bar{x}(\tau), \bar{y}(\tau), \bar{z}(\tau), \bar{r}(\tau), \tau) d\tau \quad (5.4,34b)
\]
We note that if $u'$ is the minimizing solution for (5.4.34a) and $u'$ is the maximizing solution for (5.4.34b), then they are the minimax solutions and

$$J^*_1 = J^*_2 = J^*. \quad (5.4.35)$$

For the moment only (5.4.34a) will be considered. The development for (5.4.34b) follows in parallel.

By subdividing the interval in (5.4.34a), and using the principle of optimality (see Appendix F, Section F.1), one obtains

$$J^*_1(\xi(t), t) = \min_{\xi(t)} \left\{ \int_t^{t+\Delta t} g(\xi(\tau), u(\tau), u'(\tau), \tau) d\tau + J^*_1(\xi(t + \Delta t), t + \Delta t) \right\}. \quad (5.4.36)$$

$J^*(\xi(t + \Delta t), t + \Delta t)$ is the value of the return function for $t + \Delta t \leq \tau \leq t_f$ with initial state $\xi(t + \Delta t)$.

With the continuity assumptions that are made in the statement of the theorem, $J^*(\xi(t + \Delta t), t + \Delta t)$ may be represented by a Taylor series about $(\xi(t), t)$. Thus (5.4.36) becomes

$$J^*_1(\xi(t), t) = \min_{\xi(t)} \left\{ \int_t^{t+\Delta t} g(\xi(\tau), u(\tau), u'(\tau), \tau) d\tau + J^*_1(\xi(t), t) + \frac{\partial J^*}{\partial \xi}(\xi(t), t) \Delta t + \left[ \frac{\partial J^*}{\partial u}(\xi(t), t) \right]^T f(\xi(t), u(t), u'(t), t) \right\} \quad (5.4.37)$$

or, for small $\Delta t$,

$$J^*_1(\xi(t), t) = \min_{\xi(t)} \left\{ g(\xi(t), u(t), u'(t), t) \Delta t + J^*_1(\xi(t), t) + \frac{\partial J^*}{\partial \xi}(\xi(t), t) \Delta t + \left[ \frac{\partial J^*}{\partial u}(\xi(t), t) \right]^T f(\xi(t), u(t), u'(t), t) \Delta t + R \right\} \quad (5.4.38)$$

where $R$ denotes a remainder term of order $\Delta t^2$ or greater. $J^*_1(\xi(t), t)$ and $\frac{\partial J^*}{\partial u}(\xi(t), t)$ terms have no $u(t)$ dependence, and thus may be taken out of the minimization in (5.4.38). Thus

$$0 = \frac{\partial J^*}{\partial \xi}(\xi(t), t) \Delta t + \min_{\xi(t)} \left\{ g(\xi(t), u(t), u'(t), t) \Delta t + \left[ \frac{\partial J^*}{\partial u}(\xi(t), t) \right]^T f(\xi(t), u(t), u'(t), t) \Delta t + R \right\} \quad (5.4.39)$$

Dividing by $\Delta t$, and taking the limit as $\Delta t \to 0$, the H-J-B equation for $J^*_1$ is obtained, i.e.

$$\frac{\partial J^*}{\partial \xi} + \min_{\xi(t)} \left\{ g(\xi, u, u', t) + \left[ \frac{\partial J^*}{\partial u} \right]^T f(\xi, u, u', t) \right\} = 0. \quad (5.4.40a)$$

Similarly, the H-J-B equation for $J^*_2$ may be shown to be
Denote the minimizing \( y \) in (5.4,40a) by \( y^* \) and the maximizing \( \eta \) in (5.4,40b) by \( \eta^* \). If
\[
y^* = y'
\]
and
\[
\eta^* = \eta'
\]
then the verification step discussed in Remark 4 following Theorem 5.3 has been satisfied and \( y', \eta' \) (equivalently, \( y^*, \eta^* \)) are the minimax solutions. Thus (5.4,35) holds and \( J_1, J_2 \) in (5.4,40a) and (5.4,40b) may be replaced by \( J^* \).

Furthermore, the separate minimization and maximization steps may now be combined into one minimax term. The H-J-B equation for the minimax problem may thus be written
\[
\frac{\partial J^*}{\partial t} + \max_{u, \eta} \left[ g(x, y, \eta, t) + \left( \frac{\partial J^*}{\partial x} \right)^T f(x, y, \eta, t) \right] = 0
\]
which is the desired result.

From equation (5.4,22) and Remark 2 following Theorem 5.3 we have, along the minimax trajectory, i.e. \( f - x = 0 \), \( H_y = 0 \), \( \eta^* = 0 \),
\[
\delta J^*(x, t) = \lambda^T(t) \delta x(t).
\]
It follows that the \( \lambda(t) \) may be identified with \( \frac{\partial J^*}{\partial x} \), i.e.
\[
\lambda(t) = \frac{\partial J^*}{\partial x}.
\]

Using (5.4,44) and the definition of the Hamiltonian (5.4,17), (5.4,42) may be written
\[
\frac{\partial J^*}{\partial t} + H^* \left( x, \frac{\partial J^*}{\partial x}, t \right) = 0
\]
where
\[
H^* \left( x, \frac{\partial J^*}{\partial x}, t \right) = \min_{u, \eta} \max_{x, y} \left[ g(x, y, \eta, t) + \left( \frac{\partial J^*}{\partial x} \right)^T f(x, y, \eta, t) \right].
\]

The results (5.4,45) and (5.4,46) with the boundary condition implicit in (5.4,33) are the desired results. This completes the proof of Theorem 5.4.

Remark 1

The condition (5.4,46) is actually a first-order necessary condition for \( y, \eta \) that are minimax, and may be added to the variational first-order conditions of Theorem 5.3. It states that for \( y, \eta \) to be minimax solutions for the differential game defined by (5.4,5), (5.4,6) and (5.4,7), there must be a GTSP for the game (not differential game) defined by the minimaximization of the Hamiltonian. A differential game is said to be normal [5.8] if the saddle points for (5.4,46) exist for all \( t \in [t_0, t_f] \) and if the resulting control strategies \( k^*_u(x, t), k^*_\eta(x, t) \) lead to admissible state trajectories.

Remark 2

If we remove the maximization arguments for \( \eta \) in the above proof, we have a proof for the H-J-B equation of optimal control (cf. (F.1,15)).
Remark 3

The partial differential equations for the return functions for Nash solutions to many player DG are obtained by extending the two player result and are given by [5.8]

\[
\frac{\partial J_i^*}{\partial t} + \min_{u_i} H_i \left\{ \mathbf{x}, t, k_i^* \ldots, k_{i-1}^*, u_i, k_{i+1}^* \ldots, k_N^*, \frac{\partial J_i^*}{\partial g} \right\} = 0
\]

(5.4,47)

\[
J_i^*(\mathbf{x}(t_f), t_f) = s_i(\mathbf{x}(t_f), t_f)
\]

(5.4,48)

where \( H_i \) is the Hamiltonian for the i-th player and is given by

\[
H_i(\mathbf{x}, k, \lambda_i, t) = g_i(\mathbf{x}, k, t) + \lambda_i^T \mathbf{f}(\mathbf{x}, k, t)
\]

(5.4,49)

the \( J_i \) are given by (5.4,9), and the system differential equations are given by (5.4,8).

Interesting examples of the application of these methods, especially to pursuit-evasion problems, are given by Bryson and Ho [4.16], among others. These pursuit-evasion examples are frequent in the literature largely because of Isaacs' emphasis on differential games of warfare and pursuit [5.9], but the applications of this theory are hardly restricted to this class of DG, as he is quick to point out.

The fact that these applications provide a richness of behaviour of the solutions in comparison with one-sided optimization problems is intrinsic to the nature of the problem (Isaacs [5.18]): "Differential games are the same step from finite matrix games as classical applied mathematics is from finite maximizing". They are a combination of game theory, the calculus of variations, and control theory. This is what is responsible for this richness.

We now turn to a particular class of differential games which are especially useful for practical computation of conflict of interest aircraft and wind controller models. This is the class of linear quadratic two-player zero-sum (LQTPZS) differential games.

5.4.2 Linear Quadratic Differential Games

As might be anticipated from experience with optimal controllers, the class of DG most amenable to analysis is that of DG played on linear dynamic systems with quadratic payoff functionals. In this section the most relevant (to this investigation) of these results are presented. They are based on two player versions of the N-player results available in Starr and Ho [5.8], Case [5.10], Lukes and Russell [5.12]*, and Friedman [5.19], with some unification as required to make them applicable to slightly more general two-player linear

* Ref. 5.12 considers a N-player Nash equilibrium where the players other than the i-th player are using open-loop strategies.
systems than those treated in each reference alone.

Before proceeding further we remark that although consideration of two player DG which are not zero-sum may be necessary for certain aircraft/wind controller conflict of interest models, the solutions in which we are interested, from the worst-case perspective, are the minimax solutions (as opposed to the Nash solutions). This follows naturally from the nature of these solutions*, and in particular from the fact that the Nash solution (in a game which is not zero-sum) does not protect the aircraft controller from a wind controller that does not follow his Nash strategy. Thus the only results that are required are those for zero-sum two player DG.

To this end consider the possibly time-varying system

\[ \dot{x} = Fx + G_1u + G_2v + f \]  
\[ x(t_0) = x_0 \quad t \in [t_0, t_f] \]  
\[ y = Mx. \] 

(5.4,50a)  
(5.4,50b)  
(5.4,51)

The payoff functional to be minimized by \( u \) and maximized by \( v \) is given by

\[ J = \|x(t_f) - x_d(t_f)\|_S^2 + \int_{t_0}^{t_f} \left( \|x(t) - x_d(t)\|^2_\gamma(t) 
+ \|y\|_R_1^2 + \|v\|_R_2^2 \right) dt \]  
(5.4,52a)

\[ \begin{cases} 
R_1 > 0 \\ R_2 < 0 
\end{cases} \]  
(5.4,52b)

and \( S, \gamma \) are symmetric, possibly sign indefinite weighting matrices. The vector \( x_d \) is the desired state trajectory. It is related to the desired output \( y_d \) by (cf. (F.1,81)) and Theorem F.11

\[ x_d = M^T [M M^T]^{-1} y_d. \]  
(5.4,53)

We are now ready to present the most important result of this section.

**Theorem 5.5**

For the two-player zero-sum linear quadratic differential game defined by (5.4,50) and (5.4,52), the minimax pure strategy is given by

\[ u^* = -R_1^{-1}(t)G_1^T(t) [P(t)R_1 + b] \]  
(5.4,54a)

\[ v^* = -R_2^{-1}(t)G_2^T(t) [P(t)R_2 + b] \]  
(5.4,54b)

provided that the solutions for the differential equations for

---

* See the definitions in the previous section.
\( P \) and \( b \) exist on \([t_0, t_f]\). These differential equations are given by

\[
\dot{P} = -PP - P'b + P \left[ G_1 R_1^{-1} G_1^T + G_2 R_2^{-1} G_2^T \right] P - Q \tag{5.4, 55a}
\]

\[
\dot{b} = -b' + P \left[ G_1 R_1^{-1} G_1^T + G_2 R_2^{-1} G_2^T \right] b + Qx_d - Pf \tag{5.4, 55b}
\]

with terminal conditions

\[
P(t_f) = \xi \tag{5.4, 56a}
\]

\[
b(t_f) = -\xi_d(t_f). \tag{5.4, 56b}
\]

The minimax value of the payoff functional is given by

\[
J^*(x(t_0), t_0) = x^T(t_0)P(t_0)x(t_0) + 2x^T(t_0)b(t_0) + c(t_0) \tag{5.4, 57}
\]

where

\[
\dot{c} = b^T \left[ G_1 R_1^{-1} G_1^T + G_2 R_2^{-1} G_2^T \right] b - x_d^T x_d - 2b^T \tag{5.4, 58}
\]

with terminal condition

\[
c(t_f) = x_d(t_f)x_d(t_f). \tag{5.4, 59}
\]

**Proof**

The proof for this theorem is similar to that for analogous optimal control problems, and may be obtained using either the variational methods of Theorem 5.3, or the generalized H-J-B equation approach (the value function approach) of Theorem 5.4. The details are left to Appendix G, Section G.2.

**Remark 1**

The existence of a minimax solution is not assured even if the system and weighting matrices are well-behaved (e.g. even if they are all continuous). This is a consequence of the nonlinearity of the Riccati equation and the associated phenomenon of *finite escape time*, i.e. under some conditions the solution for \( P \) will become unbounded in a finite time. From the minimaximization point of view of the theorem the existence of a finite escape time in the interval \([t_0, t_f]\) implies the existence of a *conjugate point* at this time.

The concept of a conjugate point arises in variational calculus and optimal control in the consideration of sufficient conditions for a weak extremum (see Gelfand and Fomin [4.43] and Bryson and Ho [4.16], as well as Theorem F.7). Sarma and Ragade [5.23] define a conjugate point as follows: "If along an optimal trajectory there are two points \( A \) and \( A' \) from which the value of the game computed is identical, then such points are known as conjugate points". From the LQTPZS DG point of view the existence of a conjugate point in \([t_0, t_f]\) (say at \( t_c \)) invalidates the solution for \( t < t_c \), i.e. infinitely negative or positive costs may result (Krikelis and Rekasius [5.20]).

The question of determining the interval \([t_c, t_f]\), in which a solution to the Riccati equation exists is a difficult
one. Sufficient conditions are considered in Krikelis and Rekasius [5.20], Schmitendorf [5.21], Varaiya and Lin [5.24] and Bernhard and Bellec [5.25], among others. Of some guidance in certain problems is the Riccati equation property* that if (Jacobson [5.26], and Bucy and Joseph [5.27])

\[ \mathbf{Q} \geq 0, \quad \mathbf{Q}(t) \geq 0 \] (5.4,60a)

\[ \mathbf{E} \triangleq \mathbf{G}_1 \mathbf{R}_1^{-1} \mathbf{G}_1^T + \mathbf{G}_2 \mathbf{R}_2^{-1} \mathbf{G}_2^T \geq 0 \] (5.4,60b)

then a solution exists for all \( t_c \leq t_f \). Such conditions can usually be achieved by appropriately choosing the weighting matrices.

Because these conditions are sufficient conditions, the resulting restrictions on the weighting matrices will usually be stronger than necessary. Thus in practice it is preferable to check for the boundedness of \( \mathbf{P} \) as the computation is being carried out. These checks depend on the numerical procedures that are employed and will be discussed further in Appendix H.

Remark 2

The fact that the solutions (5.4,54) are strategies (i.e. are in feedback form) implies a higher level of perversity for the wind controller (as well as feedback control for the aircraft controller). Thus if the aircraft controller deviates from the minimax strategy (5.4,54a), the disturbing wind controller will change \( \mathbf{u}(t) \) through (5.4,54b) in such a way that the value of \( J \) that results will be greater than \( J^* \), i.e. the wind controller takes advantage of any nonoptimal play by the aircraft controller. Such would not be the case if the wind controller were restricted to a minimax open-loop wind history.

Remark 3

The fully linearized aircraft equations of motion can generally be written

\[ \dot{x} = \mathbf{F}x + \mathbf{G}_1 \mathbf{y} + \mathbf{G}_2 \mathbf{u} + \mathbf{G}_3 \dot{u} + f(t) \] (5.4,61)

\[ x(t_0) = x_0 \]

where \( \mathbf{y} \) corresponds to the control vector and \( \mathbf{u} \) and \( \dot{u} \) are wind and wind rate of change inputs at the aircraft centre-of-mass. The presence of \( \dot{u} \) in (5.4,61) makes these differential equations incompatible with (5.4,50a). This is not a serious obstacle, however, since we may write (5.4,61) as

* This property accounts for the existence of a solution for all \( t_0 \) in the linear quadratic one-sided optimal control problem.
This augmented system is in the form (5.4,50a) and the results of Theorem 5.5 may now be applied.

Remark 4

The sign-definiteness properties of $R_1$ and $R_2$ (see (5.4,52b) and (5.4,52c)) cast the two-player problem as a conflict of interest problem, and act as an indirect constraint on the control and disturbance inputs. With regards to the aircraft control inputs it may be sufficient to adjust $R_1$, so that it is improbable that control limits will be exceeded. Alternatively one may use limiters, thus accepting a suboptimal application of the aircraft controller's minimax control law.

With regards to the disturbing wind controller, one may also consider indirect constraints as implied by $R_2$ or wind limiters in the form of wind and wind shear magnitude constraints. The latter are undesirable, however, for two reasons (see also the discussion in Section 5):

1. The resulting wind histories are physically unrealistic because of the discontinuities created by the limiters.
2. The use of limiters destroys the optimality of the wind controller and thus, in a sense, reduces the perversity of the wind. This is not compatible with the objective of finding maximizing "worst-case" wind controllers.

More suitable hard constraints are integral (isoperimetric) wind constraints of the form used in Section 5 (see equation (5.1)). From the form of the aircraft equations of motion (5.4,61) it follows that the two isoperimetric constraints appropriate to the problem are

$$\int_{t_0}^{t_f} \sum_{i=1}^{n} \left( u_i^2(t) - \frac{E}{R_2} \right) dt = E - \frac{S}{R_2}, \quad E > 0, \quad S > 0 \quad (5.4,64)$$

$$\int_{t_0}^{t_f} \sum_{i=1}^{n} \left( \frac{\dot{u}_i(t)^2}{R_2} - \frac{S_2}{R_2} \right) dt = S - \frac{E}{R_2}, \quad S > 0, \quad E > 0 \quad (5.4,65)$$

if we divide $E$ and $S$ by $t_f - t_0$, then the resulting quantities are, respectively, the time averaged mean square input wind velocity and acceleration (shear). It is readily apparent how such constraints are more natural than inequality constraints.
These constraints are incorporated in the conflict of interest framework in the following theorem.

Theorem 5.6

Given the system dynamics (5.4,61), the payoff functional

\[ J = \left\| x(t_f) - x_d(t_f) \right\|^2 + \int_{t_0}^{t_f} \left[ \left\| x - x_d \right\|^2 + \left\| y \right\|_{R_2}^2 \right] dt \]  

(5.4,66)

\[ S, \quad Q \geq 0, \quad R_1 > 0 \]  

(5.4,67)

and the isoperimetric constraints (5.4,64) and (5.4,65), then the minimax feedback strategies are given by

\[ u^* = -R_1^{-1} G_1 [ \hat{x} + \hat{b} ] \]  

(5.4,68a)

\[ \hat{b}^* = -\frac{1}{\mu_2} R_2^{-1} G_3 [ \hat{x} + \hat{b} ] \]  

(5.4,68b)

provided that the solutions for the differential equations for \( \hat{p} \) and \( \hat{b} \) exist on \([t_0, t_f]\). These differential equations are given by

\[ \frac{d\hat{p}}{dt} = -\hat{p}^T \hat{p} - \hat{p}^T \hat{P} + \hat{p} \left[ G_1 R_1^{-1} G_1^T + \frac{1}{\mu_2} G_3 R_2^{-1} G_3^T \right] \hat{P} - \hat{Q} \]  

(5.4,69a)

\[ \frac{d\hat{b}}{dt} = -\hat{b}^T \hat{P} + \hat{b} \left[ G_1 R_1^{-1} G_1^T + \frac{1}{\mu_2} G_3 R_2^{-1} G_3^T \right] \hat{P} + G \hat{x}_d - \hat{P} \hat{f} \]  

(5.4,69b)

with terminal conditions

\[ \hat{p}(t_f) = \hat{S} \]  

(5.4,70a)

\[ \hat{b}(t_f) = -\hat{S} \hat{x}_d(t_f) \]  

(5.4,70b)

The minimax value of the payoff functional is given by

\[ J^* = \hat{x}_d^T(t_0) \hat{P}(t_0) \hat{x}_d(t_0) + 2 \hat{S}^T(t_0) \hat{b}(t_0) + \hat{c}(t_0) + u_1 E + u_2 S \]  

(5.4,71)

where

\[ \hat{c} = E^T \left[ G_1 R_1^{-1} G_1 + \frac{1}{\mu_2} G_3 R_2^{-1} G_3 \right] \hat{b} - \hat{x}_d \hat{x}_d - \hat{x}_d^T \]  

(5.4,72)

with terminal condition

\[ \hat{c}(t_f) = \hat{x}_d(t_f) \hat{x}_d(t_f) \]  

(5.4,73)

Here \( \hat{x}, \hat{p}, \hat{c}, \hat{g}_3, \hat{g}_1, \hat{g}_3 \) and \( \hat{f} \) are given by (5.4,63),

\[ \hat{S} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \]  

(5.4,74)

\[ \hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & \mu_1 B_2 \end{bmatrix} \]  

(5.4,75)

and

\[ \hat{x}_d^T = \begin{bmatrix} T & \hat{x}_d \end{bmatrix} \]  

(5.4,76)

\[ B_2 = -B_2 \]  

(5.4,77)
\( \begin{align*} 
E_{21} &= -E'_{21}. 
\end{align*} \)

(5.4,78)

\( \mu_2 \) is a constant, positive parameter and \( \mu_1 \) is a constant, sign-indefinite parameter. These are adjusted so that the isoperimetric constraints are satisfied.

Proof

Bernhard and Bellee [5.25] consider simpler versions of this theorem, but do not give their proofs.

The isoperimetric constraints (5.4,64) and (5.4,65) are treated in the usual way for such problems, i.e. by considering an augmented, unconstrained problem (cf. with proof for Theorem 5.1). Since there are two isoperimetric constraints, two augmenting equations are required to replace the isoperimetric constraints, namely

\[
\begin{align*}
\gamma_1(t) &= \mu^T(t) E_2(t) \eta(t) \\
\gamma_1(t_0) &= -E, \quad \gamma_1(t_f) = 0
\end{align*}
\]

(5.4,79a)

and

\[
\begin{align*}
\gamma_2(t) &= \mu^T(t) E_{21}(t) \eta(t) \\
\gamma_2(t_0) &= -S, \quad \gamma_2(t_f) = 0.
\end{align*}
\]

(5.4,80a)

The augmented Hamiltonian becomes

\[
H_a = \|x - x_d\|^2 + u^2 + \mu_1^T \left[ P x + Q_1 u + Q_2 u + Q_3 \eta \right] + \mu'_1 \eta^T B'_2 \eta + \mu'_2 \eta^T B'_{21} \eta.
\]

(5.4,81)

where \( \mu'_1 \) and \( \mu'_2 \) are Lagrange multipliers associated with the constraining equations (5.4,79) and (5.4,80). The augmented unconstrained payoff functional is

\[
J_a = \|x(t_f) - x_d(t_f)\|^2 + \int_{t_0}^{t_f} \left[ \|x - x_d\|^2 + u^2 + \mu_1 \eta^2 + \mu_2 \eta^2 \right] dt
\]

(5.4,82)

where \( E_2 \) and \( E_{21} \) are given by (5.4,77) and (5.4,78) and

\[
\begin{align*}
\mu_1 &= -\mu'_1 \\
\mu_2 &= -\mu'_2.
\end{align*}
\]

(5.4,83a)

\( \mu_1 \) and \( \mu_2 \) must be selected so that (5.4,64) and (5.4,65) are satisfied.

Now, transforming the system (5.4,61) into (5.4,62) and \( J_a \) into

\[
J_a = \|x(t_f) - x_d(t_f)\|^2 + \int_{t_0}^{t_f} \left[ \|x - x_d\|^2 + u^2 + \mu_1 \eta^2 + \mu_2 \eta^2 \right] dt
\]

(5.4,84)

where the 't' matrices and vectors are defined in the statement of this theorem, we obtain a minimax problem which may be solved by direct application of Theorem 5.5.

The time-invariance of \( \mu_1 \) and \( \mu_2 \) follows from the augmented Hamiltonian's independence from \( \gamma_1 \) and \( \gamma_2 \) and from the form of the Euler-Lagrange equations in Theorem 5.3. The condition

\[
\mu_2 > 0
\]

(5.4,85)
follows from the convexity condition

\[ H_{a} < 0 \]  

(5.4,86)

for a game minimax solution for \( H_{a} \) (see the GTSP conditions discussed in Section 5.4.1). No such inequality constraint on \( \mu_{1} \) arises in this development, but from the maximizing nature of the wind inputs, positive \( \mu_{1} \) are more appropriate. Otherwise the wind controller may maximize the payoff functional by making the \( \mu_{1}^{T} E_{2} \) term in \( J_{a} \) large, possibly without any significant aircraft state variable stimulation, which is the desired objective.

The equilibrium \( J_{a}^{*} \) value follows from (5.4,57). Removing the contributions to \( J_{a}^{*} \) due to the isoperimetric constraints, the result (5.4,71) for \( J^{*} \) is obtained. This completes the proof of Theorem 5.6.

Remark 1

As far as is known there is no way of analytically relating \( \mu_{1} \) and \( \mu_{2} \) with \( E \) and \( S \), as was the case for \( \mu \) with \( E \) in van der Vaart's theorems. This implies that an iterative procedure must be employed in determining appropriate values for \( \mu_{1} \) and \( \mu_{2} \), i.e. the augmented unconstrained problem must be solved several times with different values for \( \mu_{1} \) and \( \mu_{2} \) until the actual \( E \) and \( S \) (call them \( E_{a} \) and \( S_{a} \)) are close enough in some sense, to the desired values.

This may be done by solving for the minimax strategies of the augmented unconstrained problem via Theorem 5.6 and then evaluating the constraints (5.4,64) and (5.4,65) by direct system simulation. On the next iteration \( \mu_{1} \) and \( \mu_{2} \) may be adjusted to bring the \( E_{a} \) and \( S_{a} \) closer to the desired values, noting that from the nature of the problem smaller \( \mu_{1} \) and \( \mu_{2} \) will lead to larger \( E_{a} \) and \( S_{a} \), respectively, and vice versa.

Alternatively, if only \( \mu_{1} \) is being constrained (i.e. only the isoperimetric constraint (5.4,65) is in force), then the system simulation step may be avoided by solving a separate matrix differential equation for \( S_{a} \). Such a technique is considered by Bernhard and Bellec [5.25] for a somewhat simpler problem. The more general result is presented in the following theorem.

Theorem 5.7

Given the unconstrained augmented minimax problem (5.4,62) and (5.4,84), then the value of \( \gamma_{2} (t_{0}) \) that results when playing the pure strategies (5.4,68a) and (5.4,68b) is given by

\[ \gamma_{2} (t_{0}) \equiv -S_{a} = Z_{a}^{T} \hat{M}(t_{0}) + Z_{a}^{T} \hat{M}(t_{0}) + \hat{n}(t_{0}) \]  

(5.4,87)

where

\[ \hat{M} = -\hat{M}_{2} - Z_{a}^{T} \hat{M} - \frac{1}{2} \hat{P}_{2}^{T} \hat{P}_{2} \]  

(5.4,88a)
\[ \hat{\gamma}_2(t) = \frac{1}{\mu_2} (x^{T}T + b^{T}G_3B_21G_3^{T}[\hat{\gamma} + \hat{n}] ) . \]  

We now try a solution for \( \gamma_2(t) \) of the form 

\[ \gamma_2(t) = \hat{x}^{T}(t)\hat{\gamma}(t) + \hat{\gamma}^{T}(t)\hat{n}(t) . \]  

Substituting (5.4,92) for \( \gamma_2 \) in (5.4,91), then substituting (5.4,62) for \( \hat{x} \), (5.4,68a) and (5.4,68b) for \( \hat{n} \), and finally collocating terms by their order in \( \hat{x} \), we obtain 

\[ \hat{x} = \hat{x}^{T}\left[ m + 2\hat{\gamma} + b^{T}G_3B_21G_3^{T} \right] \hat{\gamma} \]

\[ \hat{x}^{T} = \hat{x}^{T}\left[ m - 2\hat{\gamma} + b^{T}G_3B_21G_3^{T} \right] \hat{\gamma} \]

\[ + 2\hat{\gamma} + \hat{\gamma}^{T}\hat{m} + \hat{\gamma}^{T}b^{T}G_3B_21G_3^{T} \hat{m} \]

\[ + \frac{1}{\mu_2} \hat{\gamma}^{T}G_3B_21G_3^{T} \hat{m} + \left\{ \hat{n} - \hat{\gamma}^{T}G_3B_21G_3^{T} \right\} = 0. \]  

The equation (5.4,93) must hold independently of \( \hat{x}(t) \). 

Thus, using Lemma G.1 to write 

\[ 2\hat{x}^{T}M\hat{x} = \hat{x}^{T}\left[ M\hat{x} + 2\hat{\gamma} \right] \hat{x} \]  

we obtain the differential equations (5.4,88a), (5.4,88b) and (5.4,88c). The boundary conditions (5.4,89) follow from the
boundary condition (5.4,80b) on $\gamma_2(t_f)$, and from the independence of the terms in parenthesis in (5.4,93) from $\dot{x}$. This completes the proof of Theorem 5.7.

Remark 1

The differential equations of Theorem 5.7 are quite cumbersome. For the most part there seems to be little advantage in solving for $\gamma(t_0)$ in this way over the direct simulation route, except (possibly) in cases where the system simulation equations are different from those on which the minimax solution was determined (e.g. in suboptimal applications).

This brief discussion of linear quadratic DG is concluded by noting that the usefulness of these analytical results for generating worst-case variable winds can only be established in the application. Since the theory we have discussed is linearized theory, the disturbances that are generated are functions of perturbation state variables and are perturbations about reference values about which the equations of motion were linearized (see Section 3.4). Thus if the aircraft is in the linearization reference equilibrium, and this also coincides with the desired and perverse state trajectories, then the linear quadratic differential game approach will not yield perturbing inputs.

This problem may be circumvented by formulating the problem as a nonzero sum DG in which the desired and perverse state trajectories are different from each other. This, however, is a somewhat artificial approach and is conceptually undesirable. The proper technique is to generate a worst-case reference equilibrium (which will result in a worst-case total mean wind profile) about which to perform the linearization of the equations of motion and the application of linear quadratic DG theory. This would involve the considerably more difficult problem, both analytically and computationally, of solving a nonlinear open-loop differential game, and may not lead to results which are more useful in the engineering sense.

This section and the previous section gave a brief overview of the DG analysis required for the numerical examples. Other aspects of these problems, including numerical methods for efficiently solving for the differential equations of Theorems 5.5, 5.6 and 5.7, will be considered as required.
5.5 Numerical Examples of the Worst-Case Methods

In this section the IRWC and conflict of interest methods are applied to the problem of generating worst-case wind models disturbing a two-engined STOL light transport on the landing approach. The examples are intended to demonstrate a number of formulations with which these worst-case methods may be implemented, and to provide a basis for their comparison and evaluation. Preliminary results were first presented in Ref. 5.28.

The equations of motion are a simplified subset of the SDL equations (3.10,2), (3.10,3), (3.10,4), (3.10,11) and (3.10,12). Making the further assumption that

\[ W_2 = 0, \]  
(5.5.1)

implying that

\[ \psi_e = 0, \]  
(5.5.2)

these equations decouple into the longitudinal and lateral sets commonly used to represent aircraft dynamics.

All of the examples in this section will deal with the longitudinal system only, although in principle the methods are also applicable to the lateral system. The longitudinal, decoupled equations of motion may be written in matrix form as (see also Reid et al [1.39]):

\[ A_1 \Delta \dot{X} = A_2 \Delta X + B_1 \Delta \dot{\delta} + B_2 \Delta W + B_3 \Delta \dot{W} \]  
(5.5.3)

where

\[ \Delta X^T = [\Delta u \Delta \delta \Delta \theta \Delta X_I \Delta h \Delta d] \]  
(5.5.4a)

\[ \Delta \delta^T = [\Delta \delta_e \Delta \delta_P] \]  
(5.5.4b)

\[ \Delta W^T = [\Delta W_1 \Delta W_3] \]  
(5.5.4c)

\[ A_1 = \begin{bmatrix} m & -X_w^* & 0 & 0 & 0 & 0 \\ 0 & m-Z_w^* & 0 & 0 & 0 & 0 \\ 0 & -M_w^* & I_{yy} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]  
(5.5.5)

\[ A_2 = \begin{bmatrix} X_u & X_w & X_q & -mg \cos \theta_e & 0 & 0 \\ Z_u & Z_w & Z_q + Im \delta_e & -mg \sin \theta_e & 0 & 0 \\ M_u & M_w & M_q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \cos \theta_e & \sin \theta_e & 0 & -V_e \sin \theta_e & 0 & 0 \\ \sin \theta_e & -\cos \theta_e & 0 & V_e \cos \theta_e & 0 & 0 \end{bmatrix} \]  
(5.5.6)

\[ a_{271} = \cos \theta_e \sin \gamma_G + \sin \theta_e \cos \gamma_G \]  
(5.5.7a)

\[ a_{272} = \sin \theta_e \sin \gamma_G - \cos \theta_e \cos \gamma_G \]  
(5.5.7b)

\[ a_{274} = V_e (-\sin \theta_e \sin \gamma_G + \cos \theta_e \cos \gamma_G) \]  
(5.5.7c)
of the presence of both \( \hat{W} \) and \( \hat{\dot{W}} \) terms. The procedure discussed in Remark 3 following Theorem 5.5 transforms the equations into a suitable form. This procedure augments the state vector with the wind vector and redefines the associated system matrices accordingly.

Furthermore, for smoother aircraft controller response and better wind handling characteristics, the optimal linear quadratic methods treat control rate \( \dot{\phi} \) as the control input, and \( \phi \) as part of the state vector (see Appendix F, Section F.1, Formulation 6). In the following equations this augmentation is also included, although not all of the examples make use of it. In examples where it is not used it is clear how the equations should be rearranged to remove it.

Finally, anticipating certain examples to follow where worst-case wind inputs are superimposed onto a linear wind model, defined \textit{a priori}, the wind inputs \( \dot{W} \) are written in the form

\[
W = W_0 + \hat{K} \dot{x}_2 + \dot{W}
\]

where

\[
\dot{x}_2 = [x_1 \ h \ d]^T,
\]

\[
\hat{K} = \begin{bmatrix} K & 0 \end{bmatrix},
\]

The equations of motion (5.5,3) are not compatible with the worst-case formulations of the previous sections because...
\[ W_{L0} = \begin{bmatrix} W_{L1} \\ W_{L3} \end{bmatrix}, \quad (5.5,14) \]

\[ W_{L} = W_{L0} + \hat{K}X_2. \quad (5.5,15) \]

The perturbation wind input may thus be written

\[ \Delta W = W_{L0} + \hat{K}(X_2 + \Delta X_2) + \Delta W \quad (5.5,16) \]

where

\[ \Delta W = W - W_e \quad (5.5,17) \]

and

\[ W_e^T = [W_{1e} 0]. \quad (5.5,18) \]

Augmenting the state vector (5.5,4a) with \( \Delta \hat{\phi} \) and \( \Delta \hat{w} \), and substituting (5.5,16) into (5.5,3), the augmented matrix form of the longitudinal equations of motion incorporating a linear wind model may be written

\[
\begin{bmatrix}
\Delta \dot{x}_1 \\
\Delta \dot{x}_2 \\
\Delta \dot{q}
\end{bmatrix} =
\begin{bmatrix}
A_{111} & -B_{31} & 0 & 0 \\
A_{121} & A_{122} & 0 & 0 \\
0 & 0 & I & 0
\end{bmatrix}
\begin{bmatrix}
\Delta x_1 \\
\Delta x_2 \\
\Delta \phi
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\Delta \dot{w}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_{211} & A_{212} & B_{11} & 0 \\
A_{221} & B_{22} & B_{22} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\Delta x_1 \\
\Delta x_2 \\
\Delta \phi
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\Delta \dot{w}
\end{bmatrix}
\]

\[ + \begin{bmatrix}
B_{31} \\
0 \\
0
\end{bmatrix}
\Delta \hat{w} + \begin{bmatrix}
B_{31} \hat{x}_2 \\
0 \\
B_{22} [W_{L0} + \hat{X}_2]
\end{bmatrix}. \quad (5.5,19) \]

In more compact notation this equation becomes

\[ \dot{\Delta \hat{x}} = \hat{A} \Delta \hat{x} + \hat{C}_2 \Delta \hat{\phi} + \hat{C}_2 \Delta \hat{w} + \hat{\gamma} \quad (5.5,20) \]

where

\[ \Delta \hat{x}^T = [\Delta \hat{x}_1^T \Delta \hat{x}_2^T \Delta \hat{\phi}^T \Delta \hat{w}^T] \quad (5.5,21) \]

and

\[ (\hat{A} \hat{C}_1 \hat{C}_2 \hat{\gamma}) = \hat{A}^{-1} (\hat{A}_2 \hat{B}_1 \hat{B}_2 \hat{B}) \quad (5.5,22) \]

* It is general notational policy (see Appendix A) to limit use of subscripts and superscripts wherever context makes this possible. Thus, for example, \( W_e^T = [W_{L1} W_{L3}] \) in this section, although in its general form \( W_e^T = [W_{L1} W_{L3} W_{L3}] \). The former definition is implied by the definition of \( \Delta W \) in (5.5,4c), which is also different from the general definition of \( W \).
The matrices $\hat{A}_1$ through to $\hat{b}$ on the right hand side of (5.5,22) have a one-to-one correspondence with the matrices of (5.5,19). The partitions in (5.5,19) are defined in the usual way, e.g. (see also Appendix A)

$$\hat{A}_1 = \begin{bmatrix} \hat{A}_{111} & \hat{A}_{112} \\ \hat{A}_{121} & \hat{A}_{122} \end{bmatrix}. \quad (5.5,23)$$

The partition dimensions are compatible with $\Delta x_1$, as given by

$$\Delta x_1^T = [\Delta u \Delta w \Delta q \Delta \theta] \quad (5.5,24)$$

and $\Delta x_2$, $\Delta \phi$ and $\Delta W$, as given by (5.5,12), (5.5,4b) and (5.5,4c) respectively.

A description of the example aircraft, including its open-loop dynamic characteristics, is given in Appendices C and D.

All of the examples in the following subsections are for the conditions

$$V_e = 40 \text{ ms}^{-1} \quad (5.5,25a)$$

$$\gamma_G = 7^\circ \quad (5.5,25b)$$

$$W_e = 0 \quad (5.5,25c)$$

$$h_0 = 100 \text{ m} \quad (5.5,25d)$$

$$t_f = 20.5 \text{ s.} \quad (5.5,25e)$$

The duration of the simulation, $t_f$, was chosen to be the time it would take the aircraft to descend from 100m to the ground while in the linearization reference equilibrium. This time frame is comparable to the open-loop phugoid period and is of sufficient length to demonstrate the methods and indicate trends, while minimizing computation time.

5.5.1 IRWC Method Examples

The equations of motion employed for all of the examples in this subsection are the equations (5.5,20) without the control vector augmentation of the state vector. The algorithms used to compute the worst-case wind histories and the system response are discussed in Appendix H. They are implemented with computer codes designated VVWCM and VVWCM1, where the former treats the payoff functional $J_1'$ defined by (5.2,22), and the latter treats the more general payoff functional $J_2$ defined by (5.2,8b). While the VVWCM program is just a special case of the VVWCM1 program, the two were kept separate because of the saving on computing time possible in using the less general VVWCM program for solving the $J_2$ problem. In these examples, the output vector $\gamma$ and the state vector $\dot{x}$ will be considered to be identical, i.e. the matrix $M(t)$ of (5.2,2) is just the identity matrix. Also, for simplicity we assume $V_d = 0$ and $W_e = 0$. The latter, with (5.5,25c), implies that $W = W = dw = dw$.

Example 1. van der Vaart's Technique

In this example $J_2'$, given by (5.2,22), is maximized
under the following conditions:

\[ \Delta x(0) = 0 \]
\[ \beta_f^i = 0, \quad i = 1 \]
\[ \beta_f^i = 1 \]
\[ \dot{W}_3 = 0 \]
\[ \Delta \dot{\omega}(0) = 0 \]
\[ \Delta \dot{\omega}(t) = 0, \quad t \in [0, t_f] \]

In effect this reduces the example to van der Vaart's technique, i.e. it is required to find a disturbing wind history \( \Delta W_1(t) \) which maximizes the final value of the airspeed perturbation \( \Delta u\dot{\omega}(t_f) \), subject to the isoperimetric constraint

\[ \int_0^{t_f} \dot{W}_1^2 \, dt = S_{WS}. \] (5.5,27)

Here \( S_{WS} \) is the same as \( E \) in Section 5.2 of this chapter. This modified notation is compatible with the DG examples to follow.

The interpretation of \( S_{WS} \) is facilitated by dividing \( S_{WS} \) by \( t_f \). From (5.5,27) it is apparent that the resulting quantity is just the time-averaged value of \( \dot{W}_1^2 \), i.e.

\[ \frac{S_{WS}}{t_f} = \overline{\dot{W}_1^2}. \] (5.5,28)

This may be connected to an equivalent spatial shear with the relationship

\[ \frac{dW_1}{dt} = \frac{dW_1}{dh} \frac{dh}{dt} \] (5.5,29a)

or

\[ \frac{dW_1}{dt} = \frac{dW_1}{dh} \frac{dh}{\dot{h}} \] (5.5,29b)

A typical magnitude for \( \dot{h} \) follows from the reference condition (5.5,25), (3.4,23a), and (3.4,29) to (3.4,32). For this case this is given by (see 3.4,23a)

\[ |\dot{h}| = |\dot{h}_e| = |V_e \sin \theta_e| \approx 5 \text{ m/s}. \] (5.5,30)

Thus, typical values of the equivalent spatial shear are given by

\[ \left| \left( \frac{dW_1}{dh} \right)_{eq} \right| = 0.2 \frac{\sqrt{S_{WS}}}{t_f}. \] (5.5,31)

This relationship may be used to loosely define \( S_{WS} \) in terms of the ICAO horizontal shear categories (see the table in Chapter 2, Section 2.1) as follows:

<table>
<thead>
<tr>
<th>Category</th>
<th>( \frac{\sqrt{S_{WS}}}{t_f} ) (m/s²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light</td>
<td>0 - 0.4</td>
</tr>
<tr>
<td>Moderate</td>
<td>0.4 - 0.75</td>
</tr>
<tr>
<td>Strong</td>
<td>0.75 - 1.0</td>
</tr>
<tr>
<td>Severe</td>
<td>&gt; 1</td>
</tr>
</tbody>
</table>
It is stressed that such categorization is only intended to provide a better feeling for the magnitude of the quantity \( S_{WS} \). As was already indicated in Chapter 2, the actual effects of the variable winds on the aircraft depend on many factors unrelated to the level of the variable wind activity itself.

The three values for \( S_{WS} \) used in the examples are 5.0, 10.0 and 20.0 \( m^2/s^3 \). These approximately span the moderate to severe categories of the table.

Fig. 5.5a gives the open-loop airspeed perturbations resulting from the worst-case wind inputs. \( W_1 \) and \( \dot{W}_1 \) are plotted in Figs. 5.5b and 5.5c. Because of the linearity of the dynamic system and the form of the worst-case wind inputs, the resulting response is directly proportional to \( \sqrt{S_{WS}} \), and thus once the response has been found for one \( S_{WS} \), it may conveniently be found for all others without having to directly solve for the system response and wind inputs.

The worst-case wind inputs for this example are just the time-reversed impulsive response functions of airspeed to \( W_1 \) wind disturbances multiplied by \(-1/2\mu\) (see Section 5.2 of this chapter). Because of the phugoid mode's low damping compared to the short period mode, it is advantageous for the worst-case wind inputs to stimulate the phugoid mode, and thus their period is comparable to the phugoid's period of 21.4s (see Appendix D). Also, since the longitudinal aircraft system is stable, and thus the response of \( \Delta u \) to \( \dot{W}_1 \) impulses will be damped, the worst-case wind inputs, which depend on the time-reversed impulsive response function, are of increasing amplitude.

**Example 2**

In this example \( J_2 \) given by (5.2,8b), is maximized under the following conditions:

\[
\Delta x(0) = 0 \quad (5.5,32a)
\]
\[
\beta_i = 0 \quad (5.5,32b)
\]
\[
\beta_i = 0, \quad i \neq 1 \quad (5.5,32c)
\]
\[
\dot{\beta}_i = 1 \quad (5.5,32d)
\]
\[
\dot{W}_0 = 0 \quad (5.5,32e)
\]
\[
\Delta W(0) = 0 \quad (5.5,32f)
\]
\[
\Delta \dot{\delta}(t) = 0, \quad t \in [0, t_f] \quad (5.5,32g)
\]

Thus it is required to find a \( \dot{W}_1(t) \) that maximizes the payoff functional

\[
J = \int_0^{t_f} \Delta u(t) dt \quad (5.5,33)
\]

This is a special case of Theorem 5.1.

The airspeed response and the worst-case wind inputs are given in Fig. 5.6 for the same three values of \( S_{WS} \) as in the previous example. The worst-case wind inputs are characteristically different from those of the previous example, in which \( \Delta u(t_f) \) was maximized, as may be seen by comparing Figs. 5.5 and 5.6. The property that
5.103

\[ \dot{W}_1(t_f) = 0 \] follows from the form of the worst-case wind input (see equation (5.2,10) of Theorem 5.1) and the conditions of the example \( \beta_f = 0 \).

Maximizing the integral of a state variable perturbation may be desirable in certain applications. One such application is in simulators where what is often required is a wind model that produces significant airspeed deviations throughout a flight phase rather than only at a particular time.

Example 3

In this example \( J_1' \), given by (5.2,53), is maximized under the following conditions:

\[
\begin{align*}
\Delta x(0) &= 0 \quad (5.5,34a) \\
\alpha_{f_1} &= 1 \quad (5.5,34b) \\
\alpha_{f_7} &= 0.625 \quad (5.5,34c) \\
\dot{W}_3 &= 0 \quad \text{(Case 1)} \quad (5.5,34d) \\
\dot{W}_3 &= 0 \quad \text{(Case 2)} \quad (5.5,34e) \\
\Delta \dot{W}(0) &= 0 \quad (5.5,34f) \\
S_{WS} &= 5.0 \text{ m}^2 \text{s}^{-2} \quad (5.5,34g)
\end{align*}
\]

The weightings \( \alpha_{f_1} \) and \( \alpha_{f_7} \) are chosen to take into account the relative size of the magnitude of significant airspeed and glidepath deviations (see Fig. 4.3).

The solution to this problem is given by Corollary 5.2.1. In practice the VVWCM program is used to obtain solutions for all the possible sign combinations of \( \beta_{f_1} \) and \( \beta_{f_7} \) in order to determine a set of signs that (1) satisfy (5.2,31a), and (2) yield the absolute maximum of \( J_2' \), as given by (5.2,22) with \( Y_d = 0 \). Since there are only two nonzero terminal weightings, this implies that four iterations are required to exhaust the possibilities (see the discussion following Corollary 5.2,3).

For this particular problem, the solutions are symmetric, because of the zero initial conditions, and thus only two solutions need be computed for each case.

This example incorporates a time-invariant, linear quadratic optimal aircraft controller determined using the LQDG-3 program of Appendix H, and designated LOOCI. The performance criterion and weighting conditions are defined in Tables 5.1 and 5.2. The aircraft control law that results is given by

\[
\begin{align*}
\Delta \delta_E &= 0.03174\Delta u + 0.005013\Delta w + 1.262\Delta q \\
&+ 6.588\Delta \theta + 0.06015\Delta d \\
(5.5,35) \\
\Delta \delta_T &= -0.04610\Delta u + 0.03417\Delta w - 0.01131\Delta q \\
&-1.269\Delta \theta - 0.02046\Delta d. \quad (5.5,36)
\end{align*}
\]

This example therefore differs from the previous two which represent open-loop aircraft dynamics. Although no attempt was made to optimize the performance criterion weightings, this controller's characteristics were found to be satisfactory.
The cases computed are summarized in the table below, and plotted in Figs. 5.7 and 5.8.

