Transient Diffusive

Electromagnetic Fields in

Layered Anisotropic Media
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Layered Anisotropic Media

Proefschrift

ter verkrijging van de graad van doctor aan de
Technische Universiteit Delft, op gezag van
de Rector Magnificus, Prof. drs. P.A. Schenck,
in het openbaar te verdedigen ten overstaan van
een commissie aangewezen door het College van Dekanen
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Leendert Combee

geboren te Nieuwer-Amstel
elektrotechnisch ingenieur

DELFt UNIVERSITY PRESS/1991
Dit proefschrift is goedgekeurd door de promotor
prof. dr. ir. A.T. de Hoop
In memory of my mother

To my father
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The author wishes to thank Dr. M.L. Oristaglio and Dr. B.R. Spies of Schlumberger-Doll Research, Ridgefield, Connecticut 06877-4108, U.S.A., for their hospitality to carry out research in the Electromagnetics Department and providing the necessary computer resources during the period July – October 1990.
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<tr>
<td>$A_{I,J}$</td>
<td>system's matrix</td>
</tr>
<tr>
<td>$b_{j}^{(m;\nu)}$</td>
<td>eigenvector of system's matrix</td>
</tr>
<tr>
<td>$B_{j}$</td>
<td>magnetic flux density</td>
</tr>
<tr>
<td>$B_{I}$</td>
<td>field matrix</td>
</tr>
<tr>
<td>$\mathcal{D}_{m}$</td>
<td>homogeneous subdomain of the configuration</td>
</tr>
<tr>
<td>$D_{I,N}$</td>
<td>composition matrix</td>
</tr>
<tr>
<td>$D_{M,I}^{-1}$</td>
<td>decomposition matrix</td>
</tr>
<tr>
<td>$e_{k}$</td>
<td>principal directions of material tensor</td>
</tr>
<tr>
<td>$E_{r}$</td>
<td>electric field strength</td>
</tr>
<tr>
<td>$F_{I}$</td>
<td>field matrix</td>
</tr>
<tr>
<td>$G$</td>
<td>Green's function</td>
</tr>
<tr>
<td>$g_{j}^{(m;\nu)}$</td>
<td>eigenrow of the system's matrix</td>
</tr>
<tr>
<td>$h_{m}$</td>
<td>thickness of m-th subdomain</td>
</tr>
<tr>
<td>$H_{p}$</td>
<td>magnetic field strength</td>
</tr>
<tr>
<td>$i$</td>
<td>imaginary unit, $i^2 = -1$</td>
</tr>
<tr>
<td>$i_{r}$</td>
<td>base vectors of unit length of Cartesian reference frame</td>
</tr>
<tr>
<td>$J_{k}$</td>
<td>volume density of induced electric current</td>
</tr>
<tr>
<td>$J_{k}^{e}$</td>
<td>volume source density of electric current</td>
</tr>
<tr>
<td>$K_{j}$</td>
<td>volume density of induced magnetic current</td>
</tr>
</tbody>
</table>
## Glossary of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Represents:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_j^c$</td>
<td>volume source density of magnetic current</td>
</tr>
<tr>
<td>$N_I$</td>
<td>notional source matrix</td>
</tr>
<tr>
<td>$p, \psi$</td>
<td>polar coordinate specification of $i\alpha_p$</td>
</tr>
<tr>
<td>$p(r, \psi)$</td>
<td>value of $p$ on the modified Cagniard contour</td>
</tr>
<tr>
<td>$p_0$</td>
<td>value of $p$ for which the modified Cagniard contour leaves the real $p$-axis</td>
</tr>
<tr>
<td>$r, \theta$</td>
<td>polar coordinate specification of the point of observation</td>
</tr>
<tr>
<td>$\mathcal{R}$</td>
<td>Riemann surface</td>
</tr>
<tr>
<td>$R^{(m;\pm\nu)}_{\kappa,r}$</td>
<td>reflection coefficient</td>
</tr>
<tr>
<td>$s$</td>
<td>Laplace transformation parameter</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$T$</td>
<td>normalized time</td>
</tr>
<tr>
<td>$T_{\kappa,r}^{(m;\pm\nu)}$</td>
<td>transmission coefficient</td>
</tr>
<tr>
<td>$T_{BD}$</td>
<td>diffusive equivalent of body-wave arrival time</td>
</tr>
<tr>
<td>$T_{HD}$</td>
<td>diffusive equivalent of head-wave arrival time</td>
</tr>
<tr>
<td>$V_{emf}$</td>
<td>induced voltage</td>
</tr>
<tr>
<td>$V_{emf}^\circ$</td>
<td>in-loop response of circular loop source</td>
</tr>
<tr>
<td>$V_{emf}^\Box$</td>
<td>in-loop response of square loop source</td>
</tr>
<tr>
<td>$V_{nor}$</td>
<td>normalization value of $V_{emf}$</td>
</tr>
<tr>
<td>$x$</td>
<td>vector notation of point of observation</td>
</tr>
<tr>
<td>$x_m$</td>
<td>Cartesian coordinates</td>
</tr>
<tr>
<td>$x_{3;m}$</td>
<td>interface between two homogenous subdomains</td>
</tr>
<tr>
<td>$x_{3;\xi}$</td>
<td>interface at which the source is located</td>
</tr>
<tr>
<td>$x_{\nu;\xi}$</td>
<td>vector notation of location of the source</td>
</tr>
<tr>
<td>$Y^{(m;\nu)}$</td>
<td>diffusive electromagnetic field admittance</td>
</tr>
<tr>
<td>$Z^{(m;\nu)}$</td>
<td>diffusive electromagnetic field impedance</td>
</tr>
<tr>
<td>$\alpha_\mu$</td>
<td>Fourier transformation parameter</td>
</tr>
<tr>
<td>$\gamma^{(m;\pm\nu)}$</td>
<td>eigenvalue of the system’s matrix</td>
</tr>
<tr>
<td>$\delta_{k,r}$</td>
<td>symmetric unit tensor of rank two (Kronecker tensor)</td>
</tr>
<tr>
<td>$\partial$</td>
<td>partial derivative</td>
</tr>
<tr>
<td>$\partial_m$</td>
<td>partial derivative with respect to $x_m$</td>
</tr>
<tr>
<td>$\partial_t$</td>
<td>partial derivative with respect to $t$</td>
</tr>
</tbody>
</table>
Glossary of symbols

<table>
<thead>
<tr>
<th>symbol</th>
<th>represents:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_{k,m,p}$</td>
<td>antisymmetric unit tensor of rank three (Levi-Civita tensor)</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>azimuth</td>
</tr>
<tr>
<td>$\mu^{(m)}$</td>
<td>scalar magnetic permeability</td>
</tr>
<tr>
<td>$\mu^{[k]}$</td>
<td>principal value of the magnetic permeability tensor</td>
</tr>
<tr>
<td>$\mu_0^{(m)}$</td>
<td>magnetic permeability of free space of subdomain $D_m$</td>
</tr>
<tr>
<td>$\mu_{j,p}^{(m)}$</td>
<td>tensorial magnetic permeability of subdomain $D_m$</td>
</tr>
<tr>
<td>$\sigma^{(m)}$</td>
<td>scalar electrical conductivity of subdomain $D_m$</td>
</tr>
<tr>
<td>$\sigma^{[k]}$</td>
<td>principal value of the electrical conductivity tensor</td>
</tr>
<tr>
<td>$\sigma_{k,r}^{(m)}$</td>
<td>tensorial electric conductivity of subdomain $D_m$</td>
</tr>
<tr>
<td>$\tau$</td>
<td>parameter along the modified Cagniard contour</td>
</tr>
<tr>
<td>$\phi$</td>
<td>source signature</td>
</tr>
<tr>
<td>$\hat{\phi}$</td>
<td>Laplace transform of the source signature</td>
</tr>
<tr>
<td>$\varphi, \theta, \psi$</td>
<td>angles that specify the orientation of $e_k$</td>
</tr>
<tr>
<td>$\varphi, \zeta, \chi$</td>
<td>angles that specify the orientation of $e_k$</td>
</tr>
<tr>
<td>$\chi$</td>
<td>dip</td>
</tr>
</tbody>
</table>
Ты обо мне так плачешь? Ах, если бы ты охала, как мне жалко вас всех, вас, которые останетесь!

You are crying so hard for me? Ah, if you only knew how sorry I am for all of you who remain behind!

— S.V. Kovalyakina, 'A Nihilist Girl'
Chapter 1

Introduction

Geophysical methods are used in the prospecting of minerals, gas and oil and for the mapping of unseen deposits and structures beneath the surface of the earth. By studying fields of various physical natures (seismic, electromagnetic, or gravitational), generated either by natural or man-made controlled sources, one aims to determine the distribution of physical parameters like elasticity, electrical conductivity, or volume density of mass in the earth’s interior. Close correlation between the physical parameters and the mineral compositions of rocks and sedimentary deposits makes it possible to draw from geophysical survey data conclusions about the structure and composition of the earth’s crust.

The methods of geophysical exploration can be divided into two groups:

structural methods that are used to resolve the general structure of the sedimentary cover of the earth and to compile structural maps that in their turn serve to determine the location of potential oil and gas traps,

prospecting methods in the search for local non-uniformities of the physical parameters in the earth, which can be associated, for example, with mineral deposits and metallic ores.

In the first group, the main model of the subsurface structure of the earth is a (quasi) layered medium, each of the layers being homogeneous as far as their physical parameters are concerned. In the second group, the main (background) model is again a (quasi) layered medium, which now, however, involves local inhomogeneities, the
physical parameters of which differ significantly from the ones of the surrounding sedimentary layers (Dmitriev, 1987; Nekut and Spies, 1989).

The choice of a particular geophysical exploration method depends on various factors, such as the nature of the mineral deposits and the surrounding bedrocks encountered at the exploration site (Telford et al., 1976; Spies, 1980a). Although geophysical methods of exploration that use electromagnetic fields, either from natural or from artificial sources, may have a lower resolution when applied to structural problems than seismic exploration techniques, they nevertheless are highly effective in mineral prospecting. Although especially in the past two decades electric exploration methods have been gaining importance, it must be pointed out that the first fundamental studies on electric current flow in the subsurface of the earth can be found as far back as 1920 (Schlumberger, 1920), followed by other interesting papers by Mailllet and Doll (1932) and Schlumberger et al. (1934). Later, Tikhonov (1946a, 1946b, 1950) developed effective numerical and asymptotic methods for electromagnetic sounding techniques of the earth's subsurface, both for practical exploration geophysics and for fundamental studies of the deeper structures of the earth's interior.

Electromagnetic exploration systems can be divided into three distinct classes: (i) the ones based on natural source methods such as the magnetotelluric method (MT), (ii) the ones based on frequency-domain controlled-source methods (FEM) and (iii) the ones based on time-domain controlled-source methods (TEM). Advantages of the different kinds of electromagnetic exploration techniques – of course, depending on the kind of application – can be found in, for example, Nabighian (1984), McCracken et al. (1986a, 1986b), Eaton and Hohmann (1987), Nekut and Spies (1989), and Spies (1989). Different electromagnetic sounding methods complement each other in a number of ways and in many cases a joint use of two or more exploration techniques is necessary to enhance the interpretation of their respective measurement data (see, for example, Raiche et al. 1985). It is interesting to note that the application of electromagnetic sounding techniques is certainly not limited to geophysical prospecting: other applications are the search for groundwater (Fitterman and Stewart, 1986), airborne bathymetry (Zollinger et al., 1987), and the detection of metallic relics of historical interest (Lee and Buselli, 1988).

In the present treatise we are concerned with the time-domain electromagnetic method of exploration geophysics (TEM). Clearly, this method is susceptible to space and time variations of the electromagnetic parameters of the earth. Time-domain electromagnetic exploration systems are usually based on repeatedly 'switch-
ing' on and off an electromagnetic field, and recording the resulting decaying electromagnetic field at the surface of the earth during the off-stage. The electromagnetic field is usually generated by driving an electric current, with known source signature, into a large horizontal loop of wire laid on the ground. Other possibilities are: a horizontal grounded electric dipole or a small loop – the latter being more convenient for seafloor exploration purposes in which case the source is towed by a boat (San Filipo and Hohmann, 1985; Edwards and Chave, 1986; Fitterman and Stewart, 1986). Occasionally, the source can be towed by an aircraft as with airborne exploration methods (Gupta Sarma et al., 1976; Sengpiel, 1983; Bartel and Becker, 1988; Becker, 1988). For the detection of the electromagnetic fields mostly induction coils and magnetometers for measuring the time rate of change of the magnetic flux density and the magnetic field strength, respectively, are used. Well-known source-receiver systems are the coincident-loop configuration were the transmitting loop also serves as induction coil after current switch-off and the in-loop or central-loop configuration where the receiver is located at the center of the (rectangular or circular) transmitting loop.

Electromagnetic fields generated in this way can be 'separated' into two parts: a radiative part associated with the wave propagation process of the electromagnetic field and a dissipative part (Weir, 1980; Ignatik et al., 1985). The latter shows a diffusive behavior and is therefore referred to as the transient diffusive electromagnetic field. For electromagnetic fields generated by an infinitely long line source of electric current, this separation into two parts is clearly shown by Oristaglio and Hohmann (1984). The radiative part of the electromagnetic field is an early-time phenomenon, not easily amenable to measurement and therefore presently not yet of practical significance. Insight into the diffusion properties of electromagnetic fields in a conductive environment is of importance to the interpretation of data acquired in field surveys. Through mathematical modeling we gain an understanding of the way in which these electromagnetic fields diffuse into the earth. In complex inhomogeneous structures, investigation of the behavior of electromagnetic fields can only be carried out with the aid of numerical techniques such as the finite-difference method (see, for example, Oristaglio and Hohmann, 1984) or integral-equation methods (see, for example, San Filipo and Hohman, 1985). Analytical methods can only be applied to simple structures that show certain invariance properties in the spatial distribution of the electromagnetic parameters. Such a simple structure is the plane-layered model that can be thought of as a representative for a (locally) stratified earth. The physical parameters of this model are the magnetic permeability and the electrical
conductivity, both of which can show (arbitrary) anisotropy as well as relaxation effects. The relaxation effect of the magnetic permeability is called superparamagnetism or magnetic viscosity; its influence on the transient electromagnetic field behavior was discussed by, for example, Buselli (1982) and Lee (1984). Relaxation effects of the electric conductivity are called induced polarization (IP); they are caused by electrolytic polarization at the surface of metallic mineral grains (usually described by the now famous "Cole-Cole"-model (Cole and Cole, 1941; Wait, 1958, 1987; Pelton et al., 1978)). The influence of IP on the transient electromagnetic field behavior was discussed by, for example, Rathor (1978a, 1978b), Lee (1981), Weidelt (1982) and Walker and Kawasaki (1988).

In this book, we investigate the structural problem of the diffusion of transient electromagnetic fields into a plane-layered, non-polarizable and non-superparamagnetic arbitrarily anisotropic earth. This implies that on the time scale of interest we shall assume that the electric displacement currents can be neglected with respect to the conduction currents. To solve the equations governing the behavior of diffusive electromagnetic fields, a new analytical method is developed based on the Cagniard-De Hoop, or modified Cagniard method (De Hoop, 1960). This method proved to be successful in the study of propagation of waves in lossless media (De Hoop, 1961, 1979; De Hoop and Van der Hijdren, 1984) and in special cases of dispersive media (Kooij, 1990). However, its application to strictly diffusive fields is new (De Hoop and Orestaglio, 1987). With this method, closed-form analytical expressions in the form of well-behaved integrals are obtained for the time behavior of the transient electromagnetic field at any point of the configuration. Another important feature of the modified Cagniard technique is that the theory and the resulting expressions are not essentially different for arbitrarily anisotropic and isotropic media. Of course, in the latter case we expect some simplifications of the results.

With regard to the mathematical analysis of the partial differential equations of the electromagnetic field – in this case of the diffusion type, but which, as we shall see, show a strong similarity with the equivalent wave equations (in this respect the work by Van der Hijdren (1987) is of special importance) – we wish to end this introduction with some excerpts from the correspondence between Weierstrass and Kovalevskaia on diffusion-type equations and an excerpt on the same subject from the paper in which Kovalevskaia formulates what is now known as the Cauchy-Kovalevskaia theorem (Kovalevsky, 1875):
Aber ich habe bemerkt, dass, wenn diese Reihe [i.e., the formal powerseries representation of the solution of a partial differential equation and given initial conditions, in a region where this solution is analytic] convergent sein soll, die Funktionen \( \phi^{(0)}, \ldots, \phi^{(m-1)} \) nicht willkürlich angenommen werden können, sondern solchen Beschränkungen unterworfen sind, dass man im Allgemeinen sagen kann, die Reihe

\[ \phi(x, z_1, \ldots, z_r; a, a_1, \ldots, a_r) \]

convergire an keiner Stelle \((x, z_1, \ldots, z_r)\), wie nahe man dieselbe auch der Stelle \((a, a_1, \ldots, a_r)\) annehmen kann. Ich begnüge mich aber hier, dies an einem Beispiel nachzuweisen. Es sei die Differentialgleichung

\[ \frac{\partial \phi}{\partial z} = \frac{\partial^2 \phi}{\partial y^2} \]

gegeben. Wenn \( \phi_0(y|b) \) irgend eine Potenzreihe von \( y - b \) ist, so genügt die Reihe

\[ \sum_{\nu=0}^{\infty} \frac{d^{2\nu} \phi_0(y|b)}{d y^{2\nu}} \frac{(x - a)\nu}{\nu!} \]

dieser Differentialgleichung formell und geht für \( z = a \) in \( \phi_0(y|b) \) über, sie besitzt aber nur bei einer ganz besonderen Wahl von \( \phi_0(y|b) \) einen Convergensbezirk, während im Allgemeinen sie für kein Werthsystem \((x, y)\) eine bestimmte endliche Summe hat. Es sei z.B. \( a = 0, b = 0 \)

\[ \phi_0(y|b) = \frac{1}{1 - y}. \]

Dan ist

\[ \frac{d^n \phi_0}{d y^n} = \frac{n!}{(1 - y)^{n+1}}, \]

und die obige Reihe geht in

\[ \sum_{\nu=0}^{\infty} \frac{2\nu!}{\nu!} \frac{x^{\nu}}{(1 - y)^{2\nu+1}} \]

über, von der es leicht zu sehen ist, dass sie divergent ist, wie klein auch \( x, y \) angenommen werden.

Weierstrass greatly valued this result. In a letter to Kovalevskaiia he discusses the
problem of representing the solution of the diffusion equation with \( \phi(x, 0) = \phi_0(x) \) as an integral, and adds:

Du siehst, teuerste Sonja, wie Deine – Dir so einfach scheinende – Be-
merkung über die Eigentümlichkeit partieller Differentialgleichungen, daß
eine unendliche Reihe einer solchen Differentialgleichung _formell_ genügen
kann, ohne doch für irgend welche Wertsysteme ihrer Veränderlichen zu kon-
vergieren, für mich der Ausgang von Untersuchungen, die viel Interessantes
haben und manche Aufklärung verschaffen, geworden ist.\(^1\)

Our final fragment concerns the advise by Weierstrass on subjects for a course on
partial differential equations to be given by Kovalevskaja in which he emphasizes
the apparent similarity of the diffusion and the wave equations:

Dagegen würde ich Dir sehr anraten, einige partielle Differentialgleichun-
gen aus dem Gebiete der mathematischen Physik ausführlicher zu behandeln,
obwohl deren Integration mit dem ersten Teil Deiner Vorlesung kaum etwas
Namentlich ist von großem Interesse die Gleichung

\[
\frac{\partial \phi}{\partial t} = a^2 \frac{\partial^2 \phi}{\partial x^2}.
\]

Wenn man annimmt daß für \( t = 0 \), \( \phi = f(x) \) gegeben sei, so kann \( f(x) \) eine
ganz willkürliche, nur integrierbare Funktion sein, und dann wird \( \phi(x, t) \) für je-
den positiven Wert von \( t \) eine _analytische_ Funktion von \( x \), läßt sich aber nicht
für negative Werte von \( t \) definieren, wobei an die Eigentümlichkeit dieser Dif-
fferentialgleichungen, welche Du in Deiner Dissertation bemerkt hast, erinnert
werden kann. Der enorme Unterschied in dem Charakter der beiden äußerlich
so verwandten Differentialgleichungen

\[
\frac{\partial \phi}{\partial t} = a^2 \frac{\partial^2 \phi}{\partial x^2}, \quad \frac{\partial^2 \phi}{\partial t^2} = a^2 \frac{\partial^2 \phi}{\partial x^2},
\]

ist sehr frappant und belehrend.\(^2\)

(excerpts taken from Kochina-Polubarinova, 1973; see also Mittag-Leffler, 1923)

\(^1\)Letter no. 35, dated May 6, 1874
\(^2\)Letter no. 68, dated December 27, 1883
Chapter 2

Basic electromagnetic field relations

2.1 Description of the configuration

The configuration in which we are going to investigate the diffusive electromagnetic field, consists of a finite number $ND$ of homogeneous subdomains $\{D_m \subset \mathbb{R}^3; m = 1, \ldots, ND\}$ with mutually parallel interfaces. To specify the position of observation in the configuration we employ the coordinates $\{z_1, z_2, z_3\}$ with respect to a Cartesian reference frame with the origin $O$ and the three mutually perpendicular base vectors $\{i_1, i_2, i_3\}$ of unit length each. The time coordinate is denoted by $t$. The subscript notation for Cartesian vectors and tensors is used and the summation convention applies to repeated lowercase Latin subscripts; the latter range over the values 1, 2 and 3. Whenever appropriate, the position is also specified by the position vector $z = z_p i_p$. Partial differentiation is denoted by $\partial$. Differentiation with respect to $z_m$ is denoted by $\partial_m$; $\partial_t$ is a reserved symbol for differentiation with respect to $t$. The $z_3$-axis is taken perpendicular to the interfaces. This direction is denoted as the vertical direction, and in accordance with geophysical conventions the $z_3$-coordinate increases downwardly. The configuration is then shift invariant in the $z_1$- and $z_2$-directions; it is taken to be time invariant as well.

The media in the configuration are assumed to be linear, time invariant and locally reacting in their electromagnetic behavior. They are piecewise homogeneous and may be anisotropic. The electromagnetic properties of each medium are speci-
fied by its tensorial conductivity $\sigma_{k,r}$ and its tensorial permeability $\mu_{j,p}$. In the diffusive approximation the influence of the electric displacement current is neglected with respect to the one of the conduction current and the permittivity plays no role. Thus, in each subdomain all material tensors are constants: $\sigma_{k,r} = \sigma_{k,r}^{(m)}$ and $\mu_{j,p} = \mu_{j,p}^{(m)}$ for $m \in D_m$. The nomenclature of the configuration and the material tensors is given in Table 2.1 (see also Figure 2.1). The tensorial conductivity $\sigma_{k,r}$ and the tensorial permeability $\mu_{j,p}$ are symmetric tensors of rank two (i.e., $\sigma_{k,r} = \sigma_{r,k}$ and $\mu_{j,p} = \mu_{p,j}$). This symmetry follows from equilibrium thermodynamics and the Onsager relations (Smith et al., 1967).

The diffusive electromagnetic field in the configuration is generated by impulsive sources of bounded extent. The sources occupy the domain $D_{source}$. Measurements of the electromagnetic field are carried out with the aid of receivers which are located in the domain $D_{receiver}$. This is depicted in Figure 2.1. Common types of sources are the current carrying planar rectangular loop and the electric or magnetic dipole. A magnetometer (sensitive to the magnetic field strength) or a (small) coil (sensitive to time variations of the magnetic flux density) are mostly used as receivers.

<table>
<thead>
<tr>
<th>Domain</th>
<th>$z_3$-coordinate</th>
<th>thickness of layer</th>
<th>material properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>$-\infty &lt; z_3 &lt; z_{3;1}$</td>
<td></td>
<td>$\sigma_{k,r}^{(1)}, \mu_{j,p}^{(1)}$</td>
</tr>
<tr>
<td>$D_m$</td>
<td>$z_{3;m-1} &lt; z_3 &lt; z_{3;m}$</td>
<td>$h_m$</td>
<td>$\sigma_{k,r}^{(m)}, \mu_{j,p}^{(m)}$</td>
</tr>
<tr>
<td>$D_{ND}$</td>
<td>$z_{3;ND-1} &lt; z_3 &lt; \infty$</td>
<td></td>
<td>$\sigma_{k,r}^{(ND)}, \mu_{j,p}^{(ND)}$</td>
</tr>
</tbody>
</table>
2.2 The tensorial conductivity and permeability

We assume that the sources start to act at the instant \( t = t_0 \). To determine the diffusive electromagnetic field that is causally related to the action of the sources, we put the state quantities describing this electromagnetic field equal to zero in the time interval \( t < 0 \) for all \( \omega \in \mathbb{R}^3 \) (initial condition).

2.2 The tensorial conductivity and permeability

The electromagnetic properties of homogeneous material are characterized by the symmetric conductivity tensor of rank two \( \sigma_{k,r} \) and the symmetric permeability tensor of rank two \( \mu_{k,p} \). These tensors can be represented mathematically by symmetric 3-by-3 matrices and geometrically by second-degree surfaces. For the electrical conj-
ductivity and permeability tensors, this geometric surface is an *ellipsoid* (Nye, 1972). The directions of the representation surface's major axes are the principal directions of the relevant tensor. If the reference coordinate axes are parallel to the major axes of the ellipsoid, the off-diagonal terms in the representation matrix vanish. The remaining terms on the main-diagonal are the principal values of the tensor.

The highest tensor symmetry is obtained when the material is isotropic. In this case, all principal values are equal and the geometric surface is a sphere. If two of the principal values are equal but differ from the third, the geometric surface is an *ellipsoid of revolution*. This situation has cylindrical symmetry. Examples of cylindrical symmetry are thin-bedded sequences of alternating high- and low-resistivity layers. On a microscopic scale this situation might be observed for certain gneisses and schists, due to a structural situation where two of the principal strain values are equal. This kind of anisotropy is commonly denoted as *transverse isotropy* (TI): the medium properties in a certain plane (strike) are the same but different from the medium properties in a direction normal to that plane. Special cases of transverse isotropy are TIV (vertical axis of symmetry) and TIH (horizontal axis of symmetry (Winterstein, 1990)). If all three principal tensor values are different, the geometric surface is a *triaxial ellipsoid*. This situation has orthorhombic symmetry, for it is the symmetry of the holohedral orthorhombic crystal class (Hill, 1972). Orthorhombic symmetry might be due to crossbedding or flow structures. It could also represent the structural situation where the deformational strain tensor had three different principal values.

In geophysical exploration, one is more likely interested in the principal values and the principal directions of the material tensors than the value of a particular component of these tensors. For this reason, as well as for numerical investigations, it proves to be advantageous to express each of the component of these tensors in terms of the principal directions and the corresponding principal values. As an example, we consider the electrical conductivity tensor $\sigma_{kr}$. Let us denote by $e_k$, $k = 1, 2, 3$ the principal directions of this tensor and let $\sigma^{[k]}$, $k = 1, 2, 3$ denote the corresponding principal values. Each of the principal directions $e_k$ can be decomposed into the three base vectors $\{i_1, i_2, i_3\}$ of our Cartesian reference frame, i.e.,

$$
e_k = \alpha_{kr} i_r,$$

(2.1)
2.2 The tensorial conductivity and permeability

the result of this being that the components of the electrical conductivity tensor \( \sigma_{k,r} \) can be expressed in terms of the \( e_{k,r} \) as follows

\[
\sigma_{k,r} = \alpha_{m,k} \alpha_{n,r} \sigma^{[m]} \delta_{m,n}.
\] (2.2)

The \( \alpha_{k,r} \) are the directional cosines of the \( e_k \) relative to the \( i_r \), respectively. Since only three of the \( \alpha_{k,r} \) can be chosen independently, this representation is not useful for practical purposes. We seek for a method of expressing the \( \alpha_{k,r} \) into a certain set of three independent angles that are appropriate for geophysical applications and yield simple expression for the \( e_k \) and \( \sigma_{k,r} \). For this purpose we shall introduce two representations of the \( e_k \) into the \( i_r \). Depending on our application we shall use either one of the two.

(i) Upon introducing the rotation angles \( \theta, \psi \) and \( \varphi \) that specify the orientation of the principal directions \( e_k \) of the conductivity tensor relative to the fixed Cartesian reference frame \( i_r \) (see Figure 2.2), we arrive at

\[
e_1 = i_1 \cos \varphi \cos \theta - i_3 (\cos \varphi \sin \theta \sin \psi + \sin \varphi \cos \psi) + i_3 (\sin \varphi \sin \psi - \cos \varphi \cos \psi), \] (2.3)

\[
e_2 = i_1 \sin \varphi \cos \theta - i_3 (\sin \varphi \sin \theta \sin \psi - \cos \varphi \cos \psi) - i_3 (\cos \varphi \sin \psi + \sin \varphi \sin \theta \cos \psi), \] (2.4)

\[
e_3 = i_1 \sin \theta + i_2 \cos \theta \sin \psi + i_3 \cos \theta \cos \psi. \] (2.5)

The angle \( \varphi \) is defined such that \( e_{2,1} = 0 \) if \( \varphi = 0 \), while \( e_{1,1} = 0 \) if \( \varphi = \pi/2 \). Substitution of Eqs. (2.3)-(2.5) into Eq. (2.2) yields the following explicit expressions for the elements of the symmetric conductivity tensor \( \sigma_{k,r} \) in terms of its principal values and its principal directions:

\[
\sigma_{1,1} = (\sigma^{[1]} \cos^2 \varphi + \sigma^{[2]} \sin^2 \varphi) \cos^2 \theta + \sigma^{[3]} \sin^2 \theta, \] (2.6)

\[
\sigma_{2,2} = \sigma^{[1]} (\cos \varphi \sin \theta \sin \psi + \sin \varphi \cos \psi)^2 + \sigma^{[2]} (\sin \varphi \sin \theta \sin \psi - \cos \varphi \cos \psi)^2 + \sigma^{[3]} \cos^2 \theta \sin^2 \psi, \] (2.7)
\[
\sigma_{3,3} = \sigma_{1}^{[1]}(\cos\varphi \sin\theta \cos\psi - \sin\varphi \sin\psi)^2 + \\
\sigma_{2}^{[2]}(\sin\varphi \sin\theta \cos\psi + \cos\varphi \sin\psi)^2 + \sigma_{2}^{[3]}\cos^2\theta \cos^2\psi, \\
(2.8)
\]

\[
\sigma_{1,2} = (\sigma_{2}^{[2]} - \sigma_{1}^{[1]})\sin\varphi \cos\varphi \cos\theta \cos\psi - \\
(\sigma_{1}^{[1]} \cos^2\varphi + \sigma_{2}^{[2]} \sin^2\varphi)\sin\theta \cos\theta \sin\psi + \sigma_{2}^{[3]} \sin\theta \cos\theta \sin\psi, \\
(2.9)
\]

\[
\sigma_{1,3} = (\sigma_{1}^{[1]} - \sigma_{2}^{[2]})\sin\varphi \cos\varphi \cos\theta \sin\psi - \\
(\sigma_{1}^{[1]} \cos^2\varphi + \sigma_{2}^{[2]} \sin^2\varphi)\sin\theta \cos\theta \cos\psi + \sigma_{2}^{[3]} \sin\theta \cos\theta \cos\psi, \\
(2.10)
\]

\[
\sigma_{2,3} = (\sigma_{1}^{[1]} - \sigma_{2}^{[2]})(\cos^2\varphi - \sin^2\varphi)\sin\varphi \cos\varphi \sin\theta + \\
(\sigma_{1}^{[1]} - \sigma_{2}^{[2]})(\cos^2\varphi - \sin^2\varphi) \sin\psi \cos\psi - \\
(\sigma_{1}^{[1]} \cos^2\varphi + \sigma_{2}^{[2]} \sin^2\varphi) \cos^2\theta \sin\psi \cos\psi + \sigma_{2}^{[3]} \cos^2\theta \sin\psi \cos\psi. \\
(2.11)
\]

Figure 2.2.: Cartesian reference frame \{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\} and principal directions \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} of the conductivity tensor. The principal directions are specified through the angles \(\theta, \psi\) and \(\varphi\).
(ii) Another way of specifying the orientation of the principal axes is obtained by introducing the dip $\chi$ of the $e_1, e_2$ plane relative to the horizontal plane. Further we introduce the azimuth $\zeta$. The definition of $\varphi$ remains the same. See Figure 2.3. Upon Comparing Figures 2.2-2.3 we easily obtain the following relations between $\{\theta, \psi\}$ and $\{\chi, \zeta\}$

\[
\begin{align*}
\cos \chi &= \cos \theta \cos \psi, & 0 \leq \chi \leq \pi/2, \\
\sin \zeta &= \sin \theta / \sin \chi, & 0 \leq \zeta < 2\pi.
\end{align*}
\]

\[2.12\]
\[2.13\]

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure2_3}
\caption{Cartesian reference frame \{i_1, i_2, i_3\} and principal directions \{e_1, e_2, e_3\} of the conductivity tensor. The principal directions are specified through the angles $\chi$ (dip), $\zeta$ (azimuth) and $\varphi$.}
\end{figure}

From Eqs. (2.21)-(2.22) we can compute $\theta$ and $\psi$ in terms of $\chi$ and $\zeta$. Subsequent substitution of $\theta, \psi$ into Eqs. (2.6)-(2.11) yields the $\sigma_{kr}$ in terms of $\chi$ and $\zeta$. Since no essential simplifications of the relevant expressions for the $\sigma_{kr}$ are obtained, we shall not repeat those expressions again. Finally, in view of investigations on azimuthal ($\zeta$) dependence we require that the vertical components $e_{k,3}$ of each of the principal directions only depend on the value of $\chi$ and remain unaltered as $\zeta$ is changed. Note
that \( e_{3,3} \) is already independent of \( \zeta \). For this purpose we write \( \phi \) as a function of \( \zeta \) and \( \chi \). From Eqs. (2.3)-(2.4) we obtain that \( \phi \) should satisfy the relations

\[
\sin^2 \chi \cos(\phi - \phi_0) = \sin \psi \sin \psi_0 + \sin \theta \sin \theta_0 \cos \psi \cos \psi_0, \tag{2.14}
\]

\[
\sin^2 \chi \sin(\phi - \phi_0) = \cos \psi \sin \psi_0 \sin \theta - \sin \psi \cos \psi_0 \sin \theta_0, \tag{2.15}
\]

where, according to Eqs. (2.12)-(2.13), \( \psi \) and \( \theta \) follow from \( \chi \) (fixed) and \( \zeta \). \( \psi_0, \theta_0 \) and \( \phi_0 \) correspond to the values of \( \chi, \zeta \) and \( \phi \) for which we specify the values of \( e_{3,3} \) that should remain constant as \( \zeta \) is varied. Note that \( \psi_0 \) and \( \theta_0 \) correspond to one value of \( \chi \) and hence, if \( \chi \) is changed we must change the values \( \psi_0 \) and \( \theta_0 \) in accordance with Eqs. (2.12)-(2.13). Without loss of generality we can choose \( \theta_0 = 0 \). Hence, \( \phi_0 \) is to be specified for \( \zeta = 0 \) (cf. Eq. (2.13); see Figure 2.4). From Eq. (2.12) we obtain that in this case \( \psi_0 = \chi \) and consequently

\[
\cos(\phi - \phi_0) = \frac{\sin \psi}{\sin \chi}, \tag{2.16}
\]

\[
\sin(\phi - \phi_0) = \cos \psi \sin \zeta. \tag{2.17}
\]

Figure 2.4.: Cartesian reference frame \( \{i_1, i_2, i_3\} \) and principal directions \( \{e_1, e_2, e_3\} \) of the conductivity tensor. The principal directions are specified through the angles \( \chi \) (dip), \( \zeta \) (azimuth) and \( \phi_0 \) (fixed).
2.3 Electromagnetic field equations and constitutive relations

The partial differential electromagnetic field equations governing the behavior of diffusive electromagnetic fields are

\[-\varepsilon_{k,m,p} \partial_m H_p + J_k = -J_k^e, \quad (2.18)\]
\[\varepsilon_{j,m,r} \partial_m E_r + \partial_j B_j = -K_j^e, \quad (2.19)\]

where \(\varepsilon_{k,m,p}\) is the completely antisymmetric unit tensor of rank three (Levi-Civita tensor):

\[\varepsilon_{k,m,p} = \begin{cases} +1 & \text{when } \{k, m, p\} \text{ is an even permutation of } \{1, 2, 3\}, \\ 0 & \text{when not all subscripts are different}, \\ -1 & \text{when } \{k, m, p\} \text{ is an odd permutation of } \{1, 2, 3\}. \end{cases} \quad (2.20)\]

The field quantities occurring in Eqs. (2.18) and (2.19) are

\[E_r = \text{electric field strength (Vm}^{-1}),\]
\[H_p = \text{magnetic field strength (Am}^{-1}),\]
\[B_j = \text{magnetic flux density (T)},\]
\[J_k = \text{volume density of induced electric current (Am}^{-2}),\]
\[J_k^e = \text{volume source density of electric current (Am}^{-2}),\]
\[K_j^e = \text{volume source density of magnetic current (Vm}^{-2}).\]

The electromagnetic constitutive relations constitute the relationship between \(\{J_k, B_j\}\) and \(\{E_r, H_p\}\). These relations are representative for the macroscopic electromagnetic properties of the materials present in the configuration. For our case of general
linear, anisotropic, locally reacting and time-invariant media, we have

\begin{align}
J_k(\omega, t) &= \sigma_{k, r}(\omega) E_r(\omega, t), \\
B_j(\omega, t) &= \mu_{j, p}(\omega) H_p(\omega, t),
\end{align}

in which

\begin{align*}
\sigma_{k, r}(\omega) &= \text{tensorial conductivity (Sm}^{-1}), \\
\mu_{j, p}(\omega) &= \text{tensorial permeability (Hm}^{-1}).
\end{align*}

For the case of an isotropic medium we have

\begin{align}
\sigma_{k, r}(\omega) &= \sigma(\omega) \delta_{k, r}, \tag{2.23} \\
\mu_{j, p}(\omega) &= \mu(\omega) \delta_{j, p}, \tag{2.24}
\end{align}

where \( \delta_{k, r} \) is the symmetric unit tensor of rank two (Kronecker tensor):

\begin{equation}
\delta_{k, r} = \begin{cases} 
1 & \text{when the subscripts are equal}, \\
0 & \text{when the subscripts are different}, 
\end{cases} \tag{2.25}
\end{equation}

and

\begin{align*}
\sigma(\omega) &= \text{scalar conductivity (Sm}^{-1}), \\
\mu(\omega) &= \text{scalar permeability (Hm}^{-1}).
\end{align*}

Substitution of Eqs. (2.23) and (2.24) in Eq. (2.21) and Eq. (2.22) leads to the following constitutive relations for the case of a general linear, isotropic, locally reacting and time-invariant medium

\begin{align}
J_k(\omega, t) &= \sigma(\omega) E_k(\omega, t), \tag{2.26} \\
B_j(\omega, t) &= \mu(\omega) H_j(\omega, t). \tag{2.27}
\end{align}
2.4 The boundary conditions

From a physical point of view, only solutions of the differential equations (2.18) and (2.19) that are causally related to the action of the sources are acceptable. Let the sources start to act at the instant \( t = t_0 \), then we take \( \{E_r, H_p, J_k, B_j\} = 0 \) when \( t < t_0 \) for all \( \omega \).

2.4 The boundary conditions

Across the interfaces where \( \sigma_{k,r} \) and/or \( \mu_{j,p} \) show a jump discontinuity, the tangential components of the electric and magnetic field strengths are to be continuous. Since the interfaces are horizontal and parallel to the \( x_1, x_2 \)-plane, we obtain the following boundary conditions

\[
\{E_1(\omega, t), E_2(\omega, t)\} \quad \text{continuous across an interface,} \tag{2.28}
\]

\[
\{H_1(\omega, t), H_2(\omega, t)\} \quad \text{continuous across an interface,} \tag{2.29}
\]

or

\[
\lim_{x_3 \uparrow x_{3;m}} \{E_1(\omega, t), E_2(\omega, t)\} = \lim_{x_3 \downarrow x_{3;m}} \{E_1(\omega, t), E_2(\omega, t)\} \quad \text{for } m = 1, \ldots, ND - 1, \tag{2.30}
\]

\[
\lim_{x_3 \uparrow x_{3;m}} \{H_1(\omega, t), H_2(\omega, t)\} = \lim_{x_3 \downarrow x_{3;m}} \{H_1(\omega, t), H_2(\omega, t)\} \quad \text{for } m = 1, \ldots, ND - 1. \tag{2.31}
\]

In order to solve the diffusive electromagnetic field equations and their accompanying boundary conditions in the configuration discussed in Section 2.1, we subject them to a succession of integral transformations that are compatible with the time invariance of the configuration and its shift invariance in the horizontal plane.
2.5 The transform-domain equations

To analyze the electromagnetic fields in the configuration under consideration, it is advantageous to employ the invariance properties of the configuration. This is most easily done by carrying out appropriate integral transformations. First, we take a one-sided Laplace transformation with respect to time with a real and positive transformation parameter \( s \). To show our notation, we give the expression for the electric field strength, taking \( t = t_0 = 0 \) as the instant at which the sources start to act:

\[
\hat{E}_r(\omega, s) = \int_0^\infty \exp(-st)E_r(\omega, t) \, dt. \tag{2.32}
\]

Since sources of bounded strengths generate fields of bounded magnitudes, the right-hand side of Eq. (2.32) exists for \( s > 0 \). The boundedness of the Laplace transform for real and positive \( s \) ensures causality (Widder, 1946).

Further, to take advantage of the shift invariance of the configuration in the horizontal plane, we introduce the Fourier transformation with respect to the horizontal coordinates \( x_1 \) and \( x_2 \). For later convenience, we use for this transformation the parameters \( \alpha_1 s^{1/2} \) and \( \alpha_2 s^{1/2} \), with \( \alpha_1 \in \mathbb{R} \) and \( \alpha_2 \in \mathbb{R} \). Again the notation is shown by giving the expression for the electric field strength

\[
\tilde{E}_r(\alpha_1, \alpha_2, x_3, s) = \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} \exp(i\alpha_2 x_2 s^{1/2})\hat{E}_r(\omega, s) \, dx_1, \tag{2.33}
\]

in which \( i \) is the imaginary unit, \( i^2 = -1 \). Here, and in what follows, Greek subscripts are used to indicate the horizontal (i.e., the \( x_1\)- and \( x_2\)-) components of vectors and tensors; they are assigned the values 1 and 2. It is assumed that the integral at the right-hand side of Eq. (2.33) exists. The transformation inverse to Eq. (2.33) is given by

\[
E_r(\omega, s) = \frac{s}{4\pi^2} \int_{-\infty}^{\infty} \, d\alpha_2 \int_{-\infty}^{\infty} \exp(-i\alpha_2 x_2 s^{1/2})\tilde{E}_r(\alpha_1, \alpha_2, x_3, s) \, d\alpha_1. \tag{2.34}
\]

Application of Eqs. (2.32) and (2.33) to Eqs. (2.18) and (2.19) and taking into account that \( \partial_1 \rightarrow -i\alpha_1 s^{1/2} \), \( \partial_2 \rightarrow -i\alpha_2 s^{1/2} \) and \( \partial_i \rightarrow s \), leads to the transform-domain electromagnetic field equations.
\[ -\varepsilon_{k,3,\mu} \partial_3 \tilde{H}_p = -i s^{1/2} \varepsilon_{k,\mu,\nu} \alpha_\mu \tilde{H}_p - \tilde{J}_k^\nu, \quad (2.35) \]
\[ \varepsilon_{j,3,\nu} \partial_3 \tilde{E}_r = i s^{1/2} \varepsilon_{j,\mu,\nu} \alpha_\mu \tilde{E}_r - s \tilde{B}_j^\nu. \quad (2.36) \]

Application of Eqs. (2.32) and (2.33) to Eqs. (2.21) and (2.22) yields the transform-domain constitutive relations
\[ \tilde{J}_k(\alpha_1, \alpha_2, x_3, s) = \sigma(x_3) \tilde{E}_r(\alpha_1, \alpha_2, x_3, s), \quad (2.37) \]
\[ \tilde{B}_j(\alpha_1, \alpha_2, x_3, s) = \mu(x_3) \tilde{H}_p(\alpha_1, \alpha_2, x_3, s). \quad (2.38) \]

Substitution of Eqs. (2.37) and (2.38) into Eqs. (2.35) and (2.36) leads to the following transform-domain electromagnetic field equations in which only \{\tilde{E}_r, \tilde{H}_p, \tilde{J}_k^\nu, \tilde{K}_j^\nu\} do occur
\[ -\varepsilon_{k,3,\mu} \partial_3 \tilde{H}_p = -i s^{1/2} \varepsilon_{k,\mu,\nu} \alpha_\mu \tilde{H}_p - \sigma \tilde{E}_r - \tilde{J}_k^\nu, \quad (2.39) \]
\[ \varepsilon_{j,3,\nu} \partial_3 \tilde{E}_r = i s^{1/2} \varepsilon_{j,\mu,\nu} \alpha_\mu \tilde{E}_r - s \mu \tilde{H}_p - \tilde{K}_j^\nu. \quad (2.40) \]

From these equations it is clear that only the differentiation with respect to \( x_3 \) is left. In the case of isotropic media, a combination of Eqs. (2.23), (2.24), (2.39) and (2.40) results into
\[ -\varepsilon_{k,3,\mu} \partial_3 \tilde{H}_p = -i s^{1/2} \varepsilon_{k,\mu,\nu} \alpha_\mu \tilde{H}_p - \sigma \tilde{E}_k - \tilde{J}_k^\nu, \quad (2.41) \]
\[ \varepsilon_{j,3,\nu} \partial_3 \tilde{E}_r = i s^{1/2} \varepsilon_{j,\mu,\nu} \alpha_\mu \tilde{E}_r - s \mu \tilde{H}_j - \tilde{K}_j^\nu. \quad (2.42) \]

The boundary conditions at an interface between two adjacent subdomains remain unchanged. Application of Eqs. (2.32) and (2.33) to Eqs. (2.30) and (2.32) yields
\[ \lim_{x_3 \uparrow x_{3,m}} \{ \tilde{E}_1, \tilde{E}_2 \} = \lim_{x_3 \downarrow x_{3,m}} \{ \tilde{E}_1, \tilde{E}_2 \} \quad \text{for} \ m = 1, \ldots, ND - 1, \quad (2.43) \]
\[ \lim_{x_3 \uparrow x_{3,m}} \{ \tilde{H}_1, \tilde{H}_2 \} = \lim_{x_3 \downarrow x_{3,m}} \{ \tilde{H}_1, \tilde{H}_2 \} \quad \text{for} \ m = 1, \ldots, ND - 1. \quad (2.44) \]
Furthermore, the transform-domain field quantities must remain bounded as $x_3 \to \infty$ and as $x_3 \to -\infty$.