<table>
<thead>
<tr>
<th>CASE</th>
<th>$B_{f_1}$</th>
<th>$B_{f_7}$</th>
<th>$\Delta u(\tau_f)$</th>
<th>$\Delta d(\tau_f)$</th>
<th>$J_1$</th>
<th>$J_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>1.0</td>
<td>0.625</td>
<td>2.154</td>
<td>1.551</td>
<td>3.12</td>
<td>3.12</td>
</tr>
<tr>
<td>1b</td>
<td>-1.0</td>
<td>0.625</td>
<td>-2.121</td>
<td>1.431</td>
<td>3.02</td>
<td>3.02</td>
</tr>
<tr>
<td>2a</td>
<td>1.0</td>
<td>0.625</td>
<td>3.186</td>
<td>25.62</td>
<td>19.20</td>
<td>19.20</td>
</tr>
<tr>
<td>2b</td>
<td>-1.0</td>
<td>0.625</td>
<td>2.225</td>
<td>25.36</td>
<td>18.08</td>
<td>13.62</td>
</tr>
</tbody>
</table>

For Case 1 both iterations satisfy the sign conditions implied by (5.2.31a), and are thus candidates for the global worst-case solution. By inspection, however, iteration 1a, and its counterpart for $B_{f_1} = -1$ and $B_{f_7} = -0.625$, are the desired solutions, both leading to a $J_1$ value of 3.12.

In fact, for this case all four possible iterations are candidate solutions, but only two lead to the global maximum.

In contrast, for Case 2 two of the iterations (iteration 2b and its counterpart for $B_{f_1} = 1.0$, $B_{f_7} = -0.625$) drop out immediately because they do not satisfy the conditions (5.2.31a).

Thus iteration 2a and its counterpart for $B_{f_1} = -1$, $B_{f_7} = -0.625$ are the desired solutions.

These two cases not only demonstrate Corollary 5.2.1, but also provide examples of the iterative nature of the solution procedure, and of the multiplicity of solutions, some of which may be only locally maximizing (see Corollary 5.2.3 and the discussion following it).

Fig. 5.9 gives the approach paths of the aircraft for the two cases, and compares them to an open-loop case analogous to Case 2a. As might be anticipated, glidepath deviation is more readily maximized if $W_3$ inputs are included. Even though Case 2a is a closed-loop case, the $W_3$ wind input steadily increases to a very strong updraft at times near the terminal time ($W_3 \approx 10 \text{ m/s}$, see Fig. 5.8c). This makes it impossible for the controller to keep the aircraft close to the glidepath with realistic inputs of control energy and deflection, although there is considerable improvement in comparison to the open-loop example.

Example 4

In this final example of the application of the IRWC method, the payoff functional $J_2$, given by (5.2.8b), is maximized under the following conditions:

\begin{align*}
\Delta \xi(0) &= 0 & (5.5.37a) \\
B_{f_1} &= \beta_1 = 1 & (5.5.37b) \\
B_{f_7} &= \beta_7 = 0.625 & (5.5.37c) \\
\Delta \eta(0) &= 0 & (5.5.37d) \\
S_{WS} &= 5.0 \text{ m}^2\text{s}^{-3} & (5.5.37e)
\end{align*}
The aircraft is under closed-loop control by the aircraft controller L00C1 discussed in the previous example.

This is a more general problem requiring the simultaneous maximization of the weighted integrals of \( \Delta u \) and \( \Delta d \), and the terminal values \( \Delta u(t_f) \) and \( \Delta d(t_f) \). It may be solved with an application of Theorem 5.1 through the VVWCML program. The resulting airspeed response, worst-case wind inputs, and approach path are given in Figs. 5.10, 5.11 and 5.12, and are compared with the results for some other cases where only the integral of \( \Delta u(t) \) is maximized.

This example may also be compared with the previous example where only the weighted values of \( \Delta u(t_f) \) and \( \Delta d(t_f) \) were maximized (compare with Case 2a of Figs. 5.8 and 5.9). The most significant differences between these two examples arise in the worst-case \( W_1 \) inputs.

It is also worth noting that the worst-case solution for this example satisfies the sign conditions (5.2,31a) and (5.2,31b), and thus must also be at least a local maximum for the payoff functional \( J_1 \) of equation (5.2,8a). This state of affairs is fortuitous. In general, the sign conditions implied by (5.2,31b) are difficult to satisfy (see Example 2 for a case where they are not satisfied). Even if a solution were found satisfying (5.2,31b), this would in general result in discontinuities in the worst-case disturbance inputs at the points where \( \delta \) changes sign, not a desirable property for obtaining physically plausible solutions.

5.5.2 Indirect Method Examples

This method, also referred to as the perversity function technique, was introduced in Section 5.3.1 of this chapter. Its objective is to determine a worst-case wind input that forces the aircraft state to track, in the quadratic sense of the payoff functional (5.3,2), a perversity vector \( X_p(t) \) that has been specified in advance. By appropriately defining \( X_p(t) \), a minimisation of (5.3,2) will lead to undesirable wind inputs.

Both examples in this subsection are for the general conditions (5.5,24), and are for controls fixed (i.e. open-loop). They were generated using the LQDG-1 (Example 1) and LQDG-3 (Example 2) programs. The equations of motion are the equations (5.5,20). The quadratic weighting strategy is summarized in Table 5.1.

Example 1. Phugoid Perversity Functions

The payoff functional 3 of Table 5.1 is minimized under the weighting conditions defined in Table 5.3. The positive parameter \( \mu \) is given by (\( \mu \) may be incorporated directly into the weighting matrix, i.e. \( R = \mu R_{21} \), as is more appropriate for the discussion in Section 5.3.1)

\[
\mu = 0.05 \quad (5.5,38)
\]

and the initial conditions on the simulation are (\( \Delta X_{OP} \) and \( \Delta W_{OP} \) are defined in Table 5.3)

\[
\Delta X_{OP} = 0 \quad (5.5,39)
\]
There is no linear mean wind onto which the worst-case results are superimposed, i.e.,

\[ W_L = 0. \quad (5.5,41) \]

The perversity function vector \( \mathbf{\hat{x}}_p(t) \) and the weighting vector \( \mathbf{\hat{w}}_{\text{max}} \) (see Table 5.3) are based on the frequency, magnitude, and phase relationships implied by the open-loop phugoid mode eigenvalues and eigenvectors of the STOL transport. \( \mathbf{\hat{x}}_p(t) \) is given by

\[
\begin{bmatrix}
5.0 \sin(\omega_{\text{ph}} t) \\
0.5279 \sin(\omega_{\text{ph}} t - 2.9972) \\
0.04618 \sin(\omega_{\text{ph}} t - 0.03240) \\
0.1551 \sin(\omega_{\text{ph}} t - 1.779)
\end{bmatrix}
\quad (5.5,42)
\]

where

\[
\omega_{\text{ph}} = 0.293 \text{ rad./s}. \quad (5.5,43)
\]

Thus the overall objective of this example is seen to be to generate wind inputs that stimulate the phugoid mode of the aircraft, within the weighting constraints implied by the payoff functional. This is a case of considerable interest in view of recent simulator studies which suggest that the presence of naturally occurring wind inputs that stimulate the phugoid mode produce particularly undesirable responses in aircraft on the landing approach (Turkel and Frost [5.29]).

The aircraft response and the perverse state trajectories, and the worst-case wind inputs are given in Figs. 5.13 and 5.14 respectively. For \( t > 10s \) excellent tracking of \( \mathbf{\hat{x}}_p(t) \) is achieved with relatively little variation in \( W_1 \) and \( W_3 \). Both \( W_1 \) and \( W_3 \) exhibit the initial portion of a long period, low amplitude oscillation, with the \( W_1 \) oscillation being more emphasized. This is to be expected in view of the low damping, long period of the phugoid mode. The wind velocity variation can be interpreted in terms of an initially large rate of change that stimulates the phugoid, followed by small wind variations tuned to the phugoid mode that maintain the phugoid oscillation, i.e., cancel the small open-loop damping that exists. If this problem had been solved for an aircraft with a SAS that increased phugoid damping, a significantly increased wind activity would have been required to maintain the oscillations.

Fig. 5.15 gives the worst-case tailwind that results for a formulation where \( \mathbf{W}_3 \) inputs are not included. The resulting system response is similar to Fig. 5.13, and is not given. \( J_a \) (equivalent to \( J \) of (5.3,2)) was found to be 0.43 for this case, while \( J_a \) for the previous case, with \( W_3 \) included, was 0.38. Thus the loss of performance (as measured by \( J_a \)) in not including \( \mathbf{W}_3 \) inputs is not large, for this example. This insensitivity to \( \mathbf{W}_3 \) can be explained in terms of the strong airspeed response to \( \mathbf{W}_1 \) disturbances, and the large contribution of
airspeed, relative to the other dynamic components, to the phugoid response of the example aircraft.

Example 2. Hypertangent Perversity Functions

In this example the payoff functional 3 of Table 5.1 is minimized under the weighting conditions defined in Table 5.3, and:

\[ u = 0.05 \tag{5.5,44a} \]
\[ \Delta x_1(0) = 0 \tag{5.5,44b} \]
\[ \Delta y(0) = \begin{bmatrix} -5 \text{ms}^{-1} \\ 0 \end{bmatrix} \tag{5.5,44c} \]
\[ \hat{W}_L = 0 \tag{5.5,44d} \]
\[ \hat{x}_p(t) = \begin{bmatrix} -2.5[\tanh(t/3) + 1] \\ 0 \\ 0 \\ -7.5[\tanh(t/3) + 1] \\ 0 \\ 0 \end{bmatrix} \tag{5.5,44e} \]

The perversity functions for this example were chosen to

\[ \hat{x}_p(t) \] has a one-to-one correspondence with \( \Delta x_{OP} \) of Table 5.3.

smoothly approximate step deviations from the approach airspeed and the glidepath. With the signs specified as in (5.5,44e), this example becomes one where the wind inputs attempt to put the aircraft in a low and slow condition.

Fig. 5.16 gives the airspeed and glidepath deviations that result for the worst-case inputs generated in the solution of this problem. The wind inputs are given in Fig. 5.17 as an altitude profile. After an initial period of approximately six seconds, the glidepath deviation stays close to the deviation commanded by \( d_p \), but the airspeed does not approach \( u_p(t) \) until \( t \) becomes larger.

In general, the characteristics of the worst-case solutions are strongly influenced by the choice of weighting matrices. In this case, the selection of a stronger weighting on \( \Delta u(t) \) (i.e. \( \Delta x_{OP,\max} \) is made smaller) would have resulted in wind inputs that cause \( \Delta u(t) \) to stay closer to \( u_p(t) \), possibly at the expense of \( d(t) \) tracking of \( d_p(t) \). Better \( u_p \) tracking could also have been achieved by reducing the weighting on \( \hat{W}_1 \) relative to \( \hat{W}_3 \). In this way it would be more cost effective for \( \hat{W}_1 \), which strongly influences \( \Delta u(t) \), to stimulate airspeed deviations that more closely track \( u_p(t) \).

From Fig. 5.17 it is seen that the wind profiles exhibit an initial shear that is followed by a segment of relatively low wind variability. The initial activity compensates for the relatively large initial separations between \( \Delta u \) and \( u_p \), and between \( d \) and \( d_p \). Once cost effective tracking has
been established, however, the wind variability reduces to a level that is just sufficient to maintain tracking. This is particularly true for $W_3$ inputs, since $W_3$ strongly influences $d$, and $d$ tracks $d_p$ very closely.

The wind inputs for the perversity function method are of the form (5.1,82), i.e., in more compact notation,

$$\Delta \dot{W} = \Delta x_{OP} + \Delta k(t).$$

Thus the closed-loop (with respect to the wind controller) system equations are of the form

$$\Delta \dot{x}_{OP} = \Delta A_{CL} \Delta x_{OP} + \Delta C_{2} \Delta k(t)$$

where

$$\Delta A_{CL} = A' + KC'.$$

The matrices $A'$ and $C_2'$ may be obtained from the matrices $A$ and $C_2$, as defined in (5.5,22), by deleting all rows and columns from the latter that do not correspond to the elements of the vector $\Delta x_{OP}$. Since the perversity function technique, as developed here, is formally a linear quadratic tracking problem, the matrix $\Delta A_{CL}'$, characterizing the closed-loop response, has eigenvalues whose real parts are negative. This result not only emphasizes the fact that the worst-case nature of the solution enters the problem solely through the perversity functions, but also makes this method unsuitable as a feedback method, i.e. in general the system will be more stable with the wind controller than without it. Thus this method is most useful as a technique for generating wind altitude profiles (e.g. Fig. 5.17) and/or wind time histories (e.g. Fig. 5.14) which may then be used as disturbance inputs in aircraft dynamic problems (e.g. in flight simulators).

Another potential application of this method is that of determining wind inputs that force the aircraft state along a trajectory that was observed during flight. While the wind inputs that are determined are not necessarily those that were actually present, they will lead to similar aircraft behavior. It is improbable, however, that this technique would have any advantages over the conventional method of determining wind conditions by subtracting air velocities from inertial velocities, if the latter and the Euler angles are available.

5.5.3 Direct Method Examples

As discussed in Section 5.3.2 of this chapter, this method looks for wind inputs that maximize a payoff functional of the form (5.3,5). In terms of the notation of this section, and with the further restriction that

$$\Delta x_d = 0,$$

(5.5,48)

(5.3,5) becomes a type 4 payoff functional, as defined in Table 5.1, i.e.
subject to the condition

\[ B_{21} < 0. \]  

(5.5,50)

The sign definiteness of \( S \) and \( Q \) will depend on the way in which \( \Delta X_{OP} \) is specified, a point that will be discussed further as it arises in the examples. The parameter \( \mu \) is a positive parameter that may be adjusted to obtain the desired level of wind activity (cf. \( \mu_2 \) of Theorem 5.6), as measured by

\[ S_{WS} = \int_0^{t_f} \| \Delta \omega \|^2_B dt \]  

(5.5,51)

for some positive definite matrix \( B'_{21} \). For these examples \( B'_{21} \) will be one of

\[ B'_{21} = I \]  

(5.5,52a)

or

\[ B'_{21} = -B_{21} \]  

(5.5,52b)

The examples in this subsection were obtained using the LQDG-1 program of Appendix H, under the general conditions (5.5,25). The equations of motion are the equations (5.5,20). Some of the examples will include cases where the worst-case solution is superimposed onto a linear wind field specified \( a \) priori, such as in (5.5,16).

**Example 1**

The payoff functional 4 of Table 5.1 is maximized under the weighting conditions defined in Table 5.3, and

\[ \Delta X_{OP}(0) = \Delta X_{OP trim} + \Delta X_{OP trim} \]  

(5.5,53a)

\[ \Delta X_{OP trim} = [0, -0.125 \text{ms}^{-1}, 0, 0.0259 \text{rad.}] \]  

(5.5,53b)

\[ \Delta X_{OP trim} = [-2 \text{ms}^{-1}, 0, 0, 0] \]  

(5.5,53c)

\[ \Delta \dot{\xi} = 0 \text{ (controls fixed)} \]  

(5.5,53d)

\[ \Delta \dot{\xi} = [-8.57 \times 10^{-4} \text{rad., 0.0557}] \]  

(5.5,53e)

\[ \Delta \omega^T(0) = [-5 \text{ms}^{-1}, 0], \dot{\omega}_3 = 0 \]  

(5.5,53f)

\[ \Delta \omega(t) = \omega_L + \Delta \omega \]  

(5.5,53g)

\[ \omega_L = \begin{bmatrix} -0.035 h_e \\ 0 \end{bmatrix} \]  

(5.5,53h)

\( \omega_L \) as defined here is equivalent to \( \omega_L \) of (5.5,15) with \( \omega_2 \) replaced by \( \omega_2e \) and \( \dot{\omega} = 0 \) except for \( K_{12} = -0.035 \).

In effect this example amounts to maximizing the dynamic subset of the general decoupled linearized equations (5.5,20) with \( \omega_1 \) inputs alone. The initial conditions consist of trim conditions \( \Delta X_{OP trim} \) which were determined on the
assumption of trimmed flight on a $7^\circ$ glide slope in a wind that existed at the starting height $h_0$ (i.e. $\mathbf{W}_0(k_x) + \mathbf{W}(0)$), plus a component $\Delta \mathbf{x}_{\text{trim}}$ representing an initial out of trim perturbation required to excite the system. The controls are fixed at the trim values. 

Five cases were run for different values of the parameter $\mu$, as summarized in the table below. The resulting system response and worst-case wind inputs are given in Fig. 5.18 and 5.19 for four of the cases.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$J_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.6</td>
<td>5.02</td>
</tr>
<tr>
<td>10.9</td>
<td>1.60</td>
</tr>
<tr>
<td>11.2</td>
<td>0.91</td>
</tr>
<tr>
<td>12.0</td>
<td>0.45</td>
</tr>
<tr>
<td>16.0</td>
<td>0.25</td>
</tr>
</tbody>
</table>

From the response curves in Fig. 5.18, it can be seen that the wind variability, as measured by $S_{\text{WS}}(t_f)$, and the resulting level of system response change considerably as $\mu$ is varied in a relatively small range. In fact, what is occurring is a progressive destabilization of the open-loop aircraft dynamic system by the wind controller as $\mu$ decreases towards its conjugate point value.* If the wind control law, which may be written in compact notation as

$$\dot{\mathbf{W}} = \mathbf{K}(t)\Delta \mathbf{x}_{\text{OP}} + \mathbf{k}(t),$$

is incorporated in the equations of motion (cf. the procedure used in Example 2 of Section 5.5.2 of this chapter), closed-loop eigenvalues may be found for the resulting dynamic system. This was done for $\mathbf{K}(0)$, and the resulting phugoid mode eigenvalues are given in Fig. 5.20 for the different $\mu$ values used. It is seen that as $\mu$ moves towards its conjugate point value, the phugoid mode becomes more unstable and of increasing frequency.

The short-period mode eigenvalues were found to vary negligibly. Since the short-period mode of the aircraft exhibits much greater damping than the phugoid mode (see Appendix D), it is not cost-effective for a maximizing solution to destabilize the short-period mode. This characteristic is further emphasized by limiting the wind inputs to $\mathbf{W}_1$ inputs for this example. Since $\dot{\mathbf{W}}_1$ strongly affects airspeed response, and airspeed response is strong in the phugoid mode, this will make it even more cost-effective for the wind controller to ignore the short-period mode.

* For further discussion of the trim conditions, see Chapter 3, Section 3.4 and Appendix H.

* For this example the conjugate point is bounded by $10.4 < \mu_{\text{CP}} < 10.6$, but was not determined any more accurately.
Fig. 5.21 gives the variation of $S_{WS}(t_f)$ with $\mu$. It is seen that as $\mu$ approaches its conjugate point value, the rate of change of $S_{WS}$ becomes large, and it becomes more difficult to determine a $\mu$ that yields a particular value of $S_{WS}(t_f)$.

Since the system and weighting matrices for this example are time-invariant, from equation (5.4.68b) it can be seen that the time dependence of $\mathbf{P}(t)$ arises strictly from the matrix $\mathbf{P}(t)$. $\mathbf{P}(t)$ is a solution of the matrix Riccati equation for this problem, and exhibits the property of finite escape time (see Remark 1 following Theorem 5.5) through which the conjugate point reveals itself.

In Fig. 5.22, the trace of $\mathbf{P}(t)$, defined by

$$\text{tr}\{\mathbf{P}(t)\} = \sum_{i=1}^{n} P_{ii}(t)$$

(5.5.55)

is plotted as a function of time. As $\mu$ approaches its conjugate point value, $\mathbf{P}(t)$ approaches a finite escape condition for $t = 0$, and the quantity $\text{tr}\{\mathbf{P}(t)\}$ approaches infinity.

In general $\text{tr}\{\mathbf{P}(t)\}$ is seen to be small relative to its value for times near the finite escape time. In effect this means that the magnitude of the gains on the worst-case wind control law are also small relative to the gains near the finite escape time, and the amount of destabilization achieved by the wind control law, for times near the initial time, will be at a maximum. An example of the phugoid mode eigenvalue movement with time for a particular case is given in Fig. 5.23.

While the worst-case wind inputs are time histories, they may also be expressed in terms of the aircraft altitude to create a wind altitude profile provided that the altitude is monotonically decreasing for all $t \in [0, t_f]$. This condition guarantees that the wind velocity is a single-valued function of height.

The worst-case wind profiles are given in Fig. 5.24. The altitude coordinate has been shifted 100 m to permit plotting of the stronger wind cases for more of their altitude range.* The plots terminate at either the altitude of the aircraft when $t = 20.5$ s or the altitude where $\dot{h}$ becomes greater than or equal to zero, whichever occurs first.

Fig. 5.25 gives the wind inputs as a function of $h_e$, the linearization reference equilibrium altitude trajectory (see equation (3.4.23b)). Since $h_e$ is constant, $h_e$ may be thought of as a scaled time parameter, and thus Fig. 5.18g and Fig. 5.25 may be compared directly.

All of the cases presented so far are for the open-loop response. Introducing the L00C2 optimal aircraft controller reduces the effects of the wind significantly, as can be seen in Fig. 5.26. The worst-case wind inputs also change markedly. It is stressed that the aircraft controller was introduced after the maximizing wind controller had been determined, i.e. the wind controller is not worst-case for the closed-loop system, and is thus suboptimal in the application. This is not a

* The simulations where for fixed time rather than to a specified decision height, and thus for certain wind conditions negative altitudes resulted.
limitation of the direct method, but rather of the example. One could just as readily determine a maximizing wind controller for the closed-loop system.

The optimal aircraft controller was determined using the LQDG-I program of Appendix H, and is designated LOOC2. The control law is given by

\[
\Delta \delta_E = 0.01677\Delta u - 0.02499\Delta w + 0.3405\Delta q + 1.682\Delta e + 0.01222\Delta d - 1.776\Delta \delta_E + 0.01571\Delta \delta_T \quad (5.5,56a)
\]

\[
\Delta \delta_T = -0.1036\Delta u + 0.05788\Delta w - 0.09133\Delta q - 2.710\Delta e - 0.03902\Delta d + 0.5128\Delta \delta_E - 2.305\Delta \delta_T \quad (5.5,56b)
\]

The performance criterion is a type 2 criterion, as defined in Table 5.1. The weighting conditions are defined in Table 5.2.

Example 2

In this example the payoff functional 4 of Table 5.1 is maximized under the weighting conditions of Table 5.3, and

\[
\Delta X_1(0) = \Delta X_{1,\text{trim}} + \Delta X_{1,\Delta \text{trim}} \quad (5.5,57a)
\]

\[
\Delta X_{1,\text{trim}}^T = [0, -0.125 \text{ ms}^{-1}, 0, 0.0259 \text{ rad.}] \quad (5.5,57b)
\]

\[
\Delta \delta = 0 \quad \text{ (controls fixed)} \quad (5.5,57d)
\]

\[
\Delta \delta^T = [-8.57 \times 10^{-6} \text{ rad.}, 0.0557] \quad (5.5,57e)
\]

\[
\Delta \omega_{\text{OP}}^T (0) = [0] \quad (5.5,57f)
\]

\[
\Delta \omega = \omega_L + \Delta \omega \quad (5.5,57g)
\]

\[
\omega_L = \begin{bmatrix} -0.035h_e \\ 0 \end{bmatrix} \quad (5.5,57h)
\]

This example is completely analogous to the previous example except that now the weighting on $\Delta \omega_1(t_f)$ is nonzero. The latter is motivated by the previous example's wind profiles (see, e.g. Fig. 5.25) in which $\Delta \omega_1(t_f)$ (i.e. $\Delta \omega_1$ near the ground) is generally nonzero. This characteristic is not desirable for physically realistic wind models.

Two cases were run corresponding to $\mu = 11.25$ and $\mu = 10.6$. The resulting wind profiles are given in Figs. 5.27 and 5.28 and are compared to cases from the previous example with similar $S_{WS}$ values. The two wind profiles for the terminally weighted cases achieve $\Delta \omega_1(t_f)$ values near zero. The flight paths that resulted in the presence of the wind profiles of Figs. 5.27 and 5.28 are given in Fig. 5.29.

In general, achieving a desired $S_{WS}$ and terminal wind velocity requires simultaneously varying the constant $\mu$ and
the terminal weighting. For this formulation forcing $\Delta w_1(t_f)$ to be near zero was relatively straightforward; $\mu$ could be varied almost independently of the terminal weighting in order to achieve a desired $S_{WS}$ value. This is not necessarily the case for all formulations with terminal wind velocity weightings, as will be demonstrated in Example 2 of the differential game cases to follow.

5.5.4 Differential Game Examples

This method looks for minimax solutions that yield worst-case wind controllers and best-case aircraft controllers. The payoff functional that is minimaximized is of type 5 of Table 5.1. The equations of motion are the equations (5.5,20). The two examples of this subsection were obtained using the LQDG-1 program of Appendix H. Only one positive parameter $\mu$ was left in the formulation to provide a degree-of-freedom for determining solutions that result in a desired $S_{WS}$ (cf. Theorem 5.6 where $\mu_1$ and $\mu_2$ were included so that the isoperimetric constraints (5.4,64) and (5.4,65) could be satisfied simultaneously).

Example 1

The payoff functional 5 of Table 5.1 is minimaximized under the weighting conditions defined in Table 5.3, and the initial conditions (5.5,53a), (5.5,53b), (5.5,53c), (5.5,53e) and (5.5,53f) with $\dot{w}_3 \neq 0$. A number of cases were computed, as summarized in the following table:

<table>
<thead>
<tr>
<th>CASE</th>
<th>$\mu$</th>
<th>$\bar{w}_L$</th>
<th>$S_{WS}(t_f)$ (m$^2$s$^{-3}$)</th>
<th>$J^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.7584</td>
<td>0</td>
<td>0.05467</td>
<td>0.04774</td>
</tr>
<tr>
<td>2</td>
<td>0.7424</td>
<td>0</td>
<td>14.32</td>
<td>0.8552</td>
</tr>
<tr>
<td>3</td>
<td>0.7424</td>
<td>[-0.035h]</td>
<td>0.4033</td>
<td>0.1485</td>
</tr>
</tbody>
</table>

The worst-case wind inputs and aircraft response are given for cases 1 and 2 and 2 and 3 in Figs. 5.30 and 5.31 respectively. The following points are noted with regard to these results:

1. Near the conjugate point* the solutions are quite sensitive to changes in $\mu$ (see Fig. 5.30). In particular they are considerably more sensitive to $\mu$ variation than the direct method examples.

---

* Note that control rate $\Delta \dot{\alpha}$ is being treated as a control input vector in the optimization process. This creates no difficulty in obtaining a minimax solution and may be dealt with analogously to disturbance rates, i.e. by augmenting the state vector with $\Delta \dot{\alpha}$.

* The conjugate point occurs in the interval $0.7384 < \mu_{CP} < 0.7424$. 
2. The solution characteristics change significantly with and without $W_L$ for a constant $\mu$. This is not completely unexpected in that the optimization problem is different for the two cases. Ideally the comparison should be made between cases with identical values of $S_{WS}(t_f)$.

3. Case 2 exhibits strong wind disturbances for $t > 15s$. This can be explained in terms of the matrix $P(t)$ that characterizes the minimax solution. The trace of $P(t)$ has a peak near $t = 16s$, as is evident from Fig. 5.32. This peak is much greater for Case 2. Note also that $\text{tr}\{P(t)\}$ for this example exhibits markedly different behavior than the trace for the first example of the direct method solutions (see Fig. 5.23).

4. The fact that this is a minimax solution implies that not only is the wind controller worst-case but that the aircraft controller is "best-case". This conflict tends to increase sensitivity to $\mu$ in that only near the conjugate point will the wind controller be able to achieve significant gains, as measured by $J_a$, over the aircraft controller. However, to obtain such good performance, the aircraft controller may require unrealistically large control deflections (e.g. throttle setting $\Delta \delta_T$ in Figs. 5.30 and 5.31).

5. While a condition analogous to (5.4,60b) was not satisfied for this problem, Fig. 5.32 suggests that for the values of $\mu$ chosen, the problem has an infinite terminal time solution. This was verified numerically by computing $P(t)$ to $t = -t_f$. The phugoid mode eigenvalues of the system for this infinite terminal time solution are summarized below.

<table>
<thead>
<tr>
<th>PARTICULARS</th>
<th>EIGENVALUES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open-loop</td>
<td>$-0.0521 \pm 10.293$</td>
</tr>
<tr>
<td>Aircraft controller</td>
<td>$-0.423 \pm 10.346$</td>
</tr>
<tr>
<td>alone</td>
<td></td>
</tr>
<tr>
<td>Wind controller</td>
<td>$0.109 \pm 10.302$</td>
</tr>
<tr>
<td>alone</td>
<td></td>
</tr>
<tr>
<td>Minimax roots</td>
<td>$-0.262 \pm 10.272$</td>
</tr>
</tbody>
</table>

The wind controller alone leads to an unstable system, but the minimax solution itself is stable, with considerably greater damping than the open-loop damping. This suggests that the infinite terminal time aircraft controller is doing well against the worst-case wind controller.

Example 2

The payoff functional 5 of Table 5.1 is minimaximized under the weighting conditions defined in Table 5.3, and

$\Delta x_1(0) = \Delta x_1^{\text{trim}} + \Delta x_1^{\Delta \text{trim}}$ (5.5,58a)
This example differs from the previous one in that it considers only $\Delta w_1$ disturbances and includes glidepath deviations in the minimaximization.

A number of cases were considered, as summarized in the following table:

$$S_{WS}(t_f) = - \int_{t_0}^{t_f} \Delta w_1^2(t) \, dt$$

<table>
<thead>
<tr>
<th>CASE</th>
<th>$\mu$</th>
<th>$W_L$</th>
<th>$S_{WS}(t_f)$ (m$^2$s$^{-3}$)</th>
<th>$J^*_a$</th>
<th>$\Delta w_{OP}(t_f)$ WEIGHTING</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0</td>
<td>0</td>
<td>0.116</td>
<td>0.125</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1.0</td>
<td>0</td>
<td>1.24</td>
<td>0.145</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0.752</td>
<td>0</td>
<td>12.0</td>
<td>0.177</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0.752</td>
<td>[0.035$e_i$]</td>
<td>43.9</td>
<td>0.568</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1.504</td>
<td>[0.035$e_i$]</td>
<td>1.25</td>
<td>0.320</td>
<td>[-1]</td>
</tr>
<tr>
<td>6</td>
<td>0.752</td>
<td>[0.035$e_i$]</td>
<td>1.53</td>
<td>0.368</td>
<td>[-1]</td>
</tr>
</tbody>
</table>

Cases 1, 2 and 3 contain no linear mean wind and no $\Delta w_1(t_f)$ weighting and are intended to show the effects of $\mu$ variation on $S_{WS}(t_f)$ and $J^*_a$. Case 4 introduces a linear mean wind and is intended for comparison with Case 3. Cases 5 and 6 introduce $\Delta w_1(t_f)$ weightings in the payoff functional and are intended for comparison with Case 4.

Fig. 5.33 compares the aircraft response and worst-case wind inputs for Cases 1, 2 and 3. As in the previous example, $S_{WS}(t_f)$ is quite sensitive to $\mu$ changes for $\mu$ close to its conjugate point value. There is also increased dynamic response for $t$ near $t_f$. The latter is a consequence of the behavior of $P(t)$ for $t$ near $t_f$ (see Fig. 5.34). It is noted, however, that while in the previous example the initial aircraft response to the minimax control and wind inputs did not change for the $\mu$ values used, such is not true for this example. In the previous example, the solutions for small $t$ approached identical time-invariant infinite terminal time solutions. In this example, the solutions for $P(t)$ do not approach steady state solutions even though the $\text{tr}(P(t))$ are almost constant for small $t$. This conclusion was verified computationally. Furthermore, for small $t$ the $P(t)$ are not identical.

Fig. 5.35 compares Case 1 with a solution where the LOOC2 optimal controller of Table 5.2, with control law (5.5,56), is used in place of the minimax aircraft controller. The wind controller for the latter is the minimax controller of Case 1. The loss of optimality by going to an aircraft
controller that is not minimax is apparent from the response curves. In more quantitative terms, the extremum value of $J_a$ for the minimax solution is 0.125, while for the latter suboptimal solution it is 0.326, i.e., the wind controller has considerably increased its return at the expense of the aircraft controller.

Fig. 5.36 compares the aircraft response and worst-case inputs for cases 3 and 4. The latter includes a linear mean wind while the former does not. A similar comparison was made in Fig. 5.31 for the previous example. In this example including a linear mean wind increased wind activity (for constant $\mu$), while in the previous example it did not.

Fig. 5.37 compares the aircraft response and worst-case wind inputs for cases 4, 5 and 6, where cases 5 and 6 include terminal wind velocity weightings and case 4 does not. In contrast to the second example of the direct method (Figs. 5.28 and 5.29), introducing $\Delta W(t_f)$ weightings considerably reduces the wind activity. An iterative procedure simultaneously varying $\mu$ and the terminal wind velocity weighting would be required to obtain a solution that yields a desired value of $S_{WS}(t_f)$ and $S_{W1}(t_f)$. While such a solution exists for this formulation, it was not pursued further because of the sensitivity of the solutions to $\mu$ variation in the region of greatest interest, i.e., $\mu$ near its conjugate point value.

This completes the presentation of the numerical examples of the worst-case wind modeling techniques proposed in this investigation. Some further examples for larger $t_f$ will be given in the curved glidepath approach simulations of Chapter 6.

5.6 Discussion of Functional Maximization
Worst-Case Wind Modeling Techniques

In this chapter existing worst-case wind modeling techniques have been reviewed, and a number of generalizations have been proposed for the class of methods where the form of the wind inputs is not specified a priori. A block diagram classification of these methods is given in Fig. 5.38.

Van der Vaart's method was extended to more general problems. Numerical examples of the application of the generalized method, referred to as the impulsive response worst-case (IRWC) method, have been presented.

Most of the discussion, however, has focused on methods of implementing the intelligent adversary wind modeling concept. These methods determine the worst-case disturbances as a function of the aircraft state. In particular, detailed consideration has been given to the application of linear quadratic one-sided and two-sided optimization theory to determining the worst-case wind controllers. A number of formulations of this class, with accompanying numerical examples, have been presented. These include examples of the indirect (perversity function) methods, the direct (one-sided maximization) methods, and the differential game (two-sided minimaximization) methods.

Some of the advantages and disadvantages of these methods
were apparent from the theoretical development, while others became obvious only after their application to a number of numerical examples. Still others became apparent only by considering the usefulness of these methods for specific aeronautical applications such as flight simulator wind modeling and aircraft certification. These advantages and disadvantages are summarized in the ensuing discussion.

5.6.1 General Comparison of the Worst-Case Methods

The essence of the direct and DG techniques lies in their conceptualization of the wind as an intelligent adversary that acts through a wind controller. This not only suggests worst-case behavior in a very appealing way, but also permits a diverse body of mathematical and conceptual tools to be applied to determining the worst-case wind models. In the examples optimal linear quadratic methods were employed in a number of formulations that illustrate the flexibility of the wind controller concept. The worst-case wind inputs were specified as a function of state, and thus once this control law was determined, wind disturbances were available for all possible states. In the application, the wind inputs could be treated either as being determined by the state during the simulation or as time histories specified a priori for a particular set of initial conditions.

The IRWC technique, on the other hand, determines worst-case wind disturbance time histories, i.e. it is a programming rather than a controller method. It has eliminated a number of the disadvantages of van der Vaart's method (see Section 5.2 of this chapter) at some increase in complexity (e.g. worst-case solutions for integral payoff functionals of the type (5.2,8b) are available). Absolute value payoff functionals of the form (5.2,8a) lead to solution procedures that are iterative for the general problem; the resulting worst-case disturbances will generally be discontinuous at a number of points in the time interval of interest.

The perversity function method yields worst-case solutions in closed-loop form but is, for wind modeling purposes, a programming method. This is a consequence of the minimizing nature of the formulation in which the perversity enters the problem through the specification of a suitable perversity function rather than through a maximizing wind controller (see also the discussion in Example 2 of Section 5.5.2 of this chapter).

All of the numerical examples were obtained for linear systems. Since for some cases the wind disturbances were large enough to cause substantial deviations from the linearization reference equilibrium, an assessment of the validity of the linearized dynamic model is required, particularly as to its effects on the characteristics of the worst-case wind models. In principle this may be done by comparison with results obtained for more faithful nonlinear models. This was not pursued further in this study, and is recommended for future work.
5.6.2 Specification of Wind Characteristics

A quantity of particular interest in specifying the wind variability is the integral $S_{WS}$ (see (5.5,51)). In the IIRWC method the parameter $\mu$ may be determined a priori so that a specified $S_{WS}$ value results, i.e. via Theorem 5.1 or 5.2 ($y = S_{WS}^{-1}$). In the indirect, direct and DG methods an analogous parameter $\mu$ must also be specified so that a desired $S_{WS}$ results. However, no analytical formulation exists for specifying $\mu$ a priori, and thus an iterative procedure must be used for determining a suitable value. Theorem 5.7 may be used if desired. Furthermore, for cases where the wind disturbances are determined with a wind controller, the appropriate value for $\mu$ will change for different system initial conditions.

Finding a suitable value for $\mu$ for the direct and DG methods is also complicated by the presence of a conjugate point in the time interval of interest for certain values of $\mu$. While the conjugate point may always be removed (for well posed problems) by making $\mu$ large enough, the quantity $S_{WS}$ was generally found to be very sensitive to $\mu$ variation for $\mu$ values near the conjugate point value. This was particularly true for the differential game formulations. Also, typical values for $\mu$ changed markedly from example to example.

An unusual characteristic of some of the linear quadratic direct and DG formulations is that the worst-case disturbances are zero if initial conditions are zero. This is true for problems where (1) the equations of motion do not have a forcing function $f(t)$ (see (5.4,61)) and where (2) $x_d$ in the payoff functional $J_a$ (see (5.4,84)) is zero. The worst-case wind inputs that result are zero for all $t$ since they are determined from a control law of the form

$$\Delta w(t) = \frac{1}{\mu} K(t) \Delta x(t)$$

and $\Delta x$ has no forcing input to take it out of its reference equilibrium state $\Delta x = 0$. Such a solution, of course, does not satisfy an $S_{WS}$ integral constraint, other than the trivial case $S_{WS} = 0$. In fact $\mu + \mu_{CP} (\rightarrow 0$ if no conjugate point exists) for initial conditions $\Delta x(0)$ such that $\|\Delta x(0)\| < \epsilon$, $\epsilon \rightarrow 0$, $\epsilon > 0$ and $\mu$ chosen to produce a specified $S_{WS}$ value. This singularity seems to be a property of linear quadratic formulations and does not appear resolvable without either permitting impulsive disturbances at $t = 0$ to move the aircraft to a nonzero initial state or altering the form of the payoff functional.

One may by-pass this difficulty with linear quadratic formulations by applying a number of techniques, including the following:

1. Having nonzero initial conditions, as was done in most of the examples. For practical purposes, this is not a restrictive technique in that the equilibrium $\Delta x = 0$ is a mathematical idealization and is impossible to realize exactly.
2. Formulating the problem in a way that superimposes the worst-case wind inputs determined by the optimization onto wind inputs that are specified a priori. The examples that included linear mean winds are examples of this technique. Numerous other possibilities exist, including superimposing direct method or DG method solutions onto IRWC method disturbances specified in advance.

3. Specifying nonzero $\mathbf{x}_d(t)$, i.e. the desired state space trajectories of certain components of the state are not always zero for the time interval of interest.

4. Superimposing the direct method or DG method solutions onto a worst-case solution that is obtained using a programming method for a global maximization (or minimization) of the underlying nonlinear problem from which the linearized equations of motion were obtained. This technique is conceptually appealing and involves defining the linear quadratic problem in a form that yields neighbouring solutions to the nonlinear solution. Such a procedure generally leads to a time-varying linear quadratic problem, but may be worth pursuing for certain applications (e.g. aircraft certification) as will be discussed briefly in the sequel (see also the discussion following Remark 1 of Theorem 5.7).

Brief consideration was also given to weighting the terminal values of the wind velocities in order to reduce the wind speed magnitude near the ground. The IRWC method is not well suited to finding solutions that satisfy such constraints, but in principle the linear quadratic formulations are not thus limited. The problem, however, becomes one of simultaneously determining values for several parameters, using iterative procedures, such that the $S_{\text{WS}}$ and $\Delta W(t_f)$ constraints are satisfied to a desired degree of accuracy. For certain formulations this is a computationally expensive procedure.

5.6.3 Computation Times

Execution CPU times for a number of cases are compared in Table H.1. In general the IRWC and linear quadratic programs were found to have comparable execution times for a single iteration. The linear quadratic formulations, however, require more iterations to find a desired solution.

5.6.4 Application to Flight Simulator Wind Modeling

One may, in principle, implement any wind time-history or position profile to act as a wind model on a flight simulator. Any of the worst-case techniques discussed previously, as well as a number of other nonoptimal techniques, as summarized in Chapter 2, may be used for this purpose. The wind controller methods, however, have the particular advantage of specifying the disturbing inputs as a function of state and thereby removing, to a certain degree, the predictability of a particular wind model. The probability of obtaining the same disturbance inputs for a particular flight
is of the same order as the probability that the state trajectory for that flight is the same as for any other. Furthermore, worst-case wind controller models tend to penalize poor pilot performance, a feature that has interesting implications for flight training.

A preliminary assessment of these wind generation models for flight simulation purposes is currently in progress at UTIAS on a three-degree-of-freedom fixed-base simulation of the STOL transport used in the numerical examples.* One of the wind models that has been implemented in this study is the infinite terminal time version of the differential game wind controller discussed in Example 1 of Section 5.5.4 of this chapter. Some of the simulation results for this model are currently available, and will be presented in the following in order to provide an example of a human pilot flying against an optimal wind controller.

The simulation dynamics and aerodynamics were linearized about the following reference conditions:

\[ V_e = 40 \text{ ms}^{-1} \]  
\[ Y_G = 7^\circ \]  
\[ W_{1e} = 0 \].

The flight task consisted of a precision instrument approach tracking a 7° glidepath, with glidepath intercept occurring at an altitude of 460m (1500 ft.) above ground level (AGL). The wind model was engaged at an altitude of 365m (1200 ft.) AGL. Wind speed limiters were included that prevented the wind magnitude from exceeding certain maximum profiles that were specified in advance. These limiters were engaged smoothly, i.e. \( \Delta W_1 \) and \( \Delta W_3 \) did not have discontinuities. The worst-case wind inputs were superimposed onto a linear headwind model that was specified a priori.

The controls and instruments that were available to the pilot consisted of throttle, elevator, pitch trim, airspeed indicator, altimeter, \% power indicator, fast-slow bug, glidepath deviation bug and an artifical horizon.

The pilot for these runs was an experienced IFR rated engineering test pilot, with considerable pilot in command time on aircraft of the type simulated.

The pilot was asked to continue the approach as best as able to a decision height of 60m (100 ft.). These instructions included situations where he would normally have executed a go-around.

The airspeed response, glidepath deviation, and worst-case wind profiles for five runs are given in Figs. 5.39, 5.40, and 5.41. A summary of some important root-mean-square values is given in Table 5.4, as normalized with respect to the minimum value for each variable presented.

The most notable feature of these results is the different

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* The complete results of the study will be presented in Ref. 5.30.
horizontal wind inputs that result from run to run. Furthermore, the wind variability, as measured by $S_{WS}$, also changes markedly. The wind velocity changes tend to be low frequency in nature, a characteristic which is to be expected from such wind models, i.e. they will tend to stimulate the more weakly damped phugoid mode of the pilot-aircraft system.

The $W_3$ wind component tends to increase to its limiting value, and then follow the limiter velocity profile as altitude decreases. This, too, is not totally unexpected in that $W_3$ cost effectively influences glidepath deviation (kinematically) while $W_1$ more cost effectively influences airspeed disturbances (dynamically through $W_1$). In pursuing these objectives $W_1$ will show more oscillatory behavior than $W_3$.

For all five runs the pilot felt the wind inputs were useful for training purposes. In general the wind inputs were assessed as difficult to fly in the context of the flight task at hand. In four of the runs the pilot commented he would have executed a go-around at an altitude of approximately 150m (500 ft.).

In this relatively straightforward implementation of a wind controller model, the pilot is in effect flying an aircraft with modified stability, where the wind controller will in general be destabilizing. With enough runs against a given wind controller, the pilot might be expected to adapt to the model, and improve his performance. He will, however, obtain this improved performance at some increase in workload.

Such learning of the model is at a different level from learning to recognize a discrete wind input. Since these wind controller models are ultimately intended for hazard definition, the pilot will be confronted with them only occasionally (say once or twice in a given session), rather than continuously, where he would eventually adapt to flying a modified dynamic system. Since for every encounter the wind inputs are different (cf. discrete wind models), there is a lower likelihood that the pilot would recognize a particular wind controller model than there is of him recognizing a discrete wind model. In more sophisticated models, one may also randomize the level of destabilization or link it to a real time, objective measure of pilot workload.

5.6.5 Potential Application to Autoland Certification*

Certification wind models currently in use include both power spectral density (e.g. Dryden spectra) and discrete (e.g. 1-cosine type gust) models. For example, these have been applied to the structural certification of aircraft.

The advent of autoland systems has made it imperative to also define wind models for autopilot certification. The FAA is currently accumulating data and proposals that will

* The author is indebted to Professor B. Etkin of UTIAS for a number of discussions held on this topic.
ultimately lead to the specification of a wind model acceptable for this purpose. These will improve on the very basic model that is summarized in Advisory Circular 20-57A [5.31], and which is acceptable as a means of installation compliance for Category II operations.

Of considerable importance to this autoland assessment are wind models that realistically define hazardous wind activity. The optimal worst-case techniques discussed in this work seem naturally suited to such an application. Other techniques, such as Jone's statistical discrete gust method [1.69, 5.32], are also promising.

To obtain a clearer insight into how the worst-case techniques may be better than current methods, a brief description of the application of the wind model of AC-57A should first be given. The model provides statistical distributions for wind speeds, and power spectra for turbulence. These may be used to generate a large number of realizations of the wind field which are then inputted into a nonlinear dynamic model of the aircraft-autopilot system to obtain the aircraft response to each wind realization. The statistics of the aircraft response are determined from a "large enough" collection of responses. Particular realizations that are observed to result in hazardous behavior may be used to define hazardous discrete wind models for the system. In general a nonlinear dynamic model is required because of the need to realistically assess the worst-cases, i.e. cases that lead to large deviations of the aircraft from its dynamic and aerodynamic reference equilibria.

Such a Monte Carlo technique is laborious and computationally expensive. A significant improvement would result if one could develop a method that reduces the number of simulations required, e.g. with one of the worst-case techniques.

To indicate how one might proceed in this regard with the functional maximization methods discussed in this study, a candidate procedure is outlined for an autoland assessment. This is as follows:

1. Obtain a suitable data base for typical and extreme levels of wind variability (e.g. as measured by $S_{WS}$).

2. Define a payoff functional whose minimization suitably and realistically reflects the autoland task. This step is iterative, and may involve validation, with Monte Carlo techniques, of classes of payoff functionals for different aircraft categories.

3. Define a dynamic model of the aircraft and autoland controller that suitably models the aircraft response in large disturbance cases.

4. Use functional maximization theory to obtain a wind model maximizing the payoff functional specified in Step 2 for a typical set of initial conditions. This worst-case solution should be constrained by the wind variability levels determined in Step 1, as well as by whatever other wind constraints are found necessary for a realistic solution.
5. Use the worst-case solution of the previous step to define a time-varying linear problem about which neighbouring extremals may be found. These neighbouring solutions may be expressed in wind controller form, and may be obtained using the linear quadratic theory discussed in this chapter (also, see Theorem F.7 of Appendix F) once the nonlinear solution is known for one set of initial conditions.

6. Evaluate the autopilot performance in the presence of this worst-case wind model. If it is unsatisfactory, return to step 3, and introduce a modified autoland control algorithm. Repeat steps 3, 4, 5 and 6 until a suitable autoland controller is found.

This procedure involves the difficult task of obtaining a solution to a nonlinear maximization problem with all the attendant complexities that the nonlinearity may introduce, i.e. local maxima, discontinuities, and so forth. Nevertheless, in view of the already complex and laborious nature of the Monte Carlo technique, such methods may still reduce the amount of computation required, and are consequently worth pursuing further so that a complete, rational evaluation of their usefulness to the certification process may be made.
REFERENCES — CHAPTER 5


5.5 Baron, Sheldon, Differential Games and Manual Control, NASA TN D-3659, October 1966.


### TABLE 5.1

**PAYOFF FUNCTION CLASSIFICATION USED IN EXAMPLES**

<table>
<thead>
<tr>
<th>TYPE</th>
<th>PAYOFF FUNCTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[ J = \int_0^\infty \left( | \Delta x_{OP}(t) |<em>Q^2 + | \Delta \delta |</em>{R_1}^2 \right) dt ]</td>
</tr>
<tr>
<td></td>
<td>[ q_{ij} = \frac{1}{\Delta x_{OP_{\text{max},i}}} \delta_{ij} ; \quad r_{11} = \frac{1}{\Delta \delta_{\text{max},i}} \delta_{ij} ]</td>
</tr>
</tbody>
</table>

| 2    | \[ J = \int_0^\infty \left( \| \Delta x_{OP}(t) \|_Q^2 + \| \Delta \delta \|_{R_{11}}^2 \right) dt \] |
|      | \[ q_{ij} = \frac{1}{\Delta x_{OP_{\text{max},i}}} \delta_{ij} ; \quad r_{21} = \frac{1}{\Delta \delta_{\text{max},i}} \delta_{ij} \] |

| 3    | \[ J_a = \| \Delta x_{OP}(t_f) \|_Q \delta_{ij}^2 + \left( \| \Delta x_{OP}(t_f) - \Delta x_{OP}(t) \|_Q \right)^2 \] |
|      | \[ + \| \Delta \delta_{\text{max},i} \|_{R_{21}} \delta_{ij}^2 \] |
|      | \[ + \mu \| \Delta \delta_{\text{max},i} \|_{R_{21}}^2 \delta_{ij}^2 \] |
|      | \[ s_{ij} = \frac{1}{\Delta x_{OP_{\text{max},i}}} \delta_{ij} ; \quad q_{ij} = \frac{1}{\Delta \delta_{\text{max},i}} \delta_{ij} \] |
|      | \[ r_{21} = \frac{1}{\Delta \delta_{\text{max},i}} \delta_{ij} \] |

The weighting matrices are defined using a technique similar to that suggested by Bryson and Ho [4.16]. \( \delta_{ij} \) is the Kronecker delta function, \( t_w \) is a time that roughly corresponds to the dominant time constant of the system, and \( \Delta x_{OP_{\text{max},i}} \).
\( \Delta \delta_{\text{max}} \), etc. are the maximum acceptable magnitudes of the components of the corresponding vectors. Because of system coupling and inherent stability, not all of the state variables need be weighted, i.e. some of the components of \( \Delta \delta_{\text{max}} \) may be infinite. \( \text{sgn}(s_{ij}) \) and \( \text{sgn}(q_{ij}) \) characterize the nature of the solution. In this study, wind related quantities will generally be weighted negatively, and vice versa for aircraft state variables. This is compatible with the minimizing nature of the aircraft controller and the maximizing nature of the wind disturbances. Because of the \( \delta_{ij} \) terms, \( \text{sgn}(s_{ij}) \) and \( \text{sgn}(q_{ij}) \) need only be defined for the diagonal elements of \( \mathbf{S} \) and \( \mathbf{Q} \) respectively, i.e. only for \( i = j \).

### TABLE 5.1 (continued)

<table>
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<tr>
<th>Controller Designation / Computer Program</th>
<th>( \Delta \delta_{\text{OP}} )</th>
<th>( \Delta \delta )</th>
<th>( J )</th>
<th>( t_\text{w} )</th>
<th>( \Delta \delta_{1} )</th>
<th>( \Delta \delta_{1,\text{max}} )</th>
<th>( \Delta \delta_{\text{max}} )</th>
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<td>[( \Delta u ) ]</td>
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<td>0.35rad.</td>
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<td></td>
<td></td>
<td>( \Delta \delta )</td>
<td></td>
<td></td>
<td>[( \Delta w ) ]</td>
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<td></td>
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<td>( \Delta \delta )</td>
<td></td>
<td></td>
<td>[( \Delta q ) ]</td>
<td>=</td>
<td></td>
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<td></td>
<td></td>
<td>( \Delta \delta )</td>
<td></td>
<td></td>
<td>[( \Delta \theta ) ]</td>
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<td></td>
<td></td>
<td>( \Delta \delta )</td>
<td></td>
<td></td>
<td>[( \Delta \phi ) ]</td>
<td>5.0m</td>
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<td></td>
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<td>( \Delta \delta )</td>
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<td>20.5s</td>
<td>[( \Delta u ) ]</td>
<td>[5.0ms(^{-1})]</td>
<td>0.35rad.</td>
<td>[0.07rad./s(^{-1})]</td>
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<td></td>
<td></td>
<td>( \Delta \delta )</td>
<td></td>
<td></td>
<td>[( \Delta w ) ]</td>
<td>2.0ms(^{-1})</td>
<td>0.2</td>
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<td></td>
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<td>( \Delta \delta )</td>
<td></td>
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<td>[( \Delta q ) ]</td>
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<td></td>
<td></td>
<td>( \Delta \delta )</td>
<td></td>
<td></td>
<td>[( \Delta \theta ) ]</td>
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<tr>
<td></td>
<td></td>
<td>( \Delta \delta )</td>
<td></td>
<td></td>
<td>[( \Delta \phi ) ]</td>
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<td>20.5s</td>
<td>[( \Delta u ) ]</td>
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<td>( \Delta \delta )</td>
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<td>[( \Delta \phi ) ]</td>
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N/A = Not Applicable.
### TABLE 5.2 (continued)

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<th>$\Delta \delta$</th>
<th>$J$</th>
<th>$t_w$</th>
<th>$\Delta \phi_1$</th>
<th>$\Delta \phi_{\text{max}}$</th>
<th>$\Delta \phi_{\text{max}}$</th>
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<td>$[0.35 \text{rad.}]$</td>
<td>$[0.2 \text{rad.}/\text{s}]$</td>
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<td>$[\Delta \delta_T]$</td>
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<td></td>
<td>$[\Delta \phi]$</td>
<td>$[2.0 \text{ms}^{-1}]$</td>
<td>$[0.1 \text{rad.}]$</td>
<td>$[0.5 \text{ms}^{-1}]$</td>
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</table>

### TABLE 5.3

WEIGHTING CONDITIONS FOR LINEAR QUADRATIC WORST-CASE WIND MODELING EXAMPLES

<table>
<thead>
<tr>
<th>EXAMPLE</th>
<th>$\Delta \phi_{\text{OP}}$</th>
<th>$\Delta \phi_{\text{max}}$</th>
<th>$\Delta \phi_{\text{max}}$</th>
<th>$\Delta \phi_{\text{max}}$</th>
<th>$\Delta \phi_{\text{max}}$</th>
<th>$\Delta \phi_{\text{max}}$</th>
<th>$\Delta \phi_{\text{max}}$</th>
<th>$\Delta \phi_{\text{max}}$</th>
<th>$\Delta \phi_{\text{max}}$</th>
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</thead>
<tbody>
<tr>
<td>INDIRECT METHOD, EXAMPLE 1</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
</tr>
<tr>
<td>3</td>
<td>20.5s</td>
<td>5.0ms</td>
<td>0.35rad.</td>
<td>0.5</td>
<td>1.2rad./s</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
</tr>
<tr>
<td>INDIRECT METHOD, EXAMPLE 2</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
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<td>$[1.0 \text{ms}^{-2}]$</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
</tr>
<tr>
<td>DIRECT METHOD, EXAMPLE 1</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
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<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
</tr>
<tr>
<td>4</td>
<td>20.5s</td>
<td>5.0ms</td>
<td>0.35rad.</td>
<td>0.5</td>
<td>1.2rad./s</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
</tr>
<tr>
<td>DIRECT METHOD, EXAMPLE 2</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
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<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
<td>$[\Delta \phi]$</td>
</tr>
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<td>4</td>
<td>20.5s</td>
<td>5.0ms</td>
<td>0.35rad.</td>
<td>0.5</td>
<td>1.2rad./s</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
<td>$[1.0 \text{ms}^{-2}]$</td>
</tr>
<tr>
<td>Run</td>
<td>$\Delta \hat{u}/\Delta \hat{u}_{\text{min}}$</td>
<td>$\Delta \dot{\delta}<em>{p}/\Delta \dot{\delta}</em>{p,\text{min}}$</td>
<td>$\Delta \hat{\delta}<em>{E}/\Delta \hat{\delta}</em>{E,\text{min}}$</td>
<td>$S_{\text{WS}}/S_{\text{WS,\text{min}}}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-----</td>
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<td></td>
<td></td>
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<tr>
<td>1</td>
<td>1.72</td>
<td>1.11</td>
<td>1</td>
<td>1.35</td>
<td>2.63</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>2</td>
<td>1.22</td>
<td>1</td>
<td>1.09</td>
<td>1.01</td>
<td>1.58</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1.26</td>
<td>1.32</td>
<td>1.12</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1.05</td>
<td>1.22</td>
<td>1.30</td>
<td>1</td>
<td>1.52</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.22</td>
<td>1.03</td>
<td>1.19</td>
<td>1.14</td>
<td>1.40</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\Delta \hat{u}_{\text{min}} = 1.08 \text{ms}^{-1}$

$\Delta \dot{\delta}_{p,\text{min}} = 9.52 \text{ms}^{-1}$

$\Delta \hat{\delta}_{E,\text{min}} = 0.165$

$\Delta \hat{\delta}_{E,\text{min}} = 0.270 \text{rad.}$

$S_{\text{WS,\text{min}}} = 5.29 \text{m}^2\text{s}^{-3}$

$S_{\text{WS}} = \int_0^T (W_1^2 + W_2^2) \text{d}t$

\[ ^* \text{ See TABLE 5.1.} \]

\[ ^* \text{ See TABLE 5.1.} \]

\[ ^* \text{ in this column refers only to those terms of } \mathbf{a} \text{ that weight } \Delta \hat{u}_{\text{min}}(\mathbf{a}). \]

\[ ^* \text{ All other } \Delta \hat{u}_{\text{min}}(\mathbf{a}) = 1. \text{ All } \Delta \hat{u}_{\text{min}}(\mathbf{a}) = 1. \]

\[ ^* \text{ N/A = Not Applicable.} \]
FIG. 5.1 MAGNITUDE AND INTEGRAL CONSTRAINED DISTURBANCE INPUTS THAT MAXIMIZE $x_i(c_j)$

$\eta_K = \text{sgn}(h_K) K_K$

$\eta_K = h_K(t_f - t)$

FIG. 5.2 CONFLICT OF INTEREST WIND AND AIRCRAFT CONTROLLER MODELING: THE INTELLIGENT ADVERSARY CONCEPT

Switches A, B and C are included to indicate that not all of the feedbacks are necessary in the conflict of interest scenario.
FIG. 5.3a: A DISCRETE GAME EXAMPLE WITH A
MINIMAX SOLUTION
Bryson and Ho [4.16]

<table>
<thead>
<tr>
<th>$u_1$</th>
<th>$u_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{11} = 4$</td>
<td>$g_{12} = 7$</td>
</tr>
<tr>
<td>$g_{21} = 5$</td>
<td>$g_{22} = 9$</td>
</tr>
</tbody>
</table>

$J = g(u,v) = \frac{1}{2}(u^2 - \eta^2); \quad -1 \leq u \leq 1; \quad -1 \leq \eta \leq 1$,

$\frac{\partial g}{\partial u} = 0, \quad \frac{\partial g}{\partial \eta} = 0 \Rightarrow u^* = 0, \quad \eta^* = 0$,

$\frac{\partial^2 g}{\partial u^2} = 1 > 0, \quad \frac{\partial^2 g}{\partial \eta^2} = -1 < 0, \quad \frac{\partial^2 g}{\partial u \partial \eta} = \frac{(\frac{\partial^2 g}{\partial u \partial \eta})^2}{\frac{\partial^2 g}{\partial \eta^2}} = -1 < 0$.

FIG. 5.4a: EXAMPLE OF BOTH A GAME-THEORETIC SADDLE POINT
AND A CALCULUS SADDLE POINT
Bryson and Ho [4.16].

FIG. 5.3b: A DISCRETE GAME EXAMPLE WITHOUT
A MINIMAX SOLUTION
Bryson and Ho [4.16]

$J = g = u^2 - 3u_1 + 2\eta^2; \quad -1 \leq u \leq 1; \quad -1 \leq \eta \leq 1$,

$\frac{\partial g}{\partial u} = 0, \quad \frac{\partial g}{\partial \eta} = 0 \Rightarrow u^* = 0, \quad \eta^* = 0$,

$\frac{\partial^2 g}{\partial u^2} = 1 > 0, \quad \frac{\partial^2 g}{\partial \eta^2} = -1 < 0$,

but $\frac{\partial^2 g}{\partial u \partial \eta} = \frac{\partial^2 g}{\partial \eta^2} = 4 > 0$.

FIG. 5.4b: EXAMPLE OF A CALCULUS SADDLE POINT WHICH IS
NOT A GAME-THEORETIC SADDLE POINT
Bryson and Ho [4.16].
\[ \gamma_6 = 7^\circ \quad V_e = 40 \text{ ms}^{-1} \quad \text{Wie} = 0 \]

Open-Loop

Zero IC

\[ \Delta u (\text{ms}^{-1}) \]

\[ W_1 (\text{ms}^{-2}) \]

\[ W_1 (\text{ms}^{-2}) \]

**SYMBOL** | **SwS (m^2 s^{-3})** | **\sqrt{SwS/t_f (ms^{-2})}**
---|---|---
\( \Delta \) | 20 | 1.0
\( \circ \) | 10 | 0.7
\( \square \) | 5 | 0.5

\[ J = \Delta u(t_f) \]

**FIG. 5.5** IRWC METHOD, EXAMPLE 1
\[ \gamma_G = 7^\circ \]
\[ V_e = 40 \text{ ms}^{-1} \]
\[ \text{Open-Loop} \]
\[ \text{Zero IC} \]

\[ \Delta u \quad (\text{ms}^{-1}) \]

\[ W_i \quad (\text{ms}^{-1}) \]

\[ J = \int_0^{t_f} \Delta u(t) \, dt \]
\[ Y_G = 7^\circ \quad W_{le} = 0 \quad S_{WS} = 5 \, m^2 s^{-3} \]
\[ V_e = 40 \, ms^{-1} \quad W_3 = 0 \quad \sqrt{S_{WS}/t_f} = 0.5 \, ms^{-2} \]

\[ \Delta u \quad (ms^{-1}) \]

\[ t (sec) \]

- \( \beta_{f_1} = 1, \beta_{f_7} = 0.625 \) (Case 1a)
- \( \beta_{f_1} = -1, \beta_{f_7} = 0.625 \) (Case 1b)

\[ d \quad (m) \]

\[ t (sec) \]

\[ W_l \quad (ms^{-1}) \]

\[ t (sec) \]

**FIG. 5.7** IRWC METHOD, EXAMPLE 3
\[ \gamma_G = 7^\circ \]
\[ W_{le} = 0 \]
\[ S_{ws} = 5 \text{ m}^2\text{s}^{-3} \]
\[ V_e = 40 \text{ ms}^{-1} \]
\[ \sqrt{S_{ws}/t_f} = 0.5 \text{ ms}^{-2} \]

\[ \Delta u \quad (\text{ms}^{-1}) \]

\[ t \quad (\text{sec}) \]

\[ \square \beta_{f_1} = 1, \quad \beta_{f_7} = 0.625 \quad \text{(Case 2a)} \]

\[ \circ \beta_{f_1} = -1, \quad \beta_{f_7} = 0.625 \quad \text{(Case 2b)} \]

\[ W_1 \quad (\text{ms}^{-1}) \]

\[ W_3 \quad (\text{ms}^{-1}) \]

FIG. 5.8 IRWC METHOD, EXAMPLE 3
FIG. 5.9 APPROACH PATHS IN PRESENCE OF IRWC WIND INPUTS

- $V_0 = 40 \text{ ms}^{-1}$
- $\gamma_0 = 7^\circ$
- $S_{WS} = 5 \text{ m}^2\text{s}^{-3}$
- $\sqrt{S_{WS}/T_f} = 0.5 \text{ ms}^{-2}$

GLIDEPATH

- $W_0 = 0$, Closed-loop (Case 1a)
- $W_1$ and $W_3$ Inputs, Closed-loop (Case 2a)
- $W_1$ and $W_3$ Inputs, Open-loop Analogue of Case 2a

FIG. 5.10 AIRSPEED RESPONSE FOR A NUMBER OF IRWC WIND INPUTS

- $\beta_1 = 1$, $W_3 = 0$, Open-loop
- $\beta_1 = 1$, $W_3 = 0$, Closed-loop
- $\beta_1 = 1$, $\beta_1 = \beta_7 = 0.625$, Closed-loop
- $\beta_1 = 1$, Closed-loop

$V_0 = 40 \text{ ms}^{-1}$
$\gamma_0 = 7^\circ$
$W_{10} = 0$
$S_{WS} = 5 \text{ m}^2\text{s}^{-3}$

$\sqrt{S_{WS}/T_f} = 0.5 \text{ ms}^{-2}$
Figure 5.11 IRWC Wind Inputs for a Number of Cases

- $V_e = 40 \text{ ms}^{-1}$
- $\gamma = 7^\circ$
- $W_3 = 0$
- $SW_3/tf = 0.5 \text{ ms}^{-2}$

Figure 5.12 Approach Paths in the Presence of IRWC Wind Inputs

- $V_e = 40 \text{ ms}^{-1}$
- $\gamma = 7^\circ$
- $W_3 = 0$
- $SW_3/tf = 0.5 \text{ ms}^{-2}$
\[ \Delta u \quad (\text{ms}^{-1}) \]

\[ \Delta w \quad (\text{ms}^{-1}) \]

\[ \Delta q \quad (\text{rad/s}) \]

\[ \Delta \theta \quad (\text{rad}) \]

\[ \chi_G = 7.0^\circ \quad \text{Open-Loop} \]

\[ V_e = 40.0 \text{ ms}^{-1} \quad \mu = 0.05 \]

\[ * \; \chi_p(t) \quad \text{Example 1} \]

**Fig. 5.13** Aircraft response for indirect method, Example 1
\[ \begin{align*}
V_e &= 40 \text{ ms}^{-1} \\
\chi_G &= 7^\circ \\
W_{1e} &= 0 \\
\mu &= 0.05
\end{align*} \]

Open-Loop

**FIG. 5.14** WORST-CASE \( W_1 \) AND \( W_3 \) TIME HISTORIES FOR INDIRECT METHOD, EXAMPLE 1

**FIG. 5.15** WORST-CASE \( W_1 \) TIME HISTORY FOR CASE WHERE \( W_3=0 \), INDIRECT METHOD, EXAMPLE 1
**FIG. 5.16** AIRCRAFT RESPONSE FOR INDIRECT METHOD, EXAMPLE 2

- $V_e = 40 \text{ ms}^{-1}$
- $\gamma_e = 7^\circ$
- $\mu = 0.05$

**FIG. 5.17** WIND PROFILES FOR INDIRECT METHOD, EXAMPLE 2

- $V_e = 40 \text{ ms}^{-1}$
- $\gamma_e = 7^\circ$
- $W_{1e} = W_{2e} = 0$
- $\mu = 0.05$
\( \gamma_{G} = 7^\circ \)  \( W_{e} = 0 \)

\( V_{e} = 40 \text{ ms}^{-1} \)  \( W_{3} = 0 \)

Open-Loop

\[ \Delta u \quad (\text{ms}^{-1}) \]

\( \Delta w \quad (\text{ms}^{-1}) \)

\[ \Delta \theta \quad (\text{rad}) \]

\( t (\text{sec}) \)

\( \square \mu = 16.0 \)  \( \triangle \mu = 11.2 \)

\( \circ \mu = 12.0 \)  \( + \mu = 10.9 \)

FIG. 5.18  DIRECT METHOD, EXAMPLE 1
FIG. 5.18 (continued)  DIRECT METHOD, EXAMPLE 1

\[ \Delta w_1 \quad (\text{ms}^{-2}) \]

\[ W_1 \quad (\text{ms}^{-1}) \]

\[ S_{WS}(t) = \int_{0}^{t} \Delta w_1(\tau) \, d\tau \]

\[ S_{WS} \quad (\text{m}^2\text{s}^{-3}) \]
FIG. 5.19  APPROACH PATHS IN PRESENCE OF DIRECT METHOD
WORST-CASE WIND INPUTS, EXAMPLE 1

$$V_e = 40 \text{ ms}^{-1}$$
$$\theta_0 = 7^\circ$$
$$W_3 = 0$$
$$W_{18} = 0$$

--- GLIDEPATH

FIG. 5.20  PHUGOID MODE ROOT LOCUS WITH MAXIMIZING WIND CONTROLLER
Direct method, example 1

FIG. 5.21  VARIATION OF $$S_{WS}(t_f)$$ WITH $$\mu$$
Direct method, example 1
\[ V_e = 40 \text{ ms}^{-1} \]
\[ \gamma_G = 7^\circ \]
\[ W_{le} = 0 \]
\[ W_3 = 0 \]

**Table**

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>⧫</td>
<td>16.0</td>
</tr>
<tr>
<td>⧫</td>
<td>12.0</td>
</tr>
<tr>
<td>△</td>
<td>11.2</td>
</tr>
<tr>
<td>×</td>
<td>10.9</td>
</tr>
<tr>
<td>+</td>
<td>10.6</td>
</tr>
</tbody>
</table>

**Figure 5.22**  
\( \text{tr} \{ P \} \) FOR SEVERAL VALUES OF \( \mu \) DIRECT METHOD, EXAMPLE 1

**Figure 5.23**  
PHUGOID MODE ROOT LOCUS WITH MAXIMIZING WIND CONTROLLER  
DIRECT METHOD, EXAMPLE 1
FIG. 5.24 WORST-CASE WIND ALTITUDE PROFILES, DIRECT METHOD, EXAMPLE 1

FIG. 5.25 WORST-CASE WIND MODELS EXPRESSED IN TERMS OF $h_e$, DIRECT METHOD, EXAMPLE 1
\[ \Delta u \quad (\text{ms}^{-1}) \]

\[ \Delta W \quad (\text{ms}^{-1}) \]

\[ \Delta q \quad (\text{rad/s}) \]

\[ \Delta \theta \quad (\text{rad}) \]

\[ \gamma_g = 7^\circ \quad W_{le} = 0 \quad W_3 = 0 \]

\[ \mu = 10.9 \quad \text{Optimal Controller} \]

\[ + \quad \text{Open-Loop} \]

\[ \mu = 10.6 \]

**FIG. 5.26 DIRECT METHOD, EXAMPLE 1**
FIG. 5.26 (continued) DIRECT METHOD, EXAMPLE 1
\[ V_e = 40 \text{ ms}^{-1} \quad W_1e = 0 \quad W_3 = 0 \]
\[ \gamma_g = 7^\circ \quad \text{Open-Loop} \]
\[ h_e = V_e \sin \theta_e \ t + h_0 \]

**FIG. 5.27** DIRECT METHOD, EXAMPLE 2

WIND PROFILES

Effect of terminal weighting on \( w_1(t_f) \) for small values of \( S_{ws} \).
$V_e = 40 \text{ ms}^{-1}$ \hspace{1cm} $W_{1e} = 0$ \hspace{1cm} $W_3 = 0$

$\gamma_g = 7^\circ$ \hspace{1cm} Open-Loop \hspace{1cm} $h_e = V_e \sin \theta_e + h_o$

![Diagram](image)

**FIG. 5.28** DIRECT METHOD, EXAMPLE 2

**WIND PROFILES**

Effect of terminal weighting on $w_1(t_f)$ for large values of $S_{ws}$. 

$S_{ws} = 4.8 \text{ m}^2\text{s}^{-3}$

$S_{ws} = 6.6 \text{ m}^2\text{s}^{-3}$

- **NO TERMINAL WEIGHTING,**
  - $\mu = 11.2$
- **TERMINAL WEIGHTING,**
  - $\mu = 10.6$
FIG. 5.29 COMPARISON OF APPROACH PATHS IN PRESENCE OF DIRECT METHOD WORST-CASE WIND INPUTS FOR EXAMPLES 1 AND 2

Symbol: Terminal Weighting $\mu$ and $S_{WS}$

- **NO** 12.0 0.77
- **NO** 11.2 4.8
- **YES** 11.3 1.6
- **YES** 10.6 6.6

$V_e = 40 \text{ ms}^{-1}$
$\gamma_G = 7^\circ$
$W_3 = 0$
$W_{1e} = 0$

--- GLIDEPATH
\[
\begin{align*}
\gamma_G &= 7^\circ &\omega_{le} &= 0 \\
V_e &= 40 \text{ ms}^{-1} &\omega_L &= 0
\end{align*}
\]

\[
\Delta u \quad (\text{ms}^{-1})
\]

\[
\Delta w \quad (\text{ms}^{-1})
\]

\[
\Delta q \quad (\text{rad/s})
\]

\[
\Delta \theta \quad (\text{rad})
\]

\[\mu = 0.7584 \quad \text{(Case 1)}\]
\[\mu = 0.7424 \quad \text{(Case 2)}\]

FIG. 5.30 DG METHOD, EXAMPLE 1
\( \gamma_E = 7^\circ \)  \( W_{le} = 0 \)

\( V_e = 40 \text{ ms}^{-1} \)  \( W_L = 0 \)

\[ \Delta \delta_E \text{ (rad)} \]

\[ \begin{array}{c}
\square \mu = 0.7584 \quad \text{(Case 1)} \\
\circ \mu = 0.7424 \quad \text{(Case 2)}
\end{array} \]

\[ \Delta \delta_T \]

\[ W_1 \text{ (ms}^{-1}) \]

\[ W_3 \text{ (ms}^{-1}) \]

FIG. 5.30 (continued)  DG METHOD, EXAMPLE 1
\[ \gamma_L = 7^\circ \quad W_{le} = 0 \]

\[ \nu_L = 40 \text{ ms}^{-1} \quad \mu = 0.7424 \]

\[ \Delta u \quad (\text{ms}^{-1}) \]

- \( \square \) \( W_L \neq 0 \) (Case 3)
- \( \circ \) \( W_L = 0 \) (Case 2)

\[ \Delta w \quad (\text{ms}^{-1}) \]

\[ \Delta q \quad (\text{rad/s}) \]

\[ \Delta \Theta \quad (\text{rad}) \]

FIG. 5.31 DG METHOD, EXAMPLE 1
\[ \gamma_G = 7^\circ \]
\[ V_e = 40 \text{ ms}^{-1} \]
\[ W_{le} = 0 \]

\[ \Delta \delta_E \] (rad)

- \( W_L \neq 0 \) (Case 3)
- \( W_L = 0 \) (Case 2)

\[ \Delta \delta_T \]

\[ W_1 \] (ms\(^{-1}\))

\[ W_3 \] (ms\(^{-1}\))

FIG. 5.31 (continued) DG METHOD, EXAMPLE 1
\[ \text{tr}\{P\}/10 \]

\[ \gamma_g = 7^\circ \]
\[ V_e = 40 \text{ ms}^{-1} \]
\[ W_{le} = 0 \]

- \( \mu = 0.7584 \) (Case 1)
- \( \mu = 0.7424 \) (Case 2)

**FIG. 5.32 tr P(t) FOR TWO VALUES OF \( \mu \)**

DG METHOD, EXAMPLE 1
\[ \Delta u \quad (\text{ms}^{-1}) \]

\[ \Delta w \quad (\text{ms}^{-1}) \]

\[ \Delta q \quad (\text{rad/s}) \]

\[ \Delta \theta \quad (\text{rad}) \]

\[ \gamma_G = 7^\circ \quad \text{W_1e = 0} \quad \text{W_L = 0} \]

\[ V_e = 40 \text{ ms}^{-1} \quad W_3 = 0 \]

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>□</th>
<th>○</th>
<th>△</th>
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</thead>
<tbody>
<tr>
<td>CASE</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>( \mu )</td>
<td>2</td>
<td>1</td>
<td>0.752</td>
</tr>
</tbody>
</table>

FIG. 5.33  DG METHOD, EXAMPLE 2
\[ \gamma_g = 7^\circ \quad W_{le} = 0 \quad W_L = 0 \]

\[ V_e = 40 \text{ ms}^{-1} \quad W_3 = 0 \]

\[ \Delta d \quad (\text{m}) \]

\[ \Delta \delta_E \quad (\text{rad}) \]

\[ \Delta \delta_T \]

\[ W_l \quad (\text{ms}^{-1}) \]

**FIG. 5.33 (continued) DG METHOD, EXAMPLE 2**
$V_e = 40 \text{ ms}^{-1}$
$
\gamma_G = 7^\circ$
$W_{le} = 0$
$W_3 = 0$

- $\mu = 2$ (Case 1)
- $\mu = 1$ (Case 2)
- $\mu = 0.752$ (Case 3)

**FIG. 5.34** \( \text{tr} \{ P(t) \} \) FOR THREE VALUES OF \( \mu \)

DG METHOD, EXAMPLE 2
\[ \gamma_g = 7^\circ \]

\[ V_e = 40 \text{ ms}^{-1} \]

\[ W_{fe} = 0 \]

\[ \mu = 2 \]

\[ \Delta u \quad (\text{ms}^{-1}) \]

\[ \Delta w \quad (\text{ms}^{-1}) \]

\[ 10 \Delta q \quad (\text{rad/s}) \]

\[ \text{FIG. 5.35 DG METHOD, EXAMPLE 2} \]
$\gamma_G = 7^\circ$

$V_e = 40 \text{ ms}^{-1}$

$W_{le} = 0 \quad \mu = 2$

- DG Optimization (Case 1)
- Worst-Case Wind Controller
- Optimal Aircraft Controller

**FIG. 5.35 (continued) DG METHOD, EXAMPLE 2**
\[ \gamma_g = 7^\circ \quad W_{ie} = 0 \quad \mu = 0.752 \]
\[ V_e = 40 \text{ ms}^{-1} \quad W_3 = 0 \]

\[ \Delta u \quad (\text{ms}^{-1}) \]

\[ W_L \neq 0 \quad (\text{Case 4}) \]
\[ W_L = 0 \quad (\text{Case 3}) \]

\[ \Delta w \quad (\text{ms}^{-1}) \]

\[ \Delta q \quad (\text{rad/s}) \]

\[ \Delta \theta \quad (\text{rad}) \]

**FIG. 5.36**  DG METHOD, EXAMPLE 2
\[ \gamma_G = 7^\circ \quad W_{1e} = 0 \quad \mu = 0.752 \]
\[ V_e = 40 \text{ ms}^{-1} \quad W_3 = 0 \]

\[ \Delta d \quad (m) \]

\[ W_L \neq 0 \quad \text{(Case 4)} \]
\[ W_L = 0 \quad \text{(Case 3)} \]

\[ \Delta \dot{W}_l \quad (\text{ms}^{-2}) \]

\[ W_l \quad (\text{ms}^{-1}) \]

\[ S_{\text{WS}(t)} = \int_{t_0}^{t} \Delta \dot{W}_l^2(\tau) \, d\tau \]

\[ S_{\text{WS}(t)} \quad \text{m}^2 \text{s}^{-3} \]

FIG. 5.36(continued)  DG METHOD, EXAMPLE 2
\[ \gamma_e = 7^\circ \quad W_{le} = 0 \quad W_L \neq 0 \]

\[ V_e = 40 \text{ ms}^{-1} \quad W_3 = 0 \]

\[ \Delta u \quad (\text{ms}^{-1}) \]

\[ \Delta w \quad (\text{ms}^{-1}) \]

\[ \Delta \theta \quad (\text{rad}) \]

\[ \Delta d \quad (\text{m}) \]

**FIG. 5.37 DG METHOD, EXAMPLE 2**
\[ \gamma_G = 7^\circ \quad W_{le} = 0 \quad W_L \neq 0 \]
\[ V_e = 40 \text{ ms}^{-1} \quad W_3 = 0 \]

\[ \Delta \delta_E \quad \text{(rad)} \]

\[ \Delta \delta_T \]

\[ \Delta w_l \quad \text{(ms}^{-1}) \]

\[ W_l \quad \text{(ms}^{-1}) \]

**FIG. 5.37 (continued) DG METHOD, EXAMPLE 2**
FIG. 5.38 WORST-CASE WIND MODELING: CLASSIFICATION OF METHODS NOT BASED ON ATMOSPHERIC PHYSICS OR EMPIRICAL DATA.

FIG. 5.39 AIRSPEED RESPONSE FOR MANUAL APPROACHES FLOWS IN THE PRESENCE OF A DIFFERENTIAL GAME WIND CONTROLLER.
FIG. 5.40 GLIDEPATH DEVIATION FOR MANUAL APPROACHES FLOWN IN THE PRESENCE OF A DIFFERENTIAL GAME WIND CONTROLLER

FIG. 5.41 WORST-CASE WIND PROFILE FOR A HUMAN PILOT
6. CGA SIMULATION RESULTS

This chapter presents the results of the digital simulations of the light STOL transport tracking curved glidepath geometries in the presence of variable winds. The glidepath geometries are based on estimates of the $W_1$ and $W_2$ components of the wind field along the approach path in conjunction with the kinematic method discussed in detail in Chapter 3, Section 3.7.1. The usefulness of these glidepaths depends on the dynamic effects of the variable winds encountered as well as on the accuracy with which the wind velocity field at the time of the approach may be estimated. It is the purpose of these simulations to obtain a more thorough assessment of these factors.

6.1 Description of the Dynamic Model

The equations of motion are the nonlinear, body-axes equations with the AERO2 aerodynamic model, i.e. (3.9,1) through to (3.9,17). The aerodynamic data is in look-up table form (see Appendix D for more details). These equations are implemented with the NL program of Appendix H. Linearized equations were not employed because of the large deviations from trim produced by the strong variable wind conditions modeled in many of the cases to follow. Also, the curved glidepath trajectories are generally not state trajectories for the aircraft dynamic system, and thus are not suitable as a linearization reference equilibrium.

The approach autopilots were optimal controllers synthesized using the linear quadratic optimal techniques summarized in Appendix F, Formulation 6 with $X_d=0$. The longitudinal controller synthesis was based on the linearized equations of motion of Section 5.5 of Chapter 5 and the LQDG-1 program of Appendix H, while the lateral controller synthesis was based on the linearized equations of motion of Section H.5 and the LQDG-5 program, both of Appendix H. The controllers are state feedback controllers, i.e. $M=I$ is appropriate. Control rate weightings were included in the performance criteria because of the smoother response of the resulting controllers and because of the wind-proofing characteristics that are obtained from the integral feedback. Only time-invariant, infinite terminal time control laws were used. The latter is justifiable because the simulation durations are much greater than the dominant time-constants of the closed-loop system.

Before a suitable control law was determined, several performance criteria of the type 2 of Table 5.1 were tried within the overall weighting strategy summarized in this table. The weightings on control rate were adjusted so that the control law, which may be written in the form (cf. (F.1,92) where $b_{12}=0$ since $x_d=0$)

$$\begin{align*}
T_N \dot{\Delta} + \Delta \dot{\Delta} &= KAx,
\end{align*}$$

has diagonal elements of the matrix $T_N$ in the range 0.25s to 0.35s. Kleinman and Baron [6.1] identify these elements with the neuromuscular lags associated with their optimal control theory models of the human pilot, and suggest values
of 0.1 seconds. Since the models in this study do not include a pure time delay (typically 0.1s to 0.2s) that is included in Kleinman's and Baron's model, and since others have used \( T_N \) values much larger than 0.1s (e.g. Vinge and Miller [6.2], as reported by Kleinman and Baron, use a \( T_N \) value of 0.35s for a VTOL hovering task), the range used in this study was considered to be representative of a human pilot or a good autopilot, and was found to give good controller characteristics.

Strictly speaking, the identification of the diagonal elements of \( T_N \) with the \( T_N \) time constants in describing function models of the human pilot assumes that the off-diagonal elements are zero. While this was not true for the control laws used in this study, the off-diagonal elements were generally much smaller than the diagonal elements, and thus it was felt that these guidelines were still useful for selecting appropriate values for the \( T_N_{ii} \).

The controllers that were finally used in the simulations are the LOOC3 and LAOC1 controllers corresponding to the weighting conditions defined in Table 5.2. The LOOC4 controller, whose weighting conditions are also summarized in this table, was eventually eliminated because it was felt that its lighter weighting on throttle rate and throttle deflection resulted in throttle movement which was too rapid and throttle changes which were too large to be compatible with realistic controller response (see Figs. 6.1a and 6.1b). The LOOC2 controller, which has been used in some of the numerical examples of the previous chapter (see equations (5.5,56a) and following), was only used for testing of the NL program because its \( T_N \) values were out of the desired range.

These controllers, their \( T_N \) characteristics, and the resulting closed-loop characteristics are summarized in Tables 6.1, 6.2 and 6.3.

While these controllers are optimal for the linear system for which they were synthesized, their application to the CGG simulations is suboptimal. This is a consequence of the nonlinearity of the equations of motion, of the presence of disturbance inputs, of control input limiters (see Appendix C), and because of trim conditions that are not the same as the conditions under which the control law optimization was carried out, i.e. trim conditions not corresponding to (5.5,24a), (5.5,24b) and (5.5,24c). Nevertheless, the controller response was found to be satisfactory.

In general, the command inputs to the autopilot will change as the spectrally low frequency components of the wind velocity field encountered change. In practice these changes from the aircraft's original trim conditions accommodate observed, nearly steady state offsets in pitch attitude and crab angle that are required to maintain the glidepath and localizer. Such a command generator generally consists of a low pass filter, e.g.

\[
T_0 \dot{\phi}_B^C + \dot{\phi}_B^C = \phi_B
\]

(6.1,2)

where \( \phi_B^C \) is the commanded pitch attitude.

In order to minimize the order of the simulation, such filters and optimal estimators, their modern control theory analogue, were not implemented in this study. Rather, two
idealization were employed. The first of these is the set of time-invariant trim conditions obtained using the relationships (3.4,29) through to (3.4,32), and (3.4,38) and (3.4,39) for \( W_1e, W_2e \) specified \textit{a priori} and \( V_e = V_A \), the nominal approach airspeed. These trim conditions were used as command inputs to the autopilot, i.e. \( u_{\text{trim}}, w_{\text{trim}} \) and so forth (see Table 6.1). They will be referred to as the APTCl conditions.

As will be discussed further in the sequel, these command inputs are particularly relevant to curved approaches.

The second type of such conditions, to be referred to as APTC2, is obtained analogously to the first, but it is now an instantaneous calculation of trim conditions based on the actual values of the \( W_1(t) \) and \( W_2(t) \) wind components encountered on the approach, the nominal approach airspeed \( V_A \), and the local value of the glidepath angle. In general this procedure will lead to time-varying trim conditions. In other words, at any time on the approach, \( u_{\text{trim}}, w_{\text{trim}} \) and so forth of Table 6.1 are the aircraft trim conditions required to maintain the glidepath in the presence of the \( W_1 \) and \( W_2 \) wind conditions present at that time. It will be shown that the APTC2 command inputs are required on conventional approaches in order to obtain satisfactory glidepath tracking.

Some details pertaining to these autopilot command inputs for the curved and conventional approach cases are summarized in the following table. We note that for curved approach cases where the wind conditions on which the curved approach was estimated are identical to the actual wind conditions encountered (i.e. \( W_{1c} = W_1 \) and \( W_{2c} = W_2 \)), then in the limit of \textit{perfect} glidepath tracking the APTC2 conditions will become time-invariant and identical to the APTCl conditions.

<table>
<thead>
<tr>
<th>CURVED APPROACHES</th>
<th>CONVENTIONAL APPROACHES</th>
</tr>
</thead>
<tbody>
<tr>
<td>APTCl</td>
<td></td>
</tr>
<tr>
<td>• time-invariant</td>
<td>• time-invariant</td>
</tr>
<tr>
<td>• based on ( W_{1c}(h_0) ), ( W_{2c}(h_0) ) and on ( \gamma_G ) or, equivalently, on ( W_{1c}(h_0) ) ( W_{2c}(h_0) ) and on ( \gamma_G(h_0, h_0) )</td>
<td>• based on ( W_1(h_0) ) and ( W_2(h_0) ) and on ( \gamma_G )</td>
</tr>
<tr>
<td>APTC2</td>
<td></td>
</tr>
<tr>
<td>• time-varying</td>
<td>• time-varying</td>
</tr>
<tr>
<td>• based on ( W_1(t) ), ( W_2(t) ) and on local value of ( \gamma_G )</td>
<td>• based on ( W_1(t) ), ( W_2(t) ) and on ( \gamma_G )</td>
</tr>
</tbody>
</table>

It is stressed that these conditions are autopilot command conditions. They \textit{need not} agree with the initial conditions of the simulation.

A number of wind models were used in this study. These are based on the family of wind profiles discussed in Chapter 2, Section 2.2.1 and on a model similar to the wind conditions present at the time of the JFK accident [6.3]. In the latter the downdraft velocities were uniformly reduced by a factor of 0.75 to make them more compatible with the climb capabilities of the STOL transport, and the \( W_2 \) wind component was set to zero. These wind models are given in Fig. 6.2.

A number of curved glidepath geometries, based on a sampling of the wind models of Fig. 6.2, were generated. These are summarized in Fig. 6.3. Only one glidepath based on a tailwind is included as virtually all landing approaches are made into the wind.
Comparisons were made between the response predictions of the dynamics package with the look-up table aerodynamics and with a linearized quasisteady aerodynamic model based on the look-up table data (for more details see Appendix D). As might be anticipated, for light wind shear conditions very minor differences between the two models' response predictions were observed (see Fig. 6.4), but for a strong shear, the response predictions differed substantially (see Fig. 6.5); the reader is cautioned that the curves in Fig. 6.5 generally have different vertical scales from the corresponding curves in Fig. 6.4), e.g. some peak response predictions disagreed by as much as 20%. All of the simulations to follow were with the nonlinear look-up table AERO2 aerodynamic model.

6.2 Presentation of Results

Over 200 cases were run for a number of curved glidepath geometries and wind models spanning a broad cross-section of possible situations. As well as the usual time histories of the aircraft response, root-mean-square values were computed for the important aircraft response and control variables, as well as for the wind inputs. The latter are used as objective measures of controller performance and wind variability.

All of the simulations begin at an altitude of 300m and end at the transition height altitude of 30m. The nominal approach airspeed is 40ms$^{-1}$. The transition height curved glidepath angle and the conventional glidepath angle are 7°.

Unless otherwise stated the initial conditions are the trim conditions for flight along the initial portion of the glidepath-localizer in the wind conditions present at the beginning of the approach, where it is noted that the initial curved approach glidepath angle will generally be different from the conventional glidepath angle.

6.2.1 CGA in the Presence of WHN and WHL Wind Models

A number of the cases that were run for the WHN and WHL series of wind models will be discussed in this section, as summarized in Table 6.4.

Figs. 6.6 and 6.7 give the aircraft response and wind inputs for Cases 1a through to 5a and 1b through to 5b respectively. Cases 1a to 5a are for the glidepath geometry GP1, with the wind profile for Cases 1a and 1b being identical to the wind model on which the curved approach path was based. Cases 2 to 5 are for progressively stronger increasing headwind jets superimposed onto the WHN-1 profile. These jets introduce both strong shears and simulated errors in the estimate of the existing wind profile.

A number of important root-mean-square quantities for these cases are summarized in Table 6.5. The most notable feature of this table is the considerably smaller glidepath deviation RMS value for Case 1a than for Case 1b (i.e. the LOOC3 controller achieves better glidepath tracking for the curved approach) despite the somewhat greater wind variability encountered on the former. For a given airspeed, the rate of
descent will be greater on the curved glidepath G1 than on the conventional glidepath as the former is steeper than the latter. As a consequence, the wind profiles, which are expressed as a function of altitude, will be traversed more quickly on the curved approach, and the wind velocity changes seen by the aircraft will be correspondingly greater.

As the headwind jet increases in magnitude, however, this advantage is lost. This effect is a consequence of the progressively increasing wind variability (as measured by $\hat{W}_1$) and of the deviation of the actual wind profile from the nominal profile from which the G1 glidepath geometry was determined. Strictly speaking, the effect of increasing wind variability on controller performance depends on the frequency content of the winds encountered and on the natural mode characteristics of the aircraft-autopilot system. For the wind models considered here and for the STOL transport, however, greater rates of descent were generally found to degrade airspeed and glidepath tracking performance. It is noted that if one bases system performance on the glidepath deviation RMS values normalized by $\hat{W}_1$, then for all of these cases tracking the CGG is advantageous.

There are a number of noticeable differences in the response curves of Figs. 6.6 and 6.7. The increasing headwind jets tend to produce a positive airspeed deviation as the aircraft enters them, with the opposite effect occurring upon exit. The airspeed deviations for the curved approach are somewhat larger, as might be expected from the RMS values of Table 6.4.

The major qualitative differences, however, arise in the glidepath deviations of the two series of cases. While both series exhibit negative glidepath deviations during the increasing headwind jet encounters, the curved approaches re-establish good glidepath tracking, while the conventional glidepath approaches pass through the glidepath and exhibit approximately 2.5m positive deviations by the end of the simulations.

These characteristics were found to be a consequence of the time-invariance of the APTC1 conditions. For the curved approaches, because of their constant attitude nature, such time-invariant conditions are appropriate, particularly for the nominal Case 1a. This is not so for the conventional glidepath approaches, where in a nonuniform wind field the appropriate autopilot command conditions will be time-varying. The APTC1 conditions only approximate the ideal conditions APTC2, and as a result the aircraft controller performance is degraded. This has the greatest effect on glidepath tracking performance.

Since glidepath tracking is crucial to a complete assessment of curved versus conventional glidepaths, the conventional glidepath Cases 1b to 5b were rerun with the APTC2 conditions (Cases 1c to 5c of Table 6.4). The resulting aircraft response is given in Fig. 6.9, where it is seen that considerably improved glidepath tracking has been achieved. It is noted, however, that both the $\theta_B$ and d responses are qualitatively different from those of Fig. 6.7. In particular,
the entry into the increasing headwind jet produces a deviation above the glidepath, while the exit results in a deviation below. In the former the deviations in the jet region tended to be below the glidepath.

The most marked differences, however, are observed in the improvement of glidepath tracking performance as measured by $\hat{d}$ (see Table 6.4, and Figs. 6.8a and 6.8b). This improvement is so dramatic that other than for the nonjet Case 1c, where performance is only slightly better than for the curved approach Case 1a, the performance of the autopilot is generally better on the conventional glidepath. This includes airspeed RMS values and elevator and throttle activity as measured by $\hat{e}_E$ and $\hat{e}_T$. Some improvement in the performance tracking the curved glidepath could be obtained by using the APTC2 conditions for the Cases 2a to 5a (Case 1a is the nominal case for which the APTC2 conditions are nearly identical to the APTC1 conditions, see the discussion in Section 6.1), i.e. the points corresponding to Cases 2a to 5a of Figs. 6.8a and 6.8b would shift to lower RMS values. While such curved approach simulations were not run for the GP1 glidepath, similar cases were run with the WHL-1 to WHL-5 horizontal wind speed models of Fig. 6.2b and the corresponding GP2 glidepath of Fig. 6.3. The RMS value for two of these runs and the corresponding conventional approach results are summarized in Tables 6.6 and 6.11 (Cases 1la and 1lb and 20a and 20b of Table 6.4). For these cases the RMS values for glidepath deviation on the conventional approaches are seen to be substantially smaller than those for the corresponding curved approaches. This is true even for Cases 1la and 1lb where the wind inputs were identical to the wind inputs on which the GP2 glidepath was based. These linear wind cases will be discussed further in the sequel.

This failure of the curved approach technique to provide any significant advantage over the conventional approaches (for these cases) needs to be examined more closely in order to determine the causes. This is done in conjunction with Fig. 6.10.

One of the curves in this figure represents the trim throttle settings (as determined with the APTC2 technique) for the conventional glidepath tracking cases in the presence of the nominal 0.16 power law headwind profile of Cases 1a, 1b, and 1c. This curve is determined from strictly equilibrium considerations, and does not take into account the dynamic effects of the wind. The amount of throttle required varies from 0.28 at the beginning of the approach to 0.24 at the end. The trim throttle setting for the curved glidepath geometry GP1, on the other hand, is constant, a characteristic which is the major advantage, in principle, of the curved approach technique. If one may take advantage of this characteristic, some reduction of control activity should result.