### 2.6 Time Laplace-transform domain field reciprocity theorem

In this section the time Laplace-transform domain reciprocity theorem is discussed. A reciprocity relation interrelates in a specific manner the field quantities of two non-identical physical states that can occur in one and the same (bounded) domain in space. This domain is denoted by $\mathcal{D}$ with an enclosing surface $\partial \mathcal{D}$. The unit vector along the normal to $\partial \mathcal{D}$ is denoted by $\nu$ (with components $\nu_i$); it is pointing away from $\mathcal{D}$. The two diffusive electromagnetic states that can occur in the domain $\mathcal{D}$ are referred to as states $A$ and $B$. Neither the media nor the sources present in the two states need be the same (see Figure 2.5).

\[ \nu \]

**Figure 2.5:** The two states $A$ and $B$ in the domain $\mathcal{D}$ in reciprocity considerations.

State $A$ is characterized by the diffusive electromagnetic field

\[ \{ \hat{E}_r, \hat{H}_p \} = \{ \hat{E}^A_r, \hat{H}^A_p \}, \quad (2.45) \]
the constitutive parameters
\[ \{ \sigma, \mu \} = \{ \sigma_k^A, \mu_{j,p}^A \}, \] (2.46)

and the volume source distributions
\[ \{ \tilde{J}_k, \tilde{K}_j \} = \{ J_k^{\text{ei}A}, K_j^{\text{ei}A} \}. \] (2.47)

The time Laplace-transform domain diffusive electromagnetic field of State \( A \) satisfies the partial differential equations
\[ -\varepsilon_{k,m,p} \partial_m \hat{H}_p^A + \sigma_{k,r}^A \hat{E}_r^A = -\tilde{J}_k^{\text{ei}A}, \] (2.48)
\[ \varepsilon_{j,m,r} \partial_m \hat{E}_r^A + \varepsilon \mu_{j,p}^A \hat{H}_p^A = -\tilde{K}_j^{\text{ei}A}. \] (2.49)

Similarly, State \( B \) is characterized by the diffusive electromagnetic field
\[ \{ \hat{E}_r, \hat{H}_p \} = \{ \hat{E}_r^B, \hat{H}_p^B \}, \] (2.50)

the constitutive parameters
\[ \{ \sigma, \mu \} = \{ \sigma_k^B, \mu_{j,p}^B \}, \] (2.51)

and the volume source distributions
\[ \{ \tilde{J}_k, \tilde{K}_j \} = \{ \tilde{J}_k^{\text{ei}B}, \tilde{K}_j^{\text{ei}B} \}. \] (2.52)

The time Laplace-transform domain diffusive electromagnetic field of State \( B \) satisfies the partial differential equations
\[ -\varepsilon_{k,m,p} \partial_m \hat{H}_p^B + \sigma_{k,r}^B \hat{E}_r^B = -\tilde{J}_k^{\text{ei}B}, \] (2.53)
\[ \varepsilon_{j,m,r} \partial_m \hat{E}_r^B + \varepsilon \mu_{j,p}^B \hat{H}_p^B = -\tilde{K}_j^{\text{ei}B}. \] (2.54)
The fundamental interaction quantity between the two states to be considered is the divergence of the vectorial quantity \( \varepsilon_{r,m,p}(\hat{E}_r^A \hat{H}_p^B - \hat{E}_r^B \hat{H}_p^A) \). With the aid of Eq. (2.20) this can be written as

\[
\varepsilon_{r,m,p} \partial_m (\hat{E}_r^A \hat{H}_p^B - \hat{E}_r^B \hat{H}_p^A) = -\varepsilon_{p,m,r} \hat{H}_p^B \partial_m \hat{E}_r^A + \varepsilon_{r,m,p} \hat{E}_r^A \partial_m \hat{H}_p^B + \varepsilon_{p,m,r} \hat{H}_p^A \partial_m \hat{E}_r^B - \varepsilon_{r,m,p} \hat{E}_r^B \partial_m \hat{H}_p^A. \tag{2.55}
\]

The four terms on the right-hand side of Eq. (2.55) can be rewritten using the differential equations (2.48), (2.49), (2.53) and (2.54) for the different states. Collecting the results, we end up with the local form of the time Laplace-transform domain reciprocity theorem for the fields of State \( A \) and State \( B \) as

\[
\varepsilon_{r,m,p} \partial_m (\hat{E}_r^A \hat{H}_p^B - \hat{E}_r^B \hat{H}_p^A) = s(\mu_{j,p}^A - \mu_{p,j}^B) \hat{H}_p^B \hat{H}_p^A - (\sigma_{k,r}^A - \sigma_{r,k}^B) \hat{E}_k^B \hat{E}_r^A + \hat{E}_r^A \hat{k}_{r}^B - \hat{E}_r^B \hat{j}_{r}^A - \hat{H}_p^B \hat{k}_{p}^A + \hat{H}_p^A \hat{k}_{p}^A. \tag{2.56}
\]

The first two terms at the right-hand side of Eq. (2.56) are representative for the differences in the electromagnetic properties of the media present in the two states. They vanish at those locations where

\[
\sigma_{k,r}^A = \sigma_{r,k}^B \tag{2.57}
\]

\[
\mu_{j,p}^A = \mu_{p,j}^B. \tag{2.58}
\]

In case the latter conditions hold, the media of the two states are denoted as each other adjoints. In this case, the interaction between the two states is only related to the source distributions, i.e.,

\[
\varepsilon_{r,m,p} \partial_m (\hat{E}_r^A \hat{H}_p^B - \hat{E}_r^B \hat{H}_p^A) = \hat{E}_r^A \hat{j}_{r}^B - \hat{E}_r^B \hat{j}_{r}^A - \hat{H}_p^B \hat{k}_{p}^A + \hat{H}_p^A \hat{k}_{p}^A. \tag{2.59}
\]
If also these source distributions vanish in some domain, the relevant interaction quantity in that domain is equal to zero, i.e.,

\[ \varepsilon_{r,m,p} \partial_m (\hat{E}_r^A \hat{H}_p^B - \hat{E}_r^B \hat{H}_p^A) = 0. \tag{2.60} \]

### 2.7 Transform-domain field reciprocity theorem

In this section the transform-domain reciprocity theorem is discussed. This reciprocity relation interrelates in a specific manner the transform-domain field quantities of two non-identical physical states \( A \) and \( B \) that can occur in one and the same bounded vertical subregion of the configuration shown in Figure 2.1. Neither the media nor the sources present in the two states need be the same; the media in the two states must, however, be time-invariant and shift-invariant in the horizontal plane. State \( A \) is characterized by the transform-domain diffusive electromagnetic field \( \{ \hat{E}_r^A, \hat{H}_p^A \} \) that satisfies the ordinary differential equations

\[ -\varepsilon_{k,3,p} \partial_3 \hat{H}_p^A = -i \alpha_{k,\mu,p} \alpha_{\mu} \hat{H}_p^A - \sigma_{k,r} \hat{E}_r^A - j^e_{k}^A, \tag{2.61} \]

\[ \varepsilon_{j,3,r} \partial_3 \hat{E}_r^A = i \alpha_{j,\mu,r} \alpha_{\mu} \hat{E}_r^A - \mu_{j,p} \hat{H}_p^A - k_j^e^A. \tag{2.62} \]

Similarly, State \( B \) is characterized by the transform-domain diffusive electromagnetic field \( \{ \hat{E}_r^B, \hat{H}_p^B \} \) that satisfies the ordinary differential equations

\[ -\varepsilon_{k,3,p} \partial_3 \hat{H}_p^B = -i \alpha_{k,\mu,p} \alpha_{\mu} \hat{H}_p^B - \sigma_{k,r} \hat{E}_r^B - j^e_{k}^B, \tag{2.63} \]

\[ \varepsilon_{j,3,r} \partial_3 \hat{E}_r^B = i \alpha_{j,\mu,r} \alpha_{\mu} \hat{E}_r^B - \mu_{j,p} \hat{H}_p^B - k_j^e^B. \tag{2.64} \]

Next, we define the adjoint field \( \{ \hat{E}_r^B, \hat{H}_p^B \} \) of State \( B \) as the solution of the ordinary differential equations

\[ -\varepsilon_{k,3,p} \partial_3 \hat{H}_p^B = i \alpha_{k,\mu,p} \alpha_{\mu} \hat{H}_p^B + \sigma_{k,r} \hat{E}_r^B - j^e_{k}^B, \tag{2.65} \]

\[ \varepsilon_{j,3,r} \partial_3 \hat{E}_r^B = -i \alpha_{j,\mu,r} \alpha_{\mu} \hat{E}_r^B + \mu_{j,p} \hat{H}_p^B - k_j^e^B. \tag{2.66} \]
Note that Eqs. (2.65) and (2.66) still admit solutions that vanish as \( |z_3| \rightarrow \infty \).

The fundamental interaction quantity between the two states to be considered is now the derivative with respect to the \( z_3 \)-coordinate of the quantity \( \varepsilon_{r,3,p}(\tilde{E}_r^A \tilde{H}_p^B + \tilde{E}_r^B \tilde{H}_p^A) \). With the aid of Eq. (2.20) this can be written as

\[
\varepsilon_{r,3,p} \partial_3 (\tilde{E}_r^A \tilde{H}_p^B + \tilde{E}_r^B \tilde{H}_p^A) = -\varepsilon_{r,3,p} \tilde{H}_p^B \varepsilon_{r,3} \tilde{E}_r^A + \varepsilon_{r,3,p} \tilde{E}_r^B \varepsilon_{r,3} \tilde{H}_p^B
\]

\[
- \varepsilon_{r,3,p} \tilde{H}_p^A \varepsilon_{r,3} \tilde{E}_r^B + \varepsilon_{r,3,p} \tilde{E}_r^B \varepsilon_{r,3} \tilde{H}_p^A. \tag{2.67}
\]

The four terms on the right-hand side of Eq. (2.67) can be rewritten using the differential equations (2.61)-(2.62) and (2.65)-(2.66) for the two different states. Collecting the results, we end up with the local form of the transform-domain reciprocity theorem for the field of State A and the adjoint field of State B as

\[
\varepsilon_{r,3,p} \partial_3 (\tilde{E}_r^A \tilde{H}_p^B + \tilde{E}_r^B \tilde{H}_p^A) = s(\mu_{p,j}^A - \mu_{p,j}^B) \tilde{H}_p^B \tilde{H}_r^A + (\sigma_{r,k}^A - \sigma_{r,k}^B) \tilde{E}_k^B \tilde{E}_r^A
\]

\[
+ \tilde{E}_r^A \tilde{J}_r^c;B + \tilde{E}_r^B \tilde{J}_k^c;B + \tilde{H}_p^A \tilde{K}_j^c;B + \tilde{H}_p^B \tilde{K}_p^c;A. \tag{2.68}
\]

The first two terms at the right-hand side of Eq. (2.68) are representative for the differences in the electromagnetic properties of the media present in the two states. They vanish at those locations where

\[
\sigma_{r,k}^A = \sigma_{r,k}^B \tag{2.69}
\]

\[
\mu_{p,j}^A = \mu_{p,j}^B. \tag{2.70}
\]

In case the latter conditions hold, the media of the two states are denoted as each other adjoints. In this case, the interaction between the two states is only related to the source distributions, i.e.,

\[
\varepsilon_{r,3,p} \partial_3 (\tilde{E}_r^A \tilde{H}_p^B + \tilde{E}_r^B \tilde{H}_p^A) = \tilde{E}_r^A \tilde{J}_k^c;B + \tilde{E}_r^B \tilde{J}_r^c;A
\]

\[
+ \tilde{H}_p^A \tilde{K}_j^c;B + \tilde{H}_p^B \tilde{K}_p^c;A. \tag{2.71}
\]
If also these source distributions vanish in some domain, the relevant interaction quantity in that domain is equal to zero, i.e.,

\[ \varepsilon_{r,3,p} \partial_3 (\tilde{E}_r^A \tilde{H}^\dagger_B + \tilde{E}_r^\dagger_B \tilde{H}_r^A) = 0. \]  

(2.72)

The term \( \varepsilon_{r,3,p} (\tilde{E}_r^A \tilde{H}^\dagger_B + \tilde{E}_r^\dagger_B \tilde{H}_r^A) \) is then an invariant in the \( z_3 \)-direction and is referred to as the transform-domain diffusion invariant of a source-free anisotropic medium. This diffusion invariant is a fundamental quantity that has important consequences for the solutions of the transform-domain diffusive electromagnetic field equations in anisotropic media (Kenneth 1983; Kenneth et al. 1990).

With the aid of diffusion invariants we shall derive some basic properties of electromagnetic fields in arbitrarily anisotropic conducting media, prior to any evaluation or computation of the electromagnetic field from Eqs. (2.22) and (2.23). Note that the diffusion invariant contains only the horizontal components of the transform-domain diffusive electromagnetic field and not the vertical components.

Let us introduce the symbolic notations \( S^A \) and \( S^\dagger_B \) for the horizontal components of the transform-domain electric and magnetic field strengths of State \( A \) and the adjoint electric and magnetic field strengths of State \( B \), respectively. Further, we shall denote by \( \langle S^A, S^\dagger_B \rangle \) the diffusion invariant that is related to the transform-domain diffusive electromagnetic fields of the two states \( A \) and \( B \), i.e.,

\[ \langle S^A, S^\dagger_B \rangle = \varepsilon_{r,3,p} (\tilde{E}_r^A \tilde{H}^\dagger_B + \tilde{E}_r^\dagger_B \tilde{H}_r^A). \]

(2.73)

Note that \( \langle S^A, S^\dagger_B \rangle = \langle S^\dagger_B, S^A \rangle \), while in general \( \langle S^A, S^\dagger_B \rangle \neq \langle S^\dagger_A, S^B \rangle \). We shall consider diffusion invariants that are related to electromagnetic fields in an unbounded homogeneous medium.

**Unbounded homogeneous medium**

We consider an unbounded homogeneous medium in which the transient diffusive electromagnetic field is generated by sources of bounded extent. Further, we assume that all sources are located outside the region \( z_3_{\text{min}} < z_3 < z_3_{\text{max}} \). The electromagnetic field in this source-free region that propagates in the direction of increasing \( z_3 \) is denoted symbolically by \( S^+ \). The electromagnetic field in this source-free region that propagates in the direction of decreasing \( z_3 \) is denoted symbolically by \( S^- \). See
Figure 2.6. The diffusion invariant of field $S^+$ and its adjoint $S^+\dagger$ is given by (cf. Eq. (2.73))

$$\langle S^+, S^+\dagger \rangle = \mathcal{N}^+,$$  \hspace{1cm} \mathcal{N}^+ \neq 0, \hspace{1cm} (2.74)$$

where $\mathcal{N}^+$ is independent of $x_3$ for $x_{3,\text{min}} < x_3 < x_{3,\text{max}}$. In the same way we have for the diffusion invariant of field $S^-$ and its adjoint $S^-\dagger$

$$\langle S^-, S^-\dagger \rangle = \mathcal{N}^-, \hspace{1cm} \mathcal{N}^- \neq 0, \hspace{1cm} (2.75)$$

where $\mathcal{N}^-$ is independent of $x_3$ for $x_{3,\text{min}} < x_3 < x_{3,\text{max}}$. In general there is no specific relationship between $\mathcal{N}^+$ and $\mathcal{N}^-$.  

![Diagram showing the unbounded homogeneous domain with source-free region $x_{3,\text{min}} < x_3 < x_{3,\text{max}}$. The electromagnetic field propagating in the direction of increasing $x_3$ is symbolically denoted as $S^+$, the electromagnetic field propagating in the direction of decreasing $x_3$ is symbolically denoted as $S^-$.]

Finally, we shall apply our transform-domain reciprocity relation (2.51) to the
electromagnetic field $S^+$ and the adjoint field $S^-$ of both previous states in this source-free domain. We shall consider the diffusion invariant $\langle S^+, S^\dagger_- \rangle$. In view of Eqs. (2.74) and (2.75) and the fact that $S^+$ and $S^-$ represent electromagnetic fields propagating in opposite directions, we infer that in order to $\langle S^+, S^\dagger_- \rangle$ remain constant on the relevant interval of $x_3$, this diffusion invariant must vanish, i.e.

$$\langle S^+, S^\dagger_- \rangle \equiv 0 \quad \text{when} \quad x_{3,\text{min}} < x_3 < x_{3,\text{max}}.$$  \hspace{1cm} (2.76)

A similar result is obtained for $\langle S^\dagger_+, S^- \rangle$. This propagation property implies the existence of orthogonal solutions of Maxwell's equations in the sense of the transform-domain reciprocity theorem and the diffusion invariants as defined by Eq. (2.73). We shall need this fundamental property later on in Sections 3.2 and 3.3 where we derive explicit expressions for these orthogonal solutions.
Chapter 3

Diffusive electromagnetic fields in homogeneous, anisotropic media

3.1 The transform-domain electromagnetic field matrix

In this section we consider the solution of the transform-domain electromagnetic field equations (2.39) and (2.40) in a homogeneous, unbounded, anisotropic medium. These equations consist of the four first-order ordinary linear differential equations in the independent variable $x_3$

\[
\begin{align*}
\delta_3 \vec{H}_2 &= -is^{1/2}\alpha_2 \vec{H}_3 - \sigma_{1,r} \vec{E}_r - \vec{J}_1^c, \\
-\delta_3 \vec{H}_1 &= is^{1/2}\alpha_1 \vec{H}_3 - \sigma_{2,r} \vec{E}_r - \vec{J}_2^c, \\
-\delta_3 \vec{E}_2 &= is^{1/2}\alpha_2 \vec{E}_3 - s\mu_{1,p} \vec{H}_p - \vec{K}_1^c, \\
\delta_3 \vec{E}_1 &= -is^{1/2}\alpha_1 \vec{E}_3 - s\mu_{2,p} \vec{H}_p - \vec{K}_2^c,
\end{align*}
\]

and the two linear, algebraic equations

\[
\begin{align*}
\sigma_{3,r} \vec{E}_r &= -is^{1/2}\alpha_1 \vec{H}_2 + is^{1/2}\alpha_2 \vec{H}_1 - \vec{J}_3^c,
\end{align*}
\]
\[ s\mu_{3,p}\vec{H}_p = is^{1/2}\alpha_1\vec{E}_2 - is^{1/2}\alpha_2\vec{E}_1 - \vec{K}_3^s. \] (3.6)

From the boundary conditions (2.43) and (2.44) we see that the field components occurring at the left-hand sides of the differential equations (3.1)-(3.4), viz. the horizontal electric and magnetic field components, are just the ones that are to be continuous across an interface in a layered configuration. The vertical field components \( \{\vec{E}_3, \vec{H}_3\} \) jump across such an interface. As a consequence, it will be advantageous to eliminate the latter two field components from Eqs. (3.1)-(3.4). This is achieved by rewriting the linear algebraic equations (3.5) and (3.6) as

\[ \sigma_{3,3}\vec{E}_3 = -\sigma_{3,p}\vec{E}_p - is^{1/2}\alpha_1\vec{H}_2 + is^{1/2}\alpha_2\vec{H}_1 - J_3^s, \] (3.7)

\[ s\mu_{3,3}\vec{H}_3 = -s\mu_{3,s}\vec{H}_s + is^{1/2}\alpha_1\vec{E}_2 - is^{1/2}\alpha_2\vec{E}_1 - \vec{K}_3^s, \] (3.8)

and substituting the expressions for \( \{\vec{E}_3, \vec{H}_3\} \) resulting from Eqs. (3.7) and (3.8) into Eqs. (3.1)-(3.4). This procedure leads to four first-order linear differential equations with \( \{\vec{E}_1, \vec{E}_2, \vec{H}_1, \vec{H}_2\} \) as the dependent variables. We shall write the four resulting differential equations as a single matrix differential equation. To this end we introduce the transform-domain electromagnetic 4-by-1 field matrix \( F_J \),

\[ F_J = (\vec{E}_1, \vec{E}_2, is^{1/2}\vec{H}_2, -is^{1/2}\vec{H}_1)^T. \] (3.9)

Here, \( T \) indicates the transposed matrix. To the uppercase Latin subscripts the summation convention applies; they take on the values from 1 to 4. Since the electromagnetic field matrix only occurs in the transform domain we omit the tilde. The system of four differential equations (3.1)-(3.4) can then be written as the first-order ordinary matrix differential equation

\[ \partial_3 F_I = -s^{1/2}A_{I,J}F_J + N_I, \] (3.10)

where \( A_{I,J} \) is the 4-by-4 system's matrix and \( N_I \) is the 4-by-1 notional source matrix. The system's matrix \( A_{I,J} \) is independent of the Laplace transformation parameter \( s \); it is only a function of the real Fourier transformation parameters \( \alpha_1, \alpha_2 \), and the material tensors \( \sigma_{k,r} \) and \( \mu_{j,p} \). From Eqs. (3.1)-(3.4), (3.7) and (3.8) it is easily
verified that the system’s matrix $A_{I,J}$ can be written as

$$
A_{I,J} = \begin{pmatrix}
A^{(EE)}_{\sigma,\tau} & A^{(EH)}_{\sigma,\tau} \\
A^{(HE)}_{\sigma,\tau} & A^{(HH)}_{\sigma,\tau}
\end{pmatrix},
$$

(3.11)

where $A^{(EH)}_{\sigma,\tau}$, $A^{(HE)}_{\sigma,\tau}$, $A^{(EE)}_{\sigma,\tau}$ and $A^{(HH)}_{\sigma,\tau}$ are the following 2-by-2 submatrices

$$
A^{(EH)}_{\sigma,\tau} = \frac{1}{\sigma_{3,3}} \begin{pmatrix}
\alpha_1^2 + D^2(\bar{\mu}_{2,2}^2 - \bar{\mu}_{2,3}^2) & \alpha_1\alpha_2 - D^2(\bar{\mu}_{1,2} - \bar{\mu}_{1,3}\bar{\mu}_{2,3}) \\
\alpha_1\alpha_2 - D^2(\bar{\mu}_{1,2} - \bar{\mu}_{1,3}\bar{\mu}_{2,3}) & \alpha_2^2 + D^2(\bar{\mu}_{1,1} - \bar{\mu}_{1,3}^2)
\end{pmatrix},
$$

(3.12)

$$
A^{(HE)}_{\sigma,\tau} = \frac{1}{\mu_{3,3}} \begin{pmatrix}
\alpha_2^2 + D^2(\bar{\sigma}_{1,1} - \bar{\sigma}_{1,3}^2) & -\alpha_1\alpha_2 + D^2(\bar{\sigma}_{1,2} - \bar{\sigma}_{1,3}\bar{\sigma}_{2,3}) \\
-\alpha_1\alpha_2 + D^2(\bar{\sigma}_{1,2} - \bar{\sigma}_{1,3}\bar{\sigma}_{2,3}) & \alpha_1^2 + D^2(\bar{\sigma}_{2,2} - \bar{\sigma}_{2,3}^2)
\end{pmatrix},
$$

(3.13)

and

$$
A^{(EE)}_{\sigma,\tau} = A^{(HH)}_{\sigma,\tau} = \begin{pmatrix}
-i\alpha_1(\bar{\sigma}_{1,3} + i\alpha_2\bar{\mu}_{2,3}) & -i\alpha_1(\bar{\sigma}_{2,3} - \bar{\mu}_{2,3}) \\
-i\alpha_2(\bar{\sigma}_{1,3} - \bar{\mu}_{1,3}) & -i\alpha_1\bar{\mu}_{1,3} + i\alpha_2\bar{\sigma}_{2,3}
\end{pmatrix},
$$

(3.14)

with $D = (\mu_{3,3}\sigma_{3,3})^{1/2}$, $\bar{\sigma}_{k,r} = \sigma_{k,r}/\sigma_{3,3}$ and $\bar{\mu}_{k,r} = \mu_{k,r}/\mu_{3,3}$. Note that the submatrix $A^{(HH)}_{\sigma,\tau}$ is the transpose of the submatrix $A^{(EE)}_{\sigma,\tau}$. The properties of the system’s matrix $A_{I,J}$ as a function of the Fourier transformation parameters $\alpha_1$ and $\alpha_2$ will be investigated in detail in Section 3.2.

The source matrix $N_J$ is obtained as

$$
N_J = (N^{(E)}_1, N^{(E)}_2, N^{(H)}_1, N^{(H)}_2)^T,
$$

(3.15)
where

\[
N_{r}^{(E)} = \begin{pmatrix}
-\tilde{K}_{\gamma}^{\varepsilon} + \tilde{\mu}_{1,3} \tilde{K}_{3}^{\varepsilon} + i \alpha_{1} \frac{\kappa_{1}^{1/2} \tilde{j}_{3}^{\varepsilon}}{\sigma_{3,3}} \\
\tilde{K}_{1}^{\varepsilon} - \tilde{\mu}_{1,3} \tilde{K}_{3}^{\varepsilon} + i \alpha_{2} \frac{\kappa_{1}^{1/2} \tilde{j}_{3}^{\varepsilon}}{\sigma_{3,3}}
\end{pmatrix}
\]  

(3.16)

and

\[
N_{r}^{(H)} = \begin{pmatrix}
-\kappa_{1}^{1/2} (\tilde{j}_{1}^{\varepsilon} - \tilde{\sigma}_{1,3} \tilde{j}_{3}^{\varepsilon}) + i \alpha_{2} \frac{\tilde{K}_{3}^{\varepsilon}}{\tilde{\mu}_{1,3}} \\
-\kappa_{1}^{1/2} (\tilde{j}_{2}^{\varepsilon} - \tilde{\sigma}_{2,3} \tilde{j}_{3}^{\varepsilon}) - i \alpha_{1} \frac{\tilde{K}_{3}^{\varepsilon}}{\tilde{\mu}_{1,3}}
\end{pmatrix}
\]

(3.17)

3.2 General properties of the system's matrix

In this section we investigate the properties of the system's matrix \(A_{r,j}\) as a function of the Fourier transformation parameters \(\alpha_{1}\) and \(\alpha_{2}\). Of particular interest are the symmetry properties of \(A_{r,j}\), its four eigenvalues and the orthogonality relations between the different eigenvectors of \(A_{r,j}\). Knowledge of these properties enables us to determine the appropriate solution \(F_{r}(x_{j})\) of the matrix differential equation (3.10) for all combinations of (real) \(\alpha_{1}\) and \(\alpha_{2}\). Subsequent application of the inverse Fourier transformation (2.37) to \(F_{r}\) yields the time Laplace transform of the horizontal field components.

Symmetry properties of \(A_{r,j}\)

From Eqs. (3.12)-(3.14) we observe that the submatrices \(A_{r,r}^{(EE)}\), \(A_{r,r}^{(EH)}\), \(A_{o,r}^{(HE)}\) and \(A_{o,o}^{(HH)}\) satisfy the symmetry relations

\[
A_{r,r}^{(EE)}(\alpha_{\nu}) = A_{o,o}^{(HH)}(\alpha_{\nu}),
\]

(3.18)

\[
A_{r,r}^{(EH)}(\alpha_{\nu}) = A_{o,o}^{(EH)}(\alpha_{\nu}),
\]

(3.19)
3.2 General properties of the system's matrix

\[
A^{(HE)}_{\tau,\sigma}(\alpha_\nu) = A^{(HE)}_{\sigma,\tau}(\alpha_\nu), \quad (3.20)
\]
\[
A^{(HH)}_{\tau,\sigma}(\alpha_\nu) = A^{(EE)}_{\sigma,\tau}(\alpha_\nu). \quad (3.21)
\]

These symmetry relations between the four submatrices of \( A_{I,J} \) imply the invariance of \( A_{I,J} \) under the following similarity transformation (Keith and Crampin, 1977; Kenneth, 1983; Van der Hieden, 1987):

\[
A_{I,J}(\alpha_\nu) = H^{g}_{I,M} A_{M,N}(\alpha_\nu) H^{g}_{N,J}, \quad (3.22)
\]

where \( H^{g}_{I,J} \) is the symmetric 4-by-4 matrix

\[
H^{g}_{I,J} = \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}, \quad (3.23)
\]

where \( I \) is the 2-by-2 unit matrix. Note that \( H^{g}_{I,J} \) satisfies the property

\[
H^{g}_{I,M} H^{g}_{M,J} = \delta_{I,J}. \quad (3.24)
\]

Recall that in view of Eq. (3.24), \( H^{g}_{I,J} \) is said to be involutive (Frazer and Fryer, 1989). As we shall show in Section 3.3, there exists a close relationship between this similarity transformation of \( A_{I,J} \) and the transform-domain diffusion invariant of Section 2.7. From Eqs. (3.12)-(3.14) it is also observed that the submatrices \( A^{(EE)}_{\sigma,\tau}, A^{(EH)}_{\sigma,\tau}, A^{(HE)}_{\sigma,\tau}, \) and \( A^{(HH)}_{\sigma,\tau} \) satisfy the symmetry relations

\[
A^{(EE)}_{\tau,\sigma}(-\alpha_\nu) = -A^{(HH)}_{\sigma,\tau}(\alpha_\nu), \quad (3.25)
\]
\[
A^{(EH)}_{\tau,\sigma}(-\alpha_\nu) = A^{(EH)}_{\sigma,\tau}(\alpha_\nu), \quad (3.26)
\]
\[
A^{(HE)}_{\tau,\sigma}(-\alpha_\nu) = A^{(HE)}_{\sigma,\tau}(\alpha_\nu), \quad (3.27)
\]
\[
A^{(HH)}_{\tau,\sigma}(-\alpha_\nu) = -A^{(EE)}_{\sigma,\tau}(\alpha_\nu), \quad (3.28)
\]
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or

\[ A_{I,I}(-\alpha_\nu) = H_{I,M}^a A_{M,N}(\alpha_\nu) H_{N,J}^a, \]  

(3.29)

where \( H_{I,J}^a \) is the antisymmetric 4-by-4 matrix

\[ H_{I,J}^a = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \]  

(3.30)

Note that \( H_{I,J}^a \) satisfies the property

\[ H_{I,M}^a H_{M,J}^a = -\delta_{I,J}. \]  

(3.31)

Although it is assumed that \( \alpha_1 \) and \( \alpha_2 \) are real, we also investigate the symmetry properties of \( A_{I,J}(\alpha_\nu) \) for complex values of \( \alpha_1 \) and \( \alpha_2 \). From Eqs. (3.12)-(3.14) it is observed that the submatrices \( A^{(EE)}_{\sigma,\tau}, A^{(EH)}_{\sigma,\tau}, A^{(HE)}_{\sigma,\tau} \) and \( A^{(HH)}_{\sigma,\tau} \) satisfy the following relations for arbitrary complex \( \alpha_\nu \)

\[ A^{(EE)}_{\tau,\sigma}(\alpha_\nu^*) = -A^{(HH)*}_{\sigma,\tau}(\alpha_\nu), \]  

(3.32)

\[ A^{(EH)}_{\tau,\sigma}(\alpha_\nu^*) = A^{(HE)*}_{\sigma,\tau}(\alpha_\nu), \]  

(3.33)

\[ A^{(HE)}_{\tau,\sigma}(\alpha_\nu^*) = A^{(HE)*}_{\sigma,\tau}(\alpha_\nu), \]  

(3.34)

\[ A^{(HH)}_{\tau,\sigma}(\alpha_\nu^*) = -A^{(EE)*}_{\sigma,\tau}(\alpha_\nu), \]  

(3.35)

or

\[ A_{I,I}(\alpha_\nu^*) = H_{I,M}^a A_{M,N}(\alpha_\nu) H_{N,J}^a. \]  

(3.36)

Here, the asterisk (*) denotes complex conjugate.

Finally, from Eqs. (3.12)-(3.14) it is also observed that the four submatrices of \( A_{I,J} \) satisfy the following symmetry relations for arbitrary complex \( \alpha_\nu \)

\[ A^{(EE)}_{\tau,\sigma}(-\alpha_\nu^*) = A^{(HH)*}_{\sigma,\tau}(\alpha_\nu), \]  

(3.37)
3.2 General properties of the system's matrix

\[ A_{\tau\sigma}^{(EH)}(-\alpha_\nu^*) = A_{\nu,\tau}^{(EH)*}(\alpha_\nu), \]  
(3.38)

\[ A_{\tau\sigma}^{(HE)}(-\alpha_\nu^*) = A_{\nu,\tau}^{(HE)*}(\alpha_\nu), \]  
(3.39)

\[ A_{\tau\sigma}^{(EE)}(-\alpha_\nu^*) = A_{\nu,\tau}^{(EE)*}(\alpha_\nu), \]  
(3.40)

or

\[ A_{J,I}(-\alpha_\nu^*) = H_{I,M}^\delta A_{M,N}^{*}(\alpha_\nu) H_{N,J}^\delta. \]  
(3.41)

The eigenvalues of \( A_{I,J} \)

In this subsection we investigate some important properties of the eigenvalues of the system's matrix \( A_{I,J} \). These properties are the result of the symmetry relations (3.28), (3.36) and (3.41) between \( A_{I,J}(\pm\alpha_\nu) \) and \( A_{I,J}(\pm\alpha_\nu^*) \). The four eigenvalues of \( A_{I,J}(\alpha_\nu) \) are the solutions \( \gamma = \gamma^{(N)}, \ N = 1, \ldots, 4 \), of the determinantal equation

\[ \det(A_{I,J}(\alpha_\nu)-\gamma\delta_{I,J}) = 0. \]  
(3.42)

Since the left-hand side of this equation is a polynomial of the fourth degree in \( \gamma \), \( \gamma \) itself is a four-valued function of its coefficients and hence, a four-valued function of the Fourier transformation parameters \( \alpha_1 \) and \( \alpha_2 \).

Let \( \gamma = \gamma(\alpha_\nu) \) be one of the four eigenvalues of \( A_{I,J} \). From Eq. (3.31) and the symmetry relation (3.28) it follows that

\[ \det(A_{I,J}(\alpha_\nu)-\gamma\delta_{I,J}) = 0 \quad \Rightarrow \quad \det(A_{I,J}(-\alpha_\nu)+\gamma\delta_{I,J}) = 0. \]  
(3.43)

Consequently, if \( \gamma(\alpha_\nu) \) is an eigenvalue of \( A_{I,J}(\alpha_\nu) \), then \( -\gamma(\alpha_\nu) \) is an eigenvalue of \( A_{I,J}(-\alpha_\nu) \) for arbitrary complex \( \alpha_\nu \). In particular, when \( \alpha_1 = \alpha_2 = 0 \), we find that the four eigenvalues of \( A_{I,J}(\alpha_\nu) \) occur in two pairs of opposite values. From Eq. (3.31) and the symmetry relation (3.36) it follows that

\[ \det(A_{I,J}(\alpha_\nu)-\gamma\delta_{I,J}) = 0 \quad \Rightarrow \quad \det(A_{I,J}(\alpha_\nu^*)+\gamma^*\delta_{I,J}) = 0. \]  
(3.44)

Consequently, if \( \gamma(\alpha_\nu) \) is an eigenvalue of \( A_{I,J}(\alpha_\nu) \), then \( -\gamma^*(\alpha_\nu) \) is an eigenvalue of \( A_{I,J}(\alpha_\nu^*) \) for arbitrary complex \( \alpha_\nu \). In particular, when \( \text{Im}(\alpha_1) = \text{Im}(\alpha_2) = 0 \), we find
that the four eigenvalues of $A_{I,J}(\alpha_\nu)$ occur in two pairs of opposite complex conjugate values. Finally, from Eq. (3.30) and the symmetry relation (3.41) it follows that

$$\det (A_{I,J}(\alpha_\nu) - \gamma \delta_{I,J}) = 0 \implies \det (A_{I,J}(-\alpha_\nu^*) - \gamma^* \delta_{I,J}) = 0. \quad (3.45)$$

Consequently, if $\gamma(\alpha_\nu)$ is an eigenvalue of $A_{I,J}(\alpha_\nu)$, then $\gamma^*(\alpha_\nu)$ is an eigenvalue of $A_{I,J}(-\alpha_\nu^*)$ for arbitrary complex $\alpha_\nu$. In particular, when $\text{Re}(\alpha_1) = \text{Re}(\alpha_2) = 0$, we find that the four eigenvalues of $A_{I,J}(\alpha_\nu)$ occur in pairs of real values and/or pairs of complex conjugate values.

In Appendix A we derive a very important property of the eigenvalues of $A_{I,J}(\alpha_\nu)$ for real $\alpha_\nu$. There, it is shown that for $\alpha_1 = \alpha_2 = 0$ all four eigenvalues are real, while for arbitrary $\alpha_1 \in \mathbb{R}$ and $\alpha_2 \in \mathbb{R}$, none of the eigenvalues can have a real part equal to zero. In view of Eqs. (3.43), (3.44) and the results of Appendix A, we shall denote the eigenvalues of $A_{I,J}(\alpha_\nu)$ as $\gamma = \gamma^{(\pm \tau)}(\alpha_1, \alpha_2)$, $\tau = 1, 2$ such that

$$\gamma^{(+\tau)}(0,0) = -\gamma^{(-\tau)}(0,0) \quad \text{is real and positive}, \quad (3.46)$$

$$\gamma^{(+1)}(0,0) \geq \gamma^{(+2)}(0,0), \quad (3.47)$$

and

$$\gamma^{(+\tau)}(\alpha_1, \alpha_2) = -\gamma^{(-\tau)*}(\alpha_1, \alpha_2) \quad \text{for} \quad \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}, \quad (3.48)$$

where

$$\text{Re}(\gamma^{(+\tau)}(\alpha_1, \alpha_2)) > 0 \quad \text{for} \quad \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}, \quad (3.49)$$

$$\text{Re}(\gamma^{(-\tau)}(\alpha_1, \alpha_2)) < 0 \quad \text{for} \quad \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}. \quad (3.50)$$

**Orthogonality relations between the eigenvectors of $A_{I,J}$**

To each eigenvalue of the system's matrix there corresponds an eigenvector. The eigenvector of $A_{I,J}$ corresponding to $\gamma^{(-1)}$ is denoted as $b^{(-1)}_I$, the one corresponding
3.2 General properties of the system’s matrix

...to $\gamma^{(-2)}$ as $b_j^{(-2)}$, and so on. The eigenvectors of $A_{I,J}$ are obtained from

$$A_{I,J}b_j^{(\pm r)} = \gamma^{(\pm r)}b_j^{(\pm r)}. \quad (3.51)$$

Premultiplication of the left- and right-hand sides of Eq. (3.51) by $b_j^{(\pm \nu)}H_{I,M}^\theta$ results into

$$b_i^{(\pm \nu)}H_{I,M}^\theta A_{M,J}b_j^{(\pm r)} = \gamma^{(\pm r)}b_i^{(\pm \nu)}H_{I,J}^\theta b_j^{(\pm r)}. \quad (3.52)$$

Replacing the superscript $\pm r$ in Eq. (3.51) by $\pm \nu$ and taking the transpose yields

$$b_i^{(\pm \nu)}A_{J,I} = \gamma^{(\pm \nu)}b_j^{(\pm \nu)}. \quad (3.53)$$

Postmultiplication of the left- and right-hand sides of Eq. (3.53) by $H_{M,J}^\theta b_j^{(\pm r)}$ results into

$$b_i^{(\pm \nu)}A_{M,J}H_{M,J}^\theta b_j^{(\pm r)} = \gamma^{(\pm \nu)}b_i^{(\pm \nu)}H_{I,J}^\theta b_j^{(\pm r)}. \quad (3.54)$$

From the similarity transformation (3.22) we infer that $A_{M,J}H_{M,J}^\theta = H_{I,M}^\theta A_{M,J}$. Hence, the left-hand side of Eq. (3.52) is equal to the left-hand side of Eq. (3.54). Subtraction of Eq. (3.54) from Eq. (3.52) yields the orthogonality relation

$$(\gamma^{(\pm r)} - \gamma^{(\pm \nu)})b_i^{(\pm \nu)}H_{I,J}^\theta b_j^{(\pm r)} = 0. \quad (3.55)$$

For distinct eigenvalues of $A_{I,J}$, the inner product $b_i^{(\pm \nu)}H_{I,J}^\theta b_j^{(\pm r)}$ must vanish and the associated eigenvectors are said to be $H^\theta$-orthogonal (Frazer and Fryer, 1989). For equal eigenvalues, the first factor on the left-hand side of Eq. (3.55) vanishes and the inner product $b_i^{(\pm r)}H_{I,J}^\theta b_j^{(\pm r)}$ is only determined by the normalization of the eigenvectors. We normalize the eigenvectors of Eq. (3.51) in such a way that for all (complex) $\alpha_1$ and $\alpha_2$

$$b_i^{(+r)}H_{I,J}^\theta b_j^{(+r)} = +1, \quad (3.56)$$

$$b_i^{(-r)}H_{I,J}^\theta b_j^{(-r)} = -1, \quad (3.57)$$
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while in view of Eq. (3.55) we have

$$b_I^{(+r)} H_{I,J} \cdot b_J^{(-r)} = 0.$$  \hspace{1cm} (3.58)

The fact that the eigenvectors of the system's matrix $A_{I,J}$ are $H^g$-orthogonal is a direct consequence of the similarity transformation (3.22) that $A_{I,J}$ obeys. However, as already indicated in Section 2.7, this orthogonality property is also directly related to the existence of transform-domain diffusion invariants. This will be shown explicitly in the next section where we prove that Eqs. (3.55) and (2.72) are equivalent. Further, from this we conclude that the similarity transformation (3.22) and the transform-domain reciprocity relation (2.68) are also equivalent.

Upon using Eqs. (3.22), (3.36), (3.44), (3.56) and (3.57) we obtain the following relation between $b_I^{(-r)}(\alpha_1, \alpha_2)$ and $b_J^{(+r)}(\alpha_1, \alpha_2)$ for real $\alpha_1$ and $\alpha_2$

$$b_I^{(-r)}(\alpha_1, \alpha_2) = L_{I,J} \cdot b_J^{(+r)}(\alpha_1, \alpha_2) \quad \text{for} \quad \alpha_1 \in \mathbb{R}, \ \alpha_2 \in \mathbb{R},$$  \hspace{1cm} (3.59)

where $L_{I,J}$ is the symmetric 4-by-4 matrix

$$L_{I,J} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$  \hspace{1cm} (3.60)

Note that $L_{I,J}$ satisfies the relation

$$L_{I,M} \cdot L_{M,J} = \delta_{I,J}.$$  \hspace{1cm} (3.61)

In Chapter 6 we shall consider how the eigenvalues and eigenvectors of the system's matrix $A_{I,J}$ simplify for isotropic media. i.e., as $\sigma_{k,r} \to \sigma \delta_{k,r}$ and $\mu_{j,p} \to \mu \delta_{j,p}$. There, we will show that due to the degeneracy of the isotropic case, the eigenvectors $b_{I}^{(\pm \nu)}$ do not necessarily obey an uniform limit as $\sigma_{k,r} \to \sigma \delta_{k,r}$, $\mu_{j,p} \to \mu \delta_{j,p}$ and there exists some kind of freedom in actual choice of the eigenvectors of $A_{I,J}$. 
3.3 Transform-domain diffusive field constituents

In this section we consider the solution $F_j$ of the matrix differential equation (3.10)

$$\partial_3 F_j = -s^{1/2} A_{i,j} F_i + N_I.$$  \hspace{1cm} (3.62)

$F_I$ is the 4-by-1 transform-domain electromagnetic field matrix, $N_I$ the 4-by-1 notional source matrix and $A_{i,j}$ the 4-by-4 system’s matrix. We assume that the Fourier transformation parameters $\alpha_1$ and $\alpha_2$ are real. To elucidate the structure of the solution of Eq. (3.62), we carry out a linear transformation on the electromagnetic field matrix $F_I$ and, through it, want to arrive at a field-vector-formalism in which a decomposition of $F_I$ into up- and downward diffusing fields is manifest. Let $W_N$ be the 4-by-1 matrix that is related to $F_j$ via the transformation

$$F_j = D_{J,N} W_N.$$  \hspace{1cm} (3.63)

$D_{J,N}$ is subject to a convenient choice and is denoted as the composition matrix. On the assumption that $D_{J,N}$ is non-singular, the decomposition inverse matrix $D_{M,I}^{-1}$ of $D_{J,N}$ exists. The relation inverse to (3.63) is

$$W_M = D_{M,I}^{-1} F_I.$$  \hspace{1cm} (3.64)

Substitution of Eq. (3.63) into Eq. (3.62) and premultiplication by $D_{M,I}^{-1}$ yields

$$\partial_3 W_M = -s^{1/2} \Lambda_{M,N} W_N + D_{M,I}^{-1} N_I,$$  \hspace{1cm} (3.65)

where the 4-by-4 matrix $\Lambda_{M,N}$ is given by the matrix product

$$\Lambda_{M,N} = D_{M,I}^{-1} A_{I,J} D_{J,N}.$$  \hspace{1cm} (3.66)

The desired structure of Eq. (3.65) is arrived at if the four differential equations of Eq. (3.65) are mutually uncoupled, i.e., if $\Lambda_{M,N}$ is a diagonal matrix. From the theory of matrices it follows that this is accomplished by taking $D_{J,N}$ to be the
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eigencolumn matrix of \( A_{I,J} \). In that case, \( D^{-1}_{M,I} \) is the eigenrow matrix of \( A_{I,J} \) and \( \Lambda_{M,N} \) is the diagonal matrix of the corresponding eigenvalues of \( A_{I,J} \).

The eigenvalues and corresponding eigenvectors of the system's matrix \( A_{I,J} \) are denoted as \( \gamma^{(\pm \tau)} \) and \( b^{(\pm \tau)}_j \), respectively. Their properties have been discussed in detail in Section 3.2. The four eigenvectors of \( A_{I,J} \) compose the eigencolumn matrix \( D_{J,N} \). They are ordered in the following manner

\[
D_{J,N} = ( b^{(-1)}_j, b^{(-2)}_j, b^{(+1)}_j, b^{(+2)}_j ) .
\]  

(3.67)

The matrix \( D^{-1}_{M,I} \), inverse to \( D_{J,N} \), is the eigenrow matrix of \( A_{I,J} \). The rows of \( D^{-1}_{M,I} \) are denoted as \( g^{(\pm \tau)}_i \) and are ordered in a similar way as the columns of \( D_{J,N} \), viz.

\[
D^{-1}_{M,I} = ( g^{(-1)}_i, g^{(-2)}_i, g^{(+1)}_i, g^{(+2)}_i )^T .
\]  

(3.68)

In view of the orthogonality relations (3.56)-(3.58) between the eigenvectors \( b^{(\pm \tau)}_j \) of \( A_{I,J} \), it is easily verified that \( D_{J,N} \) itself satisfies the following orthogonality relation

\[
D_{I,M} H^{\delta}_{I,J} D_{J,N} = L_{M,N} ,
\]  

(3.69)

where \( L_{M,N} \) is the symmetric involutive matrix given by Eq. (3.60). Since \( L_{I,M} L_{M,J} = \delta_{I,J} \) (cf. Eq. (3.61)), we obtain from the orthogonality relation (3.69) the following explicit expression for the inverse matrix \( D^{-1}_{M,I} \) of \( D_{J,N} \)

\[
D^{-1}_{M,I} = L_{M,N} D_{J,N} H^{\delta}_{I,J} ,
\]  

(3.70)

which is equivalent to

\[
g^{(\pm \tau)}_i = \pm b^{(\pm \tau)}_j H^{\delta}_{I,J} .
\]  

(3.71)

Equation (3.70) is of great importance since it allows an easy and direct computation of the eigenrow matrix \( D^{-1}_{M,I} \) from \( D_{J,N} \) without resorting to any numerical matrix inversion procedure. Finally, \( \Lambda_{M,N} \) is the diagonal matrix (cf. Eq. (3.66))

\[
\Lambda_{M,N} = \text{diag} ( \gamma^{(-1)}, \gamma^{(-2)}, \gamma^{(+1)}, \gamma^{(+1)} ) .
\]  

(3.72)
Since the equations in Eq. (3.65) are now mutually uncoupled, their solutions can be written in terms of four linear independent functions $\Gamma^{(\pm \nu)}$, which, as we will see, express the decomposition in up- and downward diffusing fields. In accordance with Eqs. (3.49), (3.50) and (3.72) we write

$$\Gamma_{M,K} = \text{diag}\left(\Gamma^{(-1)}, \Gamma^{(-2)}, \Gamma^{(+1)}, \Gamma^{(+2)}\right),$$

(3.73)

where now, the $\Gamma_{M,K}(z_3)$ are the solutions of the differential equations (cf. Eq. (3.65)):

$$\partial_z \Gamma_{M,K} = -s^{1/2} \Lambda_{M,N} \Gamma_{N,K} + I_{M,K} \delta(z_3),$$

(3.74)

such that $\Gamma_{M,K}(z_3) \to 0$ as $|z_3| \to \infty$ for all real $\alpha_1$ and $\alpha_2$. $I_{M,K}$ denotes the 4-by-4 unit matrix. In view of Eq. (3.72) it is easily verified that the diagonal elements $\Gamma^{(\pm \nu)}(z_3)$ of $\Gamma_{M,K}(z_3)$ are given by

$$\Gamma^{(-\nu)}(z_3) = -\exp(-s^{1/2} \gamma^{(-\nu)} x_3) H(-x_3),$$

(3.75)

and

$$\Gamma^{(\nu)}(z_3) = \exp(-s^{1/2} \gamma^{(\nu)} x_3) H(x_3).$$

(3.76)

Here, $H(x_3)$ denotes the Heaviside unit step function of argument $x_3$.

We assume that the source matrix $N_I$ contains only terms that correspond to a concentrated source of the electric or magnetic current type that acts at a single level. Later on, we shall consider diffusive electromagnetic fields generated by more complicated sources such as the rectangular loop source. Let the concentrated source be located at $s = w_s$. Without loss of generality we take $w_s = (0, 0, x_{3;S})$. The source matrix $N_I$ is then of the form

$$N_I = \hat{\phi}(s) \hat{k}_{\text{source}}(s) X_I \delta(x_3 - x_{3;S}),$$

(3.77)

where $\delta(x_3 - x_{3;S})$ is the one-dimensional dirac distribution acting at the source level, $\hat{\phi}(s)$ the Laplace transform of the source signature $\phi(t)$ and $X_I$ a 4-by-1 matrix that depends on the nature of the source and the Fourier transformation parameters $\alpha_1$.
and \( \alpha_3 \). From Eqs. (3.15)-(3.17) we find that the function \( \hat{k}_{\text{source}}(s) = s^{1/2} \) when the source is of the electric current type (i.e., \( \hat{K}_f = 0 \)) and \( k(s) = 1 \) when the source is of the magnetic current type (i.e., \( \hat{J}_f = 0 \)). By combining Eqs. (3.65), (3.73) and (3.77), we arrive at the following expression for \( W_N(x_3) \)

\[
W_N(x_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) \Gamma_{N,M}(x_3 - x_{3,s}) D^{-1}_{M,I} X_I. \tag{3.78}
\]

The transform-domain electromagnetic field matrix \( F_J \) is related to \( W_N \) via the linear transformation (3.63). Consequently,

\[
F_J(x_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) D_{J,N} \Gamma_{N,M}(x_3 - x_{3,s}) D^{-1}_{M,I} X_I. \tag{3.79}
\]

The right-hand side of (3.79) can be recognized as the superposition of four terms, each term corresponding to a (transform-domain) diffusive field. We shall refer to each of these diffusive fields as diffusive field constituents. The general shape of such a single diffusive field constituent is (cf. Eqs. (3.71), (3.75) and (3.76))

\[
F_J(x_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) B_j^{(\pm \tau)} \exp(-s^{1/2} \gamma^{(\pm \tau)}(x_3 - x_{3,s})), \tag{3.80}
\]

where

\[
B_j^{(\pm \tau)} = \pm b_j^{(\pm \tau)} g_i^{(\pm \tau)} X_I. \tag{3.81}
\]

In accordance with Eqs. (3.49), (3.50), (3.75) and (3.76), the two diffusive field constituents with superscripts \((-\tau)\) correspond to fields diffusing away from the source in the upward direction, while the other two diffusive field constituents with superscripts \((+\tau)\) correspond to fields diffusing away from the source in the downward direction. The use of the + and − signs with the superscripts in (3.80) and (3.81) enables us to indicate in an elegant manner whether a quantity corresponds to diffusion in the direction of increasing \( x_3 \) (downward) or to diffusion in the direction of decreasing \( x_3 \) (upward).

In Section 3.4 we shall discuss the transformation of the the right-hand side of (3.79) from the transform-domain back to the space-time domain. This transformation is carried out with aid of the Cagniard-De Hoop or modified Cagniard method.
3.3 Transform-domain diffusive field constituents

(de Hoop, 1960; de Hoop and Oristaglio, 1987) and applies to each diffusive field constituent separately. However, back transformation of the elements of $F_J$ will lead only to appropriate space-time domain expressions for the horizontal components of the electric field strength, not of the magnetic field strength. This is due to the inclusion of the factor $s^{1/2}$ with $\vec{H}_1$ and $\vec{H}_2$ in $F_J$, Eq. (3.9). For this reason, we rewrite Eq. (3.80) as the generalized diffusive field constituent

$$F(x_3) = \phi(s) \hat{k}_{source}(s) \hat{k}_{field}(s) B^{(\pm)} \exp(-s^{1/2} \gamma^{(\pm)}(x_3-x_3; s)),$$  

(3.82)

such that the factor $s^{1/2}$ present at the left-hand side of (3.80) with $F_3$ and $F_4$ is now included in the function $\hat{k}_{field}(s)$. From Eq. (3.9) we find that the function $\hat{k}_{field}(s) = 1$ when the electric field strength is considered and $\hat{k}_{field}(s) = s^{-1/2}$ when the magnetic field strength is considered. For notational simplicity we shall denote by $\hat{k}(s)$ the product of $\hat{k}_{source}(s)$ and $\hat{k}_{field}(s)$, i.e. $\hat{k}(s) = \hat{k}_{source}(s) \hat{k}_{field}(s)$.

The four possible functions $\hat{k}(s)$ are denoted as $\hat{k}^{(EJ)}(s)$, $\hat{k}^{(EK)}(s)$, $\hat{k}^{(HJ)}(s)$ and $\hat{k}^{(HK)}(s)$. The letters $E$, $H$, $J$ and $K$ refer to the electric field strength, magnetic field strength, source of the electric current type and source of the magnetic current type, respectively. These functions are given by

$$\hat{k}^{(EJ)}(s) = s^{1/2},$$  

(3.83)

$$\hat{k}^{(EK)}(s) = 1,$$  

(3.84)

$$\hat{k}^{(HJ)}(s) = 1,$$  

(3.85)

$$\hat{k}^{(HK)}(s) = s^{-1/2}.$$  

(3.86)

The transform-domain vertical components of the electric and magnetic field strength are related to the transform-domain horizontal field components via the linear algebraic equations (3.7) and (3.8). In view of Eqs. (3.82)-(3.86) it is obvious that the transform-domain vertical field components can also be written as generalized diffusive field constituents of the form (3.82). For the transform-domain vertical component of the electric field strength we have $F = \vec{E}_3$ with the corresponding
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\[ B^{(\pm r)} \text{ given by} \]
\[ B^{(\pm r)} = -\sigma_{3,1}B_1^{(\pm r)} - \sigma_{3,2}B_2^{(\pm r)} - (i\alpha_1 B_3^{(\pm r)} + i\alpha_2 B_4^{(\pm r)})/\sigma_{3,3}. \]  \hfill (3.87)

For the transform-domain vertical component of the magnetic field strength we have \( F = \bar{H}_3 \) with the corresponding \( B^{(\pm r)} \) given by

\[ B^{(\pm r)} = \mu_{3,1}B_4^{(\pm r)} - \mu_{3,2}B_3^{(\pm r)} + (i\alpha_1 B_2^{(\pm r)} - i\alpha_2 B_1^{(\pm r)})/\mu_{3,3}. \]  \hfill (3.88)

To conclude this section we consider the relation between the transform-domain reciprocity theorem and the diffusive field constituents of Eq. (3.80). In view of Eqs. (2.72) and (3.9) we have

\[ \varepsilon_{r,3,\rho} \delta_3(\tilde{E}_i^A \tilde{H}_p^B + \tilde{E}_i^B \tilde{H}_p^A) = 0 \implies \delta_3(F_i^A H_j^B) = 0, \]  \hfill (3.89)

where \( F_i^A = (\tilde{E}_1^A, \tilde{E}_2^A, s^{1/2} \tilde{H}_2^A, -s^{1/2} \tilde{H}_1^A)^T \) and \( F_j^B = (\tilde{E}_1^B, \tilde{E}_2^B, s^{1/2} \tilde{H}_2^B, -s^{1/2} \tilde{H}_1^B)^T \) are the transform-domain field matrices of some State A and the adjoint of State B, respectively. Following Section 2.7, State A and State B can be identified with two of the diffusive field constituents. Upon using Eqs. (2.65), (2.66) and (3.10) and replacing \( s^{1/2} \) by \(-s^{1/2}\) in the exponent of the expression for \( F_j^B \) (cf. Eq. (3.80)) in order to obtain \( F_j^B \), we end up with

\[ \delta_3(F_i^A H_j^B) = 0 \implies \]
\[ (\gamma^{(\pm v)} - \gamma^{(\pm r)}) b_j^{(\pm v)} H_i^B j_j^{(\pm r)} \exp(-s^{1/2}(\gamma^{(\pm v)} - \gamma^{(\pm r)})z_3) = 0. \]  \hfill (3.90)

From this equation we immediately recognize the orthogonality relation (3.55). Whereas in Section 3.2 we obtained the orthogonality properties of the eigenvectors \( b_j^{(\pm v)} \) of the system's matrix \( A_{i,j} \) by means of the similarity transformation (3.22), we now have shown explicitly that these orthogonality properties follow from the transform-domain field reciprocity theorem and are directly related to the existence of diffusion invariants.
3.4 The transformation back to the space-time domain

In this section we consider the transformation of \( F(\alpha_1, \alpha_2, z_3, s) \) as given by Eq. (3.82) from the transform domain back to the space-time domain. Without loss of generality we can assume the concentrated source to be located at the origin \( O \) of the Cartesian coordinate system. Inverse Fourier transformation of \( F(\alpha_1, \alpha_2, z_3, s) \) to the space-Laplace domain yields (cf. Eq. (2.34))

\[
F(\omega, s) = s \hat{\phi}(s) \hat{G}(\omega, s),
\]

(3.91)

where \( \hat{G}(\omega, s) \) is the space-Laplace domain Green’s function

\[
\hat{G}(\omega, s) = \frac{k(s)}{4\pi^2} \int_{-\infty}^{\infty} d\alpha_2 \int_{-\infty}^{\infty} \exp(-s^{1/2}(i\alpha_\mu x_\mu + z_3 \gamma^{(\pm\nu)})) B^{(\pm\nu)} d\alpha_1.
\]

(3.92)

In Eq. (3.92), the superscript + holds when \( z_3 > 0 \) and the superscript - holds when \( z_3 < 0 \). In view of Eq. (3.91) and Lerch’s theorem (Widder, 1946), the space-time domain expression for \( F(\omega, t) \) is obtained as the time derivative of the convolution of \( \phi(t) \) and \( G(\omega, t) \), i.e.,

\[
F(\omega, t) = \frac{\partial}{\partial t} \int_{0}^{t} \phi(\tau) G(\omega, t - \tau) d\tau.
\]

(3.93)

Here, \( G(\omega, t) \) is the inverse Laplace transform of \( \hat{G}(\omega, s) \). We shall cast the integral on the right-hand side of Eq. (3.92) in such a form that \( G(\omega, t) \) can be found by inspection. Equation (3.90) represents the contribution from a single diffusive field constituent, viz. either an upward diffusing field containing \( \gamma^{(-\nu)} \) and \( B^{(-\nu)} \) or a downward diffusing field containing \( \gamma^{(+\nu)} \) and \( B^{(+\nu)} \), depending on the location of the point of observation with respect to the location of the source.

We start with a change of integration variables in Eq. (3.92). We replace the Fourier transformation variables \( \alpha_1 \) and \( \alpha_2 \) by the polar variables of integration \( p \) and \( \psi \) defined through

\[
i \alpha_1 = p \cos(\theta + \psi),
\]

(3.94)
\[ i \alpha_2 = p \sin(\theta + \psi), \]  

(3.95)

with \(0 \leq p < \infty, \ 0 \leq \psi < 2\pi\). The angle \(\theta\) follows from the polar-coordinate specification of the point of observation, i.e.,

\[ x_1 = r \cos(\theta), \]  

(3.96)

\[ x_2 = r \sin(\theta), \]  

(3.97)

with \(0 \leq r < \infty, \ 0 \leq \theta < 2\pi\). Since \(d\alpha_1 d\alpha_2 = -pd\psi dp\) and \(i\alpha_2 x_\mu = p \cos(\psi)\), we can rewrite Eq. (3.92) as

\[ \hat{G}(s, \omega) = -\frac{k(s)}{4\pi^2} \int_0^{2\pi} d\psi \int_0^{\infty} \exp(-s^{1/2}(p \cos(\psi) + x_3 \tilde{\gamma}^{(\pm \nu)})) \tilde{B}^{(\pm \nu)} dp, \]  

(3.98)

in which \(\tilde{\gamma}^{(\pm \nu)}(p, \psi)\) and \(\tilde{B}^{(\pm \nu)}(p, \psi)\) have been obtained from \(\gamma^{(\pm \nu)}(\alpha_1, \alpha_2)\) and \(B^{(\pm \nu)}(\alpha_1, \alpha_2)\) by the substitutions in Eqs. (3.94) and (3.95). Since the integrand of Eq. (3.98) is a periodic function of \(\psi\) with periodicity \(2\pi\), we can decompose, if desired, the integral with respect to \(\psi\) on the right-hand side of this equation into the following two parts

\[ \int_0^{2\pi} d\psi \int_0^{\infty} \ldots dp = \int_{-\pi/2}^{\pi/2} d\psi \int_0^{\infty} \ldots dp + \int_{\pi/2}^{3\pi/2} d\psi \int_0^{\infty} \ldots dp. \]  

(3.99)

By performing the substitutions \(\psi \rightarrow \psi + \pi\) and \(p \rightarrow -p\) in the second integral on the right-hand side of Eq. (3.99), we achieve that, with the latter, \(\psi\) also takes on values between \(-\pi/2\) and \(\pi/2\), while, in view of Eqs. (3.94) and (3.95), the integrand itself is not altered. Hence, the two integrals can be taken together and we arrive at

\[ \hat{G}(s, \omega) = -\frac{k(s)}{4\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \left( \int_0^{\infty} + \int_{-\infty}^{0} \right) \]  

\[ \times \exp(-s^{1/2}(p \cos(\psi) + x_3 \tilde{\gamma}^{(\pm \nu)})) \tilde{B}^{(\pm \nu)} dp. \]  

(3.100)
Now, from Eqs. (3.45), (3.94) and (3.95) we conclude that \( \bar{\gamma}^{(\pm \nu)} \) satisfies the relation

\[
\bar{\gamma}^{(\pm \nu)}(p^*, \psi) = \bar{\gamma}^{(\pm \nu)*}(p, \psi),
\]

(3.101)

while \( \bar{\gamma}^{(\pm \nu)}(p, \psi) \) is real for \( p = 0 \). Since \( \bar{B}^{(\pm \nu)}(p, \psi) \) is just a function of \( p \) and \( \psi \) with real coefficients, it satisfies a similar relation as \( \bar{\gamma}^{(\pm \nu)}(p, \psi) \), i.e.,

\[
\bar{B}^{(\pm \nu)}(p^*, \psi) = \bar{B}^{(\pm \nu)*}(p, \psi).
\]

(3.102)

Thus, \( \bar{\gamma}^{(\pm \nu)} \) and \( \bar{B}^{(\pm \nu)} \) satisfy Schwarz's reflection principle. Consequently, the integrand at the right-hand side of Eq. (3.100) satisfies Schwarz's reflection principle as well. Using this property, we can combine the integral along the positive imaginary \( p \)-axis and the integral in the opposite direction along the negative imaginary \( p \)-axis and rewrite them as a single integral. Since \( C + C^* = 2\text{Re}(C) \), we end up with

\[
\hat{G}(\omega, s) = -\frac{k(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \text{Re} \left( \int_0^{i\infty} \exp(-s^{1/2}(p\cos(\psi) + x_{3\nu}\bar{\gamma}^{(\pm \nu)})) \bar{B}^{(\pm \nu)}(p)dp \right).
\]

(3.103)

Next, we want to carry out the integration with respect to \( p \) along a certain contour in the complex \( p \)-plane that deviates from the imaginary \( p \)-axis. To this end, we extend the definition of the relevant integrand of Eq. (3.103) into the complex \( p \)-plane by analytical continuation away from the imaginary \( p \)-axis. For this, we need the locations of the singularities of the integrand in the complex \( p \)-plane. The latter coincide with the singularities of \( \bar{\gamma}^{(\pm \nu)} \).

For the properties of \( \bar{\gamma}^{(\pm \nu)}(p, \psi) \) in the complex \( p \)-plane we refer to Appendix A. From this Appendix we learn that the only singularities of \( \bar{\gamma}^{(\pm \nu)}(p, \psi) \) are: (i) at least four branch points located on the real \( p \)-axis where the \( \bar{\gamma}^{(\pm \nu)} \) of up- and downwardly diffusing fields have equal values and (ii) possible branch points off the real \( p \)-axis where the \( \bar{\gamma}^{(\pm \nu)} \) of either two upwardly or two downwardly diffusing fields have equal values. In order to keep the four branches \( \bar{\gamma}^{(\pm \nu)} \) single-valued throughout the complex \( p \)-plane we introduce branch cuts, either as straight line segments along the real \( p \)-axis joining two corresponding branch points, or as straight lines from the relevant branch point to infinity. Furthermore, the \( \bar{\gamma}^{(\pm \nu)} \) have real values on parts of the real \( p \)-axis, including an interval that contains the origin \( p = 0 \). The function
\( B^{(\pm \nu)}(p, \psi) \) that occurs in the integrand of Eq. (3.103) also has to be scrutinized for possible singularities. From its definition, Eq. (3.81), it is obvious that the only singularities in the complex \( p \)-plane of \( B^{(\pm \nu)}(p, \psi) \) are the branch points of the corresponding \( \tilde{\gamma}^{(\pm \nu)} \).

An understanding of the behavior of the \( \tilde{\gamma}^{(\pm \nu)}(p, \psi) \) on the complex \( p \)-plane and the impact of this behavior on the mathematical treatment of contour deformation can be made much easier by adapting the approach of Riemann for describing and understanding the behavior of multivalued analytic functions (Riemann, 1953; Springer, 1981). If we consider the \( \gamma^{(\pm \nu)}(p, \psi) \) as the branches of the function \( \gamma(p, \psi) \) (cf. Eq. (3.42)), then \( \tilde{\gamma}(p, \psi) \) is four-valued on the complex \( p \)-plane. The basic idea of Riemann was to modify in a suitable manner the domain of definition of the multivalued function \( \gamma(p, \psi) \) (i.e., the complex \( p \)-plane), such that a single-valued function \( \tilde{\gamma}(p, \psi) \) is obtained on the new domain \( \mathcal{R} \), \( p \in \mathcal{R} \). \( \mathcal{R} \) is called a Riemann surface. The Riemann surface \( \mathcal{R} \) can be represented as a four-sheeted (covering) surface of the complex \( p \)-plane in a way such that each sheet can be identified with a copy of the complex \( p \)-plane on which \( \tilde{\gamma}(p, \psi) \) equals one of the branches of \( \gamma(p, \psi) \). An interconnection between two sheets of the Riemann surface takes place only along a common branch cut of corresponding branches of \( \gamma(p, \psi) \). Crossing a branch cut in the complex \( p \)-plane implies going from one sheet of the Riemann surface \( \mathcal{R} \) to another (see Figure 3.1).

The integral along the positive imaginary \( p \)-axis in Eq. (3.103) is evaluated with the aid of the modified Cagniard method which consists of deforming the path of integration away from the imaginary \( p \)-axis into the complex \( p \)-plane in such a way such that the resulting expression for \( \tilde{G}(z, s) \) can be recognized as the Laplace transform of some known space-time function \( G(z, t) \), without making use of an inverse Laplace transformation (De Hoop, 1960). The new path of integration in the complex \( p \)-plane is called the modified Cagniard contour. Along this contour, the argument \( p r \cos(\psi) + x_3 \tilde{\gamma}^{(\pm \nu)} \) of the exponential function in the integrand of Eq. (3.103) should be real and positive. The modified Cagniard contour is parametrized by the variable \( \tau \), such that for fixed \( \psi \), \(-\pi/2 \leq \psi < \pi/2 \) we have

\[
p r \cos(\psi) + x_3 \tilde{\gamma}^{(\pm \nu)}(p, \psi) = \tau, \quad \text{with } \tau \text{ real and positive.} \tag{3.104}
\]

Note that \( r \cos(\psi) \geq 0 \) in the interval of \( \psi \)-integration. For every value of \( \psi \) and any
location of the observation point with respect to the source level there are two different modified Cagniard contours. One contour corresponds to \( \nu = 1 \), the other contour corresponds to \( \nu = 2 \). Along the modified Cagniard contour the exponential term in the integrand of Eq. (3.103) reduces to \( \exp(-s^{1/2}\tau) \), which function can easily be transformed back to the space-time domain.

The modified Cagniard contour \( p(\tau) \)

Since \( \tilde{\gamma}^{(\pm)}(p, \psi) \) is real-valued along some parts of the real \( p \)-axis including an interval that contains the origin \( p = 0 \) of the complex \( p \)-plane, it is easy to see that the modified Cagniard contour starts at \( p = 0 \) and then follows a part of the real \( p \)-axis. At some point \( p = p_0(\psi) \), the modified Cagniard contour will leave the real \( p \)-axis to finally approach a complex asymptote as \( \tau \to \infty \). The value of \( \tau \) at \( p = 0 \)
is denoted as \( \tau = T_{\text{min}} \). Its value follows from Eq. (3.104) and is given by

\[
T_{\text{min}} = x_3 \tilde{\gamma}^{(\pm)}(0, \psi) = x_3 \gamma^{(\pm)}(0, 0). \tag{3.105}
\]

Note that \( T_{\text{min}} \) is independent of \( \psi \) and is always greater than 0. Since we want \( \tau \) to increase monotonously along the path, the modified Cagniard contour leaves the real \( p \)-axis at the point \( p = p_0(\psi) \) where \( \tau \) reaches a local maximum value. At this point, the derivative of \( \tau \) with respect to \( p \) equals zero, i.e.,

\[
\frac{\partial \tau}{\partial p} = r \cos(\psi) + x_3 \frac{\partial \gamma^{(\pm)}(p, \psi)}{\partial p} = 0 \quad \text{for} \quad p = p_0(\psi) \in \mathbb{R}. \tag{3.106}
\]

Whether \( p_0(\psi) \) is positive or negative depends on the value of \( \frac{\partial \tau}{\partial p} \) at \( p = 0 \):

\[
\begin{cases}
\text{if} \quad \frac{\partial \tau}{\partial p} > 0 \quad \text{at} \quad p = 0 \quad \text{then} \quad p_0(\psi) > 0, \\
\text{if} \quad \frac{\partial \tau}{\partial p} < 0 \quad \text{at} \quad p = 0 \quad \text{then} \quad p_0(\psi) < 0.
\end{cases} \tag{3.107}
\]

The value of \( \tau \) at \( p = p_0(\psi) \) is denoted as \( \tau = T(\psi) \). Its value follows from Eq. (3.104) and is given as

\[
T(\psi) = p_0(\psi) r \cos(\psi) + x_3 \tilde{\gamma}^{(\pm)}(p_0(\psi), \psi), \quad T(\psi) \geq T_{\text{min}}. \tag{3.108}
\]

By using Eqs. (3.94) and (3.95), we observe that \( p_0(\psi) \) satisfies the relationship

\[
p_0(\pi/2) = -p_0(-\pi/2). \tag{3.109}
\]

Upon substituting Eq. (3.109) into Eqs. (3.104) and (3.106) we find that the value of \( T(\psi) \) for \( \psi = \pi/2 \) is equal to its value for \( \psi = -\pi/2 \). We denote this value as \( T_{\text{ver}} \). From Eq. (3.108) we have

\[
T_{\text{ver}} = T(\pi/2) = x_3 \tilde{\gamma}^{(\pm)}(p_0(\pi/2), \pi/2) = T(-\pi/2) = x_3 \tilde{\gamma}^{(\pm)}(p_0(-\pi/2), -\pi/2). \tag{3.110}
\]
3.4 The transformation back to the space-time domain

From Eqs. (3.106)-(3.107) we infer that for real values of \( p \), \( \partial^2 \tau / \partial p^2 \neq 0 \) at \( p = p_0(\psi) \). Consequently, \( p = p_0(\psi) \) is a saddle point of the real part of \( p \tau \cos \psi + x_3 \bar{\gamma}^{(\pm \nu)}(p, \psi) \) as a function of \( p \). Hence, in order that \( \tau \) increases monotonously along the path, the modified Cagniard contour must leave the real \( p \)-axis perpendicular at \( p = p_0(\psi) \) to finally reach a complex asymptote in the upper half of the complex \( p \)-plane. Occasionally, the modified Cagniard contour can turn back to the real \( p \)-axis, follow a part of it and leave the real \( p \)-axis again. This situation may arise in the case where the relevant \( \bar{\gamma} \) has three branch points on both positive and negative parts of the real \( p \)-axis (see Appendix A, Figures A.3 and A.4). In this case \( \bar{\gamma} \) is real-valued along three distinct parts of the real \( p \)-axis.

Further, in case of branch points off the real \( p \)-axis, the modified Cagniard contour can make a turn around such an off-axis branch point to cross itself again. In fact, in the latter case the modified Cagniard contour meets an off-axis branch cut and goes from one Riemann sheet of the Riemann surface \( \mathcal{R} \) to another (e.g., from the Riemann sheet connected with \( \bar{\gamma}^{(+1)} \) to the Riemann sheet connected with \( \bar{\gamma}^{(+2)} \)).

The complex asymptote of \( p(\tau) \) as \( \tau \to \infty \) is determined by the asymptotic behavior of \( \bar{\gamma}^{(\pm \nu)}(p, \psi) \) as \( |p| \to \infty \). From Appendix A we learn that \( \bar{\gamma}^{(\pm \nu)}(p, \psi) \) is approximately proportional to \( p \) when \( |p| \to \infty \). In view of Eqs. (3.48)-(3.50) we obtain

\[
\bar{\gamma}^{(\pm \nu)}(p, \psi) = -i p C^{(\pm \nu)}(\psi) + \mathcal{O}(|p|^{-1}) \quad \text{as } |p| \to \infty, \tag{3.111}
\]

where

\[
C^{(-\nu)}(\psi) = -C^{(+\nu)^*}(\psi), \quad \text{Re}(C^{(+\nu)}(\psi)) > 0. \tag{3.112}
\]

The \( C^{(\pm \nu)}(\psi) \) follow from the asymptotic behavior of the determinantal equation Eq. (3.42), Eq. (A.26) as \( |p| \to \infty \). In Appendix A we describe how the \( C^{(\pm \nu)} \) can be obtained explicitly from this fourth-order polynomial equation. Using Eq. (3.111), Eq. (3.104) reduces asymptotically to

\[
p \left( r \cos(\psi) - i x_3 C^{(\pm \nu)}(\psi) \right) \sim \tau \quad \text{as } \tau \to \infty, \tag{3.113}
\]
from which we obtain

$$p \sim \tau \left( r \cos(\psi) - i x_3 C^{(\pm \nu)}(\psi) \right)^{-1} \quad \text{as } \tau \to \infty. \quad (3.114)$$

Further we know that for each of the diffusive field constituents that contributes to the solution we always have $\Re(x_3 C^{(\pm \nu)}(\psi)) > 0$. Consequently, the modified Cagniard contour has as its asymptote a straight line through the origin that is located in the upper-half of the complex $p$-plane. In Figure 3.2 we have depicted an example of a modified Cagniard contour for an arbitrarily anisotropic medium.

![Diagram](image)

*Figure 3.2: Example of the modified Cagniard contour for an arbitrary anisotropic medium. The dashed line indicates the asymptote as $\tau \to \infty$."

From Eqs. (3.94)-(3.95), (3.101) and (3.102) we conclude that the $\gamma^{(\pm \nu)}(p, \psi)$ and the $\tilde{B}^{(\pm \nu)}(p, \psi)$ satisfy the relations

$$\gamma^{(\pm \nu)}(-p^*, -\pi/2) = \gamma^{(\pm \nu)*}(p, \pi/2), \quad (3.115)$$

$$\tilde{B}^{(\pm \nu)}(-p^*, -\pi/2) = \tilde{B}^{(\pm \nu)*}(p, \pi/2). \quad (3.116)$$
3.4 The transformation back to the space-time domain

Hence, in view of Eq. (3.104) it is easily seen that the modified Cagniard contours for \( \psi = -\pi/2 \) and \( \psi = \pi/2 \) are symmetric with respect to the imaginary \( p \)-axis. This implies that the \( C(\pm p)(\pm \pi/2) \) which determine the complex asymptote of the modified Cagniard contour for \( \psi = \pm \pi/2 \), satisfy the relation

\[
C(\pm p)(-\pi/2) = C(\pm p)^*(\pi/2). \tag{3.117}
\]

Finally, it is easily verified that the real part of the left-hand side of Eq. (3.104) is always greater than zero and proportional to \(|p|\) along the circular arc joining the positive imaginary \( p \)-axis and the modified Cagniard contour at infinity.

The positive imaginary \( p \)-axis, the two modified Cagniard contours (for \( \nu = 1 \) and \( \nu = 2 \)) and the circular arcs at infinity form two closed loops on the Riemann surface \( \mathcal{R} \). The two closed loops are not necessarily located on different sheets of the Riemann surface. In case of branch points off the real \( p \)-axis, each of the two loops can be located partially on one Riemann sheet and partially on the other. Occasionally, the positive imaginary \( p \)-axis, the two modified Cagniard contours and the circular arcs at infinity form a single closed loop on the Riemann surface. Hence, we conclude that the singularities of the integrand of Eq. (3.103) form no obstruction in the process of contour deformation as long as we do sum the contributions from the two relevant diffusive field constituents (see also Van der Hidgen, 1987).

Space-time expression for the Green’s function \( G(x, t) \)

By virtue of Jordan’s lemma, Cauchy’s theorem and the previous results, we conclude that the integral along the positive imaginary \( p \)-axis of Eq. (3.103) is equal to the integral along the modified Cagniard contour. Upon introducing \( \tau \) as the variable of integration, we obtain

\[
\hat{G}(x, s) = -\frac{k(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \left( \int_{T_{\min}}^{T(\psi)} + \int_{T(\psi)}^{\infty} \right) \nonumber \times \exp(-s^{1/2}\tau) \text{Re}(B(\pm p) \frac{\partial p}{\partial \tau}) d\tau. \tag{3.118}
\]

Note that \( \tau = T(\psi) \) is a singular point of the integrand of Eq. (3.118) since \( \partial p/\partial \tau \) is infinite for this value of \( \tau \). From Eqs. (3.104) and (3.106), however, we conclude that
this singularity is a square-root singularity and hence, is integrable. For notational simplicity we shall write Eq. (3.118) as

$$\hat{G}(\omega, s) = \frac{-\hat{k}(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_{T_{\min}}^{\infty} \exp(-s^{1/2}\tau) \text{Re}(\bar{B}(\pm\nu) \frac{\partial p}{\partial \tau}) \, d\tau. \quad (3.119)$$

With Eq. (3.119) we have now arrived at the position where $G(\omega, t)$ can be found by inspection. This is possible since the part of $\hat{G}(\omega, s)$ that depends on the Laplace transformation parameter $s$ is given by $\hat{k}(s)\exp(-s^{1/2}\tau)$ while the rest of the integrand is $s$-independent. Since $\tau$ is real and positive ($T_{\min} > 0$) this function $\hat{k}(s)\exp(-s^{1/2}\tau)$ can be recognized as the Laplace transform of some time function $k(t, \tau)$. Consequently, the space-time domain expression for $G(\omega, t)$ is obtained as

$$G(\omega, t) = \left\{ \begin{array}{ll}
0, & t < 0, \\
-\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_{T_{\min}}^{\infty} k(t, \tau) \text{Re}(\bar{B}(\pm\nu) \frac{\partial p}{\partial \tau}) \, d\tau, & t \geq 0. 
\end{array} \right. \quad (3.120)$$

Note that in the final expression for $G(\omega, t)$ we have to sum the contributions from $\nu = 1$ and $\nu = -1$. The time function $k(t, \tau)$ follows from the relation

$$k(t, \tau) = \mathcal{L}^{-1}\left( \hat{k}(s)\exp(-s^{1/2}\tau) \right), \quad (3.121)$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transformation. To each of the possible functions $\hat{k}(s)$ there corresponds a different time function $k(t, \tau)$. In accordance with Eqs. (3.83)-(3.86), we denote these functions as $k^{(EJ)}(t, \tau)$, $k^{(EK)}(t, \tau)$, $k^{(HJ)}(t, \tau)$ and $k^{(HK)}(t, \tau)$ respectively. Explicit expressions for these four time functions are found as (Fodor, 1965)

$$k^{(EJ)}(t, \tau) = \left( \frac{\tau^2}{2t} - 1 \right) \frac{\exp(-\tau^2/(4t))}{2t\sqrt{\pi t}} H(t), \quad (3.122)$$

$$k^{(EK)}(t, \tau) = \tau \frac{\exp(-\tau^2/(4t))}{2t\sqrt{\pi t}} H(t), \quad (3.123)$$
3.5 Late-time behavior of the transient diffusive electromagnetic field

\[ k^{(HJ)}(t, \tau) = \tau \frac{\exp(-\tau^2/(4t))}{2t\sqrt{\pi t}} H(t), \quad (3.124) \]
\[ k^{(HK)}(t, \tau) = \frac{\exp(-\tau^2/(4t))}{\sqrt{\pi t}} H(t). \quad (3.125) \]

Here, \( H(t) \) denotes the Heaviside unit step function of argument \( t \). Note that
\[ k^{(HJ)}(t, \tau) = -\frac{\partial k^{(HK)}(t, \tau)}{\partial \tau}, \quad (3.126) \]
\[ k^{(EJ)}(t, \tau) = -\frac{\partial k^{(EK)}(t, \tau)}{\partial \tau}. \quad (3.127) \]

3.5 Late-time behavior of the transient diffusive electromagnetic field

In this section we investigate the asymptotic behavior of the transient diffusive electromagnetic field for late times, i.e.

\[ G(x, t) \quad \text{as} \quad t \to \infty. \]

Particular interest will be paid to the case where the source is of the electric current type. Knowledge of the late-time limit is of importance when considering the secondary electromagnetic field due to an electric current current switch-off excitation. In this case, the Green's function prior to the instant where the electric current is switched off has to be taken into account in Eq. (3.93). This is just the indicated limit. Here, we shall consider only the late-time behavior of the magnetic field strength. The same procedure can be used to compute the late-time behavior of the electric field strength.

In view of Eq. (3.120), we write the space-time domain expression for the Green's
function \( G(\omega, t) \) as

\[
G(\omega, t) = -\frac{1}{2\pi^2} \int_{\tau_{\text{min}}}^{\tau'} k(t, \tau) g(\tau) \, d\tau,
\]

(3.128)

where

\[
g(\tau) = \int_{-\pi/2}^{\pi/2} \text{Re}(\bar{B}(z^\nu) p \frac{\partial p}{\partial r}) \, d\psi.
\]

(3.129)

Note that \( g(\tau_{\text{min}}) = 0 \) since \( p = 0 \) for \( \tau = \tau_{\text{min}} \) (cf. Eq. (3.105)). With the evaluation of \( \lim_{t \to \infty} G(\omega, t) \) it proves to be advantageous to rewrite Eq. (3.128) as the sum of two integrals, one on the interval \( \tau_{\text{min}} \leq \tau < \tau' \), \( \tau' > \tau_{\text{BD}} \) and one on the interval \( \tau' \leq \tau < \infty \), i.e.,

\[
G(\omega, t) = -\frac{1}{2\pi^2} \int_{\tau_{\text{min}}}^{\tau'} k(t, \tau) g(\tau) \, d\tau - \frac{1}{2\pi^2} \left( \int_{\tau'}^{\infty} k(t, \tau) g(\tau) \, d\tau \right),
\]

(3.130)

Here, we assume that \( \tau' \gg \tau_{\text{BD}} \). Only on the first interval of \( \tau \)-values, \( g(\tau) \) possibly has square-root singularities, all of which are integrable. In our further analysis we shall consider separately the asymptotic behavior of the two integrals on the right-hand side of Eq. (3.130) as \( \tau \to \infty \).

Since with the first integral on the right-hand side of Eq. (3.130) the interval of integration is finite and independent of the time \( t \), we can replace the exponential term of \( k(t, \tau) \) by a Taylor expansion about \( t = \infty \) (cf. Eq. (3.124)), i.e.,

\[
k(t, \tau) = \frac{1}{2t\sqrt{\pi}t} \left\{ \tau - \frac{\tau^3}{4t} + \cdots \right\}.
\]

(3.131)

Substitution of the Taylor expansion (3.131) of \( k(t, \tau) \) into the first integral on the right-hand side of Eq. (3.130) yields:

\[
\int_{\tau_{\text{min}}}^{\tau'} k(t, \tau) g(\tau) \, d\tau = \frac{1}{2t\sqrt{\pi}t} \int_{\tau_{\text{min}}}^{\tau'} \tau g(\tau) \, d\tau + O(t^{-5/2}) \quad \text{as} \quad t \to \infty.
\]

(3.132)
3.5 Late-time behavior of the transient diffusive electromagnetic field

In order to investigate the asymptotic behavior of the second integral on the right-hand side of Eq. (3.130) as \( t \to \infty \), we shall write \( k(t, \tau) \) as the product of a function that depends only on the ratio \( \tau / \sqrt{4t} \) and a function that depends only on the time \( t \). Further we introduce the new variable of integration \( u \) given by \( u = \tau / \sqrt{4t} \). In view of Eq. (3.124) we rewrite \( k(t, \tau) \) as

\[
k(t, \tau) = \frac{\tau \exp(-\tau^2/(4t))}{2t \sqrt{\pi t}} H(t) = \frac{k(\tau/\sqrt{4t})}{t \sqrt{\pi}} H(t).
\]  

(3.133)

Here, \( H(t) \) denotes the Heaviside unit step function of argument \( t \). From Eq. (3.133) we find that \( k(\tau/\sqrt{4t}) = \bar{k}(u) \) is given by:

\[
\bar{k}(u) = u \exp(-u^2).
\]  

(3.134)

Upon performing the change of the variable of integration from \( \tau \) to \( u \) in the relevant integral of Eq. (3.130) we arrive at

\[
\int_{T'}^{\infty} k(t, \tau) g(\tau) \, d\tau = \frac{2}{\sqrt{\pi}} \int_{\tau/\sqrt{4t}}^{\infty} \bar{k}(u) \frac{g(u\sqrt{4t})}{\sqrt{t}} \, du.
\]  

(3.135)

We now make use of the fact that the function \( g(\tau) \) we consider here – i.e. corresponding to the magnetic field generated by an electric current source – is an algebraic function of \( \tau \) having the following important property

\[
\lim_{t \to \infty} \frac{g(u\sqrt{4t})}{\sqrt{t}} = u \, S(\omega) \quad \text{for fixed value of } u.
\]  

(3.136)

In fact, Eq. (3.136) states that \( g(\tau) \) is proportional to \( \tau \) as \( \tau \to \infty \). We shall prove this property in the next subsection. \( S(\omega) \) is independent of \( u \) and \( t \), and depends only on the point of observation relative to the location of the source, the material parameters \( \sigma_{k,\nu} \) and \( \mu_{j,\nu} \) that characterize the electrical properties of the homogeneous, anisotropic medium and the component of the magnetic field strength that is considered.

We have now arrived at the position that the behavior of \( G(\omega, t) \) as \( t \to \infty \) can
be obtained. From the indicated behavior of \( g(\tau) \) as \( \tau \to \infty \), we have that

\[
\int_{T'/\sqrt{4t}}^{\infty} \frac{k(u)}{t^\lambda} \frac{g(u\sqrt{4t})}{u^\lambda} du = S(\omega) \int_{T'/\sqrt{4t}}^{\infty} \frac{k(u)}{u} du
\]

\[
= S(\omega) \left( \frac{\sqrt{\pi}}{2} - \frac{T'^3}{24t\sqrt{t}} \right) + O(t^{-5/2}) \quad \text{as } t \to \infty,
\]

(3.137)

provided that \( T' > T_{BD} \) and \( T'/\sqrt{4t} \ll 1 \). Upon combing Eqs. (3.132) and (3.137), we obtain for the asymptotic behavior of the transient diffusive magnetic field Green's function \( G(\omega, t) \) as \( t \to \infty \) the following expression:

\[
G(\omega, t) = G^{(0)}_\infty(\omega) + \frac{G^{(1)}(\omega)}{2t\sqrt{\pi t}} + O(t^{-5/2}) \quad \text{as } t \to \infty,
\]

(3.138)

where

\[
G^{(0)}_\infty(\omega) = -\frac{S(\omega)}{2\pi^2},
\]

(3.139)

\[
G^{(1)}(\omega) = S(\omega) \frac{T'^3}{12 \pi^2} - \frac{1}{2\pi^2} \int_{T_{\min}}^{T'} \tau g(\tau) d\tau.
\]

(3.140)

From Eq. (3.137) we observe that the term proportional to \( t^{-3/2} \) (and higher order terms) in the asymptotic behavior of \( G(\omega, t) \) as \( t \to \infty \) can only be determined with a full knowledge of \( g(\tau) \) for all values of \( \tau > T_{\min} \).

Asymptotic behavior of \( g(\tau) \) as \( \tau \to \infty \)

In this subsection we investigate in detail the function \( g(\tau) \) for large values of \( \tau \), i.e. \( \tau > T_{BD} \). According to Eq. (3.129), the function \( g(\tau) \) is given by

\[
g(\tau) = \int_{-\pi/2}^{\pi/2} \text{Re}(\tilde{B}(\tau)(p, \psi) p \frac{\partial p}{\partial \tau}) d\psi.
\]

(3.141)

Note that with the integral on the right-hand side of Eq. (3.141) the integrand must be considered as a function of \( \psi \) while \( \tau \) is held fixed at some (large) value. The
asymptotic behavior of $g(\tau)$ as $\tau \to \infty$ is determined by the asymptotic behavior of $p(\tau)$, $\frac{\partial p}{\partial \tau}$ and $B^{(\pm \nu)}(p, \psi)$ as $\tau \to \infty$. We shall consider the asymptotic behavior of these functions in this order.

As $\tau \to \infty$, the modified Cagniard contour $p(\tau, \psi)$ approximates its straight complex asymptote in the complex $p$-plane (cf. Eq. (3.114), i.e.,

$$p(\tau) \sim \tau \left( \tau \cos(\psi) - i\sigma \beta^{(\pm \nu)}(\psi) \right) \quad \text{as} \quad \tau \to \infty.$$  

(3.142)

From this expression we easily obtain the derivative of $p(\tau, \psi)$ with respect to $\tau$ for large values of this parameter as

$$\frac{\partial p}{\partial \tau} = \left( \tau \cos(\psi) - i\sigma \beta^{(\pm \nu)}(\psi) \right)^{-1} \quad \text{as} \quad \tau \to \infty.$$  

(3.143)

Now, $B^{(\pm \nu)}(p, \psi)$ represents one of the elements of $B^{(\pm \nu)} = \pm \bar{Y}^{(\pm \nu)} \bar{f}^{(\pm \nu)} X_j$ (cf. Eq. (3.81)) corresponding to the magnetic field strength (i.e., $J = 1$ or $J = 2$). If we take for the source a horizontal electric current dipole, then we find by close examination of the equations that determine $B^{(\pm \nu)}$, $\bar{f}^{(\pm \nu)}$ and $X_j$ (cf. Eqs. (3.16)-(3.17), (3.53), (3.71) and (3.77)) that

$$B^{(\pm \nu)}(p, \psi) = a^{(\pm \nu)}(\psi) + O(p^{-2}) \quad \text{as} \quad |p| \to \infty,$$  

(3.144)

Upon substituting Eq. (3.142)-(3.144) into Eq. (3.141), we obtain the following expression for the function $g(\tau)$ as $\tau \to \infty$

$$g(\tau) = \tau \int_{-\pi/2}^{\pi/2} \operatorname{Re} \left( \frac{a^{(\pm \nu)}(\psi)}{\tau \cos(\psi) - i\sigma \beta^{(\pm \nu)}(\psi)} \right) d\psi \quad \text{as} \quad \tau \to \infty,$$  

(3.145)

i.e., $g(\tau)$ is proportional to $\tau$ as $\tau \to \infty$. This result has been used in Eq. (3.136).
Diffusive electromagnetic fields in layered anisotropic media

4.1 Introduction

In this chapter we consider transient diffusive electromagnetic fields in layered anisotropic media. These electromagnetic fields are generated by a localized impulsive source situated at the interface between two adjacent layers (subdomains) of the stratified configuration. The domain occupied by the \( m \)th layer is denoted as \( D_m, m = 1, \ldots, ND \). The electromagnetic properties of the anisotropic material of the subdomain \( D_m \) are specified by its tensorial conductivity \( \sigma^{(m)}_{kr} \) and its tensorial permeability \( \mu^{(m)}_{js} \) (see Table 2.1). The source is located at the level \( z_3 = z_{3; s} \). Figure 4.1 shows this.

\[ \begin{align*}
D_{s-1} & \quad \{\sigma^{(s-1)}_{kr}, \mu^{(s-1)}_{js}\} \\
D_s & \quad \{\sigma^{(s)}_{kr}, \mu^{(s)}_{js}\} \\
D_{s+1} & \quad \{\sigma^{(s+1)}_{kr}, \mu^{(s+1)}_{js}\}
\end{align*} \]

\[ \begin{align*}
z_3 & = z_{3; s-1} \\
z_3 & = z_{3; s}
\]

Figure 4.1: The layered configuration with the localized source.
4.2 Transform-domain diffusive field constituents in layered media

In this section we consider the solution of the transform-domain field equations (2.22) and (2.23) that describe the transform-domain electromagnetic field in the layered configuration of Figure 4.1. In our method of solution we follow the same procedure as in Section 3.1, i.e., we eliminate from the transform-domain field equations those field components that are discontinuous across an interface between two media with different electromagnetic properties. The resulting differential equations are written in the form of a first-order ordinary matrix differential equation (cf. Eq. (3.10)). The field matrix $F_j$ representative of the transform-domain electromagnetic field in the subdomain $D_m$ is denoted by $F_j^{(m)}$ and defined as (cf. Eq. (3.9))

$$F_j^{(m)} = (\tilde{E}_1^{(m)}, \tilde{E}_2^{(m)}, s^{1/2} \tilde{H}_2^{(m)}, -s^{1/2} \tilde{H}_1^{(m)})^T.$$  \hspace{1cm} (4.1)

The first-order linear matrix differential equation satisfied by $F_j^{(m)}$ is

$$\partial_3 F_j^{(m)} = -s^{1/2} A_{i,j}^{(m)} F_j^{(m)} \quad \text{for} \quad x_{3:m-1} < x_3 < x_{3:m}, \quad m=2,\ldots, ND-1$$ \hspace{1cm} (4.2)

while

$$\partial_3 F_j^{(1)} = -s^{1/2} A_{i,j}^{(1)} F_j^{(1)} \quad \text{for} \quad -\infty < x_3 < x_{3:1}, \quad (4.3)$$

$$\partial_3 F_j^{(ND)} = -s^{1/2} A_{i,j}^{(ND)} F_j^{(ND)} \quad \text{for} \quad x_{3:ND-1} < x_3 < \infty. \quad (4.4)$$

In Eq. (4.2), $A_{i,j}^{(m)}$ is the system’s matrix corresponding to the anisotropic medium in the subdomain $D_m$; it is given by Eqs. (3.11)-(3.14).

Across a source-free interface where $\sigma_{k,r}$ and/or $\mu_{j,p}$ show a jump discontinuity, the horizontal components of the electric and magnetic field strengths are to be continuous. In view of Eq. (4.1) this implies that the electromagnetic field matrix
4.2 Transform-domain diffusive field constituents in layered media

$F_j$ must be continuous across such an interface

$$\lim_{x_3 \downarrow x_{3;m}} F_j^{(m+1)} - \lim_{x_3 \uparrow x_{3;m}} F_j^{(m)} = 0 \quad \text{for} \quad m \neq s. \quad (4.5)$$

At the source level $x_3 = x_{3;s}$ where the localized source is situated, $F_j$ jumps by a finite amount. From Eq. (3.10) it follows that for $x_3 = x_{3;s}$, $F_j$ shows a jump discontinuity of magnitude $\tilde{N}_J$ (cf. Eqs. (3.15)-(3.17))

$$\lim_{x_3 \downarrow x_{3;s}} F_j^{(s+1)} - \lim_{x_3 \uparrow x_{3;s}} F_j^{(s)} = \tilde{N}_J. \quad (4.6)$$

Furthermore, $F_j^{(1)}$ and $F_j^{(ND)}$ must satisfy the conditions

$$F_j^{(1)}(x_3) = 0 \quad \text{as} \quad x_3 \to -\infty, \quad (4.7)$$

$$F_j^{(ND)}(x_3) = 0 \quad \text{as} \quad x_3 \to \infty. \quad (4.8)$$

To reveal the structure of the solution of Eqs. (4.2)-(4.8) we carry out a linear transformation on each of the ND electromagnetic field matrices $F_j^{(m)}$ and, through it, we want to arrive at a field-vector-formalism in which a decomposition of $F_j^{(m)}$ into up- and downward diffusing fields is manifest. Let $W_N^{(m)}$ be the 4-by-1 matrix that is related to $F_j^{(m)}$ via the linear transformation (cf. Eq. (3.63))

$$F_j^{(m)} = D_{J,N}^{(m)} W_N^{(m)}, \quad (4.9)$$

where the composition matrix $D_{J,N}^{(m)}$ is subject to a convenient choice. In view of the results of Section 3.3 we take for $D_{J,N}^{(m)}$ the eigencolumn matrix of the system matrix $A_{J,F}^{(m)}$, i.e.,

$$D_{J,N}^{(m)} = \begin{pmatrix} b_j^{(m;-1)} & b_j^{(m;-2)} & b_j^{(m;+1)} & b_j^{(m;+2)} \end{pmatrix}, \quad (4.10)$$
where the $\delta_j^{(m;\pm\nu)}$ denote the four eigenvectors of the system matrix $A_{i,j}^{(m)}$. For later convenience we denote the elements of $W_N^{(m)}$ as $W_U^{(m;\pm)}$ with the convention

$$W_N^{(m)} = \left(W_1^{(m;-)}, W_2^{(m;-)}, W_1^{(m;+)}, W_2^{(m;+)}\right)^T.$$

(4.11)

The relation inverse to Eq. (4.9) is

$$W_M^{(m)} = D_{M,I}^{(m)-1} F_I^{(m)}.$$  

(4.12)

From the orthogonality of the eigenvectors of $A_{i,j}^{(m)}$ (cf. Eq. (3.69)) we obtain the following explicit expression for the decomposition inverse matrix $D_{M,I}^{(m)-1}$ of $D_{J,N}^{(m)}$

$$D_{M,I}^{(m)-1} = L_{M,N} D_{J,N}^{(m)} H_{J,I}^\delta,$$

(4.13)

where the symmetric involutive matrix $L_{M,N}$ is given by Eq. (3.60). Equation (4.12) is of great importance since it allows an easy and direct computation of each of the $ND$ eigenrow matrices $D_{M,I}^{(m)-1}$ from the $D_{J,N}^{(m)}$, without resorting to any numerical matrix inversion procedure. Substitution of Eq. (4.9) into Eq. (4.2) and premultiplication by $D_{M,I}^{(m)-1}$ yields the desired matrix differential equation for $W_N^{(m)}$:

$$\partial_3 W_M^{(m)} = -s^{1/2} \Lambda_{M,N}^{(m)} W_N^{(m)},$$

(4.14)

where $\Lambda_{M,N}^{(m)}$ is now the diagonal matrix of the eigenvalues $\gamma^{(m;\pm\nu)}$ of $A_{i,j}^{(m)}$, i.e.,

$$\Lambda_{M,N}^{(m)} = \text{Diag} \left(\gamma^{(m;-1)}, \gamma^{(m;-2)}, \gamma^{(m;+1)}, \gamma^{(m;+2)}\right).$$

(4.15)

Carrying out the indicated procedure, we have achieved that for each subdomain $D_m$ the system of four first-order linear differential equations (4.2) is now mutually uncoupled. The field matrix $W_N^{(m)}(z_3)$ of the subdomain $D_m$ is coupled to the field matrix $W_N^{(m+1)}(z_3)$ of the subdomain $D_{m+1}$ through the boundary conditions (4.5) at the interface $z_3 = z_{3;m}$ between $D_m$ and $D_{m+1}$, viz.

$$\lim_{z_3 \downarrow z_{3;m}} D_{J,N}^{(m+1)} W_N^{(m+1)} = \lim_{z_3 \uparrow z_{3;m}} D_{J,N}^{(m)} W_N^{(m)} = 0 \text{ for } m \neq s,$$

(4.16)
while at the source level \( x_3 = x_{3;i} \) we have
\[
\lim_{x_3 \uparrow x_{3;i}} D_{J,N}^{(s+1)} W_{N}^{(s+1)} - \lim_{x_3 \downarrow x_{3;i}} D_{J,N}^{(s)} W_{N}^{(s)} = \tilde{N}_J. \tag{4.17}
\]

Since there are \( ND - 1 \) interfaces, Eqs. (4.16) and (4.17) represent \( 4 \ast (ND - 1) \) linear algebraic equations of which the four equations that follow from Eq. (4.17) are inhomogeneous. Next, for each subdomain \( D_m \), the field matrix \( W_N^{(m)}(x_3) \) of Eq. (4.14) will be written in terms of the linear independent functions \( \Gamma^{(m;\pm \nu)}(x_3) \) that express the decomposition in up- and downward diffusing fields in \( D_m \). In accordance with Eq. (3.73), Eq. (4.11) and Eq. (4.15) we write

\[
\Gamma_{M,K}^{(m)} = \text{diag} \left( \Gamma^{(m;\pm 1)}, \Gamma^{(m;\pm 2)}, \Gamma^{(m;+1)}, \Gamma^{(m;+2)} \right), \tag{4.18}
\]

where now, the \( \Gamma_{M,K}^{(m)}(x_3) \) are the solutions of the differential equations (cf. Eq. (4.2))
\[
\partial_3 \Gamma_{M,K}^{(m)} = -s^{1/2} \Lambda_{M,N}^{(m)} \Gamma_{N,K}^{(m)} \tag{4.19}
\]

In view of Eq. (4.15) it is easily verified that the diagonal elements \( \Gamma^{(m;\pm \nu)}(x_3) \) of \( \Gamma_{M,K}^{(m)}(x_3) \) are given by:
\[
\begin{align*}
\Gamma^{(m;\pm \nu)}(x_3) &= \exp(-s^{1/2} \gamma^{(m;\pm \nu)}(x_3 - x_{3;m})) \quad \text{for} \quad x_{3;m-1} < x_3 < x_{3;m}, \tag{4.20}
\Gamma^{(m;\pm \nu)}(x_3) &= \exp(-s^{1/2} \gamma^{(m;\pm \nu)}(x_3 - x_{3;m-1})) \quad \text{for} \quad x_{3;m-1} < x_3 < x_{3;m}. \tag{4.21}
\end{align*}
\]

We have used the convention that in every layer of the configuration, the diffusive fields have zero phase at the interface from which they originate.

We now assume that the source matrix \( N_I \) contains only terms that correspond to a concentrated impulsive source of either the electric or the magnetic current type. The source matrix \( N_I \) is then of the form (cf. Eq. (3.77))
\[
N_I = \tilde{N}_I \delta(x_3 - x_{3;i}) = \hat{\phi}(s) \hat{k}_{source}(s) X_I \delta(x_3 - x_{3;i}), \tag{4.22}
\]

where \( \delta(x_3 - x_{3;i}) \) is the one-dimensional dirac distribution acting at the source level, \( \hat{\phi}(s) \) is the Laplace transform of the source signature \( \phi(t) \) and \( X_I \) is a 4-by-1
matrix that depends on the nature of the source. From Eqs. (3.15)-(3.17) it follows that the function \( \hat{k}_{\text{source}}(s) = s^{1/2} \) when the source is of the electric current type and \( \hat{k}_{\text{source}}(s) = 1 \) when the source is of the magnetic current type. By combining Eqs. (4.14), (4.16)-(4.18) and (4.22), we arrive at the following expression for \( W_N^{(m)}(x_3) \):

\[
W_N^{(m)}(x_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) H_{N,M}^{(m)}(x_3) W_M^{(m)}.
\]  

(4.23)

The coefficients \( W_M^{(m)} \) express the action of the source as well as the influence of the different electromagnetic properties of each layer. In some way they must follow from the boundary conditions (4.16) and (4.17) at each of the \( ND - 1 \) interfaces. This point will be elaborated later on.

The transform-domain electromagnetic field matrix \( F_j^{(m)} \) of the subdomain \( D_m \) is related to \( W_N^{(m)} \) via the linear transformation (4.9). Consequently,

\[
F_j^{(m)}(x_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) D_{j,N}^{(m)} H_{N,M}^{(m)}(x_3) W_M^{(m)}.
\]  

(4.24)

The right-hand side of Eq. (4.24) can be recognized as the superposition of four terms, each term corresponding to a (transform-domain) diffusive field. In agreement with Eqs. (4.20) and (4.21) two of these terms correspond to fields diffusing in the downward direction in \( D_m \) while the other two terms correspond to fields diffusing in the upward direction in \( D_m \). Note the resemblance between Eqs. (3.78)-(3.79) and Eqs. (4.23)-(4.24), the only difference being that with Eqs. (4.23) and (4.24) we still have to establish a relation between \( W_N^{(m)} \) and \( X_f \), whereas with Eqs. (3.78) and (3.79) this relation directly follows from Eq. (3.65).