Such a beneficial effect was not observed to occur for these cases as a direct consequence of the dynamic effects of the wind shear. The decreasing headwind tends to produce slightly low airspeed conditions that the aircraft controller compensates for by increasing the throttle (see Fig. 6.10). In the case of the conventional approach, this increase cancels out some of the decrease that is required from strictly
trim considerations (see the throttle trim curve in Fig. 6.10 for the conventional approach), while in the case of the curved approach, this steady throttle increase takes the aircraft out of its ideal constant throttle approach. The latter effect is magnified by the greater rate of descent on the GPl glidepath and the consequently increased dynamic effects of the wind shear.

From Fig. 6.10 it is apparent that the throttle changes that are required for these approaches are still relatively small (approximately 3% of full throttle for the curved approach and approximately 1% for the conventional approach), the power law profile having changed by the relatively small velocity increment of 6m s\(^{-1}\) between the altitudes of 300m and 30m. It is informative to also consider cases where the mean wind changes are considerably larger, and where the wind shear is still relatively small and steady. Such criteria are satisfied by the linear WHL-1 profile of Fig. 6.2b (Hindson and Gould [1.60] based their arguments for wind based CGG on wind profiles of this type). This profile was implemented as a headwind profile (the associated curved glidepath is the curve GP2 of Fig. 6.3), and simulation runs were made (Cases 11a and 11b of Table 6.4). The RMS values for these runs are summarized in Table 6.6, where, as noted previously, it is seen that the conventional approach is still advantageous over the curved approach. A plot analogous to Fig. 6.10, as given in Fig. 6.11, however, clearly indicates that most of the throttle activity in the curved approach case comes from the initial adjustment to the nonzero, constant shear condition, whereas in the case of the conventional approach there is an initial adjustment followed by a period of slow, continuous throttle changes to compensate for the decreasing headwind while tracking a rectilinear glidepath.

The better performance in terms of airspeed tracking seems to be a consequence of the smaller wind variability, as seen by the aircraft, for the lower rate of descent conventional approach. These results suggest that at least for certain types of wind conditions, curved glidepath geometries may indeed reduce pilot workload, as measured by \(I_T^c\) and \(I_E^C\), although some performance measures, such as airspeed and glidepath RMS deviations, may be larger because of the larger variable wind effects.

Simulation runs were also made for decreasing headwind jets, i.e. the WHN-6 to WHN-9 wind speed models of Fig. 6.2a with the wind direction model 0-1 of Fig. 6.2e. These results showed the same trends as the increasing headwind cases and will not be discussed further.

To obtain a measure of the kinematic and dynamic effects of the winds on the glidepath tracking task, a number of cases were run where there was no glidepath feedback, with the trim conditions being the time-invariant APTCl conditions (note that the APTCl conditions are different for the curved and conventional approaches, see the table in Section 6.1. These results were run for the WHN-1 to WHN-5 headwind profiles (Cases 6 to 10 of Table 6.4). The aircraft response is given in Figs. 6.12 and 6.13 for, respectively, the curved and conventional glidepaths. The resulting RMS values are summarized in Table 6.7.
In these simulations the aircraft controller was dedicated solely to the task of maintaining the specified trim attitude, namely that attitude in which the aircraft was trimmed to fly along the initial portion of the glidepath in the wind that existed at the initial altitude $h_0$. For the nominal curved approach of Case 6a, this attitude does not change throughout the approach, and should result in good glidepath tracking.* This was indeed what was observed, as may be seen from Table 6.7 by comparing the results for Cases 6a and 6b. As before, however, the wind shear effects continue to be stronger on the steeper curved approach, and consequently airspeed tracking performance is worse (see $\hat{a}$) even though control activity as measured by $\hat{c}_E$ and $\hat{c}_T$ is approximately 30% greater.

Another interesting characteristic of these simulations is that for the increasing headwind jet cases the curved approaches cease to be advantageous in terms of glidepath deviation RMS values (see Fig. 6.14). This result, however, appears to be largely due to the controller actions taken to compensate for the airspeed, angle of attack and pitch angle disturbances produced by the wind. These actions tend to take the aircraft below both the curved and conventional glidepaths while at the same time the 0.16 power law profile tends to take the aircraft above the conventional glidepath, and leave it on the curved glidepath. This results in a net compensating effect that reduces the conventional glidepath deviation RMS values. It is noted that after the dip in the RMS values of the conventional approach cases, both types of approaches show approximately parallel, linear increases of $\hat{a}$ with $\hat{W}_1$.

The results discussed to this point suggest that the major reason for the limited gains made in using the curved glidepath geometries is the steeper glidepath angles, and consequently greater rates of descent that are required to fly the curved approaches. This situation occurs when there is a decreasing headwind with decreasing height wind condition present.

To show what would happen if the opposite were true, a number of cases were run where the 0.16 profile is a tailwind model, for which the corresponding glidepath geometry is GP3. From Fig. 6.3 it is seen that the glidepath angles for this approach are less than for the conventional glidepath. It follows that one would anticipate the wind shear effects for a given wind profile to be less because of the lower rates of descent.

The cases that were run are the cases 12 through to 16 of Table 6.4. Case 12 is the reference case, where the wind model used in the simulation is identical to that on which the curved approach was based. Cases 13 through to 16 consist of increasing tailwind jets of progressively greater intensities. The RMS values that result for a number of response variables and for the wind rate of change of velocity are given in Table 6.8.

The performance of the controller is now generally better for the curved approach than for the conventional approach.

*In principle the nonzero value of $\hat{a}$ for the curved approach Case 6a represents deviation caused solely by the dynamic effects of the variable winds.
The better performance for the nominal case is for all of the RMS values except throttle rate. This exception is a consequence of the particular way in which the tailwind power law profile affects the aircraft response, the corresponding controller response making the throttle setting more variable for the curved approach than for the conventional approach (see Fig. 6.15).

For the increasing tailwind jet cases, the advantage in glidepath performance disappears (see Fig. 6.16 and Table 6.8). All the other response variable RMS values continue to favour the curved approaches.

This improvement in performance compared to the analogous headwind Cases 1 through to 5 stems from the reduced dynamic effects of the wind shear relative to the corresponding conventional approaches. This may be seen by comparing the $\Delta V$ and $\dot{W}_1$ values for Cases 1 through to 5 (Table 6.5) with those for Cases 12 through to 16 (Table 6.8). While the RMS values for the latter are generally larger than those of the former, for the tailwind cases the curved approach produces lower RMS values while the conventional approach produces larger values, and vice versa for the headwind cases. This is more clearly depicted in Fig. 6.17 where the airspeed and glidepath deviation RMS values versus average rate of descent are compared for the Cases 1 and 12. These results are compatible with the way $|\delta_{AV}|$ might be expected to vary for different winds and different glidepaths through strictly kinematic considerations, i.e. in a given wind a steeper glidepath will lead to a larger $|\delta_{AV}|$ and for a given glidepath a tailwind leads to a larger $|\delta_{AV}|$ than a headwind.

To this point in the discussion, only $W_1$ wind disturbances have been considered. It is also of interest to consider deviations from the nominal wind profile that are lateral in nature.

A number of simulations of this type were run where the wind models also have lateral components, as implied by the wind direction models D-2 and D-4 of Fig. 6.2e. For these simulations the wind models on which the curved glidepaths were based were still strictly longitudinal in nature.* Thus these simulations examine the effects of large directional errors in estimating the wind profile on which the curved glidepaths were based.

Figures 6.18 and 6.19 present the response of the aircraft tracking, respectively, the glidepath GP3 and the conventional glidepath in the presence of wind profiles that shift from a tailwind with a crosswind component to a tailwind alone as the landing approach proceeds according to the direction profile D-4 of Fig. 6.2e. These simulations were designated Cases 17, 18 and 19, and the conditions of the simulation are described in more detail in Table 6.4. The initial conditions are trim conditions that are determined so that the aircraft tracks the initial localizer and glidepath in the presence of the winds at the beginning of the approach. These initial conditions will be different for the curved and conventional approaches. The aircraft autopilot now consists of the LOOC3 and LAOCl controllers of Table 6.1.

*It is noted, however, that the APTC2 autopilot command inputs generate $\psi_{trim} \neq 0$. 
The wind model for Cases 17a and 17b corresponds to the tailwind profile on which the glidepath GP3 was based, rotated according to the direction profile D-4. The cases 18· and 19 correspond to the same tailwind profile but now with increasing tailwind jets and with the direction profile D-4.

The resulting RMS values are summarized in Table 6.9. By comparing the cases in this table with their counterparts in Table 6.8 (i.e. Case 17 with Case 12, Case 18 with Case 14 and Case 19 with Case 16), it is apparent that the presence of lateral disturbances does not significantly alter the trends that were noted in the purely longitudinal cases. Two changes that have occurred, however, are that in Case 14 $\hat{\alpha}_c$ is larger than $\hat{\alpha}$, whereas the opposite is true for Case 18, and that the elevator deflection RMS values for these lateral disturbance cases seem to show less separation between the curved and conventional runs.

Other lateral disturbance cases were run analogous to some of the purely headwind cases using the D-2 direction profile of Fig. 6.2e. These results will not be discussed here other than to note that the performance trends observed with the purely longitudinal disturbances were generally unchanged in the presence of lateral disturbances.

A number of cases were also run with the LOOC2 controller of Table 6.1. These results showed no changes that affected the conclusions obtained with the LOOC3 controller, and will not be discussed further.

6.2.2 CGA in Hazardous Shear and Downdraft Conditions

To this point the curved glidepath geometries have been based on wind models that exhibit only mild shears. In this subsection consideration is given to flight tracking a curved glidepath based on highly variable headwind components. In particular, the simulations will be for the aircraft tracking the glidepath geometry GP4 of Fig. 6.3, i.e. a curved glidepath that is based on the JFK W1 profile of Fig. 6.2d. The effects of strong updrafts and downdrafts, which were also associated with this accident, will be included for some of the runs, although the magnitude of the downdrafts is reduced to make them more compatible with the climb capabilities of the light STOL transport.

Figs. 6.20 and 6.21 give, respectively, the airspeed and flight paths for cases 23a and 23b (see Table 6.4), where Case 23a is for an approach tracking the curved approach based on the JFK profile, and Case 23b is for the conventional approach. The JFK wind model causes large airspeed and glidepath excursions. These may be qualitatively explained in terms of the type of variation in the winds. The W1 wind input consists of an initially rapidly increasing...
headwind followed by a rapid decrease to a tailwind. This produces a large airspeed increase followed by an even larger airspeed loss. The $W_3$ wind inputs have two regions with strong downdraft activity. These cause the aircraft to deviate significantly below the glidepath at two points along the approach.

The RMS values of the important response variables are summarized in Table 6.10. The curved approach Case 23a generally has larger RMS values. However, for glidepath tracking it shows a small improvement in performance over the conventional approach. Also in favour of the curved glidepath is that for a large part of the approach it is significantly above the conventional glidepath and thus the downdraft does not take the aircraft to as low an altitude as for the conventional approach. This particular advantage disappears below 100m of altitude where the curved and conventional glidepaths are almost identical.

A number of downdraft cases were also run with the WV-2 and WV-3 models of Fig. 6.2c (in conjunction with strong $W_1$ jets) for the GP1, GP2 and GP3 glidepaths, and these were compared with the results for conventional approaches. The RMS results generally continued to favour conventional approaches. However, as might be expected, glidepath deviation RMS values showed considerable increases over the no downdraft values. Typically one has the values summarized in Table 6.11 for the GP2 curved slidepath (Cases 20, 21 and 22 of Table 6.4). It is noted that for these cases the curved approach did not show better performance for glidepath tracking, as was observed with the JFK runs.

A number of cases were run with only the $W_1$ component of the JFK wind model (Cases 24, 25 and 26 of Table 6.4). In these cases the throttle lag was varied from 0 to 8 seconds in order to examine its effect on the performance of the autopilot. The resulting RMS values are summarized in Table 6.12. For these cases the optimal controller continued to be the LOOC3 controller of Table 6.1, i.e. it was not adjusted to take into account the throttle lag.

The conventional approach continues to show superior performance over the curved approach. In fact, the introduction of throttle lag tends to produce more marked degradation of performance in the curved approach cases than in the conventional approaches. This trend is very apparent in Fig. 6.22.

6.2.3 CGA in the Presence of Worst-Case Wind Controllers

The examples in the previous sections have demonstrated the advantages and limitations of the curved approach technique using wind profiles that were specified a priori. In this section these cases will be supplemented with simulations carried out in the presence of worst-case wind controllers. The purpose of these simulations is twofold:

1. To compare the curved and conventional approaches

   -
in the presence of wind disturbances modeled as controllers.

2. To provide examples of the application of the worst-case wind controllers to simulations of a considerably longer duration than those of the examples of Chapter 5. The worst-case wind controllers have response characteristics that are specified in time domain terms. Since a conventional approach takes longer than a curved approach* (for the same initial altitude), a direct comparison based on a particular decision height is not as appropriate as simulations for a specified time interval. For these cases the worst-case wind inputs are superimposed over a 0.16 headwind power law profile (on which the curved approach glidepath will be based), which was not taken into account in the linear quadratic optimization procedure that determined the wind controllers.

The power law profile causes aircraft perturbations that feed back through the wind controller. This results in degradation of the aircraft controller performance beyond what would have occurred with the wind controller alone. If the power law profile shear is large enough this effect may lead to a loss of control by the autopilot. These difficulties were avoided by staying in the low shear region of the power law profile, i.e. above an altitude of 100m. Thus some of the RMS results to follow will be presented for simulations down to an altitude of 100m rather than the 30m level used in the previous examples. Also, some of the examples will be compared on the basis of equal time of application of the wind controller.

Figs. 6.23 and 6.24 give the aircraft response for Cases 27 to 31 (see Table 6.4). In these cases the wind inputs generated from the wind control law*

\[
\begin{align*}
\hat{W}_{1DG} &= -0.2992\Delta u + 0.007219\Delta w + 0.3255\Delta q \\
&+ 1.077\Delta \theta + 0.8211\Delta \delta_E - 0.7302\Delta \delta_T^C \quad (6.2,1a) \\
\hat{W}_{3DG} &= 0.01356\Delta u - 0.04040\Delta w + 0.1428\Delta q \\
&+ 1.456\Delta \theta - 0.8336\Delta \delta_E - 0.03436\Delta \delta_T^C \quad (6.2,1b)
\end{align*}
\]

are superimposed over the power law headwind profile. This wind controller is the steady state differential game solution of Case 2 of Example 1 of Section 5.5.4, Chapter 5.

The simulations for the curved approach cases were carried out to the usual 30m altitude with the simulations for the conventional approaches lasting for the time required for the corresponding curved approach to reach the 30m altitude. In general this resulted in the conventional approach simulations ending substantially above the 30m altitude.

* This is for the usual decreasing headwind with decreasing altitude cases.
To make the results comparable it is desirable to make the wind variability similar between corresponding curved and conventional cases. This may be done by varying the constant $\mu$ (see Chapter 5) and finding a wind controller that produces the desired wind variability, as measured by $S_{WS}$ or $\hat{\bar{W}}_1$, $\hat{\bar{W}}_3$. Such a procedure is quite iterative and is unnecessarily rigorous for this demonstration. Rather, the initial condition on $u_B$ was varied to obtain different values of $\hat{\bar{W}}_1$ and $\hat{\bar{W}}_3$.

The initial conditions used are summarized in the following table.

<table>
<thead>
<tr>
<th>CASE</th>
<th>$\Delta u_B(0) = u_B(0) - u_B_{(1m)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>27a, 27b</td>
<td>0</td>
</tr>
<tr>
<td>28a, 28b</td>
<td>-1</td>
</tr>
<tr>
<td>29a, 29b</td>
<td>-2</td>
</tr>
<tr>
<td>30a, 30b</td>
<td>-3</td>
</tr>
<tr>
<td>31a, 31b</td>
<td>-4</td>
</tr>
</tbody>
</table>

The wind inputs that result deviate substantially from the headwind profile, indicating that the wind controller is quite active. This is particularly true for cases with larger $|\Delta u_B(0)|$. The headwinds initially decrease rapidly, but show a rapid increase near the end of the simulations. The vertical wind activity initially shows a modest downdraft followed by a rapidly increasing updraft towards the end of the simulations. The general trends in the response curves between the curved and conventional approaches are similar.

The RMS values for these simulations will be compared in two ways. In Table 6.13 they are presented in terms of RMS levels based on the simulation results down to an altitude of 100m. In Table 6.14 they are presented in terms of RMS levels based on simulation results for a specified time interval, namely the time required for the aircraft tracking the curved approach to reach an altitude of 100m.

The most notable feature of Table 6.13 is that the glidepath deviations are generally greater for the conventional approach cases than for the curved approach cases. This is a consequence of the larger $\hat{\bar{W}}_3$ wind velocities that have developed by the time the aircraft tracking the conventional glidepath passes through the 100m altitude (this may be seen by comparing corresponding points on the 100m altitude isolines in Figs. 6.23g and 6.24g). These discrepancies largely disappear in Table 6.14 where the RMS values are computed on the basis of equal times. This may be seen more clearly in Figs. 6.25a and 6.25b where the RMS values for airspeed deviation and glidepath deviation are plotted versus $\hat{\bar{W}}_1$ and $\hat{\bar{W}}_3$ respectively.

A number of simulations were also carried out with the worst-case wind controller corresponding to the initial time of the $\mu = 10.9$ case of Example 1 of the direct method examples of Chapter 5. This wind control law is given by
\[
\dot{W}_{DG} = -0.3650\Delta u - 0.1954\Delta w + 2.190\Delta q + 12.88\Delta \theta 
\]  
(6.2a)

\[
\dot{W}_{3DG} = 0 . 
\]  
(6.2b)

The cases that will be discussed are Cases 32a and 32b of Table 6.4. The initial condition on \( \Delta u_B \) was zero.

The optimization procedure that provided this wind control law was applied to the open-loop longitudinal system of the aircraft, and was posed in such a way so as to destabilize the phugoid mode. From Figs. 6.26a and 6.26b it is seen that this is indeed what happens despite the presence of the optimal aircraft controller. The airspeed and glidepath deviations are sinusoidal and tend to increase in amplitude with time. The latter is a consequence of the LOOC3 aircraft controller's inability to adequately stabilize the system in the presence of the wind controller and of the additional destabilizing influence of the wind shear encountered from the power law profile. The curved approach deviations for airspeed and glidepath are generally larger than the corresponding peaks for the conventional approach case. This is a consequence of the somewhat greater effect of the power law profile shear for the former because of the greater rate of descent.

Fig. 6.27 presents the wind inputs that resulted from these simulations as altitude profiles, and compares them to the power law profile. The actual wind inputs deviate substantially from the power law profile, and tune themselves to the aircraft's dynamic response to produce the large deviations that resulted. An interesting feature of the wind profile corresponding to the curved approach is that it has a striking similarity to the headwind component of the JFK profile of Fig. 6.2d.

Table 6.15 gives the RMS values of the aircraft response and wind inputs for Cases 32a and 32b. The RMS values for the curved approach are generally greater than those for the conventional approach. This is a consequence of the greater levels of wind variability (see the \( \dot{W}_1 \) values) for the former.

6.3 Discussion of Results

The results presented in the previous sections have demonstrated a number of advantages and limitations of curved glidepath geometries defined with the kinematic method of Chapter 3, Section 3.7.1, and using APTC1 and APTC2 autopilot command inputs. In general the comparison with conventional approaches has not been as favourable as purely kinematic calculations might suggest. This result is a consequence of the dynamic effects of the variable winds. These effects tend to make the aircraft controller take the aircraft out of its ideal constant attitude, constant throttle configuration. While conventional approaches also require such changes, the dynamic and
kinematic effects of the wind tend to oppose each other, and for many of the simulations presented resulted in less throttle activity for the conventional approach than for the corresponding curved approach (e.g. see Figs. 6.10 and 6.11). This was particularly true for curved approaches based on decreasing headwind profiles. For such wind profiles the rate of descent on the curved approach is somewhat larger than the rate of descent on the corresponding conventional approach, and as a result the wind shear effects tend to be greater on the former (see Fig. 6.17).

There are two types of wind conditions, however, where curved approaches have some advantage over conventional approaches. The first type is decreasing headwind cases where the wind shear is nearly constant throughout the approach, i.e. a linear wind profile such as in Cases 1ia and 1ib of Table 6.4. For such curved approach cases the aircraft controller will initially adjust throttle to compensate for the wind shear, and will subsequently require very little throttle activity (see Fig. 6.11). It is noted that for these cases airspeed holding and glidepath tracking will still be slightly worse than the corresponding values for the conventional approach because of the greater rate of descent on the curved approach.

The second type are increasing headwind or decreasing tailwind with decreasing altitude cases. For such cases the curved approach rates of descent will be less than the corresponding rates of descent for the conventional approaches, and consequently the dynamic effects of the variable winds will be less (e.g. Cases 12 to 16 of Table 6.4). However, such wind conditions are not typical of most landing approaches; in the atmospheric boundary layer wind speeds tend to decrease with decreasing height and the approaches are normally made into the wind.

These advantages and disadvantages were generally found to hold even for cases where the winds deviated substantially from the horizontal wind on which the curved approach was based. This included cases which had strong downdraft activity. Curved approaches based on highly variable decreasing headwind wind profiles, such as the JFK Cases 23 to 26, showed no particular advantage over the corresponding conventional approaches in terms of controller performance.

The effects of highly variable wind conditions around the nominal profile on which the curved approach was based were also investigated with two wind controllers obtained from the worst-case method examples of Chapter 5 (Cases 27 to 32 of Table 6.4). The results obtained strongly suggest that there are only minor differences between controller performance tracking curved and conventional approaches given nearly identical levels of wind activity (see Figs. 6.25a and 6.25b) and time-invariant autopilot command conditions. These results
support those obtained with fixed wind profiles, where controller performance for conventional approaches was generally better for cases where the rate of descent, and consequently the wind shear effects, were less than for the corresponding curved approach cases.

While no cases were run for different transition height glidepath angles, it is possible at this point to make some predictions on the effects of such changes. For decreasing $\gamma_G$, the rates of descent required will decrease (for a given airspeed), and consequently the dynamic effects of the wind shear will also decrease. This reduction of dynamic wind effects will generally work in favour of the curved approach technique, which is based on strictly kinematic considerations. It is noted, however, that the trim control changes required on the corresponding conventional approach will not have to be applied as quickly. All of these effects will be reversed for greater $\gamma_G$.

The results presented in this chapter also suggest a number of modifications that might be made to the curved approach technique to improve the controller's performance. From the point of view of reducing airspeed and glidepath deviations, improvement in performance could be obtained for the usual decreasing headwind cases by specifying the curved glidepath geometry in a way which results in rates of descent that are compatible with conventional glidepath tracking at the beginning of the approach rather than at the transition height $h_D$. This amounts to removing the condition (3.7,15) matching the curved and conventional glidepath angles at the transition height and replacing it with a similar condition for the beginning of the approach, i.e.

$$\left(\frac{\dot{h}}{\dot{x}}\right) = -\tan \gamma_G.$$  (6.3,1)

The changes that would result in the curved glidepath geometry from applying (6.3,1) rather than (3.7,15) are qualitatively depicted in Fig. 6.28 for a decreasing headwind case. Two important changes that have occurred are that the curved glidepath geometry is now below the conventional glidepath and that the aircraft trim condition at the transition height $h_D$ will make the aircraft susceptible to overshoots. The latter tendency may be minimized in a number of ways including a smooth transition glidepath to the conventional glidepath or a matching condition for the touchdown aim points of the curved and conventional glidepaths. These types of curved glidepaths were not considered further in this investigation and are recommended for future work.

From the point of view of reducing control activity for kinematically defined curved glidepath approaches, a technique which might be pursued in mild wind shears is one where some airspeed tracking and (to a lesser extent) glidepath tracking performance is sacrificed by fixing the throttle at the level required to maintain the constant rate of descent appropriate for the curved approach. This may be done with or without
the glidepath geometry modifications suggested in the previous paragraphs. In this way the longitudinal control task is reduced to one of maintaining the nearly constant attitude that would be required to track the glidepath, and changing throttle settings only when airspeed deviations become too large to be compatible with flight safety.
REFERENCES – CHAPTER 6


### TABLE 6.1

**CGA SIMULATION CONTROL LAWS**

<table>
<thead>
<tr>
<th>CONTROL LAW DESIGNATION</th>
<th>CONTROL LAW</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>LOOC2</strong></td>
<td></td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>( 0.01677\Delta u - 0.02499\Delta w + 0.3405\Delta q + 1.682\Delta \theta )</td>
</tr>
<tr>
<td></td>
<td>+0.0122( \text{d}_c ) - 1.776( \Delta \delta_E ) + 0.01571( \Delta \delta_c )</td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>-0.1036( \Delta u ) + 0.05788( \Delta w ) - 0.09133( \Delta q ) - 2.710( \Delta \theta )</td>
</tr>
<tr>
<td></td>
<td>-0.03902( \text{d}_c ) + 0.5128( \Delta \delta_E ) - 2.305( \Delta \delta_c )</td>
</tr>
<tr>
<td><strong>LOOC3</strong></td>
<td></td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>( 0.03188\Delta u - 0.05271\Delta w + 1.043\Delta q + 4.418\Delta \theta )</td>
</tr>
<tr>
<td></td>
<td>+0.03522( \text{d}_c ) - 3.174( \Delta \delta_E ) - 0.004027( \Delta \delta_c )</td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>-0.1452( \Delta u ) + 0.08381( \Delta w ) - 0.01053( \Delta q ) - 3.600( \Delta \theta )</td>
</tr>
<tr>
<td></td>
<td>-0.05687( \text{d}_c ) - 0.03625( \Delta \delta_E ) - 3.308( \Delta \delta_c )</td>
</tr>
<tr>
<td><strong>LOOC4</strong></td>
<td></td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>0.02350( \Delta u ) - 0.04886( \Delta w ) + 1.004( \Delta q ) + 4.167( \Delta \theta )</td>
</tr>
<tr>
<td></td>
<td>+0.03260( \text{d}_c ) - 3.117( \Delta \delta_E ) - 0.01957( \Delta \delta_c )</td>
</tr>
<tr>
<td>( \Delta E )</td>
<td>-0.2784( \Delta w ) + 0.2434( \Delta w ) + 0.4497( \Delta q ) - 9.197( \Delta \theta )</td>
</tr>
<tr>
<td></td>
<td>-0.1391( \text{d}_c ) - 0.7047( \Delta \delta_E ) - 3.207( \Delta \delta_c )</td>
</tr>
</tbody>
</table>

\*\( \text{d}_c \) = normal deviation from curved glidepath.

### TABLE 6.1 (continued)

<table>
<thead>
<tr>
<th>CONTROL LAW DESIGNATION</th>
<th>CONTROL LAW</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>LAOC1</strong></td>
<td></td>
</tr>
<tr>
<td>( \Delta \delta_A )</td>
<td>-0.1286( \Delta v ) - 0.5477( \Delta p ) - 0.6421( \Delta x ) - 3.185( \Delta \phi )</td>
</tr>
<tr>
<td></td>
<td>-5.271( \Delta \phi ) - 2.806( \Delta y ) - 2.903( \Delta \delta_A ) - 0.1712( \Delta \delta_R )</td>
</tr>
<tr>
<td>( \Delta \delta_R )</td>
<td>-0.01868( \Delta v ) - 0.1692( \Delta p ) + 2.766( \Delta x ) - 0.07327( \Delta \phi )</td>
</tr>
<tr>
<td></td>
<td>+1.969( \Delta \phi ) + 0.01061( \Delta y ) - 0.1712( \Delta \delta_A ) - 3.630( \Delta \delta_R )</td>
</tr>
</tbody>
</table>

Note: The LOOC3 control law was the final choice for a longitudinal approach controller.

For the purposes of the simulations:

\( \Delta u = u - u_{\text{trim}} \)
\( \Delta q = q \)
\( \Delta v = v \)
\( \Delta p = p \)
\( \Delta \theta = \theta - \theta_{\text{trim}} \)
\( \Delta \delta = \delta - \delta_{\text{CGG}} \)
\( \Delta r = r \)
\( \Delta \phi = \phi \)
\( \Delta y = y \)
\( \Delta \delta_{\text{I}} = \delta_{\text{I}} \)
\( \Delta \delta_{\text{A}} = \delta_{\text{A}} \)
\( \Delta \delta_{\text{R}} = \delta_{\text{R}} \)
### TABLE 6.2
STOL TRANSPORT CLOSED-LOOP MODE CHARACTERISTICS

<table>
<thead>
<tr>
<th>MODE</th>
<th>( \zeta )</th>
<th>( w_n ) (rad./s)</th>
<th>( w ) (rad./s)</th>
<th>( T_N^p ) (sec)</th>
<th>( T ) (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PHUGOID</td>
<td>0.699</td>
<td>0.664</td>
<td>0.475</td>
<td>1.49</td>
<td>13.2</td>
</tr>
<tr>
<td>SHORT-PERIOD</td>
<td>0.546</td>
<td>3.06</td>
<td>2.56</td>
<td>0.415</td>
<td>2.46</td>
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<tr>
<td>LOPK</td>
<td>---</td>
<td>---</td>
<td>0</td>
<td>2.38</td>
<td>---</td>
</tr>
<tr>
<td>LOCM1</td>
<td>---</td>
<td>---</td>
<td>0</td>
<td>0.227</td>
<td>---</td>
</tr>
<tr>
<td>LOCM2</td>
<td>---</td>
<td>---</td>
<td>0</td>
<td>0.300</td>
<td>---</td>
</tr>
<tr>
<td>DUTCH ROLL</td>
<td>0.440</td>
<td>2.61</td>
<td>2.34</td>
<td>0.603</td>
<td>2.68</td>
</tr>
<tr>
<td>ROLLING MODE</td>
<td>---</td>
<td>---</td>
<td>0</td>
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<td>---</td>
</tr>
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<td>SPIRAL MODE</td>
<td>---</td>
<td>---</td>
<td>0</td>
<td>0.299</td>
<td>---</td>
</tr>
<tr>
<td>LAPK</td>
<td>0.703</td>
<td>0.410</td>
<td>0.292</td>
<td>2.41</td>
<td>21.5</td>
</tr>
<tr>
<td>LACM</td>
<td>0.754</td>
<td>1.45</td>
<td>0.954</td>
<td>0.634</td>
<td>6.58</td>
</tr>
</tbody>
</table>

\( V_e = 40\text{ms}^{-1} \)

\( Y_G = 7^\circ \)

\( W_{1e} = 0 \)

LOOC3 Longitudinal Controller
LAOC1 Lateral Controller

* See Table D.2 for comparison with open-loop characteristics.

### TABLE 6.3
\( T_N \) CHARACTERISTICS OF OPTIMAL CONTROLLERS

<table>
<thead>
<tr>
<th>CONTROLLER DESIGNATION</th>
<th>( T_{N0E} ) (sec)</th>
<th>( T_{N0T} ) (sec)</th>
<th>( T_{N0A} ) (sec)</th>
<th>( T_{N0R} ) (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOCC2</td>
<td>0.5639</td>
<td>0.4343</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>LOCC3</td>
<td>0.3151</td>
<td>0.3023</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>LOCC4</td>
<td>0.3213</td>
<td>0.3123</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>LAOC1</td>
<td>N/A</td>
<td>N/A</td>
<td>0.3454</td>
<td>0.2762</td>
</tr>
</tbody>
</table>

\( V_e = 40\text{ms}^{-1} \)

\( Y_G = 7^\circ \)

\( W_{1e} = 0 \)

* NOTE: N/A = Not Applicable.

LOPK - Longitudinal Path-Keeping Mode
LOCM1 - First Longitudinal Controller Mode
LOCM2 - Second Longitudinal Controller Mode
LAPK - Lateral Path-Keeping Mode
LACM - Lateral Controller Mode
### TABLE 6.4

**CGG SIMULATION CASES**

<table>
<thead>
<tr>
<th>CASE</th>
<th>WIND MODEL</th>
<th>GLIDEPATH</th>
<th>TRIM CONDITIONS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$W_H$</td>
<td>$\zeta$</td>
<td>$W_V$</td>
</tr>
<tr>
<td>1a</td>
<td>WHN-1</td>
<td>D-1</td>
<td>WV-1</td>
</tr>
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<td>2a</td>
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$^\dagger$ See Fig. 6.2.

$^*$ See Fig. 6.3.

ConGP = Conventional Glidepath.

---

### TABLE 6.4 (continued)

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$^\dagger$ See Fig. 6.2.

$^*$ See Fig. 6.3.

ConGP = Conventional Glidepath.
### TABLE 6.4 (continued)

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† See Fig. 6.2.
* See Fig. 6.3.

Cases 27 and on superimpose a wind controller model over the fixed profile specified.

ConGP = Conventional Glidepath.

### TABLE 6.5

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<tr>
<th>CASE</th>
<th>CGG*</th>
<th>$\Delta v$ (ms$^{-1}$)</th>
<th>$\dot{\alpha}$, $\dot{\alpha}_C$ (m)</th>
<th>$10^3 \dot{\delta}_E$ (rad./s)</th>
<th>$10^5 \dot{\delta}_2$ (s$^{-1}$)</th>
<th>$\ddot{\omega}_1$ (ms$^{-2}$)</th>
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* See Fig. 6.3.

ConGP = Conventional Glidepath.
### TABLE 6.6
**SUMMARY OF RMS VALUES FOR CASES 11a AND 11b**

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<th>( 10^3 \Delta \beta (\text{rad./s}) )</th>
<th>( 10^3 \dot{\beta} (\text{s}^{-1}) )</th>
<th>( \dot{\psi}_1 (\text{ms}^{-2}) )</th>
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* See Fig. 6.3.

ConGP = Conventional Glidepath.

### TABLE 6.7
**SUMMARY OF RMS VALUES FOR CASES 6 TO 10**

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<th>( 10^3 \dot{\beta} (\text{s}^{-1}) )</th>
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* See Fig. 6.3.

ConGP = Conventional Glidepath.

### TABLE 6.8
**SUMMARY OF RMS VALUES FOR CASES 12 TO 16**

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<th>( 10^3 \dot{\beta} (\text{s}^{-1}) )</th>
<th>( \dot{\psi}_1 (\text{ms}^{-2}) )</th>
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* See Fig. 6.3.

ConGP = Conventional Glidepath.

### TABLE 6.9
**SUMMARY OF RMS VALUES FOR CASES 17 TO 19**

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<th>( \Delta \dot{\alpha} (\text{m}) )</th>
<th>( 10^3 \Delta \beta (\text{rad./s}) )</th>
<th>( 10^3 \dot{\beta} (\text{s}^{-1}) )</th>
<th>( \dot{\psi}_1 (\text{ms}^{-2}) )</th>
</tr>
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<td>19a</td>
<td>GP3</td>
<td>2.358</td>
<td>1.009</td>
<td>8.784</td>
<td>66.04</td>
<td>1.183</td>
<td>0.8550</td>
</tr>
<tr>
<td>19b</td>
<td>ConGP</td>
<td>2.531</td>
<td>1.102</td>
<td>8.784</td>
<td>66.04</td>
<td>1.287</td>
<td>0.9324</td>
</tr>
</tbody>
</table>

* See Fig. 6.3.

ConGP = Conventional Glidepath.
### Table 6.10
**Summary of RMS Values for Cases 23a and 23b**

<table>
<thead>
<tr>
<th>CASE</th>
<th>CUG</th>
<th>$\Delta V$ (m/s)</th>
<th>$\Delta \dot{h}$ (m)</th>
<th>$\dot{h}^2$ (rad/s)</th>
<th>$\ddot{h}^2$ (rad/s$^2$)</th>
<th>$\dot{W}_I$ (m/s$^2$)</th>
<th>$\dot{W}_2$ (m/s$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>23a</td>
<td>GP4</td>
<td>3.159</td>
<td>10.76</td>
<td>23.45</td>
<td>57.5</td>
<td>1.287</td>
<td>0.8208</td>
</tr>
<tr>
<td>23b</td>
<td>ConGP</td>
<td>2.462</td>
<td>11.59</td>
<td>12.65</td>
<td>34.90</td>
<td>0.8934</td>
<td>0.5386</td>
</tr>
</tbody>
</table>

* See Fig. 6.3.

ConGP = Conventional Glidepath.

### Table 6.11
**Summary of RMS Values for Cases 20 to 22**

<table>
<thead>
<tr>
<th>CASE</th>
<th>CUG</th>
<th>$\Delta V$ (m/s)</th>
<th>$\Delta \dot{h}$ (m)</th>
<th>$\dot{h}^2$ (rad/s)</th>
<th>$\ddot{h}^2$ (rad/s$^2$)</th>
<th>$\dot{W}_I$ (m/s$^2$)</th>
<th>$\dot{W}_2$ (m/s$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20a</td>
<td>GP2</td>
<td>2.359</td>
<td>1.499</td>
<td>6.999</td>
<td>22.76</td>
<td>0.9130</td>
<td>0</td>
</tr>
<tr>
<td>20b</td>
<td>ConGP</td>
<td>1.702</td>
<td>1.046</td>
<td>3.915</td>
<td>13.22</td>
<td>0.6299</td>
<td>0</td>
</tr>
<tr>
<td>21a</td>
<td>GP2</td>
<td>2.701</td>
<td>6.702</td>
<td>10.26</td>
<td>29.00</td>
<td>0.9987</td>
<td>0.1674</td>
</tr>
<tr>
<td>21b</td>
<td>ConGP</td>
<td>1.837</td>
<td>4.064</td>
<td>4.149</td>
<td>13.39</td>
<td>0.6415</td>
<td>0.1621</td>
</tr>
<tr>
<td>22a</td>
<td>GP1</td>
<td>2.984</td>
<td>11.88</td>
<td>1.308</td>
<td>34.44</td>
<td>1.066</td>
<td>0.3426</td>
</tr>
<tr>
<td>22b</td>
<td>ConGP</td>
<td>1.593</td>
<td>7.44</td>
<td>1.402</td>
<td>16.19</td>
<td>0.6929</td>
<td>0.2544</td>
</tr>
</tbody>
</table>

* See Fig. 6.3.

ConGP = Conventional Glidepath.

### Table 6.12
**Summary of RMS Values for Cases 24 to 26**

<table>
<thead>
<tr>
<th>CASE</th>
<th>CUG</th>
<th>$\Delta V$ (m/s)</th>
<th>$\Delta \dot{h}$ (m)</th>
<th>$\dot{h}^2$ (rad/s)</th>
<th>$\ddot{h}^2$ (rad/s$^2$)</th>
<th>$\dot{W}_I$ (m/s$^2$)</th>
<th>$\dot{W}_2$ (m/s$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>24a</td>
<td>GP4</td>
<td>2.743</td>
<td>1.713</td>
<td>3.261</td>
<td>13.18</td>
<td>1.058</td>
<td>0.7263</td>
</tr>
<tr>
<td>24b</td>
<td>ConGP</td>
<td>1.992</td>
<td>1.020</td>
<td>3.010</td>
<td>10.30</td>
<td>1.048</td>
<td>0.7635</td>
</tr>
<tr>
<td>25a</td>
<td>GP4</td>
<td>3.498</td>
<td>3.408</td>
<td>9.813</td>
<td>156.6</td>
<td>1.070</td>
<td>0.7167</td>
</tr>
<tr>
<td>25b</td>
<td>ConGP</td>
<td>2.394</td>
<td>1.175</td>
<td>3.671</td>
<td>16.86</td>
<td>1.070</td>
<td>0.7167</td>
</tr>
<tr>
<td>26a</td>
<td>GP4</td>
<td>4.286</td>
<td>4.991</td>
<td>12.45</td>
<td>194.5</td>
<td>1.107</td>
<td>0.7099</td>
</tr>
<tr>
<td>26b</td>
<td>ConGP</td>
<td>2.642</td>
<td>1.795</td>
<td>4.342</td>
<td>23.08</td>
<td>1.107</td>
<td>0.7099</td>
</tr>
</tbody>
</table>

* See Fig. 6.3.

ConGP = Conventional Glidepath.

### Table 6.13
**Summary of RMS Values for Cases 27 to 31**

<table>
<thead>
<tr>
<th>CASE</th>
<th>CUG</th>
<th>$t_{\text{RMS}}$(s)</th>
<th>$\Delta V$(m/s)</th>
<th>$\Delta \dot{h}$ (m)</th>
<th>$\dot{h}^2$ (rad/s)</th>
<th>$\ddot{h}^2$ (rad/s$^2$)</th>
<th>$\dot{W}_I$ (m/s$^2$)</th>
<th>$\dot{W}_2$ (m/s$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>27a</td>
<td>GP1</td>
<td>58</td>
<td>0.3855</td>
<td>0.8893</td>
<td>0.1380</td>
<td>0.7805</td>
<td>0.1569</td>
<td>0.007507</td>
</tr>
<tr>
<td>27b</td>
<td>ConGP</td>
<td>74</td>
<td>0.2860</td>
<td>2.171</td>
<td>0.2575</td>
<td>1.786</td>
<td>0.08192</td>
<td>0.02792</td>
</tr>
<tr>
<td>28a</td>
<td>GP1</td>
<td>55</td>
<td>0.5068</td>
<td>1.883</td>
<td>1.474</td>
<td>5.586</td>
<td>0.1732</td>
<td>0.02223</td>
</tr>
<tr>
<td>28b</td>
<td>ConGP</td>
<td>68</td>
<td>0.5746</td>
<td>2.920</td>
<td>1.340</td>
<td>5.359</td>
<td>0.1511</td>
<td>0.03936</td>
</tr>
<tr>
<td>29a</td>
<td>GP1</td>
<td>51</td>
<td>0.8440</td>
<td>2.511</td>
<td>3.220</td>
<td>10.71</td>
<td>0.2489</td>
<td>0.03324</td>
</tr>
<tr>
<td>29b</td>
<td>ConGP</td>
<td>63</td>
<td>0.8718</td>
<td>3.420</td>
<td>2.081</td>
<td>10.71</td>
<td>0.2533</td>
<td>0.04698</td>
</tr>
<tr>
<td>30a</td>
<td>GP1</td>
<td>49</td>
<td>1.183</td>
<td>3.161</td>
<td>5.150</td>
<td>18.17</td>
<td>0.3319</td>
<td>0.05427</td>
</tr>
<tr>
<td>30b</td>
<td>ConGP</td>
<td>59</td>
<td>1.148</td>
<td>3.682</td>
<td>4.608</td>
<td>16.52</td>
<td>0.3066</td>
<td>0.05138</td>
</tr>
<tr>
<td>31a</td>
<td>GP1</td>
<td>46</td>
<td>1.525</td>
<td>3.508</td>
<td>7.395</td>
<td>25.14</td>
<td>0.4200</td>
<td>0.04936</td>
</tr>
<tr>
<td>31b</td>
<td>ConGP</td>
<td>56</td>
<td>1.422</td>
<td>3.842</td>
<td>6.724</td>
<td>22.65</td>
<td>0.3802</td>
<td>0.05473</td>
</tr>
</tbody>
</table>

* See Fig. 6.3.

† The RMS values were based on time to descend to $h = 100m$.

ConGP = Conventional Glidepath.
### TABLE 6.14
**SUMMARY OF RMS VALUES FOR CASES 27 TO 31**

**DIFFERENTIAL GAME WIND CONTROLLER**

<table>
<thead>
<tr>
<th>CASE</th>
<th>CGG*</th>
<th>t_RMS(s)</th>
<th>(\Delta V) (ms^{-1})</th>
<th>(\Delta \dot{a}, \Delta \dot{a}_c) (m)</th>
<th>(10^3 \ddot{E}) (rad./s)</th>
<th>(10^3 \dddot{C}_T) (m^{-2})</th>
<th>(\ddot{W}_1) (ms^{-2})</th>
<th>(\ddot{W}_3) (ms^{-2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>27a</td>
<td>GPI</td>
<td>58</td>
<td>0.3855</td>
<td>0.8893</td>
<td>0.1390</td>
<td>0.7805</td>
<td>0.1569</td>
<td>0.007507</td>
</tr>
<tr>
<td>27b</td>
<td>ConGP</td>
<td>58</td>
<td>0.2045</td>
<td>1.194</td>
<td>0.1761</td>
<td>1.129</td>
<td>0.08520</td>
<td>0.01337</td>
</tr>
<tr>
<td>28a</td>
<td>GPI</td>
<td>55</td>
<td>0.5068</td>
<td>1.883</td>
<td>1.474</td>
<td>5.586</td>
<td>0.1732</td>
<td>0.02223</td>
</tr>
<tr>
<td>28b</td>
<td>ConGP</td>
<td>55</td>
<td>0.4818</td>
<td>1.895</td>
<td>1.462</td>
<td>5.603</td>
<td>0.1479</td>
<td>0.02401</td>
</tr>
<tr>
<td>29a</td>
<td>GPI</td>
<td>51</td>
<td>0.8440</td>
<td>2.571</td>
<td>3.220</td>
<td>11.78</td>
<td>0.2489</td>
<td>0.03324</td>
</tr>
<tr>
<td>29b</td>
<td>ConGP</td>
<td>51</td>
<td>0.6917</td>
<td>2.308</td>
<td>3.204</td>
<td>11.67</td>
<td>0.2381</td>
<td>0.03096</td>
</tr>
<tr>
<td>30a</td>
<td>GPI</td>
<td>49</td>
<td>1.183</td>
<td>3.161</td>
<td>5.150</td>
<td>18.17</td>
<td>0.3319</td>
<td>0.04287</td>
</tr>
<tr>
<td>30b</td>
<td>ConGP</td>
<td>49</td>
<td>1.163</td>
<td>2.649</td>
<td>5.133</td>
<td>17.97</td>
<td>0.3236</td>
<td>0.03737</td>
</tr>
<tr>
<td>31a</td>
<td>GPI</td>
<td>46</td>
<td>1.525</td>
<td>3.508</td>
<td>7.395</td>
<td>25.14</td>
<td>0.4200</td>
<td>0.04936</td>
</tr>
<tr>
<td>31b</td>
<td>ConGP</td>
<td>46</td>
<td>1.498</td>
<td>2.747</td>
<td>7.410</td>
<td>24.85</td>
<td>0.4111</td>
<td>0.04137</td>
</tr>
</tbody>
</table>

* See Fig. 6.3

† The RMS values were based on the time to descend to \(h = 100\) m for Cases 27a to 31a, and for the corresponding times for Cases 27b to 31b.

ConGP = Conventional Glidepath.

### TABLE 6.15
**SUMMARY OF RMS VALUES FOR CASES 32a AND 32b**

**DIRECT METHOD WIND CONTROLLER**

<table>
<thead>
<tr>
<th>CASE</th>
<th>CGG*</th>
<th>t_RMS(s)</th>
<th>(\Delta V) (ms^{-1})</th>
<th>(\Delta \dot{a}, \Delta \dot{a}_c) (m)</th>
<th>(10^3 \ddot{E}) (rad./s)</th>
<th>(10^3 \dddot{C}_T) (m^{-2})</th>
<th>(\ddot{W}_1) (ms^{-2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>32a</td>
<td>GPI</td>
<td>80</td>
<td>1.725</td>
<td>1.781</td>
<td>2.263</td>
<td>9.682</td>
<td>0.6409</td>
</tr>
<tr>
<td>32b</td>
<td>ConGP</td>
<td>80</td>
<td>1.494</td>
<td>1.268</td>
<td>1.922</td>
<td>8.038</td>
<td>0.5492</td>
</tr>
</tbody>
</table>

* See Fig. 6.3.

† The RMS values were computed over the first 80s of the simulation (see Fig. 6.27).

ConGP = Conventional Glidepath.
FIG. 6.1a  AIRSPEED END GLIDEPATH DEVIATION
RESPONSE COMPARISON BETWEEN THE LOOC3 AND LOOC4 CONTROLLERS
Computed with LQDG-1 program.