In accordance with Eq. (4.11) we write \( W_M^{(m)} \) as

\[
W_M^{(m)} = \left( W_1^{(m;-)}, W_2^{(m;-)}, W_1^{(m;+)} \right)^T.
\]  

(4.25)

In view of the conditions (4.7) and (4.8) imposed on \( F_j(x_3) \) as \( |x_3| \to \infty \), it is clear that \( W_\nu^{(1;+)} \) and \( W_\nu^{(ND;+)} \) satisfy the conditions

\[
W_\nu^{(1;+)} = 0,
\]  

(4.26)
4.2 Transform-domain diffusive field constituents in layered media

\[ \mathcal{W}_\nu^{(ND-1)} = 0. \]  \hspace{1cm} (4.27)

This leaves a total of \( 4 \times (ND-1) \) unknown coefficients \( \mathcal{W}_\nu^{(m;\pm)} \), being just equal to the number of linear algebraic equations (4.16) and (4.17). In the next subsection we present a method for solving these linear equations such that after substitution of the \( \mathcal{W}_\nu^{(m;\pm)} \) into Eq. (4.24), \( F_j^{(m)} \) can be transformed back from the transform-domain to the space-time domain with the modified Cagniard method (De Hoop, 1960; see Section 3.4).

The scattering matrix formalism

To solve the system of linear equations (4.16) and (4.17) and obtain expressions for \( \mathcal{W}_\nu^{(m;\pm)} \) in a form which is suitable for our purpose, we apply the scattering matrix formalism at each of the \( ND-1 \) interfaces. Through this formalism, the boundary conditions (4.15) and (4.16) are rewritten such that the amplitudes of the fields diffusing away from both sides of an interface are expressed in terms of the amplitudes of the fields diffusing towards that interface.

We start with applying the scattering formalism to a source-free interface \( x_3 = x_{3;m} \) with \( m \neq a \). Premultiplication of Eq. (4.16) by \( D_{M,l}^{(m)} \) yields the following relation between the elements \( W_\nu^{(m;\pm)} \) and \( W_\nu^{(m+1;\pm)} \) of \( W_N^{(m)} \) and \( W_N^{(m+1)} \):

\[ W_\nu^{(m;-)}(x_{3;m}) = C_{\nu,r}^{(m;11)} W_\tau^{(m+1;-)}(x_{3;m}) + C_{\nu,r}^{(m;12)} W_\tau^{(m+1;+)}(x_{3;m}), \]  \hspace{1cm} (4.28)

\[ W_\nu^{(m;+)}(x_{3;m}) = C_{\nu,r}^{(m;21)} W_\tau^{(m+1;-)}(x_{3;m}) + C_{\nu,r}^{(m;22)} W_\tau^{(m+1;+)}(x_{3;m}), \]  \hspace{1cm} (4.29)

where

\[ C_{\nu,r}^{(m;11)} = -b_j^{(m;-\nu)} H_{r,j}^\theta b_j^{(m+1;-\nu)}, \]  \hspace{1cm} (4.30)

\[ C_{\nu,r}^{(m;22)} = b_j^{(m;+\nu)} H_{r,j}^\theta b_j^{(m+1;++\nu)}, \]  \hspace{1cm} (4.31)

\[ C_{\nu,r}^{(m;12)} = -b_j^{(m;-\nu)} H_{r,j}^\theta b_j^{(m+1;++\nu)}, \]  \hspace{1cm} (4.32)

\[ C_{\nu,r}^{(m;21)} = b_j^{(m;+\nu)} H_{r,j}^\theta b_j^{(m+1;-\nu)}. \]  \hspace{1cm} (4.33)
Premultiplication of Eq. (4.16) by \( D_{M,I}^{(m+1)} \) yields a pair of relations between the \( W_\nu^{(m;\pm)} \) and \( W_\nu^{(m+1;\pm)} \) that are equivalent to Eqs. (4.28) and (4.29):

\[
W_\nu^{(m+1;\pm)}(x_{3;m}) = C^{(m;11)}_{r,\nu} W_\tau^{(m;\pm)}(x_{3;m}) + C^{(m;21)}_{r,\nu} W_\tau^{(m+1;\pm)}(x_{3;m}), \quad (4.34)
\]

\[
W_\nu^{(m+1;\pm)}(x_{3;m}) = C^{(m;12)}_{r,\nu} W_\tau^{(m;\pm)}(x_{3;m}) + C^{(m;22)}_{r,\nu} W_\tau^{(m+1;\pm)}(x_{3;m}). \quad (4.35)
\]

If we rewrite equations (4.28), (4.29), (4.34) and (4.35) and use Eqs. (4.11), (4.23) and (4.25) to express the value of \( W_\nu^{(m;\pm)}(x_{3;m}) \) in terms of the \( W_\nu^{(m;\pm)} \) and \( \Gamma_\nu^{(m;\pm)} \), we obtain the following linear relations between the amplitudes \( W_\nu^{(m;\pm)} \), \( W_\nu^{(m+1;\pm)} \) of the fields diffusing away from the interface and the amplitudes \( W_\nu^{(m;\pm)} \), \( W_\nu^{(m+1;\pm)} \) diffusing towards that interface.

\[
W_\nu^{(m;\pm)} = R^{(m;\pm)}_{\nu,\mu} \Gamma_\mu^{(m;\pm)}(x_{3;m}) W_\tau^{(m;\pm)} + T^{(m;\pm)}_{\nu,\mu} \Gamma_\mu^{(m+1;\pm)}(x_{3;m}) W_\tau^{(m+1;\pm)} \quad \text{for } m \neq s, \quad (4.36)
\]

\[
W_\nu^{(m+1;\pm)} = T^{(m;\pm)}_{\nu,\mu} \Gamma_\mu^{(m;\pm)}(x_{3;m}) W_\tau^{(m;\pm)} + R^{(m;\pm)}_{\nu,\mu} \Gamma_\mu^{(m+1;\pm)}(x_{3;m}) W_\tau^{(m+1;\pm)} \quad \text{for } m \neq s. \quad (4.37)
\]

Here, \( \Gamma_\mu^{(m;\pm)} \) and \( \Gamma_\mu^{(m;\pm)} \) denote the upper and lower 2-by-2 diagonal matrices of \( \Gamma^{(m)}_{M,K} \), respectively (cf. Eq. (4.18)). In Eqs. (4.36)-(4.37), \( R^{(m;\pm)}_{\nu,\mu} \) and \( T^{(m;\pm)}_{\nu,\mu} \) denote the 2-by-2 reflection and transmission matrices that characterize the interaction of the diffusive fields at the interface \( x_3 = x_{3;m} \). From Eqs. (4.28)-(4.35) we obtain

\[
T^{(m;\pm)}_{\nu,\mu} = C^{(m;11)}_{\nu,\mu} C^{(m;22)}_{\tau,\tau}^{-1}, \quad (4.38)
\]

\[
T^{(m;\pm)}_{\nu,\mu} = C^{(m;11)}_{\nu,\mu} C^{(m;22)}_{\tau,\tau}^{-1}, \quad (4.39)
\]

\[
R^{(m;\pm)}_{\nu,\mu} = C^{(m;12)}_{\nu,\beta} C^{(m;22)}_{\tau,\tau}^{-1} \quad (4.40)
\]

\[
R^{(m;\pm)}_{\nu,\mu} = -C^{(m;12)}_{\beta,\nu} C^{(m;11)}_{\tau,\beta}^{-1}. \quad (4.41)
\]
4.2 Transform-domain diffusive field constituents in layered media

From these equations it follows that between the reflection and transmission matrices $R_{\nu,\tau}^{(m;\pm)}$ and $T_{\nu,\tau}^{(m;\pm)}$ the following relationship exists

$$R_{\beta,\nu}^{(m;+)} R_{\beta,\tau}^{(m;+)} + T_{\beta,\nu}^{(m;+)} T_{\beta,\tau}^{(m;+)} = \delta_{\nu,\tau},$$  \hspace{1cm} (4.42)

$$R_{\beta,\nu}^{(m;-)} R_{\beta,\tau}^{(m;-)} + T_{\beta,\nu}^{(m;-)} T_{\beta,\tau}^{(m;-)} = \delta_{\nu,\tau}.$$  \hspace{1cm} (4.43)

Finally, from Eqs. (4.29) and (4.34) we see that if the amplitudes of the fields diffusing towards the interface $x_3 = x_{3;m}$ are equal to zero (i.e., when $W_{\nu}^{(m;+)} \equiv 0$ and $W_{\nu}^{(m+1;\pm)} \equiv 0$), both $C_{\nu,\tau}^{(m;11)} W_{\tau}^{(m;-)} \equiv 0$ and $C_{\nu,\tau}^{(m;22)} W_{\tau}^{(m+1;\pm)} \equiv 0$ which, in view of Eqs. (3.56) and (3.57), is only possible if both $W_{\nu}^{(m;-)} \equiv 0$ and $W_{\nu}^{(m+1;\pm)} \equiv 0$. In fact this means that $\det (C_{\nu,\tau}^{(m;11)})$ and $\det (C_{\nu,\tau}^{(m;22)})$ can never be equal to zero. The interaction of the diffusive field constituents at an interface is illustrated in Figure 4.2.

![Figure 4.2: Interaction of diffusive field constituents at the interface between $D_m$ and $D_{m+1}$. $R_{\nu,\tau}^{(m;\pm)}$ and $T_{\nu,\tau}^{(m;\pm)}$ are the reflection and transmission coefficients, respectively.](image)

A similar procedure can be applied to the interface $x_3 = x_{3;\delta}$ at which the concentrated source is located. In this case, the amplitudes $W_{\nu}^{(s;-)}$ and $W_{\nu}^{(s+1;+)}$ of the fields diffusing away from the interface are related to the amplitudes $W_{\nu}^{(s;+)}$ and $W_{\nu}^{(s+1;-)}$ of the fields diffusing towards that interface through the reflection and transmission matrices $R_{\nu,\tau}^{(s;\pm)}$ and $T_{\nu,\tau}^{(s;\pm)}$, while the source manifests itself by additional source
Diffusive electromagnetic fields in layered anisotropic media

terms $\mathcal{W}_{\nu}^{(s;\pm)}$. From Eqs. (4.17) we obtain

$$\begin{align*}
\mathcal{W}_{\nu}^{(s;-)} &= R_{\nu,\mu}^{(s;+)} \Gamma_{\mu,\tau}^{(s;+)}(x_{3;\delta}) \mathcal{W}_{\mu}^{(s;+)} \\
&\quad + T_{\nu,\mu}^{(s;-)} \Gamma_{\mu,\tau}^{(s;+)}(x_{3;\delta}) \mathcal{W}_{\mu}^{(s;+)} + \mathcal{W}_{\nu}^{(s;-)}, \quad (4.44)
\end{align*}$$

$$\begin{align*}
\mathcal{W}_{\nu}^{(s;+)} &= T_{\nu,\mu}^{(s;+)} \Gamma_{\mu,\tau}^{(s;+)}(x_{3;\delta}) \mathcal{W}_{\mu}^{(s;+)} \\
&\quad + R_{\nu,\mu}^{(s;+)} \Gamma_{\mu,\tau}^{(s;+)}(x_{3;\delta}) \mathcal{W}_{\mu}^{(s;+)} + \mathcal{W}_{\nu}^{(s;+)}.
\end{align*} \quad (4.45)$$

Here, $\Gamma^{(s;+)}$ and $\Gamma^{(s;-)}$ denote the upper and lower 2-by-2 diagonal matrices of $\Gamma_{\nu,\mu}$, respectively (cf. Eq. (4.18)). The source terms $\mathcal{W}_{\nu}^{(s;\pm)}$ follow from Eqs. (4.17), (4.23) and (4.38)-(4.39)

$$\begin{align*}
\mathcal{W}_{\nu}^{(s;-)} &= - \sum_{\tau=1}^{2} T_{\nu,\tau}^{(s;-)} g_{I}^{(s+1;-\tau)} X_{I}, \quad (4.46) \\
\mathcal{W}_{\nu}^{(s;+)} &= \sum_{\tau=1}^{2} T_{\nu,\tau}^{(s;+)} g_{I}^{(s+\tau)} X_{I}.
\end{align*} \quad (4.47)$$

With the aid of the scattering matrix formalism we have rewritten the boundary conditions (4.16) and (4.17) into a form that expresses the coupling between diffusive fields on either side of an interface in terms of a reflection and a transmission. In the next subsection we show that by solving the relevant equations for the $\mathcal{W}_{\nu}^{(m;\pm)}$ by a recurrence method, the solution will be of a form that is appropriate for our purpose, i.e., it admits a transformation of $F_{cl}(x_{3})$, as given by Eq. (4.24), from the transform-domain back to the space-time domain with the modified Cagniard method.

Recurrence method of solution and generalized diffusive field constituents

In Section 3.3 we have introduced the term diffusive field constituent to denote the upward and downward diffusing fields in an unbounded medium (cf. Eq. (3.80)). In the transformation back to the space-time domain, as well as for the interpretation of the results, it will be advantageous to introduce the concept of generalized diffusive field constituents in layered media. With this concept, the diffusion process
of electromagnetic fields through layered media is described in terms of upwardly and downwardly diffusing fields within each layer. These generalized diffusive constituents closely resemble the generalized-ray constituents used to describe wave phenomena in layered media (Spencer, 1960; Van der Hulden, 1988).

We introduce the modified reflection and transmission matrices \( R_{\nu,\tau}^{(m,\pm)} \) and \( T_{\nu,\tau}^{(m,\pm)} \) by including in them the exponential terms \( \Gamma_{\mu,\tau}^{(m,\pm)}(z_{3,m}) \) and \( \Gamma_{\mu,\tau}^{(m+1,\pm)}(z_{3,m}) \) that occur in Eqs. (4.36)-(4.37). Accordingly, we rewrite these equations as

\[
W_{\nu}^{(m,-)} = R_{\nu,\tau}^{(m,+)\nu} W_{\tau}^{(m,+)} + T_{\nu,\tau}^{(m,-)\nu} W_{\tau}^{(m+1,-)} \quad \text{for } m \neq s, \quad (4.48)
\]

\[
W_{\nu}^{(m+1,+)} = T_{\nu,\tau}^{(m,+)} W_{\tau}^{(m,+)} + R_{\nu,\tau}^{(m,-)} W_{\tau}^{(m+1,-)} \quad \text{for } m \neq s, \quad (4.49)
\]

while (cf. Eqs. (4.26) and (4.27))

\[
W_{\nu}^{(1,+)} \equiv 0 \quad \text{and} \quad W_{\nu}^{(ND,-)} \equiv 0. \quad (4.50)
\]

At the source level we have (cf. Eqs. (4.44) and (4.45))

\[
W_{\nu}^{(s,-)} = R_{\nu,\tau}^{(s,+)} W_{\tau}^{(s,+)} + T_{\nu,\tau}^{(s,-)} W_{\tau}^{(s+1,-)} + W_{\nu}^{(s,-)}, \quad (4.51)
\]

\[
W_{\nu}^{(s+1,+)} = T_{\nu,\tau}^{(s,+)} W_{\tau}^{(s,+)} + R_{\nu,\tau}^{(s,-)} W_{\tau}^{(s+1,-)} + W_{\nu}^{(s,+)}. \quad (4.52)
\]

We look for a method to solve this system of equations in a way such that (i) after substitution of the \( W_{\nu}^{(m,\pm)} \) into Eq. (4.24), \( F^{(m)}_{\nu} \) can be transformed back from the transform-domain to the space-time domain with the modified Cagniard method and (ii) the physical interpretation of this solution is clear. To this end, we solve Eqs. (4.48)-(4.52) recurrently. With each step of this recurrence scheme, a new term is obtained from the one that resulted from the previous step through linear recurrence relations. Let us denote by \( [W_{\nu}^{(m,\pm)}]_{(i+1)}^{(i)} \) the term that is obtained in the \( i^{\text{th}} \) step of the recurrence scheme, \( i = 0, 1, 2, \ldots \). The recurrence relations follow from Eqs. (4.48) and (4.49) as

\[
[W_{\nu}^{(m,-)}]_{(i+1)}^{(i)} = R_{\nu,\tau}^{(m,+)} [W_{\tau}^{(m,+)}]_{(i)}^{(i)} + T_{\nu,\tau}^{(m,-)} [W_{\tau}^{(m+1,-)}]_{(i)}^{(i)}, \quad (4.53)
\]

\[
[W_{\nu}^{(m+1,+)}]_{(i+1)}^{(i)} = T_{\nu,\tau}^{(m,+)} [W_{\tau}^{(m,+)}]_{(i)}^{(i)} + R_{\nu,\tau}^{(m,-)} [W_{\tau}^{(m+1,-)}]_{(i)}^{(i)}. \quad (4.54)
\]
The recurrence scheme of solving Eqs. (4.48)-(4.52) starts with the initial term $[W^{(m;\pm)}_{\nu}]^{(0)}$ that is obtained from Eqs. (4.51) and (4.52) and follows as

$$[W^{(s;-)}_{\nu}]^{(0)} = W^{(s;-)} \quad \text{and} \quad [W^{(s+1;+)}_{\nu}]^{(0)} = W^{(s;+)}.$$  

(4.55)

All other $[W^{(m;\pm)}_{\nu}]^{(0)}$ are equal to zero. Substitution of the initial term into the right-hand side of the recurrence relations (4.53) and (4.54) yields the first term $[W^{(m;\pm)}_{\nu}]^{(1)}$ of the recurrence solution. Repeated substitution of each term into the recurrence relations yields all the subsequent terms $[W^{(m;\pm)}_{\nu}]^{(i)}$. As can easily be verified, the solution $W^{(m;\pm)}_{\nu}$ of Eqs. (4.48)-(4.52) is equal to the summation of all terms that results from this recurence scheme, i.e.,

$$W^{(m;\pm)}_{\nu} = \sum_{i=0}^{\infty} [W^{(m;\pm)}_{\nu}]^{(i)}.$$  

(4.56)

Note that Eq. (4.56) can also be interpreted as an Neumann iterative solution of the system of equations (4.48)-(4.52) and is unconditionally convergent since we can always chose $s^{1/2}$ large enough to ensure an exponential decay of $[W^{(m;\pm)}_{\nu}]^{(i)}$ as $i \to \infty$ (cf. Eq. (4.53)-(4.54)).

In view of Eqs. (4.53) and (4.54) we see that each recurrence term $[W^{(m;\pm)}_{\nu}]^{(i)}$ is nothing else but a summation of products of modified reflection and transmission matrices in a unique order, multiplied by a source term. A convenient property of this formulation is its physical interpretation. From Eqs. (4.23) and (4.55) we infer that, in agreement with the transform-domain diffusion matrix in an homogeneous medium Eq. (3.78), the initial term $[W^{(m;\pm)}_{\nu}]^{(0)}$, Eq. (4.55), of the recurrence scheme corresponds to the direct diffusive field, i.e. an upwardly diffusing field in $D_2$ and a downwardly diffusing field in $D_{a+1}$. In the same way, the first term $[W^{(m;\pm)}_{\nu}]^{(1)}$ of the recurrence scheme gives the diffusive field due to a single interaction at the neighboring interfaces $z_{a;\pm-1}$ and $z_{a;\pm+1}$ and similarly, the second term generates the diffusive field due to two interactions at the interfaces. Hence, each term in this summation can be regarded as a diffusive field constituent determined by the interactions at the interfaces and the diffusion across the layers. We call such a term a generalized diffusive field constituent. Since in each layer there are two upwardly and two downwardly diffusing fields, the number of generalized diffusive field constituents
that contribute to the field at a certain point of observation increases rapidly with increasing value of $i$. For example, the term corresponding to $i = 0$ represents two generalized diffusive field constituents in $D_2$, the term corresponding to $i = 1$ represents four generalized diffusive field constituents in $D_2$, and the term corresponding to $i = 2$ represents sixteen generalized diffusive field constituents in $D_2$. The process of successive reflection and transmission at the interfaces is illustrated in Figure 4.3.

![Diagram](image)

Figure 4.3: The generalized diffusive field constituents that are included in the terms with $i = 0$, $i = 1$ and $i = 2$ for a layered configuration with a source located at the interface between $D_2$ and $D_{2+1}$.

Upon combining Eq. (4.56) with Eqs. (4.23) and (4.25) we arrive at the final expres-
sion for the diffusive field matrix $W^{(m)}_N(x_3)$ as

$$W^{(m)}_N(x_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) \sum_{i=0}^{\infty} \Gamma^{(m)}_{N,M}(x_3) [W^{(m)}_M]^i.$$  

(4.57)

The transform-domain electromagnetic field matrix $F^{(m)}_j$ in the subdomain $D_m$ is related to $W^{(m)}_N$ via the linear transformation (4.9). Consequently,

$$F^{(m)}_j(x_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) D^{(m)}_{j,N} \sum_{i=0}^{\infty} \Gamma^{(m)}_{N,M}(x_3) [W^{(m)}_M]^i.$$  

(4.58)

The total electromagnetic field represented by the field matrix $F^{(m)}_j$ can be decomposed into the contributions from the individual generalized diffusive field constituents by taking all individual terms from the right-hand side of Eq. (4.58). Each generalized diffusive field constituent has the general shape

$$F^{(m)}_j(x_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) B_j \exp(-s^{1/2} \sum_j h_j \gamma^{(j)}),$$  

(4.59)

where for notational simplicity we have written

$$\sum_j h_j \gamma^{(j)} = \sum_{j,\nu} h_j (\pm \gamma^{(j, \nu)}),$$  

(4.60)

in which the + and - signs go together and $j$ and $\nu$ take on all relevant values that correspond to the particular generalized diffusive field constituent under consideration. $B_j$ is the corresponding product of reflection and transmission coefficients multiplied by a source term.

Upon comparing the expression for the electromagnetic field matrix $F_j$ pertaining to a homogeneous anisotropic medium, Eq. (3.80), with the one pertaining to layered anisotropic media, Eq. (4.59), we see that they have a similar structure, where the former is nothing but the initial term of the recurrence scheme of Eqs. (4.48)-(4.52). Obviously, all the higher-order terms that follow from the recurrence scheme are equal to zero in the case of an unbounded homogeneous anisotropic medium since then no reflection takes place.

Through Eq. (4.59) we have established that each single term that contributes to the
4.3 The transformation back to the space-time domain

The total solution \( F^{(m)}(x_3) \) of Eq. (4.2) admits a transformation from the transform domain back to the space-time domain with the aid of the modified Cagniard method. However, such a transformation of the elements of the \( F^{(m)}_j \) will only lead to appropriate space-time domain expressions for the horizontal components of the electric field strength, not of the magnetic field strength. This is due to the inclusion of the factor \( s^{1/2} \) in \( \tilde{H}_1^{(m)} \) and \( \tilde{H}_2^{(m)} \) in \( F^{(m)}_j \) (cf. Eq. (4.1)). For this reason we rewrite Eq. (4.59) as the generalized diffusive field constituent (cf. Eq. (3.82))

\[
F^{(m)}(x_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) \hat{k}_{\text{field}}(s) B^{(m)} \exp(-s^{1/2} \sum_j h_j \gamma^{(j)}),
\]

(4.61)

such that the factor \( s^{1/2} \) present at the left-hand side of Eq. (4.59) in \( F_3 \) and \( F_4 \) is now included in the function \( \hat{k}_{\text{field}}(s) \). From Eq. (3.9) we find that \( \hat{k}_{\text{field}}(s) = 1 \) when the electric field strength is considered while \( \hat{k}_{\text{field}}(s) = s^{1/2} \) when the magnetic field strength is considered. For notational simplicity we shall denote by \( \hat{k}(s) \) the product of \( \hat{k}_{\text{source}}(s) \) and \( \hat{k}_{\text{field}}(s) \), i.e., \( \hat{k}(s) = \hat{k}_{\text{source}}(s) \hat{k}_{\text{field}}(s) \). The different functions \( \hat{k}(s) \) are given by Eqs. (3.83)-(3.85).

4.3 The transformation back to the space-time domain

In this section we consider the transformation of \( F(\alpha_1, \alpha_2, x_3, s) \) as given by Eq. (4.61) from the transform domain back to the space-time domain. Without loss of generality we can assume the concentrated source to be located at the origin \( \mathcal{O} \) of the Cartesian coordinate system. Inverse Fourier transformation of \( F(\alpha_1, \alpha_2, x_3, s) \) to the space-Laplace domain yields (cf. Eq. (2.34))

\[
F(\omega, s) = s \hat{\phi}(s) \hat{G}(\omega, s),
\]

(4.62)

where \( \hat{G}(\omega, s) \) is the space-Laplace domain Green's function

\[
\hat{G}(\omega, s) = \frac{\hat{k}(s)}{4\pi^2} \int_{-\infty}^{\infty} d\alpha_2 \int_{-\infty}^{\infty} \exp(-s^{1/2}(i\alpha_2 x_3 + \sum_j h_j \gamma^{(j)})) B \, d\alpha_1.
\]

(4.63)
In view of Eq. (4.62) and Lerch's theorem (Widder, 1946), the space-time domain expression for $F(\omega, t)$ is obtained as the time-derivative of the convolution of the source signature $\phi(t)$ and the Green's function $G(\omega, t)$, i.e.,

$$F(\omega, t) = \partial_t \int_0^t \phi(\tau) G(\omega, t - \tau) \, d\tau. \quad (4.64)$$

Here, $G(\omega, t)$ is the inverse Laplace transform of $\hat{G}(\omega, s)$. Note that Eq. (4.63) represents the contribution from a single generalized diffusive field constituent. We shall cast the integral on the left-hand side of Eq. (4.63) in such a form that $G(\omega, t)$ can be found by inspection. To this end, we follow the same procedure as with Section 3.4.

We start with a change of integration variables in Eq. (4.63). We replace the Fourier transformation variables $\alpha_1$ and $\alpha_2$ by the polar variables of integration $p$ and $\psi$ defined through

$$i\alpha_1 = p \cos(\theta + \psi), \quad (4.65)$$
$$i\alpha_2 = p \sin(\theta + \psi), \quad (4.66)$$

with $0 \leq p < \infty$, $0 \leq \psi < 2\pi$. The angle $\theta$ follows from the polar-coordinate specification of the point of observation, i.e.,

$$z_1 = r \cos(\theta), \quad (4.67)$$
$$z_2 = r \sin(\theta), \quad (4.68)$$

with $0 \leq r < \infty$, $0 \leq \theta < 2\pi$. Since $d\alpha_1 d\alpha_2 = -pd\psi d\psi$ and $i\alpha_1 z_1 = pr \cos(\psi)$, we can rewrite Eq. (4.63) as

$$\hat{G}(\omega, s) = -\frac{\hat{b}(s)}{4\pi^2} \int_0^{2\pi} \, d\psi \int_0^{\infty} \exp(-s^{1/2}(p r \cos(\psi) + \sum_j h_j \tilde{\gamma}^{(j)})) \hat{B}(p) \, dp, \quad (4.69)$$

in which $\tilde{\gamma}^{(j)}(p, \psi)$ and $\hat{B}(p, \psi)$ have been obtained from $\gamma^{(j)}(\alpha_1, \alpha_2)$ and $B(\alpha_1, \alpha_2)$ by the substitutions in Eqs. (4.65) and (4.66). Remind that $\tilde{\gamma}^{(j)}$ is a simplified notation for $\pm \tilde{\gamma}^{(j;i \pm \psi)}$ (cf. Eq. (4.60)). Since the integrand of Eq. (4.69) is a periodic function
of $\psi$ with periodicity $2\pi$, we can decompose the integral with respect to $\psi$ on the right-hand side of this equation into the following two parts

$$
\int_0^{2\pi} d\psi \int_0^{i\infty} \ldots dp = \int_{-\pi/2}^{\pi/2} d\psi \int_0^{i\infty} \ldots dp + \int_{\pi/2}^{3\pi/2} d\psi \int_0^{i\infty} \ldots dp.
$$

(4.70)

By performing the substitutions $\psi \rightarrow \psi + \pi$ and $p \rightarrow -p$ in the second integral on the right-hand side of Eq. (4.70), we achieve that with the latter $\psi$ also takes on values between $-\pi/2$ and $\pi/2$ while, in view of Eqs. (4.65) and (4.66), the integrand itself is not altered. Hence, the two integrals can be taken together and we arrive at

$$
\hat{G}(s, s) = -\frac{\hat{k}(s)}{4\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \left( \int_0^{i\infty} + \int_0^{-i\infty} \right) \exp(-s^{1/2}(p\rho\cos(\psi) + \sum_j h_j \bar{\gamma}^{(j)})) \bar{B} pdp.
$$

(4.71)

Since $\bar{\gamma}^{(j)}(p, \psi)$ and $\bar{B}(p, \psi)$ satisfy Schwarz's reflection principle (cf. Eqs. (3.101)-(3.102)), the integrand at the right-hand side of Eq. (4.71) satisfies Schwarz's reflection principle as well. Using this property, we can combine the integral along the positive imaginary $p$-axis and the integral in the opposite direction along the negative imaginary $p$-axis and rewrite them as a single integral. Since $C + C^* = 2\text{Re}(C)$, we end up with

$$
\hat{G}(s, s) = -\frac{\hat{k}(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \text{Re} \left( \int_0^{i\infty} \exp(-s^{1/2}(p\rho\cos(\psi) + \sum_j h_j \bar{\gamma}^{(j)})) \bar{B} pdp \right).
$$

(4.72)

Next, we want to carry out the integration with respect to $p$ along a certain contour in the complex $p$-plane that deviations from the imaginary $p$-axis. To this end, we extend the definition of the relevant integrand of Eq. (4.72) into the complex $p$-plane by analytical continuation away from the imaginary $p$-axis. For this, we need the singularities of the integrand in the complex $p$-plane. The latter coincide with the singularities of $\bar{\gamma}^{(j, \pm \nu)}$.

For the properties of $\bar{\gamma}^{(j, \pm \nu)}(p, \psi)$ in the complex $p$-plane we refer to Appendix A. From this Appendix we learn that the only singularities of $\bar{\gamma}^{(j, \pm \nu)}(p, \psi)$ are branch
points located on the real $p$-axis where the $\tilde{\gamma}^{(j;\pm \nu)}$ of up- and downwardly diffusing fields have equal values and possible branch points off the real $p$-axis where the $\tilde{\gamma}^{(j;\pm \nu)}$ of either two upwardly or two downwardly diffusing fields have equal values. In order to keep the branches $\tilde{\gamma}^{(j;\pm \nu)}$ single-valued throughout the complex $p$-plane we introduce branch cuts, either as straight line segments along the real $p$-axis joining two corresponding branch points, or as straight lines from the relevant branch point to infinity. Furthermore, the $\tilde{\gamma}^{(j;\pm \nu)}$ have real values on parts of the real $p$-axis, including an interval that contains the origin $p = 0$.

Let $N_L$ denote the number of different $\tilde{\gamma}^{(j;\pm \nu)}$ that occur as argument of the exponential function at the right-hand side of Eq. (4.72). As can easily be conjectured from Eq. (4.59), there are $N_T = 2^{N_L}$ generalized diffusive field constituents that contribute to the total Green's function of which the argument of the exponential function closely resembles the one of Eq. (4.72), the only difference being that where one constituent has $\tilde{\gamma}^{(j;\pm 1)}$ the other constituent has $\tilde{\gamma}^{(j;\pm 2)}$ for one or more values of $j$. Consequently, we can regard $\Sigma_{j,\nu} \pm h_j\tilde{\gamma}^{(j;\pm \nu)}(p, \psi)$ as a branch of a $N_T$-valued function $\Sigma_j h_j\tilde{\gamma}^{(j)}(p, \psi)$ on the complex $p$-plane. Following the approach of Riemann, we can modify in a suitable manner the domain of definition of this multi-valued function (i.e., the complex $p$-plane), such that a single-valued function $\Sigma_j h_j\tilde{\gamma}^{(j)}(p_R, \psi)$ is obtained on the new domain $\mathcal{R}$, $p_R \in \mathcal{R}$. $\mathcal{R}$ is called a Riemann surface. The Riemann surface $\mathcal{R}$ can be represented as a $N_T$-sheeted (covering) surface of the complex $p$-plane in a way such that each sheet can be identified with a copy of the complex $p$-plane on which $\Sigma_j h_j\tilde{\gamma}^{(j)}(p_R, \psi)$ equals one of the $N_T$-branches $\Sigma_{j,\nu} h_j(\pm \tilde{\gamma}^{(j;\pm \nu)}(p, \psi))$. An interconnection between two sheets of the Riemann surface takes place only along a common branch cut of corresponding branches of $\tilde{\gamma}^{(j;\pm \nu)}(p, \psi)$. Crossing a branch cut in the complex $p$-plane implies going from one sheet of the Riemann surface $\mathcal{R}$ to another (see Figure 3.1).

The function $\tilde{B}(p, \psi)$ that occurs in the integrand of Eq. (4.72) also has to be scrutinized for possible singularities. From its definition, Eqs. (4.58)- (4.59), it is obvious that the only singularities in the complex $p$-plane of $\tilde{B}(p, \psi)$ are the branch points of the corresponding $\tilde{\gamma}^{(j;\pm \nu)}$. Further, upon considering the behavior of the reflection and transmission coefficients on the Riemann surface $\mathcal{R}$, Eqs. (4.38)-(4.41), we observe that they are single-valued functions as well. Consequently, the integrand of Eq. (4.72) is a single-valued function on the Riemann surface $\mathcal{R}$.

The integral along the positive imaginary $p$-axis in Eq. (4.72) is evaluated with the aid of the modified Cagniard method which consists of deforming the path of inte-
4.3 The transformation back to the space-time domain

The transformation away from the imaginary \( p \)-axis into the complex \( p \)-plane in a way such that the resulting expression for  \( \hat{G}(\omega, s) \) can be recognized as the Laplace transform of some known space-time function \( G(\omega, t) \), without making use of an inverse Laplace transformation (De Hoop, 1960). The new path of integration in the complex \( p \)-plane is called the modified Cagniard contour. Along this contour, the argument \( p \cos(\psi) + \sum_j h_j \tilde{\gamma}^{(j)} \) of the exponential function in the integrand of Eq. (4.72) should be real and positive. The modified Cagniard contour is parametrized by the variable \( \tau \), such that for fixed \( \psi \), \( -\pi/2 \leq \psi < \pi/2 \) we have

\[
p \cos(\psi) + \sum_j h_j \tilde{\gamma}^{(j)}(p, \psi) = \tau, \quad \tau \text{ is real and positive.} \tag{4.73}
\]

Note that \( p \cos(\psi) \geq 0 \) in the interval of \( \psi \)-integration. Along the modified Cagniard contour the exponential term in the integrand of Eq. (4.72) reduces to \( \exp(-\sigma^{1/2}\tau) \), which function can easily transformed back to the space-time domain.

The modified Cagniard contour \( p(\tau) \)

Since \( \tilde{\gamma}^{(j \pm \nu)}(p, \psi) \) is real valued along some parts of the real \( p \)-axis including an interval that contains the origin \( p = 0 \) of the complex \( p \)-plane, it is easy to see that the modified Cagniard contour starts at \( p = 0 \) and then follows a part of the real \( p \)-axis. At some point \( p = p_0(\psi) \), the modified Cagniard contour will leave the real \( p \)-axis to finally approach a complex asymptote as \( \tau \to \infty \). The value of \( \tau \) at \( p = 0 \) is denoted as \( \tau = T_{\text{min}} \). Its value follows from Eq. (4.73) and is given by

\[
T_{\text{min}} = \sum_j h_j \tilde{\gamma}^{(j)}(0, \psi) = \sum_j h_j \gamma^{(j)}(0, 0) > 0. \tag{4.74}
\]

Note that \( T_{\text{min}} \) is independent of \( \psi \) and is always greater than 0. Since we want \( \tau \) to increase monotonously along the path, the modified Cagniard contour leaves the real \( p \)-axis at the point \( p = p_0(\psi) \) where \( \tau \) reaches a local maximum value. At this point, the derivative of \( \tau \) with respect to \( p \) equals zero, i.e.

\[
\frac{\partial \tau}{\partial p} = \frac{\partial}{\partial p} \left[ p \cos(\psi) + \sum_j h_j \tilde{\gamma}^{(j)} \right] = 0 \quad \text{for} \quad p = p_0(\psi) \in \mathbb{R}. \tag{4.75}
\]
Whether \( p_0(\psi) \) is positive or negative depends on the value of \( \frac{\partial \tau}{\partial p} \) at \( p=0 \)

\[
\begin{aligned}
&\text{if } \frac{\partial \tau}{\partial p} > 0 \text{ at } p=0 \text{ then } p_0(\psi) > 0, \\
&\text{if } \frac{\partial \tau}{\partial p} < 0 \text{ at } p=0 \text{ then } p_0(\psi) < 0.
\end{aligned}
\]

(4.76)

The value of \( \tau \) at \( p=p_0(\psi) \) is denoted as \( \tau = T(\psi) \). Its value follows from Eq. (4.73) and is given as

\[
T(\psi) = p_0(\psi) \tau \cos(\psi) + \sum_j h_j \tilde{\gamma}^{(j)}(p_0(\psi), \psi) \geq T_{\text{min}}.
\]

(4.77)

By using Eqs. (4.65) and (4.66) we observe that \( p_0(\psi) \) satisfies the relationship

\[
p_0(\pi/2) = -p_0(-\pi/2).
\]

(4.78)

Upon substituting Eq. (4.78) into Eqs. (4.73) and (4.75) we find that the value of \( T(\psi) \) for \( \psi = \pi/2 \) is equal to its value for \( \psi = -\pi/2 \). We denote this value as \( T_{\text{ver}} \).

From Eq. (4.77) we have

\[
T_{\text{ver}} = T(\pi/2) = \sum_j h_j \tilde{\gamma}^{(j)}(p_0(\pi/2), \pi/2)
\]

\[
= T(-\pi/2) = \sum_j h_j \tilde{\gamma}^{(j)}(p_0(-\pi/2), -\pi/2),
\]

(4.79)

From Eqs. (4.75)-(4.76) we infer that for real values of \( p \), \( \partial^2 \tau / \partial p^2 \neq 0 \) at \( p=p_0(\psi) \). Consequently, \( p=p_0(\psi) \) is a saddle point of the real part of \( p \cos \psi + \sum h_j \tilde{\gamma}^{(j)}(p, \psi) \) as a function of \( p \). Hence, in order that \( \tau \) increases monotonously along the path, the modified Cagniard contour must leave the real \( p \)-axis perpendicularly at \( p=p_0(\psi) \) to finally reach a complex asymptote in the upper half of the complex \( p \)-plane. Occasionally, the modified Cagniard contour can turn back to the real \( p \)-axis, follow a part of it and leave the real \( p \)-axis again. This situation may arise in the case where one or more of the relevant \( \tilde{\gamma}^{(j)} \) has three branch points on the positive and negative parts of the real \( p \)-axis (see Appendix A, Figures A.3 and A.4). In this case, the relevant \( \tilde{\gamma}^{(j)} \) is real-valued along three distinct parts of the real \( p \)-axis.
Further, in case of branch points off the real $p$-axis, the modified Cagniard contour can make a turn around such an off-axis branch point to cross itself again. In fact, in the latter case, the Cagniard contour meets an off-axis branch cut and goes from one Riemann sheet of the Riemann surface $\mathcal{R}$ to another (e.g., from the Riemann sheet connected with $\tilde{\gamma}^{(j;+1)}$ to the Riemann sheet connected with $\tilde{\gamma}^{(j;+2)}$).

The complex asymptote of $p(\tau)$ as $\tau \to \infty$ is determined by the asymptotic behavior of $\tilde{\gamma}^{(j;\pm\nu)}(p,\psi)$ for $|p| \to \infty$. From Appendix A we learn that $\tilde{\gamma}^{(j;\pm\nu)}(p,\psi)$ is approximately proportional to $p$ when $|p| \to \infty$. In view of Eqs. (3.48)-(3.50) we obtain

\[
\tilde{\gamma}^{(j;\pm\nu)}(p,\psi) = -ipC^{(j;\pm\nu)}(\psi) + O(p^{-1}) \quad \text{as } |p| \to \infty, \tag{4.80}
\]
where

\[
C^{(j;\pm\nu)}(\psi) = -C^{(j;\pm\nu)*}(\psi), \quad \text{Re}(C^{(j;\pm\nu)}(\psi)) > 0. \tag{4.81}
\]

The $C^{(j;\pm\nu)}(p,\psi)$ follow from the asymptotic behavior of the determinantal equation Eq. (3.42), Eq. (A.26) as $|p| \to \infty$. In Appendix A we describe how the $C^{(\pm\nu)}$ can be obtained explicitly from this fourth-order polynomial equation. For notational simplicity we shall write $C^{(j)}$ instead of $C^{(j;\pm\nu)}$. With this notation, Eq. (4.80) is written as

\[
\tilde{\gamma}^{(j)}(p,\psi) = -ipC^{(j)}(\psi) + O(p^{-1}) \quad \text{as } |p| \to \infty. \tag{4.82}
\]

Using Eq. (4.82), Eq. (4.73) reduces asymptotically to

\[
p \left( r \cos(\psi) - i \sum_j h_j C^{(j)}(\psi) \right) \sim \tau \quad \text{as } \tau \to \infty, \tag{4.83}
\]
from which we obtain

\[
p \sim \tau \left( r \cos(\psi) - i \sum_j h_j C^{(j)}(\psi) \right)^{-1} \quad \text{as } \tau \to \infty. \tag{4.84}
\]

Further we know that for each of the diffusive field constituents that contributes to the solution we always have that $\text{Re}(\sum_j h_j C^{(j)}(\psi)) > 0$. Consequently, the modified
Cagniard contour has as its asymptote a straight line through the origin that is located in the upper-half of the complex $p$-plane. In Figure 4.4 we have depicted an example of a modified Cagniard contour for an arbitrarily anisotropic medium. See also Figure 3.2.

![Figure 4.4: Example of the modified Cagniard contour for an anisotropic medium. The dashed line indicates the asymptote as $r \to \infty$.](image)

From Eqs. (4.65)-(4.66), (3.101) and (3.102) we conclude that $\tilde{\gamma}^{(j)}(p, \psi)$ and $\tilde{B}(p, \psi)$ satisfy the relations

$$
\tilde{\gamma}^{(j; \pm \nu)}(-p^*, -\pi/2) = \tilde{\gamma}^{(j; \pm \nu)*}(p, \pi/2), \quad (4.85)
$$

$$
\tilde{B}(-p^*, -\pi/2) = \tilde{B}^*(p, \pi/2). \quad (4.86)
$$

Hence, in view of Eq. (4.73) it is easily seen that the modified Cagniard contours for $\psi = -\pi/2$ and $\psi = \pi/2$ are symmetric with respect to the imaginary $p$-axis. This implies that the $C^{(j)}(\pm \pi/2)$ which determine the complex asymptote of the the Cagniard contour for $\psi = \pm \pi/2$, satisfy

$$
C^{(j)}(-\pi/2) = C^{(j)*}(\pi/2). \quad (4.87)
$$
Finally, it is easily verified that the real part of the left-hand side of Eq. (4.73) is always greater than zero and proportional to $|p|$ along the circular arc joining the positive imaginary $p$-axis and the modified Cagniard contour at infinity.

The positive imaginary $p$-axis, the $N_T$ modified Cagniard contours (one contour for each of the $N_T$ different generalized diffusive field constituents) and the circular arcs at infinity form $N_T$ closed integration contours on the Riemann surface $\mathcal{R}$. These closed contours are not necessarily located on different sheets of the Riemann surface. In case of branch points off the real $p$-axis, some of the contours can be located on two or more sheets of the Riemann surface. Hence, we conclude that the singularities of the integrand of Eq. (4.73) form no obstruction in the process of contour deformation as long as we do sum the contributions from the relevant $N_T$ generalized diffusive field constituents.

*Space-time expression for the Green's function $G(\omega,t)$*

By virtue of Jordan's lemma, Cauchy's theorem and the previous results, we conclude that the integral along the positive imaginary $p$-axis of Eq. (4.72) is equal to the integral along the modified Cagniard contour. Upon introducing $\tau$ as variable of integration, we obtain

$$
\hat{G}(\omega,s) = -\frac{\hat{k}(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \left( \int_{T_{\min}}^{T(\psi)} + \int_{T(\psi)}^{\infty} \right) \\
\times \exp(-s^{1/2}\tau) \text{Re}(\hat{\beta}_p \frac{\partial p}{\partial \tau}) \, d\tau.
$$

Note that $\tau = T(\psi)$ is a singular point of the integrand of Eq. (4.88) since $\partial p/\partial \tau$ is infinite for this value of $\tau$. From Eqs. (4.73) and (4.75), however, we conclude that this singularity is a square-root singularity and is integrable. Further, in case the modified Cagniard contour returns to the real $p$-axis, $\partial p/\partial \tau$ has additional square-root singularities at those values of $\tau$ for which the modified Cagniard contour meets the real $p$-axis, and at those values of $\tau$ for which the modified Cagniard contour leaves the real $p$-axis again. However, these singularities are integrable as well. For
notational simplicity we shall write Eq. (4.88) as

\[ \hat{G}(\omega, s) = -\frac{\hat{k}(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_0^\infty \exp(-s^{1/2}\tau) \text{Re}(\vec{B} \frac{\partial p}{\partial \tau}) d\tau. \]  

(4.89)

With Eq. (4.89) we have now arrived at the position where \( G(\omega, t) \) can be found by inspection. This is possible since the part of \( \hat{G}(\omega, s) \) that depends on the Laplace transformation parameter \( s \) is given by \( \hat{k}(s) \exp(-s^{1/2}\tau) \) while the rest of the integrand is \( s \)-independent. Since \( \tau \) is real and positive (\( T_{\text{min}} > 0 \)) this function \( \hat{k}(s) \exp(-s^{1/2}\tau) \) can be recognized as the Laplace transform of some time function \( k(t, \tau) \). \( \hat{k}(s) \) depends on the source-type and whether the electric or magnetic field strength is considered (cf. Eqs. (3.83)-(3.86)). The to \( \hat{k}(s) \exp(-s^{1/2}\tau) \) corresponding time functions are given in Eqs. (3.122)-(3.125).

Consequently, the space-time domain expression for \( G(\omega, t) \) is obtained as

\[ G(\omega, t) = \begin{cases} 
0, & t < 0, \\
-\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_0^\infty k(t, \tau) \text{Re}(\vec{B} \frac{\partial p}{\partial \tau}) d\tau, & t \geq 0.
\end{cases} \]  

(4.90)

Note that Eq. (4.90) represents the contribution from a single generalized diffusive field constituent. In order to obtain the total Green's function, we do have to sum the contributions from all generalized diffusive constituents.
Chapter 5

Diffusive electromagnetic fields generated by loop sources

5.1 Introduction

So far we have only considered diffusive electromagnetic fields in arbitrarily anisotropic media that are generated by sources of an infinitesimal extent (e.g., the electric current dipole). The spatial distribution of these sources is represented by a delta function at the location of the source (cf. Eq. (4.22)). However, practical geophysical exploration systems based on the transient electromagnetic method of investigation, commonly employ sources of non-vanishing dimensions such as the multi-turn circular or square loop. As we shall show in Section 5.2, the modified Cagniard method provides an efficient way of accounting for the non-vanishing dimensions of a rectangular loop source. In Section 5.3 we consider the in-loop configuration where the point of observation is at the center of a square loop.

5.2 The rectangular loop source

In this section we derive an expression for the diffusive electromagnetic field in a horizontally stratified, arbitrarily anisotropic medium, generated by a plane, rectangular loop source. Without loss of generality we assume that the loop is located in the plane \( z_3 = 0 \) with its center at the origin \( O \) of the Cartesian reference frame. Furthermore, the sides of loop are to be parallel to the \( z_1 \)- and \( z_2 \)-axis, the side
that is parallel to the $z_1$-axis having a length $L_1$ and the side that is parallel to the $z_2$-axis having a length $L_2$. The vertices of the square loop are denoted as $A$, $B$, $C$ and $D$. The coordinates of the vertices are given by

\[ \begin{align*}
\mathbf{a}_A &= (x_{1;A}, x_{2;A}, x_{3;A}) = (-L_1/2, -L_2/2, 0), \\
\mathbf{a}_B &= (x_{1;B}, x_{2;B}, x_{3;B}) = (-L_1/2, L_2/2, 0), \\
\mathbf{a}_C &= (x_{1;C}, x_{2;C}, x_{3;C}) = (L_1/2, L_2/2, 0), \\
\mathbf{a}_D &= (x_{1;D}, x_{2;D}, x_{3;D}) = (L_1/2, -L_2/2, 0). \end{align*} \tag{5.1} \]

The diffusive electromagnetic field is generated by an electric current $I(t) = I_0 \phi(t)$ supplied to the loop source. The point of observation is given by $\mathbf{a} = (x_1, x_2, x_3)$. See Figure 5.1.

![Figure 5.1: The plane, rectangular loop source with excitation current $I(t)$. The point of observation is $\mathbf{a} = (x_1, x_2, x_3)$. $A$, $B$, $C$ and $D$ denote the vertices of the loop.](image)

The electromagnetic field generated by an infinitesimal part $dl$ of the loop source is integrated along the four sides of the loop in order to obtain the total transient electromagnetic field at a certain point of observation. Such a small part of the loop can be considered as an electric current dipole oriented in the direction tangential
to the loop and of source strength \( J_0 = I_0 \, dt \). We carry out the integration along the loop in the space-Laplace domain rather than the conventional way in the space-time domain (Raiche et al., 1986; West, 1987). As will become clear later on, this will reduce the numerical computations to be carried out significantly.

The integration along side \( BC \) of the loop source is carried out in detail. The space-Laplace domain Green’s function that corresponds to this side is denoted as \( \hat{G}^{BC}(\mathbf{r}, s) \) and is given by (cf. Eq. (4.63))

\[
\hat{G}^{BC}(\mathbf{r}, s) = \frac{\kappa(s)}{4\pi^2} \int_{-l_{1/2}}^{l_{1/2}} da_1' \int_{-\infty}^{\infty} da_2 \int_{-\infty}^{\infty} \exp(-s^{1/2}i\alpha_1(x_1 - x_1')) \\
\times \exp(-s^{1/2}(i\alpha_2(x_2 - x_{2;B}) + \sum_j h_j^\gamma(j))) \, B^{(1)}(\alpha_1) \, da_1.
\]

(5.2)

Here, \( B^{(1)}(\alpha_1, \alpha_2) \) denotes the function \( B(\alpha_1, \alpha_2) \) corresponding to a generalized diffusive field constituent, due to an electric current dipole oriented in the \( x_i \) direction and of source strength \( I_0 \) (cf. Eqs. (4.59) and (4.61)). The integral along side \( BC \) of the loop source in Eq. (5.2) is carried out prior to the transformation back to the space-time domain is performed. This yields the space-Laplace domain expression for the diffusive electromagnetic field generated by the loop segment \( BC \) as

\[
\hat{G}^{BC}(\mathbf{r}, s) = \frac{\kappa'(s)}{4\pi^2} \int_{-\infty}^{\infty} da_2 \int_{-\infty}^{\infty} \left( \frac{\exp(-s^{1/2}i\alpha_1\tilde{z}_{1;B}) - \exp(-s^{1/2}i\alpha_1\tilde{z}_{1;C})}{i\alpha_1} \right) \\
\times \exp(-s^{1/2}(i\alpha_2\tilde{z}_{2;B} + \sum_j h_j^\gamma(j))) \, B^{(1)}(\alpha_1) \, da_1.
\]

(5.3)

Here, \( \tilde{z}_{1;B}, \tilde{z}_{1;C} \) are shorthand notations for \( z_1 - x_{1;B} \) and \( z_1 - x_{1;C} \), respectively. The function \( \kappa'(s) \) denotes \( \kappa(s)/s^{1/2} \). Note that \( \alpha_1 = 0 \) is not a singularity of the integrand at the right-hand side of Eq. (5.3).

The transformation from the transform-domain back to the space-time domain of \( \hat{G}^{BC}(\mathbf{r}, s) \) is carried out with the aid of the modified Cagniard method. We follow the same procedure as in Section 4.3, i.e., we start with a change of variables of integration. We introduce the new variables of integration \( p \) and \( \psi \) defined as

\[
i\alpha_1 = p \cos(\psi),
\]

(5.4)

\[
i\alpha_2 = p \sin(\psi),
\]

(5.5)
where \(0 \leq p < \infty\), \(0 \leq \psi < 2\pi\). Since \(d\alpha_1 d\alpha_2 = -pd\psi d\phi\), we can rewrite Eq. (5.3) as

\[
\hat{G}^{BC}(s, s) = -\frac{\hat{k}^t(s)}{4\pi^2} \int_0^{2\pi} \frac{d\psi}{\cos(\psi)} \int_0^{\infty} \left( \frac{\exp(-s^{1/2} p \tilde{z}_{1:1:1} \cos(\psi)) - \exp(-s^{1/2} p \tilde{z}_{1:1:1} \cos(\psi))}{\cos(\psi)} \right)
\times \exp(-s^{1/2} (p \tilde{z}_{2:1:1} \sin(\psi) + \sum_j h_j \tilde{z}_j(s))) \hat{B}^{(1)}(p, \psi) dp.
\]

Here, \(\tilde{z}_j(s, \psi)\) and \(\hat{B}^{(1)}(p, \psi)\) have been obtained from \(\gamma^{(j)}(\alpha_1, \alpha_2)\) and \(\hat{B}^{(1)}(\alpha_1, \alpha_2)\) by the substitutions in Eqs. (5.4) and (5.5). Note that \(\psi = \pi/2\) and \(\psi = 3\pi/2\) are not singular points of the integrand of Eq. (5.6). In order to apply the modified Cagniard method to Eq. (5.6), we have to separate the two exponential terms \(\exp(-s^{1/2} p \tilde{z}_{1:1:1} \cos(\psi))\) and \(\exp(-s^{1/2} p \tilde{z}_{1:1:1} \cos(\psi))\) of the integrand on the right-hand side of Eq. (5.6). However, since \(\psi = \pi/2\) and \(\psi = 3\pi/2\) are singular points of these separate parts, we must replace the \(\psi\)-integrals by their principal-value integrals

\[
\hat{G}^{BC}(s, s) = -\frac{\hat{k}^t(s)}{4\pi^2} \int_0^{2\pi} \frac{d\psi}{\cos(\psi)} \int_0^{\infty} \exp(-s^{1/2} p r_C \cos(\psi - \theta_C))
\times \exp(-s^{1/2} \sum_j h_j \tilde{z}_j(s)) \hat{B}^{(1)}(p, \psi) dp.
\]

\[
+ \frac{\hat{k}^t(s)}{4\pi^2} \int_0^{2\pi} \frac{d\psi}{\cos(\psi)} \int_0^{\infty} \exp(-s^{1/2} p r_B \cos(\psi - \theta_B)) \times
\times \exp(-s^{1/2} \sum_j h_j \tilde{z}_j(s)) \hat{B}^{(1)}(p, \psi) dp.
\]

Here, we have made use of \(\tilde{z}_{1:1:1} \cos \psi + \tilde{z}_{2:1:1} \sin \psi = r_B \cos(\psi - \theta_B)\), \(\tilde{z}_{1:1:1} \cos \psi + \tilde{z}_{2:1:1} \sin \psi = r_C \cos(\psi - \theta_C)\) where \(\{r_B, \theta_B\}\) and \(\{r_C, \theta_C\}\) represent the polar-coordinate specification of the point of observation relative to the vertices \(B\) and \(C\), respectively, i.e.,

\[
\tilde{z}_{1:1:1} = r_B \cos(\theta_B), \quad \tilde{z}_{2:1:1} = r_B \sin(\theta_B), \quad \tilde{z}_{1:1:1} = r_C \cos(\theta_C), \quad \tilde{z}_{2:1:1} = r_C \sin(\theta_C).
\]

Since \(\tilde{z}_j(s, \psi)\) and \(\hat{B}^{(1)}(p, \psi)\) satisfy Schwarz's reflection principle, the integrands at the right-hand side of Eq. (5.7) satisfies Schwartz's reflection principle as well. Using
this property and the periodicity of the integrand with respect to \( \psi \) with period \( 2\pi \), we rewrite Eq. (5.7) as

\[
G^{BC}(w, s) = -\frac{k'(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} \frac{d\psi}{\cos(\psi + \theta_C)} \text{Re} \left( \int_0^{\infty} \exp(-s^{1/2}p_C r_C \cos(\psi)) \right. \\
\times \exp(-s^{1/2} \sum_j h_j \tilde{\tau}^{(j)}(\tilde{B}^{(1)} dp) \\
+ \frac{k'(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} \frac{d\psi}{\cos(\psi + \theta_B)} \text{Re} \left( \int_0^{\infty} \exp(-s^{1/2} p_B r_B \cos(\psi)) \right)
\times \exp(-s^{1/2} \sum_j h_j \tilde{\tau}^{(j)}(\tilde{B}^{(1)} dp) 
\right)
\]

(5.10)

where \( \tilde{\tau}^{(j)} = \tilde{\tau}^{(j)}(p, \psi + \theta_{B,C}) \) and \( \tilde{B}^{(1)} = \tilde{B}^{(1)}(p, \psi + \theta_{B,C}) \). The integrals on the right-hand side of Eq. (5.10) are almost of the same form as Eq. (4.72), the only difference being the principal-value integrals of \( \psi \) and the \( \cos^{-1} \) functions. From this we conclude that the space-Laplace domain Green's function \( \hat{G}^{BC}(w, s) \) admits a transformation back to the space-time domain with the aid of the modified Cagniard method. The integrations with respect to \( p \) are carried out along a certain contour in the complex \( p \)-plane that deviates from the imaginary \( p \)-axis and for which the argument of the exponential functions is real (modified Cagniard contour). Obviously, to each of the two \( p \)-integrals of Eq. (5.10) there corresponds a modified Cagniard contour. Following the same procedure of Section 4.3, we arrive at the space-time domain expression for the Green's function \( G^{BC}(w, t) \) as

\[
G^{BC}(w, t) = \begin{cases} 
0, & t < 0, \\
-\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} \frac{d\psi}{\cos(\psi + \theta_C)} \int_{\tau_{\min}}^{\infty} k'(t, \tau) \text{Re} \left( \tilde{B}^{(1)}(p_C, \psi + \theta_C) \frac{\partial p_C}{\partial \tau} \right) d\tau \\
+ \frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} \frac{d\psi}{\cos(\psi + \theta_B)} \int_{\tau_{\min}}^{\infty} k'(t, \tau) \text{Re} \left( \tilde{B}^{(1)}(p_B, \psi + \theta_B) \frac{\partial p_B}{\partial \tau} \right) d\tau, & t \geq 0,
\end{cases}
\]
where \( p_B(\tau) \) and \( p_C(\tau) \) are the modified Cagniard contours that correspond to the vertices \( B \) and \( C \), respectively. They follow from (cf. Eq. (4.73))

\[
\begin{align*}
    p_B \gamma_B \cos(\psi) + \sum_j h_j \gamma^{(j)}(p_B, \psi + \theta_B) &= \tau, \\
p_C \gamma_C \cos(\psi) + \sum_j h_j \gamma^{(j)}(p_C, \psi + \theta_C) &= \tau,
\end{align*}
\]

where \( \tau \) is real and positive, \( \tau \geq T_{\text{min}} \).

The occurrence of principal value integrals in the space-time domain Green's function \( G^{BC}(x,t) \) is no obstruction to the numerical evaluation of this Green's function. The \( \tau \)-integral is an analytic and well-behaved function of \( \psi \) that can be computed to any desired accuracy. This enables us to use standard techniques for evaluating principal-value integrals of this type. Let \( g(\psi,t) \) denote one of the \( \tau \)-integrals from Eq. (5.11). A simple and efficient way for the evaluation of the principal-value integrals of Eq. (5.11) is arrived at by making use of the following identity:

\[
\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} \frac{g(\psi,t)}{\cos(\psi + \theta)} \, d\psi = \frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} \frac{g(\psi,t) - g(\psi_0,t)}{\cos(\psi + \theta)} \, d\psi + \frac{g(\psi_0,t)}{2\pi^2} \int_{-\pi/2}^{\pi/2} \frac{d\psi}{\cos(\psi + \theta)},
\]

where \( \psi_0 \) denotes the value of \( \psi \) for which \( \cos(\psi + \theta) = 0 \), \(-\pi/2 < \psi < \pi/2\). The integrand of the first integral on the right-hand side of Eq. (5.14) is a well-behaved function of \( \psi \) for \( \psi \approx \psi_0 \) and can be computed to any desired accuracy. The second integral of Eq. (5.14) is an elementary integral, i.e.,

\[
\int_{-\pi/2}^{\pi/2} \frac{d\psi}{\cos(\psi + \theta)} = 2\ln(4 \cot(|\theta/2|)) \quad 0 < |\theta| < \pi.
\]

We use the same method to obtain space-time domain expressions for the Green's functions that correspond to the transient diffusive fields generated by the other sides of the source loop. As can be expected, these expressions are similar to Eq. (5.11).
Collecting the results yields the space-time domain Green’s function $G^t(x, t)$ for a particular generalized diffusive field constituent of the transient diffusive electromagnetic field generated by a rectangular loop source as

$$G^t(x, t) = \begin{cases} 
0, & t < 0, \\
-\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_{T_{\min}}^{\infty} k^t(t, \tau) \Re(\tilde{B}^t(p_A, \psi + \theta_A) \frac{\partial p_A}{\partial \tau}) d\tau, \\
+\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_{T_{\min}}^{\infty} k^t(t, \tau) \Re(\tilde{B}^t(p_B, \psi + \theta_B) \frac{\partial p_B}{\partial \tau}) d\tau, \\
-\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_{T_{\min}}^{\infty} k^t(t, \tau) \Re(\tilde{B}^t(p_C, \psi + \theta_C) \frac{\partial p_C}{\partial \tau}) d\tau, \\
+\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_{T_{\min}}^{\infty} k^t(t, \tau) \Re(\tilde{B}^t(p_D, \psi + \theta_D) \frac{\partial p_D}{\partial \tau}) d\tau, & t \geq 0.
\end{cases} \quad (5.16)$$

where $\tilde{B}^t(p, \psi)$ denotes the function

$$\tilde{B}^t(p, \psi) = \frac{\tilde{B}^{(1)}(p, \psi)}{\cos(\psi)} - \frac{\tilde{B}^{(2)}(p, \psi)}{\sin(\psi)}. \quad (5.17)$$

$p_A(\tau)$ and $p_D(\tau)$ are the modified Cagniard contours that correspond to the vertices $A$ and $D$ respectively. They follow from (cf. Eq. (4.73))

$$p_A(\tau) \cos(\psi) + \sum_j h_j \tilde{\gamma}_j^{(j)}(p_A, \psi + \theta_A) = \tau, \quad (5.18)$$

$$p_D(\tau) \cos(\psi) + \sum_j h_j \tilde{\gamma}_j^{(j)}(p_D, \psi + \theta_D) = \tau, \quad (5.19)$$

where $\tau$ is real and positive, $\tau \geq T_{\min} = \sum_j h_j \tilde{\gamma}_j^{(j)}(p = 0)$. 
5.3 The in-loop configuration

In this section we consider a special case of the transient diffusive electromagnetic field generated by a plane, rectangular loop source, viz. the case where the sides of the loop have equal lengths $L$ and where the point of observation is located at the center of the loop. Such a configuration is referred to as the in-loop or central-loop mode of the geophysical exploration system. It is one of the most commonly used configurations. The space-time domain Green's function of the transient diffusive electromagnetic field at the center of the loop is denoted by $\mathcal{G}(\omega, t)$. This Green's function is given by (cf. Eq. (5.16))

$$
\mathcal{G}(\omega, t) = \begin{cases} 
0, & t < 0, \\
-\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_{T_{\min}}^{\infty} k'(t, \tau) \text{Re}(\tilde{B}'(p_A, \psi + \frac{\pi}{4}) \frac{\partial p_A}{\partial \tau}) \, d\tau, \\
+\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_{T_{\min}}^{\infty} k'(t, \tau) \text{Re}(\tilde{B}'(p_B, \psi - \frac{\pi}{4}) \frac{\partial p_B}{\partial \tau}) \, d\tau, \\
-\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_{T_{\min}}^{\infty} k'(t, \tau) \text{Re}(\tilde{B}'(p_C, \psi - \frac{3\pi}{4}) \frac{\partial p_C}{\partial \tau}) \, d\tau, \\
+\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \int_{T_{\min}}^{\infty} k'(t, \tau) \text{Re}(\tilde{B}'(p_D, \psi + \frac{3\pi}{4}) \frac{\partial p_D}{\partial \tau}) \, d\tau, & t \geq 0.
\end{cases}
$$

(5.20)

Here, $p_A(\tau), p_B(\tau), p_C(\tau),$ and $p_D(\tau)$ denote the modified Cagniard contours corresponding to the vertices $A, B, C,$ and $D,$ respectively. These modified Cagniard contours are parametrized by the real and positive variable $\tau$, $\tau \geq T_{\min}$, such that for fixed $\psi$, $-\pi/2 < \psi < \pi/2$ we have

$$
p_A \tau \cos(\psi) + \sum_j h_j \tilde{\tau}(j)(p_A, \psi + \frac{\pi}{4}) = \tau, 
$$

(5.21)

$$
p_B \tau \cos(\psi) + \sum_j h_j \tilde{\tau}(j)(p_B, \psi - \frac{\pi}{4}) = \tau, 
$$

(5.22)
\[ p_c r \cos(\psi) + \sum_i h_i \tilde{r}(p_c, \psi - \frac{3\pi}{4}) = r, \]  
\[ p_D r \cos(\psi) + \sum_i h_i \tilde{r}(p_D, \psi + \frac{3\pi}{4}) = r. \]  

Here, \( r \) denotes the distance from the center of the loop to each of the vertices, i.e., \( r = L/\sqrt{2} \). In general, each of the modified Cagniard contours is different. Only in special cases of anisotropy some of modified Cagniard contours will be the same. For example, for transversely anisotropic media, the modified Cagniard contours corresponding to vertices opposite of the center \( O \) of the loop are the same \((p_A(\tau) = p_C(\tau) \text{ and } p_B(\tau) = p_D(\tau))\). Further, if the axis of symmetry is vertical, all four modified Cagniard contours are the same.

The numerical evaluation of the principal-value integrals of Eq. (5.20) can be carried out with the procedure as indicated by Eqs. (5.14) and (5.15). Now, the singularities of each integrand occur at \( \psi = \pm \pi/4 \). Owing to the symmetry of position of the point of observation relative to each of the vertices, it is possible for \( G^\omega(\omega, t) \) to simplify Eq. (5.14) even more. Let us denote by \( g^A(\psi, t), \ldots, g^D(\psi, t) \) the corresponding \( \tau \) integrals in Eq. (5.20) as a function of \( \psi \) for fixed time \( t \), times \( \cos(\psi + \pi/4) \sin(\psi + \pi/4) \). For example,

\[ g^A(\psi, t) = \int_{t_{\min}}^\infty k^f(t, \tau) \text{Re}(\tilde{B}^\omega(p_A, \psi + \frac{\pi}{4}) \frac{\partial \tilde{B}^\omega}{\partial \tau}) d\tau, \]  

where now

\[ \tilde{B}^\omega(p, \psi) = \cos(\psi) \sin(\psi) \tilde{B}(p, \psi) = \sin(\psi) \tilde{B}(1)(p, \psi) - \cos(\psi) \tilde{B}(2). \]

From Eq. (5.25) we observe that the \( g^A(\psi, t), \ldots, g^D(\psi, t) \) satisfy the following relations

\[ g^A\left(\frac{\pi}{4}, t\right) = g^D\left(-\frac{\pi}{4}, t\right), \quad g^B\left(\frac{\pi}{4}, t\right) = g^A\left(-\frac{\pi}{4}, t\right), \]

\[ g^C\left(\frac{\pi}{4}, t\right) = g^B\left(-\frac{\pi}{4}, t\right), \quad g^D\left(\frac{\pi}{4}, t\right) = g^C\left(-\frac{\pi}{4}, t\right), \]
and consequently

\[ g^A(\frac{\pi}{4},t) + g^B(\frac{\pi}{4},t) + g^C(\frac{\pi}{4},t) + g^D(\frac{\pi}{4},t) = \\
g^A(-\frac{\pi}{4},t) + g^B(-\frac{\pi}{4},t) + g^C(-\frac{\pi}{4},t) + g^D(-\frac{\pi}{4},t). \]  

(5.29)

Furthermore,

\[ \int_{-\pi/2}^{\pi/2} \frac{d\psi}{\sin(\psi+\pi/4)\cos(\psi+\pi/4)} \equiv 0. \]  

(5.30)

Upon combining Eq. (5.20) with Eqs. (5.28) and (5.29) we arrive at the following expression for the Green's function \( G^0(\omega, t) \) of the transient diffusive electromagnetic field at the center of a square loop of which the numerical implementation can be carried out in a very straightforward and simple manner:

\[
G^0(\omega, t) = \begin{cases} 
0, & t < 0, \\
-\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} \left( g^A(\psi,t) - g^A(\pi/4,t) + g^B(\psi,t) - g^B(\pi/4,t) + g^C(\psi,t) - g^C(\pi/4,t) + g^D(\psi,t) - g^D(\pi/4,t) \right) \times \\
\frac{d\psi}{\sin(\psi+\pi/4)\cos(\psi+\pi/4)} & t \geq 0.
\end{cases}
\]  

(5.31)
Diffusive electromagnetic fields in isotropic media

6.1 Introduction

The objective of the preceding chapters was to obtain closed-form expressions for the transient diffusive electromagnetic field in (stratified) arbitrarily anisotropic media, i.e., without any restrictions on the kind of anisotropy of the media involved (as in contrast with, for example, Le Masne and Vasseur, 1981; Nabulsh and Wait, 1982; Kaufman and Keller, 1983; Edwards et al., 1984 and Nobes, 1986). A consequence of this general approach is that almost none of our results can be evaluated analytically and expressed in terms of elementary or special functions. As a result, all our computations on diffusive electromagnetic fields in arbitrarily anisotropic media have to be carried out numerically.

In this chapter we investigate the much simpler case of isotropic media. This case is not only of importance from a mathematical point of view – we expect major simplifications of our results as compared with anisotropic media – but also from a practical point of view: although the detection of anisotropy in the subsurface of the earth is becoming of more and more importance in geophysical prospecting (Tittman, 1990), still anisotropy is not that frequently observed and for the interpretation of field data it is in many cases sufficient to use a model of the earth that is based on isotropic media (Spies and Raiche, 1980; Adhidjaja et al., 1985; Frischknecht and Raab, 1984; Fitterman and Stewart, 1986; Nekut, 1987). But most of all, we need the results on isotropic media for the understanding of the effects of
anisotropy on the diffusion of transient electromagnetic fields.

In the subsequent sections we shall investigate how the results from previous chapters simplify if the electromagnetic properties of the media are isotropic. In Section 6.5 we shall consider a number of simple configurations for which explicit analytic expressions for the space-time domain diffusive electromagnetic field have been obtained.

6.2 Unbounded homogeneous isotropic media

In this section we shall investigate how our results from Chapter 3 on the diffusion of transient electromagnetic fields in homogeneous anisotropic media simplify if the electromagnetic properties of the medium under consideration are isotropic, i.e., when \( \sigma_{k,r} = \sigma \delta_{k,r} \) and \( \mu_{j,p} = \mu \delta_{j,p} \). As has been indicated in the introduction already, we expect to end up with relatively simple expressions for the transform-domain electric and magnetic field strengths, the corresponding space-time domain expressions of which are elementary and special functions (Abramowitz and Stegun, 1970).

The system's matrix \( A_{I,J} \) and source matrix \( N_I \)

Upon substituting \( \sigma_{k,r} = \sigma \delta_{k,r} \) and \( \mu_{j,p} = \mu \delta_{j,p} \) into the the expression for the system's matrix \( A_{I,J} \), Eq. (3.11), we obtain (cf. Eqs. (3.11)-(3.14))

\[
A_{I,J} = \begin{pmatrix}
0 & A^{(EH)}_{\sigma,r} \\
A^{(HE)}_{\sigma,r} & 0
\end{pmatrix}, \tag{6.1}
\]

where now \( A^{(EH)}_{\sigma,r} \) and \( A^{(HE)}_{\sigma,r} \) are the following 2-by-2 submatrices

\[
A^{(EH)}_{\sigma,r} = \frac{1}{\sigma} \begin{pmatrix}
\alpha_1^2 + \mu \sigma & \alpha_1 \alpha_2 \\
\alpha_1 \alpha_2 & \alpha_2^2 + \mu \sigma
\end{pmatrix}, \tag{6.2}
\]
\[ A^{(HE)}_{r,s} = \frac{1}{\mu} \begin{pmatrix} \alpha_2^2 + \mu \sigma & -\alpha_1 \alpha_2 \\ -\alpha_1 \alpha_2 & \alpha_1^2 + \mu \sigma \end{pmatrix}. \] (6.3)

Note that the submatrices \( A^{(EE)}_{r,s} \) and \( A^{(HH)}_{r,s} \) of Eq. (3.14) are equal to zero.

As is easily verified, the product of \( A^{(EH)}_{r,s} \) and \( A^{(HE)}_{s,r} \) is equal to \( \alpha_1^2 + \alpha_2^2 + \mu \sigma \) times the 2-by-2 unit matrix, i.e.,

\[ A^{(EH)}_{r,s} A^{(HE)}_{s,r} = (\alpha_1^2 + \alpha_2^2 + \mu \sigma) \delta_{r,s}. \] (6.4)

Finally, the expression for the notional source matrix \( N_I \) now reduces to (cf. Eqs. (3.15)-(3.17))

\[ N_I = (N_1^{(E)}, N_2^{(E)}, N_1^{(H)}, N_2^{(H)})^T, \] (6.5)

where

\[ N_1^{(E)} = \begin{pmatrix} -\tilde{K}_2^e + i \alpha_1 \frac{s^{1/2} \tilde{f}_e}{\sigma} \\ \tilde{K}_1^e + i \alpha_2 \frac{s^{1/2} \tilde{f}_e}{\sigma} \end{pmatrix}, \] (6.6)

\[ N_1^{(H)} = \begin{pmatrix} -s^{1/2} \tilde{f}_1^e + i \alpha_2 \frac{\tilde{K}_3^e}{\mu} \\ -s^{1/2} \tilde{f}_2^e - i \alpha_1 \frac{\tilde{K}_3^e}{\mu} \end{pmatrix}. \] (6.7)

**Eigenvalues and corresponding eigenvectors of the system's matrix**

In Appendix A we have derived some general properties of the eigenvalues \( \gamma^{(+\nu)} \) of the system's matrix \( A_{I,J} \) as a function of the complex variables \( \alpha_1 \) and \( \alpha_2 \) for arbitrarily anisotropic media. Obviously, the results from this appendix must also hold for the eigenvalues of the system's matrix \( A_{I,J} \) as given by Eq. (6.1).

The eigenvalues follow from the determinantal equation (A.5). In view of the expression for \( A_{I,J} \), Eq. (6.1), we find that the determinantal equation is the following
algebraic equation of the second degree in $\gamma^2$

$$\gamma^4 - 2(\alpha_1^2 + \alpha_2^2 + \mu\sigma)\gamma^2 + (\alpha_1^2 + \alpha_2^2 + \mu\sigma)^2 = 0. \quad (6.8)$$

From this, it follows that

$$\gamma^{(-1)} = \gamma^{(-2)} = -\gamma = -(\alpha_1^2 + \alpha_2^2 + \mu\sigma)^{1/2}, \quad \text{Re}(\cdots)^{1/2} \geq 0, \quad (6.9)$$

$$\gamma^{(+1)} = \gamma^{(+2)} = \gamma = (\alpha_1^2 + \alpha_2^2 + \mu\sigma)^{1/2}, \quad \text{Re}(\cdots)^{1/2} \geq 0. \quad (6.10)$$

Note that this partitioning of the eigenvalues is in agreement with Eqs. (3.46)-(3.48) and Eqs. (A.23)-(A.24). From Eqs. (6.9)-(6.10) we conclude that for isotropic media, the eigenvalues always occur in two equal pairs of opposite values, irrespective of the (complex) values of $\alpha_1$ and $\alpha_2$.

Following the conventions of heat conduction in solids, we shall refer to the quantity $(\sigma\mu)^{-1}$ as the (electromagnetic) diffusion constant (Carslaw and Jaeger, 1959; De Hoop and Oristaglio, 1988). In SI-units, $(\sigma\mu)^{-1}$ has the dimension of [m/s²].

To each of the eigenvalues $\gamma^{(\pm\nu)}$ there corresponds an eigenvector $\delta_j^{(\pm\nu)}$ of the system's matrix $A_{i,j}$ (cf. Eq. (3.51)). However, since the eigenvalues occur in pairs of equal values, there is some freedom in the actual choice for the eigenvectors. Even the condition that we should construct the eigenvectors of $A_{i,j}$ is such a way that they are $H^2$-orthogonal (cf. Eq. (3.58)) does not limit the eigenvectors to a single set. For a number of reasons to become clear later on in Sections 6.3 and 6.4, we have made the following "choice" for the eigenvectors of the system's matrix $A_{i,j}$ of an isotropic medium

$$\delta_j^{(-1)} = \left( Z^{(1)}\alpha_1, Z^{(1)}\alpha_2, -\alpha_1, -\alpha_2 \right)^T / (2(\alpha_1^2 + \alpha_2^2))^{1/2}, \quad (6.11)$$

$$\delta_j^{(+1)} = \left( Z^{(1)}\alpha_1, Z^{(1)}\alpha_2, \alpha_1, \alpha_2 \right)^T / (2(\alpha_1^2 + \alpha_2^2))^{1/2}, \quad (6.12)$$

$$\delta_j^{(-2)} = \left( -\alpha_2, \alpha_1, Y^{(2)}\alpha_2, -Y^{(2)}\alpha_1 \right)^T / (2(\alpha_1^2 + \alpha_2^2))^{1/2}, \quad (6.13)$$

$$\delta_j^{(+2)} = \left( -\alpha_2, \alpha_1, -Y^{(2)}\alpha_2, Y^{(2)}\alpha_1 \right)^T / (2(\alpha_1^2 + \alpha_2^2))^{1/2}, \quad (6.14)$$
where \(Y^{(\nu)}\) and \(Z^{(\nu)} = 1/Y^{(\nu)}\) denote the transform-domain diffusive electromagnetic field "admittances" and "impedances", respectively:

\[
\begin{align*}
Y^{(1)} &= \sigma/\gamma, & Z^{(1)} &= \gamma/\sigma, \\
Y^{(2)} &= \gamma/\mu, & Z^{(2)} &= \mu/\gamma.
\end{align*}
\]

(6.15) \hspace{1cm} (6.16)

In SI-units, the \(Y^{(\nu)}\) have the dimension of \([S/s^{1/2}]\).

With this set of eigenvectors we have chosen a different normalization as the one given by Eqs. (3.56)-(3.57), i.e.,

\[
\begin{align*}
b^{(+1)}_i H^\phi_{i,j} b^{(+1)}_j &= + Z^{(1)}, & b^{(+2)}_i H^\phi_{i,j} b^{(+2)}_j &= + Y^{(2)}, \\
b^{(-1)}_i H^\phi_{i,j} b^{(-1)}_j &= - Z^{(1)}, & b^{(-2)}_i H^\phi_{i,j} b^{(-2)}_j &= - Y^{(2)}.
\end{align*}
\]

(6.17) \hspace{1cm} (6.18)

The reason for choosing this normalization will become clear in Section 6.3 where we shall consider the reflection and transmission coefficients that describe the interaction of diffusive electromagnetic fields at an interface between isotropic media with different electromagnetic properties. Clearly, the \(b^{(\pm\nu)}\) are \(H^\phi\)-orthogonal, i.e.,

\[
b^{(+\nu)}_i H^\phi_{i,j} b^{(-\nu)}_j = 0.
\]

(6.19)

Finally, the four eigenrows \(g^{(\pm\nu)}_i\) of the system’s matrix \(A_{i,j}\) are obtained as

\[
\begin{align*}
g^{(-1)}_i &= \left(Y^{(1)} \alpha_1, Y^{(1)} \alpha_2, -\alpha_1, -\alpha_2\right)^T / (2(\alpha_1^2 + \alpha_2^2))^{1/2}, \\
g^{(+1)}_i &= \left(Y^{(1)} \alpha_1, Y^{(1)} \alpha_2, \alpha_1, \alpha_2\right)^T / (2(\alpha_1^2 + \alpha_2^2))^{1/2}, \\
g^{(-2)}_i &= \left(-\alpha_2, \alpha_1, Z^{(2)} \alpha_2, -Z^{(2)} \alpha_1\right)^T / (2(\alpha_1^2 + \alpha_2^2))^{1/2}, \\
g^{(+2)}_i &= \left(-\alpha_2, \alpha_1, -Z^{(2)} \alpha_2, Z^{(2)} \alpha_1\right)^T / (2(\alpha_1^2 + \alpha_2^2))^{1/2}.
\end{align*}
\]

(6.20) \hspace{1cm} (6.21) \hspace{1cm} (6.22) \hspace{1cm} (6.23)
The transform-domain field matrix $F_I$.

In this subsection we shall combine the results of Section 3.3 with the results of the previous subsections on isotropic media in order to end up with an explicit expression for the transform-domain field matrix $F_I(x_3)$. This 4-by-1 field matrix is the solution of the linear first-order matrix differential equation (cf. Eq. (3.62))

$$
\partial_3 F_I = -s^{1/2} A_{I,J} F_J + N_I,
$$

(6.24)

where now the system's matrix $A_{I,J}$ and the notional source matrix $N_I$ are given by Eqs. (6.1)-(6.3) and Eqs. (6.5)-(6.7), respectively. In Section 3.3 we have described how the general solution of Eq. (6.24) is obtained and can be expressed in terms of transform-domain diffusive field constituents. Here, we shall only recall the main results of Section 3.3 and show how they simplify in the case of homogeneous isotropic media.

Let the concentrated source be located at $w = w_s$. Without loss of generality we take $w_s = (0, 0, x_{3,s})$. The source matrix $N_I$ is then of the form (cf. Eq. (3.77))

$$
N_I = \tilde{N}_I \delta(x_3 - x_{3,s}) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) X_I \delta(x_3 - x_{3,s}),
$$

(6.25)

where $\delta(x_3 - x_{3,s})$ is the one-dimensional dirac distribution acting at the source level, $\hat{\phi}(s)$ the Laplace transform of the source signature $\phi(t)$ and $X_I$ a 4-by-1 matrix that depends on the nature of the source and the Fourier transformation parameters $\alpha_1$ and $\alpha_2$. Further, from Eqs. (6.6)-(6.7) we find that $\hat{k}_{\text{source}}(s) = s^{1/2}$ if the source is of the electric-current type, while $\hat{k}_{\text{source}}(s) = 1$ if the source is of the magnetic-current type.

The general solution of Eq. (6.24) can be written as the superposition of four terms, each corresponding to a (transform-domain) diffusive field electromagnetic field. Each term is referred to as a diffusive field constituent. The general shape of a single diffusive field constituent is (cf. Eq. (3.80))

$$
F_I(x_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) B_j^{(\pm r)} \exp(-s^{1/2} \gamma^{(\pm r)}(x_3 - x_{3,s})),
$$

(6.26)

where

$$
B_j^{(\pm r)} = \pm \delta_j^{(\pm r)} g_l^{(\pm r)} X_I.
$$

(6.27)
In agreement with Eqs. (6.9)-(6.10), the two diffusive constituents with superscripts \((-\tau)\) correspond to fields diffusing away from the source in the upward direction, while the other two diffusive constituents with the superscripts \((+\tau)\) correspond to fields diffusing away from the source in the downward direction. Now, upon making use of the fact that the \(\gamma^{(\pm \tau)}\) occur in two equal pairs of opposite values, we can take the contributions from the two relevant diffusive constituents together to obtain

\[
F_j(x_3) = \hat{\varphi}(s) \hat{k}_{\text{source}}(s) B_j^{(\pm)} \exp\left(-s^{1/2}y|x_3 - x_{3,s}|\right),
\]

(6.28)

where

\[
B_j^{(\pm)} = \pm (b_j^{(\pm 1)} g_i^{(\pm 1)} X_i + b_j^{(\pm 2)} g_i^{(\pm 2)} X_i).
\]

(6.29)

Here, \(g_i^{(\pm \tau)}X_i\) expresses the coupling between the source and the relevant diffusive constituent. In view of Eqs. (6.6)-(6.7) and (6.19)-(6.22) we find that

(i) for a vertical electric-current dipole source \(g_i^{(\pm 2)} X_i\) is equal to zero (i.e., when all the source terms are equal to zero except for \(\tilde{J}_3\)),

(ii) for a vertical magnetic-current dipole source \(g_i^{(\pm 1)} X_i\) is equal to zero (i.e., when all the source terms are equal to zero except for \(\tilde{K}_3\)).

Hence, a vertical electric dipole is coupled only to \(b_j^{(\pm 1)}\), while a vertical magnetic dipole is coupled only to \(b_j^{(\pm 2)}\). Apparently, these diffusive constituents represent two independent polarization states and are denoted as the "toroidal" and "poloidal" parts of the (transform-domain) electromagnetic field (Chave, 1984; Edwards and Chave, 1986; Nobes, 1986).

To conclude this section we shall give explicit expressions for \(B_j^{(\pm)}\) for the horizontal electric-current dipole source. Combining Eqs. (6.6)-(6.7), (6.11)-(6.14), (6.20)-(6.23) and (6.29) yields
6.3 Stratified isotropic media

In this section we investigate how the results of Chapter 4 on the diffusion of transient electromagnetic fields in stratified anisotropic media simplify if the media involved are isotropic in their electromagnetic properties, i.e., when \( \sigma_{k,r}^{(m)} \rightarrow \sigma^{(m)} \delta_{k,r}, \mu_{j,p}^{(m)} \rightarrow \mu^{(m)} \delta_{j,p} \). Here \( \sigma^{(m)} \) and \( \mu^{(m)} \) denote the scalar electric conductivity and magnetic permeability of the subdomain \( D_m \) (cf. Eq. (2.22)-(2.23)). In particular, we expect to end up with simple expressions for the reflection and transmission coefficients of the interfaces between different media.

Reflection and transmission coefficients

Within each homogeneous subdomain \( D_m \) of the configuration, the transform-domain field matrix \( F_i^{(m)} \) satisfies the following linear first-order matrix differential equation (cf. Eq. (4.2)):

\[
\partial_3 F_i^{(m)} = -\sigma^{1/2} A_{i,j}^{(m)} F_j^{(m)} \quad \text{for} \quad x_{3m-1} < x_3 < x_{3m}, \quad m=2,\ldots, ND-1
\]  

(6.31)

while

\[
\partial_3 F_i^{(1)} = -\sigma^{1/2} A_{i,j}^{(1)} F_j^{(1)} \quad \text{for} \quad -\infty < x_3 < x_{3,1},
\]

(6.32)

\[
\partial_3 F_i^{(ND)} = -\sigma^{1/2} A_{i,j}^{(ND)} F_j^{(ND)} \quad \text{for} \quad x_{3,ND-1} < x_3 < \infty.
\]

(6.33)
Here, $A_{J,J}^{(m)}$ is the system's matrix corresponding to the isotropic medium of the subdomain $\mathcal{D}_m$ as given by Eqs. (6.1)-(6.3).

Across a source-free interface where $\sigma$ and/or $\mu$ show a jump discontinuity, the horizontal components of the (transform-domain) electric and magnetic field strengths must be continuous. This implies that the field matrix $F_j(z_3)$ must be continuous across such an interface, i.e.,

$$
\lim_{z_3 \downarrow z_3;m} F_j^{(m+1)}(z_3) = \lim_{z_3 \uparrow z_3;m} F_j^{(m)}(z_3) = 0 \quad \text{for} \quad m \neq s. \quad (6.34)
$$

At the source level $z_3 = z_{3,s}$ where the concentrated source is situated, $F_j(z_3)$ is discontinuous. From Eq. (3.10) we find that for $z_3 = z_{3,s}$, $F_j$ shows a jump discontinuity of magnitude $\tilde{N}_j$ (cf. Eqs. (4.5) and (4.22)) and hence

$$
\lim_{z_3 \downarrow z_{3,s}} F_j^{(s+1)}(z_3) - \lim_{z_3 \uparrow z_{3,s}} F_j^{(s)}(z_3) = \tilde{N}_j = \hat{\phi}(s) \tilde{\kappa}_{\text{source}}(s) X_j. \quad (6.35)
$$

Here, $\hat{\phi}(s)$ denotes the Laplace transform of the source signature $\phi(t)$, $X_j$ is a 4-by-1 matrix that depends on the nature of the source and is independent of the Laplace transformation parameter $s$. From Eqs. (6.6)-(6.7) we have $\tilde{\kappa}_{\text{source}}(s) = 1$ if the source is of the magnetic-current type and $\tilde{\kappa}_{\text{source}}(s) = s^{1/2}$ if the source is of the electric-current type.

As in Section 4.2, we decompose $F_j^{(m)}$ of Eq. (6.31) into diffusive field constituents by writing $F_j^{(m)}$ as a superposition of the four eigensolutions of this matrix differential equation, i.e.,

$$
F_j^{(m)} = D_{J,N}^{(m)} W_N^{(m)}. \quad (6.36)
$$

Here, $D_{J,N}^{(m)}$ is the eigencolumn matrix of the system's matrix $A_{J,J}^{(m)}$ (cf. Eq. (4.10)). In agreement with the notation of Chapter 4, we shall denote the elements of $W_N^{(m)}$ by $W_{\nu}^{(m;\pm)}$

$$
W_N^{(m)} = \begin{pmatrix} W_1^{(m;\pm)}, W_2^{(m;\pm)}, W_1^{(m;+)}, W_2^{(m;+)} \end{pmatrix}^T. \quad (6.37)
$$
in which (cf. Eqs. (4.20)-(4.21),(4.23) and (6.9)-(6.10))

\[
W_\nu^{(m;-)} = \hat{\phi}(s) \hat{h}_\text{source}(s) W_\nu^{(m;-)} \exp(+s^{1/2}n^{(m)}(x_3 - x_{3;m}))
\]

\[\begin{align*}
x_{3;m-1} < x_3 < x_{3;m}, & \quad (6.38) \\
W_\nu^{(m;+)} = \hat{\phi}(s) \hat{h}_\text{source}(s) W_\nu^{(m;+)} \exp(-s^{1/2}n^{(m)}(x_3 - x_{3;m-1}))
\end{align*}\]

\[\begin{align*}
x_{3;m-1} < x_3 < x_{3;m} & \quad (6.39)
\end{align*}\]

The amplitudes $W_\nu^{(m;\pm)}$ express the acting of the source as well as the influence of the different electromagnetic properties of each layer. In some way they must follow from the boundary conditions (6.34) and (6.35) at each of the interfaces. This point will be elaborated later on. See Figure 6.1

![Figure 6.1: Subdomain $D_m$. The $W_\nu^{(m;\pm)}$ denote the amplitudes of the $b_j^{(m;\pm\nu)}$ diffusive field constituents and have zero phase at the interface from which they originate.](image)

The boundary conditions at the interface $x_3 = x_{3;m}$ relate the $W_\nu^{(m;\pm)}$ of the subdomain $D_m$ to the $W_\nu^{(m+1;\pm)}$ of the subdomain $D_{m+1}$, $m = 1, 2, \ldots, ND - 1$. Upon using the scattering matrix formalism to express the amplitudes of the fields diffusing away from a source-free interface in terms of the amplitudes diffusing towards that interface and rewriting the resulting equations, we end up with (cf. Eqs. (4.36)-(4.37))
\[ W^{(m;-)}_r = R^{(m;+)}_{r,r} W^{(m;+)}_r \exp(-s^{1/2}h_m \gamma^{(m)}) \]
\[ + T^{(m;-)}_{r,r} W^{(m+1;-)}_r \exp(-s^{1/2}h_{m+1} \gamma^{(m+1)}) \text{ for } m \neq s, \quad (6.40) \]

\[ W^{(m+1;+)}_r = T^{(m;+)}_{r,r} W^{(m;+)}_r \exp(-s^{1/2}h_m \gamma^{(m)}) \]
\[ + R^{(m;-)}_{r,r} W^{(m+1;-)}_r \exp(-s^{1/2}h_{m+1} \gamma^{(m+1)}) \text{ for } m \neq s. \quad (6.41) \]

in which \( h_m = x_{3;m+1} - x_{3;m} \) is the thickness of the \( m \)th layer (see Table 2.1). We denote by \( R^{(m;\pm)}_{r,r} \) and \( T^{(m;\pm)}_{r,r} \) the 2-by-2 reflection and transmission matrices that characterize the interaction of the diffusion fields at the interface \( x_3 = x_{3;m} \). From Eqs. (6.11)-(6.14) and (6.34)-(6.41) we find that \( R^{(m;\pm)}_{r,r} \) and \( T^{(m;\pm)}_{r,r} \) are the following diagonal matrices:

\[ T^{(m;+)}_{1,1} = \frac{2Z^{(m;1)}}{Z^{(m+1;1)} + Z^{(m;1)}}, \quad T^{(m;+)}_{2,2} = \frac{2Y^{(m;2)}}{Y^{(m;2)} + Y^{(m+1;2)}}, \quad (6.42) \]

\[ R^{(m;+)}_{1,1} = \frac{Z^{(m+1;1)} - Z^{(m;1)}}{Z^{(m+1;1)} + Z^{(m;1)}}, \quad R^{(m;+)}_{2,2} = \frac{Y^{(m;2)} - Y^{(m+1;2)}}{Y^{(m;2)} + Y^{(m+1;2)}}, \quad (6.43) \]

\[ T^{(m;-)}_{1,1} = \frac{2Z^{(m+1;1)}}{Z^{(m;1)} + Z^{(m+1;1)}}, \quad T^{(m;-)}_{2,2} = \frac{2Y^{(m+1;2)}}{Y^{(m+1;2)} + Y^{(m;2)}}, \quad (6.44) \]

\[ R^{(m;-)}_{1,1} = \frac{Z^{(m;1)} - Z^{(m+1;1)}}{Z^{(m;1)} + Z^{(m+1;1)}}, \quad R^{(m;-)}_{2,2} = \frac{Y^{(m+1;2)} - Y^{(m;2)}}{Y^{(m+1;2)} + Y^{(m;2)}}, \quad (6.45) \]

The \( Z^{(m;1)} \) and \( Y^{(m;2)} \) are the transform-domain diffusive electromagnetic field "impedances" and "admittances" of subdomain \( D_m \). Conform Eqs. (6.15)-(6.16) they are given by

\[ Z^{(m;1)} = \gamma^{(m)}/\sigma^{(m)} \quad Y^{(m;2)} = \gamma^{(m)}/\mu^{(m)}. \quad (6.46) \]
The fact that the reflection and transmission matrices are diagonal, reflects once more the existence of completely independent 'polarization' states in isotropic media. The interaction of the diffusive field constituents at an interface is illustrated in Figure 4.2.

Interaction at the source level

A similar procedure can be applied to the interface \( z_3 = x_3 \) at which the concentrated source is located. In this case, the amplitudes \( \overline{W}_\nu^{(s;i;\pm)} \) and \( \overline{W}_\nu^{(s+1;i;\pm)} \) of the fields diffusing away from the interface are related to the the amplitudes \( \overline{W}_\nu^{(s;i;\pm)} \) and \( \overline{W}_\nu^{(s+1;i;\pm)} \) of the fields diffusing towards that interface through the reflection and transmission matrices \( R_{\nu,\tau}^{(s;\pm)} \) and \( T_{\nu,\tau}^{(s;\pm)} \), while the source manifests itself by additional source-excited terms \( \mathcal{W}_\nu^{(s;\pm)} \). From Eqs. (6.35) and (6.38)-(6.39) we obtain

\[
\overline{W}_\nu^{(s;i;\pm)} = R_{\nu,\tau}^{(s;i;\pm)} \overline{W}_\tau^{(s;i;\pm)} \exp(-s^{1/2}h_{s+1} \gamma^{(s)}) + T_{\nu,\tau}^{(s;i;\pm)} \overline{W}_\tau^{(s+1;i;\pm)} \exp(-s^{1/2}h_{s+1} \gamma^{(s+1)}) + \mathcal{W}_\nu^{(s;i;\pm)}, \quad (6.47)
\]

\[
\overline{W}_\nu^{(s+1;i;\pm)} = T_{\nu,\tau}^{(s;i;\pm)} \overline{W}_\tau^{(s;i;\pm)} \exp(-s^{1/2}h_{s+1} \gamma^{(s)}) + R_{\nu,\tau}^{(s;i;\pm)} \overline{W}_\tau^{(s+1;i;\pm)} \exp(-s^{1/2}h_{s+1} \gamma^{(s+1)}) + \mathcal{W}_\nu^{(s;i;\pm)}. \quad (6.48)
\]

Here, the source terms \( \mathcal{W}_\nu^{(s;\pm)} \) follow from Eqs. (6.35)-(6.39) (cf. Eqs. (4.46)-(4.47))

\[
\mathcal{W}_\nu^{(s;i;\pm)} = -\sum_{r=1}^{2} T_{\nu,\tau}^{(s;i;\pm)} g_i^{(s+1;r)} X_I, \quad (6.49)
\]

\[
\mathcal{W}_\nu^{(s;i;\pm)} = \sum_{r=1}^{2} T_{\nu,\tau}^{(s;i;\pm)} g_i^{(s;r)} X_I. \quad (6.50)
\]
6.3 Stratified isotropic media

The transform-domain field matrix \( F_j \) in stratified isotropic media

From Section 4.2 it follows that by solving Eqs. (6.40)-(6.41) and (6.46)-(6.47) for the \( \tilde{W}_\nu^{(m; \pm)} \) with a recurrence procedure, the solution thus obtained will be of a form that is appropriate for our purpose, i.e., it admits a transformation of \( F_j^{(m)}(z_3) \), Eq. (6.36), from the transform-domain back to the space-time domain with the aid of the modified Cagniard method. To this end we have introduced the concept of generalized diffusive field constituents in order to describe the diffusion process of electromagnetic fields through layered media.

For details about this recurrence procedure and the generalized diffusive field constituents we refer to Section 4.2. Here we shall only give the main results and point out how the general expressions for anisotropic media simplify in the case of isotropic media.

The general solution \( F_j^{(m)}(z_3) \) of Eqs. (6.31)-(6.33) in layered isotropic media can be written as the sum of an infinite number of terms, denoted as generalized diffusive field constituents. These terms follow from a Neumann-type solution of Eqs. (6.40)-(6.41) and (6.46)-(6.47) (cf. Eqs. (4.53)-(4.54)). The general shape of a single generalized diffusive field constituent is

\[
F_j^{(m)}(z_3) = \hat{\phi}(s) \hat{k}_{\text{source}}(s) B_j^{(m)} \exp(-s^{1/2} \sum_j \gamma_j^{(s)}),
\]

(6.51)

where \( B_j^{(m)} \) is the product of certain reflection and transmission coefficients multiplied by a source term.

To obtain the total solution of the problem, we must take into account the contributions from all individual generalized diffusive constituents. However, as compared to the general anisotropic case, the actual number of constituents that contribute to the total solution can be reduced considerably. This is due to the fact that the reflection and transmission matrices are diagonal. Further, since the \( \gamma_j^{(s)} \) occur in two equal pairs of opposite values, we can take the contributions from the constituents having the same exponential term on the right-hand side of Eq. (6.50) together, thus reducing the number of terms to be evaluated even more (cf. Eq. (6.29)).

Through Eq. (6.51) we have established that each single term that contributes to the total solution \( F_j^{(m)}(z_3) \) of Eq. (6.31) admits a transformation from the transform domain back to the space-time domain with the modified Cagniard technique. However, such a transformation of the elements of \( F_j^{(m)} \) will only lead to appropri-
ate space-time domain expressions for the horizontal components of the electric field strength, not of the magnetic field strength. This is due to the inclusion of the factor $s^{1/2}$ in $\tilde{H}_1^{(m)}$ and $\tilde{H}_2^{(m)}$ in $F_j^{(m)}$ (cf. Eq. (4.1)). For this reason we rewrite Eq. (6.51) as the generalized diffused field constituent (cf. Eq. (4.61))

$$F_j^{(m)}(x_3) = \tilde{\phi}(s) \tilde{k}_{\text{source}}(s) \tilde{k}_{\text{field}}(s) B^{(m)} \exp(-s^{1/2} \Sigma_j h_j \gamma_{j(0)}), \quad (6.52)$$

such that the factor $s^{1/2}$ present at the left-hand side of Eq. (6.51) in $F_3^{(m)}$ and $F_4^{(m)}$ is now included in the function $\tilde{k}_{\text{field}}(s)$. From Eq. (3.9) we find that $\tilde{k}_{\text{field}}(s) = 1$ when the electric field strength is considered, while $\tilde{k}_{\text{field}}(s) = s^{-1/2}$ when the magnetic field strength is considered. For notational simplicity we shall denote by $\tilde{k}(s)$ the product of $\tilde{k}_{\text{source}}(s)$ and $\tilde{k}_{\text{field}}(s)$, i.e., $\tilde{k}(s) = \tilde{k}_{\text{source}}(s) \tilde{k}_{\text{field}}(s)$. The different functions $\tilde{k}(s)$ are given by Eqs. (3.83)-(3.85).