FIG. 6.1b  CONTROL INPUT COMPARISON BETWEEN
THE LOOC3 AND LOOC4 CONTROLLERS
Computed with LQDG-1 program.
FIG. 6.2a $W_H$ MODELS: Jets superimposed onto a 0.16 neutral wind profile (see Chapter 2)

FIG. 6.2b $W_H$ MODELS: Jets superimposed onto a linear wind profile (see Chapter 2)

FIG. 6.2c DOWNDRAFT WIND MODELS BASED ON EQUATION (2.2, 10)
FIG. 6.4 COMPARISON BETWEEN LINEAR AND NONLINEAR AERODYNAMIC MODEL RESPONSE PREDICTIONS FOR FLIGHT IN THE PRESENCE OF A 0.16 POWER LAW HEADWIND PROFILE (WM-1, D-1 WIND MODEL, FIG. 6.2)
FIG. 6.5 COMPARISON BETWEEN LINEAR AND NONLINEAR AERODYNAMIC MODEL RESPONSE PREDICTIONS FOR FLIGHT IN THE PRESENCE OF A 0.16 POWER LAW HEADWIND PROFILE WITH A STRONG INCREASING HEADWIND JET (WHN-5, D-1 WIND MODEL, FIG. 6.2)
FIG. 6.6 SIMULATION RESULTS FOR CASES 1a TO 5a
$V_A=40\text{ms}^{-1}$ LOOC3 CONTROLLER

GP1 APTC1

$\delta_P$ (rad.)

$\delta_C$ (rad.)

FIG. 6.6e $t$(sec)

FIG. 6.6f $t$(sec)

$w_1$ (ms$^{-1}$)

$w_2=w_3=0$

FIG. 6.6g $t$(sec)

FIG. 6.6(continued) SIMULATION RESULTS FOR CASES 1a TO 5a

SYMBOl

CASE 1a 2a 3a 4a 5a
FIG. 6.7 SIMULATION RESULTS FOR CASES 1b TO 5b
$V_A = 40 \text{ms}^{-1}$

LOOC3 CONTROLLER

CONVENTIONAL GP

APTC1

$\delta_E$ (rad.)

$\delta_C$

$W_2 = W_3 = 0$

$W_1$ (ms$^{-1}$)

FIG. 6.7a

FIG. 6.7f

FIG. 6.7g

FIG. 6.7(continued) SIMULATION RESULTS FOR CASES 1b TO 5b

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>□</th>
<th>○</th>
<th>△</th>
<th>+</th>
<th>×</th>
</tr>
</thead>
<tbody>
<tr>
<td>CASE</td>
<td>1b</td>
<td>2b</td>
<td>3b</td>
<td>4b</td>
<td>5b</td>
</tr>
</tbody>
</table>
FIG. 6.8a  GLIDEPATH DEVIATION RMS VALUES FOR CASES 1 TO 5

FIG. 6.8b  GLIDEPATH DEVIATION RMS VALUES FOR CASES 1 TO 5 IN TERMS OF WIND VARIABILITY AS MEASURED BY $\hat{W}_1$
$V_a=40\text{ms}^{-1}$

CONVENTIONAL GP

LOOC3 CONTROLLER

APTC2

**FIG. 6.9a**

**FIG. 6.9b**

**FIG. 6.9c**

**FIG. 6.9d**

**FIG. 6.9** SIMULATION RESULTS FOR CASES 1c TO 5c
$V_A = 40 \text{ms}^{-1}$

**LOOC3 CONTROLLER**

CONVENTIONAL GP  APTC2

---

**FIG. 6.9e**

$^{\cdot}E$ (rad.)

**FIG. 6.9f**

$^{\cdot}\delta_T$

**FIG. 6.9g**

$W_1 = W_2 = 0$

**W**

$W_1$ (ms$^{-1}$)

**t (sec)**

---

**SIMULATION RESULTS FOR CASES 1c TO 5c**

<table>
<thead>
<tr>
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<th>CASE</th>
</tr>
</thead>
<tbody>
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<td></td>
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</tr>
<tr>
<td></td>
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<tr>
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<td>3c</td>
</tr>
<tr>
<td></td>
<td>4c</td>
</tr>
<tr>
<td></td>
<td>5c</td>
</tr>
</tbody>
</table>
FIG. 6.10 DYNAMIC EFFECTS ON THROTTLE RESPONSE OF 0.16 HEADWIND POWER LAW PROFILE
Hatched region represents the discrepancy between actual throttle settings and the trim settings

FIG. 6.11 DYNAMIC EFFECTS ON THROTTLE RESPONSE OF CONSTANT WIND SHEAR
Hatched region represents the discrepancy between the actual throttle settings and the trim settings
FIG. 6.12 SIMULATION RESULTS FOR CASES 6a TO 10a

No glidepath deviation feedback
FIG. 6.12(continued)  SIMULATION RESULTS FOR CASES 6a TO 10a

No glidepath deviation feedback
\( V_A = 40 \text{ms}^{-1} \)

**CONVENTIONAL GP**

\[ \Delta V (\text{ms}^{-1}) \]

**LOOC3 CONTROLLER**

\[ 10^\theta_B (\text{rad.}) \]

- **FIG. 6.13a**
- **FIG. 6.13b**
- **FIG. 6.13c**
- **FIG. 6.13d**

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>□</th>
<th>○</th>
<th>△</th>
<th>+</th>
<th>×</th>
</tr>
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<tbody>
<tr>
<td>CASE</td>
<td>6b</td>
<td>7b</td>
<td>8b</td>
<td>9b</td>
<td>10b</td>
</tr>
</tbody>
</table>

**FIG. 6.13** SIMULATION RESULTS FOR CASES 6b TO 10b

No glidepath deviation feedback
**FIG. 6.13e**

\[ V_A = 40 \text{ms}^{-1} \]

**CONVENTIONAL GP**

**LOOC3 CONTROLLER**

**APTc1**

\[ \delta E \text{ (rad.)} \]

**FIG. 6.13f**

\[ W_2 = W_3 = 0 \]

\[ W_1 \text{ (ms}^{-1}) \]

**FIG. 6.13g**

**SYMBOL**

<table>
<thead>
<tr>
<th>CASE</th>
<th>6b</th>
<th>7b</th>
<th>8b</th>
<th>9b</th>
<th>10b</th>
</tr>
</thead>
</table>

**FIG. 6.13(continued)**  SIMULATION RESULTS FOR CASES 6b TO 10b

No glidepath deviation feedback
FIG. 6.14a GLIDEPATH DEVIATION RMS VALUES FOR CASES 6 TO 10 IN TERMS OF WIND VARIABILITY AS MEASURED BY $\tilde{W}_t$

**SYMBOLS REPRESENT COMPUTED POINTS**

- CGG GP, APTCI, CASES 6a - 10a
- CONVENTIONAL GP, APTCI, CASES 6b - 10b

FIG. 6.14b GLIDEPATH DEVIATION RMS VALUES FOR CASES 6 TO 10 IN TERMS OF WIND VARIABILITY AS MEASURED BY $\tilde{W}_t$

**SYMBOLS REPRESENT COMPUTED POINTS**

- CGG GP, APTCI, CASES 6a - 10a
- CONVENTIONAL GP, APTCI, CASES 6b - 10b

FIG. 6.15 DYNAMIC EFFECTS ON THROTTLE RESPONSE OF 0.16 TAILWIND POWER LAW PROFILE

Hatched region represents the discrepancy between actual throttle settings and the trim settings.
Fig. 6.16a Glidepath deviation RMS values for cases 12 to 16.

Symbols represent computed points:
- CGG GP3, APTC2, cases 12a-16a
- Conventional GP, APTC2, cases 12b-16b

Fig. 6.16b Throttle rate RMS values for cases 12 to 16.

Symbols represent computed points:
- CGG GP3, APTC2, cases 12a-16a
- Conventional GP, APTC2, cases 12b-16b

Fig. 6.17 Comparison of airspeed and glidepath deviation RMS values for headwind and tailwind 0.16 power law profiles.
FIG. 6.18 SIMULATION RESULTS FOR CASES 17a TO 19a
$V_A = 40 \text{ms}^{-1}$

$5E$ (rad.)

$\delta_0$

$W_1 (\text{ms}^{-1})$

$W_2 (\text{sec})$

SIMULATION RESULTS FOR CASES 17a TO 19a
FIG. 6.18 (continued)  SIMULATION RESULTS FOR CASES 17a TO 19a
$V_A = 40 \text{ms}^{-1}$

CONTROLLERS

GP3

LAOC1

APTC2

$\delta_A$ (rad.)

FIG. 6.18f

$\omega_2$ (ms$^{-1}$)

FIG. 6.18m

$W_3 = 0$

FIG. 6.18n

SYMBOL

CASE 17a 18a 19a
FIG. 6.19 SIMULATION RESULTS FOR CASES 17b TO 19b
$V_A = 40 \text{ms}^{-1}$

LOOC3
LAOC1
CONTROLLERS

CONVENTIONAL GP
APTC2

FIG. 6.19e

$\theta_e$ (rad.)

FIG. 6.19f

$\omega_0$ (ms$^{-1}$)

FIG. 6.19g

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>□</th>
<th>○</th>
<th>△</th>
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<tbody>
<tr>
<td>CASE</td>
<td>17b</td>
<td>18b</td>
<td>19b</td>
</tr>
</tbody>
</table>

FIG. 6.19 (continued)  SIMULATION RESULTS FOR CASES 17b TO 19b
FIG. 6.19 (continued)  SIMULATION RESULTS FOR CASES 17b TO 19b
FIG. 6.19

V_A = 40 m/s

$\delta_A$ (rad.)

CONVENTIONAL GP

APTC2

FIG. 6.19l

$\delta_R$ (rad.)

FIG. 6.19m

$W_2$ (ms$^{-1}$)

FIG. 6.19n

W_3 = 0

TABLE

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<td>○</td>
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<td>○</td>
<td>18b</td>
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<tr>
<td>△</td>
<td>19b</td>
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</tbody>
</table>

FIG. 6.19 (continued) SIMULATION RESULTS FOR CASES 17b TO 19b
**FIG. 6.20** AIRSPEED RESPONSE FOR FLIGHT THROUGH JFK WIND PROFILES

- CURVED APPROACH, CASE 23a
- CONVENTIONAL APPROACH, CASE 23b

\[ V_A = 40 \text{ ms}^{-1} \]

**FIG. 6.21** FLIGHT PATH IN THE PRESENCE OF THE JFK PROFILES

- CURVED APPROACH, CASE 23a
- CONVENTIONAL APPROACH, CASE 23b

\[ V_A = 40 \text{ ms}^{-1} \]
FIG. 6.22 EFFECT OF THROTTLE LAG VARIATION FOR THE JFK CASES
FIG. 6.23 SIMULATION RESULTS FOR CASES 27a TO 31a (WIND CONTROLLER CASES)
\[ V_A = 40 \text{ms}^{-1} \]

LOOC2 CONTROLLER

\[ \delta_E (\text{rad.}) \]

GP1

APTC1

\[ h = 100 \text{m} \]

FIG. 6.23e

\[ W_2 = 0 \]

\[ t (\text{sec}) \]

\[ W_1 (\text{ms}^{-1}) \]

\[ h = 100 \text{m} \]

FIG. 6.23g

\[ \omega_n (\text{rad.} \text{sec}^{-1}) \]

\[ h = 100 \]

FIG. 6.23f

\[ W_3 (\text{ms}^{-1}) \]

\[ t (\text{sec}) \]

\[ h = 100 \text{m} \]

FIG. 6.23h

FIG. 6.23(continued) SIMULATION RESULTS FOR CASES 27a TO 31a (WIND CONTROLLER CASES)
FIG. 6.24 SIMULATION RESULTS FOR CASES 27b TO 31b (WIND CONTROLLER CASES)

$V_A = 40 \text{m s}^{-1}$

CONVENTIONAL GP

LOOC3

APTC1

\[ \Delta V (\text{m s}^{-1}) \]

\[ \theta_B (\text{rad.}) \]

\[ d (\text{m}) \]

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>$\square$</th>
<th>$\bigcirc$</th>
<th>$\triangle$</th>
<th>$+$</th>
<th>$\times$</th>
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<tbody>
<tr>
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<td>28b</td>
<td>29b</td>
<td>30b</td>
<td>31b</td>
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</tbody>
</table>

FIG. 6.24a

FIG. 6.24b

FIG. 6.24c

FIG. 6.24d
FIG. 6.24e

\[ \theta_E (\text{rad.}) \]

\[ h = 100 \text{m} \]

FIG. 6.24f

\[ \delta_n \]

\[ h = 100 \text{m} \]

FIG. 6.24g

\[ W_1 (\text{ms}^{-1}) \]

\[ W_2 = 0 \]

FIG. 6.24h

\[ W_3 (\text{ms}^{-1}) \]

\[ h = 100 \text{m} \]

FIG. 6.24 (continued)  SIMULATION RESULTS FOR CASES 27b TO 31b (WIND CONTROLLER CASES)
FIG. 6.25a AIRSPEED DEVIATION RMS VALUES IN PRESENCE OF WORST-CASE WIND CONTROLLER BASED ON DG METHOD, EXAMPLE 1

FIG. 6.25b GLIDEPATH DEVIATION RMS VALUES IN PRESENCE OF WORST-CASE WIND CONTROLLER BASED ON DG METHOD, EXAMPLE 1

FIG. 6.26a AIRSPEED RESPONSE IN THE PRESENCE OF A WIND CONTROLLER BASED ON EXAMPLE 1 OF DIRECT METHOD EXAMPLES
FIG. 6.26b  GLIDEPATH DEVIATION IN THE PRESENCE OF A WIND 
CONTROLLER BASED ON EXAMPLE 1 OF THE DIRECT METHOD EXAMPLES
FIG. 6.27 WIND PROFILES RESULTING FROM APPLICATION OF A WIND CONTROLLER BASED ON EXAMPLE 1 OF THE DIRECT METHOD EXAMPLES

FIG. 6.28 QUALITATIVE EFFECTS OF AN ALTERNATE MATCHING CONDITION ON THE CURVED GLIDEPATHS (DECREASING HEADWIND CASES)
7. **SUMMATION AND RECOMMENDATIONS FOR FUTURE WORK**

Analytical and computational techniques that may be applied to predicting aircraft response to hazardous variable winds encountered on the landing approach and to modeling such wind conditions have been investigated. An extensive literature review has identified a number of areas requiring further study. Of these areas, worst-case wind modeling techniques and modified glidepath geometries based on an estimate of the existing wind profile have been pursued extensively.

A number of extensions and new formulations have been developed for worst-case wind modeling techniques where the form of the wind inputs is not specified a priori. These include the following:

1. **The conceptual artifice of viewing the wind as an intelligent adversary that opposes the objectives of the aircraft autopilot or human pilot** - This intelligence manifests itself through a control law which uses information about the aircraft's and aircraft controller's behavior to generate the worst possible wind inputs given certain soft constraints on the wind characteristics. This is tantamount to closing the loop on the wind.

2. The implementation of this intelligent adversary concept with one-sided and two-sided (differential game) optimization theory to generate worst-case wind control laws, and to examine the response of the aircraft system under the influence of these control laws — This step involved the development of software to solve linear quadratic differential game problems for aircraft dynamic systems.

3. The consideration of the potential application of such worst-case models to flight simulator wind modeling — Preliminary results from the flight simulator application have been presented.

4. The extension of van der Vaart's technique to the more general impulsive response worst-case wind modeling techniques.

The recommendations for future work in the worst-case wind modeling area are as follows:

1. A thorough assessment of the usefulness of the worst-case wind controller models to flight simulator wind modeling including assessments of pilot workload, model realism, usefulness for training purposes, and the development of real time techniques of controlling the wind caused workload to which the pilot is exposed. These flight simulator applications are currently being pursued at UTIAS.

2. A thorough assessment of the application of the worst-case functional maximization wind modeling methods (not necessarily wind controller methods) to certain aspects
of autoland certification - This assessment requires considera-
tion of nonlinear techniques and would ultimately require
comparison and validation with existing methods (e.g. Jone's
statistical discrete gust method and Monte Carlo techniques).
The application of these methods to other aspects of air-
craft certification (e.g. structural certification) and
indeed to other dynamic problems, may also prove to be
fruitful.

3. The development of theoretical and computational
techniques to aid in finding worst-case wind models that are
constrained to certain wind velocity conditions near the
ground, as well as satisfying certain isoperimetric con-
straints (the latter type of constraint has been considered
in some detail in this study).

4. An assessment of the loss of optimality (sensitivity)
of the worst-case wind models in suboptimal applications.

5. The application of the worst-case methods to air-
craft dynamic systems that are transformed into modal
coordinates - In this way worst-case solutions may be determined
that stimulate particular dynamic modes. This technique is
motivated by the recent work of Turkel and Frost [5.29] and
van der Vaart [1.61], as well as the results of this investi-
gation, that suggest that such disturbances (particularly
those that stimulate the phugoid mode) are the most
hazardous.

6. The application of the worst-case techniques
developed in this study to crosswind modeling.

7. The application of optimal stochastic techniques to
worst-case wind modeling.

The simulation results for aircraft tracking curved
glidepaths based on an estimate of the existing wind profile
and with glidepath matching conditions at the transition
height have shown that for many wind scenarios the curved
glidepath approaches have no performance or workload advantage
over conventional approaches. The exceptions to this include
(1) increasing headwind or decreasing tailwind (with
decreasing altitude) cases and (2) decreasing headwind cases
where the wind shear is nearly constant throughout the
approach (e.g. linear wind profiles). In the former the
advantage is obtained in airspeed and glidepath tracking
performance, and in the latter the advantage is obtained in
the reduced throttle activity required once the aircraft
controller has compensated for the nearly constant wind shear.

The major factor contributing to the better performance
on conventional glidepaths was identified as the generally
lower rates of descent of the aircraft compared to the
the corresponding curved approaches (for the usual decreasing
headwind scenario). For a given approach airspeed and wind
profile this results in greater wind shear effects on the
aircraft making the curved approach. These results continued
to hold for highly variable wind conditions, including strong
downdrafts (e.g. the JFK profile and wind controller model...
examples of Chapter 6).

A number of possible extensions to the curved approach work were identified for future consideration. These include the following:

1. An assessment of wind-based CGG that replace the glide slope matching condition (3.7,15) with (6.3,1). This should also include an assessment of the touchdown dispersion of aircraft tracking such curved glidepath geometries, and would thus require suitable flare models.

2. An assessment of the glidepath tracking performance for curved approaches made with the throttle fixed to maintain a constant rate of descent. The assessment of airspeed tracking performance and the effects of incorrect estimates of the existing winds are also very important to this type of an approach procedure.

3. An evaluation of wind-based CGG with classical controller models including dynamic command generators.

4. A parametric study of the effects of transition height glidepath angles and different approach airspeeds.

5. The development of simplified methods of generating curved glidepath geometries that take into account some of the dynamic effects of the variable winds.

6. A flight simulator evaluation of wind-based curved glidepath geometries, including pilot opinion studies, to expand on the preliminary airborne simulator work of Hindson and Smith [1.61].

As well as the recommendations arising directly from the worst-case wind modeling and the curved glidepath geometry work, a number of areas for further investigation have been identified through the literature review. These include the following:

1. Suboptimal use of aerodynamic potential in strong variable wind conditions has been implicated in a number of aircraft accidents both in the go-around and take-off phases of flight. The FAA is currently evaluating modified go-around procedures in flight simulators (see the discussion in Chapter 4, Section 4.2) that take advantage of available aerodynamic potential but temporarily allow airspeed to fall below the reference values. Such go-around procedures should also be investigated with dynamic simulations including both automatic and human pilots. In particular, consideration should be given to the synthesis of suitable wind shear go-around modes in flight director and autopilot systems. This might also include the development of approach limits (based, e.g. on some function of altitude, inertial and air velocities) that when exceeded cause the automatic pilot to go into the wind shear go-around mode. This work is currently being pursued at UTIAS.

2. The development and validation of nondimensional payoff functionals which may be used to generate good optimal
controllers for a class of aircraft — These controllers may then be used to compare the closed-loop behavior of a large number of aircraft to variable wind conditions, e.g. as would be required in a parametric study of aircraft closed-loop variable wind response characteristics versus aircraft characteristics.

3. Luers and Haine [1.51], in a recent NASA study, have suggested that an important factor in some wind related accidents may have been the degradation of aerodynamic capability and the loss of momentum caused by the heavy rains associated with the highly variable wind conditions. A continuing assessment of this factor is thus both timely and important, and involves the difficult task of predicting the aerodynamic and dynamic effects of heavy rain on aircraft.
APPENDIX A

NOTATION CONVENTIONS

The following summarizes the notation conventions used in this study. In order to enhance the readability of the text, most notations are also defined at the point of their first appearance.

A.1. Reference Frames

Reference frames will be denoted with $F_s$; subscripts will act as descriptors. Thus, e.g., the reference frame $A$ is denoted by $F_A$.

A.2. Vectors

1. Vectors will be denoted with an underscore, e.g. $\mathbf{a}$. Subscripts will act as descriptors. The *dot product* will be denoted by $\cdot$, e.g. $\mathbf{a} \cdot \mathbf{b}$, and the *cross product* by $\times$, e.g. $\mathbf{a} \times \mathbf{b}$.

2. The dextral, orthonormal vector triad associated with a particular reference frame $F_A$ will be denoted $(\mathbf{i}_A, \mathbf{j}_A, \mathbf{k}_A)$. In certain cases where the reference frame intended is clear, the subscripts may be omitted for the sake of brevity.

3. The gradient vector will be denoted $\nabla$. A subscript, e.g. $\nabla_A$, will denote the reference frame with respect to which the gradient is taken.

A.3. Matrices

1. Matrices will be denoted by an underscore, e.g. the matrix $A$ will be written $\mathbf{A}$. The *transpose* of $\mathbf{A}$ will be denoted $\mathbf{A}^T$.

2. All of the elements of a matrix will be referred to by $[a_{ij}]$, where $i$ is the row position and $j$ is the column position of an element of the matrix. The $(i,j)$-th element of $\mathbf{A}$ will be referred to as $a_{ij}$.

3. Column matrices (sometimes referred to as vectors) will normally be written with lower case letters, e.g. $\mathbf{a}$. Exceptions will be clear from the context of the text.

4. Column matrices which express the components of a vector in a particular reference frame will be written with the reference frame identifier as a superscript. Thus the components of $\mathbf{a}$, expressed in $F_A$ will be written $\mathbf{a}^A$. Any vector descriptors remain as subscripts on the corresponding column matrix (e.g. the matrix equivalent of $\mathbf{a}$ written as components in $F_A$ would be $\mathbf{a}^A_A$).

5. Rotation matrices relating the components of a vector in one reference frame to the components in a second will be written $\mathbf{L}_{BA}$ where

$$
\mathbf{v}^B = \mathbf{L}_{BA} \mathbf{v}^A
$$

(A.3.1)

6. The *identity matrix* will usually be written $\mathbf{I}$. The *zero matrix* will be written $\mathbf{0}$. 
7. Inertia matrices will generally be written $A^B$ where $A$ is a descriptor relating to the object whose inertia is being specified and $B$ is the reference frame descriptor in which the inertia components are specified. In certain cases, where the intended meaning is clear from the context, one or both descriptors may be dropped.

8. The matrix equivalent of the vector cross product will be denoted with an overscript $\cdot$. Thus the matrix equivalent of $\mathbf{a} \times \mathbf{b}$, written as components in $F_A$, is $A_a^A b^A$ where

$$A_a^A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

and $(a_1, a_2, a_3)$ are the components of $\mathbf{a}$ in $F_A$.

9. Matrix partitions will be written

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{bmatrix}$$

where the dashed lines are added only when required for clarity.

The carat symbol $\hat{\cdot}$ will denote augmented matrices, e.g. $\hat{\mathbf{a}}^T = [\mathbf{a}_1^T \mathbf{a}_2^T]$. A carat symbol over a nonmatrix variable will denote the root-mean-square value of that variable.

10. By

$$\hat{A} = \text{diag}[\hat{a}] = \text{diag}[\hat{a}^T],$$

where $\hat{a}$ is an $n \times 1$ column matrix, it is meant the diagonal matrix $\hat{A}$ such that

$$A_{ij} = \delta_{ij} a_i, \quad i, j = 1, n$$

where $\delta_{ij}$ is the Kronecker delta function defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j. \end{cases}$$

The $\text{sgn}$ function is defined by

$$\text{sgn}[\mathbf{A}(t)] = \{\text{sgn} a_{ij}(t)\}, \quad i = 1, m; \quad j = 1, n$$

for a general, time-varying $m \times n$ matrix $\mathbf{A}$, where

$$\text{sgn}[a_{ij}(t)] = \begin{cases} +1, & a_{ij}(t) > 0, \text{ or } a_{ij}(t) = 0 \text{ with } a(t-\varepsilon) > 0 \\ -1, & a_{ij}(t) < 0, \text{ or } a_{ij}(t) = 0 \text{ with } a(t-\varepsilon) < 0 \\ 0, & \text{otherwise} \end{cases}$$

for arbitrarily small positive $\varepsilon$. 
With these definitions it follows that

\[ |x| = \text{diag}[\text{sgn}(x)]x \]  
(A.3,9)

and

\[ |x^T| = x^T \text{diag}[\text{sgn}(x^T)] 
= x^T \text{diag}[\text{sgn}(x)]. \]  
(A.3,10)

11. The following matrix differentiation conventions will be employed (see also Kirk [4.17]):

(i) \[ \frac{\partial A}{\partial x} = [\partial a_{ij}/\partial x] \]  
(A.3,11)

(ii) \[ \frac{\partial s(y)}{\partial y} = \begin{bmatrix} \frac{\partial s(y)}{\partial y_1} \\ \vdots \\ \frac{\partial s(y)}{\partial y_m} \end{bmatrix} \]  
(A.3,12)

where \( y \) is \( m \times 1 \).

(iii) \[ \frac{\partial [y^T M z]}{\partial y} = M z \]  
(A.3,13)

where \( y \) is \( m \times 1 \), \( z \) is \( m \times 1 \) and \( M \) is \( m \times m \).

(iv) \[ \frac{\partial}{\partial y}[y^T M y] = M y + M^T y. \]  
(A.3,14)

12. A symmetric matrix \( A \) will be called positive-definite if

If \( M \) is symmetric, then

\[ \frac{\partial}{\partial y}[y^T M y] = 2M y. \]  
(A.3,15)

\[ \frac{\partial}{\partial y}[y^T M y] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix} \]  
(A.3,16)

where \( a \) is \( m \times 1 \), \( y \) is \( m \times 1 \) and \( z \) is \( m \times 1 \).

\[ \frac{\partial^2 s(y)}{\partial y^2} = \begin{bmatrix} \frac{\partial^2 s}{\partial y_1^2} & \frac{\partial^2 s}{\partial y_1 \partial y_2} & \cdots & \frac{\partial^2 s}{\partial y_1 \partial y_m} \\ \frac{\partial^2 s}{\partial y_2 \partial y_1} & \frac{\partial^2 s}{\partial y_2^2} & \cdots & \frac{\partial^2 s}{\partial y_2 \partial y_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 s}{\partial y_m \partial y_1} & \frac{\partial^2 s}{\partial y_m \partial y_2} & \cdots & \frac{\partial^2 s}{\partial y_m^2} \end{bmatrix} \]  
(A.3,17)

where \( s \) is a scalar function of the \( m \times 1 \) matrix \( y \).

(vii) If \( R \) is a real, symmetric \( m \times m \) matrix and \( y \) is an \( m \times 1 \) matrix, then

\[ \frac{\partial^2}{\partial y^2}[y^T R y] = 2R. \]  
(A.3,18)
\( z^T A z > 0 \) \hspace{1cm} (A.3.19)

for all \( z \neq 0 \), and will be written
\( A > 0 \). \hspace{1cm} (A.3.20)

If
\( z^T A z \geq 0 \) \hspace{1cm} (A.3.21)

for all \( z \), then \( A \) is positive semi-definite, and will be
written
\( A \geq 0 \). \hspace{1cm} (A.3.22)

Analogous definitions apply for negative-definite and negative
semi-definite matrices.

13. The Euclidean norm of a column matrix will be denoted
by \( \| \cdot \| \). Thus
\[ \| y \| \hat{=} (y^T y)^{1/2}. \] \hspace{1cm} (A.3.23)

The weighted norm will be defined
\[ \| y \|_A \hat{=} (y^T A y)^{1/2}. \] \hspace{1cm} (A.3.24)

14. It will be general notational policy for matrices to
limit the use of subscripts and superscripts as much as possible.

Thus, for example, \( W_L = [W_L, W_L] \) in Chapter 5, Section 5.5,
and \( \overline{W}_L = [W_L, W_L, W_L] \) in Chapter 2, Section 2.2.1, as is
respectively appropriate for the longitudinal and general
definitions. Similarly, \( A_2 \) is used to refer to the state
distribution matrix of both the longitudinal and lateral
equations of motion. This practice is not followed in sections
where it is not clear from the context what the order and
content of these matrices should be.

15. The subscript \( \infty \) will denote the value of a time
varying matrix as \( t \) approaches infinity, e.g. \( L_{\infty} \).

A.4. The subscript \( e \) on a quantity will refer to the lineariza-
tion reference equilibrium value of that quantity, e.g. \( V_e \).
The subscript \( c \) will refer to the value of a quantity
related to curved glidepath geometries, e.g. \( \theta_c \). A super-
script \( c \) will refer to a commanded control input, e.g.
\( \delta_T^c \).

A.5. Perturbation quantities from the linearization reference
equilibrium will be preceded by \( \Delta \) unless otherwise stated
and will be defined
\[ \Delta a \hat{=} a - a_e. \] \hspace{1cm} (A.5.1)

In classical controller synthesis it is common practice
to define deviations from a desired value as
A.6. The Laplace transform of a function \( f(t) \) will be denoted by \( \mathcal{L}[f(t)] \) or by \( f(s) \), or simply \( f \) if the context is clear. The Laplace variable will be denoted by \( s \).

A.7. Nondimensionalized quantities will be denoted with an overscore \( \sim \), e.g.
\[
\tilde{u} = u/V_e \quad \text{(A.7,1)}
\]

A.8. Aerodynamic Derivatives

Nondimensional aerodynamic derivatives will be written with a \( C \). Superscripts will indicate the reference frame in which the stability derivatives are expressed. For example, one has
\[
\frac{\partial \tilde{C}^B_e}{\partial \tilde{u}_B} \quad \text{(A.8,1)}
\]

The superscript will be omitted for aerodynamic derivatives written in \( F_S \).

Dimensional aerodynamic derivatives omit the \( C \), e.g.
\[
X_u = \frac{\partial X}{\partial u} \quad \text{(A.8,2)}
\]

The dimensional derivatives so defined are not normalized with mass or inertia moments, as is done by some authors (e.g. McRuer et al. [3.3]).

A.9. Optimal and Differential Games Theory

1. Optimal quantities will be denoted with a superscript \( \ast \), e.g. \( y^\ast \). In differential games a superscript \( \ast \) will denote quantities related to the Nash equilibrium. An overscore \( \tilde{\ast} \) will denote a minimax solution (e.g. \( \tilde{y} \)). In the two player, zero-sum sense the Nash and minimax solutions are the same and only the \( \ast \) notation is required. Some variables using a superscript \( \ast \) (e.g. \( L^\ast \)) will not imply an optimized quantity. This will be clear from the context.

2. The variation of a quantity will be denoted by \( \delta \), e.g. \( \delta J \).

3. Performance criteria, payoff functions and so forth will be denoted by \( J \). Subscripts and superscripts will be used as identifiers if required. A subscript \( \prime \) will denote an augmented payoff functional, e.g. \( J_a \).

4. The Hamiltonian will be denoted by \( H \). An augmented Hamiltonian will be denoted by \( H_a \).

5. By \( t^+ \) it is meant the time just after \( t \) and by \( t^- \) the time just before (more precisely \( t + \epsilon \) and \( t - \epsilon \) respectively for arbitrarily small positive \( \epsilon \)).

6. Round brackets will mean open intervals, and square brackets closed intervals, e.g.
\[
t \in [t_1, t_2) \text{ means } t_1 \leq t < t_2.
\]
APPENDIX B

SIMPLIFYING ASSUMPTIONS AND THEIR JUSTIFICATION

Section 3.1 of Chapter 3 lists a number of fundamental assumptions which are usually made in studying aircraft rigid body response on the landing approach. In this appendix their validity is considered in more detail.

1. The Earth is flat and any Earth-fixed reference frame is inertial.

These assumptions are intuitively acceptable because of the long rotational period and large radius of the Earth compared to typical aircraft approach times and approach distances (see also the comments in Roskam [B.1], Chapter 2). One may numerically check inertiality of the Earth reference frame by considering order of magnitude estimates of the real and apparent accelerations that an aircraft will experience in the rotating Earth reference frame and comparing them to technically important aircraft accelerations. The apparent accelerations may be determined from the kinematical expression

\[ \ddot{\mathbf{r}} = \ddot{\mathbf{r}}^\text{e} + \dot{\omega} \times \mathbf{r} + 2 \dot{\omega} \times \dot{\mathbf{r}} + \omega \times (\omega \times \mathbf{r}) \]  

(B.1)

where \( \ddot{\mathbf{r}} \) is the acceleration of the aircraft with respect to the inertial reference frame, \( \ddot{\mathbf{r}}^\text{e} \) is the acceleration with respect to the rotating reference frame, \( \mathbf{r} \) is the aircraft position vector and \( \omega \) is the angular velocity of the Earth with respect to the inertial reference frame. Similarly, one may check curvature effects by considering typical approach distances and determining the arc that they subtend on the surface of the Earth. Etkin [3.1] considers these effects in a more precise way by looking at the corresponding terms in the general equations of motion.

To summarize, the following order of magnitude estimates may be made:

(i) Earth's angular acceleration effects \( -(\dot{\omega} \times \mathbf{r}) \) - This is completely negligible in the context of aircraft flight.

(ii) Centrifugal acceleration \( -(\omega \times (\omega \times \mathbf{r})) \) - We have \( \omega \approx 7 \times 10^{-5} \text{ rad/s} \) and \( O(r) = 10^6 \text{ m} \) and thus this term is negligible also.

(iii) Coriolis acceleration \( -2\dot{\omega} \times \dot{\mathbf{r}} \) - We have \( O(\dot{r}) = 10^2 \text{ ms}^{-1} \) for landing approach and thus this term is also quite small in this context. However this is not the case at high subsonic and supersonic speeds.

* The notation \( O(a) \) denotes the order of magnitude of the bracketed symbol. A subscript \( e \) means that the corresponding subscripted symbol represents an Earth property.
(iv) Curvature effects are negligible over landing approach distances \( (10^5 \text{ m}) \) as can be seen by comparing these distances with the radius of the Earth \( (0(r_\oplus) = 10^7 \text{ m}) \). This effect is not negligible for longer flight distances, e.g. as in range calculations.

2. The aircraft is a rigid body.

If flexibility effects are to be dynamically taken into account, additional equations of motion for the extra degrees of freedom are required, and this considerably complicates the system to be studied from both the analytical and the computational perspectives. The extra accuracy that would be gained in using such an analysis depends very much on the characteristics of the particular aircraft being studied. It is frequently the case, however (Etkin [B.2], p. 135 and following) that the natural frequencies of the structural modes and the rigid body modes are far apart. In such cases the method of \textit{quasistatic deflections} may be used. In this method it is assumed that structural distortions are taken into account by equilibrium changes in the aerodynamic stability derivatives normally associated with the rigid body equations. Thus no extra equations are required for the structural dynamics. There also appears to be no evidence to indicate that dynamic flexibility effects have been a significant factor in the types of low altitude hazardous wind encounters being studied in this investigation.

3. Air density, temperature and pressure, and the gravitational field are constant.

Of interest to this study is flight through the lower 300 meters or so of the Earth's atmosphere. Across this region the density change is approximately 3%. Thus, for example, the corresponding change in the lift coefficient required to maintain constant lift going from \( h = 300 \text{ m} \) to \( h = 0 \) will be approximately \(-3\% \) relative to the original value. Similar arguments may be made for temperature and pressure, and thus such effects will be assumed to be negligible.

The value of the acceleration due to gravity \( (g) \) as a function of altitude is given by

\[
g = \frac{GM_\oplus}{(r_\oplus + h)^2} \tag{B,2}
\]

where \( G \) is the universal constant of gravitation, \( M_\oplus \) is the mass of the Earth and \( r_\oplus \) is the radius of the Earth. Thus at the surface

\[
\left. \frac{dg}{dh} \right|_{h=0} = \frac{GM_\oplus}{r_\oplus^3} \tag{B,3}
\]

It follows that

\[
\frac{\Delta g}{g} = \frac{\Delta h}{r_\oplus} \tag{B,4}
\]
But \( \Delta h = 10^2 \text{m} \ll \Delta (r_g) = 10^7 \text{m} \), and thus \( \Delta g \) may be considered to be negligible.

4. The mass properties of the aircraft are constant.

Taking into account mass property changes in the aircraft, usually due to fuel usage, complicates the equations of motion by introducing time varying masses and inertias. To model these effects properly would require equations describing mass usage and distribution. This is very much an aircraft dependent proposition.

A typical 3 degree glide slope jet transport landing approach at 80 ms\(^{-1}\) over a 300 m altitude takes approximately 75 s. In this time, a 747 type aircraft will burn approximately 300 kg of fuel. This may be compared to an overall mass of approximately \( 2 \times 10^5 \text{kg} \), depending on the payload and remaining fuel. Thus for the landing approach such effects will be considered to be negligible. Furthermore, small overall mass change also implies a small inertia change provided that there are no significant mass shifts (e.g. fuel sloshing, intentional fuel and payload redistribution and so forth).

5. The aircraft has a plane of symmetry.

This is true for the vast majority of aircraft and for the example aircraft considered. In cases where it is not, this assumption may be readily removed by not making the symmetry assumptions on the inertia matrix given in Chapter 3, Section 2.2.

6. The effects of internal and external rotors, articulation of the controls, and fuel sloshing are negligible to the gross rigid body motions of the aircraft.

Previous experience has shown this to be a valid approximation for most CTOL and STOL fixed-wing aircraft (see the comments in Roskam [B.1], Chapter 2), particularly in low altitude flight. McRuer and Wolkovitch [B.3] develop linearized equations of motion applicable to aircraft where the internal angular momenta cannot be neglected. While the major analytical consequence is the coupling between the longitudinal and lateral linearized equations, of more practical concern is the problem of accurately estimating the contributions of the rotors to the total angular momenta. For engine contributions this would require a knowledge of the moments of inertia of the rotating parts as a function of at least throttle setting.
REFERENCES — APPENDIX B


All of the digital simulations are for a two-engined light STOL transport whose mass and geometric properties are summarized below, and whose aerodynamic characteristics will be summarized in Appendix D. The simulated aircraft is not intended to accurately model an existing aircraft, but merely to preserve the general response characteristics of this class of flight vehicle. The data is based on that given in Refs. 1.39 and C.1, and on DeHavilland of Canada company data for a preliminary design of a two-engined STOL transport. While general data trends were usually preserved, much of the data (particularly aerodynamic data) was altered for the purposes of this study.

The aircraft mass and geometric characteristics are as follows:

\[ m = 4990 \text{ kg} \quad (341.9 \text{ slugs}) \]
\[ S = 39.0 \text{ m}^2 \quad (420 \text{ ft}^2) \]
\[ c = 2.0 \text{ m} \quad (6.5 \text{ ft}) \]
\[ b = 20.0 \text{ m} \quad (65 \text{ ft}) \]
\[ AR = 10 \]

nominal longitudinal centre of mass position = 0.2 \( \bar{c} \)
centre of mass position below wing:

aerodynamic centre = 0.43 \( \bar{c} \)
thrust line above centre of mass position = 0.45 \( \bar{c} \)

\[ \epsilon_T = 0^\circ \]
\[ \Gamma = 3^\circ \]
\[ l_1 = 7.62 \text{ m} \quad (25 \text{ ft}) \]
\[ l_2 = 7.83 \text{ m} \quad (25.7 \text{ ft}) \]
\[ I_{xx}^B = 21621 \text{ kg} \cdot \text{m}^2 \quad (15947 \text{ slug} \cdot \text{ft}^2) \]
\[ I_{yy}^B = 31824 \text{ kg} \cdot \text{m}^2 \quad (23472 \text{ slug} \cdot \text{ft}^2) \]
\[ I_{zz}^B = 48857 \text{ kg} \cdot \text{m}^2 \quad (36035 \text{ slug} \cdot \text{ft}^2) \]
\[ I_{xz}^B = 1482 \text{ kg} \cdot \text{m}^2 \quad (1093 \text{ slug} \cdot \text{ft}^2) \]
\[ P_{max} = 4.8470 \times 10^5 \text{ w} \quad (650 \text{ HP}) \text{ per engine} \]
\[ T \delta_T = 0 \]

CONTROL LIMITS

| \( \delta_E \) | minimum \(-0.436 \text{ rad.} / -25^\circ\) | maximum \(0.279 \text{ rad.} / 16^\circ\) |
| \( \delta_T \) (throttle fraction) | 0 | 1 |
| \( \delta_A \) | minimum \(-0.349 \text{ rad.} / -20^\circ\) | maximum \(0.349 \text{ rad.} / 20^\circ\) |
| \( \delta_R \) | minimum \(-0.349 \text{ rad.} / -20^\circ\) | maximum \(0.349 \text{ rad.} / 20^\circ\) |
REFERENCE — APPENDIX C

APPENDIX D

STOL TRANSPORT AERODYNAMIC CHARACTERISTICS AND NATURAL MODES

This appendix presents the STOL aircraft's aerodynamic characteristics in forms suitable for application to the AERO1 and AERO2 models of Chapter 3, Section 3.5.2. Relationships between the two models and between the nondimensional and the dimensional aerodynamic derivatives are also summarized.

D.1 The AERO1 Model in More Detail

The dimensional aerodynamic derivatives in (3.5,10) through to (3.5,15) may be written in terms of their nondimensional counterparts as follows (see also Etkin [B.2] and Reid, Markov and Graf [1.39]):

Longitudinal Stability Axes Dimensional Derivatives

\[ X_u = \rho V e \tan \delta_e + \frac{1}{2} \rho V e \xi_x \delta_u \]  
(D.1,1)

\[ X_w = \frac{1}{2} \rho V e \xi_x \delta_w \]  
(D.1,2)

\[ X_q = t q e \xi_x \delta_q \]  
(D.1,3)

\[ Z_u = -\rho V e \xi SC_{L_\delta} + \frac{1}{2} \rho V e \xi z_u \]  
(D.1,4)

\[ Z_w = \frac{1}{2} \rho V e \xi z_w \]  
(D.1,5)

\[ Z_q = t q e \xi z_q \]  
(D.1,6)

\[ Z_{\delta} = \frac{1}{2} t q e \xi \delta T \]  
(D.1,7)

\[ M_u = \frac{1}{2} \rho V e \xi m_u \]  
(D.1,8)

\[ M_w = \frac{1}{2} \rho V e \xi m_w \]  
(D.1,9)

\[ M_{\delta} = \frac{1}{2} t q e \xi \delta m \]  
(D.1,10)

\[ M_{\delta T} = t q e \xi \delta T \]  
(D.1,11)

\[ \delta_E = q e \xi \delta E \]  
(D.1,12)

\[ \delta_T = q e \xi \delta T \]  
(D.1,13)

\[ \delta_{\delta} = -q e \xi \delta_{\delta T} \]  
(D.1,14)

\[ \delta_{\delta T} = -q e \xi \delta_{\delta T} C_{\delta T} \]  
(D.1,15a)

\[ \delta_{\delta T} = -q e \xi \delta_{\delta T} \]  
(D.1,15b)
\[ M_E = q_e s \delta_E \delta_E \]  
\[ M_T = q_e s \delta_C \alpha_C \delta_T \]  

where 

\[ q_e = \frac{1}{2} \rho v_e^2 \]  
\[ C_{L_e} = C_{W_e} \cos \theta_e \]  
\[ t^* = \frac{\ddot{c}}{2v_e} \]  
\[ \delta_a = \alpha_e + \epsilon_T \]  
\[ C_{\alpha CT} = -C_{D\alpha CT} \]  

or 

\[ C_{\alpha CT} = -\left[C_{D\alpha CT} - \cos(\delta a)\right] \]  
\[ C_{xu} = -2 C_{\alpha CT} C_{T_e} \]  
\[ C_{z u} = 2\left[C_{LCT} + \sin(\delta a)\right] C_{T e} \]  

or 

\[ C_{z u} = 2C_{\alpha CT} L_{CT e} \]  
\[ C_{m u} = -2 C_{m CT} C_{T e} \]  

\[ C_{x a} = C_{L a} - C_{D a} \]  
\[ C_{x q} = -C_{D q} \]  
\[ C_{z a} = -(C_{u a} + C_{D a}) \]  
\[ C_{z q} = -C_{L q} \]  
\[ C_{x \delta E} = C_{D \delta E} \]  
\[ C_{T \delta T} = C_{T \delta T} (V_e) \]  
\[ C_{x \delta E} = -C_{L \delta E} \]  

Lateral Stability Axes Dimensional Derivatives

\[ Y_v = \frac{1}{2} \rho v_e^2 SC_y \]  
\[ L_v = \frac{1}{2} \rho v_e^2 SbC_{\delta \beta} \]  
\[ N_v = \frac{1}{2} \rho v_e^2 SbC_{n \beta} \]  
\[ Y_p = t^* q_e SC_{\gamma p} \]
\[ L_p = t^* q_p S_b C_{p} \] (D.1.38)

\[ N_p = t^* q_p S_b C_{n_p} \] (D.1.39)

\[ Y_r = t^* q_e S C y_r \] (D.1.40)

\[ L_r = t^* q_e S_b C_{p} \] (D.1.41)

\[ N_r = t^* q_e S_b C_{n_r} \] (D.1.42)

\[ Y_A^o = q_e S C y_A \] (D.1.43)

\[ L_A^o = q_e S_b C_{l_A} \] (D.1.44)

\[ N_A^o = q_e S_b C_{n_A} \] (D.1.45)

\[ Y_R^o = q_e S C y_R \] (D.1.46)

\[ L_R^o = q_e S_b C_{l_R} \] (D.1.47)

\[ N_R^o = q_e S_b C_{n_R} \] (D.1.48)

where for the lateral derivatives

\[ t^* = b/(2V_\theta) \] (D.1.49)

**Remark 1**

\[ C^*_{DCT} \text{ and } C^*_{LCT} \text{ include thrust contributions, while } C^*_{DCT} \text{ and } C^*_{LCT} \text{ do not.} \]

**Remark 2** (see also Etkin [B.2])

Nonzero values for the stability derivatives \( C_{xu}, C_{zu} \), \( C_{m_u} \) arise because of thrust coefficient changes due to changes in airspeed. Mach number and Reynolds number effects are assumed to be negligible.