To conclude this section we give explicit expressions for the $B_j^{(m)}$ due to a horizontal electric current dipole oriented in the $i_2$-direction and located at the interface between two isotropic media. Upon combining the expressions for the source terms $\mathcal{W}_\nu^{(\alpha)}$, Eqs. (6.49)-(6.50), with Eqs. (6.36)-(6.39) and (6.52), we obtain

$$B_1^{(1)} = B_1^{(2)} = -J_2^{s} \frac{\alpha_1 \alpha_2}{2 \gamma^{(1)} \sigma^{(2)}} \Theta_0 \Theta_1, \quad (6.53)$$

$$B_2^{(1)} = B_2^{(2)} = -J_2^{s} \frac{\alpha_2^2}{2 \gamma^{(1)} \sigma^{(2)}} \Theta_0 \Theta_1 - J_2^{s} \frac{\mu^{(2)}}{2 \gamma^{(2)}} \Theta_2, \quad (6.54)$$

$$B_3^{(1)} = B_3^{(2)} = -J_2^{s} \frac{\alpha_1 \alpha_2 (\eta_2 - \eta_1)}{4 \gamma^{(1)} \gamma^{(2)}} \Theta_1 \Theta_2, \quad (6.55)$$

$$B_4^{(1)} = J_2^{s} \frac{\gamma^{(1)}}{2 \gamma^{(2)}} \Theta_2 - J_2^{s} \frac{\alpha_2^2 (\eta_2 - \eta_1)}{4 \gamma^{(1)} \gamma^{(2)}} \Theta_1 \Theta_2, \quad (6.56)$$

$$B_4^{(2)} = -J_2^{s} \frac{1}{2} \Theta_2 - J_2^{s} \frac{\alpha_2^2 (\eta_2 - \eta_1)}{4 \gamma^{(1)} \gamma^{(2)}} \Theta_1 \Theta_2. \quad (6.57)$$

Here, $\eta_1, \eta_2, \Theta_0, \Theta_1$ and $\Theta_2$ are given by

$$\eta_1 = \sigma^{(1)} / \sigma^{(2)}, \quad (6.58)$$
\[ \eta_2 = \frac{\mu^{(2)}}{\mu^{(1)}}, \tag{6.59} \]
\[ \Theta_0 = \frac{(\gamma^{(1)} + \eta_2 \gamma^{(2)})}{(\gamma^{(2)} + \eta_2 \gamma^{(1)})}, \tag{6.60} \]
\[ \Theta_1 = 2 \gamma^{(1)}/(\gamma^{(1)} + \eta_1 \gamma^{(2)}), \tag{6.61} \]
\[ \Theta_2 = 2 \gamma^{(2)}/(\gamma^{(2)} + \eta_2 \gamma^{(1)}). \tag{6.62} \]

Obviously, if \( \sigma^{(1)} = \sigma^{(2)} \) and \( \mu^{(1)} = \mu^{(2)} \), the expressions (6.52)-(6.56) for the \( B_j^{(m)} \) reduce to the ones given by Eq. (6.30) for homogeneous isotropic media.

In most geophysical applications, the upper half-space will represent the air, while the lower half-space represents the homogeneous isotropic subsurface of the earth, i.e., \( \mu^{(1)} = \mu^{(2)} = \mu_0 \), the magnetic permeability of free-space, while \( \sigma^{(1)}/\sigma^{(2)} \to 0 \). In this case, Eqs. (6.53)-(6.57) reduce to

\[ B_1^{(1)} = B_1^{(2)} = -\tilde{J}_2 \frac{\alpha_1 \alpha_2}{\gamma^{(1)} \gamma^{(2)}}, \tag{6.63} \]
\[ B_2^{(1)} = B_2^{(2)} = -\tilde{J}_2 \frac{\alpha_2}{\gamma^{(1)} \gamma^{(2)}} - \tilde{J}_2 \frac{\mu_0}{\gamma^{(1)} + \gamma^{(2)}}, \tag{6.64} \]
\[ B_3^{(1)} = B_3^{(2)} = -\tilde{J}_2 \frac{\alpha_1 \alpha_2}{\gamma^{(1)} + \gamma^{(2)}}, \tag{6.65} \]
\[ B_4^{(1)} = \tilde{J}_2 \frac{\gamma^{(1)}}{\gamma^{(1)} + \gamma^{(2)}} - \tilde{J}_2 \frac{\alpha_2}{\gamma^{(1)}} \frac{1}{\gamma^{(1)} + \gamma^{(2)}}, \tag{6.66} \]
\[ B_4^{(2)} = -\tilde{J}_2 \frac{\gamma^{(2)}}{\gamma^{(1)} + \gamma^{(2)}} - \tilde{J}_2 \frac{\alpha_2}{\gamma^{(1)}} \frac{1}{\gamma^{(1)} + \gamma^{(2)}}, \tag{6.67} \]
6.4 Transformation back to the space-time domain

The transformation of a single diffusive field constituent \( F(\alpha_\nu, x_3, s) \) from the trans-
form-domain to the space-time domain has been described in detail in Sections 3.4 and 4.3 for (layered) arbitrarily anisotropic media. By rewriting the inverse Fourier transformation in an appropriate manner and applying an analytic continuation of \( F(\alpha_\nu, x_3, s) \) away from the real \( \alpha_1 \)- and \( \alpha_2 \)-axes, we have shown that space-time domain expressions for the electric and magnetic field components can be found by inspection. To this end we introduced the modified Cagniard contour that describes how the integrations with respect to real \( \alpha_\nu \) of the inverse Fourier transformation (2.34) are deformed into integrations along complex values of \( \alpha_\nu \).

Since there are no essential differences in the transformation from the transform-
domain back to the space-time domain if the media involved are isotropic rather than arbitrarily anisotropic, we shall only repeat the main results of Sections 3.4 and 4.3 and investigate how they simplify in the case of isotropic media. Of special interest are those cases for which an explicit expression for the modified Cagniard contour can be obtained.

Without loss of generality we can assume that the concentrated source is located on the \( x_3 \)-axis, i.e., \( x_s = (0,0,x_3,s) \). Inverse Fourier transformation of \( F(\alpha_\nu, x_3, s) \) to the space-Laplace domain yields (cf. Eq. (2.34))

\[
F(\omega, s) = s \hat{\varphi}(s) \hat{G}(\omega, s),
\]

(6.68)

where \( \hat{G}(\omega, s) \) is the space-Laplace domain Green's function

\[
\hat{G}(\omega, s) = \frac{\hat{k}(s)}{4\pi^2} \int_{-\infty}^{\infty} d\alpha_2 \int_{-\infty}^{\infty} \exp(-s^{1/2}(i\alpha_\nu x_\nu + \sum_j h_j \gamma^{(j)})) B d\alpha_1.
\]

(6.69)

Obviously, the summation over \( j \) only runs through those values of \( j \) that correspond to the particular generalized diffusive field constituent under consideration.

In view of Eq. (6.68) and Lerch's theorem (Widder, 1946), the space-time domain expression for \( F(\omega, s) \) is obtained as the time derivative of the convolution of the source
pulse shape $\phi(t)$ and the Green’s function $G(\omega, t)$, i.e.,

$$F(\omega, t) = \partial_t \int_0^t \phi(\tau) G(\omega, t - \tau) d\tau. \quad (6.70)$$

Here, $G(\omega, t)$ is the inverse Laplace transform of $\hat{G}(\omega, s)$. Equation (6.69) represents the contribution from a single generalized diffusive field constituent. In order to obtain the total Green’s function, the contributions from all generalized diffusive field constituents should be summed.

To arrive at an expression for $\hat{G}(\omega, s)$ the space-time domain counterpart of which can be recognized, we follow the same procedure as in Section 4.3. We start with a change of integration variables in Eq. (6.69). We replace the Fourier transformation variables $\alpha_1$ and $\alpha_2$ by the polar variables of integration $p$ and $\psi$ defined through (cf. Eqs. (4.65)-(4.66))

$$i\alpha_1 = p \cos(\theta + \psi), \quad (6.71)$$
$$i\alpha_2 = p \sin(\theta + \psi), \quad (6.72)$$

with $0 \leq p < \infty$, $0 \leq \psi < 2\pi$. The angle $\theta$ follows from the polar-coordinate specification of the point of observation with respect to the source, i.e.,

$$x_1 = r \cos(\theta), \quad (6.73)$$
$$x_2 = r \sin(\theta), \quad (6.74)$$

with $0 \leq r < \infty$, $0 \leq \theta < 2\pi$. Upon substituting the expressions (6.71)-(6.72) for $i\alpha_1$ and $i\alpha_2$ into Eq. (6.69) and using the periodicity of the integrand with respect to $\psi$ as well as symmetry with respect to the real $p$-axis, we arrive at (cf. Eq. (4.72))

$$\hat{G}(\omega, s) = -\frac{\dot{k}(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \text{Re} \left( \int_0^{\infty} \exp(-s^{1/2}(p \alpha \sin(\psi))) \tilde{B} p \, dp \right),$$

$$\quad (6.75)$$
in which $\tilde{\gamma}^{(j)}(p, \psi)$ and $\tilde{B}(p, \psi)$ have been obtained from $\gamma^{(j)}(\alpha_1, \alpha_2)$ and $B(\alpha_1, \alpha_2)$ by the substitutions of Eqs. (6.71)-(6.72). In view of Eqs. (6.9)-(6.10) we have

$$\tilde{\gamma}^{(j)} = (\mu^{(j)}\sigma^{(j)} - p^2)^{1/2}, \quad \text{Re}(\cdots)^{1/2} \geq 0. \quad (6.76)$$

Hence, the branch points of $\tilde{\gamma}^{(j)}$ in the complex $p$-plane are located at $p = \pm(\mu^{(j)}\sigma^{(j)})^{1/2}$. To keep the $\tilde{\gamma}^{(j)}$ single valued throughout the entire complex $p$-plane we introduce branch cuts along the positive and negative parts of the real $p$-axis, emanating from the relevant branch point to $p = +\infty$ and $p = -\infty$, respectively. Note that for isotropic media $\tilde{\gamma}^{(j)}$ does not depend on $\psi$, i.e., $\tilde{\gamma}^{(j)} = \tilde{\gamma}^{(j)}(p)$.

The integral along the positive imaginary $p$-axis in Eq. (6.75) is evaluated with the aid of the modified Cagniard method, i.e., the path of integration is deformed away from the imaginary $p$-axis into the complex $p$-plane such that the resulting expression for $\hat{G}(w, t)$ can be recognized as the Laplace transform of some known space-time function $G(w, t)$ (De Hoop, 1960). The new path of integration in the complex $p$-plane is called the modified Cagniard contour. This contour is parametrized by the variable $\tau$, such that for fixed $\psi$, $-\pi/2 \leq \psi < \pi/2$ we have

$$p \tau \cos \psi + \sum_j h_j \tilde{\gamma}^{(j)}(p) = \tau, \quad \tau \text{ is real and positive.} \quad (6.77)$$

For details about the analytic continuation of the integrand of Eq. (6.74) into the complex $p$-plane and the general properties of the modified Cagniard contour we refer to Section 4.3. From Eqs. (6.75)-(6.76) we observe that the modified Cagniard contour starts at $p = 0$ and follows a part of the positive real $p$-axis. At some point $p = p_0(\psi) \geq 0$ the Cagniard contour leaves the real $p$-axis perpendicularly to finally approach a complex asymptote as $\tau \to \infty$. The value of $\tau$ at $p = 0$ is denoted as $\tau = T_{\min}$. Its value follows from Eq. (6.77) as

$$T_{\min} = \sum_j h_j (\mu^{(j)}\sigma^{(j)})^{1/2} > 0. \quad (6.78)$$

Since we want $\tau$ to increase monotonously along the path, the modified Cagniard contour leaves the real $p$-axis at the point $p = p_0(\psi)$ where $\tau$ reaches a local maximum.
value. At this point the derivative of $\tau$ with respect to $p$ is equal to zero, i.e.,

$$\frac{\partial \tau}{\partial p} = r \cos \psi - p \sum_j h_j / \zeta^{(j)}(p) = 0 \quad \text{for} \quad p = p_0(\psi) \geq 0. \quad (6.79)$$

Note that $p_0(\psi) = 0$ for $\psi = \pm \pi/2$ while $p_0(\psi) > 0$ for all other values of $\psi$, $|\psi| < \pi/2$. The value of $\tau$ at $p = p_0(\psi)$ is denoted as $\tau = T(\psi)$.

The complex asymptote of $p(\tau)$ as $\tau \to \infty$ is determined by the asymptotic behavior of $\zeta^{(j)}(p)$ as $|p| \to \infty$. From Eq. (6.76) we obtain

$$\zeta^{(j)}(p) = -ip + O(p^{-1}) \quad \text{as} \quad |p| \to \infty. \quad (6.80)$$

Let us denote by $H$ the total vertical path length of the generalized diffusive field constituent, i.e., $H = \sum_j h_j$. With the aid of Eq. (6.80) we obtain the following asymptotic expression for the left-hand side of Eq. (6.77) as $|p| \to \infty$

$$p (r \cos \psi - iH) \sim \tau \quad \text{as} \quad \tau \to \infty, \quad (6.81)$$

and consequently

$$p \sim \tau (r \cos \psi - iH)^{-1} \quad \text{as} \quad \tau \to \infty, \quad (6.82)$$

From the latter equation we observe that for isotropic media the modified Cagniard contour has as its complex asymptote a straight line through the origin that is located in the right part of the upper half of the complex $p$-plane. See Figure 6.2.

The space-time domain Green's function

By virtue of Jordan's lemma, Cauchy's theorem and the previous results, we conclude that the integral along the positive imaginary $p$-axis of Eq. (6.75) is equal to the integral along the modified Cagniard contour $p(\tau, \psi)$. Upon introducing $\tau$ as variable of integration, we obtain (cf. Eq. (4.88))

$$\hat{G}(z, s) = -\frac{k(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \left( \int_{T_{\min}}^{T(\psi)} + \int_{T(\psi)}^{\infty} \right) \exp(-s^{1/2}\tau) \text{Re}(\frac{\partial p}{\partial \tau}) d\tau. \quad (6.83)$$
Figure 6.2: Example of the modified Cagniard contour of the two layer problem. The dashed line indicates the asymptote of the modified Cagniard contour as \( \tau \to \infty \).

Note that \( \tau = T(\psi) \) is a singular point of the integrand of Eq. (3.119) since \( \partial p / \partial \tau \) is infinite for this value of \( \tau \). From Eqs. (6.76)-(6.77) and (6.79), however, we conclude that the this singularity is a square-root singularity and hence, is integrable. With Eq. (6.83) we have achieved that \( G(\omega, t) \) can be found by inspection. This is possible since the part of \( \hat{G}(\omega, s) \) that depends on the Laplace transformation parameter \( s \) is given by \( \hat{k}(s) \exp(s^{1/2} \tau) \), while the rest of the integrand is \( s \)-independent. Since \( \tau \) is real and positive \( (T_{\min} > 0) \) the function \( \hat{k}(s) \exp(-s^{1/2} \tau) \) can be recognized as the Laplace transform of some time-function \( k(t, \tau) \). The functions \( k(t, \tau) \) occurring in the problem are given by Eqs. (3.122)-(3.125). With this, the space-time domain expression for \( G(\omega, t) \) is obtained as

\[
G(\omega, t) = \begin{cases} 
0, & t < 0 \\
- \frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \left( \int_{T_{\min}}^{T(\psi)} + \int_{T(\psi)}^{\infty} \right) k(t, \tau) \text{Re}(\hat{B} \frac{\partial p}{\partial \tau}) d\tau, & t \geq 0.
\end{cases}
\]

(6.84)
Note that Eq. (6.84) represents the contribution from a single generalized diffusive field constituent. In order to obtain the total space-time domain Green's function, we have to sum the contributions from all generalized diffusive field constituents.

At the end of Sections 6.2 and 6.3 we have given some explicit expressions for the function \( B(\alpha_1, \alpha_2) \). If we have an explicit expression for the modified Cagniard contour \( p(\tau, \psi) \) as well, we might be able to evaluate the integrals on the right-hand side of Eq. (6.84) analytically. Unfortunately, this is in general not the case: Eq. (6.76) cannot be solved explicitly for \( p \) in terms of \( \tau \) and \( \psi \) and, just as with arbitrarily anisotropic media, we have to solve Eq. (6.77) numerically with the aid of a Newton-Rhapson iterative method (Van der Hijden, 1987). This will be discussed in more detail in Chapter 7. However, two special cases exist for which Eq. (6.77) can be solved explicitly for \( p \), viz. when only one or two different \( \tilde{\gamma}^{(j)}(p) \) occur in the summation over \( j \) on the left-hand side of Eq. (6.77). We shall refer to these two special cases as the single medium problem and the two media problem, respectively. In the two subsequent subsections we shall consider these two cases in detail.

The modified Cagniard contour of the single medium problem

In this case, the generalized diffusive field constituent under consideration corresponds to the diffusion of the electromagnetic field in one subdomain only (e.g., the direct or the once reflected field; see Sections 4.2 and 6.3). We shall assume this subdomain to be \( D_j \). The modified Cagniard contour for this case follows from (cf. Eq. (6.76))

\[
p \tau \cos \psi + h_j \tilde{\gamma}^{(j)}(p) = \tau.
\]  
\[ (6.85) \]

Upon using the explicit expression Eq. (6.76) for \( \tilde{\gamma}^{(j)}(p) \), we obtain by solving the resulting quadratic equation for \( p \):

\[
p(\tau, \psi) = \frac{\tau \tau \cos \psi - h_j (T^2(\psi) - \tau^2)^{1/2}}{h_j^2 + \tau^2 \cos^2 \psi}, \quad \tau \leq T(\psi),
\]
\[ (6.86) \]

\[
p(\tau, \psi) = \frac{\tau \tau \cos \psi + i h_j (\tau^2 - T^2(\psi))^{1/2}}{h_j^2 + \tau^2 \cos^2 \psi}, \quad \tau > T(\psi),
\]
\[ (6.87) \]
in which $T(\psi)$ is given by

$$T(\psi) = \left(\mu^{(j)} \sigma^{(j)} (h_j^2 + r^2 \cos^2 \psi)\right)^{1/2}. \tag{6.88}$$

The modified Cagniard contour leaves the real $p$-axis at $p = p_0(\psi)$. The corresponding value of $\tau$ is $\tau = T(\psi)$. From Eq. (6.86) we obtain

$$p_0(\psi) = p(T(\psi), \psi) = \frac{r \cos \psi}{(h_j^2 + r^2 \cos^2 \psi)^{1/2}} (\mu^{(j)} \sigma^{(j)})^{1/2}. \tag{6.89}$$

Further,

$$\frac{\partial p}{\partial \tau} = \frac{\tilde{z}^{(j)}(p)}{(T^2(\psi) - \tau^2)^{1/2}}, \quad \tau \leq T(\psi), \tag{6.90}$$

$$\frac{\partial p}{\partial \tau} = \frac{i \tilde{z}^{(j)}(p)}{(\tau^2 - T^2(\psi))^{1/2}}, \quad \tau > T(\psi). \tag{6.91}$$

From Eqs. (6.88)-Eq. (6.87) it is clear that the modified Cagniard contour of a single isotropic medium consists of a part of the real $p$ axis from $p = 0$ to $p = p_0(\psi)$ and a hyperbola in the complex $p$-plane. This is depicted in Figure 6.3.

**The modified Cagniard contour of the two media problem**

In this case, the generalized diffusive field constituent under consideration corresponds to diffusion of the electromagnetic field in two subdomains (e.g., the once transmitted field; see Sections 4.2 and 6.3). We shall assume the two subdomains to be $D_i$ and $D_j$, respectively. Since these two domains must be adjacent we have $|i - j| = 1$). The modified Cagniard contour for this case follows from (cf. Eq. (6.77))

$$p r \cos \psi + h_i \tilde{z}^{(i)}(p) + h_j \tilde{z}^{(j)}(p) = \tau. \tag{6.92}$$

Upon using the explicit expression for $\tilde{z}^{(j)}(p)$, Eq. (6.75), and squaring left- and right-hand sides of Eq. (6.94) twice, we obtain
Figure 6.3: Example of the modified Cagniard contour of the single medium problem. The complex part of this contour is a hyperbola in the complex p-plane (data: $r \cos \psi = h_i; \mu^{(i)} \sigma^{(j)} = 4$).

\[
\left( p^2 R_+^2 - 2 r \tau r \cos \psi + r^2 - T_+^2 \right) \left( p^2 R_-^2 - 2 r \tau r \cos \psi + r^2 - T_-^2 \right) + h_i^2 h_j^2 \Delta^2 = 0, \quad (6.93)
\]

in which

\[
R_{\pm} = \left( (h_i \pm h_j)^2 + r^2 \cos^2 \psi \right)^{1/2}, \quad (6.94)
\]

\[
T_{\pm} = \left( (h_i \pm h_j)(h_i \mu^{(i)} \sigma^{(i)} \pm h_j \mu^{(j)} \sigma^{(j)}) \right)^{1/2}, \quad (6.95)
\]

\[
\Delta = \mu^{(i)} \sigma^{(i)} - \mu^{(j)} \sigma^{(j)}. \quad (6.96)
\]

Equation (6.93) is a quartic equation in $p$. As outlined in Appendix A, this equation can be solved explicitly for $p(r, \psi)$ for arbitrary $p$ and $\psi$. Equation (6.93) has four roots. Only one of these roots corresponds to the modified Cagniard contour, i.e., the deformed path of integration away from the positive imaginary $p$-axis, such that
\( \tau \) increases monotonously along the path. With this requirement we are able to determine which of the four roots of Eq. (6.93) corresponds to \( p(\tau, \psi) \).

Unfortunately, the thus obtained expression for the modified Cagniard contour of the two media problem is a rather complicated one involving square and cubic roots. In Figure 6.4 we have plotted examples of the modified Cagniard contour for the two media problem.

![Diagram](image)

**Figure 6.4:** Example of the modified Cagniard contour of the two media problem for different ratios of \( h_i/h_j \). These contours are computed by evaluating the explicit expression for \( p(\tau, \psi) \) as follows from Eq.(6.92) and Appendix A (data: \( \tau \cos \psi = h_i + h_j; \mu^{(i)} \sigma^{(i)} = 4, \mu^{(j)} \sigma^{(j)} = 1/4 \)).
6.5 Analytic results for simple configurations

Except for a number of special cases, the space-time domain Green's function $G(\mathbf{r}, t)$ for layered isotropic media – as given by Eq. (6.84) – must be evaluated with the aid of numerical techniques, either because an explicit expression for the modified Cagniard contour cannot be obtained or because the integral on the right-hand side of Eq. (6.84) cannot be reduced to elementary functions or to simple integrals representing standard special functions. In this final section we shall concentrate on a number of simple but interesting configurations for which we can evaluate the right-hand side of Eq. (6.84) analytically. Examples are: a source in an unbounded homogeneous isotropic medium and a source at the interface of two adjacent isotropic half-spaces. In the following subsections we shall evaluate and consider in detail the space-time domain expressions for the electric and magnetic field strength for a number of these special configurations.

**Electric current dipole in an unbounded homogeneous isotropic medium**

A point source in an unbounded homogeneous isotropic medium is the simplest configuration existing and can be considered as a canonical problem of the diffusion of electromagnetic fields. Although practical importance of this configuration is limited, the expressions for the electric and magnetic field strengths will give insight in the general propagation characteristics of the diffusive electromagnetic field.

We assume the source to be a switch-on electric current dipole, located at the origin of the chosen coordinate system and oriented in the $i_2$-direction. This implies that the source pulse shape is given by $\phi(t) = H(t)$ where $H(t)$ is the Heaviside unit step function. Further, the transform-domain source density $\mathbf{J}_e$ of Eq. (6.30) can be written as $\mathbf{J}_e = \hat{\phi}(s) J_0 \delta_{k,2}$; $J_0$ is the dipole moment of the electric current dipole. In SI-units, the dimension of $J_0$ is [Am].

Upon combining Eqs. (3.9), (6.30), (6.68) and (6.84)-(6.91) it is possible to rewrite the integrals on the right-hand side of Eq. (6.84) in terms of elementary and special functions (Abramowitz and Stegun, 1970; Gradshteyn and Ryzhik, 1980). After straightforward calculations we obtain the following space-time domain expressions for the horizontal components of the electric and magnetic field strengths

$$E_1(\mathbf{r}, t) = \frac{J_0}{4\pi\sigma} \partial_1 \partial_2 \left( \frac{\text{erfc}(\frac{R/T^{1/2}}{2})}{R} \right) H(T),$$

(6.97)
\[ E_2(\mathbf{r}, t) = \frac{J_0}{4\pi \sigma} \partial_\mathbf{r} \partial_\mathbf{t} \left( \frac{\text{erfc}(R/T^{1/2})}{R} \right) H(T) - J_0 \frac{\exp(-R^2/T)}{\sigma (\pi T)^{3/2}} H(T), \] (6.98)

\[ H_1(\mathbf{r}, t) = -\frac{J_0}{4\pi} \partial_\mathbf{r} \left( \frac{\text{erfc}(R/T^{1/2})}{R} \right) H(T), \] (6.99)

\[ H_2(\mathbf{r}, t) = 0. \] (6.100)

Here, \( R \) denotes the distance between source and point of observation: \( R = (r^2 + h^2)^{1/2} \) and \( T \) denotes the time normalized with respect to the diffusion constant \( \mu \sigma \):

\[ T = \frac{4t}{\mu \sigma}. \] (6.101)

erfc \( (u) \) denotes the complementary error function defined as:

\[ \text{erfc} \ (u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-\xi^2) \ d\xi. \] (6.102)

Besides the horizontal components of the electric and magnetic field strengths we are also interested in their vertical components. For this purpose we need the transform-domain expressions \( B^{(\pm)} \) for \( \bar{E}_3 \) and \( \bar{H}_3 \) (cf. Eq. (6.30)). Upon combining Eq. (3.87) with Eq. (6.30) we obtain for the function \( B^{(\pm)} \) corresponding to the vertical component of the transform-domain electric field

\[ B^{(\pm)} = \pm J_0 \frac{i\alpha_2}{2\sigma}. \] (6.103)

Following the same procedure as with the transformation back to the space-time domain of the horizontal components of the electric field strength, we obtain after straightforward calculations the following space-time domain expression for the vertical component of the electric field strength:

\[ E_3(\mathbf{r}, t) = \frac{J_0}{4\pi \sigma} \partial_\mathbf{r} \partial_\mathbf{t} \left( \frac{\text{erfc}(R/T^{1/2})}{R} \right) H(T). \] (6.104)
Upon combining Eq. (3.88) with Eq. (6.30) we obtain for the function $B^{(\pm)}$ corresponding to the vertical component of the transform-domain magnetic field

$$B^{(\pm)} = - \frac{J_0}{2\gamma} i\alpha_1,$$  \hspace{0.5cm} (6.105)

Following the same procedure as with the transformation back to the space-time domain of the horizontal components of the magnetic field strength, we obtain after straightforward calculations the following space-time domain expression for the vertical component of the magnetic field strength:

$$H_3(\mathbf{z}, t) = \frac{J_0}{4\pi} \partial_t \left( \frac{\text{erfc}(R/T^{1/2})}{R} \right) H(T).$$  \hspace{0.5cm} (6.106)

In addition to the different field components $E_r(\mathbf{z}, t)$ and $H_r(\mathbf{z}, t)$ we wish to consider one other quantity in detail, viz. the electromotive force $V_{\text{emf}}(\mathbf{z}, t)$ induced in a ideal coil of unit area. Applying Stokes' theorem to the Maxwell equation (2.19) we observe that the induced voltage is equal to the negative of the time rate of change of the magnetic flux density linked to the coil, i.e.,

$$V_{\text{emf}}(\mathbf{z}, t) = - \partial_t \int_S \nu_j B_j(\mathbf{z}, t) \, dS,$$  \hspace{0.5cm} (6.107)

where $\nu_j$ denotes the unit vector along the normal to the surface bounded by the coil. Written in this form, Eq. (6.107) is known as the Faraday-Henry law of electromagnetic induction. For a very small (as compared to the spatial variations of the magnetic flux density) planar coil of unit area and with its axis vertical, Eq. (6.107) reduces to

$$V_{\text{emf}}(\mathbf{z}, t) = - \mu \partial_t H_3(\mathbf{z}, t).$$  \hspace{0.5cm} (6.108)

Upon using Eq. (6.106) we obtain

$$V_{\text{emf}}(\mathbf{z}, t) = \frac{2J_0}{\pi \sigma} \frac{z_1 \exp(-R^2/T)}{T^2(\pi T)^{1/2}} H(T).$$  \hspace{0.5cm} (6.109)
Of special interest is the late-time asymptotic behavior of $V_{\text{emf}}(\omega, t)$. Substitution of $T = 4t/\mu\sigma$ (cf. Eq. (6.101)) into Eq. (6.109) and letting $t \to \infty$ yields:

$$V_{\text{emf}}(\omega, t) = \frac{J_0 x_1 \mu^{5/2} \sigma^{3/2}}{16 \pi^{3/2} \delta^{5/2}} + \mathcal{O}(t^{-7/2}) \quad \text{as} \; t \to \infty. \quad (6.110)$$

For interpretation purposes it is convenient to normalize the induced voltage $V_{\text{emf}}(\omega, t)$ to some value $V_{\text{nor}}(\omega)$. In the following subsection we show that an appropriate normalization for the induced voltage is given by

$$V_{\text{nor}}(\omega) = J_0 \frac{3x_1}{2\pi \sigma R^5}. \quad (6.111)$$

In Figure 6.5 we have plotted the normalized induced voltage $V_{\text{emf}}(\omega, t)/V_{\text{nor}}(\omega)$ and its late-time asymptote as a function of $T/R^2$.

**Electric current dipole at the interface between two homogeneous, isotropic media**

A point source at the interface between two isotropic media is the simplest example of a stratified media configuration. Not only is it a canonical problem of the interaction of diffusive electromagnetic fields at the interface between two media with different electromagnetic properties, but also this configuration is representative of a simple model of the subsurface of the earth. In the latter case, the lower half-space represents a homogeneous, isotropic, electrically conducting earth, while the upper half-space represents the air if we let its conductivity go to zero. Moreover, if we take the conductivity of the upper half-space equal to the conductivity of seawater, this configuration is representative of a simple model of the sea floor (Kaufman and Keller, 1983; Fitterman and Stewart, 1986; Edwards and Chave, 1986; Cheesman et al., 1987; Edwards, 1988).

As in the previous subsection, we assume the source to be a switch-on electric current dipole, located at the interface of the two media and oriented in the $i_2$-direction. This implies that the source pulse shape is given by $\phi(t) = H(t)$ where $H(t)$ is the Heaviside unit step function. Further, the transform-domain source density $\tilde{J}_c^c$ of Eqs. (6.52)-(6.56) can be written as $\tilde{J}_c^c = \phi(z) J_0 \delta_{\alpha z}$; $J_0$ is the dipole moment of the electric current dipole. In SI-units, the dimension of $J_0$ is [Am].
Figure 6.5: Normalised induced voltage response of a vertical-axis coil of unit area due to current switch-on excitation by a horizontal electrical dipole in an unbounded homogeneous isotropic medium. The dashed line indicates the late-time asymptote of the induced voltage response. \( R \) denotes the distance from source to receiver; \( T = 4t/\mu \sigma \).

It should be emphasized that explicit expressions for the space-time behavior of the diffusive electromagnetic field of this configuration are only known for the case of a line source (two-dimensional problem, Wait, 1971; Oristaglio, 1982; Oristaglio and Hohmann, 1984), not for a point source. Even in the case of a line source, the resulting expressions are complicated and do not provide further insight in the propagation characteristics of the transient diffusive electromagnetic field without a numerical analysis. In this respect it is not surprising that upon combining Eqs. (6.52)-(6.56) and (6.84)-(6.91) it turns out to be impossible to rewrite the integrals on the righthand side of Eq. (6.84) completely in terms of standard special functions. However,
Diffusive electromagnetic fields in isotropic media

if we make the assumption that the magnetic permeability of the two media are equal (i.e., \( \mu^{(1)} = \mu^{(2)} = \mu_0 \), the permeability of free space) and that the conductivity of the upper half-space equals zero (i.e., \( \sigma^{(1)} = 0 \)) while we take the point of observation at the interface of the two media, then the integrals on the right-hand side of Eq. (6.84) can be rewritten explicitly in terms of elementary and special functions (see for example Spies and Raiche, 1980; Weir, 1980; Lee, 1982; Kaufman and Keller, 1983).

Upon combining Eqs. (6.62)-(6.66) with (6.83)-(6.90) we obtain after straightforward calculations the following space-time domain expressions for the horizontal components of the electric and magnetic field strength at the surface of a homogeneous isotropic half-space:

\[
E_1(\vec{z}, t) = \frac{J_0}{2\pi \sigma^{(2)}} \partial_1 \partial_2 \left( \frac{1}{R} \right) H(T),
\]

\[
E_2(\vec{z}, t) = \frac{J_0}{2\pi \sigma^{(2)}} \partial_2 \partial_2 \left( \frac{1}{R} \right) H(T) + \frac{J_0}{2\pi \sigma^{(2)}} \partial_2 \left( \frac{\text{erf}(u^{1/2})}{R} \right) H(T),
\]

\[
H_1(\vec{z}, t) = \frac{J_0}{4\pi R^2} \exp(-u/2) \left\{ [I_0(u/2) + 2I_1(u/2)] \cos(2\theta)
\right.
\]
\[
\left. + 2\cos^2\theta I_1(u/2) \right\} H(T),
\]

\[
H_2(\vec{z}, t) = \frac{J_0}{4\pi R^2} \exp(-u/2) \left[ I_0(u/2) + 2I_1(u/2) \right] \sin(2\theta) H(T).
\]

where

\[
u = R^2/T = R^2 \mu_0 \sigma^{(2)}/4t.
\]

In Eqs. (6.110)-(6.113), \( I_0(u) \) and \( I_1(u) \) are modified Bessel functions of the first kind (Abramowitz and Stegun, 1970), while \( R \) and \( \theta \) denote the polar coordinate specification of the point of observation \( \vec{z} = (x_1, x_2, 0) \) (cf. Eqs. (6.72)-(6.73)).

Besides the horizontal components of the electric and magnetic field strengths we are also interested in their vertical components. Since across the interface only the electrical conductivity shows a jump discontinuity, the vertical component of the magnetic field strength must be continuous across the interface, while the vertical component of the electric field strength must be discontinuous across the interface.
in accordance with the continuity of the normal component of the volume density of electric current. In particular, because \( \sigma^{(1)} = 0 \), we have \( \lim_{x_3 \downarrow 0} E_3(x, t) = 0 \).

Upon combining Eq. (3.87) and Eqs. (6.65)-(6.66), we obtain for the function \( B^{(1)} \) corresponding to the vertical component of the transform-domain electric field in the upper non-conducting halfspace

\[
B^{(1)} = -J_0 \frac{i\alpha_2 (\gamma^{(2)} - \gamma^{(1)})}{\sigma^{(2)} \gamma^{(1)}}.
\] (6.117)

Upon combining Eq. (3.87) and Eqs. (6.65),(6.67), we obtain for the function \( B^{(2)} \) corresponding to the vertical component of the transform-domain electric field in the lower conducting half-space

\[
B^{(2)} = -J_0 \frac{i\alpha_2}{\sigma^{(2)}}.
\] (6.118)

Following the same procedure as with the transformation back to the space-time domain of the horizontal components of the electric field strength, we obtain after straightforward calculations the following space-time domain expressions for the vertical component of the electric field strength in the lower half-space (cf. Eq. (6.103)-(6.104))

\[
E_3(x, t) = -\frac{J_0}{2\pi \sigma} \partial_3 \partial_2 \left( \frac{\text{erfc}(R/T^{1/2})}{R} \right) H(T),
\] (6.119)

and hence

\[
\lim_{x_3 \downarrow 0} E_3(x, t) = 0,
\] (6.120)

while

\[
\lim_{x_3 \uparrow 0} E_3(x, t) = -\frac{J_0 x_2}{\pi \sigma^{(2)} R^2} \frac{1}{T} \exp(-u/2) I_1(u/2).
\] (6.121)

From Eqs. (6.119)-(6.121) we observe that the normal component of the volume density of electric current \( \sigma^{(2)} E_3(x, t) \) just below the surface of the half-space is
zero. In practical situations however, a very small normal component of \( \sigma^{(2)} E_3(\mathbf{x}, t) \) will be present, just equal to the normal component of the displacement current above the surface (i.e., in the air, see Weir, 1980).

Upon combining Eq. (3.88) with Eqs. (6.63)-(6.64) (i.e., for arbitrary \( \sigma^{(1)} \), not necessarily equal to zero) we obtain for the functions \( B^{(1)} \) and \( B^{(2)} \) corresponding to the vertical component of the transform-domain magnetic field

\[
B^{(1)} = B^{(2)} = -J_0 \frac{i\alpha_2}{\gamma^{(1)} + \gamma^{(2)}},
\]

(6.122)

Upon comparing Eq. (6.118) with Eqs. (6.63)-(6.67) we observe that whereas the \( B_j \) of the horizontal electric and magnetic field components only reduce to 'simple' expressions if \( \sigma^{(1)}/\sigma^{(2)} \) is equal to zero (cf. Eqs. (6.63)-(6.67)), the function \( B \) that corresponds to the vertical component of the transform-domain magnetic field has a very simple structure, even when \( \sigma^{(1)}/\sigma^{(2)} \) is not equal to zero.

Following the same procedure as with the transformation back to the space-time domain of the horizontal components of the magnetic field strength (however, now for arbitrary \( \sigma^{(1)} \)) we obtain after straightforward calculations the following space-time domain expressions for the vertical component of the magnetic field strength at the interface of the two media:

\[
H_3(\mathbf{x}, t) = \frac{J_0}{4\pi R^2} \frac{\cos \theta}{1 - \eta} \left\{ \frac{3}{(\pi u)^{1/2}} \left[ \exp(-u) - \eta^{1/2} \exp(-\eta u) \right] } + \eta - 1 \\
+ (1 - \frac{3}{2u}) \text{erf}(u^{1/2}) - (\eta - \frac{3}{2u}) \text{erf}((\eta u)^{1/2}) \right\} H(T),
\]

(6.123)

Here, \( \eta \) and \( u \) are given as (cf. Eqs. (6.58) and (6.115))

\[
\eta = \sigma^{(1)}/\sigma^{(2)},
\]

(6.124)

\[
u = R^2/T = R^2 \mu_0 \sigma^{(2)}/4t.
\]

(6.125)

Obviously, as \( \eta \to 1 \), the right-hand side of Eq. (6.123) reduces to the result for the unbounded homogenous isotropic medium, Eq. (6.109).

Finally, we also wish to consider the voltage \( V_{\text{emf}}(\mathbf{x}, t) \) induced in a ideal coil of unit area located at the interface between the two isotropic media and with its axis
vertical. Applying Eq. (6.106) to Eq. (6.119) yields

$$V_{\text{emf}}(\omega, t) = \frac{J_0 \cos \theta}{\pi \sigma(2) R^4 (1 - \eta)} \left\{ \frac{u^{1/2}[\eta^{1/2}(3 + 2\eta u) \exp(-\eta u) - (3 + 2u) \exp(-u)]}{\pi^{1/2}} \right. $$
$$+ \left. \frac{3}{2} \left( \text{erf}(u^{1/2}) - \text{erf}((\eta u)^{1/2}) \right) \right\} H(T). \quad (6.126)$$

Of special interest is the late-time asymptotic behavior of $V_{\text{emf}}(\omega, t)$. Substitution of Eq. (6.125) into Eq. (6.126) and letting $t \to \infty$ yields:

$$V_{\text{emf}}(\omega, t) = \frac{J_0 x_1 \mu_0^{5/2} \sigma(2)^{3/2}}{40 \pi^{3/2} t^{5/2}} \frac{1 - \eta^{5/2}}{1 - \eta} + \mathcal{O}(t^{-7/2}) \quad \text{as} \quad t \to \infty. \quad (6.127)$$

Further, if $\sigma^{(1)} = 0$ the induced voltage response at a certain point of observation at the interface between the two media will attain a finite value immediately after the source has been switched on. Clearly, this is due to neglecting the electric displacement current term in Maxwell's equations and hence, not considering early-time wave phenomena (Weir, 1980; Ignietik et al., 1985). If $\eta = 0$ the early-time asymptotic value of $V_{\text{emf}}(\omega, t)$ is given by

$$V_{\text{emf}}(\omega, t) = J_0 \frac{3x_1}{2\pi \sigma(2) R^5} \quad \text{as} \quad t \to 0 \quad \text{and} \quad \eta = 0. \quad (6.128)$$

From Eqs. (6.127)-(6.128) we conclude that the early-time induced voltage response of an isotropic half-space is proportional to $1/\sigma$ while the late-time induced voltage response is proportional to $\sigma^{3/2}$.

For interpretation purposes it is convenient to normalize the induced voltage response by its early-time asymptotic value. Upon comparing Eq. (6.128) with Eq. (6.111) we observe that the quantity $V_{\text{nor}}(\omega, t)$ which we have used in the previous subsection to normalize the induced voltage response of the unbounded homogeneous isotropic medium is just equal to the early-time response of the corresponding half-space problem, Eq. (6.128).

In Figure 6.6 we have plotted examples of the induced voltage response $V_{\text{emf}}(\omega, t)$ for several combinations of the conductivity of the upper half-space $\sigma^{(1)}$ and conductivity of the lower half-space $\sigma^{(2)}$. 
Figure 6.6: Normalised induced voltage responses following current switch-on excitation by a horizontal electric current dipole at the interface between two isotropic half-spaces. $\sigma_1$ and $\sigma_2$ denote the conductivities of the upper and the lower half-space, respectively. Normalisation is carried out with respect to the early-time response of the isotropic half-space of the first example, i.e., of which the electrical conductivity is equal to 0.04 S/m. The distance from source to receiver is 250 m.
6.5 Analytic results for simple configurations

In-loop response of a loop source at the surface of an isotropic half-space

In this final subsection we consider the response of a homogeneous isotropic half-space due to a switch-on current excitation by a loop source. In particular, we shall concentrate on the in-loop (IL) response, i.e., when the receiver – in our case an ideal coil of unit area and its axis vertical – is located at the center of the loop. Further, special attention is paid to the differences between the in-loop response of a square loop and a circular loop.

Whereas in the literature in almost all cases the response of a circular loop is considered (taking advantage of the circular symmetry of the configuration and using Fourier-Bessel transforms rather than Fourier transforms (see, for example Wait, 1951; Lee, 1974; Lee, 1982; Wait, 1982; Newman et al., 1987)), we shall here consider the response of both kinds of loop sources. Chapter 5 describes in detail how we can profitably make use of the modified Cagniard technique to obtain space-time domain expressions for the transient diffusive electromagnetic field due to current excitation by a rectangular loop source. We have shown that by carrying out the integration along the wire segments of the loop in the space-Laplace domain rather than in the space-time domain (as has been done in Nabighan, 1979; Poddar, 1982; Goldman and Fitterman, 1987; Raiche, 1987), we end up with space-time domain expressions that only contain contributions from the four vertices of the loop. Obviously, this method can only be used for polygonal loop sources and not for circular loop sources. However, since here we only consider isotropic media and the point of observation being at the center of the loop, we can make use of symmetry properties of the configuration. The in-loop response of the circular loop is just equal to the induced voltage response due to excitation by an electric current dipole times the length of the circular loop (cf. Eq. (6.126)). Let us denote by \( V_{\text{emf}}^{\odot}(t) \) the in-loop response of a homogenous isotropic half-space due to a switch-on electric current excitation by a circular loop source of radius \( a \). From Eq. (6.126) we easily obtain

\[
V_{\text{emf}}^{\odot}(t) = \frac{I_0}{a^3 \sigma^{(2)}} \left( 3 \operatorname{erf}(u^{1/2}) - 2 \left( \frac{u}{\pi} \right)^{1/2} (3 + 2u) \exp(-u) \right) H(T), \quad (6.129)
\]

where

\[
u = a^2 / T = a^2 \mu_0 \sigma^{(2)}/4t. \quad (6.130)
\]
Here, \( I_0 \) denotes the electric current in the loop source after current switch-on. In order to compute the in-loop response due to electric current excitation by a square loop source we could make use of the results of Section 5.3. However, here we can also integrate the contribution from a single electric current dipole (cf. Eq. (6.126)) along the segments of the loop. Let us denote by \( V_{\text{emf}}(t) \) the in-loop response of a homogenous isotropic half-space due to a switch-on electric current excitation by a square loop source, the length of each side of the loop being equal to \( L \). After straightforward calculations we obtain from Eq. (6.126)

\[
V_{\text{emf}}^\pi(t) = \frac{8I_0}{\pi \sigma(2)L^3} \left( 5\sqrt{2} \text{erf}(v^{1/2}) - 2 \left( \frac{2v}{\pi} \right)^{1/2} \exp(-v) \right.
\]
\[
- 4(2 + v) \exp(-v/2) \text{erf}\left( \left( \frac{v}{2} \right)^{1/2} \right) \big) H(T).
\]  

(6.131)

where

\[
v = L^2/T = L^2 \mu_0 \sigma(2)/4t.
\]  

(6.132)

Note that \( L/\sqrt{2} \) is the distance between each of the vertices and the center of the loop. In order to compare the two in-loop responses we make the surface areas of the two loops equal to each other, i.e., \( L^2 = \pi a^2 \). Upon setting \( L = ax^{1/2} \) in equation (6.131), we find that the late-time asymptotic expansion for the two in-loop responses are equal, viz.,

\[
V_{\text{emf}}^\ominus(t) = V_{\text{emf}}^\pi(t) = \frac{I_0 a^2 \mu_0^{5/2} \sigma(2)^{3/2}}{20 \pi^{1/2} \sigma^{5/2}} + O(t^{-7/2}) \quad \text{as } t \to \infty.
\]  

(6.133)

Due to the differences in shape between the square and circular loop source, we expect the corresponding early-time responses to be different. From Eqs. (6.123)-(6.124) we find that the early-time asymptotic values of \( V_{\text{emf}}^\ominus(t) \) and \( V_{\text{emf}}^\pi(t) \) are given by

\[
V_{\text{emf}}^\ominus(t) = \frac{3I_0}{a^2 \sigma(2)} \quad \text{as } t \to 0,
\]  

(6.134)
\[ V_{\text{emf}}(t) = \frac{40\sqrt{2}I_0}{\pi\sigma(3)L^3} = \frac{40\sqrt{2}I_0}{\sigma^{3/2}\sigma(3)} \quad \text{as } t \to 0. \]  
\[ (6.135) \]

From Eqs. (6.129) and (6.131) we observe that the early-time response of the square loop source is slightly larger than the early-time response of the circular loop source. The ratio of the two early-time responses is equal to \( 40\sqrt{2}/(3\pi^{3/2}) \approx 1.07 \). In Figure 6.7 we have plotted the in-loop responses for both loop sources.

![Graph showing the normalized in-loop responses](image)

**Figure 6.7:** Normalised in-loop (IL) responses of a square loop source (dashed line) and a circular loop source (solid line) of the same surface area following current switch-on excitation. The loop sources are located at the surface of an isotropic half-space of which the electrical conductivity is equal to 0.04 S/m. The radius of the circular loop is equal to 250 m. Normalisation has been taken with respect to the early-time response of the circular loop source.
Chapter 7

Numerical results

7.1 Introduction

In the preceding chapters we developed a new technique – based on the modified Cagniard method – for computing the transient diffusive electromagnetic field generated by controlled sources in stratified arbitrarily anisotropic media. In Section 6.5 we already demonstrated the feasibilities of this technique for a number of elementary configurations. We have now arrived at the stage where we can extend the complexity of the configuration to the inclusion of a number of layers, each of which can be arbitrarily anisotropic as far as their electromagnetic properties are concerned.

However, due to the large number of degrees of freedom in layered arbitrarily anisotropic media – thirteen for each anisotropic layer (six for the conductivity tensor, six for the permeability tensor and one for the layer thickness) compared to only three for each isotropic layer – we have restricted the complexity of the configuration to a number of relatively simple, but still very interesting and for prospecting purposes relevant cases. In Section 7.2 we shall consider the transient electromagnetic field in the model for a two-layer isotropic earth and the sea floor. The results from Section 7.2 serve as a background for the simplest class of anisotropy, transverse isotropy which is the subject of Section 7.3. In this section we consider in detail the diffusion characteristics of the transient electromagnetic field in a transversely isotropic half-space. Finally, in Section 7.4 we generalize the form of anisotropy of Section 7.3 to arbitrary anisotropy. We have assumed the magnetic
permeability of the media involved to be equal to the permeability of free-space \( \mu_0 = 4\pi 10^{-7} \text{H/m} \). We want to emphasize that the diffusion of transient electromagnetic fields in anisotropic media is above all a three-dimensional process. Any "flat" representation of this process can only show a limited number of aspects and occasionally may lead to a distorted view of reality.

7.2 Isotropic media

Chapter 6 describes how the modified Cagniard method can be used to obtain closed-form expressions for the transient diffusive electromagnetic field in (stratified) isotropic media. For a number of canonical problems, the application of the modified Cagniard method yields explicit analytic expressions for the different field components. These were presented in Section 6.5.

In the present section we shall investigate the transient electromagnetic field in cases for which no explicit expressions can be obtained, i.e., where Eq. (6.84) must be evaluated numerically. Our primary objective is to consider a "small" number of relatively simple configurations – however, still distinctive and relevant for geophysical prospecting purposes – that show the main characteristics of the diffusive behavior of transient electromagnetic fields in (layered) isotropic media. Our goal is not the generation of a large amount of numerical results for a wide variety of configurations (so called "master curves"), basically because we only need relevant data for later comparison with the results applying to anisotropic media. Examples of sets of master curves for all kinds of stratified media can be found in Botros and Mahmoud, 1978; Raiche and Spies, 1981; Mundry, 1984; Spies and Parker, 1984; Adhijaja et al., 1985; Gunderson et al., 1986; Goldman and Fitterman, 1987 and others.

(i) Electric fields in the earth

In this subsection we consider the transient diffusive electromagnetic field in a homogeneous isotropic half-space, due to switch-on/off excitation by an electric current grounded dipole at the surface of the half-space. We have taken the conductivity of the half-space equal to 0.04 S/m. See Figure 7.1. To understand the transient electromagnetic field behavior in the earth it is useful to study contour patterns of the electric field strength in the earth.
Figure 7.1: Homogeneous isotropic half-space. The source is an electric current dipole at the origin \( O \), oriented in the \( i_z \) direction. \( \mathcal{D}_1 \) represents the air (\( \sigma^{(1)} \rightarrow 0 \)). The conductivity of the lower half-space is equal to 0.04 S/m. \( \mu = \mu_0 \), the magnetic permeability of free space, is assumed throughout.

In Figure 7.2 we have plotted, for a number of snap times, the electric field strength following current switch-off in the cross-section \( z_2 = 0 \) of the configuration. Note that in this case the electric field is normal to the plane \( z_2 = 0 \). For each snap time we have normalized the electric field strength with respect to its maximum value for that snap time. When the transmitter is turned off, electric currents flow in the earth to preserve the electromagnetic field, a combination of Faraday's and Lenz's laws. Initially these currents are concentrated near the source with a broad zone of return currents deeper in the earth. Then the maximum of the electric field moves downward directly beneath the source.

Extensive discussions on the behavior of the transient electromagnetic field in layered isotropic media and more complicated configurations including lateral inhomogeneities can be found in Kaufman and Keller, 1983; Adhiddaja et al., 1985; SanFilipo et al., 1986; Gunderson et al., 1987; Newman et al., 1987; Zhdanov, 1987. Further, Nabighian (1979), Spies (1980) and Hoversten and Morrison (1982) demonstrated the development and diffusion of the "smoke ring" of current in the earth after switching off a steady current in a loop at the surface of the earth.
Figure 7.2: Snapshots showing the evolution of the $i_2$-component of the electric field strength in the cross-section $z_2 = 0$ of an isotropic half-space following the electric current switch-off of a horizontal electric dipole at the origin. The electrical conductivity of the half-space is equal to 0.04 S/m. Negative field values correspond to the return currents.
(ii) **Induced voltage response of a two-layer isotropic earth**

In Section 6.5 and the preceding subsection we considered the response of an isotropic half-space to switch-on/off excitation by an electric current dipole. In the present subsection the complexity of the configuration will be extended: we shall investigate the response of a two-layer earth – i.e., an isotropic half-space with an additional isotropic surface layer – due to current switch-on/off by a grounded dipole at the surface of the earth. The receiver, in this case an ideal coil of unit area with its axis vertical, is located along the \( i_1 \)-axis at a fixed distance of 250 m from the source. The configuration is depicted in Figure 7.3.