As an example, consider \( C_{xu} \). \( C_x \) is the net non-dimensional aerodynamic force along the \( \frac{1}{2}S \) axis, including thrust contributions, and is given by (see Figs. 3.2 and 3.3)

\[ C_x = \cos(\alpha + \alpha_{T}) C_T - \cos(\alpha) C_D + \sin(\alpha) C_L. \] (D.1.50)

Thus

\[ \frac{3C_x}{3u} = \left[ \cos(\alpha + \alpha_{T}) - \cos(\alpha) \right] \frac{3C_T}{3u} + \sin(\alpha) \frac{3C_L}{3u} \] (D.1.51)

where the substitutions
\[
\frac{3c_D}{3u} = \frac{3c_D}{3c_T} \frac{3c_T}{3u} \quad (D.1,52)
\]
\[
\frac{3c_L}{3u} = \frac{3c_L}{3c_T} \frac{3c_T}{3u} \quad (D.1,53)
\]

have been made. Noting that

\[
C_{x_u} = \left( \frac{3c_t}{3u} \right)_{e} \quad (D.1,54)
\]

where

\[
\tilde{u} = u/V_e \quad (D.1,55)
\]

and that

\[
a_x = 0 \quad (D.1,56)
\]

in the reference equilibrium, (D.1,51) becomes

\[
C_{x_u} = \left[ \cos(\epsilon_T + \alpha_T) - C_{D_{T_e}} \right] C_{T_u} \quad (D.1,57)
\]

Now \(\frac{3c_T}{3u}\) may be written

\[
\frac{3c_T}{3u} = \frac{3T/3u}{1/2\rho V^2S} - \frac{2T}{1/2\rho V^2S} \frac{3V}{3u} \quad (D.1,58)
\]

But

\[
V = \sqrt{u^2 + v^2 + w^2} \quad (D.1,59)
\]

and thus

\[
\frac{3V}{3u} = u/V \quad (D.1,60a)
\]

and

\[
(\frac{3V}{3u})_e = 1. \quad (D.1,60b)
\]

It follows that

\[
\frac{\frac{3c_T}{3u}}{e} = \frac{(3T/3u)_e}{e} - \frac{2T}{1/2\rho V^2S} \quad (D.1,61)
\]

or in aerodynamic derivative form

\[
C_{T_u} = \left( \frac{3c_T}{3(u/V)} \right)_e = \frac{(3T/3u)_e}{e} - 2C_{T_e} \quad (D.1,62)
\]

For the light STOL transport, it has been assumed that

\[
\frac{3T/3V}{e} = 0. \quad (D.1,63)
\]

Thus (D.1,62) simplifies to

\[
C_{T_u} = -2C_{T_e} \quad (D.1,64)
\]

and thus (D.1,57) with the substitution (D.1,23) becomes

\[
C_{x_u} = -2C_{c_T} C_{T_e} \quad (D.1,65)
\]

which is the desired result. The expressions for \(C_{x_u}\) and
C\textsubscript{m\textsubscript{u}} may be obtained using a similar procedure. The expression (D.1.1) for \(X_u\) may be obtained by noting that \(\frac{\partial C}{\partial \mu} \) may also be written

\[
\frac{\partial C}{\partial \mu} = \frac{\partial X}{\partial \mu} = \frac{4X_{u}}{\frac{1}{2} \rho V^2 S}.
\]  

(D.1.66)

Solving for \(\partial X/\partial \mu\) and evaluating for the linearization reference equilibrium, the result is

\[
X_{u} = \frac{1}{2} \rho V \frac{S}{C} + \rho V \frac{S}{C} x_{e} = (0.1, 66).
\]

But from (3.4, 15) and noting that the nondimensional weight coefficient \(C_w = \frac{mg}{\frac{1}{2} \rho V^2 S}\),

\[
C_{x e} = C_w \sin \theta_e.
\]

(D.1.67)

and from (3.4, 39)

\[
C_w = \frac{C_{x e} \sin (\alpha_e + \epsilon_n) + C_{L e}}{\cos \theta_e}.
\]

(D.1.69a)

or

\[
C_w = \frac{C \tan \theta_e}{\cos \theta_e}.
\]

(D.1.69b)

Thus

\[
C_{x e} = C \frac{\tan \theta_e}{\tan \alpha_e}.
\]

(D.1.70)

and

\[
X_{u} = \frac{1}{2} \rho V \frac{S}{C} \frac{x_{u}}{x_{e}} + \rho V \frac{S}{C} \tan \theta_e.
\]

(D.1.71)

The expression for \(Z_u\) can be obtained by following a similar procedure.

Remark 3

The terms \(C_{T e}, C_{D e}, C_{L e}, \alpha_{e} \) and \(\theta_e\) may be obtained from (3.4, 29) to (3.4, 32), (3.4, 38), and (3.4, 39) given \(W_{1 e}, W_{2 e}, \gamma_{G}, V_{e}, \epsilon_{T}\), and an aerodynamic model specifying the relationships \(C_D(\alpha_e), C_L(\alpha_e)\). Once \(\alpha_{e} \) and \(C_T e\) are determined the other terms required in specifying the dimensional derivatives (i.e. \(C_{D CT}, C_{L CT}\) and so forth) may be obtained from the general AER02 aerodynamic model (see Chapter 3, Section 3.5.2 and the following section). It is noted that the lateral nondimensional aerodynamic derivatives obtained from the AER02 model are in body axes \(F_B\), while the expressions for the corresponding dimensional derivatives require them in \(F_S\). The two are related through a rotational transformation about the \(F_B\) axis through an angle \(-\alpha_{e} \) (\(F_B\) to \(F_S\), see (3.2, 22)). The explicit form of the transformation is given in Etkin [3.1] and McRuer, Ashkenas and Graham [3.3].

D.2 The AER02 Model in More Detail

The AER02 model of Chapter 3, Section 3.5.2 is compatible
with the aerodynamic data for the STOL transport (see also Appendix C). This data is based on that given in Refs. 1.39 and C.1, and on de Havilland of Canada company data for a preliminary design of a two-engined STOL transport. While general data trends were usually preserved, much of the aerodynamic data was altered for the purposes of this study.

The modified data is plotted in Figs. D.1 through to D.3 for the aircraft in the landing configuration ($\delta_p = 40^\circ$, landing gear down), and was used in look-up table form in the computer program designated NL (see Appendix H). Linear interpolation (extrapolation) was used to obtain parameter values between (beyond) the data points. Data was stored for

$$0.0 \leq C_T \leq 3.0$$

in 0.1 increments, and for

$$-13^\circ \leq \alpha_f \leq 15^\circ$$

in 2° increments. Some of the aerodynamic derivatives are independent of $\alpha_f$ and $C_T$ and need not be stored in the look-up table. These are given in Table D.1.

The thrust variation with throttle derivative $C_{T_T}$ was also assumed to be independent of $\alpha_f$ and $C_T$, and is given by

$$C_{T_T}(V) = 0.066719 \frac{P_m}{\frac{1}{2} \rho V^2 S}$$

where $P_m$ is the maximum power per engine (see Appendix C). The relationship (D.2.3) is unit dependent and is given for MKS units. It is compatible with the equation for $C_{T_T}$ given in Ref. 1.39 when evaluated at $V = 40.0 \text{ ms}^{-1}$, as well as with

$$\frac{\partial T}{\partial V} = 0$$

for all $V$.

The program NL also requires a quasisteady linear aerodynamic model, based on the general model, for one of its running modes. This model parallels the AER01 model, but is written for direct use in the NL program, i.e. for specification of the aerodynamic forces and moments in $F_B$. It is given by the AER02 model with all of the aerodynamic derivatives evaluated at $(C_{T_T}, \alpha_f)$. There are also derivatives implied by the $C_{D_1}, C_{L_1}$ and $C_{m_1}$ terms of (3.5,19), (3.5,20) and (3.5,21), i.e. $C_{D_{\alpha}}, C_{D_{CT}}, C_{L_{\alpha}}, C_{L_{CT}}$, $C_{m_{\alpha}}$, and $C_{m_{CT}}$. These may be approximated from the given data with finite difference models, e.g.

$$C_{D_{\alpha}} = \left[ C_{D_1}(\delta_f, C_{T_T}, \alpha_f + \epsilon_{\alpha}) - C_{D_1}(\delta_f, C_{T_T}, \alpha_f - \epsilon_{\alpha}) \right] / 2\epsilon_{\alpha}$$

$$\epsilon_{\alpha} > 0$$
for $\epsilon_a$ small enough. Thus, for example, $C_D$ becomes, for $\delta_F$
fixed,

$$
C_D = C_D e + C_D h \alpha + \frac{C_D}{2V} C_D a + \frac{C_D}{2V} C_D q
+ C_D h \delta_E + C_D C_T \Delta C_T
$$

where

$$
C_D e = C_D (\delta_F, \alpha, \alpha, C_T e) + C_D (\delta_F, \alpha, \alpha, C_T e) \delta_E
$$

(D.2.7)

$$
\Delta \delta_E = \delta_E - \delta_E e
$$

(D.2.8)

$$
\Delta \alpha = \alpha - \alpha e
$$

(D.2.9)

$$
\Delta C_T = C_T - C_T e
$$

(D.2.10)

$$
C_T = C_T e + C_T V e + C_T h \delta_T e
$$

(D.2.11)

$$
\Delta V = V - V e
$$

(D.2.12)

and, with the assumption (D.2.4), from (D.1,62)

$$
C_T = -2C_T e
$$

(D.2.13)

The results for $C_L$ and $C_m$ parallel those for $C_D$.

In the dimensionalization of $C_D$ and $C_L$ in the linear
sense, care should be taken to include terms taking into
account the variation of $L$ and $D$ with $V$. In fact, follow-
ing a procedure similar to that followed in the previous section
for $X_u$, and noting that thrust contributions are not
included in $D_v$ and $L_v$, it may readily be shown that
(recall that Reynolds number and Mach number effects are being
neglected)

$$
D_v = \rho V e S C_D e
$$

(D.2.14)

and

$$
L_v = \rho V e S C_L e
$$

(D.2.15)

Thus, e.g.,

$$
D = \frac{1}{2} \rho V^2 e S D + D_v \Delta V
$$

(D.2.16)

Note that in the linear sense such $V$ derivatives are zero for
$M_B$, $Y_B$, $L_B$ and $N_B$ since

$$
M_{B e} = Y_{B e} = L_{B e} = N_{B e} = 0
$$

(D.2.17)

in the reference equilibrium used in this study.

Table D.2 summarizes the natural mode characteristics of
the STOL transport for the landing configuration for one
linearization reference equilibrium.
### TABLE D.1

**CONSTANT AERODYNAMIC DERIVATIVES**

<table>
<thead>
<tr>
<th>DERIVATIVE</th>
<th>VALUE (rad.(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_{Lq} )</td>
<td>7.144</td>
</tr>
<tr>
<td>( C_{mq} )</td>
<td>-28.76</td>
</tr>
<tr>
<td>( C_{yq} )</td>
<td>-0.1</td>
</tr>
<tr>
<td>( C_{yr} )</td>
<td>0.5</td>
</tr>
<tr>
<td>( C_{y\delta A} )</td>
<td>0.0034</td>
</tr>
<tr>
<td>( C_{r\delta A} )</td>
<td>0.2</td>
</tr>
<tr>
<td>( C_{y\delta R} )</td>
<td>0.39</td>
</tr>
</tbody>
</table>

All other derivatives are either given in Figs. D.1, D.2 and D.3, or are assumed to be zero. In particular, Reynolds number and Mach number effects are assumed to be negligible. \( C_{L_2} \), \( C_{D_2} \) and \( C_{m_2} \) are specified in Section 3.5.2.

### TABLE D.2

**LIGHT STOL TRANSPORT NATURAL MODES**

\( V_e = 40 \text{ ms}^{-1} \)

<table>
<thead>
<tr>
<th>MODE</th>
<th>( \zeta )</th>
<th>( \omega_n ) (rad./s)</th>
<th>( \omega ) (rad./s)</th>
<th>( T_{\frac{1}{2}} ), ( T_2 ) (sec)</th>
<th>( T ) (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>PHUGOID</td>
<td>0.175</td>
<td>0.298</td>
<td>0.293</td>
<td>13.39</td>
<td>21.4</td>
</tr>
<tr>
<td>SHORT-PERIOD</td>
<td>0.651</td>
<td>2.56</td>
<td>1.94</td>
<td>0.416</td>
<td>3.24</td>
</tr>
<tr>
<td>DUTCH ROLL</td>
<td>0.304</td>
<td>1.50</td>
<td>1.43</td>
<td>1.51</td>
<td>4.38</td>
</tr>
<tr>
<td>ROLL CONVERGENCE</td>
<td>---</td>
<td>---</td>
<td>0</td>
<td>0.146</td>
<td>---</td>
</tr>
<tr>
<td>SPIRAL DIVERGENCE</td>
<td>---</td>
<td>---</td>
<td>0</td>
<td>(13.0)</td>
<td>---</td>
</tr>
</tbody>
</table>

\( \zeta \) = damping ratio  
\( \omega_n \) = undamped natural frequency  
\( \omega \) = damped natural frequency  
\( T_{\frac{1}{2}} \) or 2 = time to half or double amplitude  
\( T \) = period of oscillation
FIG. D.1a  
\[ \delta_E = 0 \]
\[ \delta_F = 40^\circ \]

FIG. D.1b  
\[ \delta_E = 0 \]
\[ \delta_F = 40^\circ \]

FIG. D.1c  
\[ \delta_E = 0 \]
\[ \delta_F = 40^\circ \]

FIG. D.1 LONGITUDINAL AERODYNAMIC DATA
FIG. D.1d

FIG. D.1e

FIG. D.1f

FIG. D.1 (continued)  LONGITUDINAL AERODYNAMIC DATA
FIG. D.1g

FIG. D.1 (continued) LONGITUDINAL AERODYNAMIC DATA

FIG. D.2a

FIG. D.2 LATERAL AERODYNAMIC DATA
FIG. D.2c

FIG. D.2d

FIG. D.2e

FIG. D.2 (continued) LATERAL AERODYNAMIC DATA
FIG. D.2 (continued)  LATERAL AERODYNAMIC DATA
FIG. D.2i

ALL $C_T$

$\delta_E = 0$

$\delta_F = 40^\circ$

FIG. D.2j

ALL $C_T$

$\delta_E = 0$

$\delta_F = 40^\circ$

FIG. D.2 (continued) LATERAL AERODYNAMIC DATA
$P_m = 4.847 \times 10^5$ Watts
$
\rho = 1.2256 \text{ Kg/m}^3$
$S = 39.019 \text{ m}^2$

FIG. D.3 $C_{T\delta_T}$ VARIATION WITH AIRSPEED

See Equation (D.2,3).
APPENDIX E
TWO EXAMPLE CGA PROFILES
DERIVED ANALYTICALLY

In this appendix the technique presented in Section 3.7.1 of Chapter 3 is used to obtain analytical solutions for the curved glidepath geometries resulting from two special forms of the wind profiles $W_1(h_c)$. For simplicity the development assumes

$$\psi_c = 0.$$ 

(E.1)

Substituting (E.1) into (3.7,11) and (3.7,13), the altitude profile and horizontal profile equations become, respectively,

$$h_c(t) = \sin \theta_c V_c t + h_{c0}$$

(E.2)

and

$$x_{Ic}(t) = \cos \theta_c V_c t + \int_0^t W_1(h_c(t)) \, dt + x_{c0}.$$ 

(E.3)

E.1 Linear Wind Profile

$W_1$ is assumed to have the form (cf. (2.2,11))

$$W_{1c} = W_{l0} + \kappa_{13} \theta_c$$

(E.1,1)

where $\kappa_{13}$ is the negative of the (1,3) element of $K$, the constant gradient matrix defined by (2.2,13). The altitude profile is unaffected by $W_1$ and is given by (E.2), while $x_{Ic}$ may be obtained by substituting (E.1,1) into (E.3), and integrating. The result is

$$x_{Ic} = \frac{1}{2} \kappa_{13} V_c \sin \theta_c t^2 + V_c \cos \theta_c t$$

(E.1,2)

$$+ W_1(h_c(t)) t + x_{c0}.$$ 

(E.1,3)

where $x_{c0}$ is given by (3.7,21), and is found to be

$$x_{c0} = x_{IT} - V_c \cos \theta_c t_T - \frac{1}{2} \kappa_{13} V_c \sin \theta_c t_T^2$$

$$- W_1(h_{c0}) t_T.$$ 

(E.1,4)

$x_{IT}$ and $t_T$ are given by (3.7,14) and (3.7,20) respectively, i.e.

$$x_{IT} = -h_T \cotan \theta_c$$

(E.1,4)

and

$$t_T = (h_T - h_{c0})/V_c \sin \theta_c.$$ 

(E.1,5)

$h_T$, $h_{c0}$ and $V_c$ are specified a priori while $\theta_c$ may be determined from (3.7,19).
E.2 Power Law Profile

For this example $W_1^c$ is assumed to have the form

(cf. (2.2,1))

$$W_1^c = -W_G \left[ \frac{h_c}{h_G} \right]^n$$  \hspace{1cm} (E.2,1)

where $n$, $h_G$ and $W_G$ are defined in Chapter 2, Section 2.2.1. Analogously to the development of the previous section, the altitude profile is given by (E,2) while $x_{IC}$ is found to be

$$x_{IC0}(t) = v_c \cos \theta_c t$$

$$+ \frac{1}{v_c \sin \theta_c (n+1)} \left[ W_1^c (h_c) h_c^0 (t) - W_1^c (h_c^0) h_c^0 \right]$$

$$+ x_{IC0}$$  \hspace{1cm} (E.2,2)

and

$$x_{IC} = x_{IT} - v_c \cos \theta_c t_T - \frac{1}{v_c \sin \theta_c (n+1)} \left[ W_1^c (h_T) h_T \right.$$

$$- W_1^c (h_c^0) h_c^0 \right].$$  \hspace{1cm} (E.2,3)
APPENDIX F

STATE FEEDBACK OPTIMAL CONTROL THEORY

F.1 Overview

In this section some important results from optimal control theory, relevant to this work, are presented, usually without proof. The objective is to outline the optimal techniques which might be applied to designing aircraft feedback control systems, rather than to outline methods which yield optimal control histories given the initial conditions (i.e. open-loop or programming methods). Most of the details are left to the cited references.

Including the adjective optimal in naming this theory is particularly appealing to the engineer, who by the very nature of his profession, is constantly in the process of optimizing designs and procedures. But in this sense the name is also somewhat misleading in that it should not be construed to mean that classical servo-synthesis techniques do not involve an optimization process. In the classical framework the optimization occurs in the trial and error process of developing and testing alternative controllers which satisfy design requirements. In the optimal control theory framework, the optimization occurs in the mathematical sense of providing a solution which minimizes a functional of the state and control vectors of the system. How well this optimization satisfies the design objectives depends in a large measure on the selection of a suitable functional whose minimization expresses these objectives.

Control theory, and in particular optimal control theory, is full of specialized terminology and a number of key results from which the engineering applications may be formulated. Some of these have been mentioned in Chapter 4. They are now placed on a more precise footing.

We will consider the general system

\[ \dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x_0. \]  

(F.1.1)

This can be thought of as a nonlinear, time-varying continuous mathematical model of the dynamics of the aircraft and associated systems (discrete systems will not be considered) in the closed interval \([t_0, t_f]\). If the system is time-invariant, then (F.1.1a) may be written

\[ \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0. \]  

(F.1.1b)

The elements of \( x \) are the state of the system, that is the minimum set of variables which contain enough information about the past history to predict all future states given the equations of motion (F.1.1) and control inputs \( u(t) \) [4.11]. The control inputs are intended to move the state of the system to a desired, possibly time-varying state. We note that disturbance inputs are not considered at this point (e.g. wind). The order of the system, denoted by \( n \), is given by the number of state variables. The number of control inputs
will be denoted by \( m \).

Related to state and control variables are the concepts of state trajectory and control history. The state trajectory is the history of the state variables in state space in the interval \([t_0, t_f]\) where the system is described by the equation (F.1.1). The control history is the history of control input values in control space in \([t_0, t_f]\). Some state trajectories may not be admissible and thus we may define a set of admissible state trajectories \((X)\) on \([t_0, t_f]\). Such constraints may arise because of the physical nature of the problem, e.g. the presence of terrain obstructions. Similarly we may define a set of admissible control histories \((U)\) on \([t_0, t_f]\). Control constraints usually arise because of physical limitations of the control system, e.g. maximum available thrust or maximum available control deflection.

If the system (F.1.1a) is linear and time-varying then it will be written

\[
\dot{x}(t) = F(t)x(t) + G(t)u(t).
\]  

(F.1.2a)

If it is linear and time-invariant*, then (F.1.2a) may be written

\[
\dot{x}(t) = Fx(t) + Gu(t) 
\]  

(F.1.2b)

* Some authors refer to such systems as stationary systems. This will not be done here in order to avoid confusion with the concept of stationarity in stochastic theory.

The system outputs are the physical quantities that can be measured. They are usually related to the state of the system but need not be the state elements themselves. They may also be related to the control inputs. In general, they may be written

\[
y(t) = m(x(t), y(t), t) .
\]  

(F.1.3a)

If \( m \) is linear then the most general form becomes

\[
y(t) = M(x(t))y(t) + N(t)y(t) 
\]  

(F.1.3b)

If the output relationships are time-invariant, then the \( t \) argument in \( m, M \) and \( N \) is dropped. The number of outputs will be denoted by \( q \). We note that for our purposes it will often be sufficient to consider \( y \) to be a function of state and time only. Thus (F.1.3a) and (F.1.3b) become, respectively,

\[
y(t) = m(x(t), t) 
\]  

(F.1.4a)

and

\[
y(t) = Mx(t) 
\]  

(F.1.4b)

In some circumstances it is useful to define a controlled variable set as well. This will be defined to be the set of quantities that are to be controlled (Ref. 4.19, p. 128). Symbolically we have

\[
\hat{z}(t) = Lx(t) .
\]  

(F.1.4c)

This set will usually be a subset of \( y \). The
number of controlled variables will be denoted by \( r \).

We are now in a position to define the optimal control problem. The objective is to find a control history \( y(t) \in U \) which minimizes a performance criterion \( J \) given by

\[
J = s(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), y(t), t) \, dt
\]  

subject to the differential equation constraints (F.1,1a) and which results in a state trajectory \( x(t) \in \mathbb{X} \). More specifically, from the closed-loop (feedback) control point of view, what is required is a control law (control strategy) \( k^*(x,t) \) which yields minimizing control histories. This concept of a control law naturally leads into the definition of a set \( K \) of admissible control laws. Thus symbolically we have

\[
y^*(t) = k^*(x,t), \quad k^* \in K.
\]  

Some of the control literature does not reserve a special symbol for the control law, i.e.

\[
y = y(x,t).
\]  

Both conventions will be used in this study, although the latter convention is less precise in that it does not emphasize the distinction between a control history and a control law. Optimal quantities will always be denoted with a superscript asterisk.

For many systems a quadratic performance criterion is of great utility because it conveniently weights negative and positive deviations equally, and for some systems (i.e. linear systems) is tractable analytically. It is given by

\[
J = \|x(t_f) - x_d\|^2 + \int_{t_0}^{t_f} \left[ \|x(t) - x_d(t)\|^2 + \|y(t)\|^2 \right] \, dt
\]  

where

\[
\|x\|_A = x^T A x.
\]  

For reasons relating to the interpretation of the problem and the existence of a finite control law solution, as well as for avoiding trivial solutions, \( S, Q \) should be real symmetric positive semidefinite matrices and, \( R \) should be a real symmetric positive definite matrix. (This is discussed further in the following sections.) Here \( x_d(t) \) is the desired state time trajectory.

This discussion forms the basis for the following definitions.

\[\text{Definition 1. Controllability} - \text{We intuitively have a feeling}\]
for this term, but it is surprising to discover that the concept was put on a more formal footing only in 1960 by Kalman [F.1]. As presented by Kirk (Ref. 4.17, p. 21) Kalman's definition is stated as follows:

"If there is a finite time \( t_1 \geq t_0 \) and a control \( u(t) \), \( t \in [t_0, t_1] \) which transfers the state \( x_0 \) to the origin at time \( t_1 \), the state \( x_0 \) is said to be controllable at time \( t_0 \). If all values of \( x_0 \) are controllable for all \( t_0 \), the system is completely controllable, or simply controllable."

In the general case of a nonlinear system of the form (F.1,1a), system controllability is very difficult to establish. But if we specialize to considering time-varying and constant linear systems, then a number of practical results are available. Two of these are stated below [4.19, F.1].

**Theorem F.1**

For the linear, time-varying system (F.1,2a) with \( F(t) \) continuous and \( G(t) \) piecewise continuous, complete controllability at time \( t_0 \) holds if and only if the nonnegative-definite symmetric matrix

\[
W(t_0, t_1) = \int_{t_0}^{t_1} \dot{x}(t)G(t)G^T(t)\dot{x}(t)dt \tag{F.1,9a}
\]

is positive definite for some \( t_1 > t_0 \). \( \dot{x}(t, t_0) \) is the transition matrix of the system and is discussed in detail by, among others, Kwakernaak and Raphael [4.19] and Zadeh and Desoer [F.2] (see also Chapter 5, Section 5.2). The continuity conditions on \( F(t) \) and \( G(t) \) guarantee that the transition matrix exists.

**Theorem F.2**

For the linear, time-invariant system (F.1,2b) complete controllability holds if and only if

\[
\text{Rank}[G, FG, F^2G, \ldots, F^{n-1}G] = n \tag{F.1,9b}
\]

in which case one may choose \( t_1 - t_0 > 0 \) as small as desired.

By the rank of the \( n \times mn \) matrix in (F.1,9b) is meant the number of nonzero rows in the row-echelon form of the matrix (equivalently, the number of nonzero columns in the column-echelon form) [F.3]. Rank is identical to the order of the largest nonzero determinant of a submatrix of the matrix being considered and for a general \( n \times m \) matrix must be less than or equal to the lesser of \( n \) and \( m \).

Other results are available and are summarized in Refs. 4.18, 4.19 and F.2.

For time-varying systems a stronger form of controllability, also introduced by Kalman [F.1], is that of uniform complete controllability. This is defined as follows [4.19]:

"The time-varying system (F.1,2a) is uniformly completely controllable if there exist positive constants \( \sigma, \alpha_0, \alpha_1, \beta_0, \) and \( \beta_1 \) such that

\[
\alpha_0 I \leq W(t_0, t_0 + \sigma) \leq \alpha_1 I \tag{F.1,9c}
\]
for all $t_0$, and

$$\beta_1 I \leq W(t_0, t_0 + \sigma)W(t_0, t_0 + \sigma)^T \leq \beta_2 I \quad (F.1.9d)$$

for all $t_0$.

Uniform controllability, as well as implying controllability, also suggests that the control energy involved in a state transfer and the transfer time are roughly independent of the initial time.

It is an immediate consequence of these definitions that for time-invariant linear systems complete controllability implies uniform complete controllability.

**Definition 2. Observability** — This is a somewhat more subtle concept than controllability and considers the question of whether by knowing the system dynamics $\dot{x}$ and observing the free $(y(t) = 0)$ system output $\bar{y}(t)$ for a finite period of time $[t_0, t_1]$, we can determine the state $x(t_0)$. In particular we have the following definition [4.17]:

"If by observing the [free] output $\bar{y}(t)$ during the finite time interval $[t_0, t_1]$ the state $x(t_0) = x_0$ can be determined, the state $x_0$ is said to be observable at time $t_0$. If all states $x_0$ are observable for every $t_0$, the system is called completely observable, or simply observable."

In parallel with controllability, observability of the general nonlinear system (F.1.1a) is very difficult to establish. Linear, possibly time-varying systems are quite manageable, however, and, in fact, we may relate controllability and observability of such systems through the very useful and elegant concept of the duality of these properties. By the dual of the general linear system (F.1.2a) with output equation (F.1.4b) with respect to an arbitrary, fixed time $t_*$ we mean the system [4.19]

$$\dot{x}(t) = \xi^T(t_* - t)x(t) + \eta^T(t_* - t)y(t) \quad (F.1.10a)$$

with output

$$\bar{y}(t) = \eta^T(t_* - t)x(t) \quad (F.1.10b)$$

The dual of (F.1.10) with respect to $t_*$ is the original system (F.1.2a), (F.1.4b).

**Theorem F.3** [4.18, 4.19, F.1]

The linear system (F.1.2a), (F.1.4b) is observable at time $t_0$ if its dual, for $t_* = t_0$, is controllable at time $t_0$.

This duality principle has considerably greater ramifications in determining the parallels between optimal deterministic control theory and optimal filtering and observer theory, but these will not be considered further at this point where the main focus is system control assuming that all of the state components are known exactly and instantaneously throughout the time interval of interest.

Complimentary to the concept of observability is that of
reconstructibility [4.19], i.e. whether one can determine the current state of the system from observations of the free outputs \( y(t) \) in a finite time interval \( [t_1, t_0] \), \( t_1 < t_0 \). This is conceptually different from observability in that observability looks at future outputs whereas reconstructability looks at past outputs in determining \( x_0 \). Mathematically speaking, however, theorems for reconstructability parallel those for observability [4.19]. For the case of time-invariant linear systems, complete reconstructability implies and is implied by complete observability.

Definition 3. Regulator problem  - The regulator problem is the problem of finding an appropriate control strategy to bring the state of a system to a desired state point, possibly in the presence of disturbances. This state point is also referred to as a set point. For linear systems this set point is very often \( x_s = 0 \), and this leads to some simplification.

Definition 4. Servo problem  - The servo problem is the problem of finding suitable control inputs to track a specified trajectory of the state or output of the system. Some authors prefer to think of it in more general terms as the problem of finding a controller to track a class of desired trajectories and refer to tracking a particular trajectory as the tracking problem [4.18]. In a related definition, if the system is to follow the response of another system to specific command inputs or classes of command inputs, then the problem is referred to as the model-following problem.

Definition 5. Hamiltonian (H)  - A scalar quantity which is found to be very useful in the solution of optimal control problems and in dynamics is defined by [4.16]

\[
H(x(t), u(t), \lambda(t), t) = g(x(t), u(t), t) + \lambda^T(t)f(x(t), u(t), t) \tag{F.1.11}
\]

and is referred to as the Hamiltonian. \( \lambda \) is a \( n \times 1 \) column vector of Lagrange multipliers, useful in dealing with constraints (i.e. in this case the \( n \) differential equation constraints), \( g \) is from the performance criterion (F.1.5) and \( f \) is from (F.1.1a). It can be shown that for optimal trajectories, the Lagrange multipliers may be identified with \( J^* \) through (Kirk [4.17], p. 419)

\[
\lambda(t) = J^*(x(t), t) \tag{F.1.12}
\]

where \( J^* \) is the return function evaluated using the optimal control \( u^*(t) \) and the resulting state response from \( t \) to \( t_f \), i.e.

\[
J^*(x(t), t) = \min_\{u(t)\} \left\{ \int_t^{t_f} g(x(\tau), u(\tau), \tau)d\tau + s(x(t_f), t_f) \right\} \tag{F.1.13}
\]

where

\[
J^* = \left[ \frac{\partial J^*}{\partial x_1}, \frac{\partial J^*}{\partial x_2}, \ldots, \frac{\partial J^*}{\partial x_n} \right]^T. \tag{F.1.14}
\]
Methods for solving the optimal control problem may now be considered. Given the system equations of motion (F.1.1) and the performance criterion (F.1.7), the optimal control inputs may be found using one of three related techniques:

1. Dynamic programming.
2. Hamilton-Jacobi-Bellman (H-J-B) equation methods.
3. Variational methods.

Dynamic programming takes advantage of the principle of optimality to reduce the number of iterations required to search out optimal control strategies (as compared to an exhaustive search procedure). This principle states that (Kirk [4.17], p. 54):

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decision must constitute an optimal policy with regard to the state resulting from the first decision."

The method treats continuous systems by discretizing the state and control spaces and using a recursive relationship to find all optimal routes to a desired state from any point in admissible state space (for further details see also Bryson and Ho [4.16], Chapter 4). The advantages of the method are two:

1. **It can be applied to virtually any optimization problem including problems with state and control constraints, to map out a feedback controller from all points in admissible state space.**
2. **It finds the absolute minimum.**

Unfortunately it has two drawbacks, one of which is very restrictive. These are:

1. The method is a discrete approximation to a continuous problem (this, of course, is not true if we are dealing with a discrete system to begin with). The accuracy of the method depends on the size of the discretization intervals and on the interpolation procedures that are employed.
2. The rapid access memory requirements for treating even third order systems with one hundred quantization levels pushes the limits of modern day computers (typically of order $10^6$ bytes). This is known as the curse of dimensionality [4.17].

Since the systems treated here are almost always of dimension greater than three, dynamic programming will not be considered as a viable alternative.

The continuous time analog of Bellman's recurrence relation is the Hamilton-Jacobi-Bellman equation. This is given by

$$J_t^*(x(t), t) + H\{x(t), y^*(x(t), J_{x}^*, t), J_{x}^*, t\} = 0$$

(F.1.15)
with boundary condition

$$J^*(x(t_f), t_f^*) = s(x(t_f), t_f^*) \quad (F.1.16)$$

where the Hamiltonian $H$ is defined in (F.1.11) and (F.1.12), $J^*$ is defined in (F.1.13), and

$$H[x(t), y^*(x(t), J^*_x, t), J^*_y, t] = \min_{y(t)} H[x(t), y(t), J^*_x, t]. \quad (F.1.17)$$

It can further be shown that [4.16] (1) $J[x(t), t]$ evaluated for any $y \in U$ satisfies (F.1.15), where we remove the minimization procedure in (F.1.17), and that (4.16, 4.17, F.1) (2) if $J'[x(t), t]$ satisfies (F.1.15) and (F.1.17), then it is the absolute extremum performance criterion (sufficiency).

To be completely precise, these arguments hold only for candidate $J^*(x(t), t)$ that are twice differentiable with respect to their arguments, and similarly $s(x(t_f), t_f^*)$ and $g(x(t), y(t), t)$ must also be twice differentiable with respect to their arguments.

The equation (F.1.15) has replaced the discrete recurrence relation with a nonlinear, partial differential equation which involves a minimization. It is generally very difficult to solve from both the analytical and computational perspectives, although for certain important cases (e.g. the linear quadratic tracking problem) the solution can be given in terms of ordinary matrix differential equations.

The third approach is distinct from the first two through its use of variational calculus to determine the optimal control strategy. Variational calculus is introduced quite lucidly in Silverman's translation of Gelfand and Fomin's book on the subject [F.4]. It deals with the maximization and minimization of functionals, where a functional is defined to be a correspondence which assigns a real number to each function belonging to some class. In an imprecise way we may think of a functional as a "function of a function". Thus

$$J(y) = \int_a^b F(t, y, \dot{y}) \, dt \quad (F.1.18)$$

is a functional which assigns a real number, namely the value of the integral, to every admissible $y(t)$ provided that the integral exists. The first important results concerning functionals were due to Euler, who derived them in the seventeen hundreds, and since that time the theory has grown in importance with numerous applications to mechanical and physical problems. (This follows intuitively from the concept of state's of equilibrium being states of minimum energy.)

The application of variational calculus to control theory is considerably more recent and parallels the great increase in interest in control theory following World War II. From the definition of the optimal control problem it is clear that it

---

*We must, of course, restrict $y(t)$ (at the very least) to be functions for which the solution to (F.1.1a) exists.*
is a problem of functional minimization, namely the minimization of (F.1,5) by control inputs \( y(t) \) subject to the differential equation constraints (F.1,1a) and any other constraints implied by the choice of the state and control sets \( X \) and \( U \).

In parallel with the classical analysis of functions, we may proceed to rigorously define functional continuity, functional differentials (i.e. variations) and so forth, but this takes us too far away from the objectives of this section. Suffice it to say that while analogies between the calculus of the minimization and maximization of functions and the calculus of variations are very often useful tools in heuristic arguments, they are nonrigorous and may lead to incorrect conclusions.

A number of variational calculus concepts useful in optimal control theory are now defined.

**Definition 6. Increment and Variation** - Let \( J(y) \) be a functional defined on a normed linear space. Then its increment corresponding to the increment \( h = h(t) \) of \( y \) is

\[
\Delta J(h) = J(y + h) - J(y) .
\]  

(F.1,19a)

If

\[
\Delta J(h) = \phi_1(h) + \epsilon \|h\|^{2}
\]  

(F.1,19b)

where \( \phi_1(h) \) is a linear functional, and \( \epsilon \to 0 \) as \( \|h\| \to 0 \), then the functional \( J(y) \) is said to be differentiable and the principal linear part of the increment \( \Delta J(h) \) is named the variation. This will be denoted by \( \delta J(h) \).

If \( \Delta J(h) \) may be written as

\[
\Delta J(h) = \phi_1(h) + \phi_2(h) + \epsilon \|h\|^{2}
\]  

(F.1,19c)

where \( \phi_1(h) \) is a linear functional, \( \phi_2(h) \) is a quadratic functional and \( \epsilon \to 0 \) as \( \|h\| \to 0 \), then \( J(y) \) is twice differentiable and \( \phi_2(h) \) is named the second variation. This will be denoted \( \delta^2 J(h) \).

These definitions may be extended to functionals of more than one independent function which themselves may be functions of several variables. These details are left to Refs. 4.17 and F.4.

**Definition 7. Extremals and the Euler Equation** - Consider the functional (F.1,18). Then it may be shown [F.4] that a necessary condition for a given \( y(t) \) to minimize (maximize) this functional is that

\[
F_y - \frac{d}{dt} F_t = 0 .
\]  

(F.1,20)

Equation (F.1,20) is known as Euler's equation and integral curves of this equation, are known as extremals. Since this is only a necessary condition, not all extremals yield minima (maxima) of \( J \). Furthermore, the resulting extremals may not
satisfy the constraints of the problem.

Definition 8. Extremum - If \( y^* \) satisfies its constraints and locally minimizes (maximizes) \( J \), then \( J(y^*) \) is referred to as a relative extremum. If this minimization (maximization) is global, then \( J(y^*) \) is referred to as a global extremum.

Definition 9. Weak and strong extrema [4.16] - From the control theory point of view, many of the variational results are obtained assuming that second order terms in the variation of state and control are small (i.e. \( \|\delta x\|^2, \|\delta y\|^2 \) are negligible). This assumption, in effect, bounds the magnitude of \( \delta x \) and \( \delta y \), and is known as the weak variation. The resulting extremum is called the weak extremum. This may not be valid if we consider arbitrary variations in \( y \) which only assure negligible \( \|\delta x\|^2 \). Such variations are known as strong variations, and the resulting extremum is the strong extremum. Conditions for a strong extremum are generally stricter and more difficult to apply, i.e. a strong extremum implies a weak extremum but not vice versa.

We will now present a number of variational control theory results that highlight the variational calculus approach to the problem. Refs. 4.16 and 4.17, among others, consider the variational optimal control techniques in greater detail.

Theorem F.4 (Pontryagin's Minimum Principle)\(^{+} \) [4.17]

A necessary condition for \( y^* \) to minimize the functional \( J \) given by (F.1,5) is that

\[
H(x^*(t), y^*(t), \lambda^*(t), t) \leq H(x(t), y(t), \lambda(t), t)
\]

(F.1,21)

for all \( t \in [t_0, t_f] \) and for all \( y \in U \).

Theorem F.5 [4.16, 4.17]

Given (1) \( y(t) \) smooth, i.e. admissible \( y(t) \) are assumed to be continuous and to have continuous first derivatives, (2) \( g(x, y, t) \) and \( s(x(t_f), t_f) \) have continuous first and second partial derivatives with respect to all their arguments, (3) admissible \( y(t) \) and \( x(t) \) are not bounded, and (4) initial conditions \( x_0 \) and initial time \( t_0 \) are specified, then the necessary conditions that the control vector \( y(t) \) produces an extremum value of the performance criterion \( J \) given by (F.1,5) (i.e. necessary conditions for a local extremum, whether minimum or maximum) subject to the differential equation constraints (F.1,1a) are as follows:

\[
\dot{x}^*(t) = \frac{\delta H}{\delta x}(x^*(t), y^*(t), \lambda^*(t), t)
\]

(F.1,22a)

\(^{+}\) This result is sometimes referred to as the maximum principle depending on the point of view taken. The result is very important for dealing with problems where there are inequality constraints.
The equations (F.1,22b), (F.1,22c) with boundary conditions (F.1,23b) are known as the Euler-Lagrange equations in optimal control. Equations (F.1,22b) are sometimes called the costate or influence equations, and the conditions (F.1,22d) are referred to as the Legendre-Clebsch conditions.

The Euler-Lagrange equations, together with (F.1,22a), which from the definition of $H$ ((F.1,11)) can be shown to be just the system equations of motion with optimal inputs, and the boundary condition (F.1,23a) form a two-point boundary value problem. Such equations have analytical solutions only for special cases and are generally difficult to solve numerically. Nevertheless they are frequently easier to implement than dynamic programming and H-J-B equation methods.

The boundary conditions are now specialized to three particular cases. Others may be obtained similarly [4.17]. We have:

1. Terminal time fixed, final state free - In this case $\delta t_f = 0$ and $\delta x_f$ is arbitrary. Thus (n equations)

$$\lambda^*(t_f) = \frac{\partial g}{\partial x}(x^*(t_f), t_f)$$

(4.1,24a)

since the coefficient of $\delta x_f$ in (F.1,23b) must be zero.

2. Terminal time free, final state free - In this case $\delta t_f$ and $\delta x_f$ are arbitrary and independent. Thus their coefficients in (F.1,23b) must be both zero. We have the condition (F.1,24a) repeated, as well as the condition (one equation)

$$\frac{\partial g}{\partial x}(x^*(t_f), t_f) + H(x^*(t_f), u^*(t_f), \lambda^*(t_f), t_f) = 0$$

(4.1,24b)

for the extra free parameter $t_f$.

3. Terminal time fixed, final state constrained by $x_k(x(t_f)) = 0$ - In this case the derivation of the boundary conditions is not as straightforward. The final result is [4.16, 4.17]

$$\lambda^*(t_f) = \left[\frac{3s}{3x} + \left(\frac{3s}{3x}\right)^T \frac{1}{t_f}\right]$$

(4.1,24c)
where \( \gamma \) is a column vector* of constant Lagrange multipliers to be adjusted so that \( \xi(x(t_f)) = 0 \) is satisfied.

In Theorem F.5 it has been assumed that the admissible \( \gamma(t) \) are smooth. This assumption may be too restrictive for some problems, i.e. in some problems it may be more appropriate to permit \( \gamma(t) \) which have only piecewise-continuous first derivatives (e.g. control problems involving constraints on the control and/or state variables). Points where \( \dot{\gamma}(t) \) is discontinuous may lead to discontinuities in the derivatives of some or all of the state variables, and are referred to as corners. Such problems require extra conditions at the corners. These are derived and are put in a form suitable for use in optimal control problems in Refs. 4.16 and 4.17. In one form they are referred to as the Weierstrass-Erdmann corner conditions.

Another important class of problems is that where there are discontinuities in the system equations at interior points. Such problems are considered by Bryson and Ho [4.16]. Their formulation is summarized in the following theorem. Theorem F.5 is a special case of this more general (and more cumbersome) result.

* \( \xi \) is of order \( k, \ k \leq n-1 \) if \( g = 0, \ k \leq n \) if \( g \neq 0 \) (see Ref. 4.16, Chapter 2). These restrictions on the order of \( \xi \) also apply to \( \xi_j \) in Theorem F.6 to follow.

**Theorem F.6**

Given the performance criterion

\[
J = s[x(t_0^-), x(t_0^+), \ldots, x(t_N^-), x(t_N^+); t_0, \ldots, t_N]
\]

subject to the constraints

\[
\dot{x} = f_i(x, u, t); \ t_{i-1} < t < t_i, \ i = 1, \ldots, N \tag{F.1,26a}
\]

\[
\xi_j[x(t_0^-), x(t_0^+), \ldots, x(t_N^-), x(t_N^+); t_0, \ldots, t_N] = 0 \tag{F.1,26b}
\]

where \( t_N = t_f \) and \( x(t_i^-) \) signifies the state vector just before \( t = t_i \) and \( x(t_i^+) \) signifies the state vector just after \( t = t_i \), then the necessary conditions that the smooth control vector \( u(t) \) produces an extremum value of the performance criterion (F.1,25) are, for \( t_{i-1}^+ < t < t_i^- \), \( i = 1, \ldots, N \), as follows (\( H_i \) is the Hamiltonian for the \( i \)-th stage, i.e. \( H_i = g_i + \frac{T_i f_i}{2} \)):

\[
\dot{\lambda} = \frac{\partial H_i}{\partial \dot{x}}(x^*(t), u^*(t), \lambda^*(t), t) \tag{F.1,27a}
\]

\[
\dot{\lambda} = -\frac{\partial H_i}{\partial x}(x^*(t), u^*(t), \lambda^*(t), t) \tag{F.1,27b}
\]

\[
\frac{\partial H_i}{\partial u}(x^*(t), u^*(t), \lambda^*(t), t) = 0 \tag{F.1,27c}
\]
\[ h_i^*(t_i) = \frac{\partial \bar{h}}{\partial t_i}(t_i), \quad i = 1, \ldots, N \] \hspace{1cm} (F.1,28a)

\[ h_i^*(t_i^+) = -\frac{\partial \bar{h}}{\partial t_i^+}(t_i^+). \] \hspace{1cm} (F.1,28b)

Furthermore, the \( t_i \) (\( i = 0, \ldots, N \)) are chosen so that

\[ \frac{\partial \bar{h}}{\partial t_i^+} + H_i(t_i^+) - H_{i+1}(t_i^+) = 0 \] \hspace{1cm} (F.1,29)

is satisfied \( (H_0 = H_{N+1} = 0) \). Here

\[ \bar{s} \bar{A} \bar{s} + \sum_{j=0}^{N} \bar{y}_j^T \xi_j \] \hspace{1cm} (F.1,30)

where the column vectors \( \bar{y}_j \) must be chosen so that the constraints \( \xi_j \) are satisfied.

The equations (F.1,27b), (F.1,27c) with boundary conditions (F.1,28a) and (F.1,28b) are the Euler-Lagrange equations for this problem. The conditions (F.1,29) arise because of the presence of (possibly) free times \( t_i \) and constraining relations \( \xi_i \) at these times, and are called the transversality conditions. If \( t_i \) is specified the corresponding condition in (F.1,29) drops out; similarly if \( \xi(t_i^+) \) is specified, the corresponding condition in (F.1,28b) drops out.

Theorems F.5 and F.6 deal only with necessary conditions for finding an optimal \( y(t) \). Sufficient conditions derived using variational calculus involve examining the behavior of neighbouring extremals to a candidate extremal, and this in turn involves the consideration of second variations of the performance criterion. Furthermore, one must distinguish between conditions which are sufficient for a weak minimum and those which are sufficient for a strong minimum. The relevant results are summarized in the following theorem [4.16].

**Theorem F.7**

Assuming that (1) \( t_f \) is fixed and that (2) the assumptions of Theorem F.5 hold, then a sufficient set of conditions for the existence of a weak minimum of (F.1,5) subject to the differential equation constraints (F.1,1a) and the constraining terminal conditions

\[ \bar{h}(t_f) = 0 \] \hspace{1cm} (F.1,31)

is

\[ H_{uu}(t) > 0 \quad \text{for} \quad t_0 \leq t \leq t_f \] \hspace{1cm} (F.1,32a)

\[ \bar{v}(t) < 0 \quad \text{for} \quad t_0 \leq t < t_f \] \hspace{1cm} (F.1,32b)

\[ \bar{v}(t) - \bar{v}(t)\bar{D}^{-1}(t)\bar{v}(t)^T \] \hspace{1cm} (F.1,32c)

finite for \( t_0 \leq t < t_f \) along with the necessary conditions (F.1,22a), (F.1,22b), (F.1,22c) and associated boundary conditions (F.1,23a) and (F.1,24c). Here \( \bar{F} \), \( \bar{E} \) and \( \bar{D} \) are obtained from the ordinary matrix differential equations
The differential equations (F.1,1a) will, in general, be nonlinear. The consideration of sufficient conditions in the previous theorem leads to the investigation of neighboring extremals, and in fact one can show (Ref. 4.16) that the sufficient conditions also imply the existence of such extremals and lead to a linear neighboring optimum feedback law. This is a very important result in that we have already indicated that dynamic programming methods, which map out all admissible optimal control inputs in admissible state space (i.e. feedback control) are not practical for most realistic problems. Thus if one needs a feedback law to a nonlinear problem, the practical compromise involves two steps:

1. Define a nonlinear open-loop (i.e. programming) problem whose solution provides the optimal control history from one point in state space.

2. Look for linear neighboring feedback laws about this nonlinear optimal programming solution.

For many engineering control problems, this approach is not only practical computationally, but also yields useful and realistic results.

We now note a theorem concerning the existence of a strong minimum [4.16].

Theorem F.8

Given the same assumptions as in Theorem F.7, then a
sufficient set of conditions for a strong minimum of (F.1,5), subject to the differential equation constraints (F.1,1a) and the terminal conditions (F.1,31), is the sufficient set of conditions for a weak minimum (Theorem F.7) plus the strengthened Weierstrass condition, namely

\[ \delta H(x, J^*_x, u_1, y^*, t) > 0 \]  \hspace{1cm} (F.1,36)

for all \( t \in [t_0, t_f] \) and admissible \( u_1 \neq y^* \). Here \( \delta H \) is given by

\[ \delta H = J^*_x \left[ f(x, u_1, t) - f(x, y^*, t) \right] + g(x, u_1, t) - g(x, y^*, t). \]  \hspace{1cm} (F.1,37)

If the strict inequality in (F.1,36) is weakened, then this is only a necessary condition for a strong minimum.

The condition (F.1,36) may be related to the Weierstrass E-function of variational calculus [F.4].

This completes the brief overview of the key variational results of optimal control. While it can be shown that dynamic programming and the minimum principle, and thus variational methods, are related through the H-J-B equation (the relationship is discussed by Kirk [4.17], Section 7.1), the two approaches do not have the same advantages and disadvantages.

The advantages of the variational approach are as follows:

1. Necessary conditions for minimizing controls of a variety of systems with a number of constraints of different types are generated in a relatively straightforward manner.

2. Variational approaches are usually more amenable to numerical techniques in terms of computer requirements.

The disadvantages of the variational approach are as follows:

1. The extension of the necessary conditions to sufficient conditions is not always straightforward and frequently leads to a significant increase in complexity, both from the analytical and the computational points of view.

2. The necessary and sufficient conditions that are available are usually of local character. Thus in many problems where it is not clear that a global minimum has been achieved, further checks will be required.

3. The minimizing controls that result are generally control histories rather than control laws, although the important category of linear quadratic problems is a notable exception.

In principle the landing approach controller synthesis problem is a problem which must be solved so that the solution results in admissible state and control trajectories. Aside from the constraints implied by the equations of motion, constraints may arise from the following:
1. The aircraft may not go below the terrain level, i.e.

\[ h > h_{\text{TER}} = h_{\text{TER}}(x_I, y_I) \]  

(F.1,38)

where \( h_{\text{TER}} \) is the height of a particular point of the terrain, defined by \((x_I, y_I)\), above a reference plane. In particular such a condition guarantees that the aircraft does not pass through the runway plane and land from below.

2. The engine thrust has a minimum (idle) and maximum value. This implies that

\[ \delta_{T_{\min}} \leq \delta_T \leq \delta_{T_{\max}} \]  

(F.1,39)

and

\[ \delta_{T_{\min}} \leq \delta_T \leq \delta_{T_{\max}} \]  

(F.1,40)

One may also assume that normal approach procedure does not permit reverse thrust deployment. In this case

\[ \delta_{T_{\min}} \geq 0. \]  

(F.1,41)

3. The aerodynamic control deflections have maximum and minimum values. Thus

\[ \delta_{E_{\min}} \leq (\delta_E, \delta_E) \leq \delta_{E_{\max}} \]  

(F.1,42a)

\[ \delta_{A_{\min}} \leq (\delta_A, \delta_A) \leq \delta_{A_{\max}} \]  

(F.1,42b)

4. Further state constraints from those discussed in one above may arise because of indirect considerations (e.g. maximum flap and gear extension speeds).

To incorporate all of these constraints into an optimal control formulation would result in a very complex problem which is computationally expensive to solve. It is more reasonable to treat such constraints by appropriate selection of the performance criteria for the various segments of the approach and landing. In some cases limiters may be required, and the resulting control laws, although suboptimal, may still be quite useful.

The terminal time can be defined in a number of ways, depending on what segment of the approach and landing is being studied. One may assume that the terminal time is given by the time the aircraft passes through the decision height, i.e.

\[ t_f = t_D. \]  

(F.1,43)

In actual flight the approach time will be known approximately, but will vary from approach to approach depending on the aircraft path, airspeed and the winds that are encountered. Thus in the strictest sense this problem is an optimal control problem where the terminal time is unspecified, but some of the positional kinematic state variables at the terminal time are specified (e.g. the simulation may end at
Nevertheless it is still quite useful to consider problems where \( t_f(t_0) \) are fixed because of the theoretical simplifications. The control laws that are generated for such simpler systems may then be used on the more general problem with terminal time unspecified (again, as suboptimal control laws).

A group of optimal control problems which are of great practical interest to this investigation are those which determine the optimal feedback law for the linear dynamic system (F.1.2a) with output (F.1.4b) and with a quadratic performance criterion (F.1.7). The results for fixed terminal time are well known. The most relevant, to this study, of these are summarized below.* The reader is referred to the cited references for further details.

**Formulation 1: Linear Regulator Problem; Zero Set Points; Fixed, Finite Terminal Time**

The zero set points imply

\[
X_d = 0
\]  
(F.1.44)

for all \( t \in [t_0, t_f] \).

---

* Two conceptually important assumptions that are tacitly made in all of these formulations are (1) that the system dynamics (i.e., \( \Gamma \) and \( \Theta \) are known exactly and (2) the state \( x(t) \) and output \( y(t) \) are known perfectly and instantaneously.

The optimal feedback control law is given by [4.16, 4.17, 4.18]

\[
u^*(t) = -K^{-1}(t)\Theta^T(t)P(t)x(t)
\]

(F.1.45)

where \( P(t) \) is an \( n \times n \) symmetric matrix which is a solution of the matrix Riccati equation

\[
\dot{P}(t) = -P(t)\Gamma(t) - \Gamma^T(t)P(t) + P(t)\Theta(t)K^{-1}(t)\Theta^T(t)P(t) - Q(t)
\]

(F.1.46)

with boundary condition

\[
P(t_f) = S.
\]

(F.1.47)

The following assumptions have been made:

1. \( \Gamma(t) \) is continuous and \( \Theta(t), \Theta(t), \Phi(t) \) are piecewise continuous, and that all of these matrix functions are bounded.

2. \( S, \Theta(t) \) are positive semidefinite symmetric matrices, with \( S \) a constant matrix. \( \Phi(t) \) is a positive definite symmetric matrix.

The extremum value of \( J^*(x,t) \) is (see also (F.1.13) and \( J \) of (F.1.7))

\[
J^*(x(t_0), t_0) = x^T(t_0)P(t_0)x(t_0)
\]

(F.1.48)

and that under these conditions \( P(t) \) exists for all
If the linear dynamic system and the quadratic performance criterion are time-invariant, then the solution for $\overline{P}(t)$ is given by solving the algebraic equation

$$\overline{P}F + F^T\overline{P} - \overline{P}GR^{-1}G^T\overline{P} + Q = 0,$$  \hspace{1cm} (F.1.53)

i.e. for this case $\overline{P}$ is a steady state solution of the matrix Riccati equation.

Before proceeding further we make the observation that the existence of a solution for $t_f \to \infty$, i.e. the existence of an optimal controller that leads to a finite $J^*$, intuitively must imply something about the stability of the resulting closed-loop system. Under mildly restrictive conditions (see Refs. 4.19, F.1) one can show that, in fact, we must have asymptotic stability.

We also observe that the performance criterion (F.1.7) is written in terms of the state $\chi$ rather than in terms of the output $\chi$, which may be the quantities of actual interest. For this study, where the output is adequately described by (F.1.4b), it will generally hold true that the optimal linear quadratic deterministic controller formulations minimizing (F.1.7) are readily transformed to results optimizing $J$ given in terms of $\chi$, i.e.

$$J = \|\chi(t_f) - \chi_d(t_f)\|^2_Q + \int_{t_0}^{t_f} \{\|\chi - \chi_d\|^2_Q + \|u\|^2_E\}dt.$$  \hspace{1cm} (F.1.54)
However the introduction of $\dot{y}$ will create some added considerations, related to observability (reconstructibility) in determining the properties of the closed-loop solutions. These details are left to the cited references but the transformations are demonstrated in the following two formulations.

Before stating the theorem concerning the stability properties of the infinite terminal time regulator problem, a number of useful stability concepts for differential equations are defined. These definitions follow those of Ref. 4.19.

**Definition 10.** Differential equation stability — The Russian mathematician Lyapunov has provided control theorists and dynamicists with the most general and powerful methods for determining the stability of systems of ordinary differential equations. His results, and subsequent results derived from them, consider arbitrary positive-definite functions of the state of the system and determine the conditions that the rate of change of these functions must satisfy along the system trajectories so that stability is guaranteed (see Hahn [F.5] for further details).

It is appropriate that one of the most general (but weak) forms of stability was also defined by Lyapunov. If we consider the system (F.1,1a) without control inputs* $y(t)$ (or equivalently $y$ is a fixed function of time) and determine a nominal solution $x_1(t)$, then we say that the nominal solution is *stable in the sense of Lyapunov* if for any $t_0$ and any $\varepsilon > 0$ there exists a $\delta(\varepsilon, t_0) > 0$ such that $\|x(t_0) - x_1(t_0)\| \leq \delta$ implies $\|x(t) - x_1(t)\| < \varepsilon$ for all $t \geq t_0$. In this study the norm will be considered to be the Euclidean norm, but other norms are also possible.

The nominal solution $x_1(t)$ is *asymptotically stable* if

1. it is stable in the sense of Lyapunov and
2. for all $t_0$ there exists a $\rho(t_0) > 0$ such that $\|x(t_0) - x_1(t_0)\| < \rho$ implies that $\lim_{t \to \infty} \|x(t) - x_1(t)\| = 0$.

The nominal solution $x_1(t)$ is *asymptotically stable in the large* if

1. it is stable in the sense of Lyapunov and
2. for any $x(t_0)$ and any $t_0$, $\lim_{t \to \infty} \|x(t) - x_1(t)\| = 0$.

These stability definitions are all given in terms of a nominal state trajectory. This is a consequence of the complex behavior of nonlinear systems. For the case of linear systems we may talk in more general terms of the stability of the system itself.

If we consider the general, time-varying autonomous version of the linear system (F.1,2a) (equivalently the control inputs are a fixed time function), then we have that the system is stable in a certain sense if the zero solution $x_1(t) \equiv 0$ is stable in that sense. Furthermore this linear system is asymptotically stable if and only if it is asymptotically stable in the large.

* Such systems are sometimes referred to as autonomous systems
Finally, this linear system is *exponentially stable* if there exist positive constants $\alpha$ and $\beta$ such that

$$\|\dot{x}(t)\| \leq e^{-\beta(t-t_0)} \|x(t_0)\|, \quad t \geq t_0 \quad \text{(F.1.55)}$$

for every initial state $x(t_0)$. A time-invariant, autonomous linear system is exponentially stable if and only if it is asymptotically stable, i.e. if all of its eigenvalues have negative real parts*.

All of these concepts are concerned with the stability of autonomous systems. One may also consider stability in the presence of control and disturbance inputs (e.g. bounded-input, bounded-output stability), but results are more difficult to obtain and are usually of less general application. They will not be required in this study.

Time-invariant linear systems may be decomposed into a *stable subspace* and an *unstable subspace* [4.19]. If the $n$-th order time-invariant linear system (F.1,2b) has $n$ distinct eigenvalues, then the stable subspace for the system is the real subspace spanned by the eigenvectors (characteristic vectors) of the system that correspond to eigenvalues (characteristic values) with strictly negative parts. The unstable subspace follows from this definition. For the case

where the eigenvalues have multiplicities greater than one, then a more abstract definition is required (see Ref. 4.19) but these concepts are preserved.

Such decompositions for linear time-invariant systems are also possible for controllability and observability (reconstructibility). Thus a linear time-invariant system is *stabilizable* if its unstable subspace is contained in its controllable subspace. Similarly, the linear time-invariant system (F.1,2b) with output (F.1,4b) ($y$ constant) is *detectable* [4.19] if its unreconstructible subspace is contained in its stable subspace.

We are now ready to state the theorem concerning the stability of the closed-loop system using the infinite terminal time controller.

**Theorem F.9**

If the time-varying linear system (F.1,2a) is either (1) uniformly completely controllable or (2) exponentially stable then the steady-state optimal control law (F.1,50) is exponentially stable. If the time-invariant system (F.1,2b) is stabilizable then the $\hat{P}$ given by the solution of (F.1,53) is a unique nonnegative-definite symmetric solution and the resulting control law (F.1,50) is asymptotically stable.

**Formulation 3: Linear Regulator Problem; Nonzero Set Points**

Nonzero set points in the context of the aircraft approach
and landing problem means that the aircraft is to be flown in a state which is different from the reference state about which the equations of motion were linearized. This problem can be dealt with by linearizing the equations of motion about the nonzero set points provided that they each form a legitimate reference equilibrium. Such will usually be the case for the aircraft system (e.g. in an aborted landing approach simulation where the landing approach equilibrium defined in Chapter 3, Section 3.4 is different from the go-around equilibrium).

These comments aside, dealing with nonzero set points is still of importance in control with constant input disturbances. Thus we will consider the time-varying system given by (F.1,2a). We have

$$\dot{x}(t) = F(t)x(t) + G(t)y(t).$$  \hfill (F.1,56a)

The controlled variables are given by

$$z(t) = L(t)x(t)$$  \hfill (F.1,56b)

and the desired controlled variable set point column matrix is given by $z_S$ (this is constant).

In order to maintain the system at this set point we must have

$$z_S = \lim_{t \to \infty} L(t)x_S$$  \hfill (F.1,57)

and

$$\lim_{t \to \infty} [F_{CL}(t)x_S + G(t)y_S] = 0$$  \hfill (F.1,58)

where $z_S$, $x_S$, and $y_S$ are constant column matrices of order $r$, $n$, and $m$ respectively, and $F_{CL}$ represents the closed-loop system dynamics.

Equations (F.1,57) and (F.1,58) need not have a solution for arbitrarily dimensioned $z_S$ even if $F_{CL}(t)$ and $G(t)$ have limits as $t \to \infty$ (i.e. not every $z_S$ leads to an $x_S$ where (F.1,58) has a solution). If this were not the case, then choosing $L$ to be the identity matrix of order $n$, we would have

$$x_S = z_S$$  \hfill (F.1,59)

and this implies that the dynamic system would be able to achieve any set point in state space as $t \to \infty$ with constant $y$ (i.e. $y_S$). If this is true then all states constitute reference equilibria, clearly not a general possibility.

The conditions under which solutions do exist are now considered. The first problem is to obtain a solution for $x_S$ in terms of $z_S$. This may be done by examining the matrix $L(t)$ in more detail. No solution exists if the limits of $L(t)$, $F_{CL}(t)$, and $G(t)$ of (F.1,57) and (F.1,58) do not exist or are infinite, and thus from the outset the problem is made meaningful by the assumption that these limits exist and are finite. Furthermore the $r \times n$ matrix* $L_\infty$ ($r \leq n$)

* The subscript $\infty$ denotes limiting values as $t \to \infty$. 
will for practical engineering problems, have linearly independent rows, i.e. the system controlled variables are independent of each other. (This will also be true for \( L(t) \), \( t \in [t_0, \infty) \).) From linear algebra this implies that \( L \) is of rank \( r \) [F.3]. If \( r = n \) then \( L_\infty \) is a nonsingular square matrix and thus \( L_\infty^{-1} \) exists and

\[
x_s = L_\infty^{-1} z_s.
\]

(In fact this means that \( z \) is a different, but equivalent set of state variables.) But if \( r < n \), no left inverses exist [F.3] and (F.1,60) is invalid. This is a very crucial step in that all of the optimal control theory has been given in terms of \( x_s \), and yet it is generally not possible to rewrite the system equations in terms of \( z \), and thus solve this problem directly.

Fortunately there is a way around this difficulty. First we note a theorem concerning the existence and uniqueness of solutions of general linear algebraic equations (Theorem 3.6 of Ref. F.3).

**Theorem F.10**

For the general linear matrix algebraic equation \( Ax = b \), one of the following possibilities must hold:

1. If the rank of the augmented matrix \([A, b]\) is greater than the rank of \( A \), the system of equations is inconsistent.
2. If the rank of \([A, b]\) is equal to the rank of \( A \), this being equal to the number of unknowns, then the equations have a unique solution.
3. If the rank of \([A, b]\) is equal to the rank of \( A \), this being less than the number of unknowns, then the equations have an infinity of solutions.

Since \( L_\infty \) is of dimension \( r \times n \) with \( r < n \) and is of rank \( r \), case 3 of this theorem applies where we have also used the fact that the rank of a general \( m \times n \) matrix cannot be greater than the lesser of \( m \) or \( n \). Thus the question that must be answered is that of which of the infinity of solutions for \( x_s \) should be chosen.

From the mathematical point of view, any of the solutions will do. From the engineering point of view, however, where most often the linear system is derived by the linearization of a nonlinear system, it is important that the magnitude of the components of \( x \) remain as small as possible. Thus it is desirable to find a solution that minimizes \( x_s \) in some sense that will also tend to minimize the magnitude of the components of the state vector. One possibility is that the solution minimizes \( x_s^T x_s \), i.e. \( \| x_s \|^2 \). Such a solution does exist and can be derived using analytical techniques for
finding the minima of functions with constraints (see, for example, Noble [F.3]).

**Theorem F.11**

If \( A \) is a \( m \times n \) matrix (\( m < n \)) of rank \( m \), then the solution of the algebraic equation \( Ax = b \) that minimizes \( x^T x \) is given by

\[
x = A^T(AA^T)^{-1}b.
\]

**Proof**

We use the Lagrange multipliers \( \lambda \) to take into account the constraints \( Ax = b \) and we note that minimizing \( x^T x \) is equivalent to minimizing

\[
J = x^T x + \lambda^T (Ax - b).
\]

For a minimum it is necessary that

\[
\frac{3J}{dx} = 2x + A^T \lambda = 0 \quad (F.1,63a)
\]
or

\[
x = -\frac{1}{2} A^T \lambda. \quad (F.1,63b)
\]

Now \( \lambda \) must be chosen such that \( Ax = b \) is satisfied, i.e.

\[
\frac{1}{2} AA^T \lambda = b \quad (F.1,64a)
\]
or

\[
\lambda = -2(\lambda)^{-1}b. \quad (F.1,64b)
\]

where the inverse exists because the \( m \times m \) matrix \( AA^T \) is of rank \( m \).

Substituting (F.1,64b) into (F.1,63b), the desired result (F.1,61) is obtained. Strictly speaking this result satisfies only necessary conditions for a minimum. But \( x^T x \) is bounded from below by zero, and Theorem F.10 implies that an infinity of solutions exist. Thus the unique solution satisfying (F.1,61) must be the desired solution. The proof is now complete.

Using Theorem F.11 to solve for \( z_s \) and substituting into (F.1,58), the result is

\[
E_{CL\infty} L_{\omega}^T (L_{\omega} L_{\omega}^T)^{-1} z_s + G u_s = 0 \quad (F.1,65a)
\]
or

\[
G u_s = -E_{CL\infty} L_{\omega}^T (L_{\omega} L_{\omega}^T)^{-1} z_s. \quad (F.1,65b)
\]

The equations (F.1,65b) are a linear system of algebraic equations which may be solved for the \( m \) unknowns in \( u_s \) given the \( r \) components of \( z_s \). Theorem F.10 may be used to establish the type of solution, if any, that exists. It is adequate for our purposes, however, to assume that a unique solution exists*. In this situation it is possible to define

* If the equations are inconsistent, then the order of \( z_s \)
an optimal regulator problem which for \( t \to \infty \) takes the system from an arbitrary initial state to the final state \( x_s \) with final control values \( u_s \). In the finite time problem the system will strive to achieve these values but will not have the time to do so. A suitable performance index is given by

\[
J = \| z(t_f) \|^2_{S_1} + \int_{t_0}^{t_f} \left( \| z'(t) \|^2_{Q_1(t)} + \| u'(t) \|^2_{R(t)} \right) dt.
\]  

(F.1,66a)

In terms of the state this performance criterion is

\[
J = \| z'(t_f) \|^2_{S} + \int_{t_0}^{t_f} \left( \| z'(t) \|^2_{Q(t)} + \| u'(t) \|^2_{R(t)} \right) dt
\]  

(F.1,66b)

where

\[
Q(t) = L_T(t)Q_1(t)L(t)
\]  

(F.1,67a)

\[
S = L_T(t_f)S_1L(t_f)
\]  

(F.1,67b)

and \( Q_1, S_1 \) are \( r \times r \) symmetric positive semi-definite matrices. The primes denote shifted state, control, and

Footnote continued

must be decreased. On the other hand, if an infinity of solutions exist, then a unique solution may usually be found by increasing the order of \( z_s \), or the special solution of Theorem F.11 may be used.

controlled variables defined by [4.19]

\[
\begin{align*}
&x'(t) = \dot{x}(t) - x_s \\
u'(t) = \dot{u}(t) - u_s \\
z'(t) = \dot{z}(t) - z_s
\end{align*}
\]  

(F.1,68a)

(F.1,68b)

(F.1,68c)

and the shifted dynamic system is given by

\[
\begin{align*}
&\dot{z}'(t) = P(t)z'(t) + G(t)u'(t) \\
&z'(t) = Lz'(t).
\end{align*}
\]  

(F.1,69a)

(F.1,69b)

The optimal feedback law to this problem is given by the method discussed previously for linear regulators with zero set points and finite terminal time. This is (cf. equation (F.1,45))

\[
u'^*(t) = -R^{-1}(t)S^T(t)P(t)x'(t).
\]  

(F.1,70a)

In terms of the original system this is

\[
u^*(t) = -R^{-1}(t)S^T(t)P(t)x + R^{-1}(t)S^T(t)P(t)x_s + u_s.
\]  

(F.1,70b)

If the infinite terminal time problem for the time-varying system is considered, then some added conditions must
be satisfied, as discussed in Formulation 2. \( S \) will generally be zero, although this need not be the case under some conditions [4.19]. The optimal feedback control law form is the same as (F.1,70b) with \( \bar{P}(t) \) replaced by \( P(t) \).

If the finite terminal time, time-invariant problem is considered, then the method of solution was discussed in Formulation 1. The optimal control law is given by (F.1,70b), but now the matrices \( q_1, q, r, f, g, l \) are time-invariant.

If the infinite terminal time, time-invariant problem is considered, then the method of solution was discussed in Formulation 2. The optimal control law is given by

\[
y(t) = \mathbf{R}^{-1} G^T \mathbf{P}_x(t) + \mathbf{R}^{-1} G^T \mathbf{P}_s + u. \tag{F.1,71}
\]

This performance criterion is repeated below:

\[
J = \|x(t_f) - x_d(t_f)\|_S^2 + \int_{t_0}^{t_f} \left[ \|x(t) - x_d(t)\|_{O(t)}^2 + \|y(t)\|_{E(t)}^2 \right] dt. \tag{F.1,73}
\]

The term \( x_d(t) \) is a desired state time trajectory.

The equations of motion are given by

\[
\dot{x}(t) = f(t)x(t) + g(t)y(t) \tag{F.1,74a}
\]

with output equations

\[
\dot{y}(t) = \mathbf{M}(t) y(t). \tag{F.1,74b}
\]

We are now dealing with the output column matrix \( \dot{y} \) rather than the controlled variable column matrix \( \dot{z} \). This separation is intended to convey the fact that as \( t \to \infty \), \( x_d(t) \) should approach system set points while the \( x_d(t) \) (to which \( x_d(t) \) corresponds in this case) can approach arbitrary limits, i.e. \( x_d(t) \) is not necessarily a legitimate system trajectory nor is its limiting value, if it exists, a legitimate system set point. This means that for general \( x_d(t) \) the system will not be able to track the trajectory arbitrarily closely, although the optimal controller will determine a control strategy which minimizes the performance criterion (F.1,73). This also implies that the infinite terminal time tracking problem will not result in finite \( J \).
for arbitrary \( y_d(t) \). Besides the usual controllability condition we must also have
\[
\lim_{t \to \infty} y_d(t) = y_{d\infty} \tag{F.1.75}
\]
where \( y_{d\infty} \) satisfies all the required conditions for the constant set points \( z_s \) of the previous formulation, i.e. all the elements of \( y_{d\infty} \) must go to zero except for certain ones which asymptote to finite values. These values must correspond to a desired controlled set point column matrix \( z_s \) for which the equations (F.1.57) and (F.1.58) can be solved. A necessary condition for this is that the number of nonzero limiting values of \( y_d(t) \) (i.e. the dimension of the corresponding \( z_d(t) \)) be equal to or less than the dimension of the control column matrix \( y(t) \).

Anderson and Moore [4.18] suggest that a more suitable performance criterion to minimize is of the form
\[
J = \|x(t_f) - x_{d}(t_f)\|^2 + \int_{t_0}^{t_f} \left\{ \frac{1}{2} x^T Q_1(t) x + y^T Q_2(t) y + \|u(t)\|^2 \right\} \, dt \tag{F.1.76}
\]
where \( Q_1(t) \) and \( Q_2(t) \) are positive semidefinite symmetric matrices. The \( \|x(t)\|^2 \) term encourages smooth response while the \( \|y(t) - y_d(t)\|^2 \) term encourages close output tracking response. The latter term without the former may result in very jerky system behavior because the system attempts to rapidly track \( y_d(t) \). On the other hand, these two terms, depending on the details of \( Q_1 \) and \( Q_2 \) may tend to oppose each other, a situation which is also unacceptable.

This problem may be overcome by carefully selecting \( Q_1 \) and \( Q_2 \). Anderson and Moore [4.18] treat this problem by separating \( x \) into two components, one which is in the range space of \( M \), i.e.
\[
x = x_1 + x_2 \tag{F.1.77}
\]
\[
M^T x_1 = 0, \quad Mx_2 = 0 \tag{F.1.78}
\]
for some vector \( x \). Their final result is that
\[
Q_1 = [I - M^T (MM^T)^{-1} M] M^T [I - M^T (MM^T)^{-1} M] \tag{F.1.79}
\]
where \( Q_3(t) \) is an arbitrary positive semidefinite symmetric matrix) is a suitable form for \( Q_1(t) \). The performance criterion is given by (F.1.73) where
\[
Q = Q_1 + M^T Q_2 M \tag{F.1.80}
\]
(\( Q_2 \) is an arbitrary positive semidefinite symmetric matrix) and
\[
x_d = M^T [MM^T]^{-1} x_d \tag{F.1.81}
\]
* Once again we use the least-squares result of Theorem F.11.
(If \( M \) is the identity matrix then \( Q_1 = 0 \) and \( Q = Q_2 \).

With this background material we are now ready to give the optimal controller result for the finite terminal time tracking problem. We have, from Refs. 4.17 and 4.18,

\[
y^*(t) = -P^{-1}(t)G^T(t)[P(t)x(t) + b(t)]
\]

where \( P(t) \) is a solution of the matrix Riccati equation

\[
\dot{P}(t) = -P(t)P(t) - P^T(t)P(t) + P(t)G(t)R^{-1}(t)G^T(t)P(t) - Q(t)
\]

and \( b(t) \) is the solution of

\[
\dot{b}(t) = -[P^T(t) - P(t)G(t)R^{-1}(t)G^T(t)]b(t) + Q(t)x_d(t)
\]

where \( x_d(t) \) is given by (F.1,81). The value of the performance criterion is given by

\[
J^*(x(t_0), t_0) = x^T(t_0)F(t_0)x(t_0) + 2x^T(t_0)b(t_0) + c(t_0)
\]

where \( c(t_0) \) is determined from

\[
\dot{c} = b^TGR^{-1}b - x_d^TQx_d
\]

with

\[
c(t_f) = x_d^T(t_f)x_d(t_f).
\]

In the previous formulations input disturbances (other than impulsive disturbances which are equivalent to nonzero initial conditions) have not been dealt with. This is of great importance to the optimal control problem inasmuch as any of the optimal control histories or control laws determined with the formulations discussed previously will not be optimal if disturbances are present, i.e. they will in general not lead to control inputs which minimize the performance criterion. It is also of particular importance to this investigation because it focuses on disturbances acting on the aircraft in the form of variable winds.

Stochastic representations of the variable winds will not be considered. Thus we are dealing with a deterministic situation, i.e. the disturbance inputs for each run are known as a function of time and position, although they may not be known to the aircraft controller itself. The fact that the winds may be nonlinear functions of the position complicates the matter, i.e. incorporating them in the equations of motion (even the linear equations) will make them nonlinear and the linear formulations discussed previously will be inapplicable.
A discussion of the methods that might be used for dealing with input disturbances is given by Bolley and Bryson [F.6]. In general the resulting controllers are conveniently categorized into three types. These are:

1. Controllers which have a priori knowledge of the input disturbance time histories (see also Huber [2.1]).

2. Controllers which estimate the disturbances from observations of the system response (see also Bossi and Bryson [F.7]).

3. Integral controllers (see also Ref. 4.18, Chapter 10).

These techniques have various advantages and disadvantages, depending on the objectives of the controller design. The first approach is least realistic inasmuch as exact knowledge of the wind disturbances on a given approach will never be available. Nevertheless this approach may be useful if good estimates are available, which may be state-of-the-art in the future. The second approach is very useful in principle, although it depends on exact knowledge of the system equations. This is a somewhat limiting factor from the engineering point of view where only nominal system parameters will be known. The third approach is probably the best from the standpoint of control law implementation, although it suffers from limitations of its own. For example, in the limit the steady state offset for first order integral controllers (i.e. only the integral of the state is used as a feedback, rather than the integral of the integral and so forth) will be zero only for constant disturbances. Nevertheless, for aircraft controllers this third alternative may be the most practical technique for dealing with variable wind effects.

The optimal control theory for techniques one and three above is now formulated in a form suitable for application to the curved approach tracking task. Approach two, which involves the design of observers, will not be considered.

Formulation 5: Linear Quadratic Tracking Problem with Exact Knowledge of the Input Disturbances

We consider linear systems of the form

$$\dot{x}(t) = F(t)x(t) + G(t)y(t) + Q(t)g(t)$$

with system outputs

$$y(t) = M(t)x(t).$$

Here $Q(t)$ is the disturbance distribution matrix and $g(t)$ is the disturbance input history which is assumed to be known a priori. We note that for the aircraft equations of motion

$$g = \begin{bmatrix} \Delta W \\ \Delta \dot{W} \end{bmatrix}$$

where we have combined the wind and wind rate of change that the aircraft sees into one column matrix. As before, the desired output history is given by $y_d(t)$. The performance
By a combination of the theory developed in Huber [2.1], Kirk [4.17] and Anderson and Moore [4.18], it may be shown that the optimal control law may be generated with a Riccati equation and an auxiliary equation similar to that developed in Formulation 4. The details are given in Section 2 of this Appendix. The control law is identical to (F.1.82), with \( P(t) \) given by (F.1.83). The equation for \( b(t) \) is altered slightly and is now given by

\[
\dot{b}(t) = -\left[P^T(t) - P(t)G(t)R^{-1}(t)G(t)^T\right]b + Q(t)x_d(t) - P(t)u(t)
\]

(F.1.89a)

and

\[
b(t_f) = -Sx_d(t_f).
\]

(F.1.89b)

The value of the optimal performance criterion is still (F.1.85) but now

\[
\dot{c} = b^T G R^{-1} g^T b - x_d^T S x_d - 2b^T u
\]

(F.1.90a)

\[
c(t_f) = x_d^T(t_f) S x_d(t_f).
\]

(F.1.90b)

The performance criterion to be minimized is given by

\[
J = \|x(t_f) - x_d(t_f)\|^2_S + \int_{t_0}^{t_f} \left\{ \|x(t) - x_d(t)\|^2_Q + \|y(t)\|^2_{R_1} + \|\dot{y}(t)\|^2_{R_2}\right\} dt
\]

(F.1.91)

where \( S, Q(t), R_1(t) \) are symmetric positive semidefinite matrices and \( R_2(t) \) is a symmetric positive definite matrix. The equations of motion and output equations are given by (F.1.74). The desired state trajectory \( x_d(t) \) is given by (F.1.81) and \( Q(t) \) is given by (F.1.80).

This problem will be attacked by transforming it in such a way as to reduce it to a linear quadratic tracking problem with no derivative constraints on the transformed control vector. The details are given in Section F.3 of this Appendix. The final result is given below for first order lag integral feedback. We have, for \( u(t) = \hat{u} \),

\[
\hat{u}^* = -R_2^{-1}P_{22} \hat{x} - R_2^{-1}P_{22}^T \hat{y}^* - R_2^{-1}b_2
\]

(F.1.92)

\[
\hat{y}^*(t_0) = y(t_0)
\]

(F.1.93)

\[
\dot{P}_{11} = -P_{11}P_{11}^T - P_{12}R_1^{-1}P_{21} + Q
\]

(F.1.94a)

\[
\dot{P}_{22} = -P_{22}G^T P_{22} + P_{22}R_2^{-1}P_{22} + R_1
\]

(F.1.94b)

\[
\dot{P}_{21} = -P_{21}G + G^T P_{21} + P_{22}R_2^{-1}P_{22} + R_1
\]

(F.1.94c)
\[ P_{11}(t_f) = S \]  
\[ P_{22}(t_f) = S_{11}. \]  

\( S_{11} \) is an \( m \times m \) symmetric positive semidefinite weighting matrix on the final values of the control \( y(t_f) \). It will usually be set to 0 for our purposes, although this is not necessary.

\[ P_{21}(t_f) = 0 \]  
\[ \dot{b}_{11} = P_{21}X_{21} - P_{12}X_{12} + QX_d \]  
\[ \dot{b}_{12} = P_{22}X_{22} - QX_{11} \]  
\[ b_{11}(t_f) = -S_{11}(t_f) \]  
\[ b_{12}(t_f) = 0 \]

\[ J^*(x(t_0), y(t_0), t_0) = x^T(t_0)P_{11}(t_0)x(t_0) + y^T(t_0)P_{21}(t_0)y(t_0) + x^T(t_0)P_{22}(t_0)x(t_0) + y^T(t_0)P_{22}(t_0)y(t_0) + 2x^T(t_0)b_{11}(t_0) + 2y^T(t_0)b_{12}(t_0) + c(t_0) \]  
where
\[ c(t_0) = c_1(t_0) \]

and \( c_1(t) \) is given by
\[ \dot{c}_1 = b_{12}^Tb_{12} - X_d^TX_d \]
\[ c_1(t_f) = X_d^TX_d(t_f) \]

This formulation results in integral feedback of all of the state variables. In Section F.4 of this Appendix an alternative formulation is presented which allows direct integral feedback for selected state variables as well as incorporating rate feedback.

This completes the presentation of the deterministic optimal state controller theory that introduces the results relevant to this study. A number of related topics, including the design of observers, output controllers, stochastic controllers, discrete controllers, and sensitivity of the controllers to system variations and suboptimal implementation, have been omitted. The theory that has been covered does provide the necessary background material for studying these other areas. Their theoretical development more often than not parallels that of the deterministic state feedback theory.

Numerical methods for solving these problems will be discussed as required in the numerical calculations.

**F.2 Derivation of Formulation 5**

Linear systems of the form
\[ \dot{x}(t) = F(t)x(t) + G(t)y(t) + Q(t)z(t) \]
with system outputs
\[ y(t) = M(t)\dot{x}(t) \quad (F.2.1c) \]
are considered. Here \( M(t) \) and \( Q(t) \) are control distribution matrices respectively, and \( y(t) \) and \( \eta(t) \) are control disturbance input histories, with \( \eta(t) \) specified \emph{a priori}. The desired output history is given by \( y^*(t) \). The performance criterion is identical to (F.1.73), with \( Q \) and \( X_d \) given by (F.1.80) and (F.1.81) respectively. It is repeated here for convenience:
\[
J = \|X(t_f) - X_d(t_f)\|^2_Q + \int_{t_0}^{t_f} \left[ \|\dot{x}(t) - x_d(t)\|^2_Q + \|y(t)\|^2_Q \right] dt.
\quad (F.2.2)
\]
An optimal solution will be found using a H-J-B equation approach. From (F.1.15) and (F.1.16), this is given by
\[
J^*(\dot{x}(t), t) + H\left(\dot{x}(t), y^*(\dot{x}(t), J^*, t), J^*, t_0\right) = 0 \quad (F.2.3a)
\]
with boundary condition
\[
J^*(\dot{x}(t_f), t_f) = s[\dot{x}(t_f), t_f] \quad (F.2.3b)
\]
where \( s \) is the terminal state weighting terms in the general performance criterion (F.1.5). Here
\[
J^*(\dot{x}(t), t) = H\left[\dot{x}(t), u^*(\dot{x}(t), J^*, t), J^*, t_0\right] = \min_{u(t)} H\left[\dot{x}(t), u(t), J^*, t_0\right] \quad (F.2.4)
\]
where, for this problem (see the discussion following (F.1.11)),
\[
H = [\dot{x}(t) - x_d(t)]^T Q(t) [\dot{x}(t) - x_d(t)] + y(t)^T R(t) y(t)
+ J^* x^T [x(t), t] \{F x + G u + G \eta\}.
\quad (F.2.5)
\]
For a minimum of \( H \) over \( u \) it is necessary that
\[
\frac{\partial H}{\partial u} = 0 \quad (F.2.6)
\]
Applying this to the Hamiltonian of (F.2.5), the result is
\[
2Ru + Q^T J^* = 0 \quad (F.2.7a)
\]
or
\[
u = -\frac{1}{2} R^{-1} Q^T J^* \quad (F.2.7b)
\]
where \( R \) symmetric is assumed. Sufficient conditions for this to be a local minimum are given by
\[
\frac{\partial^2 H}{\partial u^2} > 0 \quad (F.2.8a)
\]
or, upon differentiating (F.2.7a) once more with respect to \( y \), gives
Thus sufficient conditions that (F.2,7b) is a local minimum are that \( R(t) \) be positive definite symmetric. But (F.2,8b) holds for all \( y(t) \). Thus condition (F.2,8b) also implies that this is a global minimum of \( H \) over \( y \).

Substituting (F.2,7b) into (F.2,5), and (F.2,5) into (F.2,3a), the result is

\[
\frac{3J^*}{\dot{x}} + [x - x_d]^T Q [x - x_d] + J^* T P x - \frac{1}{2} J^* T G R^{-1} G T J^* + J^* T q = 0
\]

(F.2,9)

This is a nonlinear, partial differential equation of the first order. As is sometimes the case with such equations a separation of variables approach proves to be fruitful. One may intelligently "guess" at a reasonable form of the solution because of the nature of the terminal conditions as given in equation (F.1,85) (see also equation (F.2,17) to follow).

Thus we try

\[
J^*(x(t), t) = \alpha^T(t) \bar{P}(t) \bar{x}(t) + 2 \alpha^T(t) \bar{b}(t) + c(t)
\]

(F.2,10)

If this is indeed a solution, then using (F.2,7b) and

\[
\frac{3J^*}{\dot{x}} = 2P x + 2b
\]

(F.2,11)

the optimal feedback strategy is given by

\[
y^* = -R^{-1} S^T [P x + b].
\]

(F.3,12)

Substituting (F.2,10) into (F.2,9) (with (F.2,1a) and (F.2,11)) the result is

\[
x^T P x + 2x^T \bar{b} + c + x^T Q x - 2x^T Q x_d + x_d^T Q x_d
\]

\[
+ 2x^T P P x + 2x^T \bar{P} \bar{b} - x^T P G R^{-1} G T P x
\]

\[
- 2x^T P G R^{-1} G \bar{b} - b^T G R^{-1} G \bar{b} + 2b^T \bar{P} \bar{b}
\]

\[
+ 2b^T \bar{q} = 0.
\]

(F.2,13)

Upon combining terms of the same degree in \( x \) we obtain

\[
x^T [\bar{P} + Q + 2P P - P G R^{-1} G T P] x + x^T [2\bar{b} - 2Q x_d]
\]

\[
+ 2x^T \bar{P} - 2P G R^{-1} G \bar{b} + 2P \bar{b}] + [c + x_d^T Q x_d]
\]

\[
- b^T G R^{-1} G \bar{b} + 2b^T \bar{P} \bar{b}] = 0.
\]

(F.2,14)

The quadratic portion of (F.2,14) contains a \( 2x^T P P x \) term. Using Lemma G.1, which is proven in Section 6.3 of Appendix G, this can be written

\[
2x^T P P x = x^T (P P + P T) x.
\]

(F.2,15)

Using (F.2,15) and the observation that (F.2,14) must hold for all \( x \) (thus the \([\cdot]\) terms must all be zero
independently) three ordinary differential equations are obtained. The first term becomes

\[ \dot{P} = -P_T P + P G R^{-1} G_T P - Q. \quad (F.2,16a) \]

The second term becomes

\[ \dot{b} = -[P_T - P G R^{-1} G_T] b + Q x_d - P_0 Q. \quad (F.2,16b) \]

The third term becomes

\[ \dot{c} = [b_T G R^{-1} G_T] b - x_d Q x_d - 2 P T Q. \quad (F.2,16c) \]

The boundary conditions for the equations (F.2,16) are obtained by considering the general boundary condition (F.2,3b). In terms of the performance criterion (F.2,2) and the \( J^* \) given by (F.2,10), this is

\[ x^T(t_f) P(t_f) x(t_f) + 2 x^T(t_f) b(t_f) + c(t_f) \]

\[ = x^T(t_f) S x(t_f) - 2 x^T(t_f) S x_d(t_f) + S_d^T(t_f) S x_d(t_f). \quad (F.2,17) \]

Equation (F.2,17) must hold independently of \( x(t_f) \), since \( P \), \( b \) and \( c \) are independent of \( x \). Thus the boundary conditions must be

\[ P(t_f) = S \quad (F.2,18a) \]

\[ b(t_f) = -S x_d(t_f) \quad (F.2,18b) \]

\[ c(t_f) = S_d^T(t_f) S x_d(t_f). \quad (F.2,18c) \]

Because the boundary conditions are given at \( t_f \), the extremum value of \( J^* \) is given by working backward from \( t_f \), i.e.

\[ J^*(x(t_0), t_0) = x^T(t_0) P(t_0) x(t_0) + 2 b(t_0) x_d(t_0) + c(t_0). \quad (F.2,19) \]

The problem has thus been reduced to solving three sets of ordinary differential equations. As yet, not a great deal has been said about restrictions on \( S \), \( P \) and \( Q \), other than that \( P \) must be positive-definite symmetric. It has been assumed, however, throughout the proof, that \( P \) is symmetric. For this to be the case, (F.2,16a) and the initial condition (F.2,18a) must also be symmetric. This can only be true if \( S \) and \( Q \) are also symmetric. Finally, from considerations of the nature of the performance criterion (F.2,2), \( S \) and \( Q \) must be positive semidefinite for a minimization of deviation from a reference state trajectory. If \( S \) and \( Q \) were negative definite, for example, one could "minimize" \( J \) by inputting appropriate controls which drive \( x(t_f) \) far from \( x_d(t_f) \), thus making the first term of (F.2,2) and the first term under the integral very negative, and thus minimizing \( J \) (the control inputs ...)
are still constrained by the positive definite nature of $R$). This minimum, however, would be of $J$ (which can now be negative) and not of the deviation from $x_d$, which is what is desired.

If
\[ y(t) = 0 \]

the finite terminal time linear quadratic tracking problem result of Formulation 4 of Section F.1 of this Appendix is obtained, and if both (F.2,20) and
\[ x_d(t) = 0 \]

hold, the finite terminal time linear regulator result of Formulation 1 of the same section is obtained.

The usual design of optimal controllers leads to proportional controllers, i.e. controllers which do not take advantage of derivative (rate) and integral (memory) properties of the system state. In the following two sections we discuss and prove a number of results which remove some of these restrictions, and provide a combination of integral, rate, and proportional control.

F.3 Integral Controllers (Feedback with Memory)

Anderson and Moore [4.18] indicate that an integral controller is obtained if we consider performance criteria of the form (with $x_d = 0$)
\[
J = \| x(t_f) - x_d(t_f) \|^2_S \\
+ \int_{t_0}^{t_f} \left( \| x(t) - x_d(t) \|^2_{Q(t)} + \| u(t) \|^2_{R_1(t)} + \| \dot{u}(t) \|^2_{R_2(t)} \right) dt
\]

for the linear system
\[
\dot{x} = F(x) + G(t) y.
\]

Their treatment, however, is for systems where the desired state trajectory is zero. This restriction is not imposed in the following.

Consider the augmented system
\[
\dot{x}_1 = F_1 x_1 + G_1 u_1
\]

where
\[
x_1^T = [x^T \ u^T] \quad (F.3,4a)
\]
\[
u_1^T = u^T \quad (F.3,4b)
\]
\[
F_1 = \begin{bmatrix} F & G \\ 0 & 0 \end{bmatrix} \quad (F.3,4c)
\]
\[
G = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad (F.3,4d)
\]
The performance criterion (F.3,1), in terms of the augmented system, may be written

\[ J_1 = \| x_1(t_f) - x_1(t_f) \|_{S_1}^2 \]

\[ + \int_{t_0}^{t_f} \left( \| x_1(t) - x_1(t) \|_{Q_0(t)}^2 + \| u_1 \|_{R_2(t)}^2 \right) dt \]  

(F.3,5)

where

\[ x_{1d}^T = [ x_{d1}^T \ 0^T ] \]  

(F.3,6a)

\[ Q_0 = \begin{bmatrix} 0 & 0 \\ 0 & R_1 \end{bmatrix} \]  

(F.3,6b)

\[ S_1 = \begin{bmatrix} S & 0 \\ 0 & S_{11} \end{bmatrix} \]  

(F.3,6c)

The solution to the augmented problem exists provided that for \( t \in [t_0, t_f] \)

\[ R_2(t) > 0 \]  

(F.3,7a)

\[ Q_0(t) \geq 0 \]  

(F.3,7b)

\[ S_1 \geq 0 \]  

(F.3,7c)

and \( F_1(t), \ Q_0(t), \ R_2(t) \) are continuous, and \( C_1(t) \) is piecewise continuous. This solution is given in Formulation 4 of Section F.1 of this Appendix. This is

\[ u_1^*(t) = -R_2^{-1}(t)G_2^T(t)F(t)x_1(t) + b_2(t) \]  

(F.3,8)

where \( F(t) \) is a solution of the matrix Riccati equation

\[ \dot{F}(t) = -F(t)F_1(t) - F_1^T(t)F(t) + F(t)G_1R_2^{-1}G_1^T(t) - Q_0(t) \]  

(F.3,9a)

and

\[ \dot{b}_1(t) = -[F_1^T(t) - F(t)G_1R_2^{-1}G_1^T(t)]b_1(t) + Q_0x_1(t) \]  

(F.3,9b)

with terminal conditions

\[ F(t_f) = S_1 \]  

(F.3,10a)

\[ b_1(t_f) = -S_1x_1(t_f) \]  

(F.3,10b)

The optimal value of the performance criterion \( J_1 \) is given by

\[ J_1^*(x(t_0), t_0) = x_{1}^T(t_0)F(t_0)x_1(t_0) + 2x_{1}^T(t_0)b_1(t_0) + c_1(t_0) \]  

(F.3,11)

where

\[ c_1(t) = b_{1d}^T(t)G_{1d}R_2^{-1}G_{1d}^T - x_{1d}^TQ_0x_{1d} \]  

(F.3,12)
with terminal condition

\[ c_1(t_f) = Z_{11}^T(t_f)S_{11}X_{11}(t_f). \quad (F.3,13) \]

If (F.3,7b) and (F.3,7c) are satisfied, then

\[ s, q(t), \bar{q}(t) \geq 0 \quad (F.3,14) \]

must also hold.

What is left is to reinterpret the above solution in terms of the original matrices. Let

\[ p(t) = \begin{bmatrix} p_{11} & p_{21}^T \\ p_{21} & p_{22} \end{bmatrix} \quad (F.3,15) \]

where \( p_{11} \) is \( n \times n \), \( p_{21} \) is \( m \times n \), \( p_{22} \) is \( m \times m \), and

\[ b_1 = \begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix} \quad (F.3,16) \]

where \( b_{11} \) is \( n \times 1 \), \( b_{12} \) is \( m \times 1 \). Substitution of (F.3,15) and (F.3,16) into (F.3,8), (F.3,9), (F.3,11) and (F.3,12) with terminal conditions (F.3,10) and (F.3,13), yields, after some tedious manipulations, the results (F.1,92) through to (F.1,100) of Formulation 6 of Section F.1 of this Appendix. We note that \( s_{11} \) is an \( m \times m \) symmetric positive semidefinite weighting matrix on the final values of the control \( u(t_f) \). It will usually be set to \( Q \), although this is not necessary.

F.4 Proportional Plus Integral Plus Rate Controllers

The formulation in the previous section has two disadvantages:

1. One cannot specifically suppress integral feedback on a particular state variable.
2. No mechanism for rate feedback is available.

The following formulation circumvents these problems but leads to a considerable increase in the dimensionality of the augmented system.