The surface layer is referred to as a **resistive** overburden if its conductivity is less than the conductivity of the basement (i.e., \( \sigma^{(2)} < \sigma^{(3)} \)). In the reverse case when the conductivity of the surface layer is greater than the conductivity of the basement, we refer to the surface layer as a **conductive** overburden (i.e., \( \sigma^{(2)} > \sigma^{(3)} \)).

![Figure 7.3: The two-layer earth. The source is an electric current dipole at the origin \( O \) oriented in the \( i_2 \) direction. \( D_1 \) represents the air (\( \sigma^{(1)} \rightarrow 0 \)). The conductivity \( \sigma^{(3)} \) of the lower half-space is fixed at 0.04 S/m. The distance from source to receiver is equal to 250 m. \( h \) denotes the thickness of the surface layer \( D_2 \).](image)

We shall investigate the influence of both conductivity and layer thickness \( h \) of the surface layer on the voltage induced in the receiver coil. Initially, after the grounded dipole has been switched-off, the currents that will flow in the earth to preserve the electromagnetic field are concentrated near the source without being
influenced by the basement conductivity (Hoversten and Morrison, 1982; Goldman and Fitterman, 1987). See also Figure 7.2. Hence, at early times the induced voltage response is approximately equal to the analytic solution for the fields over a homogenous half-space of conductivity $\sigma^{(2)}$, Eq. (6.128), and is given by

$$V_{\text{emf}}(t) = \frac{3J_0}{2\pi\sigma^{(2)} R^4} \quad \text{as } t \to 0. \quad (7.1)$$

At late times, the induced voltage response will be dominated by the response of the basement as if the surface layer were not present at all (Morrison et al., 1969; Lee and Lewis, 1974; Raiche and Spies, 1981; Lee, 1982; Kaufman and Keller, 1983; Fitterman and Stewart, 1986; Newman et al., 1987). From Eq. (6.127) we obtain

$$V_{\text{emf}}(t) = \frac{J_0 R \mu_0^{5/2} \sigma^{(3)} 3/2}{40 \pi^{3/2} t^{5/2}} \quad \text{as } t \to \infty. \quad (7.2)$$

In the intermediate stage between early and late times we expect a transition from the early-time response to the late-time response (cf. Eqs. (7.1)-(7.2)).

In order to compute the induced voltage response, we have to evaluate Eq. (6.84) numerically. In this case, the number of field constituents that contribute to the response is infinite (cf. Eq. (6.51)). The first constituent corresponds to the direct electromagnetic field. The second field constituent corresponds to one interaction of the electromagnetic field at the interface of $D_2$ and $D_3$, and so on. It turns out that, after the transformation back to the space-time domain, the summation of successive field constituents yields a rapidly converging result. In almost all cases that have been investigated, it was sufficient to take not more than four constituents into account.

**Resistive overburden**

For the resistive overburden we have taken the conductivity $\sigma^{(2)}$ of the surface layer equal to 0.005 S/m. As for our previous results, the conductivity of the basement is equal to 0.04 S/m. In Figures 7.4a and 7.4b we have plotted the normalized voltage responses following the current switch-off of the grounded dipole for layer thicknesses of $h = 100 \text{ m}$ and $h = 25 \text{ m}$, respectively. Clearly, an early stage, a late stage and the transition between the two can be recognized. Further, with the surface layer of 25 m thickness the transition from early to late stage takes place at a relatively much earlier time interval than for the surface layer of 100 m thickness.
Figure 7.4: Normalised induced voltage responses of an isotropic earth covered by a resistive overburden following the electric current switch-off excitation by a grounded electric dipole. The response has been computed for layer thicknesses of (a) 100 m and (b) 25 m. The conductivity of the resistive surface layer is equal to 0.005 S/m. The conductivity of the sedimentary basement is equal to 0.04 S/m. $\mu = \mu_0$, the permeability of free space, is assumed throughout. Dashed lines indicate early- and late-time asymptotes. See also Figure 7.3.
Figure 7.5: Normalised induced voltage responses of an isotropic earth covered by a conductive overburden following the electric current switch-off excitation by a grounded electric dipole. The response has been computed for layer thicknesses of (a) 100 m and (b) 25 m. The conductivity of the conductive surface layer is equal to 0.32 S/m. The conductivity of the sedimentary basement is equal to 0.04 S/m. \( \mu = \mu_0 \), the permeability of free space, is assumed throughout. Dashed lines indicate early- and late-time asymptotes. See also Figure 7.3.
Conductive overburden

For the conductive overburden we have taken the conductivity $\sigma^{(2)}$ of the surface layer equal to 0.32 S/m. As for our previous results, the conductivity of the basement is equal to 0.04 S/m. In Figures 7.5a and 7.5b we have plotted the normalized voltage responses following the current switch-off of the grounded dipole for layer thicknesses of $h = 100$ m and $h = 25$ m, respectively. As for the resistive overburden, Figure 7.4, an early stage, a late stage and the transition between the two can be recognized. However, here we observe that the transition from early to late stage takes place at a much later instant as compared with the response of the resistive overburden. This is due to the relatively high conductivity of the surface layer in which the diffusion takes place at a much 'slower' rate (Raiche and Gallagher, 1985). Further, an 'overshoot' can be distinguished for thinner surface layers. This overshoot is a distinctive feature of the induced voltage response of a conductive overburden over a resistive basement with large-offset soundings (i.e., when $R/h \gg 1$; see, for example, Spies and Eggers, 1986; Fitterman and Anderson, 1987; Newman et al., 1987; Spies, 1989).

From Figures 7.4 and 7.5 we infer that for the detection of shallow subsurface structures the early-time response is most appropriately used, while for the detection of deeper parts of the subsurface the late-time response is more revealing (see also Botros and Mahmoud, 1978; Hoversten and Morrison, 1982; Gunderson et al., 1986; Bartel and Becker, 1988; Spies, 1989).

(iii) Induced voltage response of a two-layer isotropic sea floor

In this subsection we are concerned with the response of the subsurface of the sea floor due to the electric current switch-on/off excitation by a grounded electric dipole. The configuration representing a two-layer sea floor consists of a sedimentary basement covered by a crustal layer. The latter can be either more conductive or more resistive than the basement. The upper half-space of the configuration represents the sea or ocean. This configuration is depicted in Figure 7.6. Note that as far as the subsurface is concerned this configuration is equivalent to the one discussed in the previous subsection. However, whereas for the on-land situation (Figure 7.3) the upper half-space (representing the air) was considered as a perfect isolator ($\sigma^{(1)} = 0$), now the electrical conductivity of the upper half-space – i.e., of seawater – is much larger than the one of the sedimentary layers (typical values for the electrical conductivity of seawater are 1.0 – 4.0 S/m).
Figure 7.6: The two-layer sea floor. The source is an electric current dipole at the origin \( O \) oriented in the \( i_3 \) direction. \( D_1 \) represents the sea \( (\sigma^{(1)} = 1.0 \text{ S/m}) \). The conductivity \( \sigma^{(3)} \) of the lower half-space is fixed at 0.04 S/m. The distance from source to receiver is equal to 250 m.

We shall investigate the influence of both the conductivity and the layer thicknesses \( h \) of the crustal layer on the voltage induced in the receiver coil. Initially, after the grounded dipole has been switched-off, the electric currents that will flow in both the sea and the sea floor are concentrated near the source without being influenced by the sedimentary basement. However, since seawater is highly conductive, the initial response of the receiver coil will be equal to zero and only after a moderate time the response reaches a maximum value (compare with Figure 6.5). At late times, the induced voltage response is dominated by the conductivity \( \sigma^{(1)} \) of the sea and the conductivity \( \sigma^{(3)} \) of the basement as if the crustal layer were not present at all (Kaufman and Keller, 1983). From Eq. (6.127) we obtain

\[
V_{\text{emf}}(t) \sim \frac{J_0 R \mu_0^{5/2} \sigma^{(1)} \sigma^{(3)}}{40 \pi^{3/2} t^{5/2}} \frac{1 - \eta^{5/2}}{1 - \eta} \quad \text{as } t \to \infty. \tag{7.3}
\]
In Eq. (7.3), \( \eta \) denotes the ratio between the conductivities of the basement and seawater, i.e. \( \eta = \sigma^{(3)} / \sigma^{(1)} \). Since the value of \( \eta \) will be very small we can already conclude from Eq. (7.3) that the late-time behavior of the induced voltage response is determined primarily by the conductivity of seawater and much less by the conductivity of the sea floor subsurface.

**Resistive crustal layer**

For the resistive crustal layer we have taken the conductivity \( \sigma^{(2)} \) of the crustal layer equal to 0.004 S/m. As for with all previous configurations, the conductivity of the basement is taken equal to 0.04 S/m. The conductivity of seawater has been taken equal to 1.0 S/m. In Figures 7.7a and 7.7b we have plotted the normalized voltage responses following the electric current switch-off of the grounded dipole for layer thicknesses of \( h = 50 \) m and \( h = 25 \) m, respectively. Dashed lines indicate early-time and late-time responses. Although the early stage, late stage and the transition between the two can be recognized, we observe that the influence of the basement conductivity is not that explicitly visible as in Figure 7.4.

**Conductive crustal layer**

For the conductive crustal layer we have taken the conductivity \( \sigma^{(2)} \) of the crustal layer equal to 0.40 S/m. As for the resistive case, the conductivity of the basement is taken equal to 0.04 S/m and the conductivity of seawater is taken equal to 1.0 S/m. Figures 7.8a and 7.8b show the normalized voltage responses following the electric current switch-off of the grounded dipole for a layer-thickness of \( h = 50 \) m and \( h = 25 \) m respectively. As with the resistive overburden, Figure 7.7, an early stage, a late stage and the transition between the two can be recognized. However, not only does the transition from early to late stage take place at a much later instant as compared with the response of the resistive crustal layer, but also we observe that this transition is more sensitive to the layer thickness \( h \) of the crustal layer than for the previous case. Obviously, this is due to the relatively slower rate of diffusion of the electromagnetic field through the conductive crustal layer as compared to the resistive crustal layer. In the former case, the diffusion of the electromagnetic field takes place at about the same rate as in the seawater.

From these results we infer that this kind of source-receiver system is not useful for the prospecting of the subsurface of the sea floor. Changes in the sea floor conductivity only produce minor perturbations in what is essentially a seawater response. Only in the rare circumstance of the sea floor being (locally) more conductive
Figure 7.7: Normalised induced voltage responses of an isotropic sea floor covered by a resistive crustal layer following the electric current switch-off excitation by a grounded electric dipole. The response has been computed for layer thicknesses of (a) 50 m and (b) 25 m. The conductivity of the crustal layer is equal to 0.004 S/m. The conductivity of the sedimentary basement is equal to 0.04 S/m. The conductivity of sea water is equal to 1.0 S/m. $\mu = \mu_0$, the permeability of free space, is assumed throughout. Dashed lines indicate early- and late-time asymptotes. See also Figure 7.6.
Figure 7.8: Normalized induced voltage response of an isotropic sea floor covered by a conductive crustal layer following the electric current switch-off excitation by a grounded electric dipole. The response has been computed for layer thicknesses of (a) 50 m and (b) 25 m. The conductivity of the crustal layer is equal to 0.40 S/m. The conductivity of the sedimentary basement is equal to 0.04 S/m. The conductivity of sea water is equal to 1.0 S/m. \( \mu = \mu_0 \), the permeability of free space, is assumed throughout. Dashed lines indicate early- and late-time asymptotes. See also Figure 7.6.
than sea water will this kind of system be sensitive to the conductivity variations in the subsurface of the sea floor (for example, see Cheesman et al., 1987).

As pointed out by Edwards and Chave (1986) and Cheesman et al. (1987), the most suitable source-receiver geometries for sea floor conductivity mapping are the in-line electric dipole-dipole system and the coaxial magnetic dipole-dipole system, the typical distance between source and receiver being 100 m. However, we have computed numerical results only for the same source-receiver system used in the previous configurations since our objective is to investigate the influence of the conductivity profile of the subsurface of the earth on the diffusion characteristics of the electromagnetic field, rather than the influence of different source-receiver geometries.

7.3 Anisotropic media: transverse isotropy

In this section we are concerned with the simplest class of anisotropic media, viz. transversely isotropic (TI) media (Winterstein, 1990). For this class of anisotropy the electromagnetic properties of the medium are characterized by a scalar magnetic permeability and a tensorial electrical conductivity, the latter having only two different principal values. For isotropic media all principal values are equal, for arbitrarily anisotropic media all principal values are different; see Section 2.2. This property entails that one and the same conductivity is observed in any direction in a certain plane — called the strike — and a different conductivity is observed in the direction normal to that plane.

Following Section 2.2, we denote by \( e_1, e_2 \) and \( e_3 \) the principal directions of the conductivity tensor corresponding to \( \sigma^{[1]}, \sigma^{[2]} \) and \( \sigma^{[3]} \), respectively (see Figure 2.2). We take \( e_1 \) and \( e_2 \) in the plane of the strike and \( e_3 \) normal to the plane of the strike. Then \( \sigma^{[1]} = \sigma^{[2]} = \sigma_t \) is the transverse conductivity. Further, \( \sigma^{[3]} = \sigma_n \) is the conductivity in the normal direction. Of particular interest is the coefficient of (macro)anisotropy \( \lambda = \sqrt{\sigma_t/\sigma_n} \) that characterizes the fact that the medium conducts the electric current along the strike better (\( \lambda > 1 \)) or worse (\( \lambda < 1 \)) than in the direction normal to the strike. In the first case the geometric surface of the conductivity tensor is a prolate spheroid, in the latter case the geometric surface is an oblate spheroid (Nye, 1972; Matias and Habberjam, 1986; see also Section 2.2). The first mention of this coefficient in the literature could be traced back as far as 1920.
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(Schlumberger, 1920) and in 1934 Schlumberger and Leonardon were among the first to present realistic values for the anisotropy coefficient of transversely isotropic media (Maillet and Doll, 1932; Schlumberger et al. 1934). Recent studies on the coefficient of macroanisotropy due to (water filled) fractures or aligned grains present in the sedimentary formations show that common values of $\lambda$ are in the range of $\lambda = 0.5 - 2.0$ (Campbell, 1977; Mendelson and Cohen, 1982).

Our primary objective is to investigate the influence of the orientation of the principal directions $e_k$ on the transient electromagnetic field behavior in transversely isotropic media. For this reason it is convenient to introduce the angles $\chi$ specifying the inclination or dip of the strike with respect to the horizontal plane and $\zeta$ specifying the azimuth of the strike (see Figures 2.3 and 7.9) While keeping $\sigma_n$ and $\sigma_t$ at fixed values we shall vary $\zeta$ from $-\pi$ to $\pi$ and $\chi$ from 0 to $\pi/2$.

![Diagram](image_url)

**Figure 7.9:** Sample of a transversely isotropic medium. The principal directions are denoted by $e_1$ and $e_2$ (plane of strike) and $e_3$ (normal to the strike). The angles $\chi$ and $\zeta$ denote the dip and the azimuth of the strike. Special cases are TIV ($\chi = 0$) and TIH ($\chi = \pi/2$).
Special cases of transverse isotropy are TIV where $\chi = 0$ and $e_3 = i_3$ (i.e., a vertical axis of symmetry) and TII where $\chi = \pi/2$ and $e_2 = i_3$ (i.e., a horizontal axis of symmetry) (Crampin, 1989; Winterstein, 1990). By way of illustration we have depicted in Figure 7.10 more examples of the orientation of the principal directions and strike for a number of $\zeta$-values and the corresponding electrical conductivity tensor.

Note that if $\chi = 0$, the anisotropy is cylindrically symmetric with respect to the $i_3$-axis, implying that the transient electromagnetic field behavior is independent of $\zeta$. This special form of anisotropy has been investigated by, for example, Edwards et al. (1979), Nabulshi and Wait (1982) and Kaufman and Keller (1983), but although the importance of (steeply) dipping strikes was pointed out by Matias and Habberjam (1986) – results on the dipping strike never seem to have been published. In successive subsections we shall consider the transient electromagnetic field behavior in a homogeneous transversely isotropic half-space due to the switching off of an electric current dipole located at the surface (compare with Figure 7.1). Anisotropy of which the geometric surface is a prolate spheroid and an oblate spheroid shall be discussed separately. In the final subsection we discuss the differences between the results for the two forms of anisotropy in detail. However, we shall first give a brief outline of the numerical procedures that have been used to evaluate the relevant space-time domain expressions, Eq. (4.90).

**Numerical implementation**

We need to solve numerically the equation that describes the modified Cagniard contour $p(r, \psi)$. We start from Eq. (3.102) for a single generalized diffusive field constituent

$$p r \cos \psi + x_3 \tilde{\gamma}(p, \psi) = \tau,$$

(7.4)

where $\tilde{\gamma}(p, \psi)$ is either equal to $\tilde{\gamma}^{(+1)}$ or to $\tilde{\gamma}^{(+2)}$. Note that in our case $x_3 > 0$. Since Eq. (7.4) cannot be solved explicitly for $p = p(\tau, \psi)$ we have to resort to a numerical procedure. We use the Newton-Raphson method to solve iteratively the root $p$ of the equation

$$F(p) = p r \cos \psi + x_3 \tilde{\gamma}(p, \psi) - \tau = 0,$$

(7.5)
Figure 7.10: Examples of orientation of a dipping strike for different values of its azimuth ($\zeta =0$ and $\zeta =\pi/2$). $\chi$ denotes the dip of the strike with respect to the horizontal plane. The corresponding conductivity tensor $\sigma_{k,r}$ is given as well.
for fixed value of $\psi$ and for continuously increasing values of $\tau$ from $\tau = T_{\text{min}}$ onward (cf. Eq. (3.105)). This leads to the iterative equation

$$ p^{\text{new}} := p^{\text{old}} - F(p^{\text{old}})/\partial p F(p^{\text{old}}), $$

(7.6)

where

$$ \partial p F(p) = \tau \cos \psi + x_3 \partial p \gamma(p, \psi). $$

(7.7)

Here, $p^{\text{old}}$ and $p^{\text{new}}$ denote two successive values of $p$ in this iterative procedure (with $\tau$ fixed). For the initial guess of $p^{\text{old}}$ we use the value of $p$ that was obtained from the previous value of $\tau$.

Appendix A describes how the 'exact' eigenvalues $\gamma_k$, $k = 1, \ldots, 4$ (cf. Eqs. (A.44)-(A.47)) and corresponding $\partial p \gamma_k$ can be computed for any value of $p$. However, these $\gamma_k$ — just a set of four numbers — still need to be partitioned into the appropriate $\tilde{\gamma}(\pm \psi)$ by means of analytical continuation. The initial partitioning at $p = 0$ (i.e., $\tau = T_{\text{min}}$) is given by Eqs. (3.46)-(3.47). From this value of $p$ onward we use linear extrapolations of the $\gamma_k$ and $\partial p \gamma_k$ to match each of the $\gamma_k(p^{\text{new}})$ to the $\gamma(\pm \psi)(p^{\text{old}})$. To ensure a correct functioning of this procedure it was necessary to implement an adaptive $\tau$-stepping routine that automatically decreases the $\tau$-step size either when the modified Cagniard contour approaches a branch point (i.e., when $|\partial p \gamma(p, \psi)|$ becomes large) or in the vicinity of the point where the modified Cagniard contour leaves or meets the real $p$-axis again (i.e., when $|\partial p F(p)|$ becomes small; see also Section 3.4). Using this method for solving Eq. (7.4) it is found that about three to eight iterations are needed to get a relative error of $10^{-7}$ in the value of $p$.

(i) Transverse anisotropy, prolate spheroidal geometric surface ($\lambda > 1$)

In this subsection we consider the transient electromagnetic field behavior in a transversely isotropic half-space, the coefficient of anisotropy of which is greater than one. The source is a switch-on/off electric current dipole oriented in the $\hat{z}$-direction at the surface of the anisotropic medium, analogous to the isotropic half-space as depicted in Figure 7.1. Here, we have taken $\sigma_n = 0.08$ S/m and $\sigma_t = 0.224$ S/m ($\lambda = \sqrt{2.8} \approx 1.7$). In Figures 7.11 and 7.12 we have plotted the induced voltage response $V_{\text{emf}}(t)$ after current switch-off as a function of the time $t$ and $\zeta$, while $\chi$ is taken as parameter. The values of $\chi$ are in the interval from $\chi = 0$ (TIV) to $\chi = \pi/2$
Figure 7.11: Normalized induced voltage response of a transversely isotropic half-space following the electric current switch-off of a horizontal electric dipole at the surface. $\chi$ denotes the dip of the strike with respect to the horizontal plane, $\zeta$ denotes the azimuth. $\sigma_i = 0.224 \text{ S/m}$, $\sigma_n = 0.08 \text{ S/m}$ (i.e., the geometric surface of $\sigma_{k,r}$ is a prolate spheroid).
Figure 7.12: Normalized induced voltage response of a transversely isotropic half-space following the electric current switch-off of a horizontal electric dipole at the surface. $\chi$ denotes the dip of the strike with respect to the horizontal plane. $\zeta$ denotes the azimuth. $\sigma = 0.085 \, \text{S/m}$ (i.e., the geometric surface of $\sigma_x$ is a prolate spheroid).
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(TIH). The induced voltage is normalized in the same way as for previous results, i.e., a normalization with respect to the early-time response $V_{\text{nor}}$ of an isotropic half-space of which the conductivity is equal to 0.04 S/m. As for the previous results for isotropic media, both time and normalized voltage have been plotted using logarithmic scales.

Obviously, the voltage response is azimuth-independent if $\chi = 0$ (TIV), while the dependence on $\zeta$ is maximal for the early-time response if $\chi = \pi/2$. From Figures 7.11 and 7.12 we observe that the late time response is proportional to $t^{-5/2}$ and shows a maximum dependence on $\zeta$ when $\chi = \pi/4$.

In Figure 7.13 we have plotted for a number of snap times the $E_2(x, t)$ component of the electric field strength following current switch-off excitation, in the cross-section $x_2 = 0$ of the configuration for $\chi = 0$ (TIV). Note that in this case the electric field is normal to the plane $x_2 = 0$ and symmetric with respect to the plane $x_1 = 0$. From this figure we observe that the diffusion in the direction of the strike (i.e., the horizontal direction) takes place at a slower rate than in the direction normal to the strike, in agreement with $\sigma_1 > \sigma_n$. This causes the maximum of the return currents not to be located directly underneath the source (compare with Figure 7.2) but at both sides of the source and moving downward and outward in a straight line, the cosine of the angle between line of migration and the vertical being approximately equal to $1/\lambda = \sqrt{\sigma_n/\sigma_1}$.

Finally, in Figure 7.14 we have plotted for the same number of snap times the $E_2(x, t)$ component of the electric field strength following current switch-off in the cross-section $x_2 = 0$ of the configuration when $\chi = \pi/4$ and $\zeta = \pi/4$. Clearly, all symmetry with respect to the plane $x_1 = 0$ has disappeared and also – although not visible from this figure – the electric field strength is not normal to the plane $x_2 = 0$ anymore. Further, the single maximum of the return currents is moving approximately in the direction of the strike, i.e. downwards and outwards in the direction of decreasing $x_1$.

(ii) Transverse anisotropy, oblate spheroidal geometric surface ($\lambda < 1$)

In this subsection we consider the transient electromagnetic field behavior in a transversely isotropic half-space, the coefficient of anisotropy of which is less than one. The source is a switch-on/off electric current dipole oriented in the $i_1$-direction at the surface of the anisotropic medium, analogous to the isotropic half-space as depicted in Figure 7.1. Here, we have taken $\sigma_n = 0.224$ S/m and $\sigma_1 = 0.08$ S/m. Hence,
Numerical results

Figure 7.13: Snapshots showing the evolution of the $z$-component of the electric field strength in the cross-section $z_1 = 0$ of a transversely isotropic half-space following the electric current switch-off of a horizontal electric dipole at the origin. $\sigma_r = 0.246 S/m$, $\sigma_n = 0.033 S/m$ (i.e., the geometric surface of $\sigma_r$ is a prolate spheroid). $X = 0$ (TVV). Negative field values correspond to the return currents.
Figure 7.14: Snapshots showing the evolution of the $i_2$-component of the electric field strength in the cross-section $x_2 = 0$ of a transversely isotropic half-space following the electric current switch-off of a horizontal electric dipole at the origin. $\sigma_t = 0.224 \, \text{S/m}, \sigma_n = 0.08 \, \text{S/m}$ (i.e., the geometric surface of $\sigma_t, r$ is a prolate spheroid). $\chi = \pi/4$, $\zeta = \pi/4$. Negative field values correspond to the return currents.
in this case \( \sigma_t/\sigma_n = 1/2.8 \). The coefficient of anisotropy \( \lambda \) is now equal to \( \sqrt{1/2.8} \approx 0.6 \). In Figures 7.15 and 7.16 we have plotted the induced voltage response \( V_{\text{emf}}(t) \) following current switch-off as a function of the time \( t \) and the angle \( \zeta \). \( \chi \) is taken as parameter. As for the previous results, we have normalized the induced voltage response by the early-time response of an isotropic half-space of which the electrical conductivity is equal to 0.04 S/m.

Obviously, the voltage response is azimuth-independent if \( \chi = 0 \) (TIV), while the dependence on \( \zeta \) is maximal for the early-time response if \( \chi = \pi/2 \). From Figures 7.15 and 7.16 we observe that the late time response is proportional to \( t^{-5/2} \) and shows a maximum dependence on \( \zeta \) when \( \chi = \pi/4 \).

In the next subsection we shall discuss in detail the differences between the voltage responses for the two forms of transverse anisotropy. However, a first comparison between Figures 7.11-7.12 (prolate) and Figures 7.15-7.16 (oblate) shows that, although the early-time responses for the two forms of anisotropy are not essentially different, the late-time response for the oblate case is much stronger dependent of the value of \( \zeta \). For certain values of \( \zeta \) and \( \chi \) a sign reversal of the induced voltage response is observed. This is of major concern since the occurence of sign reversals with transient electromagnetic soundings is commonly attributed to induced polarization effects (IP) and reflection phenomena due to local lateral inhomogeneities (Spies, 1980; Lee, 1981). Yet, our results clearly show that for this kind of source-receiver system, sign reversals can also be due to (transverse) anisotropy with a dipping strike, especially when the coefficient of anisotropy \( \lambda \) is less than one. Note that for the horizontal strike (TIV) and vertical strike (TIV) no sign reversals are observed. In the next subsection we shall discuss in detail the differences between the responses for the two forms of transverse isotropy and we will try to explain why certain effects are observed for certain values of \( \zeta \) and \( \chi \).

In Figure 7.17 we have plotted for a number of snap times the \( E_2(\varpi, t) \) component of the electric field strength following current switch-off excitation, in the cross-section \( x_2 = 0 \) of the configuration for \( \chi = 0 \) (TIV). Note that in this case the electric field is normal to the plane \( x_2 = 0 \) and symmetric with respect to the plane \( x_1 = 0 \). From this figure we observe that the diffusion in the direction normal to the strike (i.e., in the vertical direction) takes place at a slower rate as in the direction of the strike, in agreement with \( \sigma_t < \sigma_n \). This causes the maximum of the return currents to be located directly underneath the source and moving downwards (compare with Figure 7.13 for \( \lambda > 1 \)). Note the changes in the shape of the contour
Figure 7.15: Normalized induced voltage response of a transversely isotropic half-space following the electric current switch-off of a horizontal electric dipole at the surface. $\chi$ denotes the dip of the strike with respect to the horizontal plane, $\zeta$ denotes the azimuth. $\sigma_t = 0.08 \, S/m$, $\sigma_n = 0.224 \, S/m$ (i.e., the geometric surface of $\sigma_{k,r}$ is an oblate spheroid).
Figure 7.16: Normalised induced voltage response of a transversely isotropic half-space following the electric current switch-off of a horizontal electric dipole at the surface. $\chi$ denotes the dip of the strike with respect to the horizontal plane, $\zeta$ denotes the azimuth. $\sigma_t = 0.08 \, \text{S/m}$, $\sigma_n = 0.224 \, \text{S/m}$ (i.e., the geometric surface of $\sigma_{h,r}$ is an oblate spheroid).
Figure 7.17: Snapshots showing the evolution of the $i_2$-component of the electric field strength in the cross-section $z_2 = 0$ of a transversely isotropic half-space following the electric current switch-off of a horizontal electric dipole at the origin. $\sigma_t = 0.08 \text{ S/m}$, $\sigma_n = 0.224 \text{ S/m}$ (i.e., the geometric surface of $\sigma_{k,r}$ is an oblate spheroid). $\chi = 0$ (TTV). Negative field values correspond to the return currents.
Figure 7.18: Snapshots showing the evolution of the $i_2$-component of the electric field strength in the cross-section $x_2 = 0$ of a transversely isotropic half-space following the electric current switch-off of a horizontal electric dipole at the origin. $\sigma_2 = 0.08 \, S/m$, $\sigma_n = 0.224 \, S/m$ (i.e., the geometric surface of $\sigma_{ik}$ is an oblate spheroid). $\chi = \pi/4$, $\zeta = \pi/4$. Negative field values correspond to the return currents.
levels of the return currents as $\lambda$ is increased from a value less than one to a value greater than one (compare Figures 7.17, 7.13 and 7.8 in this order).

Finally, In Figure 7.18 we have plotted for the same number of snap times the $E_2(\varphi, t)$ component of the electric field strength following current switch-off excitation, in the crossection $x_2 = 0$ of the configuration for $\chi = \pi/4$ and $\zeta = \pi/4$. Clearly, all symmetry with respect to the $x_1 = 0$ plane has disappeared and also, the electric field strength is not normal to the plane $x_2 = 0$ anymore. Here we observe that the maximum of the return currents is not migrating downwards and outwards with respect to the strike (i.e. to the 'left', see Figure 7.14) but approximately in a direction normal to the strike.

(iii) Discussion of the results for the two forms of transverse isotropy

In the preceding two subsections we have presented numerical results for two distinctive forms of transverse isotropy, viz. where the geometric surface of the electrical conductivity tensor is a prolate spheroid ($\sigma_i > \sigma_n$) or an oblate spheroid ($\sigma_i < \sigma_n$). Next, we compare the induced voltage responses for these two forms of anisotropy and by means of a qualitative interpretation we shall try to explain all the distinctive features observed.

Comparison of the induced voltage responses

The main differences between the induced voltage responses for the two forms of transverse anisotropy can be divided into two parts: (a) differences in the early-time response and (b) differences in the late-time response.

(a) From Figures 7.11 and 7.12 we observe that for the case where $\lambda > 1$, the early-time response is maximal for $\zeta = 0$ and $\zeta = \pm \pi$ and minimal for $\zeta = \pm \pi/2$.

From Figures 7.15 and 7.16 we observe that for the case where $\lambda < 1$, the early-time response is maximal for $\zeta = \pm \pi/2$ and minimal for $\zeta = 0$ and $\zeta = \pm \pi$ (the responses for $\zeta = -\pi$ and $\zeta = \pi$ are equal, of course).

Further, from the voltage responses for the TIH-cases ($\chi = \pi/2$) we observe that for both forms of transverse anisotropy the early-time maxima have exactly equal values (though, for different values of $\zeta$). Clearly, the same holds for the early-time minima.

(b) From Figures 7.11 and 7.12 we observe that for the case where $\lambda > 1$, the late-time response is maximal for $\zeta = -\pi/2$ and minimal for $\zeta = +\pi/2$. 
From Figures 7.15 and 7.16 we observe that for the case where $\lambda < 1$, the late-time response is maximal for $\zeta = +\pi/2$ and minimal for $\zeta = -\pi/2$. In the latter case, this minimum value is negative.

Further, from the voltage responses for the TIH-cases ($\chi = \pi/2$) we observe that for both forms of transverse anisotropy, the same late-time behavior is obtained as function of $\psi$, aside a difference of $\pi/2$ between corresponding values of $\zeta$.

Discussion and interpretation

Our goal is to find some kind of correlation between certain distinctive features of the induced voltage response of the transversely isotropic half-space and the corresponding values of $\{\chi, \zeta\}$ and $\{\sigma_i, \sigma_n\}$. Whenever convenient, we shall use particular components of the conductivity tensor $\sigma_{kr}$ rather than $\sigma_i$ and $\sigma_n$. Obviously, this analysis will be principally qualitative and not quantitative since an "exact" relationship between the $\sigma_{kr}$ and the voltage response cannot be obtained explicitly for a dipping strike – even not in the early- or late-time limit.

From Figure 7.10 and the symmetry properties of the configuration with respect to $\zeta$, we can already conclude that for a dipping strike the induced voltage response as a function of $\zeta$ for fixed $t$ will, in general, be different for opposite values of $\zeta$, except for $\zeta = 0$ and $\zeta = \pi$ for which the corresponding responses must be the same. Special cases are the TIV and TIH forms of anisotropy. In the first case the induced voltage response is independent of $\zeta$; in the latter case the induced voltage responses for opposite values of $\zeta$ are the same.

Upon comparing Figures 7.11-7.12 and 7.15-7.16 with Figure 7.10 for the TIV and TIH forms of anisotropy ($\chi = 0$ and $\chi = \pi/2$, respectively), we infer that for these two cases, the voltage response follows from $\sigma_{2,2}$ only and none of the other components of $\sigma_{kr}$. Consequently, the induced voltage response for the TIV and TIH forms of anisotropy should be equal to the $V_{\text{emf}}(t)$ as given by Eq. (6.126), now with $\sigma^{(2)} = \sigma_{2,2}$ and $\eta = 0$. This result can also be found in Kaufman and Keller (1983).

The fact that for a TIH-medium the induced voltage response of a receiver coil along the $z_1$-axis is determined by the conductivity $\sigma_{2,2}$ in the $i_2$-direction is called the "paradox of anisotropy". This 'paradox' – which of course is not a real paradox – was already observed by Schlumberger et al. in 1934.
7.3 Anistropic media: transverse isotropy

In Figures 7.19a and 7.19b we have plotted the value of $\sigma_{2,2}$ as a function of $\chi$ and $\zeta$ for $\lambda > 1$ and $\lambda < 1$, respectively. The similarity between the value of $\sigma_{2,2}$ as a function of $\chi$ and $\zeta$ and the corresponding induced voltage responses is remarkable – even for dipping strikes.

Now, the observed differences in the responses for opposite values of $\zeta$ must be caused by those components of the conductivity tensor that are odd functions of $\zeta$ and that are equal to zero if $\chi = 0$ or $\chi = \pi/2$. From Figure 7.10 and the equations (2.10)-(2.13) we find that only $\sigma_{1,3}$ satisfies this conditions. In Figures 7.19c and 7.19d we have plotted the value of $\sigma_{1,3}$ as a function of $\chi$ and $\zeta$ for $\lambda > 1$ and $\lambda < 1$, respectively. The similarity between the behavior of $\sigma_{1,3}$ as a function of $\chi$ and $\zeta$ and the corresponding behavior of the late-time part of the induced voltage response is remarkable: not only do Figures 7.19c and 7.19d exactly indicate for which combinations of $\zeta$ and $\chi$ we can expect certain anomalies from the ‘ideal’ response as dictated by $\sigma_{2,2}$ but even more, a positive value of $\sigma_{1,3}$ increases the late-time response while a negative value of $\sigma_{1,3}$ decreases the late-time response. In the latter case the result may be negative. Of course, the relation between $\sigma_{1,3}$ and the induced voltage response is complicated and certainly not strictly additive.

From Figures 7.19c and 7.19d we infer that the late-time responses will be most strongly dependent on $\zeta$ for $\chi = \pi/4$. Moreover, a sign reversal is more likely to occur if $\lambda < 1$ as compared with $\lambda > 1$ since in the former case the late-time response due to $\sigma_{2,2}$ is the smallest and hence, a decrease of the late time response due to a negative value of $\sigma_{1,3}$ will cause this case of transverse anisotropy to be the first to show a sign reversal.

It must be emphasized that the above analysis is strictly qualitative and that in the case of a dipping strike not only $\sigma_{2,2}$ and $\sigma_{1,3}$ are of importance, but the other components of the conductivity tensor as well, though their contributions may not be that explicitly noticeable. More numerical results are necessary to refine and verify our ‘empirical’ relations – especially including more combinations of $\sigma_{1}$ and $\sigma_{n}$.

(iv) The in-loop response of a transversely anisotropic half-space

To conclude this section we consider the in-loop response (IL) of a transversely isotropic half-space with a dipping strike. A very important aspect of the IL-response is whether the occurrence of a sign reversal is possible or not possible for this configuration. This problem has been investigated by Weidelt (1982) for a coincident-loop system (i.e., where the transmitter loop is also used as receiver for recording the
Figure 7.19: The components $\sigma_{2,2}$ and $\sigma_{1,3}$ of the electrical conductivity tensor as a function of the dip ($\chi$) and azimuth ($\zeta$) of the strike for $\sigma_1 = 0.224 \, S/m$, $\sigma_n = 0.08 \, S/m$ ((a) and (c)) and for $\sigma_1 = 0.08 \, S/m$, $\sigma_n = 0.224 \, S/m$ ((b) and (d)).
induced voltage after current switch-off). Weidelt shows that for this source-receiver configuration a sign reversal is impossible unless induced polarization (IP) effects are taken into account.

We investigate the in-loop response $V_{\text{emf}}^\circ(t)$ of a circular loop as a function of the time $t$ and the dip of the strike $\chi$. Obviously the IL-response is azimuthally independent since the loop is circular (note that this is not the case for a square loop). The IL-response is very easily obtained, considering that an integration around the loop (see Section 6.5) in this case is equivalent to an integration with respect to $\zeta$ from $-\pi$ to $\pi$, of the induced voltage response of the source-receiver system as considered before. Contributions from opposite parts of the circular loop correspond to a certain value of $\zeta = \zeta_0$ and $\zeta = \zeta_0 + \pi$. Hence, if the average of the responses for $\zeta_0$ and $\zeta_0 + \pi$ is a positive function of $t$ for arbitrary values of $\zeta_0$, then a sign reversal is impossible. From Eqs. (3.11)-(3.13) it can be verified that the function $\bar{B}(p, \psi)$ which appears in the integrand of the expression for $\bar{G}(\varphi, s)$ (cf. Eq. (4.72)) satisfies the following relation

$$
\bar{B}(p, \psi)\bigg|_{\zeta = \zeta_0 + \pi} = \bar{B}^*(p, \psi)\bigg|_{\zeta = \zeta_0} \quad \text{as } \text{Re}(p) = 0,
$$

(7.8)

while $\bar{B}(p, \psi)$ is real when $\sigma_{1,3}$ and $\sigma_{2,3}$ are equal to zero, i.e., for isotropic, TIV and TII media. As far as the induced voltage response is concerned, the exponential term that appears in Eq. (4.72) is independent of $\zeta$ (in this case we take the $\gamma$ of the upper half-space). The $\bar{B}$ for $\zeta_0$ and for $\zeta_0 + \pi$ can be taken together, yielding an real expression similar as to what is obtained for isotropic, TIV and TII media. On the basis of this result we expect that a sign reversal is impossible, just as with isotropic, TIV and TII media.

To verify this assumption we have computed the IL-response for both $\lambda > 1$ and $\lambda < 1$. The induced voltage responses are plotted in Figures 7.20a and 7.20b. From the results we observe that the IL-responses for the prolate as well as the oblate spheriodal form of the geometric surface of $\sigma_{k,r}$ are smooth functions of $\chi$ that do not show a sign reversal.
Figure 7.20: Normalized in-loop (IL) response of a circular loop source at the surface of a transversely isotropic half-space following an electric current switch-off excitation. Results are shown for (a) $\lambda > 1$ (geometric surface of $\sigma_{k,r}$ is a prolate spheroid) and (b) $\lambda < 1$ (geometric surface of $\sigma_{k,r}$ is an oblate spheroid). $\chi$ denotes the dip of the strike with respect to the horizontal plane. Special cases are TIV ($\chi = 0$) and TIH ($\chi = \pi/2$). The radius of the loop is equal to 250 m.
7.4 Anisotropic media: arbitrary anisotropy

In this section we are concerned with the most general class of anisotropic media, viz. arbitrarily anisotropic media. For this class of anisotropy the electromagnetic properties of the medium are characterized by a scalar magnetic permeability and a tensorial electrical conductivity, the latter having three different principal values $\sigma^4$. For isotropic media all principal values are equal, for transversely isotropic media two principal values are equal, but different from the third; see Section 2.2.

Following Section 2.2, we denote by $e_1, e_2$ and $e_3$ the principal directions of the conductivity tensor corresponding to $\sigma^{[1]}, \sigma^{[2]}$ and $\sigma^{[3]}$, respectively (see Figure 2.2). Since the electrical conductivity observed in any of the principal directions is different, we cannot define a strike and coefficient of anisotropy as for transversely isotropic media. However, to ease the discussion of the numerical results, we here introduce a "strike" as well. As for transversely isotropic media, we take $e_1$ and $e_2$ in the plane of the strike and $e_3$ normal to the plane of the strike. We want to emphasize that this choice for the strike has been made only for consistency with the notation we employ for transversely isotropic media. Then, the transverse conductivities are $\sigma^{[1]}$ and $\sigma^{[2]}$, $\sigma^{[1]} \neq \sigma^{[2]}$. Further, $\sigma^{[3]}$ is the conductivity in the normal direction.

Our objective is to investigate the influence of the orientation of the principal directions $e_k$ on the transient electromagnetic field behavior in arbitrarily anisotropic media. For this reason it is convenient to introduce the angles $\chi$ specifying the inclination or dip of the strike with respect to the horizontal plane and $\zeta$ specifying the azimuth of the strike (see Figure 2.3). Further, it is necessary to introduce a third angle $\varphi_0$ specifying the relative position of $e_1$ and $e_2$ in the plane of the strike. See Figures 2.3 and 7.21. While keeping $\varphi_0$, $\sigma^{[1]}$, $\sigma^{[2]}$ and $\sigma^{[3]}$ at fixed values we shall vary $\zeta$ from $-\pi$ to $\pi$ and $\chi$ from 0 to $\pi/2$.

By way of illustration we have depicted in Figure 7.22 more examples of the orientation of the principal directions and strike for $\varphi_0 = 0$ and $\varphi_0 = \pi/2$ for a number of $\zeta$-values. We have given the components of the corresponding electrical conductivity tensor as well.
We have computed the induced voltage response $V_{\text{emf}}(t)$ for a homogeneous arbitrarily anisotropic half-space due to the switching off of an electric current in a grounded dipole, located at the surface of the anisotropic medium and oriented in the $i_2$-direction, analogous to the isotropic half-space as depicted in Figure 7.1. Here, we have taken $\sigma_1 = 0.12 \text{ S/m}$, $\sigma_2 = 0.224 \text{ S/m}$ and $\sigma_3 = 0.08 \text{ S/m}$. 

Figure 7.21: Sample of a arbitrarily anistropic medium. The principal directions are denoted by $e_1$ and $e_2$ (plane of "strike") and $e_3$ (normal to the strike). The angles $\chi$ and $\zeta$ denote the dip and the azimuth of the strike. The angle $\phi$ denotes the relative position of $e_1$ and $e_2$ in the plane of the strike.
Figure 7.22: Examples of orientation of a dipping "strike" of an arbitrarily anisotropic medium for different values of its azimuth ($\zeta = 0$ and $\zeta = \pi/2$). $\chi$ denotes the dip of the strike with respect to the horizontal plane, $\varphi_0$ denotes the relative position of the principal directions $e_1$ and $e_2$ in the plane of the strike. The corresponding conductivity tensor $\sigma_{k,r}$ is given as well. The $\sigma^{[i]}$ denote the principal values of $\sigma_{k,r}$. 
(i) $\varphi_0 = 0$

In Figures 7.23 and 7.24 we have plotted the induced voltage response $V_{emf}(t)$ following electric current switch-off excitation as a function of the time $t$ and $\zeta$, while $\chi$ is taken as parameter. The values of $\chi$ are in the interval from $\chi = 0$ to $\chi = \pi/2$. Here, we have taken $\varphi_0 = 0$ (see also Figure 7.22). The induced voltage is normalized in the same way as for the previous results for isotropic media, i.e. a normalization with respect to the early-time response $V_{nor}$ of an isotropic half-space of which the conductivity is equal to 0.04 S/m. As for the previous results for isotropic media, both time and normalized voltage have been plotted using logarthmic scales.

Obviously, the voltage response is now azimuth-dependent if $\chi = 0$. From Figures 7.23 and 7.24 we observe that the late time response is proportional to $t^{-5/2}$ and shows a maximum dependence on $\zeta$ when $\chi = \pi/4$. Further, for certain values of $\chi$ and $\zeta$ a sign reversal of the induced voltage response is observed.

(ii) $\varphi_0 = \pi/2$

Next, we investigate the induced voltage response of the same arbitrarily anisotropic half-space as with the preceding subsection, however, now we have taken $\varphi_0 = \pi/2$. See Figure 7.22.

In Figures 7.25 and 7.26 we have plotted the induced voltage response $V_{emf}(t)$ following the electric current switch-off as a function of the time $t$ and the angle $\zeta$. $\chi$ is taken as parameter. The values of $\chi$ are in the interval from $\chi = 0$ to $\chi = \pi/2$. The induced voltage is normalized in the same way as for the previous results for isotropic media, i.e. a normalization with respect to the early-time response $V_{nor}$ of an isotropic half-space of which the conductivity is equal to 0.04 S/m.

Obviously, the voltage response is now azimuth-dependent if $\chi = 0$. From Figures 7.25 and 7.26 we observe that the late time response is proportional to $t^{-5/2}$ and shows a maximum dependence on $\zeta$ when $\chi = \pi/4$, though, this dependence is less explicit as for $\varphi_0 = 0$ (Figures 7.23 and 7.24). A very important feature is the absence of a sign reversal for the case of $\varphi_0 = \pi/2$. It is important to note that for $\varphi_0 = 0$ and $\varphi_0 = \pi/2$ we have taken the principal values the same. The only difference between these two cases is the orientation of the principal directions (i.e., $e_1$ and $e_2$). From these results we infer that, whereas for transversely isotropic media the occurrence of a sign-reversal for certain values of $\zeta$ could be predicted from the values of $\sigma_1$ and $\sigma_n$ only, we find that for arbitrarily anisotropic media the three different principal
Figure 7.23: Normalised induced voltage response of a arbitrarily anisotropic half-space following the electric current switch-off of a vertical electric dipole at the surface. $\chi$ denotes the dip of the strike with respect to the horizontal plane, $\zeta$ denotes the azimuth. $\sigma^{[1]} = 0.12 \, S/m$, $\sigma^{[2]} = 0.224 \, S/m$ and $\sigma^{[3]} = 0.08 \, S/m$. $\varphi_0 = 0$, see Figure 7.22.
Figure 7.24: Normalised induced voltage response of a arbitrarily anisotropic half-space following the electric current switch-off of a horizontal electric dipole at the surface. $\chi$ denotes the dip of the strike with respect to the horizontal plane, $\zeta$ denotes the azimuth. $\sigma^{(1)} = 0.12 \, S/m$, $\sigma^{(2)} = 0.224 \, S/m$ and $\sigma^{(3)} = 0.08 \, S/m$. $\psi_0 = 0$, see Figure 7.22.
Figure 7.25: Normalized induced voltage response of a arbitrarily anisotropic half-space following the electric current switch-off of a horizontal electric dipole at the surface. $\chi$ denotes the dip of the strike with respect to the horizontal plane, $\zeta$ denotes the azimuth. $\sigma^{[1]} = 0.12 \text{ S/m}$, $\sigma^{[2]} = 0.224 \text{ S/m}$ and $\sigma^{[3]} = 0.08 \text{ S/m}$. $\varphi_0 = \pi / 2$, see Figure 7.22.
Figure 7.26: Normalized induced voltage response of a arbitrarily anisotropic half-space following the electric current switch-off of a horizontal electric dipole at the surface. $\chi$ denotes the dip of the strike with respect to the horizontal plane, $\zeta$ denotes the azimuth. $\sigma_1 = 0.12 \text{ S/m}$, $\sigma_2 = 0.224 \text{ S/m}$ and $\sigma_3 = 0.08 \text{ S/m}$. $\varphi_0 = \pi/2$, see Figure 7.22.
values $\sigma^{[4]}$ itself do not provide sufficient information to determine whether a sign-reversal is possible or not.

In the preceding section we found that there exists a close correlation between the values of $\sigma_{2,2}$ and $\sigma_{1,3}$ and the corresponding early- and late-time induced voltage responses of a transversely isotropic medium.

Upon comparing the results for arbitrarily anisotropic media (Figures 7.23-7.26) with the results for transversely isotropic media (Figures 7.11-7.12 and 7.15-7.16) and the corresponding values of $\sigma_{k,r}$ — in particular for $\zeta = 0$ and $\zeta = \pi/2$ — we infer that for this source-receiver system, the induced voltage response of an arbitrarily anisotropic half-space is primarily determined by the values of $\sigma_{2,2}$ and $\sigma_{1,3}$. In Figures 7.27a and 7.27b we have plotted the value of $\sigma_{2,2}$ as a function of $\chi$ and $\zeta$ for $\varphi_0 = 0$ and $\varphi_0 = \pi/2$, respectively. In Figures 7.27c and 7.27d we have plotted the value of $\sigma_{1,3}$ as a function of $\chi$ and $\zeta$ for $\varphi_0 = 0$ and $\varphi_0 = \pi/2$, respectively. The similarity between the values of $\sigma_{2,2}$ and $\sigma_{1,3}$ and the corresponding early- and late-time responses is remarkable: not only do Figures 7.27c and 7.28d exactly indicate for which combinations of $\zeta$ and $\chi$ we can expect certain anomalies from the response that would follow from $\sigma_{2,2}$ only, but even more, from Figures 7.27c and 7.27d we have that the largest negative value of $\sigma_{1,3}$ occurs for $\varphi_0 = 0$, $\zeta = \pi/2$ and $\chi = \pi/4$, these values corresponding to the response that indeed shows the largest sign reversal.
Figure 7.27: The components $\sigma_{2,2}$ and $\sigma_{1,3}$ of the electrical conductivity tensor for an arbitrarily anisotropic medium as a function of the dip ($\chi$) and azimuth ($\zeta$) of the strike. $\sigma^{[1]} = 0.12$ S/m, $\sigma^{[2]} = 0.224$ S/m and $\sigma^{[3]} = 0.08$ S/m. (a) and (c) correspond to $\varphi_0 = 0$, (b) and (d) correspond to $\varphi_0 = \pi/2$. See Figure 7.22.
Appendix A

Eigenvalues of the system’s matrix for complex Fourier parameters

The equations governing the behavior of the transient electromagnetic field in an electrically conducting anisotropic homogeneous medium are the diffusive Maxwell equations (2.18) and (2.19)

\[-\varepsilon_{k,\infty} \partial_m H_p + \sigma_{k,\infty} E_r = -J_k^c, \quad (A.1)\]
\[\varepsilon_{\infty} \partial_m E_r + \mu_{\infty} \partial_t H_p = -K_j^c. \quad (A.2)\]

After carrying out a Laplace transformation with respect to time and a Fourier transformation with respect to the horizontal space coordinates \(x_1\) and \(x_2\) we obtain the transform-domain field equations (2.39) and (2.40). Elimination of the field components that show a jump discontinuity across an interface in a layered configuration, results in a system’s of four linear, first-order differential equations. We write these four differential equations as a single matrix differential equation (cf. Eq. (3.10))

\[\partial_s F_1 = -s^{1/2} A_{l,j} F_j + N_l. \quad (A.3)\]
In Eq. (A.3), $F_j$ is the 4-by-1 field matrix containing the transform-domain horizontal field components, $N_j$ the 4-by-1 notional source matrix and $A_{i,j}$ the 4-by-4 system's matrix. As has been shown in Section 3.3, the solution $F_j(x_3)$ of Eq. (A.3) can be written as the superposition of diffusive field constituents of the form $b_j \exp(-\gamma^{1/2}x_3\gamma)$ where $\gamma$ and $b_j$ are the eigenvalues and corresponding eigenvectors, respectively, of the system's matrix $A_{i,j}$, i.e.,

$$A_{i,j} b_j = \gamma b_j,$$  \hspace{1cm} (A.4)

$$\det(A_{i,j} - \gamma \delta_{i,j}) = 0.$$  \hspace{1cm} (A.5)

Since the determinantal equation (A.5) is a quartic equation in $\gamma$, $\gamma$ itself is a four-valued function of the coefficients that occur in this equation. Thus, $\gamma$ is a four-valued function of the Fourier transformation parameters $\alpha_1$ and $\alpha_2$.