We consider the time-invariant system

\[ \dot{x} =Fx + Gu \quad (F.4,1) \]

and the performance criterion

* These results may be extended to systems where \( F \) and \( G \) are time-varying with no increase in conceptual difficulty. There is an increase in analytical complexity, however, in some of the manipulations to follow.
\[ J = \| x(t_f) - x_d(t_f) \|_2^2 + \int_0^{t_f} \left( \| x(t) - x_d(t) \|_2^2 + \| y_1(t) - y_{1d}(t) \|_2^2 \right) dt \]  

where \( S, Q_1, Q_2, Q_3 \geq 0 \) time-invariant \hspace{1cm} (F.4,3a)  
\( R > 0 \) weighting matrices \hspace{1cm} (F.4,3b)  
\[ Y_1(t) = \dot{x}(t) \] \hspace{1cm} (F.4,3c)  
\[ Y_2(t) = \int_0^t x(t) dt \] \hspace{1cm} (F.4,3d)  
\[ Y_{1d}(t) = \dot{x}_d(t) \] \hspace{1cm} (F.4,3e)  
\[ Y_{2d}(t) = \int_0^t x_d(t) dt. \] \hspace{1cm} (F.4,3f)  

Thus \[ \dot{Y}_1 = \dot{x} \] \hspace{1cm} (F.4,4)  

or, upon differentiating (F.4,1) and substituting into (F.4,4), \[ \dot{Y}_1 = F \dot{x} + Gu \] (F.4,5a)  

Also \[ \dot{Y}_2 = x. \] (F.4,6)  

In an analogous argument to that of the previous section, define an augmented system \[ \dot{z}_1 = F_1 z_1 + G_1 u_1 \] \hspace{1cm} (F.4,7)  

where \[ \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \\ \dot{z}_6 \\ \dot{z}_7 \\ \dot{z}_8 \end{bmatrix}^T = \begin{bmatrix} x^T \\ z_1^T \\ z_2^T \\ z_3^T \\ z_4^T \\ z_5^T \\ z_6^T \\ z_7^T \end{bmatrix} \] \hspace{1cm} (F.4,8a)  
\[ u_1^T = \begin{bmatrix} u^T \\ u_1^T \\ u_2^T \end{bmatrix} \] \hspace{1cm} (F.4,8b)  
\[ F_1 = \begin{bmatrix} F & 0 & 0 & 0 \\ 0 & F & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & F \end{bmatrix} \] \hspace{1cm} (F.4,8c)  
\[ G_1 = \begin{bmatrix} 0 \\ G \\ 0 \\ 0 \end{bmatrix} \] \hspace{1cm} (F.4,8d)  

The performance criterion for the augmented system is
where
\[ S_1 = \begin{bmatrix} S_{11} & 0 & 0 & 0 \\ 0 & S_{22} & 0 & 0 \\ 0 & 0 & S_{33} & 0 \\ 0 & 0 & 0 & S_{44} \end{bmatrix} \] (F.4,10c)

The conditions (F.4,3a) guarantee that
\[ Q, S_1 > 0 \] (F.4,11)

if \( S_{22}, S_{33} \) and \( S_{44} \) are positive semidefinite*, which is required for a meaningful definition of the problem.

* \( S_{44} \) is a terminal weighting on the control vector \( y \).

The problem has now been cast in a form where the theory of Formulation 4 of Section 1 of this Appendix may be applied. The result is
\[ u_1^*(t) = -R^{-1}(t)Q_1(t) + P(t)x_1(t) + b_1(t) \] (F.4,12)

where \( P(t) \) is a solution of the matrix Riccati equation
\[ \dot{P} = -PF_1 - F_1T + PG_1R^{-1}G_1^T \] (F.4,13a)

and
\[ \dot{b}_1 = -[x_1^T - PG_1R^{-1}G_1^T]b + Q_1x_1 \] (F.4,13b)

with terminal conditions
\[ P(t_f) = S_1 \] (F.4,14a)
\[ b_1(t_f) = -S_1x_1(t_f) \] (F.4,14b)

The optimal value of the performance criterion \( J_1 \) is still given by (F.3,11) where now
\[ c_1 = b_1^T G_1 R^{-1} G_1^T b_1 - x_1^T Q_1 x_1 \] (F.4,15)

with the terminal condition
\[ c_1(t_f) = x_1^T(t_f) S_1 x_1(t_f) \] (F.4,16)
The interpretation of this solution in terms of the original matrices is tedious but straightforward. In any case, because of the way in which the augmented state vector was defined (i.e. it contains no control terms, cf. the approach used in Section 3 of this Appendix), such a re-interpretation would seem to be an unnecessary labour. Thus the solution is left in the augmented form.

The feedback law (F.4,12) has indeed achieved the objective of providing rate-proportional-integral control because $x_1$ contains rate and integral components of the original state. Any particular type of feedback for a particular state variable may be turned off by only including certain rate and integral components of the state in the $Y_1$ and $Y_2$ equations.

The $Q_1$, $Q_2$, and the $Q_3$ terms of the original performance criterion (F.4,2) may be somewhat contradictory in their objectives. Such difficulties can usually be overcome in an iterative way using different values for $Q_1$, $Q_2$, $Q_3$. It may also be useful to have $Y_{2d}$ set to zero as the interpretation for this term when the aircraft is on the correct trajectory is not clear. Finally, in the context of the CGG studies it may be useful to set $x_d$, $y_{1d}$, and $y_{2d}$ to zero, and then for the $x_I$, $y_I$, $z_I$ feedbacks use $x_I\xi$, $y_I\xi$, and $z_I\xi$, where these are appropriately defined deviations from the curved approach trajectory rather than from the linearization reference equilibrium. This will result in suboptimal feedback control that may still provide satisfactory performance.

This flexibility has been gained at the expense of considerably increasing the dimensionality of the system (i.e. a system of order $3n$ is required if full integral and rate feedback is considered). Since for aircraft systems $m$ is usually considerably smaller than $n$, it may be advantageous to continue to treat this problem in a fashion similar to Section F.3 of this Appendix, i.e. with integral constraints on $y$, as well as derivative constraints. In such a way the dimensionality of the system would be minimized, at the expense of some flexibility.
REFERENCES — APPENDIX F


APPENDIX G

WORST-CASE WIND CONTROLLERS

G.1 Perverse Wind Controllers - The Direct Method

As proposed in Section 5.3.2, it may be possible to generate worst-case disturbance inputs by considering performance criteria of the type (5.3,5), i.e.

\[
J = \|x(t_f) - x_d(t_f)\|_2^2 + \int_{t_0}^{t_f} \left[ \|x(t) - x_d(t)\|^2 + \|g(t)\|^2 \right] dt
\]

(G.1,1)

The equations of motion are given by (F.2,1).

The objective is to maximize or minimize \( J \) in such a way that \( \|x(t) - x_d(t)\| \) is maximized. This worst-case wind controller problem is meaningful if (G.1,1) is maximized with

\[
R(t) < 0 \quad (G.1,2a)
\]

\[
S, g(t) > 0 \quad (G.1,2b)
\]

(cf. the optimal control problem where (F.2,2) is maximized with \( Q, S > 0, R > 0 \)).

The sign definiteness of \( R \) is determined from the global minimization (maximization) of the Hamiltonian that occurs over \( \eta(t) \) in equation (F.2,6) and (F.2,8a). The solution, if it exists, is given by (F.2,10), (F.2,12), (F.2,16) and (F.2,18) with \( u \) replaced by \( \eta \), \( G \) replaced by \( S \) and \( \bar{Q} \) replaced by \( R \). The matrices \( S, Q \) and \( R \) should be symmetric for the same reasons as for the optimal controller problem discussed in Appendix F, Section 2. \( y(t) \) is a control history that is specified \( a \ priori \).

G.2 Proof of Theorem 5.5

Time-varying systems of the form (the \( t \) arguments are dropped for brevity)

\[
\dot{x} = Fx + G_1u + G_2u + \bar{f}
\]

(G.2,1a)

\[
x(t_0) = x_0 \quad t \in [t_0, t_f]
\]

(G.2,1b)

with output

\[
y = Mx
\]

(G.2,1c)

are considered. The desired output is given by \( y_d(t) \). The
resulting desired state trajectory is related to \( y_d(t) \) by (5.4,53).

The payoff functional to be minimaximized is given by (5.4,52) and is repeated here for convenience:

\[
J = \| x(t_f) - x_d(t_f) \|_2^2 + \int_{t_0}^{t_f} \left[ \| x(t) - x_d(t) \|_Q^2(t) + \| y \|_{R_1}^2 + \| n \|_{R_2}^2 \right] dt
\]

(G.2,2a)

\[
R_1 > 0 \quad \text{symmetric.}
\]

(G.2,2b)

\[
R_2 < 0
\]

(G.2,2c)

A general H-J-B equation* (value function) approach will be employed to prove Theorem 5.5. The general H-J-B equation is

\[
\frac{\partial J^*}{\partial t} + \max_{u} \min_{\pi, \nu} H(x, u, \pi, J^*, t) = 0 \quad \text{(G.2,3a)}
\]

\[
J^*(x(t_f), t_f) = \| x(t_f) - x_d(t_f) \|_2^2 \quad \text{(G.2,3b)}
\]

where the Hamiltonian is given by

\[
H = \| x - x_d \|_Q^2 + \| y \|_{R_1}^2 + \| n \|_{R_2}^2 + J^T \left[ F x + G_1 u + G_2 \nu + f(t) \right]
\]

(G.2,4)

Here the relationship (5.4,44) has been employed.

* See Theorem 5.4.

Making use of the sufficient version of the continuous game minimax conditions (5.4,3a) and (5.4,3b), it may be shown that \( H \) is globally minimaximized over \( u, \pi \) if (G.2,2b) and (G.2,2c) hold and if

\[
u^* = -\frac{1}{J^T G_2 J^*}
\]

(G.2,5a)

\[
\pi^* = -\frac{1}{J^T G_2 J^*}
\]

(G.2,5b)

Substitution of (G.2,5) into (G.2,4) and (G.2,4) into (G.2,3) yields

\[
\frac{\partial J^*}{\partial t} + (x - x_d)^T Q (x - x_d) + \frac{1}{J} J^T X G_1 J T X + \frac{1}{J} J^T G_2 J^T X
\]

\[
+ J^T F X - \frac{1}{J} J^T X G_1 G_1 J X - \frac{1}{J} J^T G_2 G_2 J X + \pi^T f = 0.
\]

(G.2,6)

This is a nonlinear, partial differential equation of first order. As in Appendix F, Section F.2, a separation of variables approach will be pursued. Thus a solution of the form (F.2,10) is considered.

If (F.2,10) is indeed a solution, then (F.2,11) must also hold, and thus from (G.2,5)

\[
u^* = -\frac{1}{J^T G_1 J} [F x + b]
\]

(G.2,7a)

\[
\pi^* = -\frac{1}{J^T G_2 J} [F x + b]
\]

(G.2,7b)
Substituting (F.2,11) into (G.2,6), rearranging the resulting equation, extracting the symmetric part of $P$ using Lemma G.1, and noting the independence of this result from $\chi$, the three sets of ordinary differential equations (5.4,55a), (5.4,55b) and (5.4,58) result. The terminal conditions (5.4,56a), (5.4,56b) and (5.4,59) follow from (F.2,10), which has now been shown to be a solution, and (G.2,3b).

It follows that the extremum value of the payoff functional is given by (F.2,19). This completes the proof of Theorem 5.5.

Remark 1

The minimax solution (G.2,7) exists on $[t_0, t_f]$ and is unique provided the solution to the ordinary differential equation (5.4,55a) with initial condition (5.4,56a) exists on $[t_0, t_f]$. If $P(t)$ exists on $[t_0, t_f]$, then so must $b$ and $c$ provided that $X_d(t)$ is continuous.

Remark 2

Strictly speaking, before the solutions (G.2,7) can be accepted as minimax, the verification steps (5.4,27) through to (5.4,30) should be carried out. This is unnecessary for this problem, however, because the details of the problem make it self-evident that the two one-sided problems will yield the same minimax solutions.

G.3 Lemma G.1

Given that (1) $A$ is a symmetric $n \times n$ matrix, (2) $B$ is a general $n \times n$ matrix, and (3) $x$ is a $n \times 1$ column vector, then

$$2x^T ABx = x^T (AB + B^T A)x.$$  \hspace{1cm} (G.3.1)

Proof

$AB$ may be written as the sum of a symmetric part and an unsymmetric part:

$$AB = \frac{1}{2} [AB + (AB)^T] + \frac{1}{2} [AB - (AB)^T].$$  \hspace{1cm} (G.3.2)

Using the transpose properties

(1) $(AB)^T = B^T A^T$

and (2) the transpose of a $1 \times 1$ matrix (scalar) equals itself, it can be shown that

$$2x^T ABx = x^T [(AB + (AB)^T) + (AB - (AB)^T)] x$$  \hspace{1cm} (G.3.4)

or by equating the right hand sides

$$x^T [AB - B^T A] x = 0.$$  \hspace{1cm} (G.3.6)
The bracketed term in this equation is just the unsymmetric part of $2AB$. Thus

\[ 2x^T Abx = x^T (Ab + b^T a)x. \quad \text{(G.3,7)} \]

This is the desired result.
This appendix contains brief descriptions of the major computer programs developed for this study. All of the programs were coded in IBM Fortran for the GI compiler, and were run on the University of Toronto IBM 3033 computer. Extensive use has been made of a number of subroutines available in the IMSL library [H.1]. Program listings and detailed documentation is available upon request.

The execution times required for the different programs and their options are compared in Table H.1.

### H.1 VVWCM1 Program

This program applies Theorem 5.1 to the longitudinal decoupled linearized equations of motion given by (5.5, 19) without the state vector augmentation by the control vector \( \Delta \xi \). Thus the matrix form of these equations becomes (cf. (5.5,20))

\[
\dot{\Delta \xi} = \hat{\Delta} \Delta \xi + \hat{C}_1 \Delta \xi + \hat{C}_2 \Delta \omega + \hat{C}
\]  

(H.1,1)

The application of the theorem is restricted to cases where

\[
N(t) = I
\]  

(H.1,2a)

implying that the state vector and the output vector are identical [see (5.2,2)], and

\[
\bar{g}(t) = \bar{g} \text{ (constant)}. \quad \text{(H.1,2b)}
\]

Since the equations of motion (H.1,1) are time-invariant, the impulsive response functions required in defining the worst-case wind inputs may be computed using a direct matrix exponential, i.e.*

\[
\hat{g}(t) = e^{\hat{A}t}
\]  

(H.1,3a)

\[
= I + \hat{A}t + \frac{1}{2!} \hat{A}^2 + \frac{1}{3!} \hat{A}^3 + \ldots
\]  

(H.1,3b)

and

\[
H(t) = \hat{g}(t)\hat{C}_2.
\]  

(H.1,4)

It is computationally more efficient, however, to evaluate \( e^{\hat{A} \Delta t} \) once for a suitably chosen time interval \( \Delta t \), and then use the properties of the matrix exponential to obtain

\[
H(k \Delta t) = (e^{\hat{A} \Delta t})^k \hat{C}_2, \quad k = 0,1,2,\ldots, N \quad \text{(H.1,5)}
\]

* \( \hat{A}^2 = \hat{A} \hat{A}, \hat{A}^3 = \hat{A} \hat{A} \hat{A} \) and so forth.
The latter procedure is used in this program. The $N+1$ time points at which the matrix $\mathcal{H}(t)$ is computed are stored on a direct access dataset. Intermediate values of $\mathcal{H}(t)$ are obtained using linear interpolation. Provision has been made for by-passing the impulsive response computations for cases where the impulsive response matrix was computed separately. If this option is in effect, the $\mathcal{H}(t_i)$ are inputted from the direct access dataset.

Provision has also been made for inputting a time-invariant proportional control law. In general such a control law may be written

$$\Delta \dot{x} = K \Delta x.$$  

Incorporating (H.1,8) into (H.1,1), the latter becomes

$$\dot{\Delta x} = \hat{A}_{CL} \Delta x + \hat{C}_2 \Delta u + \hat{c}$$  

where

$$\hat{A}_{CL} = \hat{A} + \hat{C}_1 K.$$  

If this option is used, the impulsive response matrices that are computed are for the closed-loop system, i.e. (H.1,3) with $\hat{A}$ replaced by $\hat{A}_{CL}$, permitting worst-case inputs to be defined for the closed-loop system.

The computation of the worst-case wind inputs given by equation (5.2,10) of Theorem 5.1 requires the repeated evaluation of the integral

$$I(t) = \int_{t_f}^{t} \mathcal{H}(\tau-t) \phi d\tau.$$  

For the purposes of the integration, $t$ is a constant. Thus, letting

$$\tau' = t - \tau$$

$I(t)$ transforms to

$$I(t) = -\int_{0}^{t_f-t} \mathcal{H}^T(\tau') \phi d\tau'.$$  

At $t = t_f$, $I(t) = 0$, while at $t = 0$ the integration interval is at a maximum. Thus the computationally most efficient procedure is to perform the integration in $N$ intervals from $t = t_f$ to $t = 0$. Then

$$-\int_{0}^{t_f-k\Delta t} \mathcal{H}^T(\tau') \phi d\tau' = -\sum_{i=1}^{N-k} \int_{(i-1)\Delta t}^{i\Delta t} \mathcal{H}^T(\tau') \phi d\tau'.$$  

All the required values of $I(t)$ for the $N+1$ time points...
are obtained by letting $k$ range from $k = 0$ to $k = N$, with $f(t) = 0$ for $k = N$. These values are computed once and are stored on a direct access dataset. Linear interpolation is used to obtain values intermediate to the stored points.

The integrals in (H.1,14) and in the definition of the constant $\mu$, as given by (5.2,11), are computed using a trapezoidal integration scheme. The equations of motion (H.1,1) are solved using the IMSL subroutine DGEAR [H.1]. This subroutine, depending on the options specified, uses either a variable order Adams predictor corrector method or Gear's backward differentiation methods [H.2], to find the response.

A plotting package was developed for this program, based on the Gould 5000 electrostatic printer/plotter available at the University of Toronto. This plotting package may be invoked optionnally to plot the aircraft response versus time, the worst-case wind inputs versus time, the aircraft flight path, and the worst-case wind altitude profiles.

The initial conditions for the simulation may be specified as input values, as trim conditions for flight along a glidepath of ground referenced angle $\gamma_G$ in the presence of a constant, specified headwind (this will generally be the headwind that existed at the beginning of the simulation, i.e. $W_1(h_0)$), or some combination thereof. The trim condition on $\Delta \theta$ is obtained by applying the linearization reference equilibrium theory of Section 3.4 of Chapter 3 for* $W_1(e)$ replaced by $W_1(h_0)$ to compute a $\theta_{\text{trim}}$. Thus $\Delta \theta_{\text{trim}}$ may be obtained from

$$\Delta \theta_{\text{trim}} = \theta_{\text{trim}} - \theta_e.$$  

Nonzero $\Delta \theta_{\text{trim}}$ also implies trim conditions on $\Delta w$, as well as on the control inputs $\Delta \delta_E$ and $\Delta \delta_T$. These may be determined by solving the linear algebraic system that results when one looks for a steady-state solution of the equations (H.1,1) under the constraint that the steady-state solution must be identical to the linearization reference equilibrium solution of Section 4 of Chapter 3 except for nonzero** $\Delta w$, $\Delta b$, $\Delta \delta_E$ and $\Delta \delta_T$. The algebraic equations that must be solved can be shown to be (see also Appendix A of [1.39])

$$X_w \Delta w_{\text{trim}} + X_{\delta_T} \Delta \delta_{E_{\text{trim}}} + X_{\delta_T} \Delta \delta_{T_{\text{trim}}} = mg \cos \theta_e \Delta \theta_{\text{trim}}$$  

(H.1,16a)

$$Z_w \Delta w_{\text{trim}} + Z_{\delta_T} \Delta \delta_{E_{\text{trim}}} + Z_{\delta_T} \Delta \delta_{T_{\text{trim}}} = mg \sin \theta_e \Delta \theta_{\text{trim}}$$  

(H.1,16b)

* It is noted that for the equations (H.1,1), $\dot{W}_2 = W_2 = W_e = 0$.

** There will be complete agreement if $W_1(h_0) = W_1(e)$. 
\[ M_w \Delta w_{trim} + M_{SE} \Delta \delta_{E_{trim}} + M_{ST} \Delta \delta_{T_{trim}} = 0. \] (H.1,16c)

These equations are solved for \( \Delta w_{trim} \), \( \Delta \delta_{E_{trim}} \) and \( \Delta \delta_{T_{trim}} \) using the IMSL subroutine LEQT1F. They are for open-loop trim conditions, i.e. \( K = 0 \) in (H.1,8). If \( K \neq 0 \), trim conditions were not computed, and the initial control settings are determined from the initial state with the control law (H.1,8).

The algorithm used by this program is now summarized:

1. Read input data and parameters. These include program option control flags, \( \rho, g, V_e, \gamma_G, h_0, \Delta \xi(0) \), the aircraft mass and aerodynamic characteristics, \( \delta_f, \delta, K, sgn[\mu], S_{WS}, N, t_f \) and linearization reference equilibrium data.
2. Compute initial trim conditions.
3. Define system matrices \( \mathbf{\hat{A}} \) (optionally \( \mathbf{\hat{A}_{CL}} \), \( \mathbf{\hat{C}_1}, \mathbf{\hat{C}_2} \) and \( \mathbf{\hat{C}} \).
4. Compute the perturbation matrix exponential using a direct expansion, and find the impulsive response matrix at \( N+1 \) times using (H.1,5) and associated equations (optional). Store the results of these computations on a direct access dataset for later use.
5. Compute the integral \( I(t) \) for \( N+1 \) points using trapezoidal integration and store on a direct access dataset for later use.
6. Compute the value of \( \mu \), defined by (5.2,11), using a trapezoidal integration scheme.
7. Find the system response to the worst-case wind inputs for \( t = 0 \) to \( t = t_f \) and output the results at specified time intervals. Also output worst-case wind inputs.
8. Store system response and worst-case wind inputs on a direct access dataset.
9. Invoke plotting software (optional).

A number of test cases were run. These included comparisons with the results of the VVWCM program for analogous cases, checks for proper convergence for known limiting values of special cases, comparison of the computed with the specified values of \( S_{WS} \), independent testing of the subroutines of the program, and qualitative comparisons of the worst-case results computed by this program with those computed by van der Vaart [1.67] (exact comparisons were impractical because of differences between the two formulations).

H.2 VVWCM Program

This program applies Corollary 5.1.1 to the longitudinal decoupled linearized equations of motion given by (H.1,1).

It is a special case of the VVWCM1 program, and is intended to reduce the computational costs when running IRWC formulations for which

\[ \delta = 0. \] (H.2,1)
The major saving comes from eliminating the need to compute the integral $I(t)$ given by (H.1,11).

The algorithm for this program is identical to that for the VVWCM1 program with the appropriate simplifications included to take into account (H.2,1).

H.3 LQDG-1 Program

This program applies Theorem 5.6 to a suitably chosen subset of the equations of motion (5.5,19). For convenience, the matrix form of these equations is repeated here, i.e.

$$\dot{\Delta x} = \hat{A}\Delta x + \hat{C}_1 \Delta \dot{x} + \hat{C}_2 \Delta w + \hat{C}. \quad (H.3,1)$$

From the definition of $\Delta x$ (see (5.5,21)) and from the form of (H.3,1), it is clear that the wind velocity inputs and control inputs and their rates of change may be specifically weighted in the optimization. $\Delta \dot{x}$ and $\Delta w$ are the "control" inputs through which the aircraft controller and the wind controller, respectively, influence the aircraft dynamic system.

By changing the value of certain program control flags, it may be applied to a number of one-sided and two-sided optimization problems. In particular, it will look for solutions that minimize, maximize or minimaximize an augmented payoff functional of the form

$$J_a = \|x(t_f) - x_d(t_f)\|^2 + \int_0^{t_f} \left(\|x - x_d\|_Q^2 + \|\dot{y}\|_R^2 + u^2\right) dt \quad (H.3,2)$$

subject to the differential equation constraints

$$\dot{x} = Fx + G_1 u + G_2 \dot{y} + \xi. \quad (H.3,3)$$

The system (H.3,3) is obtained from the system (H.3,1) by defining a suitable subset of (H.3,1), subject to the restriction that the subset chosen is a physically meaningful, consistent dynamic system for the aircraft problems being studied here. Thus $\Delta x$ is a subset of $\Delta \dot{x}$, $\Delta u$ is a subset of $\Delta \dot{y}$, and $\Delta w$ is a subset of $\Delta \dot{w}$. The program provides appropriate software mechanisms for such manipulations. This considerably enhances its versatility at the expense of some computational efficiency.

The program provides the following major options:

1. Optimal controller mode - An optimal aircraft controller strategy of the form

$$\dot{u}^* = -R_1^{-1}G_2[e(t)x(t) + b(t)] \quad (H.3,4)$$

is found that minimizes $J_a$ with

$$G_2 = 0. \quad (H.3,5)$$
Infinite terminal time solutions may also be found by making $t_f$ large enough.

2. Maximizing wind controller mode - A worst-case wind control strategy of the form

$$
\ddot{y}^* = -R_2^{-1}Q \dot{y}(t) + P(t) x(t) + b(t)
$$

is found that maximizes $J_a$ with

$$
G_1 = 0 .
$$

Infinite terminal time solutions, if they exist, may also be found by making $t_f$ large enough.

3. Differential game mode - A two-sided minimaximization is carried out that yields minimax $\dot{y}$ and $\ddot{y}$ strategies of the form (H.3.4) and (H.3.6) respectively.

These modes, in conjunction with program capability to define the system (H.3.3) as an arbitrary subset of (H.3.1), permits this program to generate worst-case wind controllers for the indirect (perversity function), direct, and differential game formulations discussed in Chapter 5. The positive parameter $\mu$ may be varied to provide control over the wind "energy" $S_{WS}$

$$
S_{WS}(t_f) = \int_{t_0}^{t_f} \Delta \dot{y}^T R_{21} \Delta \dot{y} \, dt .
$$

The program will also increase $\mu$ by a specified factor until either a conjugate point-free solution is found (see the discussion in Remark 1 following Theorem 5.5 and in Section H.7 of this appendix) or a user-specified maximum number of iterations is exceeded.

The aircraft longitudinal response, as governed by the equations of motion (H.3.1) with the aircraft and/or wind controllers determined in the optimization step, is found using the IMSL subroutine DGEAR (see the comments in Section H.1 of this appendix). The backward differentiation method of Gear that is an option of DGEAR was found to avoid numerical difficulties with the system differential equations when they become stiff. In particular, these problems were found to occur for certain maximizing or differential game cases. An early version of this program employed a fixed step size Runge-Kutta method, but this was found to be unsatisfactory.

The initial condition options are identical to those for the VVWCM program. In particular, trim conditions are found using (H.1.15) and (H.1.16).

The subroutine package DIFGAI is used to compute the wind and aircraft controller strategies. This package is discussed in Section H.7 of this appendix.

A plotting package similar to that discussed for the VVWCM program was also developed, and may be invoked as a user option.

The algorithm used by this program is now summarized:
1. Read input data and parameters. These include program option control flags, \( \rho, g, V_e, \gamma_g, h_0, \Delta \delta(0) \), the aircraft mass and aerodynamic characteristics, quantities that are used in specifying the weighting matrices, and linearization reference equilibrium data.

2. Compute initial trim conditions.

3. Define system matrices \( \hat{A}, \hat{C}_1, \hat{C}_2, \hat{C}, F, G_1 \) and \( G_2 \).

4. Define the weighting matrices of the payoff functional (H.3,2).

5. Find the worst-case aircraft and/or wind controllers as required by the program operating mode using the DIFGAl subroutine package. If a conjugate point is encountered, repeat the computation for a greater value of the parameter \( \mu \) until either a conjugate point-free solution is found or the maximum number of iterations permitted for this step is exceeded. In the latter case, stop execution.

6. Find the longitudinal aircraft response, as governed by (H.3,1), with the optimal aircraft and/or wind controllers determined in the previous step included.

7. Store the aircraft response, the aircraft spatial trajectory and the worst-case wind inputs on a direct-access dataset (optional).

8. Invoke plotting software (optional).

A number of test cases were run. These included comparisons of the results of this program with those of the LQDG-3 program for limiting cases where the results should have been identical, and individual testing of the subroutines of the program. The best measure of program performance, however, is to compare the theoretical value of the extremum value of \( J_a \) with that computed based on the aircraft response found by the program. The latter was computed using a relatively coarse, rectangular integration scheme. In general the computed and theoretical values were found to agree within 3%. A significant portion of this error is probably a consequence of the crudeness of the direct \( J^* \) computation.

H.4 LQDG-3 Program

This program is identical to the LQDG-1 program except that the equations of motion are given by (5.5,19) without the state vector augmentation with \( \Delta \delta \). In matrix form the equations of motion are thus given by (H.1,1). The main consequence of using (H.1,1) rather than (H.3,1) is that \( \Delta \delta \) can no longer be weighted in the optimization, i.e. (H.3,2) becomes

\[
J_a = \| x(t_f) - x_d(t_f) \|^2 + \int_0^{t_f} \left( \| x - x_d \|^2_G + \| u \|^2_{R_1} + \| \dot{u} \|^2_{B_2} \right) dt
\]

(H.4,1)

subject to the differential equation constraints

\[
\dot{x} = Fx + G_1u + G_2\delta + f.
\]

(H.4,2)
H.5 LQDG-5 Program

Formally this program is identical to the LQDG-1 program except that the equations of motion are the decoupled linearized lateral equations of motion. These equations may be written in the matrix form

\[ \mathbf{A}_1 \Delta \mathbf{x} = \mathbf{A}_2 \Delta \mathbf{x} + \mathbf{B}_1 \Delta \mathbf{x} + \mathbf{B}_2 \Delta \mathbf{x} + \mathbf{B}_3 \Delta \mathbf{x} \]  

(H.5,1)

where the elements of the matrices correspond to the equations (3.10,2), (3.10,3), (3.10,11) and (3.10,12) with

\[ \psi_e = 0 \]  

(H.5,2)

since

\[ W_{2e} = 0 \]  

(H.5,3)

Condition (H.5,3) is required to permit the longitudinal and lateral separation of the SDL equations of motion. Thus

\[ \Delta \mathbf{x}^T = [\Delta v \ \Delta p \ \Delta r \ \Delta \phi \ \Delta y] \]  

(H.5,4)

\[ \Delta \mathbf{A}^T = [\Delta A_1 \ \Delta A_2 \ \Delta A_3 \ \Delta A_4] \]  

(H.5,5)
Other manipulations introducing linear wind gradient terms and augmenting the state vector with the control vector and the wind vector may be carried out completely analogously to those for the longitudinal equations, i.e. as in equation (5.5,11) and following. The details are not given here.

The LQDG-5 program was only used to determine an optimal lateral controller for the NL program.

H.6 NL Program

This program computes the aircraft response using the body axes nonlinear equations of motion (3.9,1) through to (3.9,17). The program has three operating modes:

1. The coupled longitudinal and lateral response is computed.

2. The longitudinal response is computed assuming that lateral trim conditions are maintained.

3. The lateral response is computed assuming that longitudinal trim conditions are maintained.

The aerodynamic model is the model AER02 of Section 3.5.2 of Chapter 3. The aerodynamic forces and moments can be computed in two modes. In the first, an aerodynamic linearization reference equilibrium is found and the aerodynamic forces and moments are determined as linear perturbations about this reference state. In the second, the forces and moments are determined from look-up tables based on the data described in Appendix D. Linear interpolation is used to find the values of the aerodynamic parameters at intermediate points.

The program has provision for user defined wind inputs and aircraft controllers. In glidepath-localizer tracking tasks, the spatial trajectories to be tracked are defined offline and stored on a direct access dataset. Linear interpolation is used to find intermediate values. The system response is found using the IMSL subroutine DGEAR (see also the discussion in Section H.1 of this appendix). The initial conditions may be specified as trim conditions based on the equilibrium relationships (3.4,38) and (3.4,39), as user defined conditions, or as some combination thereof.
The algorithm that was used in this program is now summarized:

1. Read input data and parameters. These include program option control flags, \( \rho, g, V_e, \gamma_C, h_0 \) and initial conditions.
2. Compute aerodynamic and initial trim conditions.
3. Compute aircraft response and output results.
4. Store response on a direct access dataset.

Program testing consisted of comparing the response predicted by the program with that predicted by linearized equations of motion, as well as with nonlinear stability axes equations. The subroutines of the program were also tested individually.

H.7 DIFGAI Subroutine Package

This subroutine package solves the ordinary differential equations discussed in Theorem 5.5, i.e. equations (5.4,55a), (5.4,55b) and (5.4,58), for problems of arbitrary dimension. The solution to these equations is required in specifying one-sided and two-sided linear quadratic optimal solutions for payoff functionals of the form (5.4,52). This subroutine package is required in the LQDG-1, LQDG-3 and LQDG-5 programs. The solution method used restricts the application of these subroutines to time-invariant problems.

The matrix Riccati equation (5.4,55a) may be written in more compact notation as

\[
\dot{P} = -PP - P^T P + PEP - Q
\]  

(H.7,1a)

\[
P(t_f) = S
\]  

(H.7,16)

where

\[
E = G_1 R_1^{-1} G_1^T + G_2 R_2^{-1} G_2^T
\]  

(H.7,2)

The solution for this equation exists for all \( t \leq t_f \) if the conditions (5.4,60) hold. This will generally only be true for optimal control problems. In differential game and one-sided wind controller maximization problems, conjugate points will usually appear for some time \( t_c < t_f \). These manifest themselves through the finite escape time of the matrix Riccati equation (see Remark 1 following Theorem 5.5), and must be detected before or as they occur.

In an early version of this subroutine package, the Riccati equation was solved using a direct Runge-Kutta integration scheme. If such a solution method is adopted, a possible method of detecting a conjugate point is to check the size of the solution \( P(t) \) using an appropriate matrix norm. Near the conjugate point the norm will become very large.

\* \( E = G_1 R_1^{-1} G_1^T \) for optimal controller problems,

\( E = G_2 R_2^{-1} G_2^T \) for maximizing wind controller problems.
This test was found to be ineffective; for many of the problems dealt with in this study, the onset of a conjugate point was very abrupt. If the conjugate point test had a norm limit which was large, the conjugate point would not be detected unless the test was done frequently over very small time intervals. On the other hand, if the norm limit was small, peaks in the norm of \( \dot{P}(t) \) would erroneously indicate an approaching conjugate point.

In any case, the fixed step size Runge-Kutta method was found to be computationally expensive. A variable step size predictor-corrector method would probably have increased integration efficiency, but an even more efficient method for time-invariant linear quadratic problems is a slight generalization of an equivalent system approach discussed in both Anderson and Moore [4.18] and Kwakernaak [4.19] for optimal control problems. In this method

\[
P(t) = Y(t)X^{-1}(t) \quad \text{(H.7,3)}
\]

where \( X \) and \( Y \) are solutions of the linear system

\[
\begin{bmatrix}
\dot{X} \\
\dot{Y}
\end{bmatrix} = \begin{bmatrix} F & -E \\ -Q & -F^T \end{bmatrix} \begin{bmatrix}
X \\
Y
\end{bmatrix} \quad \text{(H.7,4)}
\]

\[
X(t_f) = \mathbb{I}. \quad \text{(H.7,5a)}
\]

Let the transition matrix for the system (H.7,4) be denoted by \( \chi \). Then, using the properties of the transition matrix and the equations (H.7,3) and (H.7,5), it can be shown that

\[
P(t) = [X_{21}(t, t_f) + X_{22}(t, t_f)] [X_{11}(t, t_f) + X_{12}(t, t_f)]^{-1} \quad \text{(H.7,6)}
\]

where the transition matrix has been partitioned into

\[
\chi = \begin{bmatrix}
X_{11} & X_{12} \\
X_{21} & X_{22}
\end{bmatrix} \quad \text{(H.7,7)}
\]

and the partitions correspond to (H.7,4).

For time-invariant problems the transition matrix \( \chi \) may be computed using a matrix exponential method. The time-invariance permits considerable computational saving in that the matrix exponential need only be computed once for a suitably chosen time interval \( \Delta t \), in a similar fashion to the method discussed for the VVWCM1 program (see equation (H.1,5)). \( P(t) \) may then be computed for \( N \) points using the Kalman-Englar method [4.17], i.e.

\[
P(t_{i+1}) = [X_{21}(t_{i+1}, t_{i}) + X_{22}(t_{i+1}, t_{i})P(t_{i})] \\
\cdot [X_{11}(t_{i+1}, t_{i}) + X_{12}(t_{i+1}, t_{i})P(t_{i})]^{-1}.
\quad \text{(H.7,8)}
\]
Here

\[ t_{i+1} = t_i + \Delta t, \quad i = 1, 2, \ldots, N \]  \hspace{1cm} (H.7,9)

\[ t_0 = t_f \]  \hspace{1cm} (H.7,10)

\[ \Delta t = -t_f/N \]  \hspace{1cm} (H.7,11)

and

\[ X(t_{i+1}, t_i) = e^{Z \Delta t}, \]  \hspace{1cm} (H.7,12)

where \( Z \) is the first matrix on the right hand side of (H.7,4). It is stressed that (H.7,12) need only be computed once.

It is straightforward to verify that if \( X \) and \( Y \) are solutions to (H.7,4), and if \( X^{-1} \) exists, then \( P(t) \) is indeed given by (H.7,3). It is also possible to show that \( X^{-1} \), and thus the inverse in (H.7,6) and (H.7,8), also exist in a given interval \([t_1, t_f]\), provided that \( P(t) \) exists in that interval. The details are left to Anderson and Moore [4.18].

The fact that conjugate points may be encountered in the worst-case formulations studied here complicates matters in that at a conjugate point \( \|P(t)\| \) becomes infinite, and the solution (H.7,8) becomes meaningless, even though it will continue to yield results, (i.e. it has jumped across the conjugate point) for subsequent iterations. For the type of problems considered in this study this was found to be a convenient mechanism for checking for a conjugate point in that the eigenvalues of a submatrix of the \( P(t) \) computed with (H.7,8) exhibit a marked discontinuity at the conjugate point. A suitable submatrix was found to be the matrix that corresponds to the largest positive semidefinite submatrix of the matrix \( G \), i.e. of \( P(t_f) \).

No attempt was made to formally prove this conjugate point test. It was, however, found to work for all problems solved in this study.

The \( b \) and \( c \) differential equations (5.4,55b) and (5.4,58) of Theorem 5.5 are solved using the IMSL subroutine DGEAR. The DGEAR option implementing Gear's backward differentiation methods were found to be most effective because of the stiffness of the differential equations for certain cases (see also the discussion in Section H.30 of this appendix). The conjugate point test is the eigenvalue test described in the above.

The algorithm used in this subroutine package is now summarized:

1. Compute the perturbation transition matrix (H.7,12) using a direct matrix exponential method.

2. Solve for \( P(t) \) in \( \Delta t \) intervals using (H.7,8) from \( t = t_f \) to \( t = 0 \) or until a conjugate point is encountered. Solve for \( b \) and \( c \) using the IMSL subroutine DGEAR. Symmetrize \( P(t_1) \) after each step to prevent the buildup of asymmetries in \( P(t) \).
3. Store $P(t_i)$ and $b(t_i)$ every $i_{sp}$ steps on a direct access dataset.

4. Test for a conjugate point every $i_{cp}$ stored points. If a conjugate point is encountered, return control to the calling program. The conjugate point test is optional.

5. If normal termination occurs, return control to the calling program with the values $P(t_0)$, $b(t_0)$ and $c(t_0)$.

The subroutines in DIFGAL were tested individually. A number of test cases were run comparing the numerical solutions obtained with DIFGAL with those independently computed by others. Comparisons were also made with an early version of this subroutine package that used a direct Runge-Kutta method. The Kalman-Englar method was found to be computationally less expensive. The agreement between the two solutions was to better than 4 significant digits.

A number of tests were also carried out in conjunction with the LQDG-1 program. For a more detailed discussion, see Section H.3 of this appendix.

H.8 Numerical Values of Linear System Matrices

The numerical values of the linear longitudinal and lateral system matrices are given in this section for the conditions (5.5, 25a), (5.5, 25b) and (5.5, 25c) for the light STOL transport (see Appendices C and D). These may be used independently of the worst-case wind modeling computer programs described in this appendix in order to check the numerical results.

H.8.1 Longitudinal System Matrices

(See (5.5, 3) and following.)

$$A_1 = \begin{bmatrix} 4989.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5040.4 & 0 & 0 & 0 & 0 \\ 0 & 406.6 & 31824. & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -493.79 & 747.96 & 0 & -48582. & 0 & 0 \\ -2299.2 & -5816.4 & 192811. & 5965.1 & 0 & 0 \\ 88.777 & -3838.1 & -53984. & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ .99255 & -.12187 & 0 & 4.8748 & 0 & 0 \\ -.12187 & -.99255 & 0 & 39.702 & 0 & 0 \end{bmatrix}$$
H.8.2 Lateral System Matrices

(See (H.5,1) and following.)

\[ \mathbf{B}_1 = \begin{bmatrix} 0 & 24252. \\ -20032. & -16113. \\ -156748. & -11011. \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \]  

(\(H.8,3\))

\[ \mathbf{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \]

\[ \begin{bmatrix} 0.12187 & -0.99255 \\ 1.0075 & 0 \end{bmatrix} \]  

(\(H.8,4\))

\[ \mathbf{B}_3 = \begin{bmatrix} -4952.3 & 608.07 \\ 608.07 & -4952.3 \end{bmatrix} \]

\[ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \]  

(\(H.8,5\))

\[ \mathbf{A}_1 = \begin{bmatrix} 4989.5 & 0 & 0 & 0 & 0 \\ 0 & 21846. & -2880.6 & 0 & 0 \\ 0 & -2880.6 & 48631. & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

(\(H.8,6\))

\[ \mathbf{A}_2 = \begin{bmatrix} -962.94 & -1191.1 & -19490. & 43852. & 0 & 0 \\ -1548.3 & -105836. & 64293. & 0 & 0 \\ 2256.0 & -2872.5 & -34360. & 0 & 0 \\ 0 & 1 & -0.12278 & 0 & 0 \\ 0 & 0 & 1.0075 & 0 & 0 \\ 1 & 0 & 0 & 0 & 39.702 \end{bmatrix} \]

(\(H.8,7\))

\[ \mathbf{A}_3 = \begin{bmatrix} 129.69 & 14920. \\ 152733. & 47011. \\ -18140. & -108491. \end{bmatrix} \]

(\(H.8,8\))
REFERENCES — APPENDIX H


TABLE H.1
EXECUTION CPU TIMES FOR
A NUMBER OF CASES

<table>
<thead>
<tr>
<th>PROGRAM</th>
<th>EXECUTION TIME (minutes)</th>
<th>$t_f$ (sec)</th>
<th>COMMENTS</th>
</tr>
</thead>
<tbody>
<tr>
<td>VVWCM</td>
<td>0.194</td>
<td>20.5</td>
<td>$H(t)$ not computed</td>
</tr>
<tr>
<td>VVWCM</td>
<td>0.222</td>
<td>20.5</td>
<td>$H(t)$ computed</td>
</tr>
<tr>
<td>VVWCM1</td>
<td>0.252</td>
<td>20.5</td>
<td>$H(t)$ not computed</td>
</tr>
<tr>
<td>VVWCM1</td>
<td>0.285</td>
<td>20.5</td>
<td>$H(t)$ computed</td>
</tr>
<tr>
<td>LQDG-1</td>
<td>0.408</td>
<td>20.5</td>
<td>$n_g = 6$, $n = 11$, DGC</td>
</tr>
<tr>
<td>LQDG-1</td>
<td>0.664</td>
<td>20.5</td>
<td>$n_g = 9$, $n = 11$, DGC</td>
</tr>
<tr>
<td>LQDG-1</td>
<td>0.195</td>
<td>20.5</td>
<td>$n_g = 4$, $n = 11$, PFC, WP</td>
</tr>
<tr>
<td>LQDG-1</td>
<td>0.416</td>
<td>20.5</td>
<td>$n_g = 7$, $n = 11$, DMC</td>
</tr>
<tr>
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<td>$n_g = 7$, $n = 11$, ACC</td>
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<td>LQDG-3</td>
<td>0.326</td>
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<td>$n_g = 8$, $n = 9$, ACC</td>
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<td>NL</td>
<td>0.093</td>
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<td>LOR, LQA</td>
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<tr>
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<td>107</td>
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<tr>
<td>NL</td>
<td>0.030</td>
<td>10.25</td>
<td>LAR, LQA</td>
</tr>
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</table>

$n_g$ = order of system optimized (user specified, $n_g \leq n$)
$n$ = order of equations of motion
DGC = differential game case
PFC = perversity function case
WP = with plotting
DMC = direct method case
ACC = optimal approach controller case
LOR = longitudinal response
LAR = lateral response
LQA = linear quasisteady aerodynamics
NQA = nonlinear quasisteady aerodynamics
Both analytical and computational techniques that may be applied to predicting aircraft response to hazardous variable winds encountered on the landing approach and to modeling such wind conditions have been investigated. An extensive literature review has identified a number of areas requiring further study. Of these areas, worst-case wind modeling techniques and modified glidepath geometries based on an estimate of the existing wind profile have been pursued extensively. A nonlinear longitudinal and lateral dynamic model is posed for a twin-engined STOL transport using look-up table, quasisteady, nonlinear aerodynamics. The aircraft controller is a feedback controller synthesized using linear state feedback optimal control theory for a suitable linearization of the general equations of motion. This dynamic model is employed in computer simulations predicting the aircraft dynamic behavior flying curved glidepath geometries that are based on an a priori estimate of the existing wind profile. These results are assessed to determine the effects of incorrect estimates of the wind profile and the suitability of the kinematic assumptions used in deriving the curved glidepath geometry. Worst-case wind modeling techniques where the form of the wind model is not specified a priori are considered in detail, and extensions to an existing technique are proposed. New techniques are developed using more general functional maximization techniques, and in particular the worst-case wind modeling problem is viewed as a conflict of interest between the aircraft controller and a wind controller. Numerical examples of the application of the latter class of worst-case techniques are given for several formulations posed as linear quadratic problems, and two of these worst-case wind models are applied to the curved glidepath geometry problem. Some discussion is also included on the use of these methods in the certification process and in flight simulators; preliminary results from a flight simulator application are presented. A number of recommendations for future work, based both on the results of the numerical simulations and on the literature review, are included.

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THE LANDING APPROACH IN VARIABLE WINDS: CURVED GLIDEPATH GEOMETRIES AND WORST-CASE WIND MODELING
Markov, Alexander B.

Both analytical and computational techniques that may be applied to predicting aircraft response to hazardous variable winds encountered on the landing approach and to modeling such wind conditions have been investigated. An extensive literature review has identified a number of areas requiring further study. Of these areas, worst-case wind modeling techniques and modified glidepath geometries based on an estimate of the existing wind profile have been pursued extensively. A nonlinear longitudinal and lateral dynamic model is posed for a twin-engined STOL transport using look-up table, quasisteady, nonlinear aerodynamics. The aircraft controller is a feedback controller synthesized using linear state feedback optimal control theory for a suitable linearization of the general equations of motion. This dynamic model is employed in computer simulations predicting the aircraft dynamic behavior flying curved glidepath geometries that are based on an a priori estimate of the existing wind profile. These results are assessed to determine the effects of incorrect estimates of the wind profile and the suitability of the kinematic assumptions used in deriving the curved glidepath geometry. Worst-case wind modeling techniques where the form of the wind model is not specified a priori are considered in detail, and extensions to an existing technique are proposed. New techniques are developed using more general functional maximization techniques, and in particular the worst-case wind modeling problem is viewed as a conflict of interest between the aircraft controller and a wind controller. Numerical examples of the application of the latter class of worst-case techniques are given for several formulations posed as linear quadratic problems, and two of these worst-case wind models are applied to the curved glidepath geometry problem. Some discussion is also included on the use of these methods in the certification process and in flight simulators; preliminary results from a flight simulator application are presented. A number of recommendations for future work, based both on the results of the numerical simulations and on the literature review, are included.

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