In order that the solution $F_j(x_3)$ of Eq. (A.3) can be transformed from the transform-domain back to the space-time domain, we need detailed information about the properties of the eigenvalues of $A_{i,j}$ for real as well as arbitrarily complex values of $\alpha_1$ and $\alpha_2$.

**Behavior of $\gamma$ for real values of $\alpha_1$ and $\alpha_2$**

Instead of using the determinantal equation (A.5) to derive the properties of $\gamma$ for real (and complex) values of $\alpha_n$, we shall follow a different approach. Since we know that the $x_3$ dependence of the transform-domain field quantities in a homogeneous subdomain of the configuration is proportional to $\exp(-\gamma^{1/2}x_3\gamma)$, we also can replace the differentiations with respect to $x_3$ in the transform-domain field equations (2.39)-(2.40) by a multiplication by the factor $-\gamma^{1/2}$. Now, upon writing

$$\gamma = i\alpha_3$$  \hspace{1cm} (A.6)

we arrive at full notational symmetry. If we eliminate the magnetic field components from the transform-domain field equations (2.39)-(2.40), we end up with the following determinantal equation

$$\det(-\varepsilon_{k,m,p} \hat{\mu}_{p,j} \varepsilon_{j,n,r} \alpha_m \alpha_n + \sigma_{k,r}) = 0,$$  \hspace{1cm} (A.7)
in which the tensorial impermeability \( \mu_{p,j} \) follows from

\[
\mu_{j,p} \mu_{p,i} = \delta_{i,j}.
\]
(A.8)

As far as the relation between \( \gamma \) and the \( \alpha_m \) is concerned, Eqs. (A.5) and (A.7) are completely equivalent and hence, any property of \( \gamma \) obtained from Eq. (A.7) also applies to Eq. (A.5). The tensors \( \sigma_{k,r} \) and \( \mu_{j,p} \) are real, symmetric and positive definite and hence, also \( \mu_{p,j} \) is a real, symmetric and positive definite tensor. Further, since \( \epsilon_{k,m,p} \) is real and completely antisymmetric it immediately follows that \( \epsilon_{k,m,p} \mu_{p,j} \epsilon_{j,m,r} \alpha_m \alpha_n \) is semi-positive definite for imaginary values of \( \alpha_m \) and semi-negative definite for real values of \( \alpha_m \).

Let \( \varphi_{k,m,r,n} = -\epsilon_{k,m,p} \mu_{p,j} \epsilon_{j,m,r} \), then from the previous conclusions it is obvious that \( \varphi_{k,m,r,n} \alpha_m \alpha_n \) is semi-positive definite for real values of \( \alpha_m \). Next we write

\[
S_{k,r} = \varphi_{k,m,r,n} \alpha_m \alpha_n + \sigma_{k,r},
\]
(A.9)

where \( S_{k,r} \) is positive definite for real \( \alpha_m \). The determinantal equation (A.7) implies that the system of equations

\[
S_{k,r} \alpha_r = 0,
\]
(A.10)

has a non-trivial solution. Since \( S_{k,r} \) is positive definite for arbitrarily real \( \alpha_m \), it immediately follows that Eq. (A.10) never can be satisfied when all three \( \alpha_m \) have real values. Thus, if \( \alpha_1 \) and \( \alpha_2 \) are real, \( \alpha_3 \) must be complex with a non-vanishing imaginary part, i.e. \( \gamma = i \alpha_3 \) must be complex with a non-vanishing real part.

For real values of \( \alpha_1 \) and \( \alpha_2 \), \( S_{k,r} \) satisfies Schwarz's reflection principle with respect to \( \alpha_3 \), i.e.,

\[
S_{k,r}(\alpha_1, \alpha_2, \alpha_3^*) = S_{k,r}^*(\alpha_1, \alpha_2, \alpha_3) \quad \text{for} \quad \alpha_1 \in \mathbb{R}, \ \alpha_2 \in \mathbb{R}.
\]
(A.11)

Consequently, for real \( \alpha_m \), the roots \( \alpha_3 \) of Eq. (A.7) occur in complex conjugate pairs and thus, for real \( \alpha_m \) the \( \gamma \) occur in opposite complex conjugate pairs with a non-vanishing real part. Further, from Eq. (A.9) it follows that \( S_{k,r}(-\alpha_m) = S_{k,r}(\alpha_m) \).
In particular, for $\alpha_1 = \alpha_2 = 0$ we obtain

$$S_{k,r}(0,0,\alpha_3) = S_{k,r}(0,0,-\alpha_3).$$  \hfill (A.12)

This implies that for $\alpha_1 = \alpha_2 = 0$ the roots $\alpha_3$ of Eq. (A.7) are two pairs of imaginary, complex conjugate values and thus the $\gamma$ are two pairs of real and opposite values. We denote the four branches of $\gamma$ as $\gamma^{(\pm r)}(\alpha_\nu)$ such that

$$\gamma^{(+r)}(0,0) = -\gamma^{(-r)}(0,0) \quad \text{are real and positive.} \hfill (A.13)$$

Further, we shall assume that $\gamma^{(+1)}(0,0) \geq \gamma^{(+2)}(0,0)$. Since the branches $\gamma$ of Eq. (A.7) depend continuously on $\alpha_1$ and $\alpha_2$, while the real part of $\gamma$ never can vanish we conclude that for real $\alpha_\nu$ the two branches $\gamma^{(+r)}$ of $\gamma$ must have a positive real part while the other two branches $\gamma^{(-r)}$ must have a negative real part, i.e.,

$$\gamma^{(+r)}(\alpha_\nu) = -\gamma^{(-r)*}(\alpha_\nu) \quad \text{for} \quad \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}, \hfill (A.14)$$

where

$$\text{Re}(\gamma^{(+r)}(\alpha_\nu)) > 0 \quad \text{for} \quad \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}, \hfill (A.15)$$

$$\text{Re}(\gamma^{(-r)}(\alpha_\nu)) < 0 \quad \text{for} \quad \alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}. \hfill (A.16)$$

These properties have been used in Section 3.2.

Behavior of $\gamma$ for complex values of $\alpha_1$ and $\alpha_2$

Next, we investigate the behavior of $\gamma$ for complex values of $\alpha_1$ and $\alpha_2$. To do so, we shall replace $i\alpha_1$ and $i\alpha_2$ by a polar coordinate representation in terms of $p$ and $\psi$ as defined by

$$i\alpha_1 = p \cos(\psi + \theta), \hfill (A.17)$$
$$i\alpha_2 = p \sin(\psi + \theta). \hfill (A.18)$$

with $0 \leq \psi < 2\pi$; here, $\theta$ is introduced for later convenience. From Eqs. (A.17)-(A.18) we observe that real values of $\alpha_1$ and $\alpha_2$ correspond to imaginary values of $p$. 

Hence, the properties of \( \gamma \) for real values of \( \alpha_r \) as derived in the previous subsection correspond to properties of \( \gamma \) along the imaginary \( p \)-axis in the complex \( p \)-plane. In order to investigate the properties of \( \gamma \) for values of \( p \) away from the imaginary \( p \)-axis we introduce the quantities \( \tilde{\alpha}_m \) as given by

\[
\tilde{\alpha}_m = \alpha_m / p \quad \text{for } p \neq 0
\]

i.e., \( i\tilde{\alpha}_1 = \cos(\psi + \theta) \), \( i\tilde{\alpha}_2 = \sin(\psi + \theta) \), and \( i\tilde{\alpha}_3 = \gamma / p \). Substitution of Eq. (A.19) into Eq. (A.7) yields

\[
\det (\varphi_{k,m,r,n} \tilde{\alpha}_m \tilde{\alpha}_n + \sigma_{k,r} / p^2) = 0, \quad \text{for } p \neq 0.
\]

For any solution \( \tilde{\alpha}_m \) of this equation there exists a non-trivial solution \( a_r \) of the system of equations

\[
-\varphi_{k,m,r,n} \tilde{\alpha}_m \tilde{\alpha}_n a_r = \sigma_{k,r} a_r / p^2.
\]

For such a solution, we consider the identity

\[
-\varphi_{k,m,r,n}^{*} \tilde{\alpha}_m \tilde{\alpha}_n a_r = \sigma_{k,r} a_r / p^2.
\]

The factor multiplying \( 1/p^2 \) at the right-hand side of Eq. (A.22) is always real and positive. If now \( \tilde{\alpha}_3 \) is imaginary, then the left-hand side of Eq. (A.22) is also real and positive. In this case, \( 1/p^2 \) must be real and positive. Consequently, \( \text{Im}(i\tilde{\alpha}_3) = \text{Im}(\gamma / p) \) can only be equal to zero along the real \( p \)-axis and hence, \( \text{Im}(\gamma / p) \) is of a definite sign for \( 0 < \text{Im}(p) < \infty \) and \( \text{Im}(\gamma / p) \) is of a definite sign for \( -\infty < \text{Im}(p) < 0 \). Since for any imaginary \( p \) we have \( \text{Re}(\gamma^{(+)}) > 0 \) and \( \text{Re}(\gamma^{(-)}) < 0 \), it follows that for any complex \( p \)

\[
\text{Im}(\gamma^{(+)}/p) < 0 \quad \text{and} \quad \text{Im}(\gamma^{(-)}/p) > 0 \quad \text{when } \text{Im}(p) > 0,
\]

\[
\text{Im}(\gamma^{(+)}/p) > 0 \quad \text{and} \quad \text{Im}(\gamma^{(-)}/p) < 0 \quad \text{when } \text{Im}(p) < 0.
\]

Finally, we investigate the behavior of \( \gamma \) as \( |p| \rightarrow \infty \). From Eqs. (A.5) and Eq. (A.7)
we find that $\gamma$ is proportional to $p$ as $|p| \to \infty$. Now, since for imaginary $p$, the branches $\gamma^{(\pm \tau)}$ of $\gamma$ must have a non-vanishing real part, it is easily verified that for real $p$ and $|p| \to \infty$ the four branches of $\gamma$ must have a non-vanishing imaginary part and, in particular, do occur in pairs of complex conjugate values.

With these results we are able to partition the four values of $\gamma$ for any complex $p$ unambiguously into the two branches $\gamma^{(\tau)}$ and the two branches $\gamma^{(-\tau)}$, as long as we do not cross any of the possible singularities of $\gamma$. Since the $\gamma$ are roots of an algebraic equation (i.e., a fourth-degree polynomial in $\gamma$ with coefficients which are itself polynomials of $p$), it follows that the only singularities of the $\gamma$ that can occur are branch points in the complex $p$-plane. At a branch point, two branches of $\gamma$ attain the same value while the derivative of $\gamma$ with respect to $p$ is infinite.

In order to keep the branches single-valued throughout the complex $p$-plane, branch cuts must be introduced. In order to keep the four branches $\tilde{\gamma}^{(\pm \tau)}$ single-valued throughout the complex $p$-plane, we introduce branch cuts, either as straight line segments along the real $p$-axis joining two corresponding branch points, or as straight lines from the relevant branch points to infinity. In this way we achieve that each of the four branches is analytic and single-valued in the entire complex $p$-plane with the exception of the branch points and the branch cuts connected with them.

A closer examination of the determinantal equation (A.5) reveals that there are at most twelve branch points. These branch points occur in pairs of complex conjugate values and/or pairs of real values. The location of the branch points is symmetric with respect to the origin of the complex $p$ plane. Hence, there are at most six branch points in the right-half of the complex $p$-plane and the same number of branch points in the left-half of the complex $p$-plane. It should be noted that both the number and the location of the branch points in the complex $p$-plane depend on the value of $\psi$. Further, it is possible that two branches of $\gamma$ attain the same value for a certain (complex) value of $p$ even though this is not a singular point of the respective branches (i.e., $\partial \gamma / \partial p$ is finite and a continuous function of $p$ in a neighborhood of such a point). Later on, we present some numerical examples showing the location of branch points and the behavior of the $\tilde{\gamma}^{(\pm \tau)}$ along the real $p$-axis.

We shall denote the branch points located on the real $p$-axis by $p = \pm \pi \xi$. Branch points located off the real $p$-axis are denoted by $p = \pm \pi \xi^*$ and $p = \pm \pi \xi^*$. Since $\text{Im}(\gamma^{(\tau)} / p)$ and $\text{Im}(\gamma^{(-\tau)} / p)$ differ in sign unless $p$ is real (cf. Eqs. (A.23)-(A.24)), the only place where branch points corresponding to branches with labels of oppo-
site signs can be located is on the real p-axis. In the same way we find that branch points located off the real p-axis correspond to branches of γ having labels of the same sign.

Branch points \( \pi_k \) located on the real p-axis.

For real values of \( p \), Eq. (A.7) is a fourth degree polynomial equation in \( \gamma \) with real coefficients. This implies that along the real p-axis the four roots of this equation occur in pairs of complex conjugate values and/or pairs of real values. From Eq. (A.13) we know that for \( p = 0 \) all four roots have real values. Hence, by continuity, in a neighborhood of the origin of the complex p-plane along the real p-axis all four branches are real. At a branch point \( p = \pi_k \) on the real p-axis a particular pair of the branches \( \{ \gamma^{(-1)}, \gamma^{(+1)} \}, \{ \gamma^{(-2)}, \gamma^{(+2)} \}, \{ \gamma^{(-2)}, \gamma^{(+1)} \} \) or \( \{ \gamma^{(-1)}, \gamma^{(+2)} \} \) meet and become a pair of complex conjugate values. Occasionally, such a pair of complex conjugate values of two branches can become a pair of real values again at some other branch point on the real p-axis. Whether this happens or not depends on \( \sigma_{kr}, \mu_{kr} \) and the value of \( \psi \).

In order to keep the four branches \( \gamma^{(\pm r)} \) single-valued we introduce branch cuts, either as straight line segments along the real p-axis joining two branch points corresponding to one and the same pair of branches, or as straight lines from the relevant branch point to infinity. Note that if \( p = \pi_k \) is a branch point of the branch \( \gamma^{(+r)} \), then \( p = -\pi_k \) is a branch point of the branch \( \gamma^{(-r)} \). The question of how to identify the different branches in the different intervals on the real p-axis is addressed by performing an analytic continuation of the four branches close to the real p-axis. In this procedure we make use of the fact that since the \( \gamma^{(\pm r)} \) are analytic functions of \( p \) close to real p-axis, they satisfy the Cauchy-Riemann differential equations

\[
\frac{\partial \text{Re}(\gamma^{(\pm r)})}{\partial \text{Re}(p)} = \frac{\partial \text{Im}(\gamma^{(\pm r)})}{\partial \text{Im}(p)} \quad \text{and} \quad \frac{\partial \text{Re}(\gamma^{(\pm r)})}{\partial \text{Im}(p)} = -\frac{\partial \text{Im}(\gamma^{(\pm r)})}{\partial \text{Re}(p)}. \quad (A.25)
\]

These differential equations define the proper choice of the \( \gamma^{(\pm r)} \) along the real p-axis. Figures A.1-A.3 show examples of (normalized) real and imaginary parts of the \( \gamma^{(\pm r)} \) as a function of real \( p \) for various kinds of anisotropy and different values of \( \psi \). In wave problems, the real part of \( \gamma^{(\pm r)} \) as a function of real \( p \) is usually denoted as the slowness surface (Van der Hijing, 1988) because this function represents the propagation characteristics of plane-wave solutions of the relevant wave-field equations. The examples of figures A.1-A.4 show all possible classes of diffusive
slowness surfaces that can result from the diffusive electromagnetic field equations.

**Branch points \( \lambda_k, \lambda_k^* \) located off the real \( p \)-axis**

Since \( \text{Im}(\gamma^{(+r)}/p) \) and \( \text{Im}(\gamma^{(-r)}/p) \) have different signs for all \( p \) off the real \( p \)-axis, two branches that meet at a branch point \( \lambda_k \) off the real \( p \)-axis must have a superscript with the same sign, i.e., either \( \{\gamma^{(1)}, \gamma^{(2)}\} \) or \( \{\gamma^{(-1)}, \gamma^{(-2)}\} \). These branch points occur in complex conjugate pairs. Further, if \( p = \lambda_k \) is a branch point of \( \{\gamma^{(1)}, \gamma^{(2)}\} \), then \( p = -\lambda_k \) is a branch point of \( \{\gamma^{(-1)}, \gamma^{(-2)}\} \). Consequently, there are zero, four or eight of these off-axis branch points. The branch cuts emanating from these branch points are chosen as straight lines running from the relevant branch point to infinity in a way such that none of the branch cuts intersects.

It is important to note that if, for example, \( p = \lambda_k \) is a branch point of the branches \( \{\gamma^{(1)}, \gamma^{(2)}\} \), then the branch cut emanating from the branch point \( p = \lambda_k \) interconnects these two branches and if, in the process of analytic continuation of the branches we happen to cross this branch cut, then the analytic continuation of \( \gamma^{(1)} \) at one side of the branch cut is \( \gamma^{(2)} \) at the other side of the branch cut.
Figure A.1: Examples of diffusive slowness surfaces for various kinds of anisotropy and values of $\psi$. Real and imaginary parts of $\gamma^{(\pm)}$ as a function of real $p$ are represented by thick and thin lines respectively. These examples correspond to anisotropic media, two of the principal directions of the $\sigma_{k,r}$ and $\mu_{j,p}$ tensors of which are in the horizontal plane. This implies that for real $p$ the $\gamma^{(\pm)}$ occur not only in real and/or complex conjugate pairs but also in pairs of opposite values. The first example corresponds to an isotropic medium where the two diffusive slowness surfaces are circles and do coincide.
Figure A.2: Examples of diffusive slowness surfaces for various kinds of anisotropy and values of \( \psi \). Real and imaginary parts of \( \gamma^{( \pm \tau )} \) as a function of real \( p \) are represented by thick and thin lines respectively. These examples correspond to anisotropic media, two of the principal directions of the \( \sigma_{k,r} \) and \( \mu_{j,p} \) tensors of which are in the horizontal plane. This implies that for real \( p \) the \( \gamma^{( \pm \tau )} \) occur not only in real and/or complex conjugate pairs but also in pairs of opposite values.
Figure A.3: Examples of diffusive slowness surfaces for various kinds of anisotropy and values of $\psi$. Real and imaginary parts of $\gamma^{(\pm \pi)}$ as a function of real $p$ are represented by thick and thin lines respectively. Only the first two examples correspond to anisotropic media, two of the principal directions of the $\sigma_{k,p}$ and $\mu_{j,p}$ tensors of which are in the horizontal plane. The last example corresponds to an arbitrarily anisotropic medium.
Figure A.4: Examples of diffusive slowness surfaces for various kinds of anisotropy and values of $\psi$. Real and imaginary parts of $\gamma^{(\pm \tau)}$ as a function of real $p$ are represented by thick and thin lines respectively. These examples correspond to arbitrarily anisotropic media. The last example shows the maximum number of four branch points along the positive real $p$-axis.
Explicit expressions for the eigenvalues of the system's matrix

In the previous subsections of this appendix we have investigated the basic properties of the eigenvalues $\gamma^{(\pm \nu)}$ of the system's matrix $A_{I,J}$ for real and complex values of the Fourier transformation parameters $\alpha_1$ and $\alpha_2$ (i.e., in terms of $p$ and $\psi$; Eqs. (A.17)-(A.18)). With the results thus obtained we are able to partition a given set of eigenvalues into the four $\gamma^{(\pm \nu)}(p, \psi)$ (cf. Eqs. (A.23)-(A.24)).

Yet, this does not yield a solution for a numerically accurate and efficient way of calculating the eigenvalues of $A_{I,J}$. In view of the analytic continuation of the $\gamma^{(\pm \nu)}$ and the numerical computation of the modified Cagniard contour near (off-axis) branch points, it is very important to have high-accuracy values for the $\gamma^{(\pm \nu)}$ available in those regions of the complex $p$-plane. The same holds if the modified Cagniard contour returns (almost) to the real $p$-axis again. For this purpose, standard numerical routines for solving the eigenvalue equation (A.4) (most of which are based on the so-called LR-algorithm, see Wilkinson and Reinsch, 1971) do not provide sufficient accuracy. Another approach would be solving numerically the determinantal equation (A.5). Here once again, standard numerical routines (based on a simultaneous use of Newton's method and a steepest descent method, see Grant and Hitchins, 1971) provide not enough accuracy in those regions.

Fortunately, there exists a solution to overcome this problem. The determinantal equation (A.5) is a fourth degree polynomial equation in $\gamma$ for which it is possible to find explicit expressions for its roots, the $\gamma^{(\pm \nu)}$. The general solution of quartic equations was "discovered" by Ferrari in 1541 and is now known as Cardano's formula (Cardanus, 1545; Cajori, 1985). It remained until 1826 when Abel proved that the maximum order of polynomial equations for which explicit solutions can be obtained is just four. Since the derivation of these explicit expressions is by no means trivial, we shall give an outline of how these roots are obtained.

From the determinantal equation (A.5) it is easily verified that the $\tilde{\gamma} = \tilde{\gamma}^{(\pm \nu)}$ are the roots of a fourth degree polynomial equation that is written as

$$\tilde{\gamma}^4 + 2 f_1 \tilde{\gamma}^3 + f_2 \tilde{\gamma}^2 + f_3 \tilde{\gamma} + f_4 = 0. \quad (A.26)$$

Here, we already made use of the substitutions (A.17) and (A.18). Obviously, the coefficients $f_n$ are functions of $p$ and $\psi$, i.e., $f_n = f_n(p, \psi)$. Further, it can easily be verified that $f_1(p, \psi)$ and $f_3(p, \psi)$ are first and third degree polynomials of $p$, respectively, containing only odd powers of $p$; $f_2(p, \psi)$ and $f_4(p, \psi)$ are second and
fourth degree polynomials of \( p \), respectively, containing only even powers of \( p \). Note that \( f_4(p, \psi) \) is equal to the determinant of \( A_{t, t} \). Explicit expressions for the \( f_n \) in terms of the \( \sigma_{k, \tau}, \mu_{j, q} \) and \( p \) and \( \psi \) are obtained by expanding Eq. (A.5). The roots of Eq. (A.26) can be found by rewriting this fourth degree polynomial as the product of two second degree polynomials in \( \tilde{\gamma} \), i.e.,

\[
\tilde{\gamma}^4 + 2 f_1 \tilde{\gamma}^3 + f_2 \tilde{\gamma}^2 + f_3 \tilde{\gamma} + f_4 = (\tilde{\gamma}^2 + g_1 \tilde{\gamma} + g_3/2)(\tilde{\gamma}^2 + g_2 \tilde{\gamma} + g_4/2). \tag{A.27}
\]

The factors 1/2 on the right-hand side of Eq. (A.27) have been included for later convenience. Now, the roots of Eq. (A.26) are simply the zero's of the two second degree polynomial equations on the right-hand side of Eq. (A.27). By expanding the right-hand side of Eq. (A.27) and equating the coefficients that correspond to equal powers of \( \tilde{\gamma} \) we obtain

\[
g_1 = f_1 + \rho, \tag{A.28}
\]
\[
g_2 = f_1 - \rho, \tag{A.29}
\]
\[
g_3 = f_2 - f_1^2 + \rho^2 - \left( f_3 - f_1 f_2 + f_1^3 - f_1 \rho^2 \right)/\rho, \tag{A.30}
\]
\[
g_4 = f_2 - f_1^2 + \rho^2 + \left( f_3 - f_1 f_2 + f_1^3 - f_1 \rho^2 \right)/\rho, \tag{A.31}
\]

where \( \rho^2 = z \) is a non-zero root (if any) of the third degree polynomial equation

\[
x^3 + ax^2 + bx + c = 0, \tag{A.32}
\]

in which the coefficients \( a, b \) and \( c \) are given by

\[
a = 2 f_2 - 3 f_1^2, \tag{A.33}
\]
\[
b = \left( f_2 - f_1^2 \right)^2 - 2 f_1 \left( f_1 (f_2 - f_1^2) - f_3 \right) - 4 f_4, \tag{A.34}
\]
\[
c = - \left( f_1 (f_2 - f_1^2) - f_3 \right)^2. \tag{A.35}
\]
Note that the six possible values of $\rho$ that follow from Eq. (A.32) correspond to the six different ways in which the fourth degree polynomial on the left-hand side of Eq. (A.27) can be written as the product of two second degree polynomials. Hence, the different roots from Eq. (A.32) all lead to one and the same set of eigenvalues $\tilde{\gamma}^{(\pm \nu)}$. Further, from Eqs. (A.33)-(A.35) we find that $a, b$ and $c$ are second, fourth and sixth degree polynomials of $p$, respectively, containing only even powers of $p$. Let the $z_k$, $k = 1, 2, 3$ denote the three roots of Eq. (A.32). From Eq. (A.32) we have

$$z_1 = -\frac{a}{3} + \frac{2}{3}\sqrt{d^2 + e^3} - d - e/\sqrt{d^2 + e^3} - d,$$  \hspace{1cm} (A.36)

$$z_2 = -\frac{a}{3} + \exp(i2\pi/3) \frac{2}{3}\sqrt{d^2 + e^3} - d + \exp(i\pi/3)e/\sqrt{d^2 + e^3} - d,$$  \hspace{1cm} (A.37)

$$z_3 = -\frac{a}{3} - \exp(i\pi/3) \frac{2}{3}\sqrt{d^2 + e^3} - d - \exp(i2\pi/3)e/\sqrt{d^2 + e^3} - d,$$  \hspace{1cm} (A.38)

where

$$d = (c - a b/3 + 2(a/3)^3)/2,$$  \hspace{1cm} (A.39)

$$e = (b - a^2/3)/3.$$  \hspace{1cm} (A.40)

Each of the $z_k$ will lead to the same set of eigenvalues $\tilde{\gamma}^{(\pm \nu)}$. If $c = 0$, at least one of the roots of Eq. (A.32) is equal to zero and hence, we should use one of the other two roots $z_k$ to compute the $g_n$. In the case where $c = 0$, i.e., $f_3 = f_1 (f_2 - f_4^*)$, Eqs. (A.28)-(A.31) reduce to

$$g_1 = g_2 = f_1,$$  \hspace{1cm} (A.41)

$$g_3 = f_2 - f_1^* + \sqrt{(f_2 - f_1^*)^2 - 4f_4},$$  \hspace{1cm} (A.42)

$$g_4 = f_2 - f_1^* - \sqrt{(f_2 - f_1^*)^2 - 4f_4}.$$  \hspace{1cm} (A.43)

Finally, from Eq. (A.27) the four roots $\tilde{\gamma}_k$ of Eq. (A.26) are obtained as

$$\tilde{\gamma}_1 = (-g_1 - \sqrt{g_1^2 - 2g_3})/2,$$  \hspace{1cm} (A.44)
\begin{align*}
\hat{\eta}_2 &= \frac{-g_1 + \sqrt{g_1^2 - 2g_3}}{2}, \quad (A.45) \\
\hat{\eta}_3 &= \frac{-g_2 - \sqrt{g_2^2 - 2g_4}}{2}, \quad (A.46) \\
\hat{\eta}_4 &= \frac{-g_2 + \sqrt{g_2^2 - 2g_4}}{2}. \quad (A.47)
\end{align*}

In Eqs. (A.44)-(A.47) we have denoted the roots of Eq. (A.26) as \( \hat{\eta}_k \) instead of \( \tilde{\eta}_{(\pm \nu)} \) since the appropriate partition of the roots in accordance with Eqs. (A.23)-(A.24) can not be determined a priori from these expressions, but follows from an analytic continuation away from the real \( p \)-axis.
Appendix B

Transient diffusive electromagnetic fields at the source level

In this appendix we consider the evaluation of $G(\mathbf{z}, t)$ in the case where source and receiver are located on one and the same level. An important example is the one where source and receiver are both located at the interface between two adjacent half-spaces, the upper half-space representing the air, the lower half-space representing a homogeneous anisotropic earth.

The expression (4.90) is the starting point for the computation of the contributions from the different generalized diffusive field constituents to the total space-time domain Green's function. However, for the contributions to $G(\mathbf{z}, t)$ from the generalized diffusive constituents representing the direct field (see Section 4.2) for which the total vertical path length is equal to the distance $h$ between source and receiver level, we cannot simply set $h \equiv 0$. For $h \equiv 0$, not only does Jordan's lemma not apply in our process of contour deformation with the modified Cagniard technique but also, for $\psi = \pm \pi/2$ the integral with respect to $\tau$ of Eq. (4.90) is divergent. Yet, the behavior with respect to $\psi$ of the $\tau$-integral of Eq. (4.90) must be such that as $h \to 0$ we end up with a finite result for $G(\mathbf{z}, t)$.

We describe a procedure to evaluate $G(\mathbf{z}, t)$ when $h \equiv 0$ that avoids divergent integrals. From Eq. (4.72) we have the following expression for the Laplace-domain
Green's function $\hat{G}(s, s)$ corresponding to the direct field

$$\hat{G}(s, s) = -\frac{\hat{k}(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \, \text{Re} \left( \int_0^{i\infty} \exp(-s^{1/2}(p r \cos \psi + h \tilde{\gamma})) p \tilde{B}(p, \psi) dp \right).$$

(B.1)

In this equation $h$ denotes the vertical distance between source and receiver level. In agreement with Eqs. (4.49)-(4.50), the appropriate choice for $\tilde{\gamma}$ follows from the relative position of the receiver with respect to the source, i.e. when the source is located at the interface between the subdomains $D_s$ and $D_{s+1}$ we take $\tilde{\gamma} = \tilde{\gamma}^{(s+1; -\nu)}(p, \psi)$ if the receiver is located above the source, while we take $\tilde{\gamma} = \tilde{\gamma}^{(s+1; +\nu)}(p, \psi)$ if the receiver is located below the source. There are two generalized diffusive constituents that contribute to the direct field ($\nu = 1$ and $\nu = 2$).

The procedure we shall follow consists of subtracting from the integrand of $E_\tau (B.1)$ its asymptotic part as $|p| \to \infty$ and treat this asymptotic part separately. Here, we shall only consider the magnetic field strength at the source level. The same procedure can be used to compute the electric field strength at the source level.

**Magnetic field strength at the source level**

From Eqs. (3.111), (3.117) and (3.144) we have

$$\tilde{B}(p, \psi) = a(\psi) + \mathcal{O}(p^{-2}) \text{ as } |p| \to \infty,$$

(B.2)

$$\tilde{\gamma}(p, \psi) = -i p C(\psi) + \mathcal{O}(p^{-1}) \text{ as } |p| \to \infty,$$

(B.3)

where $a(\psi)$ and $C(\psi)$ satisfy the relation

$$a(\psi + \pi) = a^*(\psi),$$

(B.4)

$$C(\psi + \pi) = C^*(\psi).$$

(B.5)

As illustration of the asymptotic behavior of $\tilde{B}(p, \psi)$, we refer to the explicit expressions for isotropic media, Eqs. (6.55)-(6.57) and (6.122). Upon using Eqs. (B.2) and (B.3), we rewrite the integral with respect to $p$ on the right-hand side of Eq. (B.1)
as

\[
\int_0^{\infty} p B \exp(-s^{1/2}(pr\cos\psi + h\bar{\gamma})) \, dp = \\
\int_0^{\infty} p \left[ B \exp(-s^{1/2}h\gamma) - a(\psi) \exp(+s^{1/2}i\, hC(\psi)) \right] \exp(-s^{1/2}pr\cos\psi) \, dp \\
+ a(\psi) \int_0^{\infty} p \exp\left(-s^{1/2}(pr\cos\psi - i\, hC(\psi))\right) \, dp.
\]  \hspace{1cm} (B.6)

In our further analysis \( h \) is assumed to be different from zero, although very small compared to the radial distance \( r \) between source and receiver. Subsequently, we shall consider the resulting expressions for \( \hat{G}(x, s) \) and \( G(x, t) \) as \( h \to 0 \).

Now, for any \( h \neq 0 \) the last integral of Eq. (B.4) can be evaluated analytically, viz.

\[
\int_0^{\infty} p \exp\left(-s^{1/2}(pr\cos\psi - i\, hC(\psi))\right) \, dp = -\frac{1}{s(hC(\psi) + i\, r\cos\psi)^2},
\]  \hspace{1cm} (B.7)

and hence

\[
\hat{G}(x, s) = -\frac{k(s)}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\psi \text{Re} \left[ \int_0^{\infty} p \left( B \exp(-s^{1/2}(pr\cos\psi + h\bar{\gamma})) \\
-a(\psi) \exp(-s^{1/2}(pr\cos\psi - i\, hC(\psi))) \right) \, dp \right] \\
+ \frac{k(s)}{2\pi^2 s} \int_{-\pi/2}^{\pi/2} \text{Re} \left[ \frac{a(\psi)}{(hC(\psi) + i\, r\cos\psi)^2} \right] \, d\psi.
\]  \hspace{1cm} (B.8)

As we shall show later on, the last integral at the right-hand side of Eq. (B.8) is finite for all \( h \neq 0 \) and approaches a finite value as \( h \to 0 \).

To the integral with respect to \( p \) along the imaginary \( p \)-axis in the right-hand side of Eq. (B.8) we shall apply a deformation of the integration contour following the modified Cagniard technique (see Sections 3.4 and 4.3). For any \( \psi \neq \pm\pi/2 \) and upon \( h \to 0 \), \( h \neq 0 \), this modified Cagniard contour in the complex \( p \)-plane will be
located along but just above the real p-axis, viz.

$$\int_0^{\infty} p \left[ \tilde{B} \exp(-s^{1/2}(p r \cos \psi + h\gamma)) - a(\psi) \exp(-s^{1/2}(p r \cos \psi - i h C(\psi))) \right] dp$$

$$= \int_0^{\infty + i \pi} p \left( \tilde{B} - a(\psi) \right) \exp(-s^{1/2} p r \cos \psi) dp + O(h) \quad \text{as} \ h \to 0 \ \text{and} \ \psi \neq \pm \pi/2. \quad (B.9)$$

The modified Cagniard contour is parametrized by the real variable $\tau$ according to

$$p r \cos \psi = \tau, \quad \tau \in \mathbb{R}, \quad \tau \geq 0, \ \psi \neq \pm \pi/2. \quad (B.10)$$

Upon introducing $\tau$ as the variable of integration in Eq. (B.9), we obtain

$$\int_0^{\infty + i \pi} p \left( \tilde{B}(p, \psi) - a(\psi) \right) \exp(-s^{1/2} p r \cos \psi) dp$$

$$= \frac{1}{r^2 \cos^2 \psi} \int_0^{\infty} \left( \tilde{B}(\frac{\tau}{r \cos \psi}, \psi) - a(\psi) \right) \tau \exp(-s^{1/2} \tau) d\tau \quad (B.11)$$

Since $\tilde{B}(p, \psi) - a(\psi) = O(|p|^{-2})$ as $|p| \to \infty$ we are led to the conclusion that the integral on the right-hand side of Eq. (B.11) is finite for all $\psi$ in the interval $-\pi/2 < \psi < \pi/2$ and approaches a finite value as $\psi \to \pm \pi/2$.

The function $\tilde{k}(s)$ for the magnetic field strength is given as $\tilde{k}(s) = 1$ (cf. Eq. (3.85)). With Eq. (B.11) we have now achieved that $G(\varpi, t)$ can be found by inspection. The time function corresponding to $\tilde{k}(s) \exp(-s^{1/2} \tau)$ is denoted as $k^{(HJ)}(t, \tau)$ and is given in Eq. (3.124). Consequently, the space-time domain expression for $G(\varpi, t)$ corresponding to the magnetic field strength as $h \to 0$ is obtained as:

$$G(\varpi, t) = -\frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} \frac{d\psi}{\tau^2 \cos^2 \psi} \int_0^{\infty} \tau \text{Re} \left[ \tilde{B}(p, \psi) - a(\psi) \right] k^{(HJ)}(t, \tau) d\tau$$

$$+ \frac{1}{2\pi^2} \int_{-\pi/2}^{\pi/2} \text{Re} \left[ \frac{a(\psi)}{h C(\psi) + i \tau \cos \psi} \right] d\psi \ H(t). \quad (B.12)$$
The smoothness with respect to $\psi$ of the integrand of the first $\psi$-integral on the right-hand side of Eq. (B.12) admits a numerically simple evaluation of its contribution to the Green's function $G(\omega, t)$.

The second integral on the right-hand side of Eq. (B.10) still shows singular behavior near $\psi = \pm \pi/2$ as $\hbar \to 0$. The procedure we follow for evaluating this integral is rewriting the integrand in a way such that the expressions obtained can be evaluated either analytically or numerically without any difficulty as $\hbar = 0$. In view of Eq. (B.4) we have $\Re[a(\psi + \pi)] = \Re[a(\psi)]$ and $\Im[a(\psi + \pi)] = -\Im[a(\psi)]$. Hence, we rewrite the second integral on the right-hand side of Eq. (B.12) as:

\[
\int_{-\pi/2}^{\pi/2} \Re \left[ \frac{a(\psi)}{(hC(\psi) + i r \cos \psi)^2} \right] d\psi =
\]

\[
\int_{-\pi/2}^{\pi/2} \Re \left[ \frac{\Re[a(\psi) - a(\pi/2) + \dot{a}(\pi/2) \sin(2\psi)/2]}{(hC(\psi) + i r \cos \psi)^2} \right] d\psi
\]

\[
+ \int_{-\pi/2}^{\pi/2} \Re \left[ \frac{i \Im[a(\psi)]}{(hC(\psi) + i r \cos \psi)^2} \right] d\psi
\]

\[
+ \Re[a(\pi/2)] \int_{-\pi/2}^{\pi/2} \Re \left[ \frac{1}{(hC(\psi) + i r \cos \psi)^2} \right] d\psi
\]

\[
- \Re[\dot{a}(\pi/2)] \int_{-\pi/2}^{\pi/2} \Re \left[ \frac{\sin(2\psi)/2}{(hC(\psi) + i r \cos \psi)^2} \right] d\psi. \tag{B.13}
\]

Here, $\dot{a}(\psi)$ denotes $da(\psi)/d\psi$. The integrand of the first integral on the right-hand side of Eq. (B.13) is a smooth function of $\psi$ near $\psi = \pm \pi/2$ and hence, we can set $\hbar = 0$ in the denominator of this integrand. Further, it can be verified that the last two integrals on the right-hand side of Eq. (B.13) are $O(h)$ as $\hbar \to 0$. Finally, the integrand of the second integral on the right-hand side of Eq. (B.11) gives only rise to a contribution for $\psi$ near $\pm \pi/2$. By replacing $\Im[a(\psi)]$ by a local Taylor expansion around $\psi = \pm \pi/2$ we obtain

\[
\lim_{\hbar \to 0} \int_{-\pi/2}^{\pi/2} \Re \left[ \frac{i \Im[a(\psi)]}{(hC(\psi) + i r \cos \psi)^2} \right] d\psi = \frac{\pi \Im[\dot{a}(\pi/2)]}{r^2}. \tag{B.14}
\]
Collecting the results, we end up with

\[
\lim_{\lambda \to 0} \int_{-\pi/2}^{\pi/2} \frac{a(\psi)}{\left(\lambda C(\psi) + i r \cos \psi\right)^2} d\psi = \frac{\pi \text{Im}[\dot{a}(\pi/2)]}{r^2}
\]

\[
- \int_{-\pi/2}^{\pi/2} \frac{\text{Re}[a(\psi) - a(\pi/2) + \dot{a}(\pi/2) \sin(2\psi)/2]}{r^2 \cos^2 \psi} d\psi.
\]

(\text{B.15})

Although \(a(\psi)\) and \(\dot{a}(\psi)\) are not explicitly available and have, in general, to be evaluated numerically, we know that they are smooth functions of \(\psi\) causing no numerical difficulties for the final evaluation of Eq. (B.15).


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Samenvatting

Het onderzoek dat in dit proefschrift wordt gepresenteerd, heeft een nieuwe methode ter berekening van transiënte elektromagnetische velden in gelaagde anisotrope media tot onderwerp. Dergelijke velden zijn van belang voor de exploratiegeofysica, d.w.z. bij de bepaling van de structuur van de aardbodem en de opsporing van de plaats waar minerale afzettingen, aardgas en aardolievelden zich bevinden. Recent toegepast zijn: onderzoek naar het voorkomen van grondwater in droge gebieden en het lokaliseren van historisch interessante metalen voorwerpen.

Het transiënte elektromagnetische veld wordt gegenereerd door een pulserende bron, bijvoorbeeld een op het aardoppervlak gelegen draadlus of spoel waarin een elektrische stroom herhaald wordt in- en uitgeschakeld. Voor het verkrijgen van meetgevens worden magnetometers of inductiespoelen als ontvangers gebruikt. De elektromagnetische eigenschappen van de aarde worden gekarakteriseerd door de tensoriële elektrische soortelijke geleiding en de tensoriële magnetische permeabiliteit. De anisotropie van de aarde komt in deze materiaaltensoren tot uitdrukking; zij is veelal het gevolg van de aanwezigheid van met water gevulde microscopisch kleine, onderling evenwijdige scheuren in sedimentaire aardlagen. Kennis omtrent de propagatie-eigenschappen van de opgewekte elektromagnetische velden in (quasi-)gelaagde anisotrope media maakt het mogelijk om op basis van uit veldonderzoek verkregen meetresultaten een beeld te vormen van de (lokale) elektrische eigenschappen van de aardbodem.

De tijdschaal waarop de responsie van de aarde ten gevolge van een gepulste excitatie wordt gemeten, is in het algemeen zodanig, dat in dat tijdinterval de diëlektrische verschuivingsstroom verwaarloosd kan worden ten opzichte van de elek-

Het doel van het in dit proefschrift beschreven onderzoek is het ontwikkelen van een analytische methode waarmee het gedrag van deze transiënte elektromagnetische diffusievelden in een gelaagd, willekeurig anisotroop medium bestudeerd kan worden. De praktische toepasbaarheid van de in deze theoretische studie afgeleide analytische methode wordt getest aan de hand van een aantal representatieve modellen van een gelaagde aardbodem.

Uitgaande van de vergelijkingen van Maxwell wordt, door achtereenvolgens een Laplacetransformatie naar de tijd en een Fouriertransformatie naar de horizontale plaatscoordinaten uit te voeren, een stelsel gewone lineaire differentiaalvergelijkingen verkregen. Hierin is de vertikale plaatscoördinaat de onafhankelijk veranderlijke en zijn de horizontale componenten van het elektrische en magnetische veld de afhankelijk veranderlijken. In Hoofdstuk 3 wordt aangetoond dat - voor een homogene medium - de oplossing van dit stelsel differentiaalvergelijkingen geschreven kan worden als de som van twee opwaarts danwel twee neerwaarts diffunderende elektromagnetische velden, afhankelijk van de positie van het punt van waarneming ten opzichte van de lokatie van de bron. De terugtransformatie van deze oplossing naar het ruimtetoijddomein wordt uitgevoerd met een aangepaste versie van de gewijzigde methode van Cagniard. Met deze methode, die reeds met succes werd gebruikt bij de bestudering van de propagatie van gepulste golven in gelaagde verliesvrije media, wordt de inverse Fouriertransformatie naar het ruimte-Laplacedomein op een zodanige wijze herschreven, dat de resulterende uitdrukking herkend kan worden als een standaardvorm van de Laplace-integraal. Op deze manier worden voor de elektromagnetische veldcomponenten integraaluitdrukkingen verkregen die op een eenvoudige wijze kunnen worden geëvalueerd.

In Hoofdstuk 4 wordt de theorie uitgebreid tot transiënte elektromagnetische velden in gelaagde anisotrope media. De randvoorwaarden die voor het elektrische en magnetische veld ter weerszijden van een scheidingsvlak tussen twee media met verschillende elektromagnetische eigenschappen gelden, leiden tot de invoering van reflectie- en transmissiecoëfficiënten. De koppeling tussen de op- en neerwaarts diffunderende elektromagnetische velden ter weerszijden van een scheidingsvlak komt in deze coëfficiënten tot uitdrukking. Op het niveau van de bron leiden de randvoorwaarden tot
extra brontermen, die de koppeling tussen de bron en de op- en neerwaarts diffunderende elektromagnetische velden tot uitdrukking brengen.

Door vervolgens op recursieve wijze opeenvolgende reflecties en transmissies in de beschouwing te betrekken, is het mogelijk een uitdrukking voor de elektromagnetische veldcomponenten af te leiden die zodanig van bouw is, dat dat de terugtransformatie naar het ruimte-tijddomein kan worden uitgevoerd met behulp van de in Hoofdstuk 3 behandelde aangepaste versie van de gewijzigde methode van Cagniard.

De in de Hoofdstukken 3 en 4 afgeleidde theorie is gebaseerd op een gepulste excitatie van het elektromagnetische diffusieveld via een elektrische of magnetische dipool, dat wil zeggen een bron van oneindig kleine afmetingen. In Hoofdstuk 5 wordt de theorie uitgebreid tot de gepulste excitatie van het elektromagnetische diffusieveld via een rechthoekige draadlus van eindige afmetingen. Door de eindige afmetingen van de bron reeds in het ruimte-Laplace-domein in rekening te brengen worden, voorafgaand aan de terugtransformatie naar het ruimte-tijddomein, uitdrukkingen voor de componenten van het elektrische en magnetische veld verkregen waarin alleen bijdragen van de vier hoekpunten van de draadlus voorkomen. De vier resulterende uitdrukkingen kunnen met de in Hoofdstuk 3 behandelde aangepaste versie van de gewijzigde methode van Cagniard terugtransformeerd worden naar het ruimte-tijddomein. Op deze wijze is aanzienlijk minder rekentijd nodig om de respons van een gelaagd anisotoop medium ten gevolge van de gepulste excitatie door middel van een draadlus uit te rekenen dan op de, uit de literatuur bekende, conventionele manier waarbij de integratie van elementaire dipooldraven langs de draadlus wordt uitgevoerd in het ruimte-tijddomein.

In Hoofdstuk 6 wordt nagegaan welke vereenvoudigingen in de theorie mogelijk zijn indien de materialen elektromagnetisch isotrop zijn. Het blijkt, dat voor alle in het Laplace-Fourierdomein voorkomende grootheden exacte en eenvoudige analytische uitdrukkingen worden verkregen. Voor een aantal eenvoudige configuraties blijkt, dat de terugtransformatie naar het ruimte-tijddomein expliciet kan worden uitgevoerd. In deze gevallen kunnen exacte uitdrukkingen voor de desbetreffende veldgrootheden worden verkregen. Tenslotte wordt een vergelijking gemaakt tussen de responsies van de aarde op de gepulste excitatie via achtereenvolgens een vierkante draadlus en via een cirkelvormige draadlus.

De in dit proefschrift beschreven analytische methode ter berekening van het tran-
Siënte elektromagnetische diffusieveld is vervolgens numeriek geïmplementeerd. In Hoofdstuk 7 worden voor een aantal voor de exploratiegoofysica relevante configuraties de verkregen numerieke resultaten gepresenteerd en geanalyseerd. Achtereenvolgens komen aan de orde:

- Isotrope media. Specifieke aandacht wordt besteed aan de invloed van karakteristieke parameters zoals elektrische soortelijke geleiding en laagdikte op de responsie van een gelaagde isotrope aarde. Onderscheid wordt gemaakt tussen het geval dat bron en ontvanger op het scheidingsvlak tussen lucht en aarde gelegen zijn en het geval dat bron en ontvanger op het scheidingsvlak tussen zeewater en zeebodem gelegen zijn.

- Transversaal isotrope media, waarbij de tensoriële elektrische soortelijke geleiding twee verschillende hoofdwaarden heeft. Het blijkt, dat onder bepaalde omstandigheden een responsie wordt verkregen die tot op heden veelal eerder aan geïnduceerde polarisatieffecten of reflectieverschijnselen dan aan anisotropie werd toegeschreven.

- Willekeurig anisotrope media, waarbij de tensoriële elektrische soortelijke geleiding drie verschillende hoofdwaarden heeft. Dit is de meest algemene vorm van anisotropie. De resultaten die voor deze klasse van anisotrope media zijn verkregen, vertonen geen nieuwe aspecten ten aanzien van die transversale isotrope media.

Op basis van de analytisch en numeriek verkregen resultaten is een kwalitatieve relatie afgeleid tussen de componenten van de tensoriële elektrische geleiding en de responsie van een homogene anisotrope halfruimte.
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During his studies he received practical training for one month at the Municipal Electricity Board of Amsterdam in 1982, and for three months at the P.T.T. Dr. Neher Research Laboratory, Leidschendam, The Netherlands, in 1986. Here, he carried out (experimental) research on optical fiber couplers.

In May 1987 Leendert Combee started his Ph.D.-research on transient diffusive electromagnetic fields, carried out in the Laboratory of Electromagnetic Research, Faculty of Electrical Engineering at Delft University of Technology, under the supervision of Prof.dr.ir. A.T. de Hoop. This project ultimately led to the present Ph.D.-thesis.

From July till November 1990, he joined the Electromagnetics Department of Schlumberger-Doll Research, Ridgefield, Connecticut 06877-4108, U.S.A., to carry out further research on the project and to implement the computer code he had developed for performing the numerical computations. He has presented his work
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Besides carrying out his research, Leendert took an active part in the organisation of employment conditions of the "Assistenten in Opleiding" (Dutch Ph.D.-students) at the Delft University of Technology. Since May 1989 he is a committee member of the Delft Association of Graduate Students (Delfts Assistenten in Opleiding Overleg). From November 1989 till July 1991 Leendert acted as a chairman of this association. In relation to this position, he took part in the International Eurodoct'91 Conference held in France in April 1991.

Among Leendert's hobbies are traveling, reading, classical music and opera. However, most of his spare time is dedicated to (alpine) rock climbing. Numerous weekends are spent at climbing areas in Belgium. He has been climbing in most European countries and in the U.S.A